Essays on Inventory, Pricing and Financial Trading
Strategies
by
Ye Lu
Submitted to the Sloan School of Management
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Abstract

In a multi-product market, if one product stocks out, consumers may substitute to competing products. In this thesis, we use an axiomatic approach to characterize a price-dependent demand substitution rule, and provide a sufficient and necessary condition for demand models where our demand substitution rule applies. Our results can serve as a link between the pricing and inventory literature, and enable the study of joint pricing and inventory coordination and competition. I demonstrate the impact of this axiomatic approach on the joint pricing and inventory coordination model by incorporating the price-dependent demand substitution rule, and illustrate that if the axiomatic approach is acceptable, the optimal strategy and corresponding expected profit are quite different than models that ignore stockout demand substitution. I use this price-dependent demand substitution rule to model the joint pricing and inventory game, and study the existence of Nash equilibrium in this game.

In the second part of this thesis, I consider the problem of dynamically trading a security over a finite time horizon. The model assumes that a trader has a “safe price” for the security, which is the highest price that the trader is willing to pay for this security in each time period. A trader’s order has both temporary (short term) and permanent (long term) impact on the security price and the security price may increase after the trader’s order, to a point where it is above the safe price. Given a safe price constraint for the current time period, I characterize the optimal policy for the trader to maximize the total number of securities he can buy over a fixed time horizon. In particular, I consider a greedy policy, which involves at each stage buying a quantity that drives the temporary price to the security safety price. I show that the greedy policy is not always optimal and provide conditions under which the greedy policy is optimal. I also provide bounds on the performance of the greedy policy relative to the performance of the optimal policy.

Thesis Supervisor: David Simchi-Levi
Title: Professor of Engineering Systems Division
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Chapter 1

Introduction

In this thesis, we consider operational problems in face of uncertainty. One problem is in the area of supply chain management focusing on the coordination of inventory and pricing strategies when competition exists. The second problem deals with investment challenges faced by large institutions whose investment may affect market price.

1.1 Demand Substitution: Motivation

The coordination of pricing and inventory decisions is challenging, in particular in an environment where the firm has substitutable products. In such an environment, the price of one product affects not only its own demand but also the demand of other products. At the same time, since products are substitutable, demand for some products may increase, when other products stockout. Unfortunately, the operations management literature on multi-product inventory and pricing coordination, that takes into account both effects, does not exist.

To illustrate the challenge, consider a retailer with substitutable products such as Pepsi and Coke. Some consumers may switch from one product to another either due to a change in price or during a period of stockout of one of the products. In fact, in both cases, the number of customers switching to another product depends on all (or remaining) product prices.
1.2 Demand Substitution: Literature Review

In recognition of these challenges, the literature in this area can be divided into two categories. In the first, see for example, Aydin and Porteus[3], Birge et al [6] and Maddah and Bish  [19], price is a decision variable but customers do not switch during stockout periods. In the second, see for example Van Ryzin and Mahajan [28], Smith and Agrawal [29] and Rajaram and Tang [26], prices are assumed to be exogenous and customers switch during stockout period according to a price-independent substitution rule.

A similar challenge exists in a decentralized system, where multiple retailers compete simultaneously on pricing and inventory. As a result, the literature here can also be classified into two categories. In one, demand is independent of price, and competition is on inventory, see Parlar [24], Lippman and McCardle [15], Mahajan and Van Ryzin [21] and Netessine and Rudi [22]. In the second, see Bernstein and Federgruen [8] and Chen et al [10], demand is a function of price and competition is on price but not on inventory.

1.3 Demand Substitution: Contributions

The literature review suggests that the coordination (and competition) of pricing and inventory decisions in an environment with multiple, substitutable products (or identical products offered by multiple retailers) remains an important challenge. This is due to the lack of price dependent substitution rule, that is a rule that suggests how consumers switch from one product (retailer) to another as a function of price during periods of stockout. This is exactly the objective of this thesis. Specifically, I use an axiomatic approach to characterize a price-dependent demand substitution rule.

The basic question I address is as follows: Given a specific demand model for n products, what is the impact of removing a subset of the products from the system on the demand of the other products. Remarkably, I show that under general demand models, it is possible to exactly characterize customer demand for the remaining products. The approach used in our analysis is an axiomatic one, that is, I make no assumption on the structure of the demand model for the remaining products and show that there is a unique price-dependent
demand substitution rule to determine the remaining products demand model. Our demand 
models are general, and include the Linear model, the Attraction model, the Logit model 
and the CES model as special cases. This part of work is done in Chapter 2. In Chapter 3, I 
apply this result to a joint pricing and inventory coordination model, and demonstrate the 
impact of our substitution rule on the optimal inventory and pricing strategy. In Chapter 4, 
I use the price-dependent demand substitution rule to model the joint pricing and inventory 
model, and study the existence of Nash equilibrium in this game.

1.4 Adaptive Safe Price: Motivation and Literature 
Review

In the second part of this thesis, I consider the problem of dynamically trading a security 
over a finite time horizon. Given the dramatic increase in institutional trading in recent 
years, there has been much interest in the optimal control of the execution costs. During the 
last few years, several studies have been done on dynamic optimal trading strategies that 
minimize the expected cost of trading a security. Specifically, a trader has to buy $Q$ units of 
a security over $N + 1$ periods. Let $q_i$ denote the trade size for the security at period $i$. Then 
this problem can be expressed as:

$$
\min_{q_i} E\left\{ \sum_{i=0}^{N} p_i q_i \right\} \quad (1.1)
$$

s.t. \hspace{1cm} \sum_{i=0}^{N} q_i = Q. \quad (1.2)

In each period, the price of the security is a function of the trader’s order size. The law 
of motion for price $p_i$ may be expressed as

$$
p_{i+1} = p_i + \theta q_i + \epsilon_i, \quad (1.3)
$$

where $\theta$ is a positive constant and $\epsilon_i$ is a random variable. This model first appeared in 
Bertsimas and Lo [9]. They show that to minimize expected execution cost, a trader should

1.5 Adaptive Safe Price: Contributions

In Chapter 5, I consider the case when the trader identifies a “safe price”, $\bar{P}$, for the security. Although the trader cannot predict the exact price of this security after $N + 1$ periods, he is confident that the security’s price should be somewhere above this safe price. Therefore, as long as the trader purchases this security at a price below the safe price, he will be able to profit at the end of the time horizon.

At the same time, this safe price, $\bar{P}$, also represents the trader’s risk aversion level, which means that $\bar{P}$ is the highest price that the trader is willing to pay for this security. Suppose the trader has a budget (available cash in hand), $Q$, for this security. Given that the trader will profit at the end of the planning horizon as long as this security is bought below the safe price, the trader can adopt a strategy that maximizes the total number of securities purchased over the $N + 1$ periods. If at some time period, the trader runs out of cash, then he can stop. If at the end of last period, there is still some cash left, the trader at least has bought as many units as possible below the safe price $\bar{P}$, which maximizes the profit he can make under the assumption that the security’s price will be somewhere above this safe price.

Traders can adjust their safe price at the beginning of each period. For example, at the beginning of period $i$, the trader has observed the security’s current price and its price motion over the last $i$ periods. Given these observations, the trader may re-predict the security’s price and adjust its safe price from $\bar{P}_{i-1}$ to $\bar{P}_i$.

It is tempting to conclude that since the trader’s goal is to maximize the number of units of security, the trader should purchase to increase price up to $\bar{P}_i$. I refer to this policy as the greedy policy. Unfortunately, I show in Section 5.1, using a counter example, that the
greedy policy is not always optimal. The following questions are therefore natural: What is the structure of the optimal policy? Under what conditions the greedy policy is optimal? And, when it is not optimal, how far is it from the optimal? These questions are answered in Chapter 5.
Chapter 2

A Price-dependent Demand Substitution Rule

This chapter is organized as follows. In Section 2.1, we present and prove the price-dependent demand substitution rule for deterministic model. In Section 2.2, we use the demand substitution rule to study how demand sensitivity and system demand depend on the number of products.

2.1 Deterministic Model

Consider a market with \( n \) products indexed by \( i = 1, 2, \ldots, n \). Let

\[
D = (d_1, \ldots, d_n)^T = \text{demand vector} \quad (2.1)
\]

\[
p = (p_1, \ldots, p_n)^T = \text{retail price vector} \quad (2.2)
\]

The demand for each product depends not only on its own retail price, but also the retail prices of the other \( n - 1 \) products. We assume that the \( n \) products are substitutable, so that if the retail price for product \( i \) is increased, not only will the demand for the \( i \)'s product decrease, but also the demand for the other products, other than \( i \), will increase.
Thus, we assume that the demand models always satisfy the following assumption.

**Assumption 2.1.1.** The demand functions $d_i(p)$, $i = 1,\ldots,n$ are continuously differentiable, $\frac{\partial d_i(p)}{\partial p_i} < 0$ and $\frac{\partial d_j(p)}{\partial p_i} \geq 0$, $j \neq i$.

In this section we answer the following question. Given $n$ competitive products having demand according to demand model $D$, what is the impact of removing $m$ of the products on customer demand for the remaining $n - m$ products?

In the analysis below, we denote $\mathcal{R} = \{m+1,\ldots,n\}$ and $-\mathcal{R} = \{1,\ldots,m\}$. For any set $F \in \mathbb{R}^n$, we let $\prod_{-\mathcal{R}} F$ and $\prod_{\mathcal{R}} F$ be the projection of $F$ onto its first $m$ variables and last $n - m$ variables, respectively. Without loss of generality, we assume that we remove the product indexed by $1,2,\ldots,m$ from the system, and our objective in this section is to determine the demand for the remaining $n-m$ products. To answer this question, we assume that the demand for the remaining $n-m$ products should satisfy the following two basic assumptions. Let $d^R_j(p_R)$, $j \in \mathcal{R}$ be the new demand function for each remaining product.

**Assumption 2.1.2.**

(a) If a subset of products with positive demand is driven out of the market, the demand for each remaining product does not decrease.

(b) If all products $i$, with $i \in -\mathcal{R}$ are removed at some price vector $(p_-R, p_R)$ such that $d_i(p_-R, p_R) = 0$, for each $i \in -\mathcal{R}$, the demand for each remaining product $j$, $j \in \mathcal{R}$, does not increase.

The first assumption is not valid for complementary products, such as PCs and Laser Printers; when one is removed, we expect a decrease in the demand for the other. However, this assumption is valid for substitutable products since when some products are removed from the market, there is no reason that the demand for remaining products will decrease. Formally, the assumption can be written as follows: For any fixed $p_R$, we denote $F_-\mathcal{R} = \{p_-\mathcal{R} \in \mathbb{R}_+^m | d_-\mathcal{R}(p_-\mathcal{R}, p_R) \geq 0\}$, $S_-\mathcal{R} = \{p_-\mathcal{R} \in \mathbb{R}_+^m | d_-\mathcal{R}(p_-\mathcal{R}, p_R) = 0\}$ and $\Gamma_-\mathcal{R} = F_-\mathcal{R} \setminus S_-\mathcal{R}$. The first assumption implies that
\[ d^R_j(p_R) \geq \max_{p_{-R} \in \Gamma_{-R}} d_j(p_{-R}, p_R), \ j \in \mathcal{R}. \quad (2.3) \]

The second assumption suggests that if no customer is willing to buy from products 1, 2, \ldots, \( m \) at a certain price vector \((p_{-R}, p_R)\), removing these products from the market won’t increase the demand for the other products. Of course, removing products \( i, i \in -\mathcal{R} \) may increase the demand for the remaining product \( j, j \in \mathcal{R} \), when customers who would have purchased products \( i, i \in -\mathcal{R} \), switch to buy the remaining product \( j, j \in \mathcal{R} \) after the removal of \(-\mathcal{R}\). But this won’t happen if there is no customer demand for products \( i, i \in -\mathcal{R} \). Formally, the second assumption implies that

\[ d^R_j(p_R) \leq \min_{p_{-R} \in S_{-R}} d_j(p_{-R}, p_R), \ j \in \mathcal{R}. \quad (2.4) \]

Thus, under the above two assumptions, we have for any demand model

\[ \max_{p_{-R} \in \Gamma_{-R}} d_j(p_{-R}, p_R) \leq d^R_j(p_R) \leq \min_{p_{-R} \in S_{-R}} d_j(p_{-R}, p_R), \ j \in \mathcal{R}, \quad (2.5) \]

Our objective is to characterize conditions under which the lower bound (2.3) and upper bound (2.4) match, i.e.,

\[ \max_{p_{-R} \in \Gamma_{-R}} d_j(p_{-R}, p_R) = \min_{p_{-R} \in S_{-R}} d_j(p_{-R}, p_R), \ j \in \mathcal{R}, \quad (2.6) \]

which implies that the demand function for the remaining products \( d^R_j(p_R) \), \( j \in \mathcal{R} \) is uniquely determined. For this purpose, we partition the set of demand models satisfying Assumption 2.1.1 into three types.

2.1.1 Demand model type I

For type I demand model, we assume that demand for a subset of the products is zero only when price for these products is infinite.

