OPERATIONS RESEARCH CENTER

working paper

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

THE GENERAL N-LOCATION
DISTRIBUTION PROBLEM

by

Nitin Patel and Uday S. Karmarkar

024-73 September, 1973

Supported in part by the Office of Naval Research
under contract 67-A-0204-0076

ABSTRACT

This paper studies the one-period, general network distribution problem with linear costs. The approach is to decompose the problem into a transportation problem that represents a stocking decision, and decoupled newsboy problems that represent the realization of demand with the usual associated holding and shortage costs. This approach leads to a characterization of optimal policies in terms of the dual of the transportation problem. Specifically, it is shown that there is a correspondence between the optimal policies and the extreme points, edges, faces etc. of the dual feasible region. This method is not directly suitable for the solution of large problems but the exact solution for small problems can be obtained. It is shown that the three location case involves 37 policies as compared to seven for the two location case. For the numerical solutions of large problems, the problem has been formulated as a linear program with column generation. This latter approach is quite robust in the sense that it is easily extended to incorporate capacity constraints and the multiproduct case. Extensions of this work are briefly discussed.

Introduction :

In this paper we consider the single product, single period, multilocation inventory problem with stochastic demands and transshipment between locations. This problem was posed and investigated by Gross [1], where exact solutions were obtained for the one location and two location cases. Gross' method of solution rapidly becomes complicated to the point of intractability as the number of locations increases, and Gross suggests that search techniques be used to obtain numerical solutions for larger problems.

Krishnan and Rao in [2] have tackled a one-period problem similar to that proposed by Gross. However while Gross' formulation considered ordering and shipping decisions made simultaneously at the start of the period, the approach here was to determine optimal ordering decisions given that transshipment decisions could be deferred till demand was realized. An additional simplification made in this paper was to assume that all transshipment costs are equal. This allows arbitrary partitioning of the locations into groups with the same transportation cost still obtaining between any two groups.

We will here examine the problem as formulated by Gross, and using an alternative approach provide efficient methods for characterising the solutions and obtaining exact and approximate solutions.

The Problem :

The problem can be stated as :

$$\min_{\{O_i\}\{x_{ij}\}} \Phi = \sum_i p_i O_i + \sum_i \sum_{j \neq i} c_{ij} x_{ij} + \sum_i \phi_i(s_i) \qquad (1)$$

$$i = 1, 2, \ldots, n$$
$$j = 1, 2, \ldots, n$$

where :

$$\phi_i(s_i) = c_{oi} \int_0^{s_i} (s_i - \xi) f_i(\xi) d\xi + c_{ui} \int_{s_i}^{\infty} (y - s_i) f_i(\xi) d\xi \qquad (1a)$$

Subject to:

$$S_i = S_i^0 + O_i + \sum_j x_{ji} - \sum_j x_{ij} \quad ; \forall i \qquad (2a)$$

$$S_i, O_i, x_{ij} \geq 0 \quad , \forall i, j \qquad (2b)$$

where :

S_i^0 : Initial inventory position at warehouse i.

o_i : Quantity ordered from central location for warehouse i.

x_{ij} : Quantity transshipped from warehouse i to warehouse j.

S_i : Total quantity stocked at warehouse i after ordering and transshipment.

p_i : Variable cost of ordering for warehouse from central location. ($/unit)

c_{ij} : Transshipment cost from warehouse i to warehouse j ($/unit).

C_{oi} : Cost of overstocking at warehouse i ($/unit)

C_{ui} : Cost of understocking at warehouse i ($/unit).

$f_i(\cdot)$: Distribution of demand at warehouse i.

The assumptions made in this model are as follows :

a) The demand at each warehouse is a continuous random variable characterised by a continuous density function.

b) Delivery of stock is immediate.

c) Setup costs of ordering and transshipping are negligible.

d) Inventory cannot be disposed of or salvaged.(This assumption is easily relaxed).

e) Purchasing, transshipping , holding and shortage costs are all linear. With respect to the last two we may make the weaker assumption that the one-period costs are convex.

f) There are no capacity restrictions on warehouses.

g) There is no restriction on the amount of supply available from the central location.

It is also assumed in [1] that

i) $c_{ij} = c_{ji}$

ii) $p_i + c_{ij} > p_j \quad ; \quad \forall i, j \; ; \; i \neq j$

The last restriction is the so-called 'triangular restriction' which says that it is always cheaper to order directly at any location, rather that ordering at another location and transshipping. It turns out that this assumption really leads to the most general case in terms of the number of different optimal policies involved, and so we will employ it.

Let us first write down the Kuhn-Tucker conditions for the problem. We note that since the problem involves minimising a convex function over a convex set, these conditions are necessary and sufficient.

i)
$$\phi_i'(s_i^*) + \lambda_i^* \geq 0 \qquad \forall i \qquad (3a)$$

$$p_i - \lambda_i^* \geq 0 \qquad \forall i \qquad (3b)$$

$$c_{ij} + \lambda_i^* - \lambda_j^* \geq 0 \qquad \forall i,j,\ i \neq j \qquad (3c)$$

ii)
$$[\phi'(s_i^*) + \lambda_i^*]\, s_i^* = 0 \qquad \forall i \qquad (4a)$$

$$[p_i - \lambda_i^*]\, o_i^* = 0 \qquad \forall i \qquad (4b)$$

$$[c_{ij} + \lambda_i^* - \lambda_j^*]\, x_{ij}^* = 0 \qquad \forall i,j;\ i \neq j \ (4c)$$

iii)
$$s_i^* - s_i^o - o_i^* + \sum_j x_{ij}^* - \sum_j x_{ji}^* = 0 \ ; \forall i \qquad (5a)$$

$$s_i^*,\ o_i^*,\ x_{ij}^* \geq 0 \ , \quad \lambda_i^* \text{ u.i.s.} \qquad (5b)$$

We now reformulate the problem by separating the objective function into a linear part and a non-linear stochastic part. The non-linear part can be decoupled into independent newsboy problems and the linear partforms a transshipment problem (with the triangular restriction, this is a transportation problem).

NLP :
$$\left.\begin{array}{l} \underset{\{s_i\}}{\text{Min}} \quad \gamma_i s_i + \phi_i(s_i) \\ \qquad\qquad s_i \geq 0 \end{array}\right\} \quad \forall i, \; i = 1, 2, \ldots, n. \tag{6}$$

LP
$$\underset{\{o_i\}\{x_{ij}\}}{\text{Min.}} \quad \sum_i p_i o_i + \sum_i \sum_{j \neq i} c_{ij} x_{ij} \tag{7}$$

$$\left.\begin{array}{ll} \text{S.T :} & o_i + \sum_j x_{ji} - x_{ij} = \bar{S}_i - S_i^{\circ}, \quad \forall i \\ & o_i, \; x_{ij} \geq 0 \end{array}\right. \tag{7a}$$

Now we can write the Kuhn-Tucker conditions for each of this pair of problems and show that under certain obvious restrictions they are equivalent to the optimality conditions for the original problem.

