THE RESPONSE OF A POINT SOURCE IN A LIQUID
LAYER OVERLYING A LIQUID HALF SPACE

by

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The Response of a Point Source in a Liquid Layer
Overlying a Liquid Half Space

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ABSTRACT

The response to a harmonic point source in a liquid layer
overlying a liquid half space is computed as a function of frequency.
Included are the contributions from all normal modes that occur,
and the branch-line integral representing the refraction arrival.
The value of the refraction arrival is given in terms of the complex
error function.

The effect of different velocity and density contrasts are consid-
ered, and the effect of source depth and the distance to the receiver
are investigated.

The results giving the behavior of the magnitude of the branch-
line show that it is much larger at the mode cutoffs than at other
values of frequency. The total amplitude of the response shows a
regular oscillation in the frequency range in which two modes are
present, and somewhat irregular high and low values over the
range in which three modes are present. This behavior reflects
the difference in amplitude at frequencies for which modes reinforce
or interfere with each other.

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The problem of a point source in a liquid layer over a liquid half space was solved in terms of normal modes by Pekeris (1948). The solution, good for horizontal distances large with comparison to the depth of the layer, was expressed in terms of normal mode contributions. Each of the modes travels with frequency dispersed phase velocity. The arrivals of each mode at a given point were overlapping in time, but the frequencies for low frequency cutoff of the modes increases with mode number. By using recording systems that accentuated with low frequency part of the spectrum the arrival of the first mode could be seen on the time recording.

In this paper the point of view will be to add the contributions from the several modes in the frequency domain. The results of the computation will give some idea how propagation through a layer affects the shape of the power spectrum of the recorded signal.

The power spectrum of the motion is given by

$$\left| \frac{g(\omega)}{\omega} \right|^2 \cdot \left| \psi(\omega, r, z) \right|^2$$

where $g(\omega)$ is the Fourier transform of the source and $\psi(\omega, r, z)$ is the response at the observation point to a harmonic point source.
In his original paper, Pekeris (1948) derived an expression for the branch-line integral, representing the refraction arrival, which was valid away from the mode cutoff frequency. Officer (1953) showed that a different expression that gave large contributions must be used at the cutoff frequencies. In this study an expression for the branch-line that is valid for all frequencies is derived.
DERIVATION OF FORMULAS
In this section will be presented a brief outline of the approach used by Pekeris (1948) to solve the problem of a point source in a liquid layer over a liquid half space. The model is shown in Figure 1.

\[ b = \frac{\rho_1}{\rho_2} \quad \text{The density contrast} \]

\[ d = \text{depth of source} \]

\[ Z = \text{depth of observation point} \quad (2.1) \]

\[ C_1 = \text{velocity in layer} \]

\[ C_2 = \text{velocity in half space} \]

\[ H = \text{layer thickness} \]

\[ r = \text{horizontal distance from source to observation point} \]

The problem was solved in terms of the velocity potential

\[ \Phi (r, Z, \omega, t) = e^{i\omega t} \psi (r, Z, \omega) \quad (2.2) \]

The pressure is given by

\[ P = \rho \frac{b \phi}{\partial t} \quad (2.3) \]
The particle velocity is given by

$$\nabla^2 = -\nabla \phi$$

(2.4)

The velocity potential obeys the wave equation. The source is taken to be a simple harmonic source of the form

$$\phi(r, Z, \omega) = \frac{1}{\sqrt{r^2 + (Z - d)^2}} \cdot \frac{2 \pi f \cdot i \cdot \omega}{\epsilon_0} \cdot \left( t - \frac{r^2 + (Z - d)^2}{C} \right)$$

(2.5)

The solution must obey the boundary conditions

i) at $Z=0$

The pressure is 0

ii) at $Z=H$

The pressure and the vertical particle velocity are continuous

iii) Energy does not come in from $-\infty$ nor does the velocity become infinite as $Z$ goes to $-\infty$ .

In the layer above the source the solution to this problem written in its integral form is

$$\psi(r, Z, \omega) = \int_0^\infty J_0(Kr) \cdot \frac{\sin \beta_i \cdot Z}{\beta_i} \cdot \left[ \frac{\beta_i \cos \beta_i (H-d) \cdot \sin \beta_i}{(H-d)} \right]$$

(2.6)
with the following definitions

\[ b = \frac{\rho_1}{\rho_2} \]

\[ J_0 (Kr) \] is the zeroth order Bessel function

\[ \beta = \frac{\beta_1 \cos \beta_1 H + \beta_2 \sin \beta_1 H}{\beta_1 \cos \beta_1 H + \beta_2 \sin \beta_1 H} \]

\[ \beta_1 = \frac{1}{\sqrt{\omega^2 c_0^2 - K}} \quad K \leq \omega c_1 \quad K \text{ real} \]

\[ -i \frac{\sqrt{K^2 - \omega^2 c_0^2}}{K \geq \omega c_1} \quad i = 1, 2 \]

The \( \beta \)'s introduce branch points at \( K = K_1 = \omega / c_1 \) and \( K = K_2 = \omega / c_2 \).

The integral for \( \psi \) will be evaluated by contour integration.

Without proof (see Ewing et. al., 1957, p. 135) we state that all real poles are between \( K_1 \) and \( K_2 \). They may be considered to be just below the real axis. Using the identity expressing the Bessel function in terms of the Hankel functions

\[ 2J_0 (Kr) = H' (Kr) + H_0^2 (Kr) \]

The integral for \( \psi \) is broken into two integrals: the one containing \( H' \) is deformed upward in the \( K \) plane; the other is deformed downward as shown in Figure 2. None of the arcs contribute because of the behavior of the Hankel functions.
The integral along the real $K$ axis has been replaced by the contours shown, plus the residuals that arose by deforming the contour containing $E$ and $D$. Lines $A$ and $B$ cancel each other using $-H_o^{(2)}(iq) = H_o^{(2)}(-iq)$. Lines $D$ and $C$ also cancel because the integral is even in $\beta$, and so equal on both sides of the branch cut. The contributions from lines $E$ and $F$ will be considered following the discussion of the pole contributions.