Assumption 2.1.3. \( d_i(p) > 0 \) for any \( p \in R^n_+ \), and \( d_i(p) = 0 \) when \( p_i = \infty \), \( i = 1, \ldots, n \).
The following demand functions satisfy this assumption.

**Attraction Models.** Attraction models are among the most commonly used market share models. The market demand achieved by a given firm $i$ is given by its attraction value divided by the industry’s total value, i.e,

$$d_i(p) = \frac{u_i(p)}{\sum_{j=0}^{N} u_j(p)}$$

(2.7)

Here $M$ is the fixed market size, $u_0$ is a constant and $u_i(p) = k_i p_i^{\alpha_i}$ or $u_i(p) = k_i e^{-\alpha_i p_i}$ for constants $\alpha_i, k_i > 0$.

**Logit Model.**

$$d_i(p) = \frac{k_i e^{-\lambda p_i}}{\sum_{j=1}^{N} k_j e^{-\lambda p_j}}$$

(2.8)

with $\lambda > 0$ and $k_i > 0$ for all $i$.

**CES Model.**

$$d_i(p) = \frac{\gamma p_i^{-r}}{\sum_{j=1}^{N} \gamma^r p_j}$$

(2.9)

with $r < 0$ and $\gamma > 0$.

For type I demand model, Assumption 2.1.1 implies that $\max_{p_{-R} \in \Gamma_{-R}} d_j(p_{-R}, p_R) = d_j(\infty, p_R)$ because $d_j(p_{-R}, p_R), j \in R$ increase with $p_{-R}$. Assumption 2.1.3 implies that $\min_{p_{-R} \in S_{-R}} d_j(p_{-R}, p_R) = d_j(\infty, p_R)$ because infinity is the only point in $S_{-R}$. Therefore, for Type I demand model, equation (2.6) holds and the demand functions for remaining products are uniquely determined by $d_j^R(p_R) = d_j(\infty, p_R), j \in R$.

Observe that for any $p_R$, infinity is the solution to $d_{-R}(p_{-R}, p_R) = 0$, i.e., the price vector such that the demand for removed products is zero. This implies that the price-dependent demand substitution rule can be described as follows:
**Price-dependent Demand Substitution Rule**: For any \( p \_R \), let \( p^*_R(p \_R) \) be the solution to \( d_{-R}(p \_R, p \_R) = 0 \), i.e., the price vector such that the demand for removed products is zero, then demand function for remaining products is

\[
d^R_j(p \_R) = d_j(p^*_R(p \_R), p \_R), \quad j \in R. \tag{2.10}
\]

An interesting question is whether this demand substitution rule, determined by equation (2.6), is appropriate for other type of demand models, beyond Type I demand model. Unfortunately, we face important challenges once we try to extend beyond Type I demand models, as is illustrated by the following examples.

**Example 2.1.1.** Let

\[
\begin{align*}
d_1(p_1, p_2, p_3) &= 9 - p_1 + (p_2 - 20) \frac{3}{2} + p_3, \\
d_2(p_1, p_2, p_3) &= 9 + p_1 - p_2 + p_3, \\
d_3(p_1, p_2, p_3) &= 9 + p_1 + p_2 - 2p_3, 
\end{align*}
\]

and notice that these demand functions satisfy Assumptions 2.1.1. Set \( p_3 = 1 \) and observe that the solution to the following system

\[
\begin{align*}
d_1(p_1, p_2, 1) &= 10 - p_1 + (p_2 - 20) \frac{\sqrt{3}}{2} = 0, \\
d_2(p_1, p_2, 1) &= 10 + p_1 - p_2 = 0,
\end{align*}
\]

\( S_{-R} \) consists of three points, \((9, 19), (10, 20)\) and \((11, 21)\). Since we have three values of \( p^*_R(p \_R) \), it is not clear any more which one shall we assign into equation (2.10)? We call this the *uniqueness problem*.

Even if \( S_{-R} \) consists of only one element, it can happen that the resulting demand model does not make practical sense.
Example 2.1.2. Consider the following demand model

\[
\begin{align*}
 d_1(p_1, p_2, p_3) &= 200 - p_1 + \frac{2}{5}p_2 + \frac{3}{5}p_3, \\
 d_2(p_1, p_2, p_3) &= 164 + \frac{1}{2}p_1 - p_2 + \frac{3}{5}p_3, \\
 d_3(p_1, p_2, p_3) &= 150 + \frac{1}{2}p_1 + \frac{1}{2}p_2 - p_3.
\end{align*}
\]

Notice that these demand functions satisfy Assumptions 2.1.1 and for any \( p_3, \quad p^*_R(p_3) = (332 + \frac{21}{20}p_3, 330 + \frac{9}{8}p_3) \). Hence, removing product 1 and product 2 yields a new demand function for product 3 satisfying,

\[
d_3^{(R)}(p) = d_2(p^*_R(p_3), p_3) = 481 + \frac{7}{80}p_3. \quad (2.12)
\]

This implies that product 3’s demand increases with its own price. We call this the consistency problem.

To address the uniqueness and consistency problem, we study the following type of demand model.

2.1.2 Demand model type II

Denote \( F = \{ p \in \mathbb{R}^n_+ | d_i(p) \geq 0 \} \). For type I demand model, \( F = \mathbb{R}^n_+ \). As you will see, this is not true in this subsection. Therefore, in this subsection we only consider the region \( F \). Recall \( \prod_F \) is the projection of \( F \) onto its last \( n - m \) variables. For demand model type II, we assume that the demand functions satisfy the following property.

Assumption 2.1.4. For any fixed \( p_R \in \prod_F \), there exists finite \( p^*_R \in \mathbb{R}^m_+ \) such that \( (p^*_R, p_R) \in F \) and \( d_{-R}(p^*_R, p_R) = 0 \).

This assumption implies that given a price vector \( p_R \) for a subset of the products \( R \), there is only finite price vector \( p^*_R \) satisfying \( d_{-R}(p^*_R, p_R) = 0 \). Both Example 5.1.1 and Example 2.1.2 satisfy Assumption 2.1.4. Therefore, to solve the uniqueness and consistency
problem for type II demand model, we need to replace Assumption 2.1.1 by a slightly stronger assumption. Before we describe this assumption, we need to introduce the notation of $M$-matrix.

**Definition 2.1.1.** A square matrix $A$ is called an $M$-matrix if all off-diagonal entries are less than or equal to zero and it satisfies any one of the following equivalent conditions.

(a) All principal minors of $A$ are positive.

(b) The leading principal minors of $A$ are positive.

(c) The diagonal entries of $A$ are positive and $AH$ is strictly diagonally dominant for some positive diagonal matrix $H$.

(d) $A$ is non-singular and the inverse of $A$ is non-negative.

For more details on $M$-matrix, see [12, §2.5]. We denote the Jacobian matrix of the demand functions by

$$J = \begin{pmatrix}
\frac{\partial d_1(p)}{\partial p_1} & \cdots & \frac{\partial d_1(p)}{\partial p_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial d_n(p)}{\partial p_1} & \cdots & \frac{\partial d_n(p)}{\partial p_n}
\end{pmatrix}. \quad (2.13)$$

We replace Assumption 2.1.1 by the following assumption.

**Assumption 2.1.5.** $-J$ is an $M$-matrix.

Notice that Assumption 2.1.5 implies Assumption 2.1.1 because all off-diagonal entries of $-J$, $-\frac{\partial d_j(p)}{\partial p_i} \leq 0$, $j \neq i$, and Definition 2.1.1 part (c) implies that its diagonal entries, $-\frac{\partial d_i(p)}{\partial p_i} > 0$.

At a first glance, Assumption 2.1.5 looks quite technical. However, we show that this Assumption is more general than both of the following, commonly used assumptions, for multi-product demand models.

**Assumption 2.1.6.** $\frac{\partial d_i(p)}{\partial p_i} < -\sum_{j \neq i} \frac{\partial d_j(p)}{\partial p_j}$, $i = 1, \ldots, n$.

**Assumption 2.1.7.** $\frac{\partial d_i(p)}{\partial p_i} < -\sum_{j \neq i} \frac{\partial d_j(p)}{\partial p_j}$, $i = 1, \ldots, n$.

Assumption 2.1.6 implies that if all products' prices increase by the same amount, the demand for each product will decrease. Assumption 2.1.7 implies that a price increase by
any one of the products results in a decrease of total sales in the market. To see that
Assumption 2.1.5 is more general than Assumption 2.1.6, Assumption 2.1.6 is equivalent to
setting $H = I$, the identity matrix, in part (c) of Definition 2.1.1. Assumption 2.1.5 is more
general than Assumption 2.1.7 because if a matrix is an $M$-matrix, its transpose is also an
$M$-matrix.

In the rest of this section, we show that (i) Assumption 2.1.5 solves the uniqueness prob-
lem, i.e., for any $p_R$, the solution to $d_{-\mathcal{R}}(p_{-\mathcal{R}}, p_R) = 0$ is unique; (ii) Assumption 2.1.5 solves
the consistency problem, i.e., demand functions $d_{j}(p_R)$, $j \in \mathcal{R}$ obtained by our demand
substitution rule (2.10) satisfy Assumptions 2.1.4 and 2.1.5; (iii) under Assumption 2.1.5,
equation (2.6) holds. Therefore, $d_{j}(p_R) = d_{j}(p^*_R(p_R), p_R)$, $j \in \mathcal{R}$, are the only demand
functions that satisfy Assumption 2.1.2; (iv) Assumption 2.1.5 is not only a sufficient condition
for (i), (ii) and (iii) to hold, but also a necessary condition for (i) and (ii) to be true.

Consider the linear demand model,

$$D = b - Ap,$$  \hspace{1cm} (2.14)

where the constant vector $b$ is the expected demand if the prices of all the products are set
to zero. Therefore, $b$ must be positive. It is easy to see that if the linear demand model satisfies
Assumption 2.1.5, it also satisfies Assumption 2.1.4. Notice that if $A$ is an $M$-matrix, its
sub-matrix $A_{-\mathcal{R}}$ is also an $M$-matrix. We write $A = \begin{pmatrix} A_{-\mathcal{R}} & A_2 \\ A_3 & A_\mathcal{R} \end{pmatrix}$. Therefore, for any
fixed $p_R \in \mathbb{R}^{m-m}$, $p^*_R = A_{-\mathcal{R}}^{-1}(b - A_2 * p_R) \geq 0$, and hence for the linear demand model, the
solution to $d_{-\mathcal{R}}(p_{-\mathcal{R}}, p_R) = 0$ is unique for any fixed $p_R$.

Generally, a system of nonlinear equations can have multiple or even positive dimensional
solutions. Interestingly, it has been shown in Gale and Nikaido[11] that for any nonlinear
functions that satisfy Assumption 2.1.5, there is at most one solution to $d_{-\mathcal{R}}(p_{-\mathcal{R}}, p_R) = 0$.
For completeness, in what follows we introduce a method to address both the uniqueness
problem and the consistency problem.

Consider the region $F$. For any fixed $p_R \in \prod_{\mathcal{R}} F$, define $F_{-\mathcal{R}} = \{p_{-\mathcal{R}} \in \mathbb{R}^m_+ | (p_{-\mathcal{R}}, p_R) \in F \}$. Let $\mathcal{R}(2) = \{2, \ldots, m\}$, we know that for any $p_{\mathcal{R}(2)} \in \prod_{\mathcal{R}(2)} F_{-\mathcal{R}}$, the projection of
$F_{-\mathcal{R}}$ onto its last $m-1$ variables, Assumption 2.1.4 ensures that there exists $p_1$ such that
\[ d_1(p_1, p_{R(2)}, p_R) = 0 \] because \((p_{R(2)}, p_R) \in \prod_{R(2) \cup R} F\). Notice that \(d_1(p_1, p_{R(2)}, p_R) = 0\) can have only one solution because \(\frac{\partial d_1(p_1)}{\partial p_1} < 0\) from Assumption 2.1.1. Therefore, \(d_1(p_1, p_{R(2)}, p_R) = 0\) defines a function from \(\prod_{R(2)} F - R\) to \(R_+\) by

\[ p_1 = p_1(p_{R(2)}). \tag{2.15} \]

After submitting this function into \(d_j(p_{R}, p_R)\), we get

\[ d_j^{(2)}(p_{R(2)}, p_R) = d_j(p_1(p_{R(2)}), p_R), \quad j \in R^{(2)}. \tag{2.16} \]

We claim that \(d_j^{(2)}(p_{R(2)}, p_R)\) has following properties.

**Property 2.1.1.** If \(p^*_{-R} \in F_{-R}\) is a solution to \(d_{-R}(p_{-R}, p_R) = 0\), then \(d_{R(2)}^{(2)}(p^*_{R(2)}, p_R) = 0\).

**Property 2.1.2.** For any fixed \(p_{R(j)} \in \prod_{R(j)} F_{-R}\) with \(R(j) = \{j, j+1, \ldots, m\}\), there exists \(p_{R(2) \setminus R(j)}^*\) such that \((p_{R(2) \setminus R(j)}^*, p_{R(j)}) \in \prod_{R(2)} F_{-R}\) and \(d_{R(2) \setminus R(j)}^{(2)}(p_{R(2) \setminus R(j)}^*, p_{R(j)}, p_R) = 0\).

