COMPLEMENTARY BASES OF A MATROID

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Reproduction in whole or in part is permitted for any purpose of the United States Government
Let $e_1, e'_1, e_2, e'_2, \ldots, e_n, e'_n$ be the elements of matroid $M$.

Suppose that $\{e_1, e'_2, \ldots, e_n\}$ is a base of $M$ and that every circuit of $M$ contains at least $m+1$ elements. We prove that there exist at least $2^m$ bases, called complementary bases, of $M$ with the property that only one of each complementary pair $e_j, e'_j$ is contained in any base.

We also prove an analogous result for the case where $E$ is partitioned into $E_1, E_2, \ldots, E_n$ and the initial base contains $|E_j| - 1$ elements from partition $E_j$. 
I. Introduction

Let \( e_1, e_1', e_2, e_2', \ldots, e_n, e_n' \) be the elements of matroid \( M \). A police force of \( n \) men wishes to patrol a base of the matroid. Suppose that policeman \( k \) is only allowed to patrol either \( e_k \) or \( e_k' \) and that \( \{ e_1, e_2, \ldots, e_n \} \) is a base of the matroid, called a patrolable (complementary) base. How many patrolable bases are there?

In [2], Dantzig considered graphic matroids. He showed that if there is one patrolable spanning tree of a graph \( G \) (with no loops or parallel edges), then there necessarily must be a second. Later Adler [1] strengthened this result by showing that in this case there must necessarily be at least four.

Here we show that these results extend to general matroids and that the number of complementary bases depends upon the circuit structure of the matroid. If every circuit of \( M \) contains more than \( m \) elements, we show that there are at least \( 2^m \) complementary bases. We also generalize this result to the case when policeman \( k \) can patrol \( d_k \) of the \( (d_k+1) \) elements of the matroid assigned to him.

Our proofs are based on Lawler's matroid intersection algorithm [4] and provide an algorithm for determining alternate complementary bases. These proofs could just as well have been based upon Theorem 2c of [3]. In fact, Dantzig's paper was motivated by a discussion held with Fulkerson concerning the results in [3]. Dantzig and Adler's results were based upon complementary pivot theory of mathematical programming. Consequently, our results strengthen the well-known link (via the assignment problem of network theory, for example) between matroid algorithms and standard results in mathematical programming.
II. Preliminaries

If $S \subseteq E$ and $e \in E$, we denote $S \cup \{e\}$ and $S \setminus \{e\}$ by respectively $S+e$ and $S-e$. Also $|S|$ denotes the cardinality of $S$. Beside the matroid intersection algorithm, we only require very basic properties of matroids, so for convenience we review them here.

For our purposes, we define a matroid $M$ to be a finite set $E$ together with a non-empty family $\mathcal{I}$ of subsets of $E$ satisfying

(i) $I \subseteq I_1 \in \mathcal{I}$ implies $I \in \mathcal{I}$

(ii) Given $A \subseteq E$ all maximal sets (with respect to inclusion) of $\mathcal{I}$ that are contained in $A$ have the same cardinality.

Maximal sets in $\mathcal{I}$ are called bases of $M$. Subsets of $E$ not contained in $\mathcal{I}$ are called dependent sets and minimal dependent sets are referred to as circuits.

These definitions easily imply:

(M1) If $S \subseteq E$ and $\mathcal{I}(S) = \{I \cap S : I \in \mathcal{I}\}$, then $(S, \mathcal{I}(S))$ is a matroid.

(M2) Suppose that $E = \bigcup_{i=1}^{n} E_i$ where the sets $E_i$ are pairwise disjoint, and that $d = (d_1, \ldots, d_n)$ is a vector with non-negative integer components. Then with $\mathcal{I} = \{I \subseteq E : |I \cap E_i| \leq d_i\}$, $(E, \mathcal{I})$ is a matroid called a $d$-partition matroid. When each $d_i = 1$, we call $(E, \mathcal{I})$ a unitary partition matroid.

(M3) If $I$ is a base of $M$ and $e \in I$, then $I + e$ contains a unique circuit.

Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two matroids over the same ground set $E$. A maximum cardinality set in $\mathcal{I}_1 \cap \mathcal{I}_2$ may be found by using the following procedure.
Matroid Intersection Algorithm [4]: Assume $E = \{e_1, ..., e_n\}$. Let $L: E \rightarrow \{0,1, ..., n+1\}$ be a labelling function to be defined during the algorithm and let $S$ be a set of "scanned elements." We call an element $e$ of $E$ labelled if $L(e) \in \{0, ..., n\}$ and scanned if $e \in S$.