NLP :
$$\phi_i'(s_i^*) + \gamma_i \geq 0 \quad ; \; \forall i \tag{8a}$$
$$[\phi_i'(s_i^*) + \gamma_i]\, s_i^* = 0 \; , \; \forall i \tag{8b}$$
$$s_i^* \geq 0 \qquad \forall i \tag{8c}$$

LP : i) Primal Constraint
$$\bar{S}_i - S_i^{\circ} + o_i^* + \sum_j x_{ij}^* - \sum_j x_{ji}^* = 0 \; ; \; \forall i \tag{9}$$

ii) Dual Constraints :
$$\pi_i^* = p_i \qquad ; \; \forall i \tag{10a}$$
$$\pi_j^* - \pi_i^* \leq c_{ij} \qquad ; \; \forall i, j \; ; \; i \neq j \tag{10b}$$

iii) Complementary Slackness :

$$[p_i - \pi_i^*]\, o_i^* = 0 \qquad \forall i \qquad (11a)$$

$$[c_{ij} + \pi_i^* - \pi_j^*]\, x_{ij}^* = 0 \qquad \forall i,j \,;\, i \neq j \qquad (11b)$$

$$o_i^*, x_{ij}^* \geq 0 \quad ; \quad \pi_i^* \ \text{u.i.s.} \qquad (11c)$$

Now if we add to these conditions the requirements that $\gamma_i = \pi_i^*$ and $\bar{S}_i = S_i^*$, we see that these conditions are identical to the Kuhn-Tucker conditions for the original problem. This immediately suggests that a naive algorithm for solving the problem might consist of alternately solving the two subproblems; using the NLP to generate target inventory levels $\bar{S}_i$ for the LP, while the LP determines shadow prices π_i that are then used in the NLP as the marginal cost of purchase γ_i. However it turns out that such a procedure will not lead to the optimum. However before we discuss algorithmic procedures in any detail let us first draw attention to the dual of the transportation subproblem. We can write the dual as :

DLP:

$$\max_{\{\pi_i\}} \sum_i (\bar{S}_i - S_i^{\circ})\, \pi_i \qquad (12)$$

$$\text{s.t.} \qquad \pi_i \leq p_i \qquad \forall i$$

$$\pi_j - \pi_i \leq c_{ij} \qquad \forall i \neq j$$

$$\pi_i \ \text{u.i.s.}$$

We see that the dual has a very simple structure as might be expected. Interpreting π_i as the marginal cost of providing an extra unit at the i'th location gives the dual constraints an intuitive meaning. The first set of constraints says that the marginal price should be less than or equal to the variable cost of ordering from the central location. The second set says that the cost of providing an extra unit at the jth location should be less than or equal to the cost of providing that unit at the ith location and then transshipping it to the jth location at a cost c_{ij} .

To obtain some insight into the structure of the problem, we now examine the two location problem in some detail .

The Two-Location Problem :

Let us assume for the sake of simplicity and without loss of generality that the transshipment costs between the two locations are equal in either direction. We will also assume that the "triangular restriction" is operative, so that :

$$\begin{aligned} c_{12} &= c_{21} = c \\ p_1 &< p_2 + c \\ p_2 &< p_1 + c \end{aligned} \tag{13}$$

Now we can visualise the problem as setting target stock levels s_1 and s_2 so as to minimise the total costs of transshipment and subsequent realization of demand. We have to set target stock levels such that the total stock in the system is not less than the starting stock, since we cannot dispose of any stock. Hence the problem can be written as :

$$M_2 \left\{ \begin{array}{ll} \min\limits_{s_1, s_2} & w(s_1, s_2) = Z(s_1, s_2) + \phi_1(s_1) + \phi_2(s_2) \\ \text{s.t.} & s_1 + s_2 \geq s_1^0 + s_2^0 \\ & s_1, s_2 \geq 0 \end{array} \right.$$

where $Z(s_1, s_2) = \min\limits_{\substack{o_1, o_2 \\ x_{12}, x_{21}}} \quad c x_{12} + c x_{21} + p_1 o_1 + p_2 o_2$

$$P_2 \left\{ \begin{array}{llll} \text{s.t.} & o_1 & - x_{12} + x_{21} & = s_1 - s_1^0 \\ & o_2 & + x_{12} - x_{21} & = s_2 - s_2^0 \\ & x_{12}, x_{21}, o_1, o_2 \geq 0 . & & \end{array} \right.$$

The above is a restatement of the problem in the two location case which emphasises that the costs can be represented in terms of the target stock vector (s_1, s_2). This is made possible by the characterisationof the transshipment costs as the function $z(s_1, s_2)$. We shall now obtain this transshipment cost function explicitly by considering the dual of the problem P:

$$
(D)\quad \left\{
\begin{array}{lllll}
\text{Max} & (s_1 - s_1^\circ)\pi_1 & + (s_2 - s_2^\circ)\pi_2 & & \\
 & -\pi_1 & +\pi_2 & \leq & c \\
 & \pi_1 & -\pi_2 & \leq & c \\
 & \pi_1 & & \leq & p_1 \\
 & & \pi_2 & \leq & p_2 \\
 & \pi_1, \pi_2 & \text{u.i.s.} & &
\end{array}
\right.
$$

The feasible region to this program is shown in Figure 1. Given a target stock vector $\underset{\sim}{s} = (s_1, s_2)$ we wish to find a feasible pair (π_1, π_2) so as to maximise the objective function. That is to say, to maximise the projection on the gradient vector of the dual objective function.

