

XXVII. NETWORK SYNTHESIS

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RESEARCH OBJECTIVES

To implement the methods of active and/or nonbilateral, as well as passive bilateral, network analysis and synthesis discussed in various articles in the Quarterly Progress Report for the past two years, in addition to the two publications mentioned in last year's objectives, several additional ones are pertinent (1, 2, 3), as are the two reports that follow.

Further interpretation of the theory discussed in these papers as well as formulation of procedures for their practical exploitation are needed in order to make these new synthesis methods useful. Our efforts for this year, as well as for several years to come, will be directed toward this goal.

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References

1. E. A. Guillemin, The normal coordinate transformation of a linear system with an arbitrary loss function, *J. Math. Phys.* 39, 97 (1960).
2. E. A. Guillemin, On the analysis and synthesis of single-element-kind networks, *Trans. IRE*, Vol. CT-8, September 1960.
3. E. A. Guillemin, Topology and linear transformation theory in active network synthesis (to be published in Vol. 10, Symposium on Active Networks and Feedback Systems, Polytechnic Institute of Brooklyn, 1960).

A. REALIZATION OF AN OPEN-CIRCUIT RESISTANCE MATRIX

The problem involved here is the dual of that discussed in Quarterly Progress Report No. 56, pages 213-235. There we were concerned with the realization of a short-circuit conductance matrix appropriate to equilibrium equations on a node basis. Now we want to develop a network realization procedure, given an open-circuit resistance matrix appropriate to equilibrium equations on a loop basis.

Terminal pairs or points of entry are of the "pliers type"; that is, they are created by cutting a set of ℓ links in the network graph. In contrast with the situation on a node basis, we cannot assume that we are always dealing with a full graph. Thus if n denotes the number of tree branches in a full graph involving $n + 1$ nodes, then for $n = 1, 2, 3, 4, 5 \dots$ we have, respectively, $\ell = 0, 1, 3, 6, 10 \dots$. For ℓ equal to some integer other than these particular ones, the network cannot be represented by a full graph.

Again, the solution to our problem depends upon our ability to construct the tree appropriate to a given resistance matrix. As in the analogous problem on a node

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basis, no solution exists unless the given matrix is appropriate to some tree configuration; and once that tree structure is determined, the total network and its branch resistance values are computed without difficulty. Existence of the tree, however, does not ensure that the branch resistances are all positive and hence (as on the conductance basis) it is a necessary, though not sufficient, condition for the existence of a solution.

When a tree exists, the total network graph (which is found by inserting the links) is unique only if a full graph is involved; that is, when $\ell = n(n-1)/2$ for integer n values. Otherwise, variants in the graph structure exist, although these are not very significant.

Again, we shall be able to construct the tree (if it exists) on the basis of the algebraic sign pattern in the given matrix; however, the procedure for accomplishing this end is now considerably more roundabout. The method consists of two steps: First, we develop a scheme whereby, from the sign matrix appropriate to the given resistance matrix, we construct a sign matrix appropriate to the corresponding conductance matrix; and from this sign matrix we then find the tree by the method given in Quarterly Progress Report No. 56.

The rows in the open-circuit resistance matrix pertain to loops formed by inserting the links, one at a time, into the tree structure. If two loops, i and k , have one or more tree branches in common, then the element R_{ik} in the matrix $[R]$ is nonzero; and its sign is plus if the pertinent link currents (according to their reference directions) traverse these common tree branches in the same direction; it is minus if these link-current directions are opposite.

If a third link current circulating on loop s also traverses one or more of these same tree branches, and if the elements R_{ik} and R_{is} are positive, then R_{sk} must likewise be positive. That is, if three link currents, i , k , and s , traverse paths that have one or more tree branches in common, and if their reference directions in these common branches coincide, then all three elements R_{ik} , R_{is} , and R_{sk} are positive.

Observe, in this connection, that the reference directions must always agree for two of the three currents; and if one of them has a contrary direction, then two of the three elements R_{ik} , R_{is} , R_{sk} (the two that pertain to the contrary link current) are negative. Consistent positiveness among the signs of these elements can be achieved, in this case, by changing all signs in the row and column of $[R]$ pertaining to the contrary current. We shall speak of this algebraic sign relationship among the elements R_{ik} , R_{is} , R_{sk} as being positive or potentially positive.

This sign relationship cannot hold for a situation in which the link currents i , k , and s traverse paths that do not have a common tree branch, because the common tree-branch condition for the three current paths is both necessary and sufficient for the sign

relationship to be positive or potentially positive. Conversely, this sign relationship is a necessary and sufficient condition to establish that the three pertinent currents traverse a common tree branch.

The same reasoning evidently applies to more than three link currents and corresponding elements in the matrix $[R]$; and so we can say that if and only if the algebraic sign of two or more elements in $[R]$ are positive or potentially positive then the pertinent link currents traverse a common tree branch. In the implied network graph, the corresponding links, together with the pertinent tree branch, form a cut set. We have thus found a way of discovering cut sets solely on the basis of algebraic sign relationships among the elements of the matrix $[R]$.

