SEISMIC BOREHOLE TOMOGRAPHY

by

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(1980)

Submitted to the Department of Earth,
Atmospheric, and Planetary Sciences
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

at the

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October, 1987

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Abstract

Seismic ray tomography and seismic diffraction tomography are tested by ultrasonic laboratory experiments simulating cross-borehole, vertical seismic profiling (VSP), and surface reflection configurations. Experimental results indicate that: (1) Both seismic ray tomography and seismic diffraction tomography are hampered by the limited view angle problem, although seismic diffraction tomography is less sensitive to this problem. (2) When the scattered field can be measured, seismic diffraction tomography is in general superior to seismic ray tomography, not only because it is less sensitive to the limited view angle problem, but also because seismic diffraction tomography can image small objects with size comparable to the wavelength of the illuminating waves. (3) The advantage of ray tomography is that reconstruction can be done using the first arrivals only, the most easily measurable quantity, and there is less restriction on the properties of the object to be imaged. (4) For seismic diffraction tomography, the Rytov approximation is valid over a wider frequency range than the Born approximation in the cross-borehole configuration.

The emphasis of this thesis is on seismic diffraction tomography, which has received attention for geophysical applications only recently. To make seismic diffraction tomography a subsurface imaging technique that provides high resolution reconstructions comparable to ultrasonic medical tomography, one of the largest differences between seismic diffraction tomography and ultrasonic medical tomography — the limited view angle problem — has to be solved. This
thesis develops two methods to solve the limited view angle problem.

The first method is to apply the minimum cross entropy estimation to seismic diffraction tomography. The minimum cross entropy method helps the limited view angle problem by making the most objective estimate of the data that can not be measured by the finite aperture seismic source – receiver array. As explained in this thesis, when the minimum cross entropy estimation is applied to seismic diffraction tomography, it has the effect of extending the source array and the receiver array and therefore it is equivalent to a finite aperture compensation. By numerical and ultrasonic laboratory tests of this method, we find that the minimum cross entropy diffraction tomography can reduce the artifacts in the reconstructions and improve the horizontal resolution of the cross-borehole tomography. This method is especially useful for objects consisting of isolated impulses in a homogeneous background medium.

The second method we develop for solving the limited view angle problem is the iterative multi-frequency diffraction tomography. This method is a combination of the multi-frequency reconstruction algorithm and the iterative least squares spectrum extrapolation algorithm. The multi-frequency method provides more measured data, the spectrum extrapolation algorithm estimates the data that can not be measured by the source – receiver array of seismic borehole tomography. Results from numerical and ultrasonic laboratory experiments indicate that for a finite extent object function in a homogeneous background medium, the iterative multi-frequency diffraction tomography can help the limited view angle problem by improving the horizontal resolution and the signal/noise ratio.

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Acknowledgements

I would like to thank my thesis advisor and co-advisor: Prof. Nafi Toksöz and Prof. Gregory Duckworth for suggesting this interesting thesis topic. Nafi and Greg kept the advisor – advisee relationship very friendly and constructive, which was a very strong support to me especially when I encountered difficulties during the development of this thesis.

The Earth Resources Laboratory at MIT is a very stimulating research environment. Many people at ERL have essential inputs to this thesis. Among them, Arthur Cheng and Roger Turpening supported this thesis by providing all the equipments I needed and showed me the real world of exploration geophysics. Ru-shan Wu helped me learn diffraction tomography. Michael Prange and Steve Gildea shared with me their experience in computers. Karl Coyner taught me many things around the laboratory, from reading the resistor color code to interfacing digital equipments. ERL also offered me opportunities to interact with people in the industry as well as inside MIT who share the common interest in exploration geophysics. Discussions with Wafik Beydoun, Cengiz Esmersoy, and Bernard Levy were especially profitable. I would also like to thank Denis Schmitt, Bob Cicerone, and Jeff Meredith, they proofread several manuscripts closely related to this thesis and gave me many valuable feedbacks. My fellow graduate students and research staff at ERL/MIT made my 5-year PhD program fruitful and enjoyable. Among them, I want to express my gratitude to Carol Blackway, John Bullit, Tianqing Cao, Karl Ellefsen, Jack Foley, Rob Fricke, Paul Huang, Fatih

Finally, and most importantly, I would like to thank my parents for always giving me good educations ever since my childhood and for their encouragements over the last five years while I was away from them.
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Chapter 1

INTRODUCTION

1.1 The Subject of the Thesis

Seismic borehole tomography has been attempted for subsurface imaging in recent years. In this study, using ultrasonic laboratory measurements and theoretical studies, we test the conditions under which tomography can be applied to geophysical problems and develop techniques to improve its performance.

Before seismic tomography can be implemented on a wide scale, several problems have to be solved. One problem is that tomographic methods are based on various physical models and assumptions on the illuminating energy, the properties of the object, and the properties of the background medium. These models and assumptions may be valid for the medical imaging environments, but whether or not they are also valid for the subsurface imaging environments, is questionable. Another problem is that the sources and the receivers for subsurface imaging can only be deployed on the
surface and/or in few boreholes and the object in the subsurface can not be probed from all the desired directions. This is the so called "limited view angle problem". Unless this problem is solved, high resolution image reconstruction obtained in medical tomography can not be expected for the seismic borehole tomography. These two problems are the foci of this thesis.

To understand the behavior of tomographic methods under the subsurface imaging environments, we conduct a series of ultrasonic laboratory experiments simulating seismic surface and borehole tomography. For the limited view angle problem, we develop two methods of approach: (1) the minimum cross entropy diffraction tomography, and (2) the iterative multi-frequency diffraction tomography. Some of the potential problems of seismic borehole tomography are revealed by the ultrasonic laboratory experiments, and some of these problems, such as the limited view angle problem, are alleviated by this thesis.

1.2 Background

Tomography is an inversion technique that determines the spatial distribution of a certain physical parameter (such as X-ray attenuation) inside the object of interest by measuring the object's response to the probing energy from many directions. Tomography was applied to medical imaging using first X-rays as the probing energy, then ultrasonic medical tomography was developed. The successful applications of ultrasonic medical tomography initiated interest in seismic tomography. In seismic tomography,
the probing energy is seismic waves and the objects under investigation are subsurface
inhomogeneities. By using seismic waves with different wavelengths, inhomogeneities
of different sizes can be imaged.

Tomographic methods can be classified into two categories. Methods based on the
geometrical optics or the ray equation are called ray tomography. Methods based on
the wave equation are called diffraction tomography. The mathematical foundation of
ray tomography was set by Radon (1917) who derived the method of representing a two
dimensional function inside a region by its line integrals along lines intersecting that
region. This method was first adopted by Bracewell (1956) to reconstruct the image of
the Sun. Since then, ray tomography has been applied to other areas such as medical
imaging (Kuhl and Edwards, 1963; Cormack, 1963) and electron microscopy (Gordon
et al, 1970). Up to now, three types of reconstruction algorithms for ray tomography
have been developed: the series expansion reconstruction algorithm, the direct Fourier
transform reconstruction algorithm, and the filtered backprojection reconstruction al-
gorithm. Kak (1985) gives an overview of these three algorithms. Ray tomography
works well when the interaction between the illuminating energy and the object under
investigation can be successfully described by the ray equation. This is generally the
case when the size of the object is large relative to a wavelength and when the velocity
variation is smooth so that gradual refraction of the rays dominates over diffraction.

When the size of the object is comparable to the wavelength of the illuminating
waves, diffraction and scattering become the dominant processes. In such cases the sys-
tem must be described by the wave equation, instead of the ray equation. Diffraction
tomography was initiated by Wolf's work in inverse optics (Wolf, 1969). Wolf derived
the relationship between the scattered wavefield and the spatial distribution of the re-
fractive index of the scatterer using the Born approximation. Based on Wolf's work,
Mueller et al. (1979, 1980) developed the constant background, acoustic diffraction
tomography. The theory of the constant background, acoustic diffraction tomography
was tested by several ultrasonic laboratory studies with 360° full coverage illumina-
tion: Kaveh et al. (1979) conducted the first laboratory test of ultrasonic diffraction
tomography; Adams and Anderson (1980) tested the reflection mode diffraction tomog-
raphy; Kaveh et al. (1981) compared the effects of the Born and Rytov approxima-
tions on diffraction tomography; Greenleaf (1983) tested the transmission mode diffrac-
tion tomography using the Rytov approximation. The theory of constant background,
acoustic diffraction tomography was then generalized to variable background case by
Levy and Esmersoy (1987) and Mora (1987), and to elastic diffraction tomography by
Beylkin and Burridge (1987). Diffraction tomography also has three types of recon-
struction algorithms: the series expansion reconstruction algorithm, the direct Fourier
transform reconstruction algorithm, and the filtered backpropagation reconstruction
algorithm (Kak, 1985). A comparison of the direct Fourier transform reconstruction
algorithm and the filtered backpropagation reconstruction algorithm for diffraction to-
mography was done using numerical examples by Pan and Kak (1983). The series
expansion algorithm for the diffraction tomography was described by Devaney (1985)
and Mohammad-Djafari and Demoment (1986). This algorithm is similar to the series expansion algorithm for the ray tomography except that the forward problem of the series expansion diffraction tomography is a matrix equation relating the object function and the scattered field and the components of this matrix are the Green’s functions.

Both ray tomography and diffraction tomography have been applied to geophysical problems. Ray tomography was first applied to subsurface imaging problem by Dines and Lytle (1979), Chiu et al. (1986) proposed the damped least square seismic ray tomography using singular value decomposition (SVD) for more stable inversion, Bregman et al. (1987) use both travel time and amplitude information for seismic ray tomography. Seismic diffraction tomography was first proposed by Devaney (1984). Witten and Long (1986) tested the effects of strong scatterers, spatial sampling density, and sonic wave frequency on seismic diffraction tomography with synthetic data. Wu and Toksöz (1987) modified Devaney’s plane wave seismic diffraction tomography by using line sources and formulated the backpropagation reconstruction algorithms for the cross-borehole, VSP, and surface reflection geometries. Harris (1987) used the plane wave synthesis method to bypass the farfield approximation made in Devaney’s formulation such that weak scattering inhomogeneities near the sources and receivers can be successfully imaged. Seismic diffraction tomography so far has only been tested by synthetic data (Devaney, 1984; Witten and Long, 1986; Wu and Toksöz, 1987) and ultrasonic laboratory data (Lo et al., 1987). Seismic ray tomography, on the other hand, has been tested by many field experiments (Kretzschmar and Witterholt, 1984;
Nercessian et al, 1984; Bishop et al., 1985; Peterson et al., 1985; Chiu et al., 1986; Cottin et al., 1986; Gustavsson et al., 1986; Ramirez, 1986; Stork and Clayton, 1986; Bregman et al., 1987).

A major difficulty of implementing seismic tomography is the limited view angle problem. Possible ways to help the limited view angle problem include maximum entropy estimation, band-limited spectrum extrapolation, deconvolution, and multifrequency reconstruction (Rangayyan et al., 1985). For seismic tomography, these methods have not yet been applied.

1.3 Outline of Thesis

Chapter 1 states the subject of this thesis, reviews the background materials about seismic tomography, and outlines the materials covered in each chapter.

Chapter 2 reviews the basic principles of seismic ray tomography and seismic diffraction tomography. For both seismic ray tomography and seismic diffraction tomography, we derive the series expansion reconstruction algorithm, the direct transform reconstruction algorithm, and the filtered backprojection / backpropagation reconstruction algorithm. The physical foundations, approximation methods, and limitations of both methods are also discussed.

In Chapter 3, we observe the behavior of ray tomography and diffraction tomography when they are both applied to the subsurface imaging problems. The observation is done by a series of ultrasonic laboratory experiments simulating cross-borehole tomog-
raphy, VSP tomography, and surface reflection tomography. First, we compare these two methods in terms of their performance under the limited view angle conditions. Second, we compare the adaptabilities of these two methods to objects of various sizes and acoustic properties. Finally, for the diffraction tomography, we compare the Born and Rytov approximations based on the induced image distortion by using these two approximation methods.

The limited view angle problem of seismic borehole tomography is observed in our experiments, solutions of this problem are given in Chapter 4 and Chapter 5.

Chapter 4 develops two image reconstruction algorithms based on the minimum cross entropy estimation method that help the limited view angle problem in seismic borehole tomography. These two algorithms are derived by combining the minimum cross entropy estimation algorithm with two image reconstruction algorithms: the direct transform reconstruction algorithm and the backpropagation reconstruction algorithm. We test these two methods by numerical and ultrasonics laboratory experiments simulating cross-borehole tomography.

In Chapter 5, the limited view angle problem is treated by an iterative multi-frequency reconstruction method. This method is a combination of the noniterative multi-frequency reconstruction method and the iterative least squares spectrum extrapolation method, with modifications on the space domain window based on resolution matrix analysis. The iterative multi-frequency diffraction tomography is also tested by numerical and ultrasonic laboratory experiments.
In Chapter 6, the noise sensitivity and the resolving power of the methods de- 
oped in Chapter 4 and Chapter 5 are evaluated. Finally, in Chapter 7, we summarize 
the major conclusions of this thesis and suggest an optimal tomographic inversion 
scheme for subsurface imaging.

1.4 Contributions of This Work

1. The results achieved by seismic tomography are not as good as those of ultrasonic 
medical tomography, although the physical foundations of these two methods 
are very similar. This thesis takes an experimental approach to examine the 
causes of the poor resolution of seismic tomography: the questionable validity 
of the weak scattering approximation and the limited view angle problem. The 
experiment described in Chapter 3 is the first laboratory test of the narrow view 
angle diffraction tomography. It is also the first laboratory test of geophysical 
diffraction tomography.

2. To counter the effects of the limited view angle in seismic borehole tomography, 
(1) we compare different image reconstruction methods in terms of their sensi-
tivities to limited view angle constraint, and show that diffraction tomography 
performs better than ray tomography with limited view angles, and (2) we de-
velop methods that can estimate the data missed by the narrow view angle of 
seismic borehole tomography using (a) minimum cross entropy estimation, and 
(b) iterative multi-frequency reconstruction techniques.
3. Two minimum cross entropy diffraction tomography methods are developed in this thesis: (1) minimum cross entropy backpropagation diffraction tomography (MCEB), and (2) minimum cross entropy direct transform diffraction tomography (MCED). The theory and reconstruction algorithm of minimum cross entropy backpropagation diffraction tomography is developed and tested by synthetic and ultrasonic laboratory data. The theory of minimum cross entropy direct transform diffraction tomography, developed by Mohammad-Djafari and Demoment (1986), was modified and tested with synthetic and ultrasonic laboratory data.

4. This thesis demonstrates the first application of iterative multi-frequency diffraction tomography to seismic imaging problem. Iterative multi-frequency diffraction tomography is a combination of two methods: the multi-frequency diffraction tomography and the iterative least squares spectrum extrapolation. This thesis modifies the conventional least squares spectrum extrapolation algorithm by using an adjustable finite extent space domain window when the object under investigation is a finite extent object.
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Chapter 2

SEISMIC RAY TOMOGRAPHY
AND SEISMIC DIFFRACTION
TOMOGRAPHY

2.1 Introduction

In this chapter, we review the basic principles of seismic ray tomography and seismic diffraction tomography. Derivation of these two imaging methods can also be found in Dudgeon and Mersereau (1984), Kak (1985), and Wu and Toksöz (1987). Seismic ray tomography and seismic diffraction tomography are based on different physical foundations, subject to different limitations, and are therefore suitable for different applications. Reconstruction algorithms for both ray tomography and diffraction tomography include: (1) series expansion reconstruction algorithm, (2) direct transform
reconstruction algorithm, and (3) filtered backprojection / backpropagation reconstruction algorithm. In section 2.2, we review the ray tomography reconstruction algorithms and discuss their applicabilities to the geophysical problems and their computation efficiency. In section 2.3, the reconstruction algorithms for diffraction tomography are reviewed. At the end of section 2.3, the limited view angle problem of seismic borehole tomography is presented. Finally, in section 2.4, the limitations and strengths of seismic ray tomography and seismic diffraction tomography are briefly discussed.

2.2 Seismic Ray Tomography

Consider the seismic ray tomography geometry shown in Figure 2-1. In this figure, \( O(x, z) \) is the object function to be determined, and is illuminated by a group of rays. The travel times along these rays form a travel time profile, called the projection, \( P(u, \theta) \). We use two coordinate systems in Figure 2-1, coordinate system \((x, z)\) is where the object function is defined, coordinate system \((u, v)\) is where the projection is defined. These two coordinate systems are related by

\[
\begin{bmatrix}
  x \\
  z
\end{bmatrix} =
\begin{bmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  u \\
  v
\end{bmatrix} \tag{2.1}
\]

Figure 2-1 shows that projection \( P(u, \theta) \) is the line integral of \( O(x, z) \) along the \( \frac{x}{2} + \theta \) direction, denoted by unit vector \( \hat{v} \), intersecting the \( U \) axis at point \( u \):

\[
P(u, \theta) = \int_{-\infty}^{\infty} O(x, z) dv = \int_{-\infty}^{\infty} O(u \cos \theta - v \sin \theta, u \sin \theta + v \cos \theta) dv \tag{2.2}
\]
Let $\tilde{P}(\Omega, \theta)$ denote the Fourier transform of $P(u, \theta)$ along the $U$ axis, then

$$\tilde{P}(\Omega, \theta) = \int_{-\infty}^{\infty} P(u, \theta) \exp\{-j\Omega u\} du$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} O(u \cos \theta - v \sin \theta, u \sin \theta + v \cos \theta) \exp\{-j\Omega u\} dvdu$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} O(x, z) \exp\{-j\Omega(x \cos \theta + z \sin \theta)\} dx dz$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} O(x, z) \exp\{-j|z(\Omega \cos \theta) + x(\Omega \sin \theta)|\} dx dz$$

$$= \tilde{O}(\Omega \cos \theta, \Omega \sin \theta)$$

Let $\tilde{O}(k_x, k_z)$ denote the two-dimensional Fourier transform of $O(x, z)$:

$$\tilde{O}(k_x, k_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} O(x, z) \exp\{-j(xk_x + zk_z)\} dx dz. \quad (2.4)$$

Comparing equations (2.3) and (2.4), we find that the one-dimensional Fourier transform of the projection, $\tilde{P}(\Omega, \theta)$, is one slice of the two-dimensional Fourier transform of the object function $\tilde{O}(k_x, k_z)$, defined on the loci: $k_x = \Omega \cos \theta, k_z = \Omega \sin \theta$. This is the "projection — slice theorem". The loci $k_x = \Omega \cos \theta, k_z = \Omega \sin \theta$ in the $(k_x, k_z)$ plane is shown in Figure 2-2. If the projection $P(u, \theta)$ is available for $\theta = 0 \rightarrow \pi$, then the whole $(k_x, k_z)$ plane is covered by $\tilde{O}(\Omega \cos \theta, \Omega \sin \theta)$, which is a complete two-dimensional Fourier transform of the object function.

To obtain $O(x, z)$ from $\tilde{O}(\Omega \cos \theta, \Omega \sin \theta)$, "direct transform reconstruction algorithm" first interpolates $\tilde{O}$ from the polar grid $(\Omega \cos \theta, \Omega \sin \theta)$ to the rectangular grid $(k_x, k_z)$, then takes an inverse Fourier transform of $\tilde{O}(k_x, k_z)$.

The "filtered backprojection reconstruction algorithm", on the other hand, does not need interpolation. To derive the filtered backprojection algorithm, we first write the
two-dimensional Fourier transform of the object function, then replace \( k_z \) by \( \Omega \cos \theta \)
and \( k_\theta \) by \( \Omega \sin \theta \):

\[
O(x, z) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \tilde{O}(k_z, k_\theta) \exp \{ j(k_z x + k_\theta z) \} dk_z dk_\theta
\]
\[= \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{\infty} \tilde{O}(\Omega \cos \theta, \Omega \sin \theta) \exp \{ j\Omega(x \cos \theta + z \sin \theta) \} \Omega d\Omega d\theta
\]
\[= \frac{1}{4\pi^2} \int_{0}^{\pi} \int_{0}^{\infty} \tilde{O}(\Omega \cos \theta, \Omega \sin \theta) \exp \{ j\Omega(x \cos \theta + z \sin \theta) \} \Omega d\Omega d\theta + \frac{1}{4\pi^2} \int_{0}^{\pi} \int_{0}^{\infty} \tilde{O}[\Omega \cos(\theta + \pi), \Omega \sin(\theta + \pi)] \exp \{ j\Omega[x \cos(\theta + \pi) + z \sin(\theta + \pi)] \} \Omega d\Omega d\theta
\]

Since

\[
\tilde{O}[\Omega \cos(\theta + \pi), \Omega \sin(\theta + \pi)] = \tilde{O}(-\Omega \cos \theta, -\Omega \sin \theta),
\]
equation (2.5) can be written as

\[
O(x, z) = \frac{1}{4\pi^2} \int_{0}^{\pi} \int_{0}^{\infty} \tilde{O}(\Omega \cos \theta, \Omega \sin \theta) \exp \{ j\Omega(x \cos \theta + z \sin \theta) \} \Omega d\Omega d\theta + \frac{1}{4\pi^2} \int_{0}^{\pi} \int_{0}^{\infty} \tilde{O}(\Omega \cos \theta, \Omega \sin \theta) \exp \{ j\Omega(x \cos \theta + z \sin \theta) \} \Omega d\Omega d\theta
\]
\[= \frac{1}{4\pi^2} \int_{0}^{\pi} \int_{0}^{\infty} \tilde{O}(\Omega \cos \theta, \Omega \sin \theta) \exp \{ j\Omega(x \cos \theta + z \sin \theta) \} \Omega d\Omega d\theta.
\]

Using the projection - slice theorem, we can replace \( \tilde{O}(\Omega \cos \theta, \Omega \sin \theta) \) in equation (2.7) by \( \tilde{P}(\Omega, \theta) \):

\[
O(x, z) = \frac{1}{4\pi^2} \int_{0}^{\infty} \int_{-\infty}^{\infty} \tilde{P}(\Omega, \theta) \exp \{ j\Omega(x \cos \theta + z \sin \theta) \} \Omega d\Omega d\theta.
\]

Equation (2.8) is the filtered backprojection reconstruction algorithm where \( \tilde{P}(\Omega, \theta) \) is the Fourier transform of the travel time profile, \( O(x, z) \) is the slowness distribution.
The series expansion reconstruction algorithms for the ray tomography, such as the algebraic reconstruction technique (ART) and the simultaneous iterative reconstruction technique (SIRT), are all based on the "projection method" for solving a system of linear equations proposed by Kaczmarz (1937). In the series expansion reconstruction algorithms, the imaging area is divided into \( j \) pixels as shown in Figure 2-3. Let \( O_j \) be the average of a certain physical parameter (such as sonic slowness) inside the \( j \)th pixel, and \( P_i \) be the line integral of that parameter along the \( i \)th ray. Then for an imaging system with \( N \) pixels and \( M \) measurements, the forward problem is:

\[
P_i = \sum_{j=1}^{N} S_{ij} O_j, \quad i = 1, 2 \ldots, M,
\]

(2.9)

where \( S_{ij} \) is the length of the segment of the \( i \)th ray intersecting the \( j \)th pixel. The series expansion reconstruction algorithms use an iterative approach to invert \( O_j \). It starts with an initial estimate of \( O_j \), denoted by \( \hat{O}_j \). From this initial estimate, the estimated line integral can be calculated by

\[
\hat{P}_i = \sum_{j=1}^{N} S_{ij} \hat{O}_j, \quad i = 1, 2 \ldots, M.
\]

(2.10)

The iterative algorithm updates the estimate \( \hat{O}_j \) by the recurrence formula:

\[
\hat{O}_j^{\text{new}} = \hat{O}_j^{\text{old}} + \Delta O_{ij}
\]

(2.11)

\[
= \hat{O}_j^{\text{old}} + S_{ij} \frac{P_i - \sum_{j=1}^{N} S_{ij} \hat{O}_j^{\text{old}}}{\sum_{j=1}^{N} (S_{ij})^2}.
\]

\( \Delta O_{ij} \) is the correction on \( \hat{O}_j^{\text{old}} \) after examining the \( i \)th ray. Equation (2.11) is actually the least square solution of the following equation:

\[
\Delta P_i = P_i - \hat{P}_i
\]

(2.12)
\[
\sum_{j=1}^{N} S_{ij}(O_j - \hat{O}_j) = \sum_{j=1}^{N} S_{ij} \Delta O_{ij}.
\]

Equation (2.11) keeps updating \( \hat{O}_j \) until the difference between the measured and the estimated line integral is smaller than a prespecified threshold.

