XXXI. COMPUTATION RESEARCH

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RESEARCH OBJECTIVES

This group provides a programming service for Laboratory Members who use computers in their research. The majority of our work is scientific programming, that is, the use of numerical methods (numerical analysis) for solving differential equations, integrals, root locus problems, and so forth. Some of the problems that we have encountered have not lent themselves to standard numerical techniques and it has been necessary to develop new methods. We are now doing research on new ways for finding complex roots of transcendental equations. This will probably be the chief emphasis of our work during the coming year.

We do some non-numerical programming, however. The availability of the time-sharing system has opened an entire new field for computer applications. We have written several programs for the researcher himself to use, and probably we shall be requested to write similar ones.

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A. NEWTON'S METHOD FOR FINDING COMPLEX ROOTS OF A TRANSCENDENTAL EQUATION*

Householder[1] classifies Newton's well-known method for finding zeros as a second-order functional iteration method that applies to transcendental as well as algebraic equations, and to complex as well as real roots. We find, however, few published examples of its use for finding complex roots. I. M. Longman[2] felt that the method did not appear to have had the application it deserved. To support his thesis he cited the following example:

$$\left(X^2+1\right)^{1/2} + \left(X^2+\frac{1}{3}\right)^{1/2} - ix = 2$$

for which he obtained the solution $x = .3688946067 + .3810680642i$ in six iterations of Newton's method, starting with an initial guess $x_o = 1 + i$. From our computations for the Plasma Electronics Group, two more examples can now be cited to support Longman's thesis. The first arose in the research of Paul Chorney,[3] the second in the research of Carlton E. Speck.[4] Because the latter computations are more recent and the more complicated of the two, they are given here.

Given real values for V_{pc}^2, $B_\perp^2$, and p, find a V_c (real and complex) such that

$$1 - \frac{1}{2}V_{pc}^2 \sum_{n=-\infty}^{\infty} \frac{J_{n-1}^2(p) - J_{n+1}^2(p)}{(V_c-n)} + \frac{V_{pc}^2 B_\perp^2}{p^2} \sum_{n=-\infty}^{\infty} \frac{n^2 J_n^2(p)}{(V_c-n)^2} = 0. \qquad (1)$$

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$J_n(p)$ is the n^{th}-order Bessel function of the first kind with argument p.

In his Master's thesis, Speck established that roots exist close to integer values of V_c, that is, $V_c \approx n$. In order to estimate the root close to the integer m, one can rewrite Eq. 1 as follows:

$$1 - \frac{1}{2} V_{pc}^2 \left[\frac{J_{m-1}^2(p) - J_{m+1}^2(p)}{V_c - m} + \frac{J_{m-2}^2(p) - J_m^2(p)}{V_c - m + 1} + \frac{J_m^2(p) - J_{m+2}^2(p)}{V_c - m - 1} + \sum_{\substack{n=-\infty \\ n\neq m, m+1, m-1}}^{\infty} \frac{J_{n-1}^2(p) - J_{n+1}^2(p)}{V_c - n} \right] + \frac{V_{pc}^2 B_\perp^2}{p^2} \left[\frac{m^2 J_m^2(p)}{(V_c-m)^2} + \frac{(m-1)^2 J_{m-1}^2(p)}{(V_c-m+1)^2} + \frac{(m+1)^2 J_{m+1}^2(p)}{(V_c-m-1)^2} + \sum_{\substack{n=-\infty \\ n\neq m, m+1, m-1}}^{\infty} \frac{n^2 J_n^2(p)}{(V_c-n)^2} \right] = 0. \qquad (2)$$

Making the approximations

$$\sum_{\substack{n=-\infty \\ n\neq m, m+1, m-1}}^{\infty} \frac{J_{n-1}^2(p) - J_{n+1}^2(p)}{V_c - n} \approx \sum_{\substack{n=-\infty \\ n\neq m, m+1, m-1}}^{\infty} \frac{J_{n-1}^2(p) - J_{n+1}^2(p)}{m - n} = S_1$$

$$\sum_{\substack{n=-\infty \\ n\neq m, m+1, m-1}}^{\infty} \frac{n^2 J_n^2(p)}{(V_c-n)^2} \approx \sum_{\substack{n=-\infty \\ n\neq m, m+1, m-1}}^{\infty} \frac{n^2 J_n^2(p)}{(m-n)^2} = S_2$$

and clearing of fractions, one can rewrite Eq. 2 as the following polynomial ($x = V_c - m$):

$$\left(1 + B_\perp^2 \frac{V_{pc}^2}{p^2} S_2 - \frac{1}{2} S_1 V_{pc}^2\right) x^6 - .5 V_{pc}^2 \left[J_{m-1}^2(p) - J_{m+1}^2(p) - J_{m+2}^2(p) + J_{m-2}^2(p)\right] x^5$$

$$\left[-2 - .5 V_{pc}^2 \left(2J_m^2(p) - J_{m+2}^2(p) - J_{m-2}^2(p) - 2S_1\right) + \frac{V_{pc}^2 B_\perp^2}{p^2} \left(m^2 J_m^2(p) + (m-1)^2 J_{m-1}^2(p) + (m+1)^2 J_{m+1}^2(p) - 2S_2\right)\right] x^4$$