The residues of the poles contribute what is known as the normal modes contributions. The poles arise at the zero's of the period equation. For $K_2 \leq K \leq K_1$. The period function may be written as

$$f(\omega, K^{(N)}) = b \sqrt{(K^{(N)})^2 - \frac{\omega^2}{C^2}} \tan H \frac{\omega^2}{C^2} - (K^{(N)})^2 = 0$$

(2.8)

The contribution from each pole is given by the Cauchy residue theorem as

$$\psi^{(N)} = \frac{2\pi}{H} \sqrt{\frac{2}{\pi H}} \psi^{(-\frac{1}{2})} G(K^{(w)}) \sin(\beta d) \sin(\beta^2)$$

$$G(K^{(w)}) = \frac{1}{\sqrt{K^{(w)}}} \frac{\beta_1 H}{\beta_1 H - \sin(\beta_1 H) \cos(\beta_1 H) \tan(\pi H)}$$

(2.9)
\( K^{(N)} \) is the Nth root of the period equation and \( \beta \) is evaluated for \( K = K^{(N)} \).

The Hankel function has been expanded in an asymptotic form good for large \( r \)

\[
H_0^{(2)}(K^{(N)}_N r) \sim \frac{2}{\pi K^{(N)}_N r} e^{i \left[ \frac{\pi}{4} - K^{(N)}_N r \right]}
\tag{2.10}
\]

The total from all the poles is obtained by summing the contributions from each pole.

\[
\psi^{\text{poles}} = \sum_{N=1}^{N(w)} \psi^{(N)}
\]

The number of poles that occur increases with \( \omega \). The number of poles \( \bar{N}(\omega) \) for a given \( \omega \) is given by the smallest integer greater than

\[
\left[ \left( \frac{\omega}{C_1} \right)^2 - \left( \frac{i \omega}{C_2} \right) \right]^{1/2} = \frac{H}{\pi} + 1/2 \tag{2.11}
\]

There is no pole contribution if the frequency is below the cutoff frequency of the first mode.
The branch line contribution is an integration path that comes in from \((K_2, -i \infty)\) to \(K_2\) on the left side of the branch line then returns on the right of the branch cut as shown in Figure 2. The value of \(\beta_1\) is the same on both sides of the cut, but \(\beta_2\) changes sign across the cut. The branch line contribution is then

\[
\psi^b = \int_{K_1}^{K_2} H_2^{(2)}(Kr) F(\beta_1, \beta_2) K dK + \int_{K_1}^{K_{1-\lambda\phi}} H_2^{(2)}(Kr) F(\beta_1, \beta_2) K dK \quad (2.12)
\]

where

\[
F(\beta_1, \beta_2) = \frac{\sin \beta_1 z}{\beta_1} \left[ \frac{\beta_1 \cos \beta_1 (H-d) + \beta_2 \sin \beta_1 (H-d)}{2} \right]
\]

Officer (1956, p. 190) shows \(\psi^b\) may be combined into a single integral.

\[
\psi^b = \int_{K_1}^{K_2} H_2^{(2)}(Kr) \frac{\beta_1 \sin \beta_1 z \cdot \sin \beta_1 d}{\beta_1^2 \cos^2 \beta_1 H + \beta_2^2 \sin^2 \beta_1 H} K dK
\]
A new variable is introduced by the relation
\[ \text{K} = \text{K}_{1} \text{r} \]
\[ \text{r} \in \mathbb{C} - iX \]

The asymptotic expansion of (2.10) is again used for the Hankel function.

\[ H(Kr) = H(K_{1} r \text{r} ) \approx \sqrt{\frac{2}{\pi K_{r} r}} e^{-i(K_{1} r - \pi/4)} e^{-K_{1} r X} \]

(2.15)

Because the Hankel function vanishes exponentially for negative imaginary parts of K, for large r almost all the contribution to the branchline integral occurs in the vicinity of the point K = K_{2}. At this point X = 0. In the change of variables only terms of the first order in X have been retained. Doing this the denominator of F is written

\[ \beta_{1}^{2} \cos^{2} \beta_{1} H + b_{1}^{2} H = K_{1}^{2} \cos^{2} K_{1} H + + i b_{1}^{2} K_{1}^{2} 2 X (c_{1}/c_{2}) \sin^{2} K_{1} H \]

(2.16)

\[ \equiv A + \lambda BX \]
where

\[
\begin{align*}
A &= K_1^2 M^2 \cos^2(K_1 MH) \\
B &= 2b^2 K_1^2 \left( \frac{C_1}{C_2} \right) \sin^2 K_1 MH \\
\mu &= (1 - \frac{C_1}{C_2})^2
\end{align*}
\] (2.17)

The branch line integral may now be written as an integral over \(X\).

\[
\psi^b = 2ib \int_0^\infty \sqrt{\frac{2 \pi c_2}{K_1 T e_i}} e^{i \left[ \frac{\pi}{4} - K_2 x \right]} e^{-K_1 r x} \frac{K_1 \Gamma(2x) c_2 c_1}{A + \lambda B x} \]

\[
\times e^{i \pi/2} \cdot (-i K_1 K_2) \, dX
\]

\[
= \frac{4ib K_2}{\sqrt{\pi K_1 T}} \frac{\sin(K_1 MU)}{\sin(K_1 U d)} \left( \int \right)
\]

\[
\int = \int_0^\infty \frac{x^{1/2} e^{-K_1 r x}}{A + \lambda B x} \, dX
\]

Pekeris (1948) evaluated the integral (2.18) under an assumption equivalent to setting \(B=0\). For much of the frequency range this is a reasonable procedure, because most of the contribution occurs for \(X\) small so

\[
A >> BX
\]
Following from this the denominator may be approximated by $A$.

Under this condition

$$J = \frac{\sqrt{\pi}}{A^2 (K_1 r)^{3/2}}$$  \hspace{1cm} (2.19)

$$\psi^b = \frac{2 b k_2 \sin(K_1 M \mu) \sin(K_1 M \mu)}{(K_1 \nu)^2 \mu^2 \cos^2 K_1 M \mu} e^{-i K_1 r}$$  \hspace{1cm} (2.20)

Officer (1953) pointed out that a different evaluation was necessary at frequencies for which $A = 0$. The frequencies for which this occurs are given by

$$\omega_N = \frac{c_i}{h} (\ln N - 1) \frac{\pi}{2} = \sqrt{1 - (c_1/c_2)^2}$$

These are the cutoff frequencies for each mode. With $A = 0$

$$\int_0^{\infty} e^{-Xk_1 r} \frac{X}{\nu^2} dX = \frac{\sqrt{\pi}}{(K_1 \nu)^{3/2}}$$  \hspace{1cm} (2.21)

and

$$\psi^b = \frac{2}{b r} \sin(K_1 M \mu) \sin(K_1 M d) e^{-i K_1 r}$$  \hspace{1cm} (2.22)
At these frequencies the magnitude of the branch line contribution decreases as $1/r$ rather than as $1/r^2$ in the case of frequencies away from the mode cutoffs.