The convergence of this algorithm can be visualized in the vector space. Equation (2.9) represents \( M \) hyperplanes in an \( N \) dimensional space, \( \Delta O_{ij} \) calculated by equation (2.11) brings the current estimate to its projection on the \( i \)th hyperplane. Therefore, this type of algorithm is actually a process of looking for the intersection of the hyperplanes by repeatedly projecting the estimate from one hyperplane to another. Simple geometry, such as the one shown in Figure 2-4, illustrates how this projecting process will approach the solution if the solution is unique. A rigorous proof of the convergence of this algorithm is given by Tanabe (1971).

The algebraic reconstruction technique (ART) is the direct implementation of equation (2.11), which means the object function is updated every time an estimated line integral is checked with a measured line integral. For the simultaneous iterative reconstruction technique (SIRT), the correction on each pixel, \( \hat{O}_j \), is the weighted average of all the corrections on that pixel, \( \Delta O_{ij} \), \( i = 1 \rightarrow M \), calculated by examining all the rays:

\[
\hat{O}_j^{\text{new}} = \hat{O}_j^{\text{old}} + \Delta O_j
\]

\[
= \hat{O}_j^{\text{old}} + \frac{1}{W_i} \sum_{i=1}^{M} \Delta O_{ij}
\]

Dines and Lytle (1979) use the ray density in each pixel as the weighting factor, \( W_j \).
Other choices of the weighting factor may also work, depending on applications.

The relationship between the forward problem and the three reconstruction algorithms described above for the seismic ray tomography is summarized in Figure 2-5. Travel time profile $P(u, \theta)$ is the data vector for the seismic ray tomography problem, it can be inverted directly for the slowness distribution $O(x, z)$ by the series expansion reconstruction algorithms such as ART and SIRT, or, it can be inverted by exploiting the projection – slice theorem: the filtered backprojection reconstruction algorithm or the direct transform reconstruction algorithm. If we use the filtered backprojection reconstruction algorithm, we first calculate the one-dimensional Fourier transform of the travel time profile, $\tilde{P}(\Omega, \theta)$, then use equation (2.8) to calculate the object function from $\tilde{P}(\Omega, \theta)$. If we use the direct transform reconstruction algorithm, we still have to calculate $\tilde{P}(\Omega, \theta)$, then use equation (2.3), the projection – slice theorem, to calculate the two-dimensional Fourier transform of the object function on the loci $k_z = \Omega \cos \theta$, $k_z = \Omega \sin \theta$ in the $(k_x, k_z)$ plane. The polar grid spectrum, $\hat{O}(\Omega \cos \theta, \Omega \sin \theta)$, is then interpolated to a rectangular grid spectrum, $\hat{O}(k_x, k_z)$, and the object function is obtained by taking an inverse Fourier transform of $\hat{O}(k_x, k_z)$.

In terms of their relative performance of these three reconstruction algorithms, the series expansion reconstruction algorithm is the one that can be formulated most easily for any source – receiver geometry, but it is also computationally the most demanding one among these three algorithms. The other two algorithms, the direct transform reconstruction algorithm and the filtered backprojection reconstruction algorithm, are
both one-step inversions. They are much faster than the series expansion reconstruction algorithm, but they require special source-receiver geometry which is difficult for geophysical applications. Among these two projection-slice theorem based methods, the direct transform method is faster than the filtered backprojection method, but it requires a two-dimensional interpolation which introduces errors.

2.3 Seismic Diffraction Tomography

In this section, we first derive the "generalized projection-slice theorem" for a diffracting source. Then, two image reconstruction algorithms based on this theorem: the direct transform reconstruction algorithm and the filtered backpropagation reconstruction algorithm are derived. Finally, we derive the series expansion reconstruction algorithm for the diffraction tomography.

Consider a scattering experiment shown in Figure 2-6. A finite extent object with varying velocity \( C(r) \) is situated in a constant background medium with a uniform velocity \( C_0 \), where \( r \) is the position vector of the object point. By the first Born approximation (see Appendix A), we can obtain the basic equation for the acoustic scattering problem:

\[
U(r_s, r_g) = -k_0^2 \int_V O(r) G(r, r_s) G(r, r_g) dV,
\]

where subscripts \( g \) and \( s \) refer to geophone and source respectively, \( U(r_s, r_g) \) is the scattered field measured at position \( r_g \) when a point source is at position \( r_s \), \( O(r) \) is
the object function defined as:

\[ O(\tau) = 1 - \frac{C_o^2}{C^2(\tau)}. \]  
\( (2.15) \)

In this chapter, we use the two-dimensional Green's function for the background medium,

\[ G(\tau, \tau') = \frac{j}{4} H_0^{(1)}(k_o | \tau - \tau' |), \]  
\( (2.16) \)

where \( H_0^{(1)} \) is the zero-order Hankel function of the first kind, and \( k_o = \frac{\omega}{c_o} \) is the wavenumber in the constant background medium. Take the Fourier transform of equation (2.14) along both the source line and the geophone line, the scattering problem in the Fourier domain is obtained:

\[ \tilde{U}(k_s, k_g) = -k_o^2 \int_V O(\tau) \tilde{G}(k_s, \tau) \tilde{G}(k_g, \tau) d\tau \]  
\( (2.17) \)

where \( \tilde{U}(k_s, k_g) \) is the Fourier transform of \( U(l_s, l_g) \):

\[ \tilde{U}(k_s, k_g) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(l_s, l_g) \exp[-j(k_s l_s + k_g l_g)] dl_s dl_g \]  
\( (2.18) \)

In equation (2.18), \( l_s \) is the distance of the source along the source line, \( l_g \) is the distance of the geophone along the geophone line. \( \tilde{G} \) in equation (2.17) is the Fourier transform of \( G \). The derivation of \( \tilde{G} \) can be found in Morse and Feshbach (1953), page 823:

\[ \tilde{G}(k_s, \tau) = \frac{j}{2} \frac{\exp(j\gamma_s d_s)}{\gamma_s} \exp(-j k_s \hat{s} \cdot \tau) \]  
\( (2.19) \)

\[ \tilde{G}(k_g, \tau) = \frac{j}{2} \frac{\exp(j\gamma_g d_g)}{\gamma_g} \exp(-j k_g \hat{g} \cdot \tau) \]  
\( (2.20) \)

\[ \gamma_s = \sqrt{k_o^2 - k_s^2} \]  
\( (2.21) \)

\[ \gamma_g = \sqrt{k_o^2 - k_g^2} \]  
\( (2.22) \)
where $d_s$ and $d_g$ are the distance from the origin of the $(x, z)$ coordinate system to the source line and the geophone line, $k_s$ and $k_g$ are the wavenumbers along the source line and the geophone line, $\gamma_s$ and $\gamma_g$ are the corresponding perpendicular wavenumbers, $\hat{s}$ and $\hat{g}$ are the unit vectors of plane waves to the source and the geophone respectively. Substitute equations (2.19) and (2.20) into equation (2.17), the generalized projection - slice theorem is derived:

$$U(k_s, k_g) = \frac{k^2}{4\gamma_s \gamma_g} \exp[j(\gamma_s d_s + \gamma_g d_g)]$$

$$\int_S O(r) \exp[-j k_o (\hat{g} + \hat{s}) \cdot r] dr$$

$$= \frac{k^2}{4\gamma_s \gamma_g} \exp[j(\gamma_s d_s + \gamma_g d_g)] \tilde{O}[k_o (\hat{g} + \hat{s})],$$

where $\tilde{O}[k_o (\hat{g} + \hat{s})]$ is the two-dimensional Fourier transform of the object function on the loci $k_o (\hat{g} + \hat{s})$, $\tilde{U}(k_s, k_g)$ is the two-dimensional Fourier transform of the measured scattered field, $U(l_s, l_g)$, along the $l_s$ and $l_g$ directions. Since we use a two-dimensional geometry, the volume integral in equation (2.17) is replaced by a surface integral.

To obtain $O(x, z)$ from $\tilde{O}[k_o (\hat{g} + \hat{s})]$, the "direct transform reconstruction algorithm" first interpolates $\tilde{O}$ from the $k_o (\hat{g} + \hat{s})$ loci to a rectangular grid $(k_z, k_z)$, then takes the inverse Fourier transform of $\tilde{O}(k_z, k_z)$ to find $O(x, z)$. Figure 2-7 shows the geometrical relationship between the loci $k_o (\hat{g} + \hat{s})$ and the rectangular grid $(k_z, k_z)$ for a single $\hat{s}$.

The "backpropagation reconstruction algorithm", on the other hand, does not need the interpolation. To derive this algorithm, we first write down the the inverse Fourier transform of $\tilde{O}(k_z, k_z)$, replace $\tilde{O}(k_z, k_z)$ by $\tilde{O}(k_o (\hat{g} + \hat{s}))$, and then use the generalized
projection – slice theorem to express $O_0(k, g + \delta)$ in terms of $U(k_s, k_g)$:

\[
O(x, z) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \tilde{O}(k_z, k_s) \exp[j(k_z x + k_z z)] dk_z dk_s
\]

\[
= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \tilde{O}[k_0(g + \delta)] \exp[j(k_z x + k_z z)] dk_z dk_s
\]

\[
= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} 4\gamma_s \gamma_g \frac{\exp[-j(\gamma_s d_s + \gamma_g d_g)]}{k_0^2} \exp[j(k_z x + k_z z)] dk_z dk_s.
\]  

\[
\tilde{U}(k_s, k_g) \exp[j(k_z x + k_z z)] dk_z dk_s.
\]

To complete the derivation, the $k_z$ and $k_s$ terms in equation (2.24) have to be expressed in terms of $k_s$ and $k_g$. In the following, we derive the Jacobian for the coordinate transformation, $J(k_z, k_s | k_s, k_g)$, for three experiment geometries: cross-borehole, VSP, and surface reflection. For the cross-borehole geometry, as shown in Figure 2-8(a),

\[
k_z = \gamma_g - \gamma_s \tag{2.25}
\]

\[
k_s = k_s + k_g
\]

\[
J(k_z, k_s | k_s, k_g) = \frac{|k_z \gamma_g + k_g \gamma_s|}{\gamma_s \gamma_g} \tag{2.26}
\]

For the VSP geometry, as shown in Figure 2-8(b),

\[
k_z = k_s + \gamma_s \tag{2.27}
\]

\[
k_s = -\gamma_s + k_g
\]

\[
J(k_z, k_s | k_s, k_g) = \frac{|k_z k_g + \gamma_s \gamma_g|}{\gamma_s \gamma_g} \tag{2.28}
\]

For the surface reflection geometry, as shown in Figure 2-8(c),

\[
k_z = k_s + k_g \tag{2.29}
\]
\[ k_z = -\gamma_s - \gamma_g \]
\[ J(k_z, k_z \mid k_s, k_g) = \frac{|k_z \gamma_g - k_s \gamma_s|}{\gamma_s \gamma_g} \]  
(2.30)

From Figure 2-8, which expresses the range of \( k_o \hat{g} \) and \( k_o \hat{s} \) for the cross-borehole, VSP, and surface reflection geometries, the spectral coverage for these geometries, defined by the loci \( k_o(\hat{g} + \hat{s}) \), can be constructed. The spectral coverage for the cross-borehole, VSP, and surface reflection geometries are shown in Figure 2-9 (a), (b), and (c) respectively.

The backpropagation reconstruction algorithm for the cross-borehole geometry is obtained by substituting equations (2.25) and (2.26) into equation (2.24):

**Cross-borehole**

\[ O(x, z) = \frac{1}{\pi} \int_{-k_o}^{k_o} \exp[jk_z z - j\gamma_s z] O_1(x, z, k_s) dk_s. \]  
(2.31)
\[ O_1(x, z, k_s) = \frac{1}{\pi} \int_{-k_o}^{k_o} \exp[jk_z z - j\gamma_g (d_g - z)] \left| \frac{k_s \gamma_g + k_g \gamma_s}{k_o^2} \right| \tilde{U}(k_s, k_g) dk_g. \]

Similarly, the backpropagation reconstruction algorithms for the VSP and surface reflection geometries are obtained by substituting equations (2.27), (2.28), (2.29), and (2.30) into equation (2.24):

**VSP**

\[ O(x, z) = \frac{1}{\pi} \int_{-k_o}^{k_o} \exp[jk_z x - j\gamma_s z] O_1(x, z, k_s) dk_s. \]  
(2.32)
\[ O_1(x, z, k_s) = \frac{1}{\pi} \int_{-k_o}^{k_o} \exp[jk_z x - j\gamma_g (d_g - z)] \left| \frac{k_s \gamma_g + \gamma_s \gamma_g}{k_o^2} \right| \tilde{U}(k_s, k_g) dk_g. \]

**Surface reflection**

\[ O(x, z) = \frac{1}{\pi} \int_{-k_o}^{k_o} \exp[jk_z x - j\gamma_s z] O_1(x, z, k_s) dk_s. \]  
(2.33)
\[ O_1(x, z, k_s) = \frac{1}{\pi} \int_{-k_o}^{k_o} \exp[jk_g x - j\gamma_g z] \frac{k_s \gamma_g - k_s \gamma_s}{k_o^2} \tilde{U}(k_s, k_g) dk_g. \]

Equations (2.31), (2.32), and (2.33) are the diffraction tomography backpropagation reconstruction algorithms for the cross-borehole, VSP, and surface reflection geometries respectively, where \( O(x, z) \) is the unknown, \( \tilde{U}(k_s, k_g) \) is the Fourier transform of the measured scattered field.

The series expansion reconstruction algorithm for the diffraction tomography is not based on the generalized projection – slice theorem. To derive this algorithm, we first discretize the basic equation for the scattering problem – equation (2.14):

\[ U_{gs} = -k_o^2 \sum_{k=1}^{N} O_k G_{gk} G_{sk} \]  

\( U_{gs} \) is the scattered field measured at the geophone with index \( g \) when the point source with index \( s \) is activated. Equation (2.34) assumes that the object function is consist of \( N \) point scatterers. \( G_{gk} \) and \( G_{sk} \) are the three-dimensional Green’s functions:

\[ G_{lk} = \frac{\exp(jk_o \mid \vec{r}_l - \vec{r}_k \mid)}{4\pi \mid \vec{r}_l - \vec{r}_k \mid} \]  

Substitute equation (2.35) into equation (2.34):

\[ U_{gs} = \sum_{k=1}^{N} \left( -k_o^2 \frac{\exp(jk_o d_{gk}) \exp(jk_o d_{sk})}{4\pi d_{gk}} \frac{\exp(jk_o d_{sk})}{4\pi d_{sk}} \right) O_k, \]  

where \( d_{gk} \) is the distance between the \( g \)th geophone and the \( k \)th point scatterer, \( d_{sk} \) is the distance between the \( s \)th source and the \( k \)th point scatterer. For each source, the scattered field measured at the geophones can be written as:

\[ U_g = \sum_{k=1}^{N} T_{gk} O_k \]  

(2.37)
Using the Kaczmarz's projection method, the recurrence formula for the diffraction tomography series expansion reconstruction algorithm can be derived from equation (2.37):

\[
T_{gk} = -k_0^2 \exp(jk_0d_{gk}) \frac{\exp(jk_0d_{sk})}{4\pi d_{sk}}
\]

The basic structures of the series expansion method diffraction tomography and the series expansion method ray tomography are very similar. However, the data kernel matrix \( S \) in the series expansion ray tomography is a very sparse matrix whereas the data kernel matrix \( T \) in the series expansion diffraction tomography is a full matrix. Also, \( T_{gk} \) is a function of the source position, receiver position, and the point scatter position only, and it's value is fixed once it is computed. The \( S_{ij} \), on the other hand, depends on the source position, receiver position, and the current object function and therefore has to be computed during each iteration step.

Figure 2-10 summarizes the relationships between the forward problem and the three reconstruction algorithms described above for the diffraction tomography. The data vector in the diffraction tomography problem is the scattered field, \( U(l_s, l_g) \). \( U(l_s, l_g) \) can be inverted directly for the object function by the series expansion method diffraction tomography reconstruction algorithm, or, it can be inverted by the filtered back-propagation reconstruction algorithm or by the direct transform reconstruction algo-
rithm. If the backpropagation reconstruction algorithm is used, \( U(l_s, l_g) \) is first Fourier transformed along the source line and the geophone line, obtaining \( \tilde{U}(k_x, k_y) \). Then, by the backpropagation reconstruction algorithms, equations (2.31), (2.32), and (2.33), \( O(x, z) \) can be computed. If we use the direct transform reconstruction algorithm, we first use the generalized projection – slice theorem, equation (2.23), to calculate the Fourier transform of the object function on the \( k_o(g + \delta) \) loci, \( \tilde{O}[k_o(g + \delta)] \), then we interpolate \( \tilde{O}[k_o(g + \delta)] \) to \( \tilde{O}(k_z, k_z) \), and the object function \( O(x, z) \) is obtained by taking the inverse Fourier transform of \( \tilde{O}(k_z, k_z) \).

As indicated by Figure 2-9, the source – receiver geometries for seismic diffraction tomography can only measure a portion (the shaded regions in Figure 2-9) of the two-dimensional Fourier transform of the object function. As can be expected, images reconstructed from these incomplete spectra have poor resolution in certain directions. For example, in cross-borehole geometry, since the available spectral data only cover the dumbbell shaped region along the \( k_z \) axis – the vertical wavenumber axis, the horizontal resolution will be poor. This is the so called "limited view angle problem". The limited view angle problem can be bypassed by simply assigning zero to the unavailable spectral data, as the conventional diffraction tomography methods do, or, it can be solved by estimating the unavailable spectral data based on estimation theory, as do the methods developed in this thesis.
2.4 Discussion

In this chapter, we reviewed six tomographic reconstruction algorithms. Among them, four have been applied to geophysical problems: the series expansion reconstruction algorithms for the ray tomography such as ART and SIRT (Dines and Lytle, 1979), the filtered backpropagation reconstruction algorithm and the direct transform reconstruction algorithm for the diffraction tomography (Devaney, 1984; Wu and Toksöz, 1987), and the series expansion diffraction tomography (Devaney, 1985). Although the projection – slice theorem based reconstruction algorithms for the ray tomography (including the direct transform reconstruction algorithm and the filtered backprojection reconstruction algorithm) are the most frequently used reconstruction algorithms in medical imaging, they have not been applied to geophysical problems so far because they require special source – receiver geometry.

Seismic ray tomography and seismic diffraction tomography have different physical foundations. Seismic ray tomography is based on the high frequency approximation, which means the size of the inhomogeneities to be imaged should be much larger than the wavelength of the illuminating wave. Also, due to the restriction from the available sources and receivers, seismic ray tomography using ART or SIRT reconstruction algorithm is usually a very ill – conditioned problem which degrades its performance. Seismic diffraction tomography is based on the Born or Rytov approximations, which means the object to be imaged should be a weak scatterer. Also, the backpropagation reconstruction algorithm derived in this chapter requires that the object function be
a finite extent object function situated in a constant background medium. Further theoretical development can release these restrictions.
2.5 REFERENCES

Morse, P. M. and Feshbach, H., 1953, Methods of Theoretical Physics, McGraw-Hill, New York.
Figure 2-1: Geometry of the seismic ray tomography. $O(x,z)$ is the object function, such as the slowness distribution, $P(u,\theta)$ is the projection, such as the travel time profile.
Figure 2-2: Projection-slice theorem: The one-dimensional Fourier transform of the travel time profile, $\tilde{P}(\Omega, \theta)$, is a "slice" of the two-dimensional Fourier transform of the object function, $\tilde{O}(k_x, k_z)$. The shaded area represents $\tilde{O}(k_x, k_z)$, the solid circles represent $\tilde{P}(\Omega, \theta)$. 
Figure 2-3: For the seismic ray tomography with series expansion reconstruction algorithm, the imaging area is divided into $N$ pixels. $O_j$ is the average of a certain physical parameter inside the $j$th pixel, $S_{ij}$ is the length of the segment of the $i$th ray intersecting the $j$th pixel.
Figure 2-4: Illustration of Kaczmarz's projection method of solving a system of linear equations. This example simulates an imaging system with two projections ($M = 2$) and 2 pixels ($N = 2$).
Figure 2-5: Relationship between the forward problem and the three reconstruction algorithms for seismic ray tomography.
Figure 2-6: Geometry of the seismic diffraction tomography. $O(\mathbf{r})$ is the object function, $U(\mathbf{r}_s, \mathbf{r}_g)$ is the scattered field measured at position $\mathbf{r}_g$ when a point source is activated at position $\mathbf{r}_s$. The vertical distances from the origin to the source line and the geophone line are $d_s$ and $d_g$ respectively. $l_s$ is the distance of the source along the source line, $l_g$ is the distance of the geophone along the geophone line.
Figure 2-7: Geometrical relationship between the loci $k_o(\hat{g} + \hat{s})$, represented by the open circles, and the Cartesian grid $(k_z, k_x)$ for a single $\hat{s}$. 
Figure 2-8: Coordinate transformation relationships between the \((k_x, k_z)\) coordinate system and the \((k_s, k_g)\) coordinate system for the cross-borehole, VSP, and surface reflection configurations.
Figure 2-9: Spectral coverage of diffraction tomography for three source – receiver geometries: (a) cross-borehole, (b) VSP, and (c) surface reflection.
Figure 2-10: Relationship between the forward problem and the three reconstruction algorithms for the seismic diffraction tomography.
Chapter 3

ULTRASONIC LABORATORY
TESTS OF SEISMIC BOREHOLE
TOMOGRAPHY

3.1 Introduction

In this chapter, we present a series of ultrasonic laboratory experiments simulating seismic borehole tomography. Seismic ray tomography has been tested in the field (Bishop et al., 1985; Peterson et al., 1985; Chiu et al., 1986; Cottin et al., 1986; Gustavsson et al., 1986), but because of the noise in the field experiments, the inaccuracy in ray tracing, and the limited view angle problem, the resolution in the results of these tests is not satisfactory. Seismic diffraction tomography, although theoretically has higher resolution and computationally more efficient than seismic ray tomography, has only
been tested by synthetic data (Devaney, 1984; Esmer soy, 1986; Wu and Toksöz, 1987) which were free of noise, and satisfied the assumptions such as weak scattering, constant background medium, and finite extent object function.