Initially, $L(e_j) = n+1$ for all $e_j$ and $S = \emptyset$. Assume that a set $I \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$ is given (possibly $I = \emptyset$).

(a) For $1 \leq j \leq n$ and $e_j \notin I$, if $I + e_j \notin \mathcal{A}_1$, set $L(e_j) = 0$.

(b) If all labelled elements have been scanned, terminate. $I$ is optimal. Otherwise go to (c).

(c) Let $e_j$ be labelled and unscanned. Change $S$ to $S + e_j$. If $e_j \in I$, set $L(e) = j$ for each unlabelled $e \in E - I$ such that $I - e_j + e \notin \mathcal{A}_1$; then go to (b). If $e_j \notin I$, then if there is a circuit $C$ of $M_2$ with $C \subseteq I + e_j$, set $L(e) = j$ for each unlabelled $e \in C$ and go to (b). If no such $C$ exists, go to (d).

(d) (Augment): Define $e_{j_1}, ..., e_{j_m} = e_j$ by $j_k = L(e_{j_k}) \neq 0$ for $k=1, ..., m-1$ and $L(j_1) = 0$. Replace $I$ with $I + e_{j_1} - e_{j_2} + ... - e_{j_{m-1}} + e_{j_m}$. Reinitiate $L(e_j) = n+1$ for all $e_j$, $S = \emptyset$, and return to (a).

III. Complementary Bases of a Matroid

Suppose that $E = \{e_1, e'_1, e_2, e'_2, ..., e_n, e'_n\}$, that $M_1 = (E, \mathcal{A}_1)$ is a matroid and that $I = \{e_1, e_2, ..., e_n\}$ is a base of $M_1$. Note that $I$ contains only one element from each of the complementary pairs $e_j, e'_j$. We will call such a base a complementary base of $M_1$. Defining $\mathcal{A}_2 = \{I \subseteq E : |I \cap \{e_j, e'_j\}| \leq 1, j=1, ..., n\}$, $M_2 = (E, \mathcal{A}_2)$ is a unitary partition matroid and complementary bases of $M_1$ coincide with maximum cardinality sets in $\mathcal{A}_1 \cap \mathcal{A}_2$. 
Let $N_{M_1}$ denote the number of complementary bases of $M_1$ and as $n$ varies let $\mathcal{M}$ denote the collection of matroids $M_1$ with the above properties. In this section we prove certain lower bounds on $N_{M_1}$. The results depend heavily upon the circuit structure of $M_1$. Accordingly, we define

$$N(m) = \min \{ N_M : M \in \mathcal{M} \text{ and every circuit of } M \text{ contains } > m \text{ elements} \}.$$ 

**Theorem 1:** $N(1) \geq 2$.

**Proof:** We consider an arbitrary $M_1(E, \mathcal{J}_1)$ contained in $\mathcal{M}$ with the property that every circuit of $M_1$ has $\geq 2$ elements and show that $N_{M_1} \geq 2$.

Let $\mathcal{J}_2$ be defined as above. Starting with $I = \{ e_1, \ldots, e_n \}$ suppose that we remove $e_1$ from the system producing the submatroids $(E, \mathcal{J}_1(E))$ and $(E, \mathcal{J}_2(E))$ where $E = E - e_1$. Apply the matroid intersection algorithm to the set $I - e_1$ contained in $\mathcal{J}_1(E) \cap \mathcal{J}_2(E)$, beginning by labelling all $e \in E$ such that $I - e + e \in \mathcal{J}_1(E)$. Note that we reach step (d) of the algorithm and produce a new complementary basis of $M_1$ if and only if $e'$ is labelled. Thus, assume $e'$ is not labelled and let $L_1$ be the set of labelled elements. Repeat, dropping $e_2, e_3, \ldots, e_n$ in turn instead of $e_1$ from $E$ producing the sets of labelled elements $L_2, \ldots, L_n$. If $e'_j \in L_j$ for any $j$, step (d) is applied providing the result. We show that it must be true that $e'_j \in L_j$ for some $j$.