The number of cut sets equals the number of tree branches. The number of links, which equals the order of the given matrix $[R]$, determines only a lower limit for the number of tree branches, since, in the formula $\ell = n(n-1)/2$ pertaining to a full graph, n must be large enough to yield a value for ℓ that is as large as, or larger than, the order of $[R]$. Hence in constructing a tree appropriate to $[R]$ we have, theoretically, an infinite number of possibilities. We shall limit our detailed discussion essentially to the tree with fewest branches, since this leads to a full or almost full graph and hence is closest to being the dual of the synthesis procedure based upon a given $[G]$ matrix.

In the expression

$$[R] = [\beta_{\ell b}] \cdot [r] \cdot [\beta_{\ell b}]_t \quad (1)$$

which we get for a purely resistive network, as discussed before (1), the tie-set matrix has the form

$$[\beta_{\ell b}] = \begin{bmatrix} u_{\ell} & \vdots & \beta_{\ell n} \end{bmatrix} \quad (2)$$

The rows of $\beta_{\ell n}$ indicate confluent tree branches traversed by the respective link currents corresponding to these rows, and the columns indicate links belonging to cut sets for the respective tree branches to which these columns pertain. Thus, since $[a]$ and $[\beta]$ fulfill the consistency condition $[a] = [\beta]_t^{-1}$, we have $\beta_{\ell n} = -(a_{n\ell})_t$, as we have previously pointed out (2), and so $\beta_{\ell n}$ is actually a cut-set and a tie-set matrix at the same time. By rows, it is a tie-set matrix (if we omit the links in the tie sets); and by columns, it is a cut-set matrix (if we omit the tree branches in the cut sets).

Taking cognizance of these facts, the algebraic sign relationship among elements of the matrix $[R]$ enables us to construct the matrix $\beta_{\ell n}$, as we shall now show in detail by means of several examples.

Consider the following sign matrix pertinent to a given $[R]$ of order 10:

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	1	2	3	4	5	6	7	8	9	10	
	+	+	+	+	+	0	+	+	+	+	1
		+	+	+	0	+	+	+	+	+	2
			+	+	-	0	+	+	+	+	3
				+	0	-	+	+	+	+	4
					+	0	+	-	+	-	5
						+	-	+	+	-	6
							+	+	+	+	7
								+	+	+	8
									+	+	9
										+	10

For convenience, the rows and columns are numbered from 1 to 10. Incidentally, we note the presence of a number of zero elements. These are rather common in open-circuit resistance matrices, in contrast to short-circuit conductance matrices. In a full graph, the latter have no zero elements, while the former do. In a less-than-full graph, the resistance matrix may have many zeros.

Scanning along the first row of matrix 3, we see by inspection that elements at the intersections of rows and columns 1, 2, 3, 4, 7, 8, 9, 10 are all positive; hence this group of links forms a possible cut set. Observe that we have omitted 5 and 6 for obvious reasons; thus 5 is not coupled to 4, and 6 is not coupled to 1. However, again scanning along row 1, we can pick out links 1, 5, 7, 9 as forming another possible cut set. Note here that 2 and 4 are not included because they are not coupled to 5. Branch 3 has a sign inconsistency with 5, as do 8 and 10, while 6 is omitted because its coupling with 5 is zero.

There are no other groups that include branch 1, so we now scan along the second row and pick out branches 2, 6, 8, 9. Here 3 has no coupling with 6; R_{46} has the wrong sign; R_{67} and $R_{6,10}$ also have wrong signs. We can pick other possible groups out of row 2 – for instance, 2, 7, 8, 9, 10 – but this is a subgroup in the very first one we picked out; and unless we are interested in constructing a tree with more than the smallest number of branches, we skip such subgroups, as we skipped similar subgroups in connection with row 1.

In the third row of matrix 3 we now pick out branches 3, 5, 8, 10, for which all pertinent R_{sk} are positive after we multiply row and column 5 by -1. As they stand, these elements are potentially positive. Row 3 reveals no other groups, so we next consider row 4 and find the group 4, 6, 7, 10 for which the R_{sk} are potentially positive.

The remaining rows contain only subgroups in the ones already found; and we note, moreover, that the five groups that we have picked out imply a tree with $n = 5$, which,

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Branch resistance values are easily determined from a given numerical matrix $[R]$, beginning with those tree branches that singly represent nondiagonal elements in $[R]$. For example, the resistance of branch 11 must equal the element R_{12} or R_{34} or R_{23} or R_{14} . The resistance of branch 14 must equal $-R_{35}$ or $-R_{58}$ or $-R_{5,10}$,

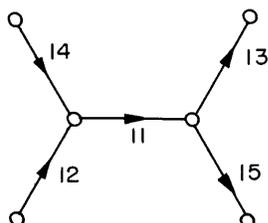


Fig. XXVII-1.