The ultrasonic laboratory experiments presented in this chapter test seismic ray tomography and seismic diffraction tomography with the same data set, collected by source-receiver configurations simulating three geophysical field operations: cross-borehole, VSP, and surface reflection. The objectives of these experiments are: (1) evaluate the relative performance of ray tomography and diffraction tomography when they are both applied to the subsurface imaging problem, and (2) observe the behavior of the physical models (diffraction) and approximations (high frequency approximation for ray tomography, Born and Rytov approximations for diffraction tomography), which are the bases of the tomographic methods, when they are applied to the subsurface imaging problem. Well known reconstruction algorithms such as the simultaneous iterative reconstruction technique (SIRT) and the filtered backpropagation reconstruction algorithm will be used in this chapter. The new reconstruction techniques developed in this thesis are presented in Chapter 4 and Chapter 5.

Kaveh et al. (1981) conducted a laboratory experiment comparing the effects of the Born and Rytov approximations on diffraction tomography. In their study, the object was evenly illuminated from all directions. Our experiments, on the other hand, studies the effects of the Born and Rytov approximations when the object is illuminated only from limited view angles.
3.2 Ultrasonic Experiments

3.2.1 Laboratory setup

Ultrasonic experiments simulating seismic borehole tomography are carried out in a modelling tank. This tank is $100\text{cm} \times 60\text{cm} \times 50\text{cm}$ in dimension and is equipped with computer-based control and data acquisition systems. Water is used as a constant velocity background medium. The imaging area is a $24\text{cm} \times 24\text{cm}$ square region in the center of the water tank. For 50 KHz acoustic waves in water, the imaging area is about 8 wavelengths $\times$ 8 wavelengths. Objects of various sizes and acoustic properties are used as targets to be imaged. Table 3-1 lists the P wave velocity and density of the testing materials used in this chapter and Chapter 4 and 5. We use two broad-band hydrophones as the source (Celesco LC-34) and the receiver (International Transducer Corporation ITC-1089D). Both hydrophones are made of PZT spheres. The frequency range in our experiments is 10 KHz to 200 KHz. The source and the receiver can be moved independently in three-dimensional space by six stepping motors. Each step is equal to 0.064 mm. The translation scanning scheme of the hydrophones is controlled by a SLO-SYN stepping motor controller. The ultrasonic wave is generated at the source hydrophone by a Panametrics 5055PR pulser. The received signals are filtered by a Krohn-Hite 3202R filter, amplified by a Panametrics 5660B preamplifier, and digitized by a Data Precision DATA6000 digital oscilloscope with 12-bit amplitude resolution. The digital oscilloscope and the stepping motor controller are interfaced with the IBM
PC-AT computer by the IEEE-488 interface bus. Digitized data are transmitted to a VAX 11/780 computer for image reconstruction. Images are displayed on a Comtal image processor. A block diagram of the laboratory setup is shown in Figure 3-1.

In our tomography experiments, we simulate three source-receiver configurations frequently used in geophysical field operations: VSP, cross-borehole, and surface reflection. The source and receiver configurations used in our experiments are shown in Figure 3-2. In most of the experiments, the object is a gelatin cylinder 90 mm in diameter. P-wave velocity and density of this gelatin cylinder are 1.55 km/sec and 1.24 g/cc respectively. The difference of the P-wave velocity between the object and the background medium is only 4%. This small velocity difference is designed to satisfy the constraints for using the Born and Rytov approximations (see Appendix A).

For the diffraction tomography experiments, objects are reconstructed by the filtered backpropagation algorithms listed in Table 3-2. The input to these algorithms is the scattered wavefield produced by the object. We use a dual-experiment method to measure the scattered wavefield. First, we put the object inside the water tank, scan the source and the receiver around it, and measure the total wavefield. Then, we remove the object, repeat the same scanning procedure to obtain the incident wavefield. The difference between these two sets of data is the scattered wavefield due to the object. This dual-experiment method also helps eliminate the interference from the experiment setup such as the finite-sized source and receiver hydrophones.

For the ray tomography experiments, we use the same data set collected in the
diffraction tomography experiments. The travel times of the first-arrived P waves of the total field waveforms are used to invert the velocity of the object by the simultaneous iterative reconstruction technique described in Chapter 2.

### 3.2.2 Reconstruction algorithms

For seismic diffraction tomography experiments, we use the backpropagation reconstruction algorithms for the cross-borehole, VSP, and surface reflection geometries derived in Chapter 2. For easy reference, they are summarized in Table 3-2.

In Chapter 2, we use two dimensional geometry to derive the backpropagation reconstruction algorithm. Our experiments, however, are two and half dimensional. The validity of approximating $2\frac{1}{2}D$ scattering problem by $2D$ scattering problem is discussed in Appendix B. By numerical examples, Appendix B indicates that in the farfield, the $2\frac{1}{2}D$ scattering problem can be approximated by the $2D$ scattering problem.

For the case of reconstruction based on the Rytov approximation, all the reconstruction formulae are the same except that the $\tilde{U}(k_g, k_s)$ in Table 3-2 is replaced by $\tilde{\Phi}(k_g, k_s)$, which is the two-dimensional Fourier transform of the complex phase function $\Phi(\mathbf{r}_g, \mathbf{r}_s)$ defined by

$$
\Phi(\mathbf{r}_g, \mathbf{r}_s) = U_i(\mathbf{r}_g, \mathbf{r}_s) \log \frac{U_i(\mathbf{r}_g, \mathbf{r}_s) + U(\mathbf{r}_g, \mathbf{r}_s)}{U_i(\mathbf{r}_g, \mathbf{r}_s)}
= U_i(\mathbf{r}_g, \mathbf{r}_s) \phi_d(\mathbf{r}_g, \mathbf{r}_s) \quad (3.1)
$$

where $U_i(\mathbf{r}_g, \mathbf{r}_s)$ is the incident field, $\phi_d(\mathbf{r}_g, \mathbf{r}_s)$ is the complex phase difference between the total field and the incident field. Equation (3.1) is derived in Appendix A.
For the seismic ray tomography experiments, we use the simultaneous iterative reconstruction technique (SIRT) which is also derived in Chapter 2:

\[
\hat{O}_{ij}^{new} = \hat{O}_{ij}^{old} + \frac{1}{W_{ij}} \sum_{i=1}^{M} \Delta O_{ij},
\]

(3.2)

\[
\Delta O_{ij} = S_{ij} \frac{P_{i} - \sum_{j=1}^{N} S_{ij} \hat{O}_{ij}^{old}}{\sum_{j=1}^{N} (S_{ij})^2}.
\]

(3.3)

\(S_{ij}\) in equation (3.3) is calculated by ray tracing. In our experiment, we use a two-dimensional ray tracing algorithm similar to the algorithm described by Anderson and Kak (1982). This algorithm is derived by first expressing the position vector of the ray \(r\) in a Taylor series and discarding the third and higher order terms:

\[
r(l + \Delta l) = r(l) + \frac{dr}{dl} \Delta l + \frac{1}{2} \frac{d^2r}{dl^2}(\Delta l)^2,
\]

(3.4)

where \(l\) is the distance along the ray and \(\Delta l\) is the step size. We then write the ray equation as:

\[
\frac{d}{dl}(n \frac{dr}{dl}) = \nabla n,
\]

(3.5)

where \(n\) is the refractive index. Substituting equation (3.5) into equation (3.4), we obtain the following expression for the ray which can be implemented directly as a ray tracing algorithm.

\[
r(l + \Delta l) = r(l) + \frac{dr}{dl} \Delta l + \frac{1}{2n} \left[ \nabla n - (\nabla n \cdot \frac{dr}{dl}) \frac{dr}{dl} \right] (\Delta l)^2.
\]

(3.6)

In equation (3.5), \(\frac{dr}{dl}\) is the unit vector tangent to the ray, and \(\nabla n\) is calculated by central-difference approximation. The refractive index at an arbitrary position is approximated by bilinear interpolation using the four nearest grid values. We use the
shooting method to solve the ray linking problem, and the launching angle is determined by the Newton's method.

3.3 Experimental Results

3.3.1 Cross-borehole tomography experiment

The layout of this experiment is shown in Figure 3-2(a). The source hydrophone is activated at 32 equally spaced positions along a source line, simulating 32 sources in one borehole. The receiver hydrophone records waveforms at 32 equally spaced positions along a receiver line, simulating 32 receivers in another borehole. The gelatin cylinder is placed between the source line and the receiver line. The ultrasonic wavefield generated at each source position is measured at 32 receiver positions and therefore $32 \times 32$ waveforms are recorded for measuring the incident wavefield and the same amount of data are recorded for measuring the total wavefield. The dominant frequency of our signals is 50 KHz. It corresponds to a wavelength of 30 mm in water. The sampling interval along the source line and the receiver line is 7.62 mm. Waveforms are digitized at a sampling interval of 300 nano-seconds. Figure 3-3 shows 32 waveforms recorded at 32 receiver positions with the source hydrophone at the middle of the source line. Figure 3-3(a) shows the total field waveforms, Figure 3-3(b) the incident field waveforms, and Figure 3-3(c) the scattered field waveforms. Taking the Fourier transform of the waveforms, we obtain the magnitude and phase of the total field, the incident field, and the scattered field at various frequencies. We use Tribolet's algorithm (Tribolet, 1977)
with the computation efficiency improvement made by Bonzanigo (1978) to unwrap the phase data. Using these data, we reconstruct the object with the filtered backpropagation algorithm. Reconstructions with both the Born and the Rytov approximations are calculated. Figure 3-4 shows the reconstructions with the Born and Rytov approximations at 30 KHz and 50 KHz. At the lower frequency, images reconstructed by either the Born or the Rytov approximations are about the same quality (compare Figures 3-4 (a) and (b)), whereas at higher frequency, the image reconstructed by the Rytov approximation is less distorted than the one reconstructed by the Born approximation (compare Figures 3-4 (c) and (d)). This wavelength independence property of the Rytov approximation observed in this experiment is consistent with the results of the numerical study by Slaney et al. (1984). In their work, they demonstrated that the validity of the Rytov approximation is judged by the phase change per wavelength, not by the total phase change. Therefore, as long as the velocity contrast between the object and the surrounding medium is small enough (less than a few percent, as suggested by Slaney et al., 1984), the Rytov approximation is valid without constraints on the size of object. The Born approximation, however, requires that the scattered field be small. This will be violated when the size of the weak inhomogeneity becomes large. It is also noted that in the cross-borehole configuration, the information coverage in the frequency domain is poor in the horizontal direction (see Figure 4 in Wu and Toksöz, 1987). This is consistent with the poorer horizontal resolution in the images reconstructed in our cross-borehole experiments.
The travel times of the first-arrived P waves of the total field waveforms are also measured in the cross-borehole experiment to reconstruct the gelatin cylinder by the simultaneous iterative reconstruction technique. The images reconstructed are shown in Figure 3-5. Figure 3-5(a) is an initial estimate assuming no information about the object is available. Figure 3-5(b) is the reconstruction after twenty iterations from Figure 3-5(a). Figure 3-5(c) is another initial estimate circular in shape but with a radius twice the radius of the true object. Figure 3-5(d) is the corresponding reconstruction after twenty iterations. Comparing Figures 3-5 (b) and (d) we note that ray tomography using the simultaneous iterative reconstruction technique improves significantly if the initial model approximates the true object. Further iterations did not improve the images significantly in either case.

3.3.2 Cross-borehole experiment with a more complex object

To further investigate the relative performance of the diffraction tomography and the ray tomography, we ran another cross-borehole experiment with a more complex object. This object is a gelatin cylinder with two aluminum rods inside as shown in Figure 3-6. The image reconstructed by diffraction tomography based on the Born approximation is shown in Figure 3-7(a). Both the gelatin cylinder and the two aluminum rods are successfully reconstructed. The same data are also inverted by ray tomography with the simultaneous iterative reconstruction technique. With an initial estimate such as Figure 3-5(c), the ray tomography reconstruction after twenty iterations is shown
in Figure 3-7(b). The gelatin cylinder is reasonably well reconstructed, but the two aluminum rods are poorly reconstructed. This experiment demonstrates that when the size of the object is comparable to the wavelength, diffraction tomography with the filtered backpropagation reconstruction algorithm can reconstruct such a small object better than ray tomography.

### 3.3.3 VSP tomography experiment

In this experiment, the source hydrophone is activated at 32 equally spaced positions along the source line, simulating 32 sources arranged in a straight line on the surface. The receiver hydrophone records waveforms at 32 equally spaced positions along a straight line perpendicular to the source line, simulating 32 receivers in a borehole. Examples of the waveforms recorded with a gelatin cylinder as the scatterer are shown in Figure 3-8. Figure 3-9 shows the filtered backpropagation reconstructions at 30 KHz and 50 KHz with the Born and Rytov approximations. All these four examples in Figure 3-9 reconstruct the upper right portion of the object better than the lower left portion.

Similar to the cross-borehole ray tomography experiment, the travel times of the first arrived P waves of the total field waveforms are used to reconstruct the object by the simultaneous iterative reconstruction technique. The results are shown in Figure 3-10. Figure 3-10 (a) and (c) are initial guess images. Figure 3-10 (b) and (d) are the corresponding reconstructions after one hundred iterations. Comparing Figure 3-9
and Figure 3-10, it is apparent that the diffraction tomography is less sensitive to the limited view angle problem than the ray tomography.

3.3.4 Surface reflection tomography experiment

The layout of this experiment is shown in Figure 3-2(c). The source hydrophone scans along a source line, simulating 32 sources on the surface. The receiver hydrophone scans along another line parallel to the source line, simulating 32 receivers also on the surface. Figure 3-11 is an example of the waveforms recorded in this surface reflection experiment with the source hydrophone situated in the middle of the source line, where Figure 3-11(a) is the total field, 3-11(b) the incident field, and 3-11(c) the scattered field. Since the acoustic impedance contrast between the water and the gelatin cylinder is very small, the back-scattered wavefield in this experiment is very weak. This will reduce the signal/noise ratio in this experiment and make our reconstruction very noisy. Images reconstructed by the filtered backpropagation algorithm are shown in Figure 3-12. Figure 3-12 (a) and (b) are the reconstructions at 30 KHz based on the Born and the Rytov approximations. Figure 3-12 (c) and (d) are the reconstructions at 50 KHz based on the Born and the Rytov approximations. In this example, the Born approximation performs as well or better than the Rytov approximation. As discussed by Kaveh et al. (1981) and Wu and Aki (1985) the Born approximation performs well for back scattering. In the surface reflection experiment, the dominant forward scattering component which is disturbing for the Born approximation is not received.
by the receiver. The input to the reconstruction algorithm is the relatively weak back scattering component of the scattered wavefield and this may be one of the reasons why the Born approximation works in this case.

### 3.4 Conclusions

Both the diffraction tomography and the ray tomography can be used for subsurface imaging. These two methods have different adaptability for the available source-receiver configurations and the size and properties of the object. When the source-receiver configuration is such that the insonifying waves are directly transmitted only through part of the object, such as the VSP and the surface reflection experiments, the diffraction tomography is superior to the ray tomography. If the object is uniformly illuminated, as is the case in the cross-borehole experiment, the size and the properties of the object determine the best reconstruction algorithm. In the cross-borehole configuration, if the size of the object is comparable to the wavelength, the diffraction tomography method is better than ray tomography. If the size of the object is much bigger than the wavelength, ray tomography may perform as well as diffraction tomography. It should be made clear that these conclusions are based on our laboratory setup where we can separate the scattered wavefield by measuring the background field. In field applications, this arrangement may be possible in enhanced recovery process, fracturing or other cases where "before" and "after" measuring procedure can be made.

Two factors closely related to the fidelity of the seismic diffraction tomography are
also examined in this chapter: (1) source-receiver configuration, and (2) approximation methods. Among the three source-receiver configurations tested in this study, the cross-borehole configuration gives the best result. Images reconstructed with the surface reflection configuration have very strong background noise. VSP configuration images the quadrant of the object facing the source line and the receiver line better than the opposite quadrant of the object.

The Born and Rytov approximations are also compared based on our experimental data. In this chapter, we discuss only reconstructed images with data whose wavelengths are $\frac{1}{5}$ (50 KHz examples) or $\frac{1}{18}$ (30 KHz examples) of the diameter of the object (gelatin cylinder). This makes it difficult to decide whether or not the Rytov approximation is superior to the Born approximation (Chernov, 1960; Kaveh et al., 1981; Slaney et al., 1984; Zapalowski et al., 1985) when the size of the weak inhomogeneity is much bigger than the wavelength. Our experimental results suggest that the Rytov approximation has a wider range of validity than the Born approximation for the cross-borehole experiment. This is expected since the Rytov approximation is better suited to the transmission (forward scattering) experiment than the Born approximation. For the VSP and surface reflection tomography experiments, no substantial difference between the Born and Rytov approximations is observed in this study.
3.5 REFERENCES


P-wave Velocity Density
Km/sec $g/cm^3$

<table>
<thead>
<tr>
<th>Material</th>
<th>Velocity</th>
<th>Density</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>1.50</td>
<td>1.00</td>
</tr>
<tr>
<td>Gelatin</td>
<td>1.55</td>
<td>1.24</td>
</tr>
<tr>
<td>Aluminum</td>
<td>6.30</td>
<td>2.70</td>
</tr>
<tr>
<td>Glass</td>
<td>5.64</td>
<td>2.20</td>
</tr>
<tr>
<td>Silicon rubber</td>
<td>1.04</td>
<td>1.22</td>
</tr>
</tbody>
</table>

Table 3-1. P-wave velocity and density of the materials tested in this thesis.
<table>
<thead>
<tr>
<th></th>
<th>Reconstruction formula</th>
<th>Coordinate transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cross-borehole</td>
<td>$O(x, z) = \frac{1}{\pi} \int_{-k_o}^{k_o} dk_x \exp(ik_x z - i \gamma_s x)O_1(x, z, k_s)$</td>
<td>$K_z = \gamma_g - \gamma_s, \quad K_z = k_g + k_s$</td>
</tr>
<tr>
<td></td>
<td>$O_1(x, z, k_s) = \frac{1}{\pi} \int_{-k_o}^{k_o} dk_x \exp(ik_x z - i \gamma_g(x_z - z))D(k_g, k_s)$</td>
<td>$J(K_z, K_z</td>
</tr>
<tr>
<td></td>
<td>$D(k_g, k_s) = \bar{U}(k_g, k_s)\frac{</td>
<td>k_s \gamma_g + k_g \gamma_s</td>
</tr>
<tr>
<td>VSP</td>
<td>$O(x, z) = \frac{1}{\pi} \int_{-k_o}^{k_o} dk_x \exp(ik_x x - i \gamma_s x)O_1(x, z, k_s)$</td>
<td>$K_z = k_g + \gamma_s, \quad K_z = k_g - \gamma_s$</td>
</tr>
<tr>
<td></td>
<td>$O_1(x, z, k_s) = \frac{1}{\pi} \int_{-k_o}^{k_o} dk_x \exp(ik_x z - i \gamma_g(x_z - z))D(k_g, k_s)$</td>
<td>$J(K_z, K_z</td>
</tr>
<tr>
<td></td>
<td>$D(k_g, k_s) = \bar{U}(k_g, k_s)\frac{</td>
<td>k_g k_s + \gamma_g \gamma_s</td>
</tr>
<tr>
<td>Surface reflection</td>
<td>$O(x, z) = \frac{1}{\pi} \int_{-k_o}^{k_o} dk_x \exp(ik_x x - i \gamma_g(z - z_s))O_1(x, z, k_s)$</td>
<td>$K_z = k_g + k_s, \quad K_z = -(\gamma_g + \gamma_s)$</td>
</tr>
<tr>
<td></td>
<td>$O_1(x, z, k_s) = \frac{1}{\pi} \int_{-k_o}^{k_o} dk_x \exp(ik_x z - i \gamma_g(z - z_g))D(k_g, k_s)$</td>
<td>$J(K_z, K_z</td>
</tr>
<tr>
<td></td>
<td>$D(k_g, k_s) = \bar{U}(k_g, k_s)\frac{</td>
<td>k_s \gamma_g - k_g \gamma_s</td>
</tr>
</tbody>
</table>

Table 3-2  Filtered back propagation reconstruction formulae for the cross-borehole, VSP, and surface reflection experiments.
Figure 3-1: Block diagram of the microcomputer based ultrasonic imaging system.
Figure 3-2: Top view of the actual layout of the tomography experiments. (a) Cross-borehole experiment. (b) VSP experiment. (c) Surface reflection experiment.
Figure 3-3: Examples of waveforms recorded in the cross-borehole experiment. (a) Total field waveforms. (b) Incident field waveforms. (c) Scattered field waveforms.
Figure 3-4: Images reconstructed by the filtered backpropagation algorithm in the cross-borehole experiment. The gelatin cylinder should be centered at the cross with the size and shape as shown by the white circle at the lower left corner of the figure. (a) Reconstruction with the Born approximation at 30 KHz. (b) Reconstruction with the Rytov approximation at 30 KHz. (c) Reconstruction with the Born approximation at 50 KHz. (d) Reconstruction with the Rytov approximation at 50 KHz.
Figure 3-5: Images reconstructed by the simultaneous iterative reconstruction technique in the cross-borehole ray tomography experiment. The gelatin cylinder should be centered at the cross with the size and shape as shown by the circle at the upper left corner of the figure. (a) An initial estimate assuming no information about the object function. (b) Reconstruction with an initial estimate (a) after twenty iterations. (c) An initial estimate with some a priori information about the object. (d) Reconstruction with an initial estimate such as (c) after twenty iterations.
Figure 3-6: Experimental configuration of the cross-borehole experiment with a more complex object. The object is a gelatin cylinder with two aluminum rods inside.
Figure 3-7: Images reconstructed by the filtered backpropagation reconstruction algorithm and the simultaneous iterative reconstruction technique for a gelatin cylinder with two aluminum rods in the cross-borehole experiment. (a) Image reconstructed by the filtered backpropagation algorithm with the Born approximation at 30 KHz. (b) Image reconstructed by the simultaneous iterative reconstruction technique.
Figure 3-8: Examples of waveforms recorded in the VSP experiment. (a) Total field waveforms. (b) Incident field waveforms. (c) Scattered field waveforms.
Figure 3-9: Images reconstructed by the filtered backpropagation algorithm in the VSP experiment. The gelatin cylinder should be centered at the cross with the size and shape as shown by the white circle at the lower left corner of the figure. (a) Reconstruction with the Born approximation at 30 KHz. (b) Reconstruction with the Rytov approximation at 30 KHz. (c) Reconstruction with the Born approximation at 50 KHz. (d) Reconstruction with the Rytov approximation at 50 KHz.
Figure 3-10: Images reconstructed by the simultaneous iterative reconstruction technique in the VSP ray tomography experiment. The gelatin cylinder should be centered at the cross with the size and shape as shown by the circle at the upper left corner of the figure. 
(a) An initial estimate assuming no information about the object function. 
(b) Reconstruction of the object with an initial estimate like (a) after one hundred iterations. 
(c) An initial estimate with some a priori information about the object. 
(d) Reconstruction of the object with an initial estimate like (c) after one hundred iterations.
Figure 3-11: Examples of waveforms recorded in the surface reflection experiment. (a) Total field waveforms. (b) Incident field waveforms. (c) Scattered field waveforms.
Figure 3-12: Images reconstructed by the filtered backpropagation algorithm in the surface reflection experiment. The gelatin cylinder should be centered at the cross with the size and shape as shown by the white circle at the left lower corner of the figure. (a) Reconstruction with the Born approximation at 30 KHz. (b) Reconstruction with the Rytov approximation at 30 KHz. (c) Reconstruction with the Born approximation at 50 KHz. (d) Reconstruction with the Rytov approximation at 50 KHz.
Chapter 4

MINIMUM CROSS ENTROPY

SEISMIC DIFFRACTION

TOMOGRAPHY

4.1 Introduction

Due to the finite extent of the sources and the receivers, it is always difficult to probe the object in seismic borehole tomography from all the desired directions. This problem is known as the limited view angle problem. Up to now, most seismic tomography methods have bypassed this problem by assigning zero to the "blind area" where data are not available. In this chapter, we develop two algorithms which estimate the blind area data based on the "minimum cross entropy principle".