Officer (1953) explains the significance of this as constructive addition to the refractive wave by the wave reflected in the layer. Consider a plane wave in the layer, travelling at the critical angle as in Figure 3. At $D$ energy is put into the refraction wave travelling in the half space. The wave is reflected then travels to $C$ where it is then reflected downward to $D'$. At $D'$ it again adds to the refracted wave. The phase of the refracted wave that reaches $D'$ will be the phase that was at $D$ at a time $L/C_2$ earlier. For constructive interference, the wave in the layer and the refracted wave must differ in phase by $2\pi N$ at $D'$. The phase change over the path $DCD'$ is given by

$$\frac{2H}{C_1 \cos \Theta} + \pi$$

(2.23)

The change for the refracted path is

$$\frac{2H \omega \tan \Theta}{C_2}$$

(2.24)
Their difference is

\[ \frac{\omega}{c_1} \frac{2 H}{\cos \theta} + 2 \pi - \frac{\omega}{c_2} \frac{2 H \tan \theta}{\cos \theta} \]

(2.25)

For the critical angle

\[ \sin \theta = \frac{c_1}{c_2} \]

So the condition that the phases differs by \(2 \omega \pi\) gives

\[ \frac{\omega H}{c_1} \left( \frac{1}{\sqrt{1 - \left(\frac{c_1}{c_2}\right)^2}} - \frac{c_1}{\sqrt{1 - \left(\frac{c_1}{c_2}\right)^2}} \right) = 2 \pi \left( N - \frac{1}{2} \right) \]

(2.26)

\[ \frac{\omega H}{c_1} \sqrt{1 - \left(\frac{c_1}{c_2}\right)^2} = 2 \pi \left( N - \frac{1}{2} \right) \]

(2.27)

\[ k_1 H \mu = 2 \pi \left( N - \frac{1}{2} \right) \]

(2.28)

Comparing with (2.17) this is seen to be the necessary condition for \(A=0\).

The contribution from the branch line integral is large at the cutoff frequencies but falls off away from them. An expression is desired that gives the value of the integral good for all values of frequency.
The following presents an evaluation of the branch line good for all values of A and B.

The integral in question is from (2.1c).

\[ J = \int_0^\infty \frac{x^{1/2} e^{-sx}}{A + iBx} \, dx \quad s \equiv K_1 Y \tag{2.29} \]

\[ = \int_0^\infty \frac{2y^2 e^{-sy^2}}{A + iB y^2} \, dy \quad y^2 \equiv X \tag{2.30} \]

\[ = \frac{2}{iB} \int_0^\infty \frac{y^2 e^{-sy^2}}{D + y^2} \, dy \tag{2.31} \]

\[ = \frac{2}{iB} \left[ \int_0^\infty \frac{e^{-y^2}}{D + y^2} \, dy - \int_0^\infty \frac{D e^{-y^2}}{D + y^2} \, dy \right] \]

\[ = \frac{1}{iB} \sqrt{\frac{\pi}{K}} - \frac{2}{iB} I \tag{2.32} \]

\[ I = \int_0^\infty D e^{-Sy^2} \tag{2.33} \]

The evaluation of I follows.

Consider I as a function of S.

\[ I' = \frac{dI}{dS} = \int_0^\infty -D y^2 e^{-Sy^2} \, dy \]

\[ = \frac{1}{D} I' + I = \int_0^\infty \frac{D + y^2}{D + y^2} e^{-Sy^2} \, dy \]

\[ I' - I = D = -\sqrt{\frac{\pi}{K}} \frac{D}{2} \tag{2.34} \]
assume a solution to (2.34) of the form \( I = e^{DS} \cdot U(S) \)

\[
e^{DS} \left\{ DU + U^1 - DU \right\} = -\frac{d}{2} \sqrt{\frac{\pi}{S}}
\]

\[
u^1 = -e^{-DS} \frac{d}{2} \sqrt{\frac{\pi}{S}}
\]

\[
u = -\int_0^S \frac{\sqrt{\pi D}}{2} \frac{e^{-DE}}{\sqrt{\pi}} dE + U(0)
\]

\[
= -\int_0^{VSD} \sqrt{\pi D} \frac{e^{-w^2}}{\sqrt{\pi}} dw + U(0) \quad w^2 = DE
\]

\[
u = -\frac{\pi \sqrt{D}}{2} \int_0^{VSD} e^{-w^2} dw + U(0) \quad (2.35)
\]

\[
I(0) = \int_0^\infty b^4 e^{+y^4} dy = U(0)
\]

\[
u(0) = \sqrt{D} \frac{\pi}{2} \quad (2.36)
\]

Substituting (2.36) in (2.35)

\[
I(S) = e^{DS} \left\{ \sqrt{\frac{\pi}{S}} - \frac{\pi \sqrt{D}}{2} \operatorname{erf}(VSD) \right\} \quad (2.37)
\]

Equation (2.37) is substituted into (2.32) giving

\[
J = \frac{1}{iB} \left[ \sqrt{\frac{\pi}{S}} - \sqrt{D} \pi e^{DS} \left\{ 1 - \operatorname{erf}(VSD) \right\} \right] \quad (2.38)
\]
Using this evaluation of \( J \), the general expression for the branch line integral is

\[
\Psi_b = \frac{\frac{1}{\sqrt{\pi k_1 r}}}{J} \sin(k_M d) \sin(k_M^2) \frac{k^2}{k_1} \mathcal{E}^{-i k_2 r}.
\]

This reduces to the previous formulas 2.19 or 2.21 for \( J \) if \( B \) or \( A \) is 0. For \( A=0 \)

\[
D=0
\]

and

\[
J = \frac{1}{i B} \sqrt{\frac{\pi}{s}}
\]

which is the result in (2.19)

To compare the result for \( B=0 \) with (2.21), we must find the limit as \( B \to 0 \).