We can see by direct inspection of the dual feasible region that :

1) For $s_1 - s_1^\circ > 0$, $s_2 - s_2^\circ > 0$, (p_1, p_2) is the optimal point and

$$z(s_1, s_2) = p_1(s_1 - s_1^\circ) + p_2(s_2 - s_2^\circ)$$

2) For $s_1 - s_1^\circ < 0$; $s_1 - s_1^\circ + s_2 - s_2^\circ > 0$, $(p_2 - c, p_2)$ is the optimal point and

$$z(s_1, s_2) = (p_2 - c)(s_1 - s_1^\circ) + p_2(s_2 - s_2^\circ)$$

3) For $s_2 - s_2^\circ < 0$; $s_1 - s_1^\circ + s_2 - s_2^\circ > 0$; $(p_1, p_1 - c)$ is the optimal point and

$$z(s_1, s_2) = p_1(s_1 - s_1^\circ) + (p_1 - c)(s_2 - s_2^\circ)$$

These represent the extreme points of the feasible region and we get an expression for the objective function in each case, in terms of (s_1, s_2). We can also say something about the cases where the objective function is parallel to an edge of the feasible region:

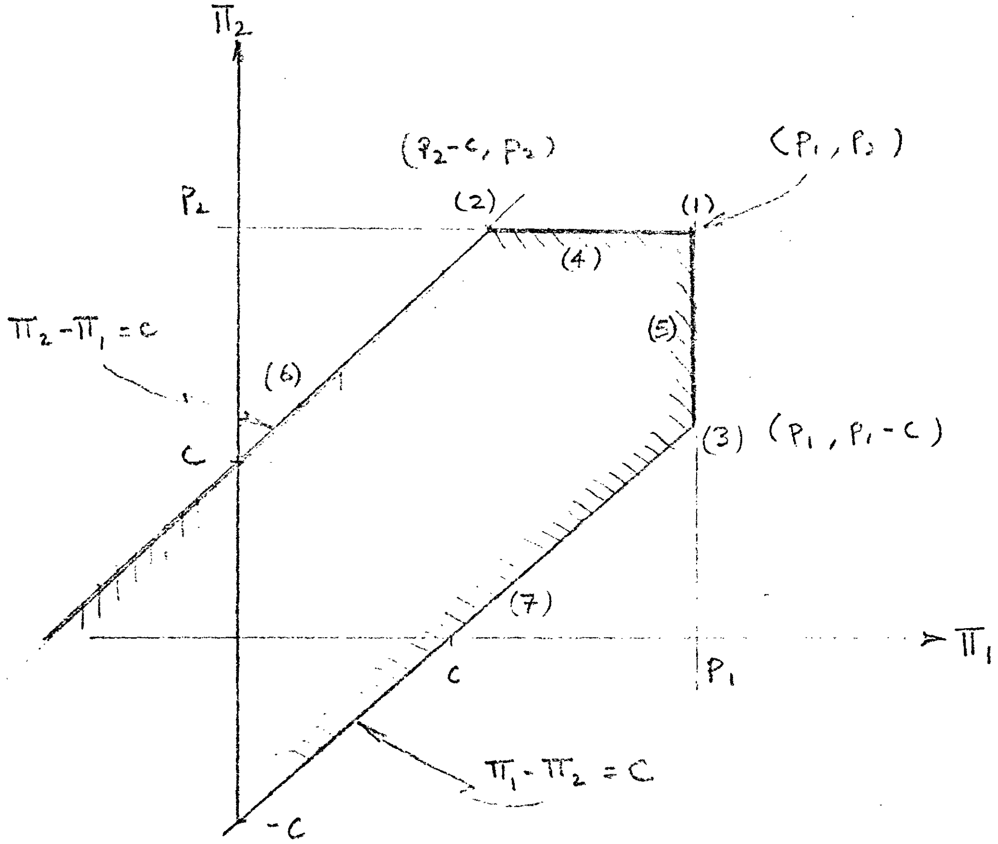

Figure 1 : The Two-Location Case
Feasible region for the Dual Problem

4) For $s_1 - s_1^o = 0, \quad s_2 - s_2^o > 0$ we have $\pi_2 = p_2 \quad , \quad p_2 - c \leq \pi_1 \leq p_1$

5) For $s_1 - s_1^o > 0 \quad , \quad s_2 - s_2^o = 0$ we have $\pi_1 = p_1 \quad ; \quad p_1 - c \leq \pi_2 \leq p_2$

6) For $s_1 - s_1^o + s_2 - s_2^o = 0 \quad ; \quad s_1 - s_1^o < 0 \ , \ s_2 - s_2^o > 0$ wehave that

$$\pi_2 - \pi_1 = c \quad ; \quad \pi_1 \leq p_2 - c \quad ; \quad \pi_2 \leq p_2$$

7) For $s_1 - s_1^o + s_2 - s_2^o = 0 \quad ; \quad s_1 - s_1^o > 0 \ , \ s_2 - s_2^o < 0$ we have that

$$\pi_1 - \pi_2 = c \quad , \quad \pi_1 \leq p_1 \quad ; \quad \pi_2 \leq p_1 - c$$

and finally:

8) For $(s_1 - s_1^o + s_2 - s_2^o) < 0$ we have an unbounded solution to (D)corresponding to an infeasible primal .

We can represent this information about the dual variable values on the (s_1, s_2)plane. This has been shown in Figure 2. The feasible region for the original problem M is the portion of the (s_1, s_2)plane such that $s_1 + s_2 \geq s_1^o + s_2^o \ ; \ s_1, s_2 \geq 0$. The regions of the boundary of the dual feasible region correspond to regions in the (s_1, s_2) plane and have been marked correspondingly with the appropriate (π_1, π_2) values shown.

We note the geometrical nature of the correspondence between the solutions to (D) and the (s_1, s_2)plane:
The extreme points 1,2 and 3 of the dual correspond to the interiors of regions 1,2 and 3 in the s-plane.
The edges of the dual correspond to boundaries between regions in the s-plane.
And finally, the starting stock position which is point 0 in the s-plane corresponds to the degenerate case of a dual objective function which is identically zero so that all points feasible in (D) are also optimal.
Thus we can view this correspondence intuitively and geometrically as point to plane, line to line, and plane to point.

If we now examine the original problem M, we see that it consists of minimising a convex function over a convex set. Thus a unique optimum exists which can fall in any of the regions marked in the s-plane,