**Property 2.1.3.**

\[
\begin{pmatrix}
\frac{\partial d_1^{(2)}(p)}{\partial p_1} & \cdots & \frac{\partial d_n^{(2)}(p)}{\partial p_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial d_1^{(2)}(p)}{\partial p_1} & \cdots & \frac{\partial d_n^{(2)}(p)}{\partial p_n}
\end{pmatrix}
\]

is an \(M\)-matrix.

**Proof.** Property 2.1.1 follows from the definition of \(d_{R(2)}^{(2)}(p_{R(2)}, p_R)\). If \(d_{-R}(p^*_{-R}, p_R) = 0\), then \(d_{R(2)}^{(2)}(p^*_{R(2)}, p_R) = d_{R(2)}^{(2)}(p_1^*(p_{R(2)}), p_{R(2)}, p_R) = 0\). Property 2.1.2 follows from Assumption 2.1.4 and Property 2.1.1. Next, we prove Property 2.1.3. From the definition of \(p_1(p_{R(2)})\), we must have \(d_1(p_1(p_{R(2)}), p_{R(2)}, p_R) = 0\). Implicit function theorem tells us that

\[ \frac{\partial p_1(p_{R(2)})}{\partial p_i} = -\frac{\partial d_1(p_1)}{\partial p_i}, \quad i \in R^{(2)}. \tag{2.17} \]

Therefore, we get
where the last inequality follows from part (a) of Definition 2.1.1. For \( i, j \in \mathbb{R}^{(2)} \) and \( j \neq i \),

\[
\frac{\partial d_i^{(2)}(p_{\mathbb{R}^{(2)}}, p_{\mathbb{R}})}{\partial p_j} = \frac{\partial d_i(p)}{\partial p_1} \frac{\partial p_1(p_{\mathbb{R}^{(2)}}, p_{\mathbb{R}})}{\partial p_j} + \frac{\partial d_i(p)}{\partial p_j} \geq 0. \tag{2.18}
\]

Notice that if a matrix is an \( M \)-matrix, its principle sub-matrix is also an \( M \)-matrix. Therefore, part (c) of Definition 2.1.1 implies that there are positive constants \( \lambda_i \), \( i = 1, \ldots, m \) such that

\[
-\sum_{j=1}^m \lambda_j \frac{\partial d_i(p)}{\partial p_j} > 0, \tag{2.19}
\]

for any \( i = 1, \ldots, m \). Therefore, for any \( i = 2, \ldots, m \), we have

\[
-\sum_{j=2}^m \frac{\lambda_j}{\lambda_1} \frac{\partial d_i^{(2)}(p_{\mathbb{R}^{(2)}}, p_{\mathbb{R}})}{\partial p_j} = \frac{\partial d_i(p)}{\partial p_1} \sum_{j=2}^m \frac{\lambda_j}{\lambda_1} \frac{\partial d_i(p)}{\partial p_j} - \sum_{j=2}^m \frac{\lambda_j}{\lambda_1} \frac{\partial d_i(p)}{\partial p_j} > 0, \tag{2.20}
\]

where the equality follows from (2.18), and the inequalities follow from (2.19). Property 2.1.3 holds. \( \square \)

The following lemma reveals an interesting relationship between \( d_{-\mathbb{R}}(p_{-\mathbb{R}}, p_{\mathbb{R}}) \) and \( d_{-\mathbb{R}^{(2)}}^{(2)}(p_{-\mathbb{R}^{(2)}}, p_{\mathbb{R}}) \).
Lemma 2.1.1. For any fixed \( p_R \in \prod_{\mathbb{R}} F \), denote \( F_{R(2)} = \{ p_{R(2)} \in \prod_{\mathbb{R}} F_{-R} | d_{\mathbb{R}(2)}^{(2)}(p_{R(2)}, p_R) \geq 0 \} \) then \( F_{R(2)} = \prod_{\mathbb{R}(2)} F_{-R} \).

Proof. " \( \subseteq \) " follows from definition of \( F_{R(2)} \). Now we prove " \( \supseteq \) " . For any \( p_{R(2)} \in \prod_{\mathbb{R}(2)} F_{-R} \), by the definition of projection, there exists \( p^*_1 \) such that \( (p^*_1, p_{R(2)}) \in F_{-R} \). Therefore, we have \( d_1(p^*_1, p_{R(2)}, p_R) \geq 0 \). Let \( p_1 = p_1(p_{R(2)}) \) be defined by equation (2.15). Then \( d_1(p_1(p_{R(2)}), p_{R(2)}, p_R) = 0 \). From Assumption 2.1.5 we know \( d_1(p) \) strictly decreases in \( p_1 \), and hence we must have \( p_1(p_{R(2)}) \geq p^*_1 \). Assumption 2.1.5 also tells us that \( d_{\mathbb{R}(2)}(p) \) increases in \( p_1 \). Therefore, we have \( d_{\mathbb{R}(2)}(p_{R(2)}, p_R) = d_{\mathbb{R}(2)}(p_1(p_{R(2)}), p_{R(2)}, p_R) \geq d_{\mathbb{R}(2)}(p^*_1, p_{R(2)}, p_R) \geq 0 \). This implies that \( p_{R(2)} \in F_{R(2)} \) and consequently \( \prod_{\mathbb{R}(2)} F_{-R} \subseteq F_{R(2)} \). Hence, \( F_{R(2)} = \prod_{\mathbb{R}(2)} F_{-R} \). \( \square \)

We can apply the same method to obtain \( d_{\mathbb{R}(j+1)}^{(j+1)}(p_{R(j+1)}, p_R) \) by submitting \( p_j = p_j(p_{R(j+1)}) \) (defined like (2.15)) into \( d_{\mathbb{R}(j+1)}^{(j)}(p_j, p_{R(j+1)}, p_R) \), i.e., \( d_{\mathbb{R}(j+1)}^{(j+1)}(p_{R(j+1)}, p_R) = d_{\mathbb{R}(j)}^{(j)}(p_j(p_j(p_{R(j+1)}), p_{R(j+1)}) \), \( p_{R(j+1)}, p_R) \), here \( \mathbb{R}(j) = \{ j, j + 1, \ldots, m \} \), \( j = 2, \ldots, m \). And by induction, we can show that \( d_{\mathbb{R}(j)}^{(j)}(p_{R(j)}, p_R) \) has the same properties that \( d_{\mathbb{R}(2)}^{(2)}(p_{R(2)}, p_R) \) has. Therefore, Lemma 4.1 implies that \( F_{\mathbb{R}(j+1)} = \prod_{\mathbb{R}(j+1)} F_{\mathbb{R}(j)} \), for \( F_{\mathbb{R}(j)} = \{ p_{\mathbb{R}(j)} \in \prod_{\mathbb{R}(j)} F_{\mathbb{R}(j-1)} | d_{\mathbb{R}(j)}^{(j)}(p_{\mathbb{R}(j)}, p_R) \geq 0 \} \) and \( j = 2, \ldots, m \) with \( F_{\mathbb{R}(1)} = F_{-R} \).

We show uniqueness and consistency properties in the following theorem.

Theorem 2.1.3. Consider any demand model that satisfies Assumptions 2.1.4 and 2.1.5.

(a) (uniqueness) Given any fixed \( p_R \in \prod_{\mathbb{R}} F \), the solution of \( d_{-R}(p_{-R}, p_R) = 0 \) is unique. We denote it by \( p^*_{-R} \) (or \( p^*_{-R}(p_R) \) since it is uniquely determined by \( p_R \)).

(b) (consistency) The demand functions obtained from the demand substitution rule, \( d_{\mathbb{R}}^{(j)}(p_R) = d_{\mathbb{R}}^{(j)}(p^*_{-R}(p_R), p_R), j \in \mathbb{R}, \) satisfy Assumptions 2.1.4 and 2.1.5.

Proof. (a) We prove by induction starting from the last element in the vector \( p^*_{-R} \). Notice that \( d_{\mathbb{R}(m)}^{(m)}(p_{R(m)}, p_R) \) consists of a single function with a single variable \( p_m \). From Property 2.1.1, we know that for any \( p^*_{-R} \) such that \( d_{-R}(p^*_{-R}, p_R) = 0 \), \( d_{\mathbb{R}(m)}^{(m)}(p^*_{R(m)}, p_R) = 0 \). From Property 2.1.3 we know \( d_{\mathbb{R}(m)}^{(m)}(p_{R(m)}, p_R) = 0 \) can have only one solution because \( \frac{\partial d_{\mathbb{R}(m)}^{(m)}(p_{R(m)}, p_R)}{\partial p_m} < 0 \). Therefore, in the set \( \{ p^*_{-R} \in R^m | d_{-R}(p^*_{-R}, p_R) = 0 \} \), the \( m \)th component, \( p^*_m \), is unique.
We now apply the same idea to \( d_{m-1}^{(m-1)}(p_{m-1}, p^*_m, p_\mathcal{R}) = 0 \) to obtain that the solution to this equation is unique and hence Property 2.1.1 implies that the \((m - 1)th\) component in the set \( \{ p^*_\mathcal{R} \in \mathbb{R}^m \mid d_{-\mathcal{R}}(p^*_\mathcal{R}, p_\mathcal{R}) = 0 \} \), \( p^*_{m-1} \), is unique. Using induction, we have that the \( kth \) component, \( p^*_k \), is unique for \( k = m - 2, \ldots, 1 \). Therefore, there is only one \( p^*_\mathcal{R} \) such that \( d_{-\mathcal{R}}(p_{\mathcal{R}(-)}, p_\mathcal{R}) = 0 \).

(b) Define \( \mathcal{R}(j) = \{ j, j+1, \ldots, n \} \), \( j = 2, \ldots, m + 1 \), and \( d_{j}^{(2)}(p_\mathcal{R}(2)) = d_j(p_1(p_\mathcal{R}(2)), p_\mathcal{R}(2)), \) \( j \in \mathcal{R}(2) \), here \( p_1(p_\mathcal{R}(2)) \) is the unique solution to \( d_1(p_1, p_\mathcal{R}(2)) = 0 \). Then, applying exactly the same method as the one used for proving Properties 2.1.1, 2.1.2 and 2.1.3, we can show that \( d_{j}^{(2)}(p_\mathcal{R}(2)), \) \( j \in \mathcal{R}(2) \) satisfies Assumptions 2.1.4 and 2.1.5. By induction, we know \( d_{j}^{(m+1)}(p_\mathcal{R}(m+1)), \) \( j \in \mathcal{R}(m+1) \) satisfies Assumptions 2.1.4 and 2.1.5.

The last step is to show that removing product one by one from the list of products is the same as removing a group simultaneously. Indeed, the property that the solution to \( d_{-\mathcal{R}}(p_{\mathcal{R}(-)}, p_\mathcal{R}) = 0 \) is unique for any fixed \( p_\mathcal{R} \in \prod_{\mathcal{R}} F \) implies that \( d_{j}^\mathcal{R}(p_\mathcal{R}) = d_j(p^*_\mathcal{R}(p_\mathcal{R}), p_\mathcal{R}) = d_{j}^{(m+1)}(p_\mathcal{R}(m+1)), \) \( j \in \mathcal{R}(m+1) \). This completes the proof part of (b).

The Theorem thus implies that Assumption 2.1.5 completely addresses the uniqueness and consistency problem. Next, we show that under Assumption 2.1.5, equation (2.6) holds. For this purpose, we need some technical results that provide a different characterization of of Assumption 2.1.4.

**Theorem 2.1.4.** For any fixed \( p_\mathcal{R} \in \prod_{\mathcal{R}} F \), there exists only finite \( p^*_\mathcal{R} \in \mathbb{R}^m_+ \) such that \( (p^*_\mathcal{R}, p_\mathcal{R}) \in F \) and \( d_{-\mathcal{R}}(p^*_\mathcal{R}, p_\mathcal{R}) = 0 \) if and only if \( F_{-\mathcal{R}} \) is nonempty and bounded.

**Proof.** We first prove “\( \Rightarrow \)”. Since \( d_1(p) \) strictly decreases in \( p_1 \), for any \( (p_1, p_\mathcal{R}(2)) \in F_{-\mathcal{R}} \), we must have \( p_1 \leq p_1(p_\mathcal{R}(2)) \) to make \( d_1(p_1, p_\mathcal{R}(2), p_\mathcal{R}) = 0 \). And since \( d_1(p_1(p_\mathcal{R}(2)), p_\mathcal{R}(2), p_\mathcal{R}) = 0 \), implicit function theorem implies that

\[
\frac{\partial p_1(p_\mathcal{R}(2))}{\partial p_j} = -\frac{\partial d_1(p)}{\partial p_1} \geq 0, \quad j \in \mathcal{R}(2).
\] (2.21)

Since \( p_1(p_\mathcal{R}(2)) \) increases in \( p_\mathcal{R}(2) \), \( F_{-\mathcal{R}} \) is bounded if \( \prod_{\mathcal{R}(2)} F_{-\mathcal{R}} \) is bounded. Lemma 4.1 shows that \( \prod_{\mathcal{R}(2)} F_{-\mathcal{R}} = F_{\mathcal{R}(2)} \). Therefore, the original problem (proving that \( F_{-\mathcal{R}} \) is bounded) is reduced to showing that \( F_{\mathcal{R}(2)} \) is bounded. However, we have shown that
$d^{(2)}_{R(1)}(p^{(2)}_R, p_R)$ has the same properties as of $d_{-R}(p_{-R}, p_R)$, and so does $d^{(j)}_{R(j)}(p^{(j)}_R, p_R), j = 3, \ldots, m$. Therefore, by induction, this problem is reduced to showing that $F_{-R(m)}$ is bounded. Notice that $d^{(m)}_{R(m)}(p^{(m)}_R, p_R)$ consists of only a single function with a single variable $p_m$.