The limited view angle problem can be treated as an underdetermined inverse prob-
lem — making estimate of the unknown (the object) which can not be fully probed by our measuring system. Given measured data in such a problem, there can be many different estimates about the unknown and all of them satisfy the measured data. The maximum entropy method and the minimum cross entropy method solve this many-to-one mapping problem by offering us a criterion to choose one estimate from all the other eligible estimates. The maximum entropy method and the minimum cross entropy method are briefly explained below. For an axiomatic derivation of these two methods, the reader is referred to Shore and Johnson (1980).

Since we are making an estimate in the limited view angle tomography problem, there is a certain amount of uncertainty associated with our estimate due to both noise and incomplete coverage. The uncertainty is a function of the estimate. For a finite amount of measurements, only a finite number of uncertainty can be reduced and there is a lower bound for the uncertainty value for a given measuring geometry. To be objective (Jaynes, 1968) about our estimate, we should avoid making an estimate with an uncertainty value lower than this bound, and if we want to make the most objective estimate, we should search for the estimate with the highest uncertainty value. Khinchin (1957) proved that entropy is the unique function of the estimate that quantitatively measures the uncertainty of the estimate. This is the essence of the maximum entropy principle: If we want to make the most objective estimate, we should search for the estimate with the maximum entropy that still matches the measured data. The minimum cross entropy principle is a generalization of the maximum entropy principle when a
priori distribution of the object function which we are trying to estimate is available. Let $O(x, z)$ denote the object function and $P(x, z)$ denotes a priori distribution of the object function. Following the definition by Shore and Johnson (1980), the entropy, $H_1$, and the cross entropy, $H_2$, of the object function can be written as:

$$H_1 = - \int O(x, z) \log O(x, z) dxdz,$$

(4.1)

$$H_2 = \int O(x, z) \log \frac{O(x, z)}{P(x, z)} dxdz.$$  

(4.2)

The maximum entropy principle is to choose the estimate with the maximum $H_1$, the minimum cross entropy principle is to choose the estimate with the minimum $H_2$. These are two different criteria. However, under one condition, these two criteria are equivalent. This condition is that $P(x, z)$ is a constant, or, no a priori distribution information is available. The intuitive argument above defines both the maximum entropy estimation and the minimum cross entropy estimation. It also justifies the applicability of these two methods to problems where we do not have enough data to uniquely specify the unknowns, such as the limited view angle tomography problem.

The maximum entropy method (MEM) and the minimum cross entropy method (MCEM) can be applied to the limited view angle tomography problem in several different ways. To avoid confusion, we first list several reconstruction algorithms and estimation algorithms for the ray tomography and the diffraction tomography in Table 4-1. Then we explain how the reconstruction algorithms and the estimation algorithms can be combined to achieve optimal solutions to the limited view angle tomography problem. As discussed in Chapter 2, the ray tomography problem has three types
of reconstruction algorithms: the series expansion method (Type 1 algorithm), the
direct transform method (Type 2 algorithm), and the filtered backprojection method
(Type 3 algorithm). The diffraction tomography problem also has three types of re-
construction algorithms: the series expansion method (Type 4 algorithm), the direct
transform method (Type 5 algorithm), and the filtered backpropagation method (Type
6 algorithm).

All these six algorithms are simply reconstruction algorithms. They can be com-
bined with the estimation algorithms listed in Table 4-1 to approach the limited view
angle problem. Two estimation algorithms are listed in Table 4-1: the minimum cross
entropy estimation (including the maximum entropy estimation) and the estimation
based on the consistency principle. Estimation based on the consistency principle is
actually a generalization of the classical Gerchberg-Papoulis band limited spectrum
extrapolation algorithm (Medoff et al., 1983). In this chapter, we only discuss how
to apply the minimum cross entropy estimation to the limited view angle tomography
problem. So far, four different ways of combining the MEM or MCEM estimation al-
gorithms with the tomographic reconstruction algorithms have been developed and we
label them as algorithm Type 7, 8, 9, and 10 in Table 4-1.

Type 7 algorithm applies the MEM estimation to the ray tomography with series
expansion reconstruction. This algorithm maximizes the entropy of the object function
under the constraints imposed by the matrix equation relating the object function and
the projection data (such as the travel time profile). Lent (1977) solved this optimiza-
tion problem as a primal problem and proved that the MART algorithm (multiplicative algebraic reconstruction technique) proposed by Gordon et al. (1970) converges to the maximum entropy solution. Minerbo (1978) solved the same optimization problem as a dual problem and developed the MENT algorithm (maximum entropy). Minerbo (1978) compared the performance of MENT and MART using synthetic data and claimed that MENT can avoid the streaking artifacts that MART usually has. Sanderson (1979) also tested the MENT algorithm with synthetic data simulating nuclear reactor fuel pin bundles and found that MENT provides accurate reconstructions of fuel pin bundles even when simulated voids are present.

Type 8 algorithm applies the MEM estimation to ray tomography with direct transform reconstruction which is possible with laboratory or clinical data acquisition geometries. This algorithm maximizes the entropy of the object function under the constraints imposed by the two-dimensional Fourier transform relationship between the object function and the measured spectrum. Wernecke and D'addario (1977) applied this algorithm to radio astronomy data as an image processing method.

The type 9 algorithm applies the MCEM estimation to the series expansion method diffraction tomography. This algorithm minimizes the cross entropy of the object function under the constraints imposed by the matrix equation relating the object function and the measured scattered field. Mohammad-Djafari and Demoment (1986) developed an algorithm solving this optimization problem and tested their algorithm with synthetic data. A Type 10 algorithm was also developed by Mohammad-Djafari and
Demoment (1986). That algorithm applies the MCEM estimation to direct transform diffraction tomography by minimizing the cross entropy of the object function under the constraints imposed by the Fourier transform relationship between the object function and the measured spectrum. Mohammad-Djafari and Demoment (1986) developed Type 9 and Type 10 algorithms by solving the constrained optimization problem as a primal problem.

In this chapter, we (1) reformulate the Type 10 algorithm and develop a minimum cross entropy direct transform reconstruction algorithm by solving a dual problem, instead of a primal problem as did Mohammad-Djafari and Demoment (1986), and (2) develop the minimum cross entropy backpropagation reconstruction algorithm (Type 11 algorithm in Table 4-1). This algorithm is equivalent to extending the coverage of the source array and the receiver array in the space domain and therefore a finite aperture compensation algorithm. Numerical and ultrasonics laboratory experiments using these two algorithms are also presented. The noise sensitivity of the methods developed in this chapter will be discussed in Chapter 6.
4.2 Theory and Algorithms

4.2.1 Minimum cross entropy direct transform reconstruction algorithm (MCED)

Using the Born approximation, the basic equation for the acoustic wave scattering problem for the cross-borehole geometry shown in Figure 4-1(a) can be written as:

\[
U(r_s, r_g) = -k_o^2 \int_V O(\tau) G(\tau, r_s) G(\tau, r_g) d\tau, \tag{4.3}
\]

where

\[
G(\tau, r') = \frac{j}{4} H_0^{(1)}(k_o |\tau - r'|).	ag{4.4}
\]

Subscripts \(s\) and \(g\) refer to source and receiver respectively, \(G\) is the Green's function for the background medium, \(H_0^{(1)}\) is the zero order Hankel’s function of the first kind, \(U(r_s, r_g)\) is the scattered field measured at position \(r_g\) when the point source is at position \(r_s\), \(k_o\) is the wavenumber in the background medium, and \(O(\tau)\) is the object function defined as:

\[
O(\tau) = \left| 1 - \frac{C_2^2}{C^2(\tau)} \right|. \tag{4.5}
\]

This definition of \(O(\tau)\) is the absolute value of the \(O(\tau)\) defined in Chapter 2, it guarantees the argument inside the log be positive, however, it can not distinguish two velocities, \(C_1(\tau)\) and \(C_2(\tau)\), if \(|C_1(\tau) - C_2(\tau)| = |C_2(\tau) - C_1(\tau)|\) but \(C_1(\tau) > C_o > C_2(\tau)\).

The regular direct transform reconstruction algorithms invert the object function
$O(\tau)$ from the measured scattered field $U(\tau_s, \tau_g)$ by the following four steps:

step 1:  
$$\tilde{U}(k_s, k_g) = F[U(\tau_s, \tau_g)]$$

step 2:  
$$\tilde{O}(k_o(\hat{\gamma} + \hat{s})) = \frac{4\pi\gamma_s}{k_o^2} \tilde{U}(k_s, k_g) \exp[-j(\gamma_s d_s + \gamma_s d_g)]$$

step 3:  
$$\tilde{O}(k_z, k_z) \text{ interpolation } \tilde{O}[k_o(\hat{\gamma} + \hat{s})]$$

step 4:  
$$O(x, z) = F^{-1}[\tilde{O}(k_z, k_z)]$$

where $k_s$ and $k_g$ are the wavenumbers along the source line and the receiver line,

$$\gamma_s = \sqrt{k_o^2 - k_s^2}, \quad \gamma_g = \sqrt{k_o^2 - k_g^2}, \quad \hat{s} \text{ and } \hat{\gamma} \text{ are the unit vectors of plane wave to source and receiver, respectively. } F \text{ and } F^{-1} \text{ denote forward and inverse Fourier transforms.}$$

This algorithm starts with taking the Fourier transform of the measured scattered field, calculating the Fourier transform of the object function on the $k_o(\hat{\gamma} + \hat{s})$ locus, interpolating $\tilde{O}$ from the $k_o(\hat{\gamma} + \hat{s})$ locus to the rectangular grid $(k_z, k_z)$, and, finally, taking the inverse Fourier transform of $\tilde{O}(k_z, k_z)$ and obtaining the object function $O(x, z)$. This is the Type 5 algorithm in Table 4-1. The equation used in step 2 is the generalized projection–slice theorem derived in Chapter 2.

There are two problems in this reconstruction algorithm. One is the the incomplete coverage, the other is interpolation error. The incomplete coverage problem is illustrated in Figure 4-1. As we can see in Figure 4-1(b), when we have a finite aperture source and receiver array, such as shown in Figure 4-1(a), the available spectrum $\tilde{O}(k_z, k_z)$ covers only two lenticular shaped regions along the $k_z$ axis. If we assign zero to the $\tilde{O}(k_z, k_z)$ outside the lenticular shaped regions and then take the inverse Fourier transform to recover $O(x, z)$, poor horizontal resolution will be introduced.
This problem can be helped by applying the minimum cross entropy estimation to the direct transform reconstruction algorithm. The minimum cross entropy direct transform reconstruction problem can be stated as the following constrained optimization problem:

minimize

$$H = \int_{z_{\min}}^{z_{\max}} \int_{x_{\min}}^{x_{\max}} O(x, z) \log \frac{O(x, z)}{P(x, z)} dx dz$$

subject to

$$\hat{O}_{\text{measured}}(k_x, k_z) = F[O(x, z)]$$

for \((k_x, k_z) \in A, \quad A : \text{support}\) 

where \(P(x, z)\) is the a priori distribution of \(O(x, z)\). The integrations in equation (4.6) are taken over the whole imaging area: from \(X_{\min}\) to \(X_{\max}\) and from \(Z_{\min}\) to \(Z_{\max}\). The support \(A\) is the region in the \((k_x, k_z)\) plane where the two-dimensional Fourier transform of the object function can be measured. Mohammad-Djafari and Demoment (1986) solved this optimization problem as a primal problem. They used the Lagrange multiplier method to connect the constraints with the penalty function (the cross entropy of the object function) and used the conjugate gradient method to search for the minimum of the penalty function. In this chapter, we solve a dual problem of this optimization problem. The strategy of our estimation algorithm is to keep the measured data \(\hat{O}(k_x, k_z)\) inside the support \(A\) unchanged and to estimate the \(\hat{O}(k_x, k_z)\) outside \(A\) such that the cross entropy \(H\) is minimized. To search for the minimum of \(H\) by changing \(\hat{O}(k_x, k_z), \quad (k_x, k_z) \notin A\), we can set the gradient of \(H\) with
respect to $\hat{O}(k_z, k_z)$, $(k_z, k_z) \notin A$ equal to zero and use the resulting equation as the "minimum cross entropy constraint":

$$\frac{\partial H}{\partial \hat{O}(k_z, k_z)} = \frac{\partial}{\partial \hat{O}(k_z, k_z)} \left[ \int_{z_{min}}^{z_{max}} \int_{x_{min}}^{x_{max}} O(x, z) \log \frac{O(x, z)}{P(x, z)} dx \right] \frac{O(x, z)}{P(x, z)} dz dz$$

$$= \frac{1}{4\pi^2} \int_{z_{min}}^{z_{max}} \int_{x_{min}}^{x_{max}} \left[ 1 + \log \frac{O(x, z)}{P(x, z)} \right] e^{-j(k_z x + k_z z)} dx dz$$

$$= 0$$

for $(k_z, k_z) \notin A$

Since $1 + \log \frac{O(x, z)}{P(x, z)}$ is real, its inverse Fourier transform is an even function, so, we can replace $k_z$ by $-k_z$ and $k_z$ by $-k_z$ in equation (4.8), multiply equation (4.8) by $4\pi^2$, and denote the result by $q(k_z, k_z)$,

$$\frac{\partial H}{\partial \hat{O}(k_z, k_z)} = \int_{z_{min}}^{z_{max}} \int_{x_{min}}^{x_{max}} \left[ 1 + \log \frac{O(x, z)}{P(x, z)} \right] e^{-j(k_z x + k_z z)} dx dz$$

$$= F[1 + \log \frac{O(x, z)}{P(x, z)}]$$

$$= q(k_z, k_z)$$

$$= 0$$

for $(k_z, k_z) \notin A$

Equation (4.9) is now the minimum cross entropy constraint. It says that the object function with the minimum cross entropy is the one whose corresponding $q(k_z, k_z)$ equals zero for $(k_z, k_z) \notin A$. This is a space domain constraint. We also have constraint on the Fourier domain which says that the $\hat{O}(k_z, k_z)$ inside the support $A$ should be equal to the measured $\hat{O}(k_z, k_z)$. The optimization problem stated in equations (4.6)
and (4.7) now becomes the problem of searching for the \(O(x, z)\) with constraints on both the space domain and the Fourier domain. We perform this search by forcing the space domain constraint and the Fourier domain constraint alternately until both constraints are satisfied within a prespecified error range. A searching algorithm of this kind has been applied to the multidimensional maximum entropy power spectrum estimation (Lim and Malik, 1981). Our minimum cross entropy direct transform reconstruction algorithm (MCED) is listed below and a flowchart of this algorithm is shown in Figure 4-2.

**Step 1:**
\[
q(k_z, k_z) = F[1 + \log \frac{O(x, z)}{P(x, z)}]
\]

**Step 2:**
\[
q(k_z, k_z) = 0
\]
for \((k_z, k_z) \notin A\)

**Step 3:**
\[
O(x, z) = P(x, z)\exp\{F^{-1}[q(k_z, k_z)] - 1\}
\]

**Step 4:**
\[
\tilde{O}_{\text{estimated}}(k_z, k_z) = F[O(x, z)]
\]

**Step 5:**
\[
\tilde{O}_{\text{new}}(k_z, k_z) = \begin{cases} 
\tilde{O}_{\text{measured}}(k_z, k_z) & \text{for } (k_z, k_z) \in A \\
\tilde{O}_{\text{estimated}}(k_z, k_z) & \text{for } (k_z, k_z) \notin A 
\end{cases}
\]

**Step 6:**
\[
O(x, z) = F^{-1}[\tilde{O}_{\text{new}}(k_z, k_z)]
\]

go to step 1

Unlike the regular direct transform reconstruction algorithm which assigns zero to the \(\tilde{O}(k_z, k_z)\) outside the lenticular shaped regions along the \(k_z\) axis, MCED fills up the \((k_z, k_z)\) plane based on the minimum cross entropy principle and, theoretically, can improve the horizontal resolution.

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4.2.2 Minimum cross entropy backpropagation reconstruction algorithm (MCEB)

To invert the object function in the diffraction tomography, we can also use the backpropagation algorithm developed by Devaney (1982, 1984) and Wu and Toksöz (1987). The backpropagation reconstruction formula for the cross-borehole configuration is equation (2.31) in Chapter 2:

\[
O(x, z) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{k_s \gamma_g + k_y \gamma_s}{k_o^2} \tilde{U}(k_s, k_y) \exp[-j \gamma_g d_y] \exp[j((\gamma_g - \gamma_s)x + (k_s + k_y)z)] dk_s dk_g
\]

Unlike the direct transform reconstruction, equation (4.10) inverts for \(O(x, z)\) without the two-dimensional interpolation.

The minimum cross entropy estimation can also be applied to the backpropagation reconstruction algorithm. We formulate the minimum cross entropy backpropagation reconstruction as the following constrained optimization problem:

minimize

\[
H = \int_{z_{\text{min}}}^{z_{\text{max}}} \int_{x_{\text{min}}}^{x_{\text{max}}} O(x, z) \log \frac{O(x, z)}{P(x, z)} dx dz
\]

subject to

\[
U(l_s, l_g)_{\text{measured}} = F^{-1}[\tilde{U}(k_s, k_y)]
\]

for \((l_s, l_g) \in B, B : \text{support}\)
where

\[
\hat{U}(k_s, k_g) = \frac{k^2}{k_s \gamma_g + k_g \gamma_s} e^{i \gamma_g d_g} \int_{z_{\min}}^{Z_{\text{max}}} \int_{x_{\min}}^{X_{\text{max}}} O(x, z) \exp\{-j[(\gamma_g - \gamma_s)x + (k_g + k_s)z]\} dx dz.
\]  

(4.13)

Equation (4.13) is the inverse of equation (4.10) and can be called the propagation operator; it can be verified by substituting it into equation (4.10). The support \( B \) is the region in the scattered field domain where the scattered field \( U(l_s, l_g) \) is available from the cross-borehole measurement. The strategy of our minimum cross entropy estimation algorithm is to estimate \( U(l_s, l_g) \) outside \( B \) (extending the source line and the receiver line) such that the cross entropy \( H \) of the object function is minimized.