Suppose not. Then every labelled element in the $n$ applications of the matroid intersection algorithm above will be scanned. But then, the nature of the optimal matroid intersection algorithm (i.e., an unlabelled element is labelled based only upon the element being scanned) implies that if $e'_j \in L_j$, then $L_j \subseteq L_1$. Also, since $I$ is a base of $M_1$ and no circuits have length 1, $I + e'_j$ contains a circuit $C$ of $M_1$ with some $e_i \in C$. Property M3 of matroids states that $I - e_i + e'_j \in \mathcal{J}_1$, thus $e'_j \in L_1$. Start with $e'_1 \in L_2$, say. When $e'_1$ is
scanned \( e_1 \) is labelled thus, \( L_1 \subseteq L_2 \). \( e_2 \in L_j \) for some \( j \neq 2 \). If \( e_2 \in L_1 \), then \( L_1 = L_2 \) so \( e_2 \in L_1 \), a contradiction. Thus, assume by relabelling if necessary that \( e_2 \in L_3 \). In general, if \( e_j \in L_{j+1}, j=1, \ldots, k, L_1 \subseteq L_2 \subseteq \cdots \subseteq L_k \), then \( e_k \not\in L_j \) for \( j < k \) as otherwise \( e_j \in L_j \). But, once we arrive at \( L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n \), \( e_1 \) must be contained in some \( L_j \), giving a contradiction. Therefore, the hypothesis that \( e_j \not\in L_j \) for some \( j \) was invalid.

**Lemma**: Suppose that every circuit in \( M_1 \) has > \( m \) elements. Then given any \( m-1 \) elements, say \( e_1, \ldots, e_{m-1} \), of the base \( I = \{e_1, \ldots, e_n\} \), there is a complementary base \( I' \neq I \) of \( M_1 \) with \( e_j \in I' \), \( j=1, \ldots, m-1 \).

**Proof**: Let \( j \in \{m, \ldots, n\} \). \( I+e_j \) contains a circuit \( C \) of \( M_1 \) since \( I \) is a base. Our hypothesis implies that this circuit contains at least \( m \) elements of \( I \). \( e_j \) is consequently labelled in the algorithm if we drop any of these \( m \) elements from the system. Thus, using the notation of the previous proof, \( e_j \in L_i \) for some \( i \in \{m, \ldots, n\} \). If \( e_j \in L_i \), we augment and have the result. But, \( e_j \not\in L_j \) for all \( j \in \{m, \ldots, n\} \) leads to a contradiction by the argument used in the previous proof.

**Theorem 2**: \( N(m) \geq 2^m \).

**Proof**: Suppose that every circuit of the matroid \( M_1 \) has > \( m \) elements. We prove a slightly stronger result than the theorem's assertion: there exists a rooted directed tree \( T_m \) whose edges are elements of \( E \) and which has the following properties.

(i) A complementary pair of elements of \( E \) originate from every node of \( T_m \).

(ii) Every maximal (with respect to inclusion) directed path originating from the root of \( T_m \) consists of \( m \) elements of \( E \) and these elements
are contained in a complementary base of $M_1$. (Note that we have not excluded an element of $E$ from appearing in $T_m$ more than once.) For example, for $m=2$ we have the picture

Note that the $2^m$ complementary bases corresponding to the directed paths of (ii) must be distinct, since in tracing back to the root from the endpoints of $T_m$ any two paths must meet for the first time at some node. The arcs following this node are complementary so the bases are distinct. Thus the stronger result proves the theorem.

Theorem 1 shows that $N(1) \geq 2$. Since there are two distinct complementary bases, one contains some $e_{j_1}$ and the other its complement $e'_{j_1}$. Thus, letting $e_{j_1}$ and $e'_{j_1}$ be two arcs originating from the root, the stronger result holds for $m=1$.

By induction the stronger result holds for $m-1$, giving the tree $T_{m-1}$. Given any path of $T_{m-1}$ with elements $S = \{e_1, e_2, \ldots, e_{m-1}\}$, say, $S$ is contained in a complementary base $I$ of $M_1$. By the previous lemma, $S$ is also contained in at least one other complementary base $I'$ of $M_1$. Since $I \neq I'$, there is an $e_{j_m} \in I$, $e'_{j_m} \in I'$. Extend $T_{m-1}$ by adding $e_{j_m}$ and $e'_{j_m}$ to the endpoint
of $T_{m-1}$ incident to $e_{m-1}$. Doing this for all $2^{m-1}$ maximal paths of $T_{m-1}$, we extend it to $T_m$ and complete the proof.