In the proof of Theorem 2.1.3, we have shown that there exists a unique $p^*_m$ such that $d^{(m)}_{R(m)}(p^*_m, p_R) = 0$. Since $d^{(m)}_{R(m)}(p_m, p_R)$ strictly decreases in $p_m$, we must have $F_{-R(m)} \subseteq [0, p^*_m]$, which is bounded. And consequently, $F_{-R}$ is bounded.

Now we prove "\(\leq\)". Since $F_{-R}$ is nonempty, there exists a $p^{0}_{-R} \in F_{-R}$. If $d_{-R}(p^{0}_{-R}, p_R) \neq 0$, we construct a sequence starting from $p^{0}_{-R}$ in the following way. We move from $p^{0}_{-R}$ to $p^{1}_{-R}$ by keeping all components of $p^{0}_{-R}$ unchanged except increasing $(p^{0}_{-R})_1$ to $(p^{1}_{-R})_1$ such that $d_1(p^{1}_{-R}, p_R) = 0$. From Assumption 2.1.1, we know that increasing $(p^{0}_{-R})_1$ will increase the value of $d_i(p), i = 2, \ldots, n$. Therefore, we are staying inside $F_{-R}$ before violating the nonnegativity constraint of $d_1(p)$. Since $F_{-R}$ is bounded, there must be a finite $(p^{1}_{-R})_1$ such that $d_1(p^{1}_{-R}, p_R) = 0$. Applying the same technique, we obtain $p^{i}_{-R}, i = 2, \ldots, m$ such that $d_i(p^{i}_{-R}, p_R) = 0, i = 2, \ldots, m$. If we don’t have $d_{-R}(p^{m}_{-R}, p_R) = 0$ after one round, we start all over again by increasing $(p^{m}_{-R})_1$ to get $p^{m+1}_{R}$ such that $d_1(p^{m+1}_{-R}, p_R) = 0$. If this algorithm stops after finite steps with a $p^*_R$ such that $d_{-R}(p^*_R, p_R) = 0$, we achieve our goal. Otherwise, we have a sequence $\{p^i_{-R}\}_{i=1}^\infty \in F_{-R}$ such that $d_j(p^{k+i}_{-R}, p_R) = 0$ for $j = 1, \ldots, m$ and $k = 0, 1, \ldots$. Notice that $\{\{p^i_{-R}\}_j\}_{i=1}^\infty$ is a nondecreasing sequence for any $j = 1, \ldots, m$. It must converge to some point $(p^*_R)_j$. Therefore, $\{p^i_{-R}\}_{i=1}^\infty$ must converge to $p^*_R$. Since $F_{-R}$ is bounded and closed (the closeness of $F_{-R}$ follows from the continuity of $d_i(p), i = 1, \ldots, m$), we must have $p^*_R \in F_{-R}$. Moreover, we know that if a sequence converges to a point, its subsequence must converge to the same point. Therefore, by continuity we have $d_j(p^*_R, p_R) = \lim_{k \to \infty} d_j(p^{k+i}_{-R}, p_R) = 0$ for any $j = 1, \ldots, m$. We have $d_{-R}(p^*_R, p_R) = 0$.

Theorems 2.1.3 and 2.1.4 motivate the following important property of type II demand model that plays a key role in our proof of equation (2.6).

**Theorem 2.1.5. (bound)** For any $p_{-R} \in F_{-R}$, we have $p_{-R} \leq p^*_R(p_R)$, where $p^*_R(p_R)$ is
the unique solution of $d_R(p_R, p_R) = 0$ (defined in Theorem 2.1.3).

Proof. From Theorem 2.1.4, we know that for any demand model satisfying Assumption 2.1.4, $F_{-R}$ is bounded. Therefore, for any $p_{-R} \in F_{-R}$, if $p_{-R} \neq p^*_{-R}(p_R)$, we can construct the same nondecreasing sequence starting from $p_{-R}$ as we did in the proof of Theorem 2.1.4. The proof of this Theorem tells us that this sequence must converge to the solution to $d_{-R}(p_{-R}, p_R) = 0$. From Theorem 2.1.3, we know $p^*_{-R}(p_R)$ is the only solution to this system of equations. Therefore, this sequence must converge to $p^*_{-R}(p_R)$. Since this is a nondecreasing sequence, we must have $p_{-R} \leq p^*_{-R}(p_R)$. 

Theorem thus implies that for any fixed $p_R$, the unique vector $p^*_{-R}(p_R)$ is an upper bound (component by component) on any price vector $p_{-R} \in F_{-R}$. Before we prove equation (2.6), we first introduce our last type of demand model.

2.1.3 Demand model type III

This demand model is a combination of type I and type II demand models. It is characterized by the following assumption.

Assumption 2.1.8. $d_i(p)$ is well-defined at $p_j = \infty$, for $i, j = 1, \ldots, n$. Part of these demand functions satisfies Assumptions 2.1.1 and 2.1.3, the other part of these demand functions satisfies Assumptions 2.1.4 and 2.1.5.

Notice that in type I demand model, for any fixed $p_R$ (can be infinity), $p^*_{-R}(p_R) = \infty$ has exactly the same properties (uniqueness and bound) as the $p^*_{-R}(p_R)$ defined in Theorem 2.1.3 for type II demand model. Therefore, we use the following theorem to summarize our main results for all types of demand models.

Theorem 2.1.6. For any fixed $p_R$, the solution of $d_{-R}(p_{-R}, p_R) = 0$ is unique. We denote this solution by $p^*_{-R}(p_R)$ (it is infinity for type I demand model, defined in Theorem 2.1.3 for type II demand model and a mixed solution consisting of infinite and finite components for type III demand model). Then, for any $p_{-R} \in F_{-R}$, we have $p_{-R} \leq p^*_{-R}(p_R)$.

We are now ready to present our main result.
Theorem 2.1.7. Consider the three types of demand models defined in this section. For any \( p_\mathbb{R} \), we have

\[
\max_{p_{-\mathbb{R}} \in \Gamma_{-\mathbb{R}}} d_j(p_{-\mathbb{R}}, p_\mathbb{R}) = \min_{p_{-\mathbb{R}} \in S_{-\mathbb{R}}} d_j(p_{-\mathbb{R}}, p_\mathbb{R}), \quad j \in \mathbb{R}. \tag{2.22}
\]

Therefore, removing all the \( m \) products in the set \(-\mathbb{R}\) from the market creates a demand function for the remaining \( n - m \) products that follows

\[
d_j^\mathbb{R}(p_\mathbb{R}) = d_j(p^*_{-\mathbb{R}}(p_\mathbb{R}), p_\mathbb{R}), \quad j \in \mathbb{R}, \tag{2.23}
\]

where \( p^*_{-\mathbb{R}}(p_\mathbb{R}) \) is defined in Theorem 2.1.6, and this is the ONLY demand function that satisfies Assumption 2.1.2.

**Proof.** From Theorem 2.1.6, we know that for any \( p_\mathbb{R} \), \( S_{-\mathbb{R}} \) consists of only one element, \( p^*_{-\mathbb{R}}(p_\mathbb{R}) \). Therefore, \( \min_{p_{-\mathbb{R}} \in S_{-\mathbb{R}}} d_j(p_{-\mathbb{R}}, p_\mathbb{R}) = d_j(p^*_{-\mathbb{R}}(p_\mathbb{R}), p_\mathbb{R}) \). We first prove \( \max_{p_{-\mathbb{R}} \in \Gamma_{-\mathbb{R}}} d_j(p_{-\mathbb{R}}, p_\mathbb{R}) \geq d_j(p^*_{-\mathbb{R}}(p_\mathbb{R}), p_\mathbb{R}) \). In the proof of Theorem 2.1.4, we know that there is always a sequence \( \{p^j_{-\mathbb{R}}\}_{k=1}^\infty \in \Gamma_{-\mathbb{R}} \) such that \( \lim_{k \to \infty} p^j_{-\mathbb{R}} = p^*_{-\mathbb{R}}(p_\mathbb{R}) \) (this property certainly also holds if \( p^*_{-\mathbb{R}}(p_\mathbb{R}) = \infty \)). Since \( d_j(p) \) is continuous,

\[
d_j(p^*_{-\mathbb{R}}(p_\mathbb{R}), p_\mathbb{R}) = \lim_{k \to \infty} d_j(p^j_{-\mathbb{R}}, p_\mathbb{R}) \leq \lim_{k \to \infty} \max_{p_{-\mathbb{R}} \in \Gamma_{-\mathbb{R}}} d_j(p_{-\mathbb{R}}, p_\mathbb{R}) = \max_{p_{-\mathbb{R}} \in \Gamma_{-\mathbb{R}}} d_j(p_{-\mathbb{R}}, p_\mathbb{R}). \tag{2.24}
\]

From Theorem 2.1.6, we know that for any \( p_{-\mathbb{R}} \in F_{-\mathbb{R}}, \ p_{-\mathbb{R}} \leq p^*_{-\mathbb{R}}(p_\mathbb{R}) \). And since for any \( j \in \mathbb{R}, d_j(p_{-\mathbb{R}}, p_\mathbb{R}) \) increases in \( p_{-\mathbb{R}} \),

\[
\max_{p_{-\mathbb{R}} \in \Gamma_{-\mathbb{R}}} d_j(p_{-\mathbb{R}}, p_\mathbb{R}) \leq d_j(p^*_{-\mathbb{R}}(p_\mathbb{R}), p_\mathbb{R}), \quad j \in \mathbb{R}. \tag{2.25}
\]

Therefore, we have \( \max_{p_{-\mathbb{R}} \in \Gamma_{-\mathbb{R}}} d_j(p_{-\mathbb{R}}, p_\mathbb{R}) = d_j(p^*_{-\mathbb{R}}(p_\mathbb{R}), p_\mathbb{R}) = \min_{p_{-\mathbb{R}} \in S_{-\mathbb{R}}} d_j(p_{-\mathbb{R}}, p_\mathbb{R}) \). Since the lower bound and upper bound of \( d_j^\mathbb{R}(p_\mathbb{R}) \) match at \( d_j(p^*_{-\mathbb{R}}(p_\mathbb{R}), p_\mathbb{R}) \), equality (2.23) holds. Moreover, this is the only demand function that satisfies Assumption 2.1.2. \( \square \)

Theorem 2.1.7 tells us that if the demand model satisfies Assumptions 2.1.4 and 2.1.5 (which is a subset of Assumption 2.1.1), then the new demand model is obtained by setting the price of the removed products such that their demand is zero and this is the only possible demand model.
We now prove that Assumption 2.1.5 is not only sufficient but also necessary. That is, we show that if the demand model satisfies Assumptions 2.1.1 and 2.1.4, and the new demand model is obtained by setting the prices of the removed products so their demand is zero, then the demand model must satisfy Assumption 2.1.5.

**Theorem 2.1.8.** If a demand model, \( D = \{d_1(p), \ldots, d_n(p)\} \), satisfies Assumptions 2.1.1 and 2.1.4 and \( d_j^{(R)}(p_R) = d_j(p^*_R(p_R), p^*_R) \) satisfies \( \frac{\partial d_j^{(R)}(p_R)}{\partial p_j} < 0 \) for \( j \in \mathbb{R} \), then \(-J\) is an \( M\)-matrix.

**Proof.** We prove the theorem by showing that the leading principle minors of \(-J\) are positive.

\[
\frac{\partial d_m^{(R)}(p_R)}{\partial p_{m+1}} = \frac{\partial d_{m+1}(p^*_R(p_R), p_R)}{\partial p_{m+1}} = \frac{\partial d_{m+1}(p^*_R(p_R), p_R)}{\partial p_R} \frac{\partial p^*_R(p_R)}{\partial p_{m+1}} + \frac{\partial d_{m+1}(p)}{\partial p_{m+1}}
\]

\[
= -\frac{\partial d_{m+1}(p)}{\partial p_R} \frac{\partial d_{m+1}(p)}{\partial p_{m+1}} + \frac{\partial d_{m+1}(p)}{\partial p_{m+1}}
\]

\[
= \det \left( \begin{array}{cc}
\frac{\partial d_{m+1}(p)}{\partial p_R} & -\frac{\partial d_{m+1}(p)}{\partial p_{m+1}} \\
-\frac{\partial d_{m+1}(p)}{\partial p_R} & -\frac{\partial d_{m+1}(p)}{\partial p_{m+1}}
\end{array} \right),
\]

where the third equality follows from (2.29). Since \( \frac{\partial d_m^{(R)}(p_R)}{\partial p_{m+1}} < 0 \), the above equality implies that the \((m+1)th\) leading principle minor of \(-J\) must have the same sign as the \(m\)th leading principle minor of \(-J\). Since \( d_1(p) \) satisfies Assumption 2.1.1, the first leading principle minor of \(-J\) must be positive. Since \( m \) can be any number from 1 to \( n - 1 \), all leading principle minors of \(-J\) are positive. Therefore, from part (b) of Definition 2.1.1, we know \(-J\) is a \( M\)-matrix. \( \square \)

We conclude that Theorem 2.1.7 and 2.1.8 imply that Assumption 2.1.5 is a necessary and sufficient requirement for Type II demand model.