Since the cross entropy corresponding to the data inside \( B \) is fixed by the measured data, we design our algorithm to minimize the cross entropy \( H \) with respect to \( U(l_s, l_g) \) outside the support \( B \). In other words, the cross entropy reaches its minimum when

\[
\frac{\partial H}{\partial U(l_s, l_g)} = 0 \quad \text{for} \quad (l_s, l_g) \notin B.
\]

\[
\frac{\partial H}{\partial U(l_s, l_g)} = \frac{\partial H}{\partial \hat{U}(k_s, k_g)} \frac{\partial \hat{U}(k_s, k_g)}{\partial U(l_s, l_g)}
\]

\[
= e^{-i(k_s l_s + k_g l_g)} \int_{z_{\min}}^{Z_{\text{max}}} \int_{x_{\min}}^{X_{\text{max}}} \partial \hat{U}(k_s, k_g) \left[ O(x, z) \log \frac{O(x, z)}{P(x, z)} \right] dx dz
\]

\[
= e^{-i(k_s l_s + k_g l_g)} \int_{z_{\min}}^{Z_{\text{max}}} \int_{x_{\min}}^{X_{\text{max}}} \left[ 1 + \log \frac{O(x, z)}{P(x, z)} \right] \frac{k_s \gamma_g + k_g \gamma_s}{\pi^2 k^2} \exp\{j[(\gamma_g - \gamma_s)x + (k_g + k_s)z] - j \gamma_g d_g\} dx dz
\]

\[
= 0
\]

\[
\text{for} \quad (l_s, l_g) \notin B
\]
Let
\[ \Phi(x, z) = 1 + \log \frac{O(x, z)}{P(x, z)}, \] (4.15)
and
\[ \psi(k_s, k_g) = \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \frac{k_s \gamma_s + k_g \gamma_g}{\pi^2 k_0^2} e^{-j \gamma_0 d_0} \exp\{j[(\gamma_g - \gamma_s)x + (k_g + k_s)z]\} dx dz, \]
\[ \text{equation (4.14) becomes} \]
\[ \frac{\partial H}{\partial U(l_s, l_g)} = e^{-j(k_s l_s + k_g l_g)} \psi(k_s, k_g) \]
\[ = 0 \]
\[ \text{for } (l_s, l_g) \notin B \]
Since \( \int \int 0 dk_s dk_g = 0 \), the minimum cross entropy constraint can be written as:
\[ \eta(l_s, l_g) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \psi(k_s, k_g) e^{-j(k_s l_s + k_g l_g)} dk_s dk_g \]
\[ = F[\psi(k_s, k_g)] \]
\[ = 0 \]
\[ \text{for } (l_s, l_g) \notin B \]
Similar to the minimum cross entropy direct transform reconstruction algorithm, we implement the minimum cross entropy backpropagation reconstruction with an iterative algorithm alternating between the scattered field domain and the space domain. In the scattered field domain, we keep the \( U(l_s, l_g) \) inside \( B \) equal to the measured scattered field and update \( U(l_s, l_g) \) outside \( B \) by the estimated scattered field. In the space
domain, we update the current object function by forcing the minimum cross entropy constraint described by equation (4.18). This algorithm starts with the scattered field $U(l_s, l_g)$ with the $U(l_s, l_g)$ outside $B$ initially set to zero. The first two steps of this algorithm calculate the current object function $O(x,z)$:

\begin{align*}
\text{step 1:} & \quad \tilde{U}(k_s, k_g) = F[U(l_s, l_g)] \\
\text{step 2:} & \quad O(x,z) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \frac{k_s \gamma_g + k_g \gamma_s}{k_g^2} \tilde{U}(k_s, k_g) e^{-i\gamma_0 d_g} \right| \\
& \quad \exp\{j[(\gamma_g - \gamma_s)x + (k_g + k_s)z]\} \, dk_s dk_g
\end{align*}

After step 2, we apply the minimum cross entropy constraint by calculating $\eta(l_s, l_g)$ from $O(x,z)$ with equations (4.15), (4.16), and (4.18) and forcing $\eta(l_s, l_g) = 0$ for $(l_s, l_g) \notin B$.

\begin{align*}
\text{step 3:} & \quad \Phi(x,z) = 1 + \log \frac{O(x,z)}{P(x,z)} \\
\text{step 4:} & \quad \psi(k_s, k_g) = \int_{z_{\min}}^{z_{\max}} \int_{x_{\min}}^{x_{\max}} \Phi(x,z) \left| \frac{k_s \gamma_g + k_g \gamma_s}{\pi^2 k_g^2} e^{-i\gamma_0 d_g} \right| \\
& \quad \exp\{j[(\gamma_g - \gamma_s)x + (k_g + k_s)z]\} \, dx \, dz \\
\text{step 5:} & \quad \eta(l_s, l_g) = F[\psi(k_s, k_g)] \\
\text{step 6:} & \quad \eta(l_s, l_g) = 0
\end{align*}

for $(l_s, l_g) \notin B$

To close the iteration loop, we need the following equation to invert equation (4.16) (step 4):

\begin{align*}
\Phi(x,z) &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \frac{k_g^2}{k_s \gamma_g + k_g \gamma_s} e^{i\gamma_0 d_g} \psi(k_s, k_g) \right| e^{-j[(\gamma_g - \gamma_s)x + (k_g + k_s)z]} \, dk_s dk_g \\
& \quad + \exp\{-j[(\gamma_g - \gamma_s)x + (k_g + k_s)z]\} \, dk_s dk_g \tag{4.19}
\end{align*}
Equation (4.19) can be proved by substituting it into equation (4.16). Using equations (4.13), (4.19), and Fourier transform, the truncated $\eta(l_s, l_g)$ can be mapped back to the scattered field domain by the following steps:

step 7:  
$$\psi(k_s, k_g) = F^{-1}[\eta(l_s, l_g)]$$

step 8:  
$$\Phi(x, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{k^2}{|k_s \gamma + k_g \gamma_s|} e^{j\gamma k} \psi(k_s, k_g)$$
$$\exp\left\{-j(\gamma_s - \gamma) x + (k_g + k_s) z\right\} dk_s dk_g$$

step 9:  
$$O(x, z) = P(x, z) \exp\{\Phi(x, z) - 1\}$$

step 10:  
$$\tilde{U}(k_s, k_g) = \frac{k^2}{|k_s \gamma + k_g \gamma_s|} e^{j\gamma k} \int_{z_{min}}^{z_{max}} \int_{x_{min}}^{x_{max}} O(x, z)$$
$$\exp\left\{-j(\gamma_s - \gamma) x + (k_g + k_s) z\right\} dx dz$$

step 11:  
$$U_{estimated}(l_s, l_g) = F^{-1}[\tilde{U}(k_s, k_g)]$$

The last step (step 12) of this algorithm is to replace the previous scattered field $U(l_s, l_g)$ by $U_{estimated}(l_s, l_g)$ calculated at step 11 for $(l_s, l_g) \notin B$ and the measured scattered for $(l_s, l_g) \in B$. Once $U(l_s, l_g)$ is updated, the algorithm starts again from step 1.

step 12:  
$$U_{new}(l_s, l_g) = \begin{cases} 
U_{estimated}(l_s, l_g) & \text{for } (l_s, l_g) \notin B \\
U_{measured}(l_s, l_g) & \text{for } (l_s, l_g) \in B 
\end{cases}$$

A flowchart of this algorithm is shown in Figure 4-3. The basic structure of this algorithm is very similar to the minimum cross entropy direct transform reconstruction algorithm. It alternately forces the space domain constraint (step 6) and the scattered field domain constraint (step 12) until both constraints are satisfied within a prespecified error range.
The minimum cross entropy backpropagation reconstruction algorithm estimates the scattered field $U(l_s, l_g)$ outside the support $B$ based on the minimum cross entropy principle. This extrapolation in the $(l_s, l_g)$ plane, as shown in Figure 4-4(a), is equivalent to adding imaginary sources and imaginary receivers in the space domain, as shown in Figure 4-4(b). However, if we examine the effect of the MCEB algorithm on the $(k_s, k_z)$ plane, as shown in Figure 4-4(c), we find that when those imaginary sources and receivers are added and the view angle increases from $\alpha$ to $\beta$, spectral coverage on the $(k_s, k_z)$ plane only expands from two smaller lenticular regions to two bigger lenticular regions along the vertical wavenumber axis; not much information is added in the vicinity of the horizontal wavenumber axis and, therefore, we can not expect the MCEB algorithm to make a major improvement on the horizontal resolution.

4.3 Applications to Test Data

We test the minimum cross entropy direct transform reconstruction algorithm (MCED) and the minimum cross entropy backpropagation reconstruction algorithm (MCEB) with numerical and ultrasonic laboratory experiments. The microcomputer based ultrasonic imaging system used for these experiments is described in Chapter 3. Chapter 3 also describes how the scattered field is measured by the dual-experiment method.
4.3.1 Experiment 1

This experiment simulates the cross-borehole tomography with 32 sources in one borehole and 32 receivers in another. The object consists of three squares with velocity 1.6 km/sec situated in a constant background medium with velocity 1.5 km/sec. Each side of the square is about one wavelength long. The geometry of the sources, receivers, and the object is shown in Figure 4-5(a). We calculate the scattered field along the receiver borehole by equation (4.3) with the Born approximation. From the scattered field, $U(r_s, r)$, we reconstruct the object function by both the regular direct transform reconstruction algorithm and the minimum cross entropy direct transform reconstruction algorithm. Two-dimensional interpolation is done by smooth surface fitting subroutine (IQHSCV) in the IMSL library. Figure 4-5(b) shows the result of the regular direct transform reconstruction, Figure 4-5(c) shows the result of the minimum cross entropy direct transform reconstruction after 100 iterations. In this test, we assume that no a priori distribution about the object function is available and therefore $P(x, z)$ is a constant. The result shown in Figure 4-5 is consistent with our expectation: the horizontal resolution is improved.

4.3.2 Experiment 2

This experiment also simulates the cross-borehole tomography but uses a complex object (crossing lines) and 64 sources and 64 receivers. The object function is shown in Figure 4-6(a). The velocity of the object function and the background medium are 1.6
km/sec and 1.5 km/sec respectively. A priori distribution $P(z, z)$ is still a constant. Figure 4-6(b) is the result of the regular direct transform reconstruction, Figure 4-6(c) is the result of the minimum cross entropy direct transform reconstruction after 100 iterations. Comparing Figures 4-6 (b) and (c), we find that the minimum cross entropy reconstruction algorithm enhanced the central portion of the object function and also reduced some artifacts. The horizontal resolution was not much improved. Notice that the object function used in this experiment is a relatively strong scatterer compared with the object function used in Experiment 1. Also, this object function is not of finite extent. Both problems may account for the poor reconstruction in this experiment.

4.3.3 Experiment 3

In this experiment, the minimum cross entropy direct transform reconstruction algorithm is tested by the data collected in the ultrasonic laboratory experiment simulating cross-borehole tomography described in Chapter 3. The geometry of this experiment is shown in Figure 3-2(a). The regular direct transform reconstruction at 50 KHz is shown in Figure 4-7(a), the minimum cross entropy direct transform reconstruction after 100 iterations is shown in Figure 4-7(b). Figure 4-7 indicates that, by using the minimum cross entropy method, the resolving power of the direct transform reconstruction algorithm is improved. A more quantitative evaluation of the resolution of the two images shown in this experiment is given in Chapter 6.
4.3.4 Experiment 4

This is an ultrasonic laboratory experiment with a cross-borehole configuration as shown in Figure 4-8. The source hydrophone is activated at 32 equally spaced positions along a source line, simulating 32 sources in one borehole. The receiver hydrophone records waveforms at 32 equally spaced positions along a receiver line, simulating 32 receivers in another borehole. The object consists of four glass rods with P wave velocity 5.64 km/sec. The background medium is water with velocity 1.5 km/sec.

From the measured scattered field, we reconstruct the object using both the regular backpropagation reconstruction algorithm (equation (4.10)) and the minimum cross entropy backpropagation reconstruction algorithm. The a priori distribution \( P(x, z) \) is a constant in this experiment. The result of the regular backpropagation reconstruction is shown in Figure 4-9(a), the result of the minimum cross entropy backpropagation reconstruction after 70 iterations is shown in Figure 4-9(b). The artifacts on the left hand side of Figure 4-9(a) are reduced in Figure 4-9(b) by using the minimum cross entropy backpropagation reconstruction algorithm. The horizontal resolution is also improved by the minimum cross entropy backpropagation reconstruction algorithm, but not as much as the improvement obtained in Experiment 1 and 3.

There are two reasons for this small horizontal resolution improvement: (1) As indicated by Figure 4-4(c), the minimum cross entropy backpropagation method only expands the spectral coverage from two smaller lenticular regions along the \( k_x \) axis to two bigger lenticular regions along the \( k_z \) axis. The spectral data along the \( k_z \) axis,
which is important for the horizontal resolution, is still unavailable. (2) The noise in the ultrasonic laboratory data may also account for the insignificant horizontal resolution improvement. The noise sensitivity of the minimum cross entropy methods will be discussed in Chapter 6.

4.3.5 Experiment 5

The only difference between this experiment and Experiment 4 is that 21 glass rods are used as the object. The top view of the object is shown in Figure 4-10(a). Figure 4-10(b) shows the result of the regular backpropagation reconstruction, Figure 4-10(c) shows the result of the minimum cross entropy backpropagation reconstruction after 60 iterations. Comparing Figure 4-10(b) and (c), we find that the minimum cross entropy reconstruction algorithm removed most of the artifacts that appeared in Figure 4-10(b). The horizontal resolution, however, did not change much by using the minimum cross entropy estimation, which can be explained by the two reasons given in the previous section.

4.3.6 Experiment 6

In this experiment, the ultrasonic laboratory data used in Experiment 3 are inverted by the minimum cross entropy backpropagation reconstruction algorithm. Figure 4-11(a) shows the regular backpropagation reconstruction at 50 KHz, Figure 4-11(b) shows the minimum cross entropy backpropagation reconstruction after 60 iterations, also at 50
KHz. Figure 4-11(b) indicates that the artifacts in the lower portion of Figure 4-11(a) are reduced by using the minimum cross entropy method. Similar to Experiment 4 and 5, the horizontal resolution is slightly improved by applying the minimum cross entropy estimation. This is explained by the two reasons given at the end of Section 4.3.4: lack of spectral information along the \( k_x \) axis, and noise in the data.

4.4 Conclusions

The minimum cross entropy estimation algorithm can be combined with the direct transform reconstruction algorithm and the backpropagation reconstruction algorithm to alleviate the limited view angle problem in diffraction tomography. The minimum cross entropy backpropagation reconstruction algorithm (MCEB) is equivalent to extending the coverage of the source array and the receiver array and therefore it is a finite aperture compensation algorithm. One limitation of this algorithm is that for the cross-borehole tomography problem, even if the source array and the receiver array are extended to infinity, the spectral coverage in the \((k_x, k_z)\) plane is still incomplete - a dumbbell shaped region along the vertical wavenumber axis and therefore can not improve the horizontal resolution much. The minimum cross entropy direct transform reconstruction algorithm (MCED), on the other hand, can fill up the \((k_x, k_z)\) plane. When the two-dimensional interpolation error is small and the Born approximation is valid, the horizontal resolution of the cross-borehole tomography can be significantly improved. Due to the presence of noise in the data, the resolution improvement ob-
tained by using the minimum cross entropy estimation may be reduced. The noise sensitivity of these two methods will be discussed in Chapter 6.

The other advantage of using these two minimum cross entropy image reconstruction algorithms is that they can reduce artifacts without losing resolution. This effect comes directly from the thermodynamic interpretation of the maximum entropy principle. As explained in Section 4.1, the minimum cross entropy estimation degenerates to the maximum entropy estimation when a priori distribution is a constant. Under this condition, the thermodynamic interpretation of the maximum entropy principle is to choose the estimate with the largest randomness, or, the most structureless estimate. The MCED/MCEB reconstruction algorithms achieve the most random reconstruction by introducing the least amount of orderliness except for the orderliness that has some contribution in the measured data. The orderliness that has contribution in the measured data is the image of the true object. This orderliness is preserved by the MCED/MCEB reconstruction algorithm. The orderliness that appears in the reconstruction but actually does not exist is called the artifact. Unlike true objects, artifacts have no contribution in the measured data (simply because they do not exist). Artifacts always appear in reconstructions, but can be reduced by the MCED/MCEB reconstruction algorithm because the MCED/MCEB reconstruction algorithm acquires more randomness by breaking down the orderliness of the artifacts.

It has been observed by other workers that the maximum entropy method works well for restoring objects consisting of isolated impulses in a uniform background, such
as astronomical images (Wernecke and D'Addario, 1977). Frieden (1983) explained this successful application of the maximum entropy estimation by proving that the maximum entropy estimation is the maximum likelihood estimation biased toward an uniformly grey object. In our experiments, the removal of the artifacts in the background (therefore approaching an uniformly grey object) and the enhancement of the glass rods can also be explained by Frieden’s argument.
4.5 REFERENCES


### Table 4-1

<table>
<thead>
<tr>
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<th>Reconstruction Algorithm</th>
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<td>Direct transform method</td>
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<td>Filtered backprojection method</td>
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<td>Type 8</td>
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<td>Estimation based on consistency principle</td>
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Figure 4-1: The limited view angle problem of the cross-borehole tomography. (a) Finite aperture source array and finite aperture receiver array in a cross-borehole configuration. (b) The spectrum that can be measured by the cross-borehole configuration. The heavily shaded regions represent the spectrum that can be measured by the finite aperture source array and receiver array, the lightly shaded regions represent the spectrum that can be measured when the source array and receiver array are extended to infinity.
Figure 4-2: Flowchart of the minimum cross entropy direct transform reconstruction algorithm.

\[ \tilde{O}_{\text{new}}(k_x, k_z) = \begin{cases} 
\tilde{O}_{\text{measured}}(k_x, k_z) & \text{for } (k_x, k_z) \in A \\
\tilde{O}_{\text{estimated}}(k_x, k_z) & \text{for } (k_x, k_z) \notin A 
\end{cases} \]

\[ O(x, z) = F^{-1}[\tilde{O}_{\text{new}}(k_x, k_z)] \]

\[ \tilde{O}_{\text{estimated}}(k_x, k_z) = F[O(x, z)] \]

\[ q(k_x, k_z) = F[1 + \log \frac{O(x, z)}{P(x, z)}] \]

\[ O(x, z) = P(x, z) \exp\{F^{-1}[q(k_x, k_z)] - 1\} \]

\[ q(k_x, k_z) = 0 \]

for \((k_x, k_z) \notin A\)
Figure 4-3: Flowchart of the minimum cross entropy backpropagation reconstruction algorithm.
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Chapter 5

ITERATIVE

MULTI-FREQUENCY

DIFFRACTION TOMOGRAPHY

5.1 Introduction

Possible ways to get around the limited view angle problem include maximum entropy estimation, band-limited spectrum extrapolation, deconvolution, and multi-frequency reconstruction (Rangayyan et al., 1985). Chapter 4 uses the maximum entropy estimation. In this chapter, we use the band-limited spectrum extrapolation and the multi-frequency reconstruction. Spectrum extrapolation is basically using the spectral data inside the support (measured data) and a priori information about the object to extrapolate the spectrum outside the support. Existing spectrum extrapolation algo-
rithms can be classified as the minimum norm least squares (MNLS) extrapolation (Jain and Ranganath, 1981) or the singular value decomposition (SVD) extrapolation (Sullivan and Liu, 1984). The algorithm proposed by Papoulis (1975) is the first MNLS type spectrum extrapolation algorithm for continuous signals. Sabri and Steenaart (1978) and Cadzow (1979) modified Papoulis' algorithm by replacing the recursive algorithm with a nonrecursive extrapolation operator. Kolba and Parks (1983) proposed the "band-limited time-concentrated" extrapolation algorithm where the time (or space) domain interval over which a good extrapolation is expected is incorporated into the formulation of the algorithm and is, therefore, an optimal extrapolation algorithm if the object is actually inside the space domain interval (finite extent object). Mefoff et al. (1983) developed a more general iterative estimation algorithm based on a "consistency principle" and showed that the MNLS type spectrum extrapolation algorithm is a special case of their algorithm. The MNLS spectrum extrapolation algorithm has been applied to the limited view angle tomography problem by Lent and Tuy (1981), Sato et al. (1981), Tam and Perez-Mendez (1981), and Tam (1982).

The singular value decomposition spectrum extrapolation algorithm does a Moore-Penrose inversion of the spectrum extrapolation via singular value decomposition. The advantage of this algorithm is that SVD allows more control over the effects of the ill conditioning of the spectrum extrapolation problem. Tsai and O'Connor (1984) presented the ill conditioned behavior of the spectrum extrapolation problem and made error analysis which determines how many singular values should be retained to reach
a balance between the truncation error and the numerical instability. Sullivan and Liu (1984) made similar error analysis and presented a numerical example of a two-dimensional spectrum extrapolation problem. The SVD spectrum extrapolation algorithm has been applied to the limited view angle tomography problem by Saito and Chiba (1985) and Mammone and Rothacker (1987). Their numerical examples indicate that this method is less sensitive to noise than the MNLS method. In this thesis, we only use the MNLS spectrum extrapolation method; the SVD spectrum extrapolation method is not used.

The forward problem of the seismic diffraction tomography can be written as:

$$G\mathbf{\tilde{O}} = \mathbf{\tilde{z}},$$

(5.1)

where $\mathbf{\tilde{O}}$ is the object function, $\mathbf{\tilde{z}}$ is the data vector, and $G$ is an operator that generates data from the object function, such as probing the object function with a seismic wave. In this chapter, the operator $G$ is called the data kernel. In the limited view angle tomography problem, we use the generalized projection slice theorem to calculate a portion of the Fourier transform of the object function, then find the object function from its incomplete spectrum.

The conventional method of applying the MNLS spectrum extrapolation algorithm to the limited view angle tomography problem is to extrapolate the available spectrum data to the blind area by alternately forcing the Fourier domain constraint and the space domain constraint. In Fourier domain, it keeps the spectrum inside the support equal to the measured spectrum and updates the spectrum outside the support by the
Fourier transform of the current object function. In space domain, it truncates the current object function by a window defined by a priori knowledge about the extent of the object. Convergence of this type of algorithm was proved by Youla (1978), Tom et al. (1981), and Schafer et al. (1981). The shortcomings of the conventional usage of the MNLS spectrum extrapolation algorithm is that it does not exploit the information about the object function obtained at each iteration step in determining the window used in the space domain truncation, and does not apply the a priori knowledge about the data kernel. In this chapter, by exploiting the resolution matrix of the spectrum extrapolation problem, we propose two windows that incorporate a priori information about both the object function and the data kernel.

Multi-frequency reconstruction can be applied to both the interpolation – direct transform reconstruction algorithm and the filtered backpropagation reconstruction algorithm. When the interpolation – direct transform method is used, the scattered field at different frequencies is first mapped into the Fourier domain by the generalized projection – slice theorem. A two-dimensional Fourier transform of the object function is then obtained by interpolating these multi-frequency data points. Once we have the two-dimensional Fourier transform of the object function, the object function can be calculated by an inverse Fourier transform. This method was used by Keune and Greenleaf (1982) for the multi-frequency diffraction tomography reconstruction. Their paper, however, emphasized the limited view angle problem and did not contrast their multi-frequency reconstruction with the single-frequency reconstruction. Since back-
propagation is a linear operator, when the multi-frequency approach is applied to the filtered backpropagation reconstruction algorithm, one can simply superpose the single-frequency scattered field data and then backpropagate the multi-frequency scattered field data to obtain the multi-frequency reconstruction. This method was described by Slaney and Kak (1985), but they did not show examples.

In this chapter, we combine the interpolation – direct transform multi-frequency reconstruction algorithm with our modified MNLS spectrum extrapolation algorithm to obtain an iterative multi-frequency diffraction tomography algorithm. This algorithm is tested by numerical and ultrasonic laboratory experiments.

5.2 THEORY

In this section, we first write the MNLS spectrum extrapolation algorithm in terms of the consistency principle estimation (Medoff et al., 1983). Then, by introducing the concept of the resolution matrix, we find the properties that an ideal space domain window used in the MNLS spectrum extrapolation algorithm should have. We then describe the noniterative multi-frequency diffraction tomography reconstruction algorithm and combine that algorithm with the MNLS spectrum extrapolation algorithm to build an iterative multi-frequency diffraction tomography reconstruction algorithm. Based on the properties that an ideal space domain window should have, we then propose two space domain windows that partially satisfy those requirements: the "adjustable finite extent window" and the "duration time ellipse window". Finally, we
present our iterative multi-frequency diffraction tomography reconstruction algorithm by a flowchart.