\[
J = \frac{1}{i B} \left[ \sqrt{\frac{\pi}{s}} - \pi V_D \int DS \left\{ 1 - \text{erf} \left( \sqrt{\frac{2}{V_D}} \right) \right\} \right]
\]

The error function is expanded in its asymptotic form good for large agreement.

\[
J = \frac{1}{i B} \left[ \sqrt{\frac{\pi}{s}} - \pi V_D \int DS \left\{ 1 - 1 + \frac{Q}{\sqrt{\pi DS}} \left( 1 - \frac{1}{2 DS} + \cdots \right) \right\} \right]
\]

\[
\approx \frac{1}{i B} \left[ \frac{\sqrt{\pi}}{2} \frac{1}{DS} \right]
\]

\[
\approx \frac{1}{2AS} \sqrt{\frac{\pi}{s}} = \frac{\sqrt{\pi}}{2A(k_1 r)^{3/2}}
\]

(2.40)
This is the value of the integral found for $B=0$ and given in (2.21).

The expression for the branch line given in (2.39) is a good approximation at all frequencies, and could be used in a numerical integration to obtain the time response from the refracted wave.
Computations were made to obtain the vertical velocity of the liquid layer at the surface for a simple harmonic source, the response at the surface is itself a sinusoidal motion. The vertical velocity is obtained by differentiating the velocity potential with respect to \( Z \) and letting \( Z = 0 \). Previously the velocity potential was derived as the sum of the normal mode contributions (2. 9) plus the branch-line contribution. (2. 36) Although it is the magnitude of the response that is to be considered, the phase of each contribution must be used in combining the terms to find the total response \( \Psi (\omega) \).

The contribution of the poles to the vertical particle velocity is given by (2. 9) as

\[
\frac{\partial \Psi^p}{\partial Z} = \frac{2\pi}{H} \sqrt{\frac{2}{\pi}} \sum \frac{G(k^{(m)}) Q [-k^{(m)} r + \pi/4]}{\sin (\beta_1 d) \cdot \beta_1} \\
\frac{G(k^{(m)})}{\sqrt{k^{(m)}}} = \frac{\beta_1 H}{\beta_1 - \sin(\beta_1 H) \cos(\beta_1 H) - \beta_1 \sin(\beta_1 H) \tan(\beta_1 H)}
\]

\[
\beta_1 = \sqrt{\omega^2 c^2 - k^{(m)}^2}
\]

The contribution from the branch-line is given by (2. 38) as

\[
\frac{\partial \Psi^b}{\partial Z} = \frac{4 i b K_k \sin(k_1 d) K_1 M}{\sqrt{\pi} K_1 r} \cdot K_1^2 Q^{-i K_1 r} \cdot \\
\cdot \frac{1}{A'B} \left[ \sqrt{\frac{\pi}{K_1 r}} - \frac{bs}{\sqrt{D}} \left\{ 1 - erf \left( \frac{1}{\sqrt{K_1 d}} \right) \right\} \right]
\]
A and B are both real and positive so that D is negative imaginary.

The complex quantity \( K_1 r D \) has a phase of \(- \pi/4\). In evaluating the branch line integral use was made of the Share subroutine BS WOFZ.

The subroutine calling sequence has been changed to allow its use by FORTRAN. This subroutine computes

\[
W(z) = e^{-\frac{z^2}{4}} \left( 1 + \frac{2z}{\pi} \int_0^\infty e^{-\frac{t^2}{2}} dt \right)
\]

From this the complex error function is obtained from

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\zeta^2} d\zeta
\]

\[
= 1 - e^{-z^2} \cdot W(iz)
\]

For each frequency the contribution from the poles and from the branch-line are computed and added together to give \(|\psi|\). The result was printed as output. In addition, the amplitude and phase of each pole, the total amplitude of all the poles, and the amplitude of the branch-line \(|\psi_b|\) were printed.

The computations were coded in FORTRAN and run on an IBM 7090 computer. Appendix A gives a listing and a flow chart of the program.
Solution of the Period Equation

Before the value of the pole contribution can be calculated, the values of $K^{(n)}$ must be obtained by solving numerically the period equation

$$ b \beta_1 \tan(\beta_1 h) + \beta_1 = 0 $$

This form of the period equation is valid for $K_1 > K > K_2$. A graph given by Ewing et. al. (1957, p. 139) suggests a method of solving this equation. The graph is given in Figure 4. The period equation is solved in terms of $\beta_1$. Both sides of the period equation are plotted in the graph. Where they are equal give values of $\beta_1$ that satisfy the period equation.

$\beta_2$ may be expressed in terms of $\beta_1$, rather than in terms of $K$.

$$ \beta_2^2 = K_1^2 - K_2^2 $$
$$ \beta_1^2 = K_1^2 - K_2^2 $$

On eliminating $K$

$$ \beta_2^2 = K_1^2 - K_2^2 - \beta_1^2 $$
$$ \beta_1^2 = \sqrt{K_1^2 - K_2^2 - \beta_1^2} $$

The period equation is then expressed in terms of $\beta_1$ as

$$ \tan(\beta_1 h) = -\beta_1 / b \sqrt{K_1^2 - K_2^2 - \beta_1^2} $$
The number of roots for a given frequency is given by
\[ \bar{N} (\omega) = \text{the smallest integer, greater than} \]
\[ \left( \sqrt{K^2 - K_i^2} / \pi \right) + \frac{1}{2} \]

From Figure 4 it is seen that the Nth root \( \beta \) \( (K^{(n)}) \) lies in the interval between \( (n - \frac{1}{2}) \pi \) and \( n \pi \). With this knowledge, the equation is solved by starting with the end points of the interval enclosing the root, then dividing it in half. Evaluating the period equation at the midpoint determines the half of the interval that contains the root. This operation is repeated 15 times so that the error in \( H^{(b)} \) is less than
\[ \left( \frac{1}{2^{15}} \right) \cdot \frac{\pi}{2} \]

The method of false position is then applied once to further decrease the error. The value of \( K^{(b)} \) is then obtained from the value of \( K_i \).
\[ K^{(n)} = \sqrt{K_i^2 - \beta_i^2} \]