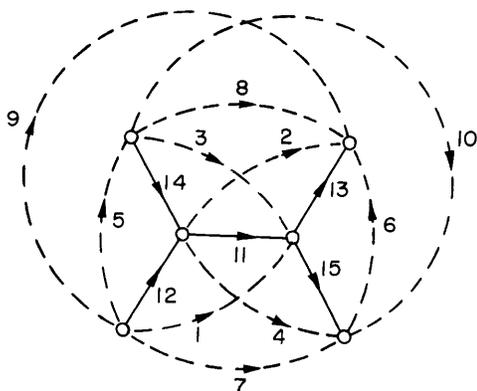


Fig. XXVII-2.

and so forth. Thus no special formulas are needed to fill in the branch resistance values, once the graph corresponding to $[R]$ is determined; and it is quite clear that the element values in this given matrix must fulfill some rather tight conditions if the resulting network is to exist. By and large, these conditions are more demanding than those pertinent to a given $[G]$ matrix on the node basis.

In this connection, it is alternatively possible, of course, to attempt the realization of a given $[R]$ matrix by computing its inverse and then applying the method pertinent to the realization of $[G]$ matrices. In fact such a procedure, if successful, realizes $[R]$ in a full graph, and hence has greater realization potentialities than the dual procedure discussed here.

If we are interested in the synthesis of a purely resistive network, or any other one-element-kind network, this

alternative scheme is undoubtedly the thing to use. But if the network is of the two- or three-element-kind variety and we are given two or all three of the open-circuit matrices $[R]$, $[L]$, and $[S]$, then this alternative scheme is not applicable, for there is no way in which we can interconnect the separate single-element-kind networks so as to obtain the desired result, while with the dual realization method there also exists a dual procedure for accomplishing the desired interconnection.

Meanwhile, we shall discuss additional examples of the open-circuit matrix realization method, in order to illustrate its features and peculiarities more adequately. Let us next consider the sign matrix for a given $[R]$ of order 10 with the form

1	2	3	4	5	6	7	8	9	10	
+	0	-	-	0	+	-	0	-	+	1
	+	0	-	-	+	+	-	0	-	2
		+	0	-	-	+	+	-	0	3
			+	0	0	-	+	+	-	4
				+	-	0	-	+	+	5
					+	-	0	0	-	6
						+	-	0	0	7
							+	-	0	8
								+	-	9
									+	10

Starting with the first row, we pick out links for which the R_{sk} elements are all positive or potentially positive. The first such group is 1, 3, 6, 7, having a potentially positive set of nondiagonal elements. It is easy to recognize this relationship by inspection, as well as to see that no other links are included in this group. Again, in row 1, we can select the group 1, 4, 9, 10 in the same manner; and it is now clear that there are no other groups containing link No. 1. In the second row the first group that we discover is 2, 4, 7, 8; and a second group is seen to be 2, 5, 6, 10. In the third row we have the group 3, 5, 8, 9. The remaining rows yield only subgroups. Hence these five groups are again used to determine a tentative tie-set schedule, namely

	11	12	13	14	15	
1	x	x				
2			x	x		
3	x				x	
4		x	x			
5				x	x	
6	x			x		
7	x		x			
8			x		x	
9		x			x	
10		x		x		

in which the columns represent cut sets formed by the stated groups of links.

To convert the x's in this schedule into plus or minus signs, we can begin by arbitrarily choosing a plus sign for the topmost element in each column. This we can do, since all signs in any column (which is a row of a cut-set matrix) can be changed at will

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anyway. Forming the Gramian determinant by rows in 7 must yield the signs in matrix 6. Carrying out this operation with the first row fixes all signs in the first two columns in matrix 7; with the second row, it fixes all elements in the 3rd and 4th columns; and with the third row, it fixes all remaining signs in the 5th column. Forming the Gramian products with all remaining rows yields signs that are consistent with the remaining ones in matrix 6 (as they must be if the implied tree structure exists).

We thus are able to replace matrix 7 by the tie-set matrix

$$\beta_{ln} = \begin{matrix} & \begin{matrix} 11 & 12 & 13 & 14 & 15 \end{matrix} \\ \begin{bmatrix} + & + & 0 & 0 & 0 \\ 0 & 0 & + & + & 0 \\ - & 0 & 0 & 0 & + \\ 0 & - & - & 0 & 0 \\ 0 & 0 & 0 & - & - \\ + & 0 & 0 & + & 0 \\ - & 0 & + & 0 & 0 \\ 0 & 0 & - & 0 & + \\ 0 & - & 0 & 0 & - \\ 0 & + & 0 & - & 0 \end{bmatrix} & \end{matrix} \quad (8)$$

Forming the Gramian determinant by columns, we now get the sign matrix for the corresponding node basis:

$$\begin{bmatrix} + & + & - & + & - \\ & + & + & - & + \\ & & + & + & - \\ & & & + & + \\ & & & & + \end{bmatrix} \quad (9)$$

Here we observe that if we multiply rows and columns 2 and 4 by minus signs, we convert matrix 9 into one in which all signs except those on the principal diagonal are minus, which is characteristic of a dominant matrix, and hence the tree is starlike. The rest of the solution is now straightforward.