5.2.1 Consistency principle estimation

We follow the derivation in Medoff et al. (1983) to explain the consistency principle estimation. Let $\bar{z}^{true}$ denote the data vector, such as the travel time or attenuation profile. $\bar{z}^{true}$ can be partitioned into two vectors $\bar{z}^{lost}$ and $\bar{y}^{obs}$ denoting the missing data and the measured data respectively.

$$\bar{z}^{true} = \begin{bmatrix} \bar{z}^{lost} \\ \bar{y}^{obs} \end{bmatrix}$$

(5.2)

Let $C_z$ denote the data constraint operator, such as forcing the travel time profile within a certain range; $C_I$ denote the image constraint operator, such as forcing positivity or a finite extent window; $S$ denote the forward problem operator; and $R$ denote the inverse problem operator. Then, if $\bar{z}^{est}$ is the "consistent" estimate of the missing data $\bar{z}^{lost}$, it should satisfy the data constraint operator, in other words, applying the data constraint operator $C_z$ to $\bar{z}^{est}$ does not change $\bar{z}^{est}$ as long as $\bar{z}^{est}$ is a consistent estimate of $\bar{z}^{true}$.

$$\begin{bmatrix} \bar{z}^{est} \\ \bar{y}^{obs} \end{bmatrix} = C_z \begin{bmatrix} \bar{z}^{est} \\ \bar{y}^{obs} \end{bmatrix}$$

(5.3)

Similarly, an image expressed as an inverse operator applied on a data vector,

$$R \begin{bmatrix} \bar{z}^{est} \\ \bar{y}^{obs} \end{bmatrix}$$
is considered a consistent estimate of the true image if it satisfies the image constraint, or, equivalently, applying the image constraint operator $C^I$ on an image does not change it as long as that image is a consistent estimate of the true image:

$$R \begin{bmatrix} \bar{x}^{est} \\ \bar{y}^{obs} \end{bmatrix} = C^I R \begin{bmatrix} \bar{x}^{est} \\ \bar{y}^{obs} \end{bmatrix} \quad (5.4)$$

Also, a consistent estimate of the lost data should satisfy the forward and inverse relationship. This means if $\bar{x}^{est}$ is a consistent estimate of $\bar{x}^{lost}$, when it is combined with $\bar{y}^{obs}$, an inverse operator $R$ followed by a forward operator $S$ should not change $\bar{x}^{est}$:

$$SR \begin{bmatrix} \bar{x}^{est} \\ \bar{y}^{obs} \end{bmatrix} = \begin{bmatrix} \bar{x}^{est} \\ \bar{y}^{obs} \end{bmatrix} \quad (5.5)$$

Combining equations (5.3), (5.4), and (5.5), we have the following relationship for a consistent estimate $\bar{x}^{est}$:

$$\begin{bmatrix} \bar{x}^{est} \\
\bar{y}^{obs} \end{bmatrix} = SC^I R C^Z \begin{bmatrix} \bar{x}^{est} \\
\bar{y}^{obs} \end{bmatrix} \quad (5.6)$$

Rewriting equation (5.6) in a recurrence form, the consistency principle estimation algorithm is obtained:

$$\begin{bmatrix} \bar{x}^{est}_{i+1} \\
\bar{y}^{obs}_{i+1} \end{bmatrix} = SC^I R C^Z \begin{bmatrix} \bar{x}^{est}_i \\
\bar{y}^{obs}_i \end{bmatrix} \quad (5.7)$$

As pointed out by Medoff et al. (1983), equation (5.7) degenerates to the conventional MNLS spectrum extrapolation algorithm for a finite extent (in time or space domain)
signal if we replace \( S \) and \( R \) by forward and inverse Fourier transform, \( C^f \) by a finite extent window (in time or space domain), and \( C^z \) by an operator that forces the spectrum inside the support to be equal to the measured spectrum \( \bar{y}^{obs} \) and update the previous estimate by the current estimate \( \bar{z}_i^{est} \) outside the support. Another special case of equation (5.7) is the limited angle tomography reconstruction algorithm proposed by Tam and Perez-Mendez (1981), where \( S \) is the forward Radon transform, \( R \) is the inverse Radon transform, and \( C^f \) and \( C^z \) are the same as the \( C^f \) and \( C^z \) used in the two-dimensional MNLS spectrum extrapolation algorithm.

### 5.2.2 Resolution analysis

In this section, we study the resolution of the conventional MNLS spectrum extrapolation algorithm. We first segment equation (5.7) into two parts: a forward problem and an inverse problem:

**forward problem**

\[
\begin{bmatrix}
\bar{z}^{est} \\
\bar{y}^{est}
\end{bmatrix}
= FC^f \bar{O}^{est}
\]  
(5.8)

**inverse problem**

\[
\bar{O}^{est} = F^{-1} C^z
\begin{bmatrix}
\bar{z}^{est} \\
\bar{y}^{est}
\end{bmatrix}
\]  
(5.9)

\[
C^z
\begin{bmatrix}
\bar{z}^{est} \\
\bar{y}^{est}
\end{bmatrix}
= \begin{bmatrix}
\bar{z}^{est} \\
\bar{y}^{obs}
\end{bmatrix}
\]
where $F$ and $F^{-1}$ are forward and inverse Fourier transforms, $\tilde{O}_{est}$ is the estimate of the object function, $\tilde{g}_{est}$ is the estimated spectrum inside the support, and $\tilde{g}_{obs}$ is the measured spectrum inside the support. Equation (5.9) can also be written as:

$$\tilde{O}_{est} = F^{-1} \begin{bmatrix} \tilde{x}_{est} \\ \tilde{g}_{obs} \end{bmatrix}$$  \hspace{1cm} (5.10)$$

Following the definition in Menke (1984), the model resolution matrix, $M$, is a measure of how close the estimated model $\tilde{O}_{est}$ (object function in this study) approaches the true model $\tilde{O}_{true}$:

$$\tilde{O}_{est} = M\tilde{O}_{true}$$  \hspace{1cm} (5.11)$$

The true model $\tilde{O}_{true}$ is related to the observed data and missing data by the data kernel $G$, which is the forward Fourier transform in the MNLS spectrum extrapolation method:

$$FG^{true} = \begin{bmatrix} \tilde{x}_{lost} \\ \tilde{g}_{obs} \end{bmatrix}$$  \hspace{1cm} (5.12)$$

Take inverse Fourier transform on both sides of equation (5.12):

$$\tilde{O}_{true} = F^{-1} \begin{bmatrix} \tilde{x}_{lost} \\ \tilde{g}_{obs} \end{bmatrix}$$  \hspace{1cm} (5.13)$$

If we let $C^I$ be an operator that rejects the vectors in the null subspace ($\tilde{O}_o$) of the model space,

$$C^I\tilde{O} = C^I(\tilde{O}_p + \tilde{O}_o)$$

$$= \tilde{O}_p.$$
Then, apply such $C'$ to equations (5.10) and (5.13),

$$
\begin{bmatrix}
0 \\
\vec{y}_{\text{obs}}
\end{bmatrix}
= C' \hat{\Omega}_{\text{true}} = \hat{\Omega}_{\text{true}}^{\text{true}}
$$

(5.15)

$$
\begin{bmatrix}
0 \\
\vec{y}_{\text{obs}}
\end{bmatrix}
= C' \hat{\Omega}_{\text{est}} = \hat{\Omega}_{\text{est}}^{\text{est}}
$$

(5.16)

Comparing equations (5.15) and (5.16), we conclude that although we cannot make the model resolution matrix equal to the identity matrix ($\hat{\Omega}_{\text{true}} = \hat{\Omega}_{\text{true}}^{\text{true}}$), we can make $\hat{\Omega}_{\text{est}} = \hat{\Omega}_{\text{est}}^{\text{true}}$ by choosing $C'$ as an operator that rejects the vectors in the null subspace of the model space. This conclusion will be exploited in determining one of the space domain windows that will be used in the iterative multi-frequency diffraction tomography reconstruction algorithm — the "duration time ellipse window".

Next, consider the data resolution matrix, $D$, defined as a measure of how close the predicted data $\vec{z}_{\text{est}}$ approach the observed data $\vec{z}_{\text{obs}}$:

$$
\vec{z}_{\text{est}} = D \vec{z}_{\text{obs}}
$$

(5.17)

From equations (5.8) and (5.12):

$$
\begin{align*}
\vec{z}_{\text{est}} &= FC' \hat{\Omega}_{\text{est}} \\
\vec{z}_{\text{obs}} &= F \hat{\Omega}_{\text{true}}.
\end{align*}
$$

We find that, to make $\vec{z}_{\text{est}} = \vec{z}_{\text{obs}}$, it is required that:

$$
\hat{\Omega}_{\text{true}} = C' \hat{\Omega}_{\text{est}}
$$

(5.18)
The image constraint $C'$ that satisfies equation (5.18) is the ideal space domain window based on the data resolution matrix consideration. Equation (5.18) will be used later to determine another space domain window, the "adjustable finite extent window", which is useful for the iterative multi-frequency diffraction tomography reconstruction algorithm.

5.2.3 Iterative multi-frequency diffraction tomography

In this section, we use the interpolation - direct transform method for multi-frequency reconstruction. We first define the object function $O(x, z)$ as

$$O(x, z) = 1 - \frac{C^2_o}{C^2(x, z)},$$

(5.19)

where $C(x, z)$ is the velocity distribution of the object function and $C_o$ is the velocity of the constant background medium. Let $U(r_s, r_g, \omega)$ denote the multi-frequency scattered field measured at position $r_g$ when the point source is at position $r_s$. We first take Fourier transform of $U(r_s, r_g, \omega)$ along the source line and the receiver line, obtaining $\tilde{U}(k_s, k_g, \omega)$, then, using the generalized projection - slice theorem derived in Chapter 2, we can calculate the Fourier transform of the object function, $\tilde{O}[\frac{\omega}{C_o}(\hat{g} + \hat{s})]$, on the loci $\frac{\omega}{C_o}(\hat{g} + \hat{s})$:

$$\tilde{O}[\frac{\omega}{C_o}(\hat{g} + \hat{s})] = \frac{4\gamma_s\gamma_{\omega}}{(\frac{\omega}{C_o})^2} \tilde{U}(k_s, k_g, \omega) \exp[-j(\gamma_s d_g + \gamma_{\omega} d_s)]$$

$$\tilde{U}(k_s, k_g, \omega) = F[U(r_s, r_g, \omega)]$$

(5.20)

$$\gamma_s = \sqrt{(\frac{\omega}{C_o})^2 - k_s^2}$$
where \( k_s \) and \( k_g \) are wavenumbers along the source line and the receiver line, \( \hat{s} \) and \( \hat{g} \) are unit vectors of plane waves to the source and receiver respectively, and \( \omega \) is the frequency of the wavefield. We use an example shown in Figure 5-1 to illustrate the benefit of using the multi-frequency reconstruction method. Figure 5-1(a) shows the geometry of the source and the receivers in this example. In this cross-borehole example, we use one source in one borehole and several receivers in another borehole. Figure 5-1(b) shows the loci of \( \frac{\omega}{C_o} (\hat{g} + \hat{s}) \) on \((k_s, k_g)\) plane at three different frequencies, \( \omega_1, \omega_2, \omega_3 \). If we only use single frequency data, \( \omega_2 \), the information coverage is only one arc as shown in Figure 5-1(c), but if we use multi-frequency data with frequency range \( \omega_1 \leq \omega \leq \omega_3 \), the information coverage is the shaded area shown in Figure 5-1(c). It is clear that, when we have only one source, single frequency data is not enough for image reconstruction whereas multi-frequency data can be used to partially reconstruct the object function. Once we locate the loci of \( \frac{\omega}{C_o} (\hat{g} + \hat{s}) \) on the \((k_s, k_g)\) plane for all frequencies, the next step is to interpolate the \( \tilde{O}[\frac{\omega}{C_o} (\hat{g} + \hat{s})] \) value from the \( \frac{\omega}{C_o} (\hat{g} + \hat{s}) \) loci shown by the open circles in Figure 5-2 to a rectangular grid \((k_s, k_g)\) shown by the solid circles in Figure 5-2. We use a two-dimensional interpolation subroutine in the IMSL library for this interpolation. For \((k_s, k_g)\) outside the support, such as point \( C' \) in Figure 5-2, the noniterative direct transform method assigns zero to those points. After interpolation, the noniterative multi-frequency reconstruction method is completed by taking the inverse Fourier transform of \( \tilde{O}(k_s, k_g) \), and obtaining the object function.
For the iterative multi-frequency diffraction tomography, the first three steps of the
reconstruction algorithm are identical to the noniterative multi-frequency diffraction
tomography: taking Fourier transform of $U(\tau, \tau, \omega)$, calculating $\hat{O}[\omega(\hat{g} + \hat{s})]$ by
the generalized projection slice theorem, and interpolating $\hat{O}[\omega(\hat{g} + \hat{s})]$ to $\hat{O}(k_x, k_z)$.
Unlike the noniterative method which assigns zero to $\hat{O}(k_x, k_z)$ for $(k_x, k_z)$ outside the
support, the iterative multi-frequency reconstruction method uses the MNLS spectrum
extrapolation algorithm to estimate $\hat{O}(k_x, k_z)$ for $(k_x, k_z)$ outside the support.

5.2.4 Least squares spectrum extrapolation

One difficulty in applying the MNLS spectrum extrapolation algorithm to the limited
view angle tomography problem is that the image constraint $C^I$ is not well defined. The
conventional MNLS spectrum extrapolation method uses a priori information about the
extent of the object as $C^I$ (Sato et al., 1981; Tam and Perez-Mendez, 1981). By doing
this, information about the data kernel and new information about the object function
obtained at each iteration are not fully used. In this section, we propose using an
image constraint which is the overlap of the following three space domain windows: (1)
the conventional fixed finite extent window, determined by a priori information about
the object function, (2) the adjustable finite extent window, determined by the current
estimate of the object function, and (3) the duration time ellipse window, determined
by the data kernel.
The adjustable finite extent window is derived by exploiting the result from the data resolution matrix consideration, equation (5.18), where we find that the ideal image constraint should be an operator that eliminates the difference between the true object function and the estimated object function. This is a redundant statement in the image reconstruction problem since the true object function is unknown. However, if the true object function is a finite extent object that only has zero components outside the object function support $B$ ($B$ is also unknown), and if we can estimate $B$, then we can make $C^I$ an operator that force the $\hat{O}^{\text{est}}$ outside $B$ equal to zero. By doing this, the left hand side of equation (5.18) becomes

$$\hat{O}^{\text{true}} = \begin{bmatrix} 0 \\ \hat{O}^{\text{true}}_{\text{inside } B} \end{bmatrix},$$

(5.21)

and the right hand side of equation (5.18) becomes

$$C^I \hat{O}^{\text{est}} = C^I \begin{bmatrix} \hat{O}^{\text{est}}_{\text{outside } B} \\ \hat{O}^{\text{est}}_{\text{inside } B} \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{O}^{\text{est}}_{\text{inside } B} \end{bmatrix}.$$  

(5.22)

Comparing equations (5.21) and (5.22), we can see that by using such $C^I$, although we can not make $\hat{O}^{\text{true}}_{\text{inside } B} = \hat{O}^{\text{est}}_{\text{inside } B}$, we can at least make $\hat{O}^{\text{true}}_{\text{outside } B} = \hat{O}^{\text{est}}_{\text{outside } B} = 0$, and partially satisfy the ideal image constraint derived from the data resolution matrix consideration – equation (5.18). In fact, such $C^I$ is the finite extent window used in the conventional MNLS spectrum extrapolation algorithm. In the conventional usage of the MNLS spectrum extrapolation algorithm, $B$ (and therefore $C^I$) is determined solely from a priori information about the extent of the object function, and $C^I$ is fixed.
throughout all the iteration steps. In this thesis, equations (5.21) and (5.22) tell us that the better we can approximate $B$, the better we can reduce the difference between the left and right hand sides of equation (5.18). Therefore, we propose adjusting $B$ (therefore $C^f$) at each iteration step based on the current estimate of the extent of the object function. We call such image constraint $C^f$ the adjustable finite extent window.

The other space domain window we propose — the duration time ellipse window — is based on the model resolution matrix consideration. Consider a scattering experiment with point scatterers in a homogeneous background medium with one source and one receiver, as shown in Figure 5-3. Figure 5-3(a) shows the locations of the source, the receiver, and the scatterer. Figure 5-3(b) shows a scattered field waveform with duration time $T$. Let the travel time from the source to point scatterer $P$ be $t_1$ and the travel time from $P$ to the receiver be $t_2$, then, as long as $t_1 + t_2 \leq T$, the scattered field caused by point $P$ is contained in the recorded waveform. The scattered field waveform shown in Figure 5-3(b) is the sum of all the individual scattered fields contributed by each point scatterer inside the elliptical region defined by $t_1 + t_2 \leq T$. Contribution to the scattered field from any point outside this elliptical region arrives later than $T$, and therefore is not recorded. This elliptical region changes its shape when the experiment is not a single source — single receiver experiment. For example, for an experiment with one source and two receivers, the region that defines points without contributions to the record is the shaded region shown in Figure 5-3(c). To simplify, we call this kind of region, that separates points with or without contributions to the record, the
"duration time ellipse window" though it can take any arbitrary shape.

The duration time ellipse window can be used to separate some vectors in the null subspace of the model space. In the seismic diffraction tomography problem, the vectors in the null subspace of the model space can be classified into three categories:

1. Zero vector.

2. Vectors that have nonzero components only outside the duration time ellipse window.

3. Vectors that have nonzero components inside the duration time ellipse window, but their response to the data kernel cancel one another and the data vector generated is a zero vector.

The ideal space domain window should be an operator that rejects all the vectors in the null subspace of the model space. Separating null subspace vectors in Category 1 is trivial, vectors belonging to Category 2 can be separated by the duration time ellipse window, and vectors belonging to Category 3 can be identified by a SVD of the data kernel. In this thesis, the algorithm we propose only rejects the null subspace vectors belonging to Category 2. The effects of using SVD for separating Category 3 null subspace vectors are not included in this thesis.

In summary, the adjustable finite extent window is determined by the current knowledge about the extent of the object function, the duration time ellipse window is determined by the knowledge about the data kernel, both windows partially satisfy the
ideal space domain window requirements imposed by the resolution matrix analysis.

In our iterative multi-frequency diffraction tomography reconstruction algorithm, the image constraint \( C' \) we use is the overlap of these two windows and the conventional fixed finite extent window. Other image constraints such as positivity, lower bound, and upper bound may also be included, depending on the applications.

5.2.5 Algorithm

The algorithm we use in this chapter is illustrated by the flowchart shown in Figure 5-4. We first take the Fourier transform of the scattered field \( U(r, \varphi, \omega) \) at each frequency. Then, we use equation (5.20) to calculate the Fourier transform of the object function at each frequency \( \tilde{O}(\varphi, \omega) \). \( \tilde{O}(\varphi, \omega) \) is then interpolated to \( \tilde{O}_{\text{measured}}(k_z, k_z) \), the Fourier transform of the object function on a rectangular grid. \( \tilde{O}_{\text{measured}}(k_z, k_z) \) is available only inside the Fourier domain support \( A \). The \( \tilde{O}(k_z, k_z) \) outside \( A \) is estimated by the iterative MNLS spectrum extrapolation algorithm described by equation (5.7). This algorithm alternately forces the Fourier domain constraint and the space domain constraint. In the Fourier domain, we force the \( \tilde{O}(k_z, k_z) \) inside the support to be equal to the measured \( \tilde{O}(k_z, k_z) \), and the \( \tilde{O}(k_z, k_z) \) outside the support to be equal to the Fourier transform of the current object function. In the space domain, we truncate the current object function by the space domain window which is the overlap of the adjustable finite extent window and the duration time ellipse window. The truncated object function is Fourier transformed to calculate the estimate of \( \tilde{O}(k_z, k_z) \) outside the
support and is also used as the feedback for determining the finite extent window for the next iteration. Since the object function defined by equation (5.21) is real, the pass band of the adjustable finite extent window $W(x, z)$ can be determined by choosing a threshold value close to the noise level of the reconstruction and let $W(x, z) = 1$ for $O(x, z)$ above this threshold value and $W(x, z) = 0$ for $O(x, z)$ below this threshold value. Determined in this way, $W(x, z)$ is an approximate to the boundary of the true object function. Notice that, this kind of space domain window can only be used when the true object function is a finite extent object, as indicated by equation (5.21).

5.3 Experiments

5.3.1 Numerical experiment

The model used in the numerical experiment is shown in Figure 5-5(a). This experiment simulates a cross-borehole tomography experiment with 32 sources in one borehole and 32 receivers in another. The dimension of the simulated imaging area is $243.84\text{mm} \times 243.84\text{mm}$ and is divided into $32 \times 32$ pixels. The object function consists of three squares, each square consists of four pixels. The velocity of the constant background medium is 1.5 km/sec, the velocity of the object function is 1.7 km/sec. The scattered field is calculated by applying the Born approximation to the solution of the inhomogeneous acoustic wave equation:

$$U(r_2, r_1, \omega) \approx -\frac{\omega}{C_0} \int_V O(r_r)G(r_r, r_1, \omega)G(r, r_2, \omega)dr$$  \hspace{1cm} (5.23)
\[ G(r, r', \omega) = \frac{j}{4} H_0^{(1)}(\frac{\omega}{c_o} |r - r'|), \]  

(5.24)

where \( G \) is the Green's function for the background medium, \( H_0^{(1)} \) is the zero order Hankel's function of the first kind. We calculate \( U(r, r', \omega) \) at three different frequencies, 75 KHz, 80 KHz, and 85 KHz, with wavelength in the background medium equal to 20 mm, 18.75 mm, and 17.65 mm (or 2.62, 2.46, and 2.32 pixels). The Fourier transform of the object function, \( \tilde{O}(k_z, k_x) \), is obtained by the noniterative part of the algorithm shown in Figure 5-4. For this cross-borehole geometry, \( \tilde{O}(k_z, k_x) \) is available only inside two lenticular shaped regions along the \( k_x \) axis. In this experiment, we compare the noniterative single frequency reconstruction with the noniterative multi-frequency reconstruction. For the single frequency experiment, we only use the data at 80 KHz whereas for the noniterative multi-frequency experiment we use data at all the three frequencies. In both cases, we assign zero to \( \tilde{O}(k_z, k_x) \) outside the Fourier domain support. Figure 5-5(b) shows the single frequency reconstruction and Figure 5-5(c) shows the noniterative multi-frequency reconstruction. Both horizontal resolution and background noise level are significantly improved by using the multi-frequency method.

The iterative multi-frequency reconstruction method is also tested in this experiment. Figure 5-6(a) shows the initial space domain window used in this experiment. The pass band of this window is square in shape, situated in the center of the imaging area, the transition band is defined by a Gaussian distribution curve. After the first iteration, our knowledge about the boundary of the object function is improved and
the finite extent window shown in Figure 5-6(a) is replaced by the finite extent window shown in Figure 5-6(b), defined by a threshold value of 10% of the peak amplitude of the first reconstruction. At each iteration hereafter, the finite extent window is adjusted according to the current estimate of the object function. After 1000 iterations, no substantial changes on both the object function and the finite extent window is observed. Figure 5-6(c) shows the final finite extent window and Figure 5-5(d) shows the final reconstruction. Comparing Figure 5-5(c) with Figure 5-5(d), we find that the horizontal resolution and the background noise level are improved.

To compare the improvements obtained by using the noniterative multi-frequency method and the iterative multi-frequency method, we define a $32 \times 32$ matrix $X$ that relates the $32 \times 32$ true object function matrix and the $32 \times 32$ estimated object function matrix:

$$O^{\text{est}} = XO^{\text{true}}$$

Since we know the true object function in our experiments, matrix $X$ can be computed by

$$X = O^{\text{est}}O^{\text{true}^{-1}}$$

This is not truly a resolution matrix, but should be more diagonal-like the better the reconstruction. Figure 5-7(a) shows the matrix $X$ of the single frequency reconstruction, Figure 5-7(b) shows the matrix $X$ of the noniterative multi-frequency reconstruction, Figure 5-7(c) shows the matrix $X$ of the iterative multi-frequency reconstruction. The ideal $X$ is an identity matrix, and by using the noniterative multi-frequency method
and the iterative multi-frequency method, a stepwise reduction of the off-diagonal components of $X$ can be observed in Figure 5-7.