The computations have all been performed with \( C_1 = 1 \) and \( H = 1 \). This allows the results to be used for other values of \( C_1 \) and \( H \) if the distance unit is taken as \( H \) and the time unit is taken as \( H/C_1 \). The model is then defined by giving \( r \), \( d \), and \( C_2 \) in terms of these units. In the computations the model is defined by giving \( r \), \( d \),
As an example of how the program could be used to calculate the response in a model in which $C_1$ or $H$ are not one, consider the model,

$$C_1 = 2 \text{ km/sec}$$
$$C_2 = 5 \text{ km/sec}$$
$$d = 1 \text{ km}$$
$$H = 3 \text{ km}$$
$$r = 100 \text{ km}$$

Use a distance unit of 3. The time unit is $\frac{H}{C_1} = \frac{2}{3} \text{ sec}$. In these units

$$H = 1$$
$$C_1 = 1$$
$$C_2 = \frac{5}{2}$$
$$d = \frac{1}{3}$$
$$r = \frac{100}{3}$$

If the response were desired at $\omega = 10 \text{ radian/sec}$, the computation would be made at the value of $\omega = \frac{10.2}{3}$. 
RESULTS AND DISCUSSION
Branch-Line

As was expected, the magnitude of the branch-line integrated was much higher at the mode cutoff frequencies than at other points of the frequency curve. This is shown very strikingly in Figure 5. In this figure as in all the amplitude curves, the cutoff frequencies for four modes appear as vertical arrows. In Figure 6 a plot is presented for the same model, showing the behavior of the branch-line contribution, near the cutoff of the third mode, using a smaller difference in frequency between points.

In the derivation of the branch-line contribution it was shown that at the mode cutoff frequencies the branch-line falls off as \(1/r\) while away from the branch-line the frequency falls off as \(1/r^2\).

Table 1 has the value of the branch-line contribution \(|\psi^b|\) for three distances and also the ratios of the contributions. This table shows the general formulas does indeed give \(1/r\) and \(1/r^2\) dependence at the limits of frequencies close to and far away from cutoff frequencies.

As a check on the computations, the magnitude of \(|\psi^b|\) has been computed using the formulas (2.20) and (2.22).

This was done for the case \(C_1 = 1.5\), \(b = .5\), \(d = .5\), and \(R = 50\), which was plotted in Figure 6.

At \(\omega = 10.53\), the formulas (2.39) used in the computed calculations gave \(|\psi^b| = .3627\). The cutoff frequency is \(\omega = 10.535\). At
this frequency formulas (2.22) gave \( |\psi^b| = 0.44381 \). Considering the rapidity at which \( |\psi^b| \) drops away from cutoff, this agreement seems satisfactory. At \( \omega = 11.31 \), a frequency far from cutoff equation (2.20) gave .0010596, while (2.39) gave .001060.

In formula (2.40) the general formula was shown to go to Pekeris' formula (2.20) away from cutoff by using a limit process. In the program this limiting process was not carefully programmed so that far from cutoff frequencies the branch-line contribution is in error due to loss of significance in the machine computation. This is seen in Figures 5 and 6, by the uneveness of some of the values having small magnitude. However, the magnitude of the contribution is so small in these ranges that it was not thought worthwhile to improve the program for the case of small values of the parameter B.
TABLE I

Expected Ratios for $1/r$

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>(20)</th>
<th>(50)</th>
<th>(100)</th>
<th>(20)</th>
<th>(100)</th>
<th>(50)</th>
<th>(100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.51</td>
<td>.6836</td>
<td>.2092</td>
<td>.07812</td>
<td>8.75</td>
<td>2.68</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.53</td>
<td>.9756</td>
<td>.3627</td>
<td>.1670</td>
<td>5.84</td>
<td>2.17</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.55</td>
<td>.8936</td>
<td>.3141</td>
<td>.1360</td>
<td>6.57</td>
<td>2.31</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.57</td>
<td>.6384</td>
<td>.1856</td>
<td>.06588</td>
<td>9.69</td>
<td>2.82</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.59</td>
<td>.4609</td>
<td>.1138</td>
<td>.03499</td>
<td>13.17</td>
<td>3.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.61</td>
<td>.3378</td>
<td>.0733</td>
<td>.02060</td>
<td>16.40</td>
<td>3.56</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.71</td>
<td>.09628</td>
<td>.01612</td>
<td>.004062</td>
<td>23.70</td>
<td>3.97</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.81</td>
<td>.04207</td>
<td>.006791</td>
<td>.001700</td>
<td>24.74</td>
<td>3.99</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Expected Ratios for $1/r^2$

<table>
<thead>
<tr>
<th></th>
<th>(20)</th>
<th>(100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected</td>
<td>25.0</td>
<td>4.0</td>
</tr>
</tbody>
</table>

Cutoff frequency for the third mode is 10.535.
Looking first at the modes individually we will consider several parts of the expression for particle velocity. The term $G(K^{(n)})$ in equation (2.9) has been discussed by Pekeris (1948). He noted that for each mode this factor is 0 at its low frequency cutoff. This occurs because at the low frequency cutoff of the $n$th mode

\[ H \beta_1 = (n - 1/2) \pi \]

makes the term $\tan \beta_1 H$ in the denominator infinitely large. This behavior is seen in Figures 7, 8, and 9 where the amplitude of each mode is 0 at cutoff; then rises rapidly as $H \beta_1$ goes from $(n - 1/2) \pi$ toward $n\pi$ with increasing frequency.

In Figure 10 a plot of $\beta_1 (K^{(n)})$ was made for the first 5 modes. Notice in each case $\beta_1 (K^{(n)})$ initially rises rapidly from its cutoff value of $(n - 1/2) \pi$. The second derivative of $\beta_1$ with respect to $\omega$ is negative; the slope of the $\beta_1$ curve decreases for increasing $\omega$. As $H \beta_1 \rightarrow n \pi$ the value of $G(K^{(n)}) \rightarrow 1/\sqrt{K^{(n)}}$. The factor $1/\sqrt{K^{(n)}}$ causes the mode amplitude to fall off slowly as the frequency increases.

\[ \beta_1 = \sqrt{\frac{c_w^2}{c_r^2} - K^{(n)}^2} \quad (3.1) \]

\[ K^{(n)} = \frac{\omega^2}{c_r^2} - K_r^2 \quad (3.2) \]
This is expanded by the binomial theorem as

\[ K^{(n)} = \frac{\omega}{C_1} - \frac{1}{2} \frac{2 C_1}{\omega} + \text{higher order in } \frac{1}{\omega} \]

Therefore

\[ \frac{1}{\sqrt{K^{(n)}}} \sim \sqrt{\frac{c}{\omega}} \] (3.3)

when \( \omega > > \beta_1 \). The frequency where this approximation is valid increases with mode number. The behavior of \( K^{(n)} \) for the first two modes is plotted in Figure 11. \( K^{(n)} \) is seen to go towards the line \( K^{(n)} = \omega \). The fact that phase velocity curves go to \( C_1 \) for large \( \omega \) reflects this.