Theorem 2.1.7 also motivates the following interesting observations.

**Observation 2.1.1.** (order independent) The demand for the remaining \( n - m \) products doesn't depend on the order in which the \( m \) products are removed.
For example, if \( m = 2 \), demand for the remaining \( n - 2 \) products is independent of whether we remove product 1 first, product 2 first, or perhaps both are removed simultaneously. The observation is valid because the expression \( p_{-R} \) as function of \( p_R \) always satisfies \( d_{-R}(p_{-R}, p_{-R}) = 0 \), and from Theorem 2.1.6 we know that the solution to this system of equations is unique. Observation 5.1 ensures that the final price-demand model doesn't depend on its forming process.

Finally, Theorem 2.1.7 also implies,

**Observation 2.1.2.** Given any specific demand model described in this section (Linear model, Attraction models, Logit model or CES model) for the original \( n \) products, the demand model for the remaining \( n - m \) products remains the same type.

### 2.2 Demand Sensitivity and System Demand

In this section we analyze the sensitivity of product demand to price before and after removing some products from the market, and also study how system demand depends on the number of products. Given product \( i \), we characterize the sensitivity of product \( i \) demand to its own price by the quantity

\[
\frac{d_i(p_i, p_{-i}) - d_i(p_i + h, p_{-i})}{d_i(p_i, p_{-i})}.
\]

This quantity measures the percentage of product \( i \)'s customers that will be lost if product \( i \)'s price is increased by \( h \) units.

We need the following definition.

**Definition 2.2.1.** Suppose \( X \in \mathbb{R} \) and \( T \in \mathbb{R}^{n-1} \). A function \( f: X \times T \rightarrow \mathbb{R} \) has increasing differences in \((x, t)\) if for all \( x' \geq x \) and \( t' \geq t \),

\[
f(x', t') - f(x, t') \geq f(x', t) - f(x, t).
\]

(2.27)

It can be easily verified that for the Linear model, the Attraction models, the Logit model and the CES model, \( \log d_i(p) \) has increasing differences in \((p_i, p_{-i})\). Thus, in these models,
for each product $i$, if $p_i' \geq p_i$ and $p_{-i}' \geq p_{-i}$, we have

$$\log d_i(p_i', p_{-i}') - \log d_i(p_i, p_{-i}') \geq \log d_i(p_i', p_{-i}) - \log d_i(p_i, p_{-i}),$$

(2.28)

and hence

$$\frac{d_i(p_i, p_{-i}) - d_i(p_i', p_{-i}')} {d_i(p_i, p_{-i})} \leq \frac{d_i(p_i, p_{-i}) - d_i(p_i', p_{-i})} {d_i(p_i, p_{-i})},$$

which implies that demand sensitivity for a specific product is not increasing with the price of other products.

The following lemma is useful in this section.

**Lemma 2.2.1.** For any $k \in -\mathbb{R}$ and $j \in \mathbb{R}$, $p_k^*(p_R)$ increases in $p_j$.

**Proof.** This property holds for Type I demand model because in this case $p_k^*(p_R) = \infty$. For demand model type II, Since $d_{-R}(p^*_R(p_R), p_R) = 0$, implicit function theorem tells us that

$$\frac{\partial p_k^*(p_R)} {\partial p_j} = -\frac{\partial d_{-R}(p)} {\partial p_k} - \frac{1} {\partial d_{-R}(p)} \frac{\partial d_{-R}(p)} {\partial p_j},$$

(2.29)

where $\frac{\partial p_k^*(p_R)} {\partial p_j} = [\frac{\partial p_k^*(p_R)} {\partial p_j}]^T$, $\frac{\partial d_{-R}(p)} {\partial p_k}$ and $\frac{\partial d_{-R}(p)} {\partial p_j} = [\frac{\partial d_{-R}(p)} {\partial p_j}]^T$, $i, k \in -\mathbb{R}$. Since $-\frac{\partial d_{-R}(p)} {\partial p_k}$ is a M-matrix, part (d) of Definition 2.1.1 implies that its inverse is nonnegative. From Assumption 2.1.5, we know $\frac{\partial d_{-R}(p)} {\partial p_j}$ is nonnegative. Therefore, we must have $\frac{\partial p_k^*(p_R)} {\partial p_j} \geq 0$.

This property must also hold for Type III demand model because it holds for both Type I and II demand models.

The next proposition reveals the impact of removing products on the sensitivity of product demand to its own price.

**Proposition 2.2.1.** Given any demand model for the original $n$ products that satisfies (2.28), the demand for each remaining product is less sensitive to its own price than before.

**Proof.** For any remaining product $j \in \mathbb{R}$,
\[
\log d_j^R (p_j, p_{R\setminus\{j\}}) - \log d_j^R (p_j + h, p_{R\setminus\{j\}})
= \log d_j (p_{-R}^* (p_j, p_{R\setminus\{j\}}), p_j, p_{R\setminus\{j\}}) - \log d_j (p_{-R}^* (p_j + h, p_{R\setminus\{j\}}), p_j + h, p_{R\setminus\{j\}})
\leq \log d_j (p_{-R}^* (p_j, p_{R\setminus\{j\}}), p_j, p_{R\setminus\{j\}}) - \log d_j (p_{-R}^* (p_j, p_{R\setminus\{j\}}), p_j + h, p_{R\setminus\{j\}})
\leq \log d_j (p_{-R}, p_j, p_{R\setminus\{j\}}) - \log d_j (p_{-R}, p_j + h, p_{R\setminus\{j\}}),
\]

where the first inequality follows from Lemma 2.2.1 and \( \log d_j (p) \) increases in \( p_{-R} \), the second inequality follows from Theorem 2.1.6 and \( \log d_j (p) \) has increasing differences in \((p_j, p_j)\).

Inequality (2.30) implies that for any remaining product \( j \in R \),

\[
\frac{d_j^R (p_j, p_{R\setminus\{j\}}) - d_j^R (p_j + h, p_{R\setminus\{j\}})}{d_j^R (p_j, p_{-j})} \leq \frac{d_j (p_j, p_{-j}) - d_j (p_j + h, p_{-j})}{d_j (p_j, p_{-j})}.
\]

Therefore, we have proved Proposition 2.2.1. \( \square \)

Next, we analyze the sensitivity of product demand to other products’ prices before and after removing some products from the market. In this case, we characterize the sensitivity of demand to other products’ prices by the quantity

\[
d_i (p_k + h, p_{-k}) - d_i (p_k, p_{-k}),
\]

for any \( k \neq i \). This quantity measures the (absolute) increase in demand for product \( i \) due to an increase in the price of product \( k \), \( k \neq i \), by \( h \) units.

It can be easily verified that for the Linear model, the Attraction models, the Logit model and the CES model, given any fixed \( p_i \), \( d_i (p) \) has increasing differences in \((p_k, p_{-\{k,i\}})\) for any \( k \neq i \). This implies that for each product \( i \), given any fixed \( p_i \), if \( p'_k \geq p_k \) and \( p'_{-\{k,i\}} \geq p_{-\{k,i\}} \), we have

\[
d_i (p_i, p'_k, p'_{-\{k,i\}}) - d_i (p_i, p_k, p_{-\{k,i\}}) \geq d_i (p_i, p'_k, p_{-\{k,i\}}) - d_i (p_i, p_k, p_{-\{k,i\}}),
\]

and therefore the sensitivity of product \( i \) demand to product \( k \) price, \( k \neq i \), increases with the price of all products but \( k \) and \( i \).
The following proposition tells us the impact of removing products on demand sensitivity to other products’ prices.

**Proposition 2.2.2.** *Given any demand model for the original n products that satisfies (2.32), the demand for each remaining product is more sensitive to the other products’ prices than before.*

*Proof.* For any remaining product $j \in \mathbb{R}$ and $k \in \mathbb{R}$,

\[
\begin{align*}
  d^R_j(p_k + h, p_{R\setminus\{k\}}) - d^R_j(p_k, p_{R\setminus\{k\}}) \\
  &= d_j(p^*_R(p_k + h, p_{R\setminus\{k\}}), p_k + h, p_{R\setminus\{k\}}) - d_j(p^*_R(p_k, p_{R\setminus\{k\}}), p_k, p_{R\setminus\{k\}}) \\
  &\geq d_j(p^*_R(p_k, p_{R\setminus\{k\}}), p_k + h, p_{R\setminus\{k\}}) - d_j(p^*_R(p_k, p_{R\setminus\{k\}}), p_k, p_{R\setminus\{k\}}) \\
  &\geq d_j(p_R, p_k + h, p_{R\setminus\{k\}}) - d_j(p_R, p_k, p_{R\setminus\{k\}}),
\end{align*}
\]

where the first inequality follows from Lemma 2.2.1 and $d_j(p)$ increases in $p_R$, the second inequality follows from Theorem 2.1.6 and (2.32). Therefore, we have proved Proposition 2.2.2. \(\square\)

Propositions 2.2.1 and 2.2.2 are intuitive. To see that, consider three products indexed by 1, 2 and 3, and remove product 1 from the market. Let’s analyze the impact of the price of products 2 and 3 on the demand for product 2. Since product 1 is not available anymore, customers who originally chose product 1, can either choose product 2 or 3. So, as we increase the price of product 2, some of the customers who would have switched to product 1, will stay with product 2. Thus, the demand for product 2 is less sensitive to the price of the product 2 than when there are three products in the market. Similarly, if the price of the product 3 increases, some customers who would have switched to product 1, will move to product 2. Hence, the demand for the product 2 is more sensitive to the price of product 3 than when there are three products in the market.

In the rest of this section, we study how system demand depends on the number of products. An empirical study reported in Iyengar and Lepper [14] shows that more products
don’t always lead to more system demand. Specifically, a study of supermarket shoppers reveals that demand is likely to be higher when consumers are offered a limited set of products.

Our model is not conclusive in this respect. Indeed, it is possible in our model that at some price vector, total demand for the remaining \( n - m \) products can be more than the total demand of the original \( n \) products. This is illustrated by following example.

**Example 2.2.1.** Consider a system with two retailers where \( b = (10, 10) \) and \( A = \begin{pmatrix} 1 & 0 \\ -10 & 1 \end{pmatrix} \). Notice that \( A \) is a \( M \)-matrix. If we set \( p_1 = p_2 = 1 \), the total demand of both products is 28. However, after product 1 is removed, the demand for product 2 is 109.

However, if the demand model satisfies Assumption 2.1.7, then this situation can not happen and system demand will decrease after some products are removed.

**Proposition 2.2.3.** For any demand model satisfying Assumption 2.1.7, the total demand of \( n \) products is always higher than the system demand obtained when \( m \) products are removed.

**Proof.** From any \( p_R \) and \( p_{-R} \in F_{-R} \), Theorem 2.1.6 implies that \( p_{-R} \leq p_{-R}^*(p_R) \). Then, Assumption 2.1.7 tells us that

\[
\sum_{i=1}^{n} d_i(p_{-R}, p_R) \geq \sum_{i=1}^{n} d_i(p_{-R}^*(p_R), p_R) = \sum_{i=m+1}^{n} d_i^R(p_R),
\]

where the equality follows from the facts that \( d_i(p_{-R}^*(p_R), p_R) = 0 \) for \( i \in -R \) and \( d_i^R(p_R) = d_i(p_{-R}^*(p_R), p_R) \) for \( i \in R \).

\[\square\]

### 2.3 Summary

In this chapter, I apply an axiomatic approach to characterize price-demand relationship after some products are removed from the market. This price-dependent demand substitution rule serves as an important building block to study joint pricing and inventory coordination as
well as retail competition. Also in this chapter, I provide insights on demand sensitivity to
price before and after removing some products from the market.
Chapter 3

Joint Pricing and Inventory Coordination

This chapter is organized as follows. In Section 3.1, we extend the price-dependent demand substitution rule to its stochastic counterpart. In Section 3.2, we apply our results to a joint pricing and inventory coordination model, and demonstrate the impact of our substitution rule on the optimal inventory and pricing strategy. Specifically, we compare two inventory-pricing models: one in which demand for a product disappear during stockout time and one where substitution occurs according to our demand substitution rule.

