

## XIV. ELECTRODYNAMICS OF MOVING MEDIA

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### RESEARCH OBJECTIVES

We are developing the fundamental equations of electrodynamics (for both electromagnetism and dynamics) of moving and deforming material, especially dielectric and magnetic material. The ultimate goal is the determination of the macroscopic force acting on a continuum, and the distribution of that force as a function of space and time. The fundamental equations must be consistent with (a) the special theory of relativity; (b) electromagnetism of free space; (c) electromagnetism of stationary bodies; (d) relativistic continuum mechanics of solids or fluids without electromagnetic fields; (e) thermodynamic measurements of the various kinds of power flow; and (f) nonlinear material properties. Thus far we have obtained equations for nondispersive conservative media. Work is now in progress on media showing dispersion both in time and/or space. We intend to extend the equations to irreversible phenomena. Also, the connection between the equations for nonlinear media and their corresponding small-signal formulation will be explored.

Use of these equations in specific situations is contemplated (a) to illustrate the equations and the various approximations, and (b) to predict novel effects, with a view toward testing for these effects experimentally.

A history of this problem was given in Quarterly Progress Report No. 70, July 15, 1963, pp. 79-82.

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### A. INTERPRETATION OF RELATIVISTIC FORCE DENSITY IN MOVING POLARIZABLE AND MAGNETIZABLE MEDIA

We have previously reported<sup>1</sup> an expression for the relativistic force density within a moving and deforming magnetizable and polarizable conservative (isotropic) fluid in an electromagnetic field. The force density is

$$\begin{aligned} \vec{f}^k = & -\nabla \pi_o + (\vec{P} \cdot \nabla) \vec{E} + (\mu_o \vec{M} \cdot \nabla) \vec{H} + \vec{v} \times (\vec{P} \cdot \nabla) \mu_o \vec{H} - \vec{v} \times (\mu_o \vec{M} \cdot \nabla) \epsilon_o \vec{E} \\ & + \left[ \frac{\partial}{\partial t} \vec{P} + \nabla \cdot (\vec{v} \vec{P}) \right] \times \mu_o \vec{H} - \left[ \frac{\partial}{\partial t} (\mu_o \vec{M}) + \nabla \cdot (\vec{v} \mu_o \vec{M}) \right] \times \epsilon_o \vec{E} \\ & - \frac{\partial}{\partial t} \left[ \frac{\gamma^2 \vec{v}}{c^2} (\pi_o + W_o) + \vec{G}_N \right] - \nabla \cdot \left\{ \vec{v} \left[ \frac{\gamma^2 \vec{v}}{c^2} (\pi_o + W_o) + \vec{G}_N \right] \right\}. \end{aligned} \quad (1)$$

Here,  $\vec{P}$  is the polarization density,  $\vec{M}$  is the magnetization density,  $\pi_o$  is the sum of the electrostrictive, magnetostrictive, and hydrostatic pressures as observed in the rest frame,  $\vec{v}$  is the velocity of the fluid, and

$$\vec{G}_N = -\frac{\gamma^2}{c^2} [(\vec{v} \cdot \vec{P})(\vec{E} + \vec{v} \times \mu_o \vec{H}) + (\vec{v} \cdot \mu_o \vec{M})(\vec{H} - \vec{v} \times \epsilon_o \vec{E})]. \quad (2)$$

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This expression has been obtained in two ways: by application of Hamilton's principle to an appropriate Lagrangian, and by the principle of virtual work applied to a fluid element in its rest frame and a subsequent relativistic transformation. For both of these derivations, thermodynamic information was used to obtain the appropriate energy function.

It is possible to obtain the same expression for the force density without the pressure term  $\pi_0$  from a simple model of a polarizable and magnetizable fluid without recourse to thermodynamics. This model gives insight into the significance of the various terms and, therefore, is presented here. For sake of brevity, and without loss of generality, we shall limit ourselves to a polarizable fluid because the way to include magnetization effects will be self-evident. We shall then show how the pressure term may be included phenomenologically.