A more important factor in determining the amplitude of the mode is the factor \( \sin \beta_1 d \). This gives the amount that the mode is excited by a source at depth \( d \). In Figure 9 a plot has been made of the amplitude versus frequency for the first six modes. The source depth in the case is .5. The amplitudes of the second and fourth modes start to fall off faster than that of the first and third modes. This occurs because \( \beta_1 d \) goes towards \( 2N\pi (0.5) = n\pi \) for the even modes.
while going toward \( (2n+1) \pi \cdot 5 = (n + 1/2) \pi \) for the odd modes.

Several other plots were made to illustrate the behavior of this factor in controlling the mode amplitude. In Figure 7 the amplitude of the first mode was plotted against frequency for the three source depths \( d = .25, .5, .75 \). At the low frequency part of the mode the deepest source gives the largest amplitude. As the frequency increases the source near the bottom become less important. The curve for the source at \( d = .5 \) gives the largest amplitude at high frequencies. The variance of the response with depth for the first two modes is examined more closely in Figure 12. In this plot the amplitude of the mode is plotted against depth of source. This has been done for four frequencies in the range between the cutoff frequencies of the second and third mode. For both modes the behavior of the amplitude is similar at each of the four frequencies. The amplitude for the first mode varies fairly smoothly with depth and has its maximum value near \( d = .65 \). The amplitude is small for a shallow source. The second mode has a small amplitude for small \( d \). This amplitude reaches a maximum near .33 then falls to a value of 0 between .6 and .7. The depth at which the minimum occurs is that where \( \beta_1 d \) is equal to \( \pi \).

As an example, at \( \omega = 7.9 \), \( \beta_1 \) for the second mode is 5. (see Figure 10). Then the amplitude of the second mode should be 0 at
\[ d = \frac{\pi}{5} \sim .63 \]

This apparently occurs as seen in Figure 12.

At depths greater than this minimum, the amplitude rises as the source is moved toward the bottom of the layer.

To see if important changes occur when the velocity \( C_2 \) in the lower layer is varied, the value of the amplitudes of the first two modes versus depth of source were plotted in Figure 13. By comparing this curve for which \( C_2 = 2 \) with Figure 12 in which \( C_1 = 1.5 \), it is seen that the behavior of the mode amplitudes are almost identical. In this figure the mode amplitudes for three depths are plotted for a density contrast of \( b = .8 \) with \( C_2 = 2 \). These points seem to indicate that this model has the same behavior as the models with density contrast of .5.

The phase of each mode contribution is determined by

\[ e^{-i \left[ K^D r + \pi/4 \right]} \]

If the first \( M \) modes are present with amplitudes \( A_i(\omega) \) and phase of \( e^{i \Theta(\omega)} \) the square of the amplitude of their sum is given by

\[
\sum_{\chi=1}^{M} A_i(\omega) + \sum_{K \neq L} A_K A_L \cos(\Theta_K - \Theta_L) \tag{3.5}
\]
For \( M=2 \), this formula gives for the square of the amplitude

\[ A_1^2 + A_2^2 + A_1 A_2 \cos (\theta_1 - \theta_2) \]  

(3.6)

The maximum value then taken is

\[ (A_1 + A_2)^2 \]  

(3.7)

occurring when the modes are in phase. The minimum is

\[ (A_1 - A_2)^2 \]  

(3.8)

which occurs when the modes are \( \pi \) radians out of phase. If the phase difference \( \theta_1 - \theta_2 \) is changing fairly uniformly with \( \omega \) then the difference between frequencies at which successive maxima of the amplitude of the sum occurs is

\[ \Delta \omega = \frac{2\pi}{\frac{d(\theta_1 - \theta_2)}{d\omega}} \]  

(3.9)

This alternating constructive and destructive interaction between the modes is the most striking feature of the curves computed. In the range above the low frequency cutoff of the second mode but below the cutoff of the third mode this behavior is seen to match the curves
almost exactly. (see figures 14 through 24). The phase of the modes are given by

\[ e^{-i \left[ K^{(n)}_r + \pi/4 \right]} \]  

(3.10)

The rate of change is given by

\[ r \frac{d(K^{(1)} - K^{(2)})}{d \omega} \]  

(3.11)

which is proportional to \( r \). It would then be expected that the frequency difference would change much faster with increasing frequency. This is born out by comparing Figures 15, 17, and 18.

The same model and source depth are used in all three curves. The horizontal distances from the source are 20, 50, and 100 respectively. The frequency difference between successive maxima are approximately 2.20 for distance 20, 90 for distance 50, and 45 for distance 100. These differences are roughly inversely proportional to the horizontal distance. It is noticed that the frequency difference between successive minima or maxima increases with frequency.

This occurs because as the frequency increases the value of \( \frac{dK^{(n)}}{d\omega} = i \) (see Figure 11). Therefore \( \frac{d(K^{(1)} - K^{(2)})}{d\omega} \) decreased with frequency.
Plotted in Figure 19 is a curve for a small density contrast of \( d = 0.8 \). This curve is quite similar to the curve in Figure 15. However, the maxima are slightly further apart.

In Figure 20 the velocity contrast in the model is large, \( C_2 = 2.5 \). Here the distance between maxima is very small when compared to Figure 17. Both of these curves are for the amplitude response at a distance 20.