3.1 Stochastic Model

In this section, we study the impact of removing \( m \) of the products on customer demand for the remaining \( n - m \) products when demand is stochastic. We use customer choice theory to characterize the stochastic demand model. Similar to the deterministic case, we show the existence of a unique demand structure for the remaining \( n - m \) products in the stochastic demand model.

We assume that customer arrivals follow some renewal process, \( N(t) \), which is the number of customer arrivals by time \( t \). For each customer \( k, k = 1, \ldots, N(t) \), denote \( A_{ki} = \{ \text{the event that customer } k \text{ chooses product } i \text{ from the group of } n \text{ products} \} \) for \( i = 1, \ldots, n \), and \( A_{ko} = \{ \text{the event that customer } k \text{ doesn’t buy from the group of } n \text{ products} \} \). Therefore,
the demand for product $i$ by time $t$ is

$$Y_i = \sum_{k=1}^{N(t)} 1_{\{A_{ki}\}}, \quad i = 1, \ldots, n,$$

(3.1)

where $1_{\{A_{ki}\}}$ is the indicator function of event $A_{ki}$. We assume customer choices among the $n$ products are independent of each other and have the same distribution. Therefore, the probability of $A_{ki}$, $a_i := \Pr(A_{ki})$ must be independent of $k$. Evidently, $a_i$ depends on the price vector $p$. If product $i$'s price $p_i$ increases, its probability of being chosen decreases. Therefore, $a_i(p)$ decreases in $p_i$. If the price of any other product increases, the probability of product $i$ being chosen increases, hence $a_i(p)$ increases in $p_j$ for $j \neq i$. Therefore, $a_i(p)$ satisfies Assumptions 2.1.1.

Similarly to the deterministic demand models, we can divide the probability functions into type I probability model that satisfies Assumptions 2.1.1 and 2.1.3, type II probability model that satisfies Assumptions 2.1.4 and 2.1.5, and Type III demand model that satisfies Assumption 2.1.8. The following theorem is similar to Theorem 2.1.6, except that it applies to probability functions rather than demand functions.

**Theorem 3.1.1.** For any fixed $p_R$, the solution of $a_{-R}(p_{-R}, p_R) = 0$ is unique. We denote this solution by $p^*_{-R}(p_R)$ (it is infinity for type I probability model, defined as in Theorem 2.1.3 for type II probability model and a mixed solution consisting of infinite and finite components for type III probability model). Then, for any $p_{-R} \in F_{-R} = \{p_{-R} \in R^n_{+} | a_{-R}(p) \geq 0\}$, we have $p_{-R} \leq p^*_{-R}(p_R)$.

What is the impact of removing $m$ of the products on the probability function of each remaining product? Denote $a^R_{j}(p_R)$, $j \in R$ to be the new probability function of product $j$, $j \in R$, where $R$ is the remaining set of products. Similarly to Assumption 2.1.2, we make the following assumption.

**Assumption 3.1.1.**
(a) If a subset of products with positive probability are driven out of the market, the probability that each remaining product is chosen does not decrease.

(b) If all products \(i, \ i \in -\mathcal{R}\) are removed at some price vector \((p_{-\mathcal{R}}, p_{\mathcal{R}})\) such that \(a_i(p_{-\mathcal{R}}, p_{\mathcal{R}}) = 0\) for each \(i \in -\mathcal{R}\), the probability that each remaining product is chosen does not increase.

The second assumption says that if the probability that a customer is willing to buy from products 1, 2, \ldots, \(m\) is zero, removing these products will not increase the probability that each remaining product is chosen by that customer. We are ready to characterize the probability function \(a^\mathcal{R}_j(p_{\mathcal{R}}), \ j \in \mathcal{R}\) for all the remaining products in \(\mathcal{R}\).

**Theorem 3.1.2.** Removing all the \(m\) products in the set \(-\mathcal{R}\) from the market creates a probability function for the remaining \(n - m\) products that follows

\[
a^\mathcal{R}_j(p_{\mathcal{R}}) = a_j(p^*_{-\mathcal{R}}(p_{\mathcal{R}}), p_{\mathcal{R}}), \ j \in \mathcal{R},
\]

where \(p^*_{-\mathcal{R}}(p_{\mathcal{R}})\) is defined in Theorem 3.1.1. This is the ONLY probability function that satisfies Assumption 3.1.1.

The proof of this theorem is identical to the proof of Theorem 2.1.7. Therefore, for any remaining product \(j, \ j \in \mathcal{R}\), the new demand function is

\[
Y^\mathcal{R}_j = \sum_{k=1}^{N(t)} 1_{(A_{kj})}, \ j = m + 1, \ldots, n,
\]

with \(\Pr(A_{kj}) = a^\mathcal{R}_j(p_{\mathcal{R}})\) defined as equation (3.2). Since the probability functions have the same properties as that of the demand functions in Section 2.1, Observations 5.1 (Order independent) is also applicable to this stochastic model.

### 3.2 Comparisons Between Two Models

Our objective in this section is to apply the substitution rule developed earlier to a stochastic multi-product joint pricing and inventory model. In such a model, stockout is possible, and hence it is important to incorporate substitution in the analysis. To understand the impact of
substitution, we evaluate two models: one in which demand for a product disappears during stockout time and one where substitution occurs according to our demand substitution rule.

Consider a retailer who sells two substitutable products during a finite time horizon of length $T$. At the beginning of the horizon, the retailer decides how many units to order and at what price to sell each of the product. The time horizon is assumed to be short, so no adjustments of price or inventory are made during the period. The retailer’s objective is to choose prices and inventory levels for both products so as to maximize total expected profit.

Assume that customer arrivals follow a poisson process with arrival rate $\lambda$. During the time horizon, one or both products may stockout. Let $y_i$ be the inventory level of product $i$, $i = 1, 2$, at the beginning of the horizon and $S_{y_i}$ be the time product $i$ stocks out. This time is of course a random variable and may be greater than or equal to $T$ which implies no stockout for product $i$.

![Figure 3-1: product 1's demand](image)

This figure illustrates how the system evolves. Customers arrive at a rate of $\lambda$ and purchase one of the products or depart the system.

Before product 2 stocks out, each customer chooses product 1 with probability $a_1(p)$, product 2 with probability $a_2(p)$ and a no-buy option with probability $a_0(p)$. After product 2 stocks out, each customer chooses product 1 with probability $a_1^{\mathcal{R}}(p_1)$ and no-buy option with probability $a_0^{\mathcal{R}}(p_1)$ where $\mathcal{R} = \{1\}$. $a_1^{\mathcal{R}}(p_1)$ and $a_0^{\mathcal{R}}(p_1)$ follow the price-dependent demand substitution rule developed in Section 3.1.
By contrast, the traditional approach when optimizing pricing and inventory decisions is to assume that there is no stockout demand substitution, i.e., product 2’s customers are completely lost once the product stocks out. This implies that after product 2 stocks out, each customer will choose product 1 with probability $a_1(p)$ and a no-buy option with probability $a_0(p) + a_2(p)$.

This is the difference between the traditional pricing coordination model, referred to below as the no stockout substitution model, and our model where we incorporate substitution following our axiomatic approach. The following example shows that this difference can affect retailer’s strategy and the corresponding expected profit.

**Example 3.2.1.** Assume that the customer arrival rate $\lambda = 10$ during a time horizon $[0,1]$. Unit ordering cost for both products $c_1 = c_2 = $100. Let the demand distribution between different products follow the attraction model with probability functions $a_1(p) = \frac{1.5 \exp^{-0.032p_1}}{5 \times 10^{-4} + 1.5 \exp^{-0.032p_1} + 5 \exp^{-0.04p_2}}$ and $a_2(p) = \frac{5 \exp^{-0.04p_2}}{5 \times 10^{-4} + 1.5 \exp^{-0.032p_1} + 5 \exp^{-0.04p_2}}$.

The following table depicts the difference in the optimal strategy and expected profit between the model with no stockout substitution and the one with the stockout substitution rule from previous sections. These values are obtained using search algorithms.

<table>
<thead>
<tr>
<th></th>
<th>Inventory 1</th>
<th>Inventory 2</th>
<th>Price 1</th>
<th>Price 2</th>
<th>Ex Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Stockout Substitution</td>
<td>7</td>
<td>0</td>
<td>$221$</td>
<td>$\infty$</td>
<td>$634$</td>
</tr>
<tr>
<td>With Stockout Substitution</td>
<td>5</td>
<td>3</td>
<td>$226$</td>
<td>$219$</td>
<td>$700$</td>
</tr>
</tbody>
</table>

The Table suggests that under no stockout substitution, the retailer should focus on selling only product one to achieve a maximum expected profit of $634. Because product two is not offered, the expected profit of $634 is obtained by setting $p_2$ to infinity and determining the expected profit for a single product model with $p_1 = 221$. If on the other hand, the retailer accepts our axiomatic approach, the retailer should order both products and increase their total expected profit to $700.
To better understand the difference between the two models, consider the cumulative distribution functions (CDF) of product 1’s demand without, and with, stockout demand substitution.

![Two Cumulative Distribution Functions](image)

Figure 3-2: CDF for product 1 demand under the two models when \( y_2 = 3, p_1 = 226 \) and \( p_2 = 219 \)

The lower (resp. upper) curve represents the cumulative distribution function of product 1’s demand by (resp. without) taking into account stockout demand substitution. As expected, the figure demonstrates that demand for product 1 when considering stockout demand substitution is stochastically greater then demand for that product when stockout substitution is not applied. This difference in demand can be significant and hence affects retailer strategy and the corresponding expected profit.

Observe that in the previous example, in some sense, "more is better." That is, selling two products provides a higher expected profit than offering a single product. The following example shows that this is not always the case.

**Example 3.2.2.** Assume that the customer arrival rate \( \lambda = 10 \), the time horizon is \([0,1]\)
and the ordering cost for both products satisfies \( c_1 = c_2 = 100 \). Let the demand distribution between different products follow the attraction model with probability functions

\[
\begin{align*}
a_1(p) &= \frac{1.5 \exp^{-0.03p_1}}{5 \times 10^{-4} + 1.5 \exp^{-0.03p_1} + 5 \exp^{-0.05p_2}} \quad \text{and} \quad a_2(p) &= \frac{5 \exp^{-0.05p_2}}{5 \times 10^{-4} + 1.5 \exp^{-0.03p_1} + 5 \exp^{-0.05p_2}}.
\end{align*}
\]

We first observe that for any \( p \), \( \lambda(a_1(p) + a_2(p)) > \lambda a_1^R(p_1) \). This implies that the total expected demand across the two products is greater than the expected demand of product 1 only. However, higher expected demand doesn’t lead to a higher profit as is demonstrated in the following table.

<table>
<thead>
<tr>
<th>Table 3.2: More is not always better!</th>
<th>inventory 1</th>
<th>inventory 2</th>
<th>price 1</th>
<th>price 2</th>
<th>profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single product</td>
<td>8</td>
<td>0</td>
<td>$230</td>
<td>\infty</td>
<td>$729</td>
</tr>
<tr>
<td>Two products</td>
<td>7</td>
<td>1</td>
<td>$232</td>
<td>$192</td>
<td>$717</td>
</tr>
</tbody>
</table>

To develop some intuition into this type of behavior, observe that when offering both products, profit margin for product 2 is smaller than that of product 1. Removing product 2 from the portfolio increases expected demand for product 1, since \( a_1(p) < a_1^R(p_1) \), and this additional demand provides a higher profit margin.

The previous example motivates a focus on situations where more products implies higher profit. Consider similar products that differentiate only along minor characteristics such as colors or flavors. It is often the case that retailers charge the same price for all these products. The following proposition shows that if the probability functions that customers choose two products are symmetric, then offering two products always produces higher expected profit than a single product.

Before we present this proposition, we need to introduce the notion of stochastic order. We say that the random variable \( X \) is stochastically larger than the random variable \( Y \), written \( X \succeq_{st} Y \), if \( P(X > a) \geq P(Y > a) \) for all \( a \). The following classical result is taken from Ross [27], see his Proposition 9.1.2.

**Lemma 3.2.1.** \( X \succeq_{st} Y \Leftrightarrow E(f(X)) \geq E(f(Y)) \) for all increasing functions \( f \).
Proposition 3.2.1. Assume that unit ordering costs are equal, i.e., $c_1 = c_2$ and that $a_1(p) + a_2(p) \geq \max\{a_1^R(p_1), a_2^R(p_2)\}$, that is, the probability that a customer purchases when the two products are available is no less than the probability that the customer will purchase when only one of the two products is available. If the probability functions are symmetric, that is, for any $p$, $a_1(p_1, p_2) = a_2(p_2, p_1)$ and $a_2(p_1, p_2) = a_1(p_2, p_1)$, then offering two products results in an expected profit no less than when offering only one of the two products.

Proof. If the retailer sells a single product, say product 1, then there is an optimal price, $p^*_1$ and optimal inventory level, $y^*_1$. In this case, demand for product 1 is a poisson random variable with mean $\lambda T a_1^R(p^*_1)$. We denote this random demand as $\text{pois}(\lambda T a_1^R(p^*_1))$.