As a third example we consider the sign matrix of an open-circuit resistance matrix given by

1	2	3	4	5	6	7	8	9	10	
+	+	+	+	+	+	+	0	0	0	1
+	+	+	+	+	+	+	+	+	0	2
+	+	+	+	+	+	+	+	+	+	3
+	+	+	+	+	+	+	+	+	+	4
+	+	+	+	+	+	+	+	+	0	5
+	+	+	+	+	+	+	+	+	+	6
+	+	+	+	+	+	+	+	+	+	7
+	+	+	+	+	+	+	+	+	+	8
+	+	+	+	+	+	+	+	+	+	9
+	+	+	+	+	+	+	+	+	+	10

(10)

As we shall see, this example illustrates some essential considerations that were not brought out by the previous ones. Picking out groups of links for which the pertinent R_{sk} elements are all positive, we start with the group 1, 2, 3, 4, 5, 6, 7, and next recognize the group 2, 3, 4, 5, 6, 7, 8, 9. In the third row we can pick the links 3, 4, 6, 7, 8, 9, 10, skipping 5 because it is not coupled with 10.

Next, in the fourth row we are inclined to pick the links 4, 6, 7, 8, 9, 10, again skipping 5 for the same reason as that involved in the formation of the previous group. However, we now observe that the group that we have just selected is a subgroup under the one picked out of the previous row. Indeed, it becomes clear that any other groups that we can pick out, from here on, are subgroups. There is nothing wrong with picking subgroups. We can select lots of them; but if we are again interested in finding the tree that has the fewest branches, we can select only certain subgroups and apparently we have no guiding principle to indicate how to proceed.

There is, however, such a principle. In the formation of a tentative schedule like matrix 7 in the previous example we can invoke an "exclusion principle" (it was not needed in the other examples) which recognizes that this schedule can have no repetitive rows, since these indicate paths for loop currents (or Kirchhoff voltage-law equations) and these must form a distinct set. Therefore, we must exclude from the formation of subgroups indicated above, all such that would yield identical rows in the tentative schedule like matrix 7.

Of course, if we do not limit the number of columns in this schedule, then we can admit all of the subgroups that we want without having to invoke this exclusion principle; but we are aiming for the minimum 5-branch tree, and so we are limited to five columns.

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With these ideas in mind, we arrive at the following tentative tie-set schedule:

	11	12	13	14	15
1	x	x			
2	x	x	x		
3	x	x	x	x	
4	x	x	x	x	x
5		x	x		
6		x	x	x	
7		x	x	x	x
8			x	x	
9			x	x	x
10				x	x

(11)

Here the column headed 12 is the first one that we picked out above, column 13 is the second, and column 14 is the third. The column headed 15 is the fourth group selected above, with links 6 and 8 discarded. In this form it is still a legitimate subgroup, as is also the one in column 11, which is a subgroup under the one in column 12. It is not hard to see that the selection of subgroups in columns 11 and 15 is unique if we wish to avoid repetitive patterns in the rows. The arrangement of the groups in columns is, of course, arbitrary, but we have chosen an arrangement that makes the distinction between successive row patterns evident in a systematic manner.

Since the signs in matrix 10 are all positive, the tie-set matrix β_{ln} is the schedule 11 with plus signs instead of x's, and zeros otherwise. Hence the sign matrix on a node basis is seen to consist of all plus signs and no zeros, which is characteristic of a linear tree.

E. A. Guillemin

References

1. E. A. Guillemin, Some generalizations of linear network analysis that are useful when active and/or nonbilateral elements are involved, Quarterly Progress Report, Research Laboratory of Electronics, M. I. T., Oct. 15, 1957, pp. 103-114; see pp. 109-110, Eqs. 14 and 20.

2. Ibid., see p. 109, Eq. 15.

B. A NORMAL COORDINATE TRANSFORMATION FOR AN ARBITRARY LINEAR PASSIVE NETWORK ON LOOP OR NODE BASIS AND ITS GEOMETRICAL INTERPRETATION

For a passive bilateral network let us choose the loop basis and begin by writing equilibrium equations in matrix form as given by

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Remembering that a determinant of order ℓ is given by a sum of $\ell!$ terms, each of which is a product of ℓ elements, we see that condition 6 is an algebraic equation of degree 2ℓ in s (after multiplying through by s^ℓ so as to eliminate negative powers of s). Roots of this equation, which is variously referred to as the determinantal or as the characteristic equation, are natural frequencies of the network (also called characteristic values or modes).

Since Eq. 6 has real coefficients, the roots will either be real or have the form of conjugate complex pairs. In the latter instance each pair defines a frequency (imaginary part) and a damping constant (real part) in a manner that is familiar to us from our experience with simple circuits.

The humble assumption 2 is thus not only justified but becomes expanded into a sum of 2ℓ terms of this form, one for each root or natural frequency. Hence solutions 2 take the form

$$i_j = \sum_{\nu=1}^{2\ell} A_j^{(\nu)} \exp(s_\nu t) \quad \text{for } j = 1, 2, \dots, \ell \quad (7)$$

in which the natural frequencies are denoted by $s_1, s_2, \dots, s_{2\ell}$. Pairs of terms involving conjugate roots have conjugate A_j -values and hence yield a real contribution to the pertinent current i_j .

For the evaluation of amplitudes $A_j^{(\nu)}$ we now return to Eqs. 4 and recognize that we have altogether 2ℓ such sets of equations to consider, one set for each s_ν -value. Since the coefficients b_{ij} in these equations are functions of s , we have a different set of coefficients for each value of s .