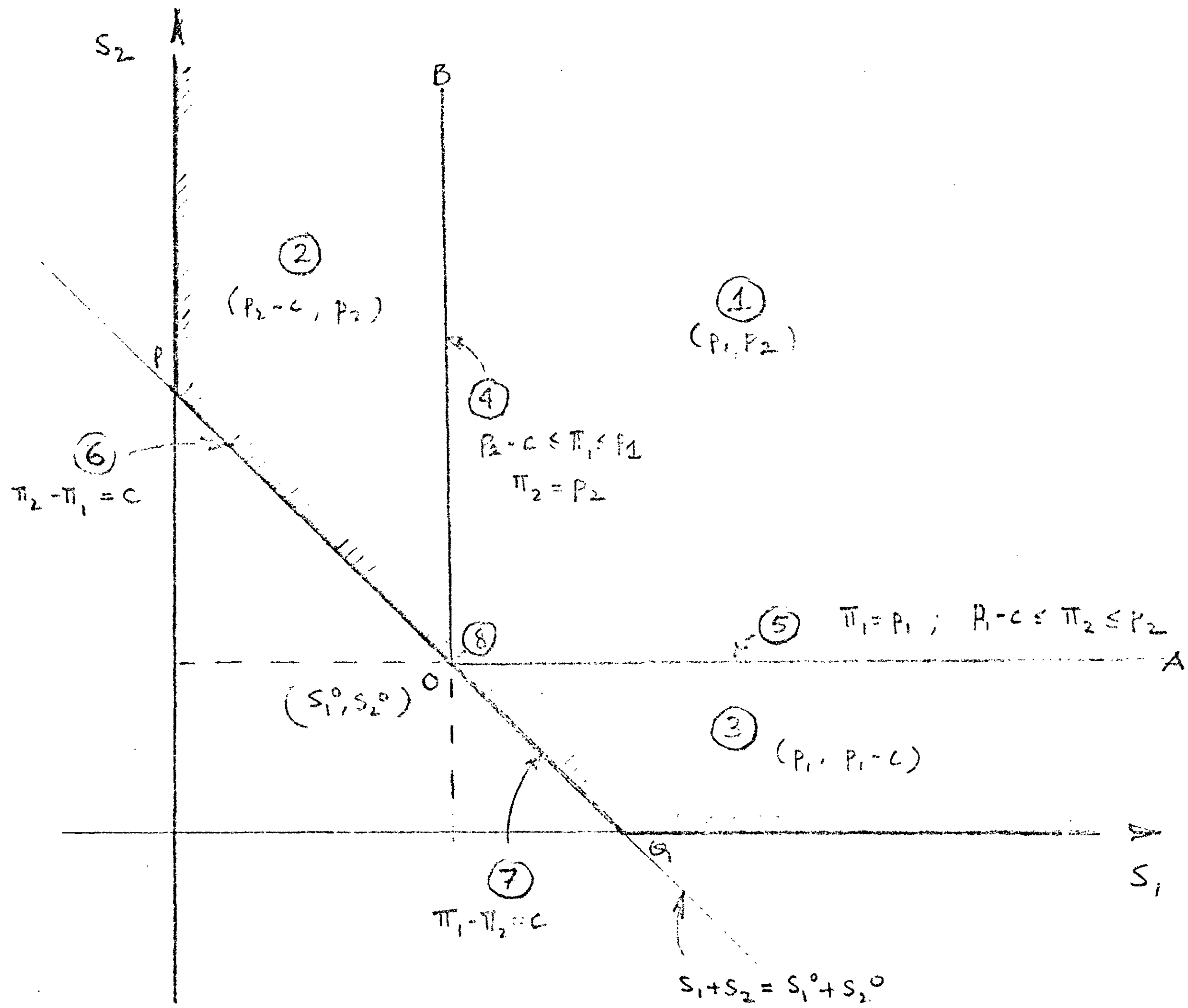

Figure 2: The Two-Location Case
Policy Regions in the (s_1, s_2) plane.

giving an optimal policy corresponding to each of the regions. Since these regions arise in the process of characterising the transportation cost as the function $z(s_1,s_2)$, the number of policies involved depends on this function, which as we saw depended on the geometrical features of the feasible region of (D). The number of distinct forms of optimal policy can thus be determined simply by examining the feasible region.

One method of explicitly determining these policies is to visualise the problem as starting at point 0 in Figure 2 with a stock position (s_1°, s_2°) and moving away from this point in a feasible direction to a new stock level that minimises the total cost $w(s_1, s_2)$. The point 0 communicates with all seven regions and it is worth moving away from it in the direction of one of these regions if the cost in that direction is decreasing. We can clearly use gradient arguments to characterise the policies, especially when we note that the derivative of the cost function in the direction of one of the axes, is independent of the other variable ; that is to say

$$\nabla w(s_1, s_2) = (\phi_1'(s_1) + \pi_1 \quad , \quad \phi_2'(s_2) + \pi_2)$$

where π_1 and π_2 depend on the region in which $\underline{\nabla w}$ is evaluated. We may also note that

a) For $s_1 > s_1^\circ$, $\pi_1 = p_1$; $s_1 < s_1^\circ$, $\pi_1 = p_2 - c$

b) For $s_2 > s_2^\circ$, $\pi_2 = p_2$; $s_2 < s_2^\circ$, $\pi_2 = p_1 - c$

It is thought that this approach will prove useful in larger problems in making numerical computations.However, to determine the exact nature of the optimal policies in this case, it is simplest to directly apply the Kuhn-Tucker optimality conditions to each region, noting that the associated values of the dual variables already satisfy conditions(9) through (11). So we have to consider only conditions (8a)-(8c) which effectively imply that either $s_i = 0$ or $\phi_i'(s_i) + \pi_i = 0$

We will assume merely for convenience that $s_i \neq 0$ although the reader may keep in mind that an optimal stock level of zero is a theoretical possibility.

Also for notational ease, we define $y = S_i^*(p)$ to be the solution of

$$\phi_i'(y) + p = 0$$

We also note that since $\phi_i(\cdot)$ is convex, $\phi_i'(\cdot)$ is a non-decreasing function of its argument and it thus follows that $S_i^*(p)$ is a non-increasing function of p . (This simply says that as the marginal cost of supplying a location increases, the optimal stock level at that location decreases).

Now applying the K-T conditions to each of the regions, we have :

Region 1 : The region is defined by $S_1 > S_1^0$, $S_2 > S_2^0$. Furthermore we have:

$$\phi_1'(S_1^*) + P_1 = 0 \quad ; \quad \phi_2'(S_2^*) + P_2 = 0$$

Therefore the optimal policy is

$$S_1^* = S_1^*(P_1) \quad ; \quad S_2^* = S_2^*(P_2)$$

and this policy applies when

$$S_1^0 < S_1^*(P_1) \quad ; \quad S_2^0 < S_2^*(P_2)$$

Region 2 : We have $\phi_1'(S_1^*) + (P_2 - c) = 0$; $\phi_2'(S_2^*) + P_2 = 0$

which imply the optimal policy

$$S_1^* = S_1^*(P_2 - c) \quad ; \quad S_2^* = S_2^*(P_2)$$

This policy is optimal for starting stock (S_1^0, S_2^0) such that

$$S_1^0 > S_1^*(P_2 - c) \quad ; \quad S_2^0 < S_2^*(P_2)$$

$$S_1^0 + S_2^0 < S_1^*(P_2 - c) + S_2^*(P_2)$$

Region 3 : As for region 2 : $\phi_1'(S_1^*) + P_1 = 0$; $\phi_2'(S_2^*) + (P_1 - c) = 0$

The optimal policy is

$$S_1^* = S_1^*(P_1) \quad ; \quad S_2^* = S_2^*(P_1 - c)$$

This policy applies if

$$S_1^0 < S_1^*(P_1) \quad ; \quad S_2^0 > S_2^*(P_1 - c)$$

$$S_1^0 + S_2^0 < S_1^*(P_1) + S_2^*(P_2 - c)$$

Region 4 : We have $S_1^* = S_1^0$ and since $\phi_2'(S_2^*) + P_2 = 0$ we have that $S_2^* = S_2^*(P_2)$