By a knowledge of the size of the mode amplitudes it is possible to make some surmise about the depth of source. However, the modes overlap in the frequency domain, so the magnitude of the contribution to \( |\psi(\omega)| \) from the different modes cannot be measured separately. Still the low excitation of a mode does show up in a curve of amplitude of \( |\psi(\omega)| \) versus frequency. As was discussed, the amplitude of the second mode was very dependent on depth (see Figure 12). If the source depth is near the depth at which the second mode is not excited then the magnitude of \( \psi(\omega) \) does not start to show fluctuating variation in amplitude as the frequency rises above the cutoff of the second mode. In Figure 22 the oscillations of the amplitude are small at frequencies above the cutoff for the second mode. The source in this case is at depth \( d = 0.65 \). At this depth the second mode is not greatly excited until well above the cutoff of the second mode, which is \( \omega = 6.32 \), for the velocity.
As the frequency gets higher the second mode does start to contribute and the amplitude starts to oscillate.

If the amplitude of the first two modes are about equal, as would occur if the source were at a depth near 0.54 or 0.70, then the modes would cancel each other when out of phase. These would then be values of $\omega$ for which the amplitude of their sum would be very small. In Figure 22 a plot made for depth 0.55 shows this to occur.

For very shallow source depth the amplitude of the first mode is small compared to that of the second. Figure 12 shows that for a depth of 0.1 the ratio of the second mode's amplitude to that of the first is approximately 4. For such a shallow source $|\psi|$ would increase greatly above the cutoff of the second mode. The difference between successive maximum and minimums would be approximately twice the magnitude of the first mode.

In Figure 24 the magnitude of $\psi$ has been plotted. From the computer output the first mode is seen to have amplitude of 0.12. Unfortunately, a run was not made that covered the region below the second mode cutoff. However, considering other results, the curve was drawn in for the section between $\omega = 5.5$ and $\omega = 6.1$. The first
mode amplitude of .12 is seen to be very close to half the difference between successive maxima and minima. The amplitude of the first mode changes very slowly at frequency well above its cutoff. The first mode amplitude, then, would have about the same value below the cutoff frequency of the second mode as it has in the frequency range where the second mode is also present. Considering this a shallow source could be detected by a sudden jump from low amplitude to a much higher amplitude with fluctuations between maximum and minimum of about twice the amplitude of the first mode.

Three plots have been made for the frequency range above the cutoff frequency for the third mode. These are presented in Figures 25, 26, and 27. The three magnitude curves show an irregular pattern of highs and lows. There is a certain regularity between the local maxima of the curves, but the size of the maxima varies. It is probable that reflects the behavior that formula (3.5) predicts. The maximum that could be obtained would be the sum of the amplitudes of the three modes. This would occur if all three modes were in phase. By comparing Figures 26 and 27 it is observed that the frequency difference between the maxima is smaller for the larger value of r, as was true in the frequency range where only two modes were present.
APPENDIX A

FLOW CHART AND
PROGRAM LISTING
FLOW CHART for program used for computations
SQRY = SQRTF(Y*S)*XI
XX = SQRY(1)
YY = SQRY(2)
CALL CERR(XX, YY, ERR1, ERR2, NO)
ERR(1) = ERR1
ERR(2) = ERR2

EFCN = ONE - EXPF(-S*Y)*ERR
F1 = EFCN - ONE

F2 = SQRY*PIE*F1*EXPF(Y*S)
F3 = (SQRTF(PIE/5)+F2)/(XI*S)

BRNCH = F3*CG
RB(JW) = BRNCH(1)
FB(JW) = BRNCH(2)
GO TO 23

C FOR B=0

RE3L = 2.*DEN*XX2
DENN = XK1(JW) * R(JR)**2*XMU
RB(JW) = REAL*COSF(PIE/2.*XX2*R(JR))/DENN
FB(JW) = -REAL*SINF(PIE/2.*XX2*R(JR))/DENN
CONTINUE

DO 40 JD = 1, ND
WRITE OUTPUT TAPE NZ, 112, C1, C2, DEN, H, R(JR), D(JD)

112 FORMAT(1H14X3HC1=F8.5, 4X3NC2=F8.5, 4X4HDEN=F8.5, 4X6HHR=F8.5, 4X2HR=F10.4, 4X6HDEPTH=F10.4//)
WRITE OUTPUT TAPE N2, 131

110 FORMAT(1H F8.5, 7X6F19.6)
DO 39 JW = 1, NW
TRP = 0.
TFP = 0.
NPD = NP(JW)
DO 32 JP = 1, NPD
DFP = SINF(BW(JW, JP) + D(JD))
RP(JP) = RPOLE(JW, JP) + DFP
FP(JP) = FPOLE(JW, JP) + DFP
POLM(JP) = SQRTF(RP(JP) ** 2 + FP(JP) ** 2)
TRP = TRP + RP(JP)

TFF = TFP + FP(JP)
TOTP = SQRTF(TRP ** 2 + TFP ** 2)
DFB = SINF(XK1(JW)*XMU*U(JD))
TR3 = RB(JW) * DFB
TFB = FB(JW) * DFB
BRMAG = SQRTF(TR3 ** 2 + TFB ** 2)
XMAG = SQRTF((TRP + TFB) ** 2 + (TFP + TFB) ** 2)

114 FORMAT(1H F8.5, 2X6(F11.4, F7.3))

CONTINUE

WRITE OUTPUT TAPE NZ, 112, C1, C2, DEN, H, R(JR), D(JD)
WRITE OUTPUT TAPE N2, 135

DO 42 K = 1, NW
WRITE OUTPUT TAPE N2, 110, W(K), TOT(K), BR(K), XMAG(K)

42 FORMAT(1H TOT(K) = TOTP, BR(K) = BRMAG, XMAG(K) = XMAG)
CONTINUE

WRITE OUTPUT TAPE NZ, 112, C1, C2, DEN, H, R(JR), D(JD)
WRITE OUTPUT TAPE N2, 135

131 FORMAT(1H F11X6HMODE 11X6HMODE 21X6HMODE 31X6HMODE 41X6HMODE 51X6HMODE 61XHMODE)

135 FORMAT(1H HOANG, FR, 11X11HTOTAL POLE5X6HBRANCH10X5HTOTAL/)
40 CONTINUE
50 CONTINUE
GO TO 1001
END
SUBROUTINE ROOTS(W, H, DEN, C1, C2, JW, NP, XK, PIE)
DIMENSION XK(100, 20), NP(100), YR(100, 20)

PERDF(BET) = BET / (SQRF(SK1 - SK2 - BET**2)) + TAN(BET * H) * DEN

1      XK1 = W / C1
      XK2 = W / C2
      SK1 = XK1**2
      SK2 = XK2**2
      NP(JW) = M * SQRF(SK1 - SK2) / PIE + 4999999
      NP = NP(JW)