If all roots of the characteristic equation are complex – and we may as well assume that they are – then we need consider only one set of equations for each pair of conjugate s_ν -values because the corresponding $A_j^{(\nu)}$ -values correspondingly occur in conjugate complex pairs.

Now let us consider one such set of Eqs. 4. It is a homogeneous set, and the pertinent determinant is zero. The matrix $[B]$ has a rank less than ℓ ; and unless we are dealing with a highly degenerate situation, the rank is exactly $\ell - 1$. In that case Eqs. 4 determine the ratios of the A_j 's to one another. Initial conditions in the network (which will not be considered in detail at this time) determine one A_k -value for each set of equations, and these equations then determine all others in terms of this one.

According to the theory of algebraic equations, the ratio of A_j -values is given by the corresponding ratio of cofactors in any row of determinant 6. For example, we may write

$$A_1 : A_2 : \dots : A_\ell = B_{11} : B_{12} : \dots : B_{1\ell} \quad (8)$$

or

Here the coefficients b_{ij} are again given by expression 3; however, they are completely known because the complex frequency s is known, being the frequency of the applied source. In contrast with Eqs. 4, set 12 is inhomogeneous and the pertinent determinant B is not zero.

By Cramer's rule we can write for the resulting current amplitude in loop j

$$I_j = \frac{B_{ij} E_i}{B} \quad (13)$$

and hence the current distribution is expressible by the set of ratios

$$I_1 : I_2 : \dots : I_\ell = B_{i1} : B_{i2} : \dots : B_{i\ell} \quad (14)$$

Now suppose that the pertinent network has a natural frequency $s_\nu = -a_\nu + j\omega_\nu$ for which $a_\nu \ll \omega_\nu$ so that a pronounced resonance occurs if we choose the frequency s of the sinusoidal source 10 equal to $j\omega_\nu$. This j -axis point in the complex s -plane is very close to the natural frequency s_ν , the separation a_ν being small compared with $|s_\nu|$. Therefore, the determinant B , which is zero for $s = s_\nu$, has a very small value, and all current amplitudes I_j given by Eq. 13 are exceptionally large, as is characteristic of a resonant response.

We are not so much interested in the magnitude of the response as we are in the distribution ratios given by Eq. 14. Again, since $s \approx s_\nu$, all of the cofactors of B have values that are practically the same as they are for $s = s_\nu$; and since (as shown by Eqs. 8 and 9) their ratios are the same for any row i , Eq. 14 shows that the current distribution is independent of the location of the source.

The quantities

$$y_{ji} = B_{ij}/B \quad (15)$$

appearing in Eq. 13 are recognized to be the short-circuit driving-point and transfer admittance functions of our network. The zeros of B , or the natural frequencies of the network, are poles of these admittance functions. A partial fraction expansion of the rational function 15 (which incidentally is a proper fraction because the denominator is of higher degree than the numerator) places the poles of y_{ji} in evidence and reads

$$y_{ji} = \frac{k_{ji}^{(1)}}{s - s_1} + \frac{k_{ji}^{(2)}}{s - s_2} + \dots + \frac{k_{ji}^{(2\ell)}}{s - s_{2\ell}} \quad (16)$$

Here the residues $k_{ji}^{(\nu)}$ are given by the familiar expression

$$k_{ji}^{(\nu)} = [(s - s_\nu) y_{ji}]_{s=s_\nu} \quad (17)$$

From what has just been said about the ratios of expression 14 being the same for

any row i , we see that the ratios of the residues

$$k_{1i}^{(\nu)} : k_{2i}^{(\nu)} : \dots : k_{\ell i}^{(\nu)} = B_{i1} : B_{i2} : \dots : B_{i\ell} \quad (18)$$

have this same property for any natural frequency designated by the superscript ν . The residue matrix

$$[K] = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1\ell} \\ k_{21} & k_{22} & \dots & k_{2\ell} \\ \dots & \dots & \dots & \dots \\ k_{\ell 1} & k_{\ell 2} & \dots & k_{\ell \ell} \end{bmatrix} \quad (19)$$

for any s_ν , therefore, has rank 1 because all rows are proportional.

Because of the reciprocity theorem, $y_{ji} = y_{ij}$ and hence $k_{ji} = k_{ij}$ also, which means that any row of $[K]$ is the same as the corresponding column; $[K]$ is symmetrical. It follows that if we know a single row or column in this matrix, we can at once write the entire matrix.

When the residue matrix $[K]$ for a set of short-circuit driving-point and transfer admittances has rank 1 for all natural frequencies s_ν , that set of functions is said to be compact. We have just shown that the $[y]$ matrix for any nondegenerate network is compact. That is to say, if a network has the property that all natural frequencies are excitable from all of its independent links, then the pertinent $[y]$ matrix is compact. Its residue matrix has rank 1.

By the completely dual procedure, beginning with equilibrium equations on a node basis, we arrive at precisely the same result with regard to a corresponding set of open-circuit driving-point and transfer impedances having the matrix $[z]$. It is compact if the pertinent network has the property that all natural frequencies are excitable from all n independent node pairs.