Now we know that the optimal value of the dual variable π_1 must be such that $P_2 - c \le \pi_1 \le P_1$. Hence this policy will apply when

$$\phi_1'(s_1^\circ) + \pi_1 = 0 \quad , \quad s_2^\circ < s_2^*(p_2)$$

$$s_1^*(p_1) \leq s_1^\circ \leq s_1^*(p_2 - c)$$

Region 5: As in region 4, the optimal policy is :

$$s_1^* = s_1^*(p_1) \quad ; \quad s_2^* = s_2^\circ$$

This policy applies when

$$s_1^\circ < s_1^*(p_1) \quad ; \quad s_2^*(p_2) \leq s_2^\circ \leq s_2^*(p_1 - c)$$

And the optimal value of the dual variable is given by

$$\phi_2'(s_2^\circ) + \pi_2 = 0$$

Region 6 : The region consists of points in the feasible region such that

$$s_1 < s_1^\circ \quad , \quad s_2 > s_2^\circ \quad ; \quad s_1 + s_2 = s_1^\circ + s_2^\circ$$

For the dual variables, we have that

$$\pi_1 \leq p_2 - c \quad , \quad \pi_2 \leq p_2 \quad ; \quad \pi_2 - \pi_1 = c$$

and we have the conditions

$$\phi_1'(s_1^*) + \pi_1 = 0 \quad ; \quad \phi_2'(s_2^*) + \pi_2 = 0$$

From the conditions on the dual variables, we have that the policy will apply when

$$s_1^\circ \geq s_1^*(p_2 - c) \quad ; \quad s_1^\circ + s_2^\circ \geq s_1^*(p_2 - c) + s_2^*(p_2)$$

The optimal policy is given by s_1^* and s_2^* such that

$$\phi_1'(s_1^*) - \phi_2'(s_2^*) = c$$

and

$$s_1^* + s_2^* = s_1^\circ + s_2^\circ$$

Region 7 : The analysis here is similar to Region 6 .The optimal policy is given by

$$\phi_2'(s_2^*) - \phi_1'(s_1^*) = c$$

$$s_1^* + s_2^* = s_1^\circ + s_2^\circ$$

This policy is optimal for starting stocks such that

$$s_2^\circ \geq s_2^*(p_1 - c) \quad ; \quad s_1^\circ + s_2^\circ \geq s_2^*(p_1 - c) + s_1^*(p_1)$$

The various optimal policy regions correspond to conditions on the starting stocks. We can thus represent these conditions on the s-plane so that depending in which region the starting stock falls, we immediately know the optimal policy. This is done in Figure 3, and this diagram will be seen to be identical with the results obtained by Gross [1]. We have thus succeeded in recovering his results.

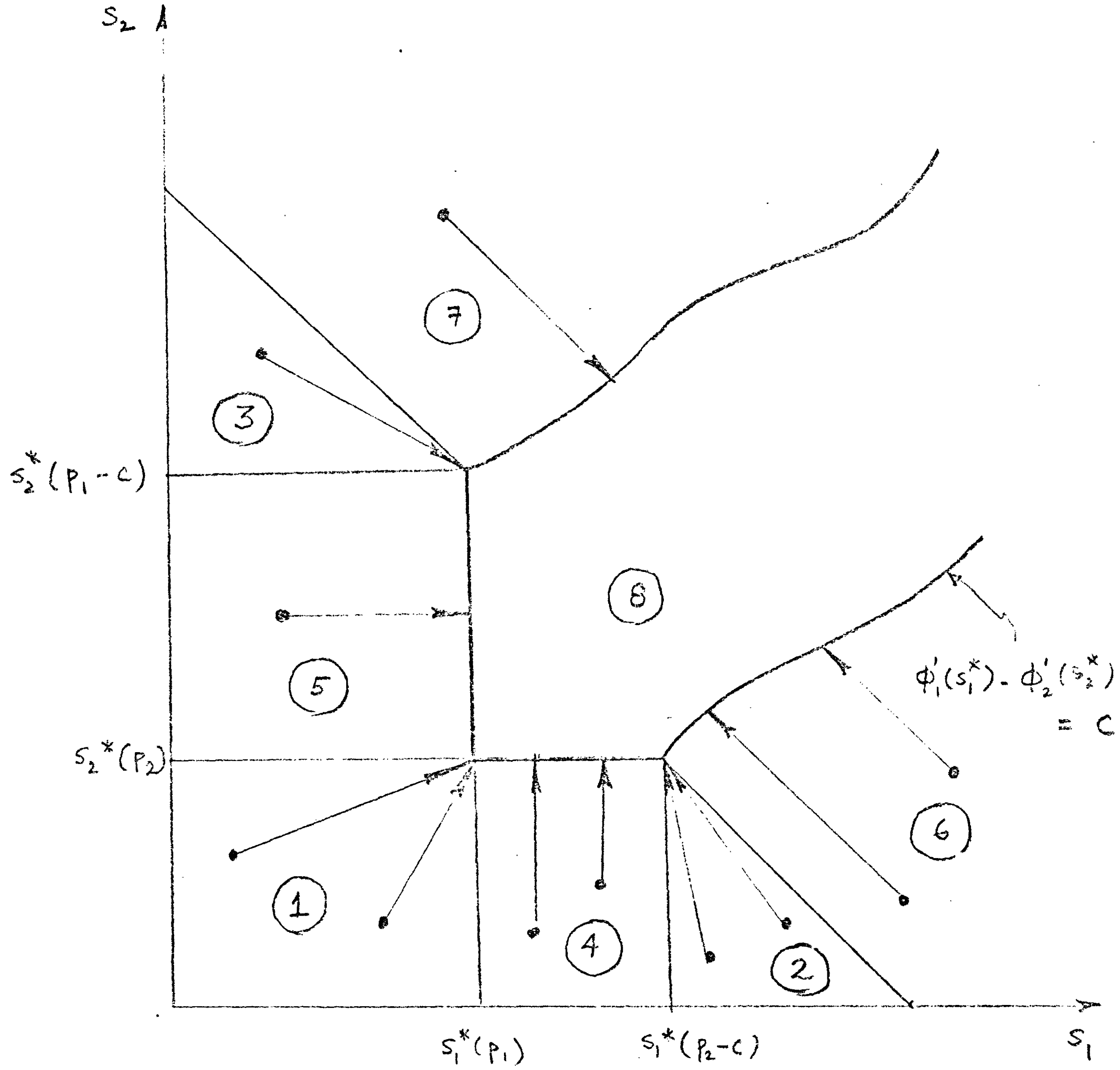

Figure 3: The Two-Location Case.
Optimal policies for different starting stock conditions.

