A. MEASUREMENT OF THE KERNELS OF A NONLINEAR SYSTEM BY CROSSCORRELATION WITH GAUSSIAN NON-WHITE INPUTS

In the Wiener theory a nonlinear system is characterized by the kernels, $h_n$, of the functionals, $G_n$. For a Gaussian white-noise input, the functionals, $G_n$, are orthogonal and their kernels, $h_n$, can be determined by crosscorrelating the output with a multidimensional delay of the input. In this report a procedure is developed for determining the kernels of a set of functionals which are orthogonal for a Gaussian non-white input process.

Consider that the system $N$ shown in Fig. VIII-1 is to be characterized with an input, $z(t)$, which is a Gaussian non-white process. We shall assume that the power density spectrum, $\Phi_{zz}(\omega)$, of the input, $z(t)$, is factorizable. It then can be written

$$\Phi_{zz}(\omega) = \Phi^+_zz(\omega) \Phi^-zz(\omega)$$

in which $\Phi^+_zz(\omega)$ is the complex conjugate of $\Phi^-zz(\omega)$; also, all of the poles and zeros of $\Phi^+_zz(\omega)$ are in the upper half of the complex $\lambda$-plane. Thus $\Phi^+_zz(\omega)$ and $1/\Phi^+_zz(\omega)$ are each realizable as the transfer function of a linear system. We can then consider the system of Fig. VIII-1 in the equivalent form shown in Fig. VIII-2, in which the transfer functions of the two linear systems, $k_1(t)$ and $k_2(t)$, are:
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\[
K_1(\omega) = \frac{1}{\Phi_{zz}(\omega)} \\
K_2(\omega) = \Phi_{zz}^+(\omega)
\]

(2)

Also, as shown, the system A is the system formed by the tandem connection of the linear system \(k_2(t)\) and the system N. We observe that the input to the system A is a Gaussian white process whose power density spectrum is 1 watt per radian per second. Thus, as previously described, the kernels, \(h_n\), of the orthogonal functionals, \(G_n\), for the system A are

\[
h_n(\tau_1, \ldots, \tau_n) = \frac{(2\pi)^n}{n!} \frac{y(t) x(t-\tau_1) \ldots x(t-\tau_n)}{x(t)}
\]

(3)

except when, for \(n > 2\), two or more \(\tau_i's\) are equal.

We therefore need to know the crosscorrelation function

\[
\phi_{yx}(\tau_1, \ldots, \tau_n) = y(t) x(t-\tau_1) \ldots x(t-\tau_n)
\]

(4)

in order to determine the kernels, \(h_n\), of the system A. Since only \(z(t)\) is available to us, we shall express the desired correlation function in terms of the crosscorrelation between the output, \(y(t)\), and a multidimensional delay of the input, \(z(t)\). By substituting the relation

\[
x(t) = \int_0^\infty k_1(\sigma) z(t-\sigma) \, d\sigma
\]

(5)

in Eq. 4, the desired correlation function can be expressed as

\[
\phi_{yx}(\tau_1, \ldots, \tau_n) = \int_0^\infty k_1(\sigma_1) d\sigma_1 \ldots \int_0^\infty k_1(\sigma_n) d\sigma_n \phi_{yz}(\tau_1-\sigma_1, \ldots, \tau_n-\sigma_n)
\]

(6)

in which

\[
\phi_{yz}(\tau_1, \ldots, \tau_n) = y(t) z(t+\tau_1) \ldots z(t+\tau_n)
\]

(7)

is the crosscorrelation between the output, \(y(t)\), and a multidimensional delay of the input, \(z(t)\). In the frequency domain, Eq. 6 can be expressed as

\[
\Phi_{yx}(\omega_1, \ldots, \omega_n) = K_1(\omega_1) K_1(\omega_2) \ldots K_1(\omega_n) \Phi_{yz}(\omega_1, \ldots, \omega_n)
\]

(8)

in which
\[ \Phi_{yz}(\omega_1, \ldots, \omega_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} e^{-j\omega_1 \tau_1} d\tau_1 \cdots \int_{-\infty}^{\infty} e^{-j\omega_n \tau_n} d\tau_n \Phi_{y\tau}(\tau_1, \ldots, \tau_n) \] (9)

and the transfer function \( K_1(\omega) \) is given by

\[ K_1(\omega) = \int_{-\infty}^{\infty} k_1(t) e^{-j\omega t} dt \] (10)

Substituting Eq. 2 in Eq. 8, we have the desired relation in the frequency domain.

\[ \Phi_{yx}(\omega_1, \ldots, \omega_n) = \frac{\Phi_{yz}(\omega_1, \ldots, \omega_n)}{\Phi_{zz}(\omega_1) \cdots \Phi_{zz}(\omega_n)}. \] (11)

Either Eq. 6 or Eq. 11 can be used to determine the kernels, \( h_n \), as given by Eq. 3 in terms of the measured crosscorrelation function between the output and a multidimensional delay of the input, \( z(t) \).

Once the kernels \( h_n \) have been determined, a representation of the system \( N \) is as given in Fig. VIII-3a which can be redrawn as shown in Fig. VIII-3b. We note from Fig. VIII-3b that the outputs of the parallel branches are orthogonal for the input \( z(t) \). Thus, we have expanded the nonlinear system \( N \) in a set of functionals that are orthogonal for Gaussian inputs with a power density spectrum of \( \Phi_{zz}(\omega) \). Note that for this procedure, we never need construct either of the linear systems \( k_1(t) \) or \( k_2(t) \). Once the representation in the form of Fig. VIII-3b is known, the functionals that are orthogonal for the input \( z(t) \) can be calculated. We shall call these functionals \( L_n \). The first few functionals are:

\[ L_1[h_1, z(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_1(\sigma_1) k_1(\tau_1 - \sigma_1) x(t - \tau_1) d\sigma_1 d\tau_1 \] (12)

\[ L_2[h_2, z(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\sigma_1, \sigma_2) k_1(\tau_1 - \sigma_1) k_1(\tau_2 - \sigma_2) x(t - \tau_1) x(t - \tau_2) d\sigma_1 d\sigma_2 d\tau_1 d\tau_2 \]

\[ - 2\pi \int_{-\infty}^{\infty} h_2(\sigma_2, \tau_2) d\sigma_2 \] (13)

\[ L_3[h_3, z(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_3(\sigma_1, \sigma_2, \sigma_3) k_1(\tau_1 - \sigma_1) k_1(\tau_2 - \sigma_2) k_1(\tau_3 - \sigma_3) x(t - \tau_1) x(t - \tau_2) x(t - \tau_3) d\sigma_1 d\sigma_2 d\tau_1 d\tau_2 d\tau_3 \]

\[ - 3(2\pi)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_3(\sigma_1, \sigma_2, \tau_2) k_1(\tau_1 - \sigma_1) x(t - \tau_1) d\sigma_1 d\sigma_2 d\tau_1 \] (14)
Fig. VIII-3. Representation of the expansion for system N.
in which
\[ k_1(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\Phi_z^+(\omega)} d\omega. \]  \hspace{1cm} (15)

In terms of these functionals, the output of the system \( N \) for an input \( z(t) \) can be written
\[ y(t) = \sum_{n=1}^{\infty} \mathbb{L}_n[h_n, z(t)]. \]  \hspace{1cm} (16)

In this manner, we have characterized the nonlinear system \( N \) with a Gaussian non-white input in terms of a set of functionals, \( \mathbb{L}_n \), that are orthogonal for the given input process.

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References