5.3.2 Ultrasonic laboratory experiment

The iterative multi-frequency diffraction tomography is also tested by a scale model ultrasonic experiment. This experiment was conducted by the ultrasonic imaging system described in Chapter 3. In this experiment, we simulate a cross-borehole tomography experiment in the field. The experiment setup is shown in Figure 5-8. The object is a semicircular cylinder made of silicon rubber RTV 3110. The P wave velocity and density of this material is 1.04 km/sec and 1.218 g/cc. The background medium is water with velocity 1.50 km/sec and density 1.0 g/cc. We assume there is no velocity variation along the axial direction of the rubber cylinder and apply our two-dimensional reconstruction algorithm to a three-dimensional imaging problem. The source hydrophone is activated at 32 equally spaced positions along the source line, simulating 32 sources in one borehole, the receiver hydrophone records waveforms at 32 equally spaced positions along the receiver line, simulating 32 receivers in another borehole. The imaging area is 243.84 mm x 243.84 mm in dimension and the radius and length of the rubber cylinder are 42 mm and 220 mm respectively. We use the dual experiment method described in Chapter 3 to measure the scattered field caused by the object. The frequency range we use starts at 25 KHz to 100 KHz with peak frequency around 54 KHz, corresponding to a wavelength of 26.3 mm in water. Due to the cross-borehole configuration, the
measured $\tilde{O}(k_x, k_z)$ only occupies two lenticular shaped regions in the $(k_x, k_z)$ plane. Figure 5-9(a) shows the single frequency reconstruction at 54 KHz, Figure 5-9(b) shows the noniterative multi-frequency reconstruction using data at 51 KHz, 54 KHz, and 57 KHz altogether. In both examples, we assign zero to $\tilde{O}(k_x, k_z)$ outside the support. It is observed in this experiment that the multi-frequency reconstruction not only delineates the shape of the object better than the single frequency reconstruction, it also reduces the background noise level. The windows used in the iterative multi-frequency reconstruction method are illustrated by Figure 5-10. Figure 5-10(a) shows the initial finite extent window which is determined by a priori information about the extent of the object. In this experiment, we use a two-dimensional Gaussian distribution surface as the initial finite extent window. Figure 5-10 (b) and (c) show the finite extent window after the first and the 1000th iteration. In this experiment, we calculate the adjustable finite extent window by a threshold value of 10 % of the peak amplitude of the current object function. The final reconstruction after 1000 iterations is shown in Figure 5-9(c). This reconstruction delineates the rubber cylinder even better than the noniterative multi-frequency reconstruction.

The matrices $X$ corresponding to the reconstructions shown in Figure 5-9 are also computed and shown in Figure 5-11. Figure 5-11(a) shows the matrix $X$ of the single frequency reconstruction, Figure 5-11(b) shows the matrix $X$ of the noniterative multi-frequency reconstruction, and Figure 5-11(c) shows the matrix $X$ of the iterative multi-frequency reconstruction. It is observed that the off-diagonal components of the $X$'s are
reduced step by step by using the noniterative multi-frequency reconstruction method and the iterative multi-frequency reconstruction method.

For the multi-frequency methods, it is reasonable to expect better reconstruction by using data covering broader frequency range. This expectation is tested by the data collected in the ultrasonic laboratory experiment simulating cross-borehole tomography described in Chapter 3. The object in this experiment is a gelatin cylinder and the background medium is water, other details of this experiment are described in Chapter 3. Figure 5-12(a) shows the single frequency direct transform reconstruction at 50 KHz, Figure 5-12(b) shows the noniterative multi-frequency direct transform reconstruction with frequency range 37 KHz — 50 KHz, and Figure 5-12(c) shows the noniterative multi-frequency direct transform reconstruction with frequency range 37 KHz — 67 KHz. The improvements obtained by using data covering broader frequency range can be observed by comparing Figure 5-12(b) and (c).

5.4 Conclusions

1. The limited view angle problem for the seismic diffraction tomography can be helped by the multi-frequency reconstruction method. Using this method, we observed improvements on both the resolution and the signal/noise ratio in our experiments.

2. The spectrum extrapolation method also helps the limited view angle problem. In this chapter, we modify the conventional minimum norm least squares (MNLS)
spectrum extrapolation algorithm by exploiting the resolution matrix of the spectrum extrapolation problem. We find that the fixed finite extent window used in the conventional MNLS spectrum extrapolation algorithm only remotely approximates the ideal space domain window, especially when a priori information about the object function is not enough. To fix this problem, we propose replacing the fixed finite extent window by an adjustable finite extent window which is a closer approximation to the ideal space domain window and is determined by the current estimate of the boundary of the object function. Notice that, this adjustable finite extent space domain window is applicable only when the object function is a finite extent object function. In the numerical and ultrasonic laboratory tests of the modified spectrum extrapolation algorithm, we find that this window approaches the ideal space domain window as the iteration proceeds.

3. We developed an iterative multi-frequency reconstruction algorithm by cascading the multi-frequency reconstruction method with the modified MNLS spectrum extrapolation algorithm. Our experimental results indicate that this algorithm can further improve the resolution and the signal/noise ratio of the noniterative multi-frequency reconstruction.
5.5 REFERENCES


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Tsai, M. J. and O'Connor, D. A., 1984, Experiments with extrapolation of band-limited signal:


Figure 5-1: Multi-frequency reconstruction. (a) A cross-borehole configuration with one source in one borehole and several receivers in another borehole. (b) The loci of $\frac{\omega}{C_0}(\hat{g} + \hat{s})$ in the $(k_x, k_z)$ plane with three frequencies, $\omega_1$, $\omega_2$, $\omega_3$. (c) For single frequency reconstruction (use $\omega_2$ only), available data only cover an arc in the $(k_z, k_x)$ plane, for multi-frequency reconstruction (use all the frequencies for $\omega_1 \leq \omega \leq \omega_3$), available data cover the shaded area in the $(k_z, k_x)$ plane.
Figure 5-2: Geometrical relationship between the $\frac{\omega}{C_p}(\hat{g} + \hat{s})$ loci (represented by open circles) and the Cartesian grid $(k_x, k_z)$ (represented by solid circles) for three frequencies.
Figure 5-3: Duration time ellipse window. (a) Duration time ellipse window for a one source – one receiver configuration. (b) A waveform record with duration time $T$. Any point scatterer outside the ellipse shown in (a), its contribution to the scattered field can not be recorded by this waveform record. (c) For an experiment with one source and two receivers, the "pass band" of the duration time ellipse window is not an ellipse.
Figure 5-4: Flowchart of the iterative multi-frequency diffraction tomography.
Figure 5-5: Numerical experiment. (a) Model. (b) Image reconstructed by the direct transform reconstruction algorithm with single frequency data. (c) Image reconstructed by the noniterative direct transform reconstruction algorithm with multi-frequency data. (d) Image reconstructed by the iterative multi-frequency direct transform reconstruction algorithm.
Figure 5-6: Windows used in the numerical data reconstruction. (a) Initial finite extent window, determined by a priori information about the outer boundary of the object. (b) Finite extent window after the first iteration. (c) Finite extent window after 1000 iterations.
Figure 5-7: Computed $X$ matrices for reconstructions shown in Figure 5-5. (a) Matrix $X$ of the single frequency reconstruction. (b) Matrix $X$ of the noniterative multi-frequency reconstruction. (c) Matrix $X$ of the iterative multi-frequency reconstruction.
Figure 5-8: Experiment setup of the ultrasonic laboratory experiment. The object is a rubber cylinder, the source-receiver configuration simulates a cross-borehole geometry, the background medium is water.
Figure 5-9: Images reconstructed in the ultrasonic laboratory experiment. (a) Image reconstructed by the single frequency direct transform reconstruction algorithm. (b) Image reconstructed by the noniterative multi-frequency direct transform reconstruction algorithm. (c) Image reconstructed by the iterative multi-frequency direct transform reconstruction algorithm.
Figure 5-10: Windows used in the ultrasonic laboratory data reconstruction. (a) Initial finite extent window, determined by \textit{a priori} information about the outer boundary of the object. (b) Finite extent window after the first iteration. (c) Finite extent window after 1000 iterations.
Figure 5-11: Computed $X$ matrices for reconstructions shown in Figure 5-9. (a) Matrix $X$ for the single frequency reconstruction. (b) Matrix $X$ for the noniterative multi-frequency reconstruction. (c) Matrix $X$ for the iterative multi-frequency reconstruction.
Figure 5-12: The effects of using broader frequency range in the multi-frequency reconstruction. The object should be centered at the cross with the size and shape as shown by the circle at the upper left corner of (a). (a) Single frequency reconstruction of a gelatin cylinder at 50 KHz. (b) Noniterative multi-frequency reconstruction with frequency range 37 KHz — 50 KHz. (c) Noniterative multi-frequency reconstruction with frequency range 37 KHz — 67 KHz.
Chapter 6

RESOLVING POWER AND NOISE SENSITIVITY OF TOMOGRAPHIC METHODS

6.1 Introduction

In Chapter 4 and Chapter 5, two methods that help the limited view angle problem of seismic borehole tomography are developed: the minimum cross entropy diffraction tomography and the iterative multi-frequency diffraction tomography. In this chapter, these two methods, as well as the conventional tomographic reconstruction methods, are evaluated in terms of (1) their noise sensitivity, and (2) their resolving power.

As shown in Figure 6-1, the tomographic imaging process can be represented by a system with its input the true object function and its output the estimated object func-
tion. For seismic diffraction tomography, this system consists of four operators: the probing operator, the source-receiver coverage operator, the noise adding operator, and the image reconstruction operator. By varying the noise adding operator or the image reconstruction operator and keeping the other three operators unchanged, the effects on the estimated object function can tell us the noise sensitivity of the tomographic methods and the resolving power of different image reconstruction algorithms.

The probing operator and the source-receiver coverage operator also introduce distortions in the estimated object function because both of them are non-ideal. In seismic borehole tomography, the probing operator is not ideal because of the experimental error and the assumptions in the forward model, such as the isotropic medium assumption and the point source assumption. The source-receiver coverage operator is not ideal because it gives us only the incomplete scattered wavefield data. In this chapter, however, we only discuss the effects of the noise-adding operator and the image reconstruction operator.

To quantitatively observe the effects on the estimated object function by varying the noise and the image reconstruction method, we characterize the estimated object function by its point spread function $S(x, z)$, which is convolved with the true object function to obtain the estimated object function:

$$O^{\text{est}}(x, z) = O^{\text{true}}(x, z) * S(x, z), \quad (6.1)$$

$$S(x, z) = F^{-1} \left\{ \frac{\tilde{O}^{\text{est}}(k_x, k_z)}{\tilde{O}^{\text{true}}(k_x, k_z)} \right\}. \quad (6.2)$$

The ideal $S(x, z)$ is a two-dimensional unit impulse. By measuring the variance of the
calculated \( S(x, z) \) along the horizontal axis and the vertical axis, we can quantify the horizontal and vertical resolution of the estimated object function.

\[
\sigma_x^2 = \int_{-\infty}^{\infty} (x - \langle x \rangle)^2 S(x, 0) dx \\
\sigma_z^2 = \int_{-\infty}^{\infty} (z - \langle z \rangle)^2 S(0, z) dz,
\]

(6.3) (6.4)

where \( \langle x \rangle, \langle z \rangle \rangle = (0, 0) \). Note that some of the distorting effects of the point spread function are due to the probing and coverage operators. For the cross-borehole tomography, due to the limited view angle problem, \( \sigma_x^2 \) is usually larger than \( \sigma_z^2 \). It is hoped that by using the minimum cross entropy diffraction tomography or the iterative multi-frequency diffraction tomography, the horizontal resolution can be improved, or that, equivalently, \( \sigma_x^2 \) can be reduced.

Chapter 6 is organized in the following manner: In Section 6.2, the noise sensitivity of various tomographic methods are tested by numerical examples. In Section 6.3, the resolution of various tomographic methods are tested by ultrasonic laboratory experiments. Concluding remarks on the noise sensitivity and resolving power of the tomographic methods tested are given in Section 6.4.

### 6.2 The Noise Sensitivity of Tomographic Methods

In this section, we run two numerical experiments to evaluate the noise sensitivity of various tomographic methods. The noise sensitivity of the regular direct transform method is evaluated in the first experiment, the noise sensitivity of other tomographic
methods are evaluated in the second experiment. In the first experiment, we add different amounts of random noise to the synthetic scattered field data, and invert the noisy scattered field data by the regular direct transform method. The results of this experiment are shown in Figure 6-2. This figure shows the regular direct transform reconstruction with 0%, 10%, 20%, and 40% of noise added to the synthetic scattered field data. Those percentages are the ratios of the average power of the added noise over the average power of the scattered field data. With increasing amount of noise in the scattered field data, the noise in the reconstructions also increase continuously. A threshold noise level above which the image abruptly degrades is not observed in this experiment.

In the second experiment, we invert a scattered field data set with 40% added noise by four different reconstruction methods: (1) the regular direct transform method, (2) the minimum cross entropy direct transform method, (3) the multi-frequency direct transform method, and (4) the minimum cross entropy – multi-frequency direct transform method. Then we contrast these four reconstructed images with the four images reconstructed from a noise free data set, also by the above four methods. One of these methods, the minimum cross entropy – multi-frequency direct transform method, is a combination of the minimum cross entropy method and the multi-frequency method. A flowchart of this method is shown in Figure 6-3. The results of the second experiment are shown in Figure 6-4, 6-5, 6-6, and 6-7. Figure 6-4 shows the four reconstructions with 40% added noise, their corresponding point spread functions are shown in Figure
The noise free reconstructions are shown in Figure 6-6, their corresponding point spread functions are shown in Figure 6-7. Examine Figure 6-4 and Figure 6-6, it is not surprising to see that for all these four reconstruction methods tested, the images reconstructed from the noisy data always deviate more from the true object than the images reconstructed from the noise-free data. Also, both the minimum cross entropy method and the multi-frequency method help reduce the noise in the reconstructions.

By examining the point spread functions in Figure 6-5 and Figure 6-7, and comparing the horizontal and vertical variances of those point spread functions, we notice that the variance difference among the four point spread functions in the noisy data experiment is less than the variance difference among the four point spread functions in the noise-free experiment. This means that the 40% noise in the scattered field data reduces the amount of resolution improvement obtained by using the minimum cross entropy method and the multi-frequency method.

For a quantitative evaluation of the noise sensitivities of the single frequency minimum cross entropy method and the multi-frequency method, the variance difference,

\[ \Delta \sigma_z^2 = \sigma_x^2, \text{ regular method } - \sigma_z^2 \]  
\[ \Delta \sigma_z^2 = \sigma_z^2, \text{ regular method } - \sigma_z^2, \]  

obtained by using these two methods is computed for both the 40% noise experiment and the noise-free experiment. The results are listed in Table 6-1. It is noticed that for the noise-free experiment, the variance differences obtained by using the minimum cross entropy method (0.03 and 0.04) are smaller than the variance differences obtained by
using the multi-frequency method (0.06 and 0.05). This means that for the noise-free experiment, the multi-frequency method improves resolution more effectively. For the 40% noise experiment, the variance differences obtained by using the minimum cross entropy method (0.02 and 0.07) are larger than the variance differences obtained by using the multi-frequency method (0.01 and 0.05). This means that when the noise is present, the minimum cross entropy method improves the resolution more significantly than the multi-frequency method.

6.3 The Resolving Power of Tomographic Methods

Eight tomographic methods developed or reviewed in this thesis are tested in this section with a common data set. Among them, four methods are based on the direct transform reconstruction: (1) the regular direct transform method, (2) the minimum cross entropy direct transform method, (3) the multi-frequency direct transform method, and (4) the minimum cross entropy – multi-frequency direct transform method. The other four are based on the backpropagation reconstruction: (5) the regular backpropagation method, (6) the minimum cross entropy backpropagation method, (7) the multi-frequency backpropagation method, and (8) the minimum cross entropy – multi-frequency backpropagation method. The last method, the minimum cross entropy – multi-frequency backpropagation method is a combination of the minimum cross entropy method and the multi-frequency method. It’s flowchart is shown in Figure 6-8.

We use the data collected in an ultrasonic laboratory experiment simulating cross-
borehole tomography to test these eight methods. The object in this experiment is a gelatin cylinder with velocity 1.55 km/sec, density 1.24 g/cc, and diameter 90 mm. The background medium is water with velocity 1.5 km/sec. The imaging area is 243 mm X 243 mm in dimension. The source – receiver geometry and other details of this experiment are described in Section 3.2.

Reconstructions by the four direct transform based methods are shown in Figure 6-9. Figure 6-9(a) is the regular direct transform reconstruction at 50 KHz, Figure 6-9(b) is the minimum cross entropy direct transform reconstruction at 50 KHz, Figure 6-9(c) is the multi-frequency direct transform reconstruction with frequency 37 KHz — 50 KHz, Figure 6-9(d) is the minimum cross entropy – multi-frequency direct transform reconstruction with frequency range 37 KHz — 50 KHz. The corresponding point spread functions are shown in Figure 6-10 (a), (b), (c), and (d), together with their horizontal variance $\sigma_x^2$ and vertical variance $\sigma_y^2$. Looking at the reconstructions in Figure 6-9 and the corresponding point spread functions in Figure 6-10, we find that the regular direct transform method gives the poorest reconstruction, with the largest $\sigma_x^2$ and $\sigma_y^2$ among the four methods tested. Both the minimum cross entropy direct transform method and the multi-frequency direct transform method improve the resolution, and using these two methods together, as shown by Figure 6-9(d) and 6-10(d), gives the reconstruction with the highest horizontal and vertical resolution among these four methods.

Images reconstructed by the four methods based on the backpropagation are shown
in Figure 6-11. Figure 6-11(a) is the regular backpropagation reconstruction at 50 KHz, Figure 6-11(b) is the minimum cross entropy backpropagation reconstruction at 50 KHz, Figure 6-11(c) is the multi-frequency backpropagation reconstruction with frequency 37 KHz — 50 KHz, and Figure 6-11(d) is the reconstruction using both the minimum cross entropy method and the multi-frequency method. The corresponding point spread functions of these four reconstructions are shown in Figure 6-12 (a), (b), (c), and (d). Among these four backpropagation based methods, the regular backpropagation method gives the poorest reconstruction, and its point spread function has the largest horizontal and vertical variance. Both the minimum cross entropy method and the multi-frequency method improve the horizontal and vertical resolution, and using these two methods together, as shown in Figure 6-11(d) and 6-12(d), gives the reconstruction with the highest resolution among these four methods.

6.4 Conclusions

1. Both the minimum cross entropy method and the multi-frequency method improve the resolving power of both the direct transform reconstruction and the backpropagation reconstruction. To solve the limited view angle problem, using these two methods together gives the best result.

2. The effects of applying the minimum cross entropy method and the multi-frequency method to the regular direct transform reconstruction are more significant than the effects of applying them to the regular backpropagation reconstruction.
3. The presence of noise reduces the amount of resolution improvement obtained by using the minimum cross entropy method and the multi-frequency method.

4. For the noise free experiment, the multi-frequency method is more effective than the minimum cross entropy method in improving resolution. When 40% noise is added in the scattered field data, the minimum cross entropy method improves the resolution more significantly than the multi-frequency method.
Noise Free Experiment

<table>
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<td>Minimum cross entropy</td>
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<td>0.04</td>
</tr>
<tr>
<td>Multi-frequency</td>
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<td>0.05</td>
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</table>

40% Noise Experiment

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<th>$\Delta \sigma_z^2$</th>
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</thead>
<tbody>
<tr>
<td>Minimum cross entropy</td>
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<td>0.07</td>
</tr>
<tr>
<td>Multi-frequency</td>
<td>0.01</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 6-1
Figure 6-1: The tomographic imaging system. The input is the true object function, the output is the estimated object function. This system consists of four operators: the probing operator, the source – receiver coverage operator, the noise adding operator, and the image reconstruction operator.
Figure 6-2: Regular direct transform reconstructions with 0%, 10%, 20%, and 40% of noise added in the synthetic scattered field data.
Figure 6-3: Flowchart of the minimum cross entropy - multi-frequency direct transform reconstruction method.
Figure 6-4: Reconstructions with 40% noise added in the scattered field data. (a) Regular direct transform reconstruction. (b) Minimum cross entropy direct transform reconstruction. (c) Multi-frequency direct transform reconstruction. (d) Minimum cross entropy – multi-frequency direct transform reconstruction.
Figure 6-5: Point spread functions of images reconstructed from noisy data. (a) Point spread function of the regular direct transform reconstruction. (b) Point spread function of the minimum cross entropy direct transform reconstruction. (c) Point spread function of the multi-frequency direct transform reconstruction. (d) Point spread function of the minimum cross entropy – multi-frequency direct transform reconstruction.
Figure 6-6: Reconstructions from noise free data. (a) Regular direct transform reconstruction. (b) Minimum cross entropy direct transform reconstruction. (c) Multi-frequency direct transform reconstruction. (d) Minimum cross entropy – multi-frequency direct transform reconstruction.
Figure 6-7: Point spread functions of images reconstructed from noise free data. (a) Point spread function of the regular direct transform reconstruction. (b) Point spread function of the minimum cross entropy direct transform reconstruction. (c) Point spread function of the multi-frequency direct transform reconstruction. (d) Point spread function of the minimum cross entropy – multi-frequency direct transform reconstruction.
Figure 6-8: Flowchart of the minimum cross entropy - multi-frequency backpropagation reconstruction method.
Figure 6-9: Direct transform reconstructions in the ultrasonic laboratory experiment. The object should be centered at the cross with the size and shape as shown by the circle at the upper left corner of (a). (a) Regular direct transform reconstruction. (b) Minimum cross entropy direct transform reconstruction. (c) Multi-frequency direct transform reconstruction. (d) Minimum cross entropy – multi-frequency direct transform reconstruction.
Figure 6-10: Point spread functions of the images reconstructed in the ultrasonic laboratory experiment with the direct transform methods.  
(a) Point spread function of the regular direct transform reconstruction.  
(b) Point spread function of the minimum cross entropy direct transform reconstruction.  
(c) Point spread function of the multi-frequency direct transform reconstruction.  
(d) Point spread function of the minimum cross entropy – multi-frequency direct transform reconstruction.
Figure 6-11: Backpropagation reconstructions in the ultrasonic laboratory experiment. The object should be centered at the cross with the size and shape as shown by the circle at the lower left corner of (a). (a) Regular backpropagation reconstruction. (b) Minimum cross entropy backpropagation reconstruction. (c) Multi-frequency backpropagation reconstruction. (d) Minimum cross entropy – multi-frequency backpropagation reconstruction.
Figure 6-12: Point spread functions of the images reconstructed in the ultrasonic laboratory experiment with the backpropagation methods. (a) Point spread function of the regular backpropagation reconstruction. (b) Point spread function of the minimum cross entropy backpropagation reconstruction. (c) Point spread function of the multi-frequency backpropagation reconstruction. (d) Point spread function of the minimum cross entropy – multi-frequency backpropagation reconstruction.
Chapter 7

GENERAL CONCLUSIONS AND THE OPTIMAL TOMOGRAPHIC INVERSION

7.1 Summary of Major Conclusions

In this thesis, we evaluate the relative performance of seismic ray tomography and seismic diffraction tomography and develop methods that solve a problem that hampers both ray tomography and diffraction tomography when they are applied to subsurface imaging — the limited view angle problem.