The approach used here, while admittedly more devious than that of Gross, is essentially simpler and more intuitive. Furthermore it can be extended to larger problems and suggests algorithmic procedures. We will discuss below the three location case, and a computational method for attacking large problems.

The Three-Location Problem :

The extension of the results obtained to the three location case is straightforward but tedious. We will briefly indicate how this might be done without deriving detailed results.

As before we can write the dual of the associated transportation subproblem as : (D_3) :

$$\max_{\{\pi_i\}} \; (s_1 - s_1^0)\pi_1 + (s_2 - s_2^0)\pi_2 + (s_3 \;\; s_3^0)\pi_3$$

$$\left.\begin{array}{rrrl} \pi_1 & & & \leq p_1 \\ & \pi_2 & & \leq p_2 \\ & & \pi_3 & \leq p_3 \end{array}\right\} \quad (a)$$

$$\left.\begin{array}{rrrl} \pi_1 & - \pi_2 & & \leq c_{21} \\ -\pi_1 & + \pi_2 & & \leq c_{12} \\ & \pi_2 & - \pi_3 & \leq c_{32} \\ & -\pi_2 & + \pi_3 & \leq c_{23} \\ \pi_1 & & - \pi_3 & \leq c_{31} \\ -\pi_1 & & + \pi_3 & \leq c_{13} \end{array}\right\} \quad (b)$$

$$\pi_1, \pi_2, \pi_3 \quad \text{u.i.s.}$$

The equations in (a) represent mutually perpendicular planes, which are perpendicular to the axes, and intersect at (p_1, p_2, p_3). The six equations in (b) form a"pipe" or parallelpiped of hexagonal cross-section that intersects the cuboid formed by the other equations . The resulting feasible region is a convex polyhedron shaped rather like a pencil, bounded by its "point" in the positive orthant, and unbounded in the negative orthant. We may again assume that $c_{ij} = c_{ji}$ without loss of generality, and we may also assume that the "triangular restriction" holds so that

$$p_i + c_{ij} > p_j \qquad \forall\, i, j \;;\; i \neq j$$

The last assumption in fact leads to the most general case since it ensures that the feasible region includes the point (p_1, p_2, p_3), i.e. the parallel piped cuts the "corner" of the cuboid. This construc- has been sketched in Figure 4.

As in the two-location case, we can characterise the transportation cost function $z(s_1,s_2,s_3)$ by the extreme points, edges and faces of the feasible region. Again, a particular extreme point (or boundary point) will become optimal if the gradient of the objective function given by $(s_1-s_1^0, s_2-s_2^0, s_3-s_3^0)$ lies in the appropriate region.As before we can plot these regions in s-space (R^3) with a geometrical correspondence obtaining as follows:

Faces (planes) of the dual correspond to lines in the s-space, edges correspond to planes and extreme points correspond to volumes. It may also be noted that an edge in Figure 4 formed by the intersection of two faces (or the linear combination of two extreme points),corresponds in the s-space to the plane generated by the vectors corresponding to the two faces (or the boundary between the two volumes corresponding to the two extreme points.) These regions in s-space are shown in Figure 5.

Recall that in the two-location case, we obtained a policy for each point and edge of the dual feasible region giving a total of seven policies. It is easily verified that in the three location case we have a maximum possible 10 extreme points, 18 edges (of which six are extreme rays) and 9 faces (of which six are unbounded and form the parallelpiped) . Thus in this case there are 37 distinct policies to be considered. We will not here list the exact nature of