Now let us consider, again, the homogeneous algebraic Eqs. 4 involving amplitudes A_j of the transient currents in Eq. 2 and coefficients given by the expression 3. As pointed out above, existence of a nontrivial and nondegenerate solution requires the pertinent matrix $[B]$, Eq. 5, to have rank $\ell - 1$. If we regard the rows of $[B]$ as defining a set of vectors b_1, b_2, \dots, b_ℓ in ℓ -dimensional space, then this vector set must occupy an $(\ell-1)$ -dimensional subspace. The desired solution, represented by a vector A with the components, A_1, A_2, \dots, A_ℓ , is simultaneously orthogonal to all the vectors b_1, b_2, \dots, b_ℓ ; that is, A is orthogonal to the $(\ell-1)$ -dimensional subspace occupied by the vector set of $[B]$. Equations 4, therefore, determine only a direction in the ℓ -dimensional space, namely, the orientation of the vector A .

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Since we have a distinct set of Eqs. 4 for each natural frequency s_ν , we find, in this way, as many directions in the ℓ -dimensional space as there are natural frequencies, recognizing that each natural frequency is defined by a conjugate pair of s_ν -values. In a general situation in which ℓ such pairs are involved, the homogeneous Eqs. 4 determine altogether ℓ directions in the ℓ -dimensional vector space. These are defined by the unit vectors

$$t_\nu = \frac{A^{(\nu)}}{|A^{(\nu)}|} \quad \text{for } \nu = 1, 2, \dots, \ell \quad (20)$$

where consecutive numbering refers, here, to one of each pair of conjugate complex s_ν -values. Components of these unit vectors are complex numbers, as are components of the vectors $b_1 \dots b_\ell$. The absolute value sign in Eq. 20 pertains to space coordinates only. We are dealing with a complex vector space.

A set of coordinate axes having the directions of these unit vectors are the normal coordinates of the given network. Since these directions are, in general, not mutually orthogonal, the normal coordinates form an affine system, although the set of reference coordinates is Cartesian.

Components

$$t_{j\nu} = \frac{A_j^{(\nu)}}{|A^{(\nu)}|} \quad (21)$$

of the unit vectors 20, which are their projections upon the reference axes, are recognized as being the direction cosines of the normal coordinates in the reference system. The matrix

$$[T] = \begin{bmatrix} t_{11} & \dots & t_{1\ell} \\ \dots & \dots & \dots \\ t_{\ell 1} & \dots & t_{\ell\ell} \end{bmatrix} \quad (22)$$

therefore characterizes the transformation

$$i_\nu] = [T] \cdot i'_\nu] \quad (23)$$

that expresses current variables i_j in the Cartesian system in terms of current variables i'_j in the affine system.

Introducing this transformation in the homogeneous form of equilibrium Eq. 1 yields the transformed set of equations

$$([L]p + [R] + [S]p^{-1}) \cdot [T] \cdot i'_\nu] = 0 \quad (24)$$

If, for the currents in the normal coordinates, we assume

$$i'_\nu = A'_\nu e^{st} \quad (25)$$

then we get for the amplitudes $A'_1 \dots A'_\ell$ a set of homogeneous algebraic equations (like Eqs. 4) with the matrix

$$[B'] = [B] \cdot [T] \quad (26)$$

For the elements of this matrix, we have

$$b'_{i\nu} = \sum_{j=1}^{\ell} b_{ij} t_{j\nu} = L'_{i\nu} s + R'_{i\nu} + S'_{i\nu} s^{-1} \quad (27)$$

Substituting from Eq. 3, we obtain

$$b'_{i\nu} = s \sum_{j=1}^{\ell} L_{ij} t_{j\nu} + \sum_{j=1}^{\ell} R_{ij} t_{j\nu} + s^{-1} \sum_{j=1}^{\ell} S_{ij} t_{j\nu} \quad (28)$$

The vectors t_ν in Eq. 20 defined by the columns of $[T]$ (the transposed vector set of $[T]$) are mutually orthogonal to all vectors defined by the rows of $[B]$ for $s = s_\nu$. More specifically, for $s = s_1$, the first column in $[T]$ is orthogonal to all rows in $[B]$; for $s = s_2$, the second column in $[T]$ is orthogonal to all rows in $[B]$, and so forth. Therefore, all elements in the ν^{th} column of $[B']$ (the elements 28 for $i = 1, 2, \dots, \ell$) contain the factor $(s-s_\nu)$, and so we can write

$$[B'] = [B_1] \cdot \begin{bmatrix} (s-s_1) & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & (s-s_\ell) \end{bmatrix} \quad (29)$$

in which an abbreviated method is used for writing a diagonal matrix involving the factors $(s-s_1) \dots (s-s_\ell)$, and the elements in $[B_1]$ are recognized to be linear functions of s (polynomials of degree one).