2      DO 9 JP = 1, NP
            NIT = 0
            RH = FLOATF(JP) * PIE
            RL = RH - 49999 * PIE
            RHMAX = SQRF(SK1 - SK2)
            IF(RH - RHMAX) 11, 10, 10
            RH = RHMAX - 0.00001
            CONTINUE

3      DO 36 ITER = 1, 15
            RT = 0.5 * (RH + RL)
            IF(PERDF(RT)) 30, 30, 31

4      RL = RT
            GO TO 36

5      RH = RT
            CONTINUE

6      PERL = PERDF(RL)
            PERH = PERDF(RH)

7      SLJAP = (PERH - PERL) / (KH - RH)
            TEMPR = RL - PERL / SLJAP
            YR(JW, JP) = PERDF(TEMPR)

8      XK(JW, JP) = SQRF(SK1 - TEMPR**2)

9      RETURN
END
MODEL

VACUUM

\[ \rho_1, C_1 \]

\[ \rho_2, C_2 \]

\[ C_2 > C_1 \]

\[ H \]

\[ d \]

\[ \text{source} \]

\[ \rightarrow \]

\[ \rightarrow \]

\[ \downarrow \]

\[ \downarrow \]

\[ Z \]

FIGURE 1

K PLANE

FIGURE 2
DIAGRAM TO EXPLAIN
REFRACTION PHENOMENA

![Diagram](image)

**Figure 3**

GRAPH ILLUSTRATING SOLUTION
OF THE PERIOD
EQUATION

![Graph](image)

**Figure 4**
FIGURE 5
Magnitude of the branch line versus frequency

\( C_2 = 1.5, \; b = 0.5, \; r = 50, \; d = 0.5 \)
FIGURE 6
Magnitude of the branch line versus frequency

$C_2 = 1.5, \quad b = .5, \quad r = 50$

$d = .5$
Magnitude of the first mode versus frequency for three values of $d$

$C_2 = 2$, $b = .8$, $d = .5$

$\psi^{\text{max}}$

$|d = .75|$

$|d = .5|$

$|d = .25|$
\[ \beta(K^{(1)}) \text{ versus frequency} \]

\( C_2 = 1.5, \quad b = 0.5 \)
FIGURE 9

Magnitude vs. frequency for 5 modes

$C_2 = 2$, $b = 5$, $r = 50$, $d = 5$
FIGURE 10

$\theta_n \left( \frac{\Omega}{\omega} \right)$ vs. frequency for 5 modes

$C_2 = 1.5, b = .5$

MODE 5

MODE 4

MODE 3

MODE 2

MODE 1

\$40\ MASS.\ AVE.,\ CAMBRIDGE,\ MASS.\$

\$40\ MASS.\ AVE.,\ CAMBRIDGE,\ MASS.\$

\$40\ MASS.\ AVE.,\ CAMBRIDGE,\ MASS.\$

\$40\ MASS.\ AVE.,\ CAMBRIDGE,\ MASS.\$
FIGURE 11

$K^N$ vs. frequency for first 2 modes

$C_2 = 1.5, \quad b = .5$
FIGURE 12
Magnitude of \(|\mathbf{N}|\) vs. depth for 2 modes. Plotted for 4 values of frequency

\(C_s = 1.5, \quad b = 0.5, \quad r = 50\)

- \(\omega = 7.9\)
- \(\omega = 9.9\)
- \(\omega = 6.9\)
- \(\omega = 8.9\)
FIGURE 13
Magnitude of $\psi_{1}(N)$ vs. depth for first two modes

$C_1 = 2$, $b = 0.5$, $r = 50$, $\omega = 7$

$\triangle$ indicate 1st mode with $b = 0.8$
$\circ$ indicate 2nd mode with $b = 0.8$
FIGURE 14
Magnitude of $\psi$ vs. frequency.

$C_2 = 1.5$, $b = 0.5$, $r = 50$, $d = 0.25$

Vertical arrow indicates mode cutoff.
FIGURE 15

Magnitude of $\psi$ vs. frequency

$c_2 = 1.5, b = .5, r = 50, d = .5$

Vertical arrow indicates mode cutoff
FIGURE 16

Magnitude of $\psi$ vs. frequency

$C_2 = 1.5$, $b = .5$, $r = 50$, $d = .75$
FIGURE 17

Magnitude of \( \psi \) vs. frequency

\( C_2 = 1.5, b = .5, r = 20, d = .5 \)

Vertical arrow indicates mode cutoff
FIGURE 18

Magnitude of $\psi$ vs. frequency
$C_2 = 1.5$, $b = .5$, $r = 100$, $d = .5$
Vertical arrow indicates mode cutoff
FIGURE 19

Magnitude of $\psi$ vs. frequency.

$C_2 = 1.5$, $b = .8$, $r = 50$, $d = .5$

Vertical arrow indicates mode cutoff
FIGURE 20

Magnitude of $\psi$ vs. frequency

$C_2 = 2.5$, $b = .5$, $r = 20$, $d = .5$
FIGURE 21

Magnitude of $\psi$ vs. frequency

$C_2 = 2.5$, $b = .5$, $r = 100$, $d = .75$
FIGURE 22

Magnitude of $\psi$ vs. frequency

$C_2 = 1.5, \ b = .5, \ r = 50, \ d = .65$
FIGURE 23

Magnitude of $\psi$ vs. frequency

$c_2 = 1.5, b = 0.5, r = 50, d = 0.55$
FIGURE 24
Magnitude of $|\phi|$ vs. frequency
$C_2 = 1.5$, $b = 0.5$, $r = 50$, $d = 5$
Vertical arrow indicates mode cutoff.
FIGURE 25

Magnitude of $|\psi|$ vs. frequency

$C_2 = 1.5, \ b = .5, \ r = 100, \ d = .75$
**FIGURE 26**

Magnitude of $|\psi|$ vs. frequency.

$C_2 = 1.5$, $b = 0.5$, $r = 100$, $d = 1.5$

Vertical arrow indicates mode cutoff.
FIGURE 27
Magnitude of $\psi$ vs. frequency
$C_2 = 1.5, b = .5, \tau = 50, d = .5$
Vertical arrow indicates mode cutoff
BIBLIOGRAPHY