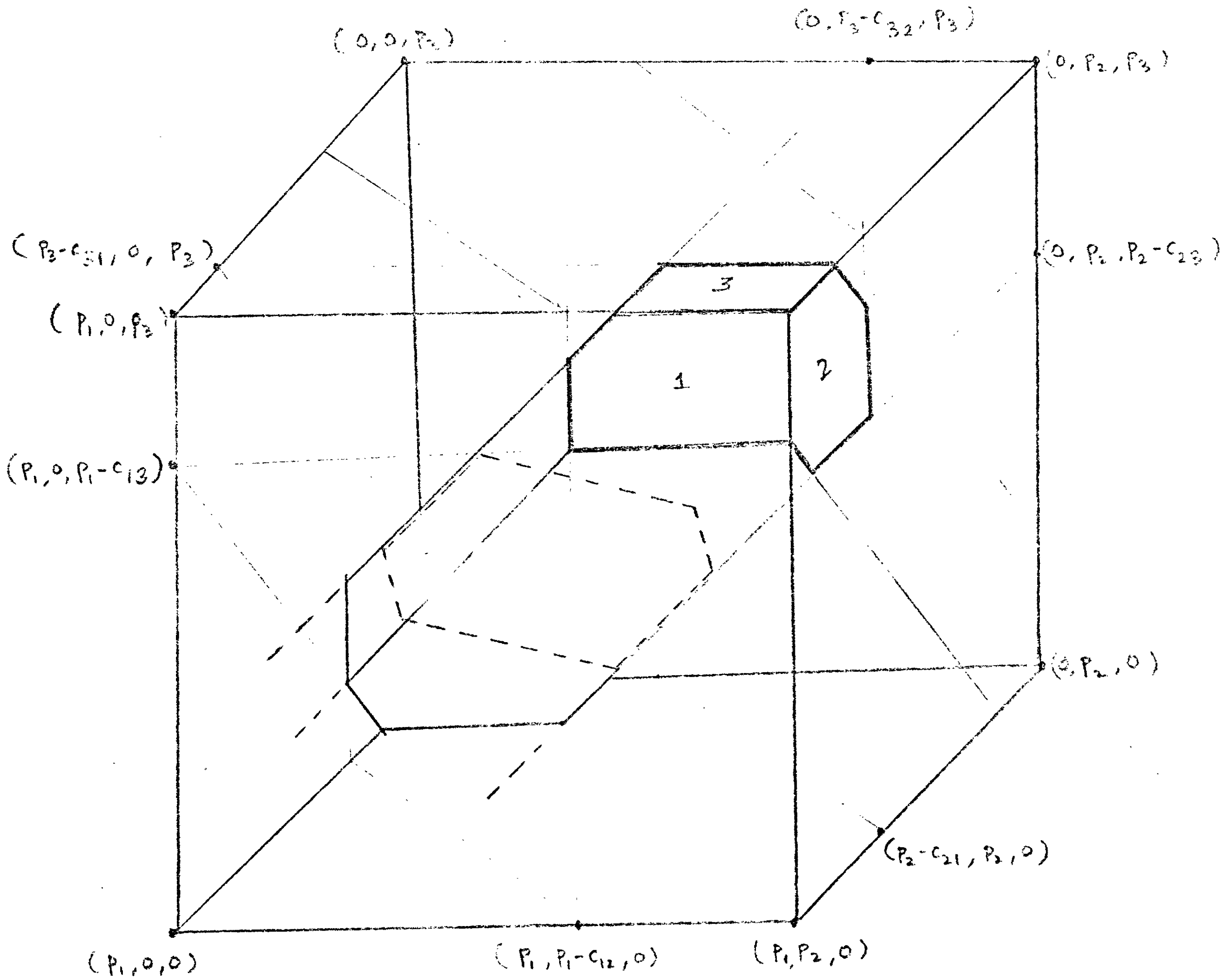

Figure 4 : The Three-Location Case
Feasible Region for the Dual Problem

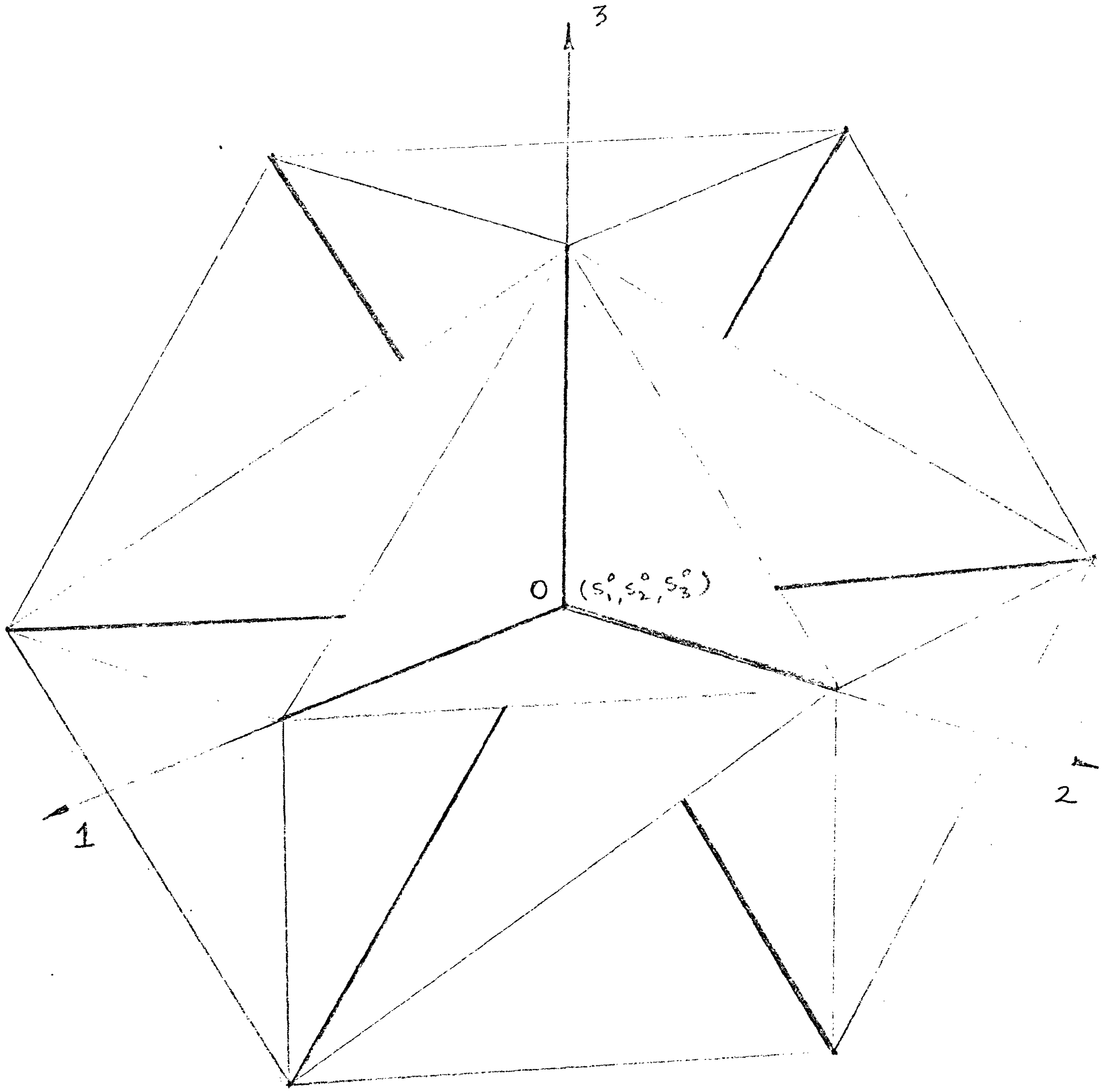

Figure 5 : The Three-Location Case

Policy Regions in the (s_1, s_2, s_3)

Computational Methods :

While the exact method used above has much to recommend it from a theoretical point of view, its practicability diminishes rapidly as the number of locations increases. For one thing geometric intuition is of little help beyond the cases already discussed and for another, the number of distinct policies to be considered becomes very large as the dimensionality of the problem increases. It would clearly be useful to have a method of computing the optimal solution for a given starting stock position without having to know the whole solution. Gross [1] has suggested the use of gradient search techniques involving n^2 variables. Presumably, more efficient search techniques could be devised using the structural properties of the problem as revealed in the analyses above. However this subject will be deferred to a later investigation.

We present below a brief description of a column generation (generalized programming) technique, which is an immediate consequence of the problem decomposition used in equations (6) and (7).

The subproblems are newsboy problems corresponding to each location, which generate proposals for the master problem. The proposals are in the form of columns with entries corresponding to a proposed target stock level for a location and the associated shortage and holding cost. These columns are then incorporated into the master problem, which is a suitably modified version of the transportation subproblem. The master problem then selects linear combinations of the proposal columns. Since the holding and shortage cost functions $\phi_i(s_i)$ are convex, the linear combination of costs from the columns' will overestimate the true costs involved. This essentially ensures that successive generation of columns will lead to better approximations of the cost functions $\phi_i(\cdot)$. As columns are generated, the linear combination of adjacent proposals will lead to an inner linear approximation of these cost functions. Since the master problem is one of minimisation, it will automatically choose adjacent proposals when forming linear combinations. This procedure is shown in Fig.6.