Since the columns in $[B']$, Eq. 26, are linear combinations of the columns in $[B]$, and since $[B]$ is symmetrical, the columns in $[\bar{T}]$ (the bar indicates the conjugate value) or the rows in $[\bar{T}]_t$ are mutually orthogonal to the columns in $[B_1]$ for $s = \bar{s}_\nu$. That is to say, the first row in $[\bar{T}]_t$ (the transposed conjugate of $[T]$) is orthogonal to all columns in $[B]$ and hence also to all columns in $[B_1]$ for $s = \bar{s}_1$; the second row in $[\bar{T}]_t$ is orthogonal to all columns in $[B_1]$ for $s = \bar{s}_2$; and so forth. Hence, in the product $[\bar{T}]_t \cdot [B_1]$, all elements of the ν^{th} row contain the factor $(s-\bar{s}_\nu)$, and so we can write

$$[\bar{T}]_t \cdot [B_1] = \begin{bmatrix} (s-\bar{s}_1) & & 0 \\ & \ddots & \\ 0 & & (s-\bar{s}_\ell) \end{bmatrix} \cdot [H] \quad (30)$$

in which the elements of $[H]$ are constants.

Equations 26, 29, and 30 now enable us to write

$$[\bar{T}]_t \cdot [B] \cdot [T] = \begin{bmatrix} (s-\bar{s}_1) & & 0 \\ & \ddots & \\ 0 & & (s-\bar{s}_\ell) \end{bmatrix} \times [H] \times \begin{bmatrix} (s-s_1) & & 0 \\ & \ddots & \\ 0 & & (s-s_\ell) \end{bmatrix} \quad (31)$$

Since the matrix $[B]$ is symmetrical and real for real values of s , the transposed conjugate of the left-hand side of Eq. 31 is equal to itself; therefore the right-hand side must have the same property, and hence $[H]$ must be real and symmetrical. [By the conjugate of this expression is meant replacing it by the analogous one with each s_ν or \bar{s}_ν replaced by its conjugate. For example, $[T]$ becomes $[\bar{T}]$ and vice versa, but $[B]$ remains unchanged because it is a polynomial with zeros s_ν and \bar{s}_ν .

From this result we get

$$[B]^{-1} = [T] \cdot \begin{bmatrix} 1 & & 0 \\ s-s_1 & & \\ & \ddots & \\ 0 & & s-s_\ell \end{bmatrix} \cdot [H]^{-1} \cdot \begin{bmatrix} 1 & & 0 \\ s-\bar{s}_1 & & \\ & \ddots & \\ 0 & & s-\bar{s}_\ell \end{bmatrix} \cdot [\bar{T}]_t \quad (32)$$

The elements of this matrix are the short-circuit driving-point and transfer admittance functions y_{ji} . Considering their partial fraction expansion, as shown in Eq. 16, we can evaluate the residue matrix $[K]$, Eq. 19, appropriate to $s = s_\nu$, by inspection of the right-hand side of Eq. 32.

Thus, for $s \rightarrow s_\nu$, the ν^{th} diagonal term in the first diagonal matrix swamps all others, and we get

$$[K]_{s=s_\nu} = \begin{bmatrix} 0 & t_{1\nu} & 0 \\ 0 & t_{2\nu} & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & t_{\ell\nu} & 0 \end{bmatrix} \cdot [H]^{-1} \cdot \begin{bmatrix} 1 & & 0 \\ s_\nu - \bar{s}_1 & & \\ & \ddots & \\ 0 & & s_\nu - \bar{s}_\ell \end{bmatrix} \cdot [\bar{T}]_t \quad (33)$$

while at the conjugate pole where $s \rightarrow \bar{s}_\nu$, we have

$$[\mathbf{K}]_{s=\bar{s}_\nu} = [\mathbf{T}] \cdot \begin{bmatrix} 1 & & & 0 \\ \bar{s}_\nu - s_1 & & & \\ & & & \\ 0 & & & 1 \\ & & & \bar{s}_\nu - s_\ell \end{bmatrix} \cdot [\mathbf{H}]^{-1} \cdot \begin{bmatrix} 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \bar{t}_{1\nu} & \bar{t}_{2\nu} & \dots & \bar{t}_{\ell\nu} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad (34)$$

which is the conjugate of expression 33, as it should be.

We also observe from expression 33 that the residue matrix has rank 1, since only the ν^{th} column of the matrix $[\mathbf{T}]$ is involved. This same circumstance shows that the columns in $[\mathbf{K}]$ are multiples of the ν^{th} column in $[\mathbf{T}]$, as the ratios expressed by Eq. 18 involving cofactors of the determinant B indicate, inasmuch as the elements 21 in the ν^{th} column of $[\mathbf{T}]$ fulfill these same ratios according to Eqs. 8 and 9.

The result expressed by Eq. 31, although not a normal coordinate transformation in the usual sense, certainly bears a close resemblance to it, and accomplishes the same end regarding isolation of the natural frequencies, since these, in the form of conjugate complex pairs, are assigned to the ℓ axes whose direction cosines are the elements in respective columns of the matrix $[\mathbf{T}]$. Transformation 31 fails, however, in the simultaneous diagonalization of the matrices $[\mathbf{L}]$, $[\mathbf{R}]$ and $[\mathbf{S}]$, although it accomplishes a sort of semi-diagonalization of unique form. The diagonalization becomes complete if $[\mathbf{H}]$ turns out to be a diagonal matrix, which condition will be found to agree with the condition under which a diagonal form for the determinant B can be achieved.