Consider a feasible pricing and inventory strategy when two products are offered. In this policy, $p_1 = p_2 = p^*_1$ and inventory levels $y_1$ and $y_2$ are chosen such that $y_1 + y_2 = y^*_1$.

Let $S_{y_1}$ and $S_{y_2}$ be the stockout time for product 1 and product 2 respectively and $S = \min\{S_{y_1}, S_{y_2}\}$, the time when stockout first happens. If $S \leq T$, then $X$, the total demand cross both products satisfies $X = \text{pois}(\lambda S(a_1(p) + a_2(p))) + \text{pois}(\lambda(T - S)a_1^R(p^*_1))$. Here, we have assumed that product 2 stocks out first. But even if product 1 stocks out first, the expression for $X$ still holds because $a_2^R(p_2) = a_1^R(p_1)$, which follows from the assumption that the probability functions are symmetric and $p_1 = p_2 = p^*_1$.

If $S > T$, $X = \text{pois}(\lambda T(a_1(p) + a_2(p)))$.

It's well known that poisson random variable is stochastically increasing in its mean (see Example 9.2(B) in Ross [27]). Therefore, it follows from the assumption $a_1(p) + a_2(p) \geq \max\{a_1^R(p_1), a_2^R(p_2)\}$ that $X$ is stochastically larger than $\text{pois}(\lambda T a_1^R(p^*_1))$. Lemma 3.2.1 implies that $E(\min\{X, y^*_1\}) \geq E(\min\{\text{pois}(\lambda T a_1^R(p^*_1)), y^*_1\})$. Hence, the profit of selling two products, $(p^*_1 - c_1)E(\min\{X, y^*_1\}) - c_1y^*_1 \geq (p^*_1 - c_1)E(\min\{\text{pois}(\lambda T a_1^R(p^*_1)), y^*_1\}) - c_1y^*_1$, the profit of selling product 1 only.

3.3 Summary

In this chapter, I extend the demand substitution rule to a stochastic environment. I demonstrate the impact of our axiomatic approach to the joint pricing and inventory coordination
model by incorporating the price-dependent demand substitution rule to capture customer behavior when one of the product stocks out. The result illustrates that if the axiomatic approach is acceptable, the optimal strategy and corresponding expected profit are quite different than models that ignore stockout demand substitution.
Chapter 4

The Joint Pricing and Inventory Game

In this chapter, I use the consumer choice theory to model retail competition among multiple retailers. For each retailer, I develop a newsvendor model that combines inventory decisions with a non-multiplicative price-demand model. I show that if the retailer’s expected demand is log-concave in its price, the profit is also log-concave in the price. This result implies that in a competitive setting where multiple retailers compete on prices, a Nash equilibrium exists in the so-called pricing game. This extends the work of Bernstein and Federgruen [8] and Chen et al [10] from multiplicative demand models to a non-multiplicative demand model. This part of work is done in Section 4.1.

One limitation of most pricing games models is the assumption that if one retailer stocks out, her costumers don’t switch to other retailers; they just exit the system. Therefore, each retailer’s inventory affects her own profit only, and hence retailers only compete on price. In Section 4.2, I relax this assumption by incorporating stockout demand substitution among retailers. This implies that if one retailer stocks out, her costumers may switch to other retailers. Hence, each retailer’s inventory level also affects other retailers’ demand and consequently their profits. Thus, retailers compete on both price and inventory, which I refer to as the *joint pricing and inventory game*. I show that the quasi-concavity of each retailer’s profit function held in the pricing game doesn’t have to hold in the joint pricing and inventory game. Hence, in general there is no Nash Equilibrium in this game. However,
I characterize conditions under which a Nash equilibrium exists in the joint pricing and inventory game.

4.1 An Extension in the Pricing Game

In this section, I assume there are \( n \) retailers in the market. Each retailer sells a similar or identical product. Customer arrivals follow a poisson process \( N(t) \) with arrival rate \( \lambda \). Without loss of generality, I assume that the time period is \([0,1]\). For each customer \( k \), \( k = 1, \ldots, N(1) \), denote \( A_{ki} = \{ \text{the event that customer } k \text{ chooses retailer } i \} \) for \( i = 1, \ldots, n \), and \( A_{k0} = \{ \text{the event that customer } k \text{ doesn't buy from all the } n \text{ retailers } \} \). Therefore, the demand for retailer \( i \) is

\[
Y_i = \sum_{k=1}^{N(1)} 1_{\{A_{ki}\}}, \quad i = 0, 1, \ldots, n.
\] (4.1)

where \( 1_{\{A_{ki}\}} \) is the indicator function of event \( A_{ki} \). I assume that different customers make i.i.d. choices among the group of \( n \) retailers. Therefore, the probability of \( A_{ki} \), \( a_i := \text{Pr}(A_{ki}) \) must be independent of \( k \). Evidently, \( a_i \) depends on the price vector \( p \). If retailer \( i \)'s price \( p_i \) increases, its probability of being chosen decreases. Therefore, \( a_i(p) \) decreases in \( p_i \). Similarly, if retailer \( j \) increases its prices the probability that a customer will chose retailer \( i \) increases.

Given this arrival process, the demand for retailer \( i \) has a poisson distribution with rate \( \lambda a_i(p) \). For large \( \lambda \), the central limit theorem implies that \( Y_i \) is a normal distribution with mean \( \mu_i(p) = \lambda a_i(p) \) and standard deviation \( \sigma_i(p) = \sqrt{\mu_i(p)} = \sqrt{\lambda a_i(p)} \), i.e., \( Y_i = \mu_i(p) + \epsilon_i \sqrt{\mu_i(p)} \), here \( \epsilon_i \) is a standard normal distribution.

Denote by \( c_i \) and \( y_i \) the per-unit order price and order quantity, respectively, of retailer \( i, i = 1, \ldots, n \). I assume that there is no salvage value, though this assumption can easily be relaxed. While a retailer's price impacts on the profits of all retailers in a pricing game, her order quantity affects her own profit only. The payoff function of retailer \( i \) is

\[
\pi_i(p, y_i) = E[p_i \min\{y_i, Y_i\} - c_i y_i].
\] (4.2)
Under the assumption of normal distribution, the optimal inventory level, denoted $y^*_i$, is given by

$$y^*_i = \mu_i(p) + z_i \sigma_i(p), \quad (4.3)$$

where

$$z_i = \Phi^{-1}(1 - \frac{c_i}{p_i}), \quad (4.4)$$

and $\Phi(z)$ denotes the c.d.f. of a standard normal random variable. I let $\phi(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$ denote the standard normal density function. Substituting the expression for $y^*_i$, in equation (4.2), I get

$$\pi_i(p) = (p_i - c_i) \mu_i(p) - \frac{p_i e^{-z_i^2/2} \sigma_i(p)}{\sqrt{2\pi}}, \quad (4.5)$$

where I have used the fact that for a standard normal random variable $Z$, $E(Z - z)+ = \phi(z) - z(1 - \Phi(z))$. I denote by $\pi_i^{det}(p) = (p_i - c_i) \mu_i(p)$ retailer $i$'s profit under the price vector $p$, in a deterministic system where no uncertainty prevails, i.e., demand for retailer $i$ equals $\mu_i(p)$. I also denote

$$f_i(p_i) = \frac{p_i e^{-z_i^2/2}}{\sqrt{2\pi}(p_i - c_i)}, \quad (4.6)$$

which is a function that only depends on $p_i$. I can rewrite equation (4.5) as

$$\pi_i(p) = \pi_i^{det}(p)(1 - f_i(p_i)g_i(p)), \quad (4.7)$$

here

$$g_i(p) = \frac{\sigma_i(p)}{\mu_i(p)} = \frac{1}{\sqrt{\mu_i(p)}}. \quad (4.8)$$

I want to show that $\log \pi_i(p)$ is strictly concave if $\log \mu_i(p)$ is concave. First, I need the following lemma.

**Lemma 4.1.1.** Let $f_i(x) : R \mapsto R^+$, $i = 1, 2$, be twice differentiable functions. If $\log f_i(x)$, $i = 1, 2$, are both convex functions, then $f_1(x) \ast f_2(x)$ is a convex function.
Proof. The convexity of \( \log f_i(x) \) implies that

\[
\frac{\partial^2 \log f_i(x)}{\partial x^2} = \frac{\partial^2 f_i(x)}{\partial x^2} f_i(x) - \left( \frac{\partial f_i(x)}{\partial x} \right)^2 \geq 0
\]

\[
\Rightarrow \frac{\partial^2 f_i(x)}{\partial x^2} \geq \frac{\left( \frac{\partial f_i(x)}{\partial x} \right)^2}{f_i(x)}.
\]

Using this inequality, I have

\[
\frac{\partial^2 f_1(x) * f_2(x)}{\partial x^2} = \frac{\partial^2 f_1(x)}{\partial x^2} f_2(x) + f_1(x) \frac{\partial^2 f_2(x)}{\partial x^2} + 2 \frac{\partial f_1(x)}{\partial x} \frac{\partial f_2(x)}{\partial x}
\]

\[
\geq \left( \frac{\partial f_1(x)}{\partial x} \right)^2 f_2(x) + f_1(x) \left( \frac{\partial f_2(x)}{\partial x} \right)^2 + 2 \frac{\partial f_1(x)}{\partial x} \frac{\partial f_2(x)}{\partial x}
\]

\[
\geq 2 \sqrt{\left( \frac{\partial f_1(x)}{\partial x} \right)^2 f_2(x) \frac{f_1(x)}{f_2(x)} \left( \frac{\partial f_2(x)}{\partial x} \right)^2 + 2 \frac{\partial f_1(x)}{\partial x} \frac{\partial f_2(x)}{\partial x}}
\]

\[
= 2 \left| \frac{\partial f_1(x)}{\partial x} \frac{\partial f_2(x)}{\partial x} \right| + 2 \frac{\partial f_1(x)}{\partial x} \frac{\partial f_2(x)}{\partial x} \geq 0.
\]

This completes the proof. \( \square \)

Whitin [31] was the first to formulate the newsvendor problem with price effect. In his model, as in ours, selling price and inventory are set simultaneously to maximize a retailer’s profit. For a review of pricing and the newsvendor problem, I refer the reader to Petruzzi and Dada [25]. Yong [32] was the first to introduce a model that combines both additive and multiplicative effects of price on demand. Unfortunately, his model has to satisfy some strong assumption in order for the expected profit to be concave and lead to a tractable model.

The following theorem states that the log-concavity of expected demand \( \mu_i(p) \) in \( p_i \) implies strict log-concavity of profit function \( \pi_i(p) \) in \( p_i \). This condition is much simpler than the conditions required in Yong [32].

**Theorem 4.1.1.** If \( \log \mu_i(p) \) is concave in \( p_i \), then \( \log \pi_i(p) \) is strictly log-concave in \( p_i \) over the region \( [1.01c_i, 100c_i] \).

Proof. From (4.7), we know that
\[
\log \pi_i(p) = \log(p_i - c_i) + \log \mu_i(p) + \log(1 - f_i(p_i)g_i(p)). \quad (4.10)
\]

We first show that \(\log f_i(p_i)\) is a convex function by showing that \(h_i(p_i) = \log f_i(c_i p_i) = \log \frac{p_i e^{-z_i^2/2}}{\sqrt{2\pi (p_i-1)}} = \log p_i - \log \sqrt{2\pi} - \log(p_i - 1) - \frac{z_i^2}{2}\), where \(z_i' = \Phi^{-1}(1 - \frac{1}{p_i})\), is a convex function. The second derivative of \(h_i(p_i)\) is

\[
\frac{\partial^2 h_i(p_i)}{\partial p_i^2} = -\frac{1}{p_i^2} + \frac{1}{(p_i - 1)^2} - \left(\frac{2\pi}{p_i^3} + \frac{2\pi}{p_i^3} z_i^2\right)e^{z_i^2} + \frac{2\sqrt{2\pi}}{p_i^3} z_i' e^{z_i^2/2}. \quad (4.11)
\]

To prove that \(h_i(p_i)\) is a convex function, we need to show \(\frac{\partial^2 h_i(p_i)}{\partial p_i^2} \geq 0\), which is equivalent to

\[
g_i(p_i) = p_i^2 * \frac{\partial^2 h_i(p_i)}{\partial p_i^2} = -1 + \frac{p_i^2}{(p_i - 1)^2} - \left(\frac{2\pi}{p_i^3} + \frac{2\pi}{p_i^3} z_i^2\right)e^{z_i^2} + \frac{2\sqrt{2\pi}}{p_i^3} z_i' e^{z_i^2/2} \geq 0. \quad (4.12)
\]