We consider a fluid consisting of noninteracting electric dipoles, each of vector strength

$$\bar{p} = q\bar{\delta}, \quad (3)$$

where  $q$  is the magnitude of the charges, and  $\bar{\delta}$  is the vector distance between them. We interpret the dipole strength (3) as that observed in the laboratory frame in which the dipole (its center of mass, to be precise) may possess an arbitrary velocity  $\bar{v}$ . If this velocity is relativistic, the dipole strength  $\bar{p}$  observed in the laboratory frame is not the same as that observed in the rest frame. The dipole strengths differ because of the relativistic contraction of the component of  $\bar{\delta}$  in the direction of motion. For simplicity, we assign the entire mass of the dipole to the negative-charge portion of the dipole and ascribe to it the velocity  $\bar{v}(\bar{r})$ . The positive charge of the dipole then moves with velocity

$$\bar{v}_+(\bar{r}) = \bar{v}(\bar{r}) + \frac{d\bar{\delta}}{dt}.$$

The forces acting on the dipoles are assumed to be long-range forces only. Thus a fluid of this kind cannot exhibit pressure or electrostrictive effects and we cannot expect to reproduce the term  $\nabla\pi_0$  in Eq. 1. The net force acting on a dipole as evaluated in the laboratory frame is obtained as the vector sum of the force acting on the positive charge of the dipole and that acting on the negative charge. We must take into account the possible variations of the electric and magnetic fields over the distance  $\bar{\delta}$ . We obtain

$$\begin{aligned} \bar{F}_d &= q[\bar{E}(\bar{r}+\bar{\delta}) + \bar{v}_+(\bar{r}) \times \mu_0 \bar{H}(\bar{r}+\bar{\delta})] - q[\bar{E}(\bar{r}) + \bar{v}(\bar{r}) \times \mu_0 \bar{H}(\bar{r})] \\ &= q\bar{\delta} \cdot \nabla \bar{E} + q\bar{v} \times \mu_0 (\bar{\delta} \cdot \nabla) \bar{H} + q \frac{d\bar{\delta}}{dt} \times \mu_0 \bar{H}. \end{aligned} \quad (4)$$

The polarization density  $\bar{P}$  of the dielectric fluid is the dipole strength per unit volume.

If  $N$  is the number of dipoles per unit volume, then

$$\bar{\mathbf{P}} = Nq\bar{\delta}. \quad (5)$$

The expression for  $\bar{\mathbf{F}}_d$  in Eq. 4 can be converted into a force density per unit volume by adding the forces acting on all dipoles per unit volume

$$\bar{\mathbf{f}}_d = N\bar{\mathbf{f}}_d = \bar{\mathbf{P}} \cdot \nabla \bar{\mathbf{E}} + \bar{\mathbf{v}} \times \mu_o \bar{\mathbf{P}} \cdot \nabla \bar{\mathbf{H}} + \left[ \frac{\partial \bar{\mathbf{P}}}{\partial t} + \nabla \cdot (\bar{\mathbf{v}} \bar{\mathbf{P}}) \right] \times \mu_o \bar{\mathbf{H}}. \quad (6)$$

Here, we have used the continuity law satisfied by the dipole number density

$$\frac{\partial}{\partial t} N + \nabla \cdot (\bar{\mathbf{v}} N) = 0. \quad (7)$$

The entire force density (6) is not utilized in the acceleration of the fluid mass density (considered here to be associated with the negative charge). The binding forces between the positive and negative charges store energy. Energy in motion acquires momentum according to the relativistic transformation laws. Furthermore, a stress is introduced on a volume element  $dv$  containing the mass associated with the negative charges. Indeed, if one draws the boundaries of this volume element in the rest frame taken to be a rectangular parallelepiped, one finds that the binding forces of all those positive charges just outside the element, whose negative partners are inside the volume element, pass through the boundary. The net force passing the boundary  $dy_o dz_o$  is equal to the product of the force ( $q\bar{\mathbf{E}}_o$ ) within each dipole times the number of dipoles piercing this boundary ( $N\delta dy_o dz_o$ ). For the net force per unit area we have