The chief reason for presenting this geometrical interpretation of the normal coordinate transformation is that we can now visualize a number of things about network behavior with greater ease and clarity. For example, in a nondegenerate situation in which all natural frequencies are excitable from all loops or links, the set of unit vectors 20, which we can crudely visualize as a bunch of arrows emanating from the origin of our Cartesian reference coordinates, have arbitrary orientations so that each has projections (components) on all of the reference axes. An excitation in any one reference axis produces oscillations in all normal coordinates, and through these (by the mechanism of projection) its effect is transmitted to all the other reference coordinates.

The mechanism of network response can thus be visualized in simple geometrical terms. Degeneracies occur if one or more of the unit vectors 20 coincide with some of the reference coordinates.

Suppose that the unit vector t_1 coincides with reference axis 1. Since the reference axes are orthogonal, t_1 is now orthogonal to all reference axes except axis 1, and hence its particular natural frequency is not excitable except from axis 1; if it is excited there, it cannot be transmitted to any of the other reference axes. Recognizing that the term

"reference axis" is the geometrical counterpart of "loop" or "link," we readily see the physical implication of this statement.

Each unit vector z_0 has various projections upon the reference axes; some large, some small, some may be zero. These projections portray a current distribution function for the network. The set of vectors z_0 give us ℓ such distribution functions, one for each complex conjugate pair of natural frequencies.

Any sinusoidal steady-state response can also be visualized in terms of the excitation of normal coordinates that act as a coupling mechanism between different loops of the network. The phenomenon of resonance occurs when the applied frequency is close to a natural frequency and thus excites an unusually large amplitude in one of the normal coordinates which then transmits this oscillation to all other loops in amounts proportional to the projections of this normal coordinate upon the corresponding reference axes.

In general, when the applied frequency is arbitrary, all normal coordinates are moderately excited; none predominates over the others. The extent to which each normal coordinate is excited or the distribution of amplitudes among these coordinates depends upon the point of excitation; and so one expects that the response in any other reference coordinate also depends upon the point of excitation. The distribution function changes as the location of the sources is varied.

For a resonance condition, however, one normal coordinate is so highly excited that we may neglect all the others by comparison. Thus, with essentially only one normal coordinate excited, the distribution of response throughout the reference coordinates is invariant to the source location. The distribution is frozen, as pointed out already. Now, we have a simple geometrical picture showing how this result comes about.

We need one more piece of information to make this picture complete, namely, that establishing the extent to which each normal coordinate becomes excited for a sinusoidal source of given frequency and location. That this distribution of excited amplitudes among the normal coordinates is not merely dependent upon the relative geometrical orientations of reference and normal coordinate axes, is strikingly evident in the resonance situation in which one normal coordinate is singled out to the almost complete exclusion of the others, notwithstanding the fact that, by geometrical projectivity alone, this condition cannot come to pass.

Evidently, this distribution function depends also upon the source frequency relative to the various natural frequencies. Precisely, this is the distribution that is placed in evidence by the partial fraction expansion 16 of the appropriate steady-state response function. Here we can assume that the first ℓ terms pertain to one of each pair of conjugate natural frequencies and the remaining ℓ terms to the other of each pair. Then the result made evident by Eq. 33 enables us to write

$$t_{j\nu} = \frac{k_{ji}^{(\nu)}}{\left[\left(k_{1i}^{(\nu)}\right)^2 + \left(k_{2i}^{(\nu)}\right)^2 + \dots + \left(k_{\ell i}^{(\nu)}\right)^2 \right]^{1/2}} = \frac{k_{ji}^{(\nu)}}{\left| k_i^{(\nu)} \right|} \quad (35)$$

for any fixed value of the index i . Except for a normalization factor, we thus see that the residues $k_{ji}^{(\nu)}$ of the steady-state response functions y_{ji} are the direction cosines of the normal coordinates!

Terms in the expression 16 yield the amplitudes with which various normal coordinates are excited. In this connection it must be remembered that with every excitation frequency $s = j\omega$ in the sinusoidal steady state there is associated the conjugate frequency $s = -j\omega$, so that we must consider both in order to obtain the resultant excitation of any normal coordinate. We need compute this resultant, however, only for the first ℓ terms in expression 16, since the corresponding results for the remaining ℓ terms merely yield a conjugate value that can be written down at once if desired.

As a result of these interpretations, the partial fraction expansion of steady-state response functions takes on a new and broader significance, not only for the evaluation of transient response but also for computation of sinusoidal steady-state response, which likewise is regarded as effected through the excitation of normal coordinates. In this regard, the resonance phenomenon, in particular, receives an interpretation of striking geometrical simplicity and clarity (1).

From a function theoretical point of view these facts are not surprising, since the partial fraction expansion is a characterization in terms of the life-giving elements (the poles) of a function, while normal coordinates afford a characterization in terms of the life-giving elements (the natural frequencies) of networks.

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References

1. For a detailed discussion along these lines see E. A. Guillemin, *Communication Networks*, Vol. I (John Wiley and Sons, Inc., New York, 1931), Chapter VII, pp. 248-283; see also Making normal coordinates coincide with the meshes of an electrical network, *Proc. IRE* 15, 935 (1927).