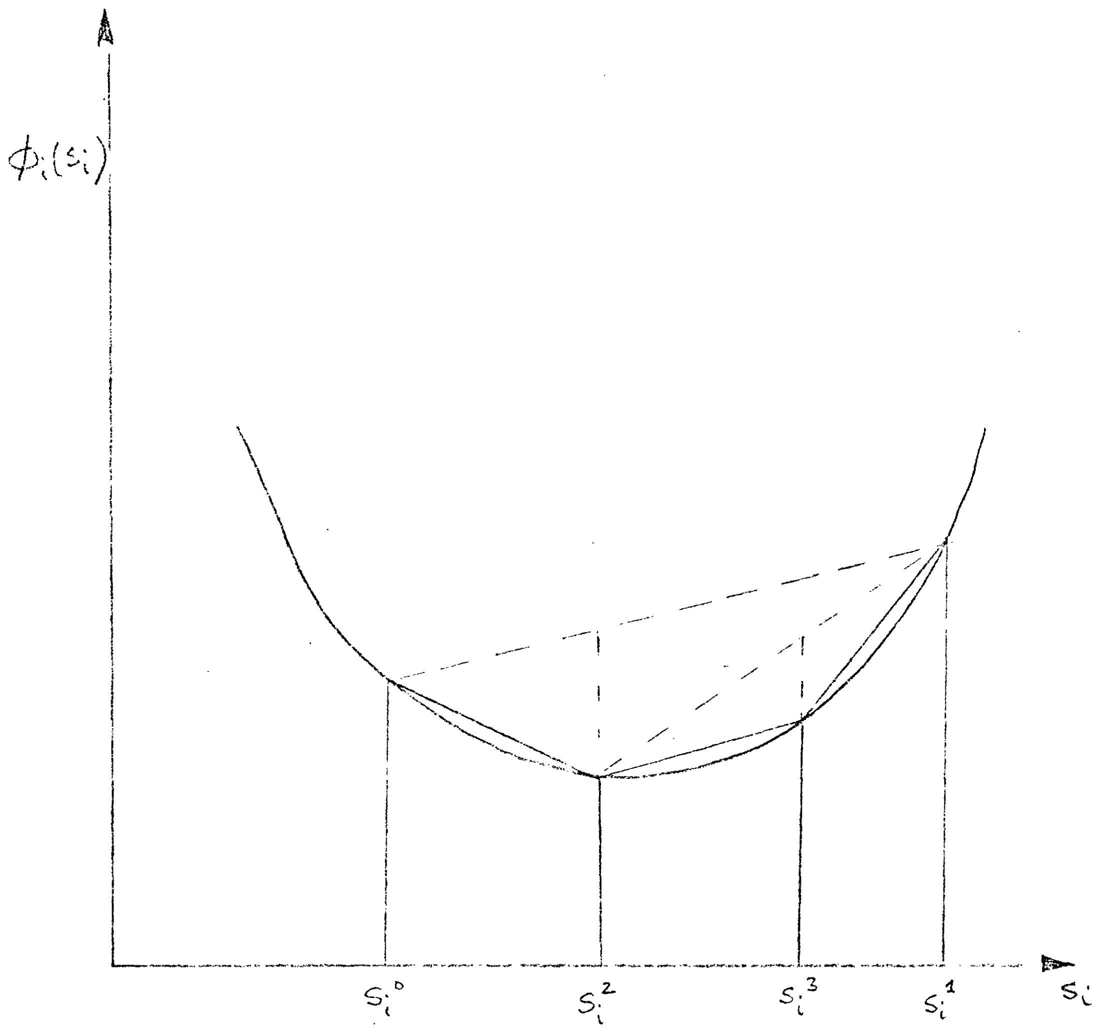

Figure 6 : Inner linear approximation of a subproblem cost function by successive columns.

The column generation procedure can be described as follows :

Initially we will generate n columns corresponding to the starting stocks at the n locations. This will guarantee the existence of a feasible solution to the master problem. The master problem at the t'th iteration is of the form :

$$\min_{\{x_{ij}\},\{o_i\},\{\lambda_{ik}\}} \sum_{i \neq j} c_{ij} x_{ij} + \sum_i p_i o_i + \sum_{i,k=1}^{t}\sum c_{ik} \lambda_{ik} \quad ; \quad \begin{matrix} i=1,\dots,n \\ j=1,\dots,n \end{matrix}$$

$$\text{s.t.} \quad \sum_j x_{ij} - \sum_j x_{ji} - o_i + \sum_{k=1}^{t} s_{ik} \lambda_{ik} = s_i^0 \quad : \pi_i$$

$$\sum_{k=1}^{t} \lambda_{ik} = 1 \quad : \mu_i$$

$$o_i , x_{ij} , \lambda_{ik} \geq 0$$

We are assuming here for the sake of simplicity that one column is generated for each subproblem at each iteration. π_i and μ_i are the shadow prices corresponding to the constraints as shown. With these shadow prices we solve n non-linear subproblems of the form

$$\min_{s_i} \phi_i(s_i) - \pi_i s_i - \mu_i \quad ; \quad s_i \geq 0 \quad ; \forall i$$

These are trivial to solve, and the solutions are then passed up to the master problem as columns of the form

$$\begin{bmatrix} c_{i(t+1)} \\ \hline 0 \\ \vdots \\ s_{i(t+1)} \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix}$$

The master is updated to include these columns and then solved again. The procedure is repeated till optimality is reached or till the solution is thought to be close enough to optimality. Optimality occurs when the same single column is picked twice by the master. For examining the nearness to optimality and upper bound on the cost can be established at any iteration, but this will not be discussed here.

There are several refinements possible in this procedure in the selection and incorporation of columns. For example we may want to immediately columns that are likely to be important corresponding to typical policies arising from extreme points of the dual. Examples of these are $s_i^*(p_i)$; $s_i^*(p_j - c_{ji})$. Since the master problem will only use two columns at a time, it may be possible to throw out some of the columns generated after f couple of iterations. These and other issues related to refinement of the procedure will also be deferred to later investigations.

Conclusions and Extensions :

We have examined a one period, general multilocation problem from both theoretical and computational points of view. The computational column generation method can be easily extended to consider the multiproduct capacitated case. It also appears that considerable improvement of the computational procedure is possible.

The major theoretical extensions to the problem that will be studied in future work are the multiperiod problem, the fixed charge case, and results for special network structures. From a computational point of view, apart from column generation methods and improved gradient search methods, a Markovian Decision Process formulation is under study. The latter would be able to handle multiperiod situations and also handle the fixed cost case. The major difficulty lies in reducing the magnitude of the problem to permit computational feasibility.

References :

[1] Gross, D.; "Centralised Inventory Control in Multilocation Supply Systems"; Chap.3 in H.Scarf, D.Gilford and M.Shelley(eds.), Multistage Inventory Models and Techniques; Stanford U.Press; Stanford, Calif.(1963).

[2] Krishnan,K.S.; V.R.K.Rao;"Inventory Control in N Warehouses"; J.IND.ENG., 16, 212-215 (1965).