$$\bar{\mathbf{n}} \cdot \bar{\mathbf{P}}_o \bar{\mathbf{E}}_o,$$

where  $\bar{\mathbf{n}}$  is the unit vector normal to the element  $dy_o dz_o$  and the subscript  $o$  indicates that the quantities are those observed in the rest frame. We conclude that the polarization of the fluid is accompanied by a stress of tensor strength  $\bar{\mathbf{P}}_o \bar{\mathbf{E}}_o$ . Relativistically, a stress in motion possesses momentum. The force density  $\bar{\mathbf{f}}_d$  in Eq. 6 must overcome the time rate of change of this momentum, that of the momentum associated with the energy stored in the dipoles, and the inertia associated with the mass of the fluid. The momentum associated with the stress energy tensor in the rest frame,

$${}^0 T_{ij}^P = \begin{bmatrix} -\bar{\mathbf{P}}_o \bar{\mathbf{E}}_o & \vdots & 0 \\ \dots\dots\dots & \vdots & \dots\dots\dots \\ 0 & \vdots & -W_o \end{bmatrix}, \quad (8)$$

can be found by conventional relativistic transformation methods for a stress-energy tensor as  $\bar{\mathbf{G}}^P = \gamma^2 W_o \bar{\mathbf{v}}/c^2 + \bar{\mathbf{G}}_N$ , where  $\bar{\mathbf{G}}_N$  is given in Eq. 2. The force density  $\bar{\mathbf{f}}^k$  producing the acceleration of the fluid is that portion of  $\bar{\mathbf{f}}_d$  that remains after the force density responsible for the total rate of change of the momentum  $\bar{\mathbf{G}}^P$  has been subtracted.

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$$\begin{aligned} \vec{f}^k = \vec{f}_d - \frac{\partial}{\partial t} \vec{G}^P - \nabla \cdot (\vec{v} \vec{G}^P) = \vec{P} \cdot \nabla \vec{E} + \vec{v} \times (\vec{P} \cdot \nabla) \mu_o \vec{H} + \left[ \frac{\partial \vec{P}}{\partial t} + \nabla \cdot (\vec{v} \vec{P}) \right] \times \mu_o \vec{H} \\ - \frac{\partial}{\partial t} \left[ \frac{\vec{v}}{c^2} \gamma^2 W_o + \vec{G}_N \right] - \nabla \cdot \left\{ \vec{v} \left[ \frac{\vec{v}}{c^2} \gamma^2 W_o + \vec{G}_N \right] \right\}. \end{aligned} \quad (9)$$

This equation agrees with (1) if magnetization effects are omitted and  $\pi_o$  is set equal to zero. In this way, we have found a simple interpretation for most of Eq. 1. The generalization to a fluid exhibiting magnetization is easy. Pressure effects and electrostrictive effects can be included phenomenologically by assigning, instead of the tensor (8), a new energy momentum tensor in the rest frame

$${}^0T_{ij}^p = \begin{bmatrix} \pi_o \delta_{ij} - \vec{P}_o \vec{E}_o & \vdots & 0 \\ \dots & \dots & \dots \\ 0 & \vdots & -W_o \end{bmatrix}. \quad (10)$$

The kinetic and electrostrictive pressure contribution  $\pi_o$  is entered in the stress part of the rest-frame energy momentum tensor. The inclusion of  $\pi_o$  has two obvious consequences. First of all, it contributes through its gradient to the force density; second, it contributes to the momentum when the tensor (10) is evaluated in the laboratory frame. This is exactly the way in which the inclusion of pressure affects Eq. 1.

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References

1. P. Penfield, Jr., Force of electromagnetic origin in fluids, Quarterly Progress Report No. 70, Research Laboratory of Electronics, M.I.T., July 15, 1963, pp. 82-88.