The comparison between seismic ray tomography and seismic diffraction tomography is based on the ultrasonic laboratory experiments. Experimental results indicate that when the scattered field can be measured, seismic diffraction tomography is supe-
rior to seismic ray tomography because seismic diffraction tomography is less sensitive to the limited view angle problem and can image small objects with size comparable to the wavelength of the illuminating waves. Seismic diffraction tomography, however, is restricted by the following limitations: weak scattering and constant background medium. The advantage of using seismic ray tomography is that reconstruction can be done using the first arrivals only, the most easily measurable quantity.

The methods we develop for the limited view angle problem are: (1) the minimum cross entropy diffraction tomography, and (2) the iterative multi-frequency diffraction tomography. Two minimum cross entropy diffraction tomography reconstruction algorithms were described in this thesis: the minimum cross entropy backpropagation reconstruction algorithm and the minimum cross entropy direct transform reconstruction algorithm. Numerical and ultrasonic laboratory experiments show that both algorithms can help the limited view angle problem by improving the horizontal resolution and reducing the artifacts in the reconstructions. They are especially powerful for objects consisting of isolated impulses in a constant background medium. One of these two algorithms, the minimum cross entropy backpropagation reconstruction algorithm, is also a finite aperture compensation algorithm.

The iterative multi-frequency diffraction tomography also helps the limited view angle problem. This method is a combination of the multi-frequency reconstruction algorithm and the iterative least square spectrum extrapolation algorithm. The multi-frequency method provides more measured data, the spectrum extrapolation algorithm
estimates the unmeasurable data. Our cross-borehole tomography experiments show that both the horizontal resolution and the signal/noise ratio in the reconstructions can be improved by using the iterative multi-frequency diffraction tomography.

The conventional minimum norm least square spectrum extrapolation algorithm is modified in this thesis. By exploiting the resolution matrix of the spectrum extrapolation problem, we develop an adjustable finite extent space domain window for the spectrum extrapolation algorithm. This window is applicable only for finite extent object functions. Numerical and ultrasonic laboratory experiments show that this modified space domain window approaches the ideal space domain window as the iteration proceeds.

The presence of noise reduces the amount of resolution improvement obtained by using the minimum cross entropy methods and the iterative multi-frequency methods.

7.2 Suggestion for the Optimal Tomographic Inversion

Both ray tomography and diffraction tomography have their strengths and limitations. It is hoped that an optimal tomographic inversion can be developed that use both ray tomography and diffraction tomography such that these two methods can compensate each other.

One of the drawbacks of the current diffraction tomography technique is that the scattered wavefield has to be separated from the total wavefield before inversion. Hu et al. (1987) developed the "post migration filtering" technique to overcome a similar
problem for the reverse time migration of the cross-borehole data. Their method is adopted here with minor modifications and is called the "post backpropagation filtering" since we replace their migration operator by the backpropagation operator. Hu et al.'s method is to migrate the total wavefield directly and the image obtained is a superposition of a high amplitude - low frequency incident field image and a low amplitude - high frequency scattered field image. This composite image is then high - pass filtered, and the low amplitude - high frequency scattered field, which is the estimated object function, is obtained. By replacing the migration operator with the backpropagation operator, the post backpropagation filter is represented by the flowchart shown in Figure 7-1. The input to this system is the total wavefield, the output of this system is a partial reconstruction of the object function — the weak scattering, high frequency component of the object function. This system will be used as one sub-system in the optimal tomographic inversion.

The purpose of the second sub-system is to compensate the first sub-system, to reconstruct the low frequency - strong scattering component of the object function. Ray tomography is suitable for this purpose because ray tomography works well when (1) the sizes of the inhomogeneities are much larger than the wavelength, and (2) the velocity (or attenuation) variation is large. The input to the second sub-system is the total wavefield data, the first arrivals and amplitude information are inverted by ART or SIRT, and the output is the low frequency - strong scattering component of the object function.
The common drawback of the seismic ray tomography and the seismic diffraction tomography is the limited view angle problem, which is treated by the third sub-system. The third sub-system can be either the minimum cross entropy – multi-frequency direct transform method, or the minimum cross entropy – multi-frequency backpropagation method, both of them are described in Chapter 6. Whether the direct transform method or the backpropagation method should be used is mainly a computation consideration. The backpropagation based method is computationally more demanding but also gives better reconstruction. The input to the third sub-system is the weak scattering – high frequency component of the object function, the output of this system is the improved version of the weak scattering – high frequency component of the object function.

The optimal tomographic inversion consisting of the three sub-systems described above, is shown in Figure 7-2. The basic idea of the optimal tomographic inversion is to decompose the object function into components that suit either ray tomography or diffraction tomography. The weak scattering – high frequency component is first reconstructed by the backpropagation operator inside sub-system I, and then refined in sub-system III with respect to the limited view angle problem. The strong scattering – low frequency component is reconstructed by ART or SIRT in sub-system II. The outputs from sub-system II and sub-system III are then combined to produce the final reconstruction.
7.3 REFERENCES

Post Backpropagation Filtering

\[ U_t (= U_i + U) \]

\[ O_t (= O_i + O) \]

High pass filter

\[ O \]

**Figure 7-1: Flowchart of the post backpropagation filter.**

- \( U_t \): Total wavefield
- \( U_i \): Incident wavefield
- \( U \): Scattered wavefield
- \( O \): Object function (low amplitude – high frequency)
- \( O_i \): Incident field image (high amplitude – low frequency)
- \( O_t \): Composite image
Figure 7-2: Flowchart of the optimal tomographic inversion. The optimal tomographic inversion consists of three sub-systems: sub-system I, the post backpropagation filter, sub-system II, the ray tomography reconstruction, and sub-system III, the minimum cross entropy - multi-frequency diffraction tomography reconstruction.
Appendix A

AN INTRODUCTION TO THE BORN AND RYTOV APPROXIMATION

A brief review of the Born and Rylov approximation is given in this appendix. For more detailed derivation, the reader is referred to Chernov (1960), Tartarskii (1971), Flatté et al. (1979), Devaney (1984), and Slaney et al. (1984).

A.1 The Born Approximation

We start with the acoustic wave equation in an inhomogeneous medium

\[
(\nabla^2 + k^2(\tau))U(\tau) = 0.
\]

(A.1)
$k(r)$ is the wavenumber at position $r$, $U_t(r)$ is the total wavefield and is assumed to be the sum of the incident field $U_i(r)$ and the scattered field $U_s(r)$:

$$U_t(r) = U_i(r) + U_s(r).$$  \hspace{1cm} (A.2)

Define the object function as:

$$O(r) = 1 - \frac{C_o^2}{C(r)^2},$$  \hspace{1cm} (A.3)

where $C_o$ is the velocity of the constant background medium, $C(r)$ is the velocity at position $r$. With this definition, $k(r)$ can be expressed as:

$$k^2(r) = k_o^2 - k_o^2 O(r),$$  \hspace{1cm} (A.4)

where $k_o$ is the wavenumber of the homogeneous background medium. Substituting equations (A.2) and (A.4) into equation (A.1), we have:

$$(\nabla^2 + k_o^2)[U_i(r) + U_s(r)] = k_o^2 O(r)[U_i(r) + U_s(r)].$$  \hspace{1cm} (A.5)

Since $(\nabla^2 + k_o^2)U_i(r) = 0$, equation (A.5) becomes:

$$(\nabla^2 + k_o^2)U_t(r) = k_o^2 O(r)[U_i(r) + U_s(r)].$$  \hspace{1cm} (A.6)

The solution of equation (A.6) can be expressed as an integral equation:

$$U_t(r) = -k_o^2 \int O(r')[U_i(r') + U_s(r')]G(r, r')dr'.$$  \hspace{1cm} (A.7)

where $G(r, r')$ is the free space Green's function. We note that equation (A.7) is a nonlinear relationship between $U_t(r)$ and $O(r)$. As long as the scattered field
caused by the inhomogeneity is weak compared with the incident field, we can use the Born approximation to achieve linearity. When \( U(r) \ll U_i(r) \), we can set \( U(r) = 0 \) for \( U(r) \) inside the integral of equation (A.7) and obtain
\[
U(r) \approx -k_0^2 \int O(r')U_i(r')G(r',r')dr'.
\]
(A.8)

For a point source, the incident field \( U_i(r) \) can be replaced by a Green's function and equation (A.8) can be written as
\[
U(r, r_s) \approx -k_0^2 \int O(r)G(r,r_s)G(r,r_g)dr.
\]
(A.9)

\( U(r, r_s) \) is the scattered wavefield measured at position \( r_g \) when the point source is located at position \( r_s \).

### A.2 The Rytov Approximation

Express the total wavefield as:
\[
U_t(r) = e^{i\phi_t(r)},
\]
(A.10)

where \( \phi_t(r) \) is the complex phase of the total field. Substitute equation (A.10) into equation (A.1), equation (A.1) becomes:
\[
\nabla^2 \phi_t(r) + [\nabla \phi_t(r) \cdot \nabla \phi_t(r)] + k^2(r) = 0.
\]
(A.11)

Use the relationship between \( k(r) \) and \( O(r) \) in equation (A.4), and omit \( r \) in \( \phi_t(r) \) and \( k(r) \) for brevity, equation (A.11) can be written as:
\[
\nabla^2 \phi_t + (\nabla \phi_t \cdot \nabla \phi_t) + k_0^2 = k_0^2 O(r)
\]
(A.12)
Assume

\[ \phi_t = \phi_i + \phi_d, \quad (A.13) \]

where \( \phi_i \) is the complex phase of the incident wavefield and \( \phi_d \) is the complex phase difference between the total field and the incident field. Substitute equation (A.13) into equation (A.12), we obtain

\[ [\nabla^2 \phi_i + \nabla \phi_i \cdot \nabla \phi_i + k_0^2] + 2(\nabla \phi_i \cdot \nabla \phi_d) + \nabla^2 \phi_d = -(\nabla \phi_d \cdot \nabla \phi_d) + k_0^2 O(\tau). \quad (A.14) \]

Since those terms inside the square brackets of equation (A.14) is just another form of the homogeneous wave equation, so

\[ \nabla^2 \phi_i + \nabla \phi_i \cdot \nabla \phi_i + k_0^2 = 0. \quad (A.15) \]

Equation (A.14) now becomes:

\[ 2(\nabla \phi_i \cdot \nabla \phi_d) + \nabla^2 \phi_d = -(\nabla \phi_d \cdot \nabla \phi_d) + k_0^2 O(\tau). \quad (A.16) \]

Exploiting the equality relationship

\[ \nabla^2 (U_i \phi_d) = \phi_d \nabla^2 U_i + 2 \nabla U_i \cdot \nabla \phi_d + U_i \nabla^2 \phi_d, \quad (A.17) \]

and remembering that \( \nabla^2 U_i = -k_0^2 U_i \), equation (A.17) can be rearranged as

\[ 2 \nabla U_i \cdot \nabla \phi_d + U_i \nabla^2 \phi_d = (\nabla^2 + k_0^2) U_i \phi_d. \quad (A.18) \]

Combine equations (A.16) and (A.18), we obtain

\[ (\nabla^2 + k_0^2) U_i \phi_d = U_i[-(\nabla \phi_d \cdot \nabla \phi_d) + k_0^2 O(\tau)]. \quad (A.19) \]
Express the solution of equation (A.19) as an integral equation:

\[ U_i(r) \phi_d(r) = - \int U_i(r') [ - \nabla \phi_d(r') \cdot \nabla \phi_d(r') + k_0^2 O(r')] G(r,r') dr'. \quad (A.20) \]

Equation (A.20) is a nonlinear relationship between the measured data \( \phi_d(r) \) and the object function \( O(r) \). The Rytov approximation can now be used to linearize equation (A.20) if \( \nabla \phi_d(r) \) is small. For small \( \nabla \phi_d(r) \), the quantity \( (\nabla \phi_d(r) \cdot \nabla \phi_d(r)) \) inside the integral can be neglected and equation (A.20) becomes:

\[ U_i(r) \phi_d(r) \approx -k_0^2 \int U_i(r') O(r') G(r,r') dr'. \quad (A.21) \]

For a point source at \( r_s \), the incident field at \( r \) is:

\[ U_i(r) = G(r,r_s). \quad (A.22) \]

Equation (A.21) can then be written as:

\[ U_i(r_s,r_g) \phi_d(r_s,r_g) \approx -k_0^2 \int O(r) G(r,r_s) G(r,r_g) dr. \quad (A.23) \]

Equation (A.23) is a linear relationship between the measured data \( U_i(r_s,r_g) \phi_d(r_s,r_g) \) and the object function \( O(r) \). Notice that the right hand side of equation (A.23) is exactly the same as the right hand side of equation (A.9). This means that although the diffraction tomography reconstruction algorithms described in Chapter 2 are derived from equation (A.9), using the Born approximation, those algorithms can be used for the case of Rytov approximation simply by changing the input to the algorithm from \( U(r_s,r_g) \) for the Born approximation to \( U_i(r_s,r_g) \phi_d(r_s,r_g) \) for the Rytov approximation.
A.3 The relationship between the Born and Rytov approximation

In the following, we first show that when $\phi_d$ is small, the Rytov approximation (equation (A.23)) reduces to the Born approximation (equation (A.9)). Then, we verify that the Born approximation is a weak scattering approximation whereas the Rytov approximation is a smooth perturbation approximation. When $\phi_d \ll 1$, we can expand

$$e^{\phi_d} \approx 1 + \phi_d.$$  \hspace{1cm} (A.24)

Therefore

$$U_i \phi_d \approx U_i (e^{\phi_d} - 1) = e^{\phi_t + \phi_d} - U_i = U_t - U_i = U.$$  \hspace{1cm} (A.25)

The left hand side of equation (A.23) becomes the scattered field $U$ when $\phi_d \ll 1$. The Rytov approximation reduces to the Born approximation as expected. If we write explicitly

$$U_t = A_t e^{i \psi_t},$$

$$U_i = A_i e^{i \psi_i},$$  \hspace{1cm} (A.26)

where $\psi_t$ and $\psi_i$ are the real phases of the total field and the incident field, $A_t$ and $A_i$ are the corresponding amplitudes. Then

$$\phi_d = \phi_t - \phi_i$$

$$= \log U_t - \log U_i$$  \hspace{1cm} (A.27)
\[ \log \left( \frac{A_t}{A_i} \right) + i(\psi_t - \psi_i) \]

The Born approximation requires the difference between \( U_t \) and \( U_i \) be small, which implies \( \phi_d \) be small and also \( \log \left( \frac{A_t}{A_i} \right) \ll 1 \), \( (\psi_t - \psi_i) \ll 1 \). Those requirements may not be satisfied for a very large (compared with wavelength) object due to the accumulative phase difference \( (\psi_t - \psi_i) \). This shows the limitations of applying the Born approximation. On the other hand, the Rytov approximation does not require the smallness of \( \phi_d \). In the derivation (from equation (A.20) to (A.21)), it only requires that the gradient of \( \phi_d \) be small, i.e. that the change of \( \phi_d \) within a wavelength be small compared to \( O(\ell^\frac{1}{2}) \). In other words, the Rytov approximation only requires the smoothness of the scattered field, not the smallness of the scattered field. Therefore, the Born approximation is a weak scattering approximation whereas the Rytov approximation is a smooth scattering (or smooth perturbation) approximation.
A.4 REFERENCES


Appendix B

TWO DIMENSIONAL AND TWO AND HALF DIMENSIONAL PROBLEM

B.1 Numerical Example

The ultrasonic laboratory experiments conducted in this thesis are 2 1/2D scattering experiments, however, in deriving the backpropagation reconstruction formula in Chapter 2, we used 2D scattering geometry. In this Appendix, we determine the applicability of 2D theory for reconstruction of 2 1/2D data. We compare the synthetic scattered wavefield calculated by using both the 2 1/2D scattering formula and the 2D scattering formula and determine how they diverge as a function of the distance of the object from the source and the receiver. We find that using the 2 1/2D scattering...
formula or the 2D scattering formula does not make substantial difference in the farfield.

Our calculation simulates our experiment setup. For the 2D calculation, we put a line scatterer between the line source and the line receiver, as shown in Figure B-1 (a); for the 2\(\frac{1}{2}\)D calculation, we put a line scatterer between the point source and the point receiver, as shown in Figure B-1 (b). Esmersoy (1986) derived the 2\(\frac{1}{2}\)D scattered wavefield:

\[
U_{2\frac{1}{2}D}(r, s) = \frac{\sqrt{2\pi}}{16\pi^2} \frac{e^{ik_0 r} O(r)}{r - r_s} \int \frac{e^{ik_0 (r - r_s) \cdot \mathbf{\ell}}}{|r - r_s|^2 |r - r_g|^2} dr
\] (B.1)

where \(r = (x, 0, z)\), is the coordinate on the source-receiver plane. In this Appendix, we calculate the 2\(\frac{1}{2}\)D scattered wavefield caused by a line scatterer perpendicular to the source-receiver plane and moving along the line connecting the point source and the point receiver, as shown in Figure B-1 (b). For such a line scatterer, the volume integral in equation (B.1) only includes a single point at position \(r\), the intersection of the line scatterer and the source-receiver plane. Also, the quantity \(|r - r_s| + |r - r_g|\) in equation (B.1) is a constant, this is because both \(r_s\) and \(r_g\) are fixed and \(r\) is always on the line connecting the point source and the point receiver. Let \(|r - r_s| + |r - r_g|\) = \(L\), the 2\(\frac{1}{2}\)D scattered wavefield of the line scatterer is:

\[
U_{2\frac{1}{2}D}(r, s, r_g, r_s) = \frac{\sqrt{2\pi}}{16\pi^2} \frac{e^{ik_0 r} O(r)}{L} \frac{e^{i k_0 (r - r_s) \cdot \mathbf{\ell}}}{|r - r_s|^2 |r - r_g|^2} \] (B.2)
The 2D scattering formula is obtained by using the 2D Green's function in equation (2.14):

\[ U^{2D}(\mathbf{r}_{s}, \mathbf{r}_{g}) = -k_{o}^{2} \int_{S} O(\mathbf{r}) G(\mathbf{r}_{s}, \mathbf{r}) G(\mathbf{r}_{g}, \mathbf{r}) d\mathbf{r} \]

\[ = -k_{o}^{2} \int_{S} O(\mathbf{r}) \frac{j}{4} H_{o}^{(1)}(k_{o} | \mathbf{r} - \mathbf{r}_{s} |) \frac{j}{4} H_{o}^{(1)}(k_{o} | \mathbf{r} - \mathbf{r}_{g} |) d\mathbf{r} \]  

(B.3)

For a line scatterer parallel to the line source and the line receiver, the surface integral in equation (B.3) include only one point, the intersection of the line scatterer and the source–receiver plane. So, equation (B.3) becomes

\[ U^{2D}(\mathbf{r}_{s}, \mathbf{r}_{g}, \mathbf{r}) = -k_{o}^{2} O(\mathbf{r}) \frac{j}{4} H_{o}^{(1)}(k_{o} | \mathbf{r} - \mathbf{r}_{s} |) \frac{j}{4} H_{o}^{(1)}(k_{o} | \mathbf{r} - \mathbf{r}_{g} |) \]  

(B.4)

We use equations (B.2) and (B.4) to calculate the \( U^{2\frac{1}{2}D} \) and 2D scattered wavefield of the line scatterer measured at the receiver position. In this calculation, the source position \( \mathbf{r}_{s} \), and the receiver position \( \mathbf{r}_{g} \) are fixed, and the relative difference between the \( U^{2\frac{1}{2}D} \) and 2D scattered wavefield, \( e(l) \), is computed as a function of the distance between the scatterer and the source, \( l \):

\[ e(l) = \frac{U^{2\frac{1}{2}D}(l) - U^{2D}(l)}{U^{2\frac{1}{2}D}(l)} \]  

(B.5)

The result of our calculation for \( l \) ranges from 30 mm to 214 mm and frequency ranges from 10 KHz to 80 KHz is shown in Figure B-2. Figure B-2 indicates that for frequencies higher than 20 KHz, when the scatterer is in the farfield with respect to both the source and the receiver, such as in the middle of the source — receiver line, using 2D scattering formula or \( U^{2\frac{1}{2}D} \) scattering formula does not make substantial
difference.

Our second calculation is to test the effect of the size (with respect to wavelength) of the scatterer on the validity of approximating $2\frac{1}{2}$D scattered wavefield by 2D scattered wavefield. We use equations (B.1) and (B.3) to calculate the $2\frac{1}{2}$D and 2D scattered wavefield of a cylindrical scatterer. Variation of the scattered wavefield caused by variation of the size of the scatterer is computed by keeping the actual size of the scatterer (limits of integration in equations (B.1) and (B.3)) fixed while varying the wavelength. For a cylindrical scatterer (in the farfield with respect to both the source and the receiver) with effective diameter ranges from $0.25\lambda$ to $4\lambda$, the relative difference between $2\frac{1}{2}$D and 2D scattered wavefield is shown in Figure B-3. This figure shows that as long as the scatterer is in the farfield with respect to both the source and the receiver, the difference between the $2\frac{1}{2}$D and 2D scattered wavefield is less than 10% for cylindrical scatterer with the diameter of several wavelengths.

The results shown in Figure B-2 and B-3 are not surprising. When the scatterer is in the farfield, or when the size of the scatterer is large compared with the wavelength, the argument of the Hankel function is large, for large argument, the Hankel function can be approximated by the first term of its asymptotic expansion:

$$
(2\pi)^{\frac{1}{2}} |k_o| |\frac{j}{4} H_{-1}^1(k_o | r_1 - r_2 |) | \approx \frac{(jk_o)^{\frac{1}{2}}}{2} e^{jk_o |r_1 - r_2|} \frac{e^{jk_o |r_1 - r_2|}}{\frac{1}{2} r_1 - r_2 |^{\frac{1}{2}}} \tag{B.6}
$$

If we replace the Hankel functions in equation (B.3) by the right hand side of equation (B.6), we find that equation (B.3), the 2D scattered wavefield is very
similar to the $2\frac{1}{2}$D scattered wavefield, equation (B.1).

**B.2 REFERENCES**

Figure B-1: (a) 2D scattering experiment. (b) 2½D scattering experiment.
Figure B-2: Relative difference between the 2D scattered wavefield and the $2\frac{1}{2}$D scattered wavefield.
Figure B-3: The effect of the size of the scatterer on the relative difference between the 2.5D and 2D scattered wavefield. In this example, the scatterer is in the farfield with respect to both the source and the receiver.