The bound given in Theorem 2 is the best possible for let $E = \{e_1', \ldots, e_m', e_1, \ldots, e_m\}$ and let $\mathcal{J}_1$ consist of all subsets of $E$ with $m$ or fewer elements. Then every $m$ element set of $E$ with one element from each complementary pair is a complementary base. Thus, in $M_1 = (E, \mathcal{J}_1)$, $N_{M_1} = 2^m$.

IV. A Generalization

Let $E = \bigcup_{j=1}^{n} E_j$, $E_j = \{e_j^1, \ldots, e_j^{d_j+1}\}$ be a partition of $E$ into pairwise disjoint sets with $d_j \geq 1$, and let $d = (d_1, \ldots, d_n)$. Suppose that $M_1 = (E, \mathcal{J}_1)$ is a matroid. A base $I$ of $M_1$ satisfying $|I \cap E_j| = d_j$ is called a $d$-complementary base. If each $d_j = 1$, a $d$-complementary base is just a complementary base as considered in the last section. In this section, we outline extensions of previous results applicable to general $d$.

Let $I = A_1 \cup A_2 \cup \ldots \cup A_n$, $A_j = \{e_j^1, \ldots, e_j^{d_j+1}\}$, be a $d$-complementary base of $M_1$. Apply the algorithm of Theorem 1 with $\mathcal{J}_2$ replaced by the $d$-partition matroid $\mathcal{J}_2' = \{I \subseteq E : |I \cap E_j| \leq d_j\}$, i.e., in turn drop $e_j^1, e_j^2, \ldots, e_j^s, \ldots, e_j^{d_j}$ from $E$ and $I$ and apply the matroid intersection algorithm to the resulting submatroid beginning with the set $I - e_j^k$. If $e_j^s$ is labelled when any of $e_j^1, \ldots, e_j^{d_j}$ have been dropped, a new $d$-complementary base has been produced. Otherwise, arguing as in the proof of Theorem 1, let $L_r^s$ be the elements labelled when dropping $e_r^s$ and applying the matroid intersection algorithm. Note that if $e_j^s$ is scanned in the algorithm each of $e_j^s \in L_r^s$ is labelled, thus $e_j^s \in L_r^s$ implies that $I_j^t \subseteq L_r^s$ for all $t \in \{1, \ldots, d_j\}$. Consequently, the argument proving Theorem 1 gives:
Theorem 1A: If there exists a d-complementary base to the matroid \( M_1 = (E, \mathcal{I}_1) \) and each circuit of \( M_1 \) contains at least two elements, then \( M_1 \) contains a second d-complementary base.

Similarly, arguments analogous to those of section III provide the following results.

Lemma: Suppose that every circuit of \( M_1 \) contains elements from at least \( m+1 \) of the partitions \( E_j \). Let \( I = A_1 \cup \ldots \cup A_n, A_j \subseteq E_j \), be a d-complementary base of \( M_1 \). Then given any \( (m-1) \) of the sets \( A_j \), say \( A_1, \ldots, A_{m-1} \), there is a d-complementary base \( I' \neq I \) of \( M_1 \) with \( A_j \subseteq I' \) \( j = 1, \ldots, m-1 \).

Theorem 2A: Let \( E = \bigcup_{j=1}^{n} E_j \) be a partition of \( E \) into pairwise disjoint sets \( E_j \), \( |E_j| = d_j + 1 \geq 2 \), and let \( M_1 = (E, \mathcal{I}_1) \) be a matroid. Suppose that every circuit of \( M_1 \) contains elements from at least \( m+1 \) of the partitions \( E_j \). Then if there is one d-complementary base of \( M_1 \), there are at least \( 2^m \).

Note: To extend the proof of Theorem 2, let the edges of \( T_m \) be subsets \( A_j \subseteq E_j \) for some \( 1 \leq j \leq n \) with \( |A_j| = d_j \) and change properties (i) and (ii) of \( T_m \) to:

(i') For any node \( v \) of \( T_m \), two edges \( A \neq B \) originate at \( v \) and both \( A \) and \( B \) are contained in \( E_j \) for some \( 1 \leq j \leq n \).

(ii') Every maximal directed path originating from the root of \( T_m \) consists of \( m \) subsets \( A_j \) of \( E \) and the union of these subsets is contained in a complementary base of \( M_1 \).
Finally, observe that the hypothesis of Theorem 2A is much stronger than the statement that every circuit of $M_1$ contains at least $m+1$ elements. We conjecture that the result is not true with this weaker hypothesis.
REFERENCES