$$+\left[-.5V_{pc}^2\left\{-2\left(J_{m-1}^2(p)-J_{m+1}^2(p)\right)+\left(J_{m+2}^2(p)-J_{m-2}^2(p)\right)\right\}\right.$$

$$\left.+\frac{2V_{pc}^2B_{\perp}^2}{p^2}\left\{-(m-1)^2J_{m-1}^2(p)+(m+1)^2J_{m+1}^2(p)\right\}\right]x^3$$

$$+\left[1-.5V_{pc}^2\left(-2J_m^2(p)+J_{m+2}^2(p)+J_{m-2}^2(p)+S_1\right)+\frac{V_{pc}^2B_{\perp}^2}{p^2}\right.$$

$$\left.\left(-2m^2J_m^2(p)+(m-1)^2J_{m-1}^2(p)+(m+1)^2J_{m+1}^2(p)+S_2\right)\right]x^2$$

$$-.5V_{pc}^2\left(J_{m-1}^2(p)-J_{m+1}^2(p)\right)x+\frac{V_{pc}^2B_{\perp}^2}{p^2}m^2J_m^2(p)=0. \qquad (3)$$

The roots of this polynomial were found and the V_c nearest m was used as the initial guess to Newton's method, which usually converged in less than 10 iterations to four-figure accuracy. Nonconvergence occurred when the guess was in the vicinity of a double root or in the vicinity of a zero of the derivative of (1) with respect to V_c. Once a root had been found by Newton's method, the values of the parameters V_{pc}^2, $B_{\perp}^2$, and p could be varied by a small amount and the root used as an initial guess, thereby by-passing the polynomial approximation. In this manner, Speck was able to obtain plots of V_c against any one of the parameters.

For this problem, Newton's method proved to be a powerful technique for finding complex roots. Because, on the average, only 5 or 6 iterations were necessary, computer time (which was mainly used to evaluate (1) and its derivative) was kept to a minimum. In this respect, Newton's method has an advantage over that described by Lieberman,[5] which requires more computer time because of the function evaluations at each grid point. One distinct advantage, however, of the latter method is its independence of zeros of the derivative.

The numerical calculations were performed on the Project MAC time-sharing system without which a guess-run procedure as described above would have been much more difficult and time-consuming.

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References

1. A. S. Householder, Principles of Numerical Analysis (McGraw-Hill Book Company, Inc., New York, 1953), p. 122.

2. I. M. Longman, "On the Utility of Newton's Method for Computing Complex Roots of Equations," Math. Comp. 14, 187-189 (1960).
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4. C. E. Speck, "Quasi-Static Theory of Plasma Instabilities at Cyclotron Harmonics," S. M. Thesis, Department of Electrical Engineering, M. I. T., August 1965.
5. M. A. Lieberman, "Dispersion Diagrams for Hot-Electron Plasmas, Quarterly Progress Report No. 77, Research Laboratory of Electronics, M. I. T., April 15, 1965, p. 141.

B. NUMERICAL EXAMPLE TO ILLUSTRATE KIZNER'S METHOD FOR SOLVING NONLINEAR EQUATIONS*

In a recent paper on solving nonlinear equations, Kizner[1] outlines a method that raises the order of convergence without calculating derivatives of higher order or requiring more accuracy than Newton's[2] method. This paper, however, does not give a numerical example. To illustrate the method, we solved the following example and compared its solution with those obtained from Newton's method and the iteration method[3]: Find a z such that $3z - \cos z - 1 = 0$.

The Kizner technique may be thought of as the integration of a differential equation by using a Runge-Kutta method. Let z_1 be the first estimate to a root of $f(z) = 0$. A root $\bar{z}$ may be written as

$$\bar{z} = \int_{f(z)}^{0} \frac{dz}{df}\, df + z. \tag{1}$$

If we assume the existence of the quantities in Eq. 1, then (1) can be written in the form

$$\frac{d\bar{z}}{df} = \frac{dz}{df} \tag{2}$$

with the initial condition z_1. Equation 2 can now be approximately solved by the Runge-Kutta[4] technique, with $h = -f(z_1)$. The $\bar{z}$ thus obtained can be used as a new estimate and the method repeated until the desired accuracy is obtained. For our example, $z_1 = 0$ was chosen. The results were

	Kizner	Newton	Iteration
x_2	.607107	.666667	.666667
x_3	.607102	.607493	.595296
x_4	.607102	.607102	.609328
x_7			.607102

$f(.607102) \approx 10^{-7}$

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Similar results were obtained when $x_1 = 1.570795$. A more stringent test would be to use the method on the problem given in Section XXXI-A.

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References

1. W. Kizner, "A Numerical Method for Finding Solutions of Nonlinear Equations," J. Soc. Industr. Appl. Math. 12, 424 (1964).
2. J. Scarborough, Numerical Mathematical Analysis (The Johns Hopkins Press, Baltimore, 1962), p. 199.
3. Ibid., p. 206.
4. Ibid., p. 331.