We prove that this is true for \(p_i \in [1.01, 100]\). Notice that the absolute value of \(z_i'\), \(|z_i'| = |\Phi^{-1}(1 - \frac{1}{p_i})|\), is decreases over \([1.01, 2]\) and increases over \([2, 100]\). Therefore, for any \(p_i \in [1.01, 100]\),

\[
|z_i'| \leq \max\{|\Phi^{-1}(1 - \frac{1}{1.01})|, |\Phi^{-1}(1 - \frac{1}{100})|\} = 2.3301. \quad (4.13)
\]

\[
\frac{\partial z_i(p_i)}{\partial p_i} = \frac{\sqrt{2\pi}}{p_i^3} e^{z_i^2/2} \quad \text{implies that}
\]

\[
\left|\frac{\partial(\frac{2\pi}{p_i^3} + \frac{2\pi}{p_i^3} z_i^2)e^{z_i^2}}{\partial p_i}\right| = \left|\left(-\frac{4\pi}{p_i^3} - \frac{4\pi}{p_i^3} z_i^2 + \frac{4\pi}{p_i^3} \frac{\sqrt{2\pi}}{p_i^3} e^{z_i^2/2}\right)e^{z_i^2} + \left(\frac{2\pi}{p_i^3} + \frac{2\pi}{p_i^3} z_i^2\right)2z_i' \frac{\sqrt{2\pi}}{p_i^3} e^{z_i^2} e^{z_i^2/2}\right|
\leq (4\pi + 4\pi|z_i'|^2 + 4\pi|z_i'| \sqrt{2\pi} e^{1|z_i'|^2/2})e^{1|z_i'|^2} + (4\pi + 4\pi|z_i'|^2)|z_i'| \sqrt{2\pi} e^{1|z_i'|^2/2}
\leq 1.90 * 10^6, \quad (4.14)
\]

where the first inequality follows from \(p_i \geq 1.01\) and the second inequality follows from
Similarly, we have
\[
\left| \frac{\partial^2 \sqrt{2\pi} z_i e^{z_i^2/2}}{\partial p_i} \right| = \left| \left( -\frac{2\sqrt{2\pi}}{p_i^2} z_i' + \frac{2\sqrt{2\pi} \sqrt{2\pi}}{p_i} e^{z_i^2/2} \right) e^{z_i^2/2} + \frac{2\sqrt{2\pi}}{p_i} z_i' e^{z_i^2/2} \frac{\sqrt{2\pi}}{p_i} e^{z_i^2/2} \right| \\
\leq \left| (2\sqrt{2\pi} |z_i'| + 2\sqrt{2\pi} \sqrt{2\pi} e^{|z_i|^2/2}) e^{|z_i|^2/2} + 2\sqrt{2\pi} z_i' e^{z_i^2/2} |z_i'| \sqrt{2\pi} e^{z_i^2/2} \right| \\
\leq 1.86 \times 10^4. 
\] (4.13)

We also have
\[
\left| \frac{\partial^2 \frac{p_i^2}{(p_i - 1)^3}}{\partial p_i} \right| = \left| \frac{2 \cdot p_i}{(p_i - 1)^3} \right| \leq 2.02 \times 10^6, 
\] (4.14)

where the inequality follows from \( p_i \geq 1.01 \).

(4.14), (4.15), (4.16) and Mean Value Theorem imply that for any \( p_i, p_i' \in [1.01, 100] \),
\[
|g_i(p_i) - g_i(p_i')| \leq (1.90 \times 10^6 + 1.86 \times 10^4 + 2.02 \times 10^6)|p_i - p_i'| < 4 \times 10^6|p_i - p_i'|. 
\] (4.17)

(4.17) indicates that if \(|p_i - p_i'| < 10^{-7}\), then \(|g_i(p_i) - g_i(p_i')| < 0.4\). Define \( S = \{ p_i | p_i = 1.01 + 10^{-7} \times k, k = 0, 1, 2, \ldots \ \text{and} \ p_i \leq 100 \} \). After calculating by computer, we find \( \min_{p_i \in S} \{ g_i(p_i) \} = 0.83 \), which implies that \( \min_{p_i \in [1.01, 100]} \{ g_i(p_i) \} \geq 0.83 - 0.4 > 0 \).

This completes the proof that \( \log f_i(p_i) \) is convex over the region \([1.01c_i, 100c_i]\).

From (4.8), \( \log g_i(p) = -\frac{1}{2} \log \mu_i(p) \), which is a convex function in \( p_i \) by the assumption that \( \log \mu_i(p) \) is concave. Therefore, Lemma 4.1.1 implies that \( f_i(p_i)g_i(p) \) is convex in \( p_i \) and consequently \( 1 - f_i(p_i)g_i(p) \) is concave in \( p_i \), which implies that \( \log(1 - f_i(p_i)g_i(p)) \) is concave in \( p_i \). Given that \( \log(p_i - c) \) is strictly concave in \( p_i \), it follows from (4.10) that \( \log \pi_i(p) \) is strictly concave in \( p_i \) over the region \([1.01c_i, 100c_i]\).

Although we only show that \( \log \pi_i(p) \) is strictly log-concave in \( p_i \) over the region \([1.01c_i, 100c_i]\) in Theorem 4.1.1, the idea of this proof can be extended to show that \( \log \pi_i(p) \) is strictly log-concave in \( p_i \) over any region which is a subset of \((c_i, +\infty)\).

The following theorem extends the pricing game model of Bernstein and Federgruen [8]

**Theorem 4.1.2.** If \( \log \mu_i(p) \) is concave in \( p_i \), a pure strategy Nash equilibrium exists in the pricing game.

*Proof.* Theorem 4.1.1 implies that there is a unique best response function \( p_i^*(p_{-i}) \) for any \( p_{-i} \). The continuity of \( p_i^*(p_{-i}) \) is implied by the continuity of \( \pi_i(p) \). By Kakutani’s fixed point theorem, there must be a fixed point for best response functions \( p_i^*(p_{-i}), i = 1, \ldots, n \), which has to be a pure strategy Nash equilibrium. \( \square \)

Notice that \( \log \mu_i(p) \) is concave in \( p_i \) if and only if \( \log \alpha_i(p) \) is concave in \( p_i \) since \( \mu_i(p) = \lambda \alpha_i(p) \). It can be easily verified that this assumption is satisfied by the attraction models and the linear model.

### 4.2 Existence of Nash Equilibrium in the Joint Pricing and Inventory Game

In literature on pricing games, including our previous analysis, a simplified assumption is that after a retailer stocks out, her customers exist the system, i.e., none of them switches to other retailers. In this section, we relax this assumption. We consider two retailers, retailer 1 and retailer 2, competing in the market. Customers’ arrival process is a poisson process with arrival rate \( \lambda \). Before retailer 2 stocks out, each arrival customer chooses retailer 1 with probability \( a_1(p) \), retailer 2 with probability \( a_2(p) \) and no-buy option with probability \( a_0(p) \).

After retailer 2 stocks out, some of retailer 2’s customers switch to retailer 1, and some of them leave the system. Specifically, for each customer arriving after retailer 2 stocks out, let \( a_1^R(p_1) \) be the probability that this customer chooses retailer 1, and \( a_0^R(p_1) \) is the probability that the customer chooses the no-buy option. We refer readers to Lu and Simchi-Levi[16] to see how \( a_1^R(p) \) and \( a_0^R(p) \) are determined by following an axiomatic approach. The same principle applies to retailer 2 if retailer 1 stocks out first. Therefore, not only the price, but also the inventory of one retailer will affect the demand for the other retailer and consequently
the retailer’s profit. We call this game the joint pricing and inventory game.

Assume that \( Y_2 \) is the inventory level of retailer 2. Denote \( S_{y_2} \) be the arrival time of its \( y_2 \)th customer, which is the stock-out time for retailer 2. Then \( S_{y_2} \) is the sum of \( y_2 \) independent exponentially distributed random variable with rate \( \lambda_{a_2}(p) \). Without loss of generality, we assume the time horizon is \([0,1]\). Denote \( N_2(1) \) to be a poisson distribution with rate \( \lambda_{a_2}(p) \) and \( Y_1 \) be the effective demand of retailer 1. Then if \( S_{y_2} < 1 \), \( Y_1 = pois(\lambda_{a_1}(p)S_{y_2}) + pois(\lambda_{a_1}(p)(1 - S_{y_2})) \). If \( S_{y_2} \geq 1 \), \( Y_1 = pois(\lambda_{a_1}(p)) \). The following is the expected effective demand for retailer 1 given any price vector \( p \) and \( y_2 \).

\[
\mu_1(p, y_2) = E(Y_1) = E(E(Y_1|S_{y_2})) = \int_{t=0}^{1} E(Y_1|S_{y_2} = t)f_{S_{y_2}}(t)dt + \int_{t=1}^{\infty} E(Y_1|S_{y_2} = t)f_{S_{y_2}}(t)dt \\
= \int_{t=0}^{1} (\lambda_{a_1}(p)t + \lambda_{a_1}(p)(1 - t))f_{S_{y_2}}(t)dt + \int_{t=1}^{\infty} \lambda_{a_1}(p)f_{S_{y_2}}(t)dt \\
= \lambda_{a_1}(p) + \lambda(\frac{a_{R_1}(p) - a_1(p)}{a_2(p)}) \int_{t=0}^{1} (1 - t)f_{S_{y_2}}(t)dt \\
= \lambda_{a_1}(p) + \lambda(\frac{a_{R_1}(p) - a_1(p)}{a_2(p)}) (P(S_{y_2} \leq 1) - \frac{y_2}{\lambda_{a_2}(p)}P(S_{y_2} + 1 \leq 1)) \\
= \lambda_{a_1}(p) + \frac{a_{R_1}(p) - a_1(p)}{a_2(p)} E(\max\{pois(\lambda_{a_2}(p)) - y_2, 0\}). \quad (4.18)
\]

The quantity \( \frac{a_{R_1}(p) - a_1(p)}{a_2(p)} \) can be interpreted as the proportion of retailer 2’s customers who will switch to retailer 1 if retailer 2 stocks out. Therefore, the expected effective demand for retailer 1 equals to its own expected demand plus expected leftover demand of retailer 2 multiplied by demand substitution rate.

We approximate \( Y_1 \) by a Poisson distribution with arrival rate \( \mu_1(p, y_2) \). Central limit theorem tells us that for large \( \lambda \), \( Y_1 = \mu_1(p, y_2) + \epsilon_1 \sqrt{\mu_1(p, y_2)} \), where \( \epsilon_1 \) is a standard normal distribution. From (4.5), we know

\[
\pi_1(p, y_2) = (p_1 - c_1)\mu_1(p, y_2) - \frac{p_1e^{-x^2/2}}{\sqrt{2\pi}} \sqrt{\frac{\mu_1(p, y_2)}{2\pi}}. \quad (4.19)
\]
In the case of the pricing game, see Section 4.1, we know that if \( \log a_1(p) \) is concave in \( p_1 \), \( \log \pi_1(p) \) is strictly concave in \( p_1 \), which implies that \( \pi_1(p) \) has only one maximal point. Unfortunately, this is not true for the joint pricing and inventory game. That is, even if \( \log a_1(p) \) is concave in \( p_1 \), \( \pi_1(p, y_2) \) can have two optimal points.

![Figure 4-1: \( \pi_1(p_1, 40, 6) \)](image)

**Example 4.2.1.** We assume that \( c_1 = 1 \) and \( \lambda = 80 \). Let \( a_1(p) = \frac{\exp^{-1.39p_1}}{0.0001 + \exp^{-1.39p_1} + \exp^{-p_2}} \), \( a_2(p) = \frac{\exp^{-p_2}}{0.0001 + \exp^{-1.39p_1} + \exp^{-p_2}} \) and \( a_1'(p) = \frac{\exp^{-1.39p_1}}{0.0001 + \exp^{-1.39p_1} + \exp^{-p_2}} \). For \( y_2 = 40 \) and \( p_2 = 6 \), Figure 4-1 shows that \( \pi_1(p_1, 40, 6) \) can have two optimal points over \( p_1 \).

To illustrate that no pure strategy Nash Equilibrium exists, fix \( y_2 \) and determine the best response of retailer 1, \( p_1^*(p_2) \) for any given price of retailer 2, \( p_2 \). Figure 2 shows that at \( p_2 = 6 \), there are two best response strategies for retailer 1, one at a higher price of $5.45, and a second at a price of $3.59. These two price levels correspond to the two optimal points in Figure 4-2. In addition, since the best response of retailer 1 is not monotone, we conclude that the game is not supermodular. Thus, a pure strategy Nash equilibrium doesn’t have to exist in general for the joint pricing and inventory game.

In the rest of this section, we characterize conditions under which a pure strategy Nash equilibrium exists for the joint pricing and inventory game.
We assume that probability functions follow the widely used attraction model $a_i(p) = \frac{k_i e^{-\lambda_i p_i}}{a_0 + k_i e^{-\lambda_1 p_1} + k_2 e^{-\lambda_2 p_2}}$, $i = 1, 2$. In this case, we know from Lu and Simchi-Levi [16] that retailer $i$'s probability of being chosen after retailer $j$ stocks out is $a_i^{\text{R}}(p) = \frac{k_i e^{-\lambda_i p_i}}{a_0 + k_i e^{-\lambda_1 p_1} + k_2 e^{-\lambda_2 p_2}}$ for $i = 1, 2$ and $j \neq i$.

This assumption allows to develop a new form for the expected effective demand faced by retailer 1, $Y_1$.

\[
\mu_1(p, y_2) = E(E(Y_1|S_{y_2})) = \int_{t=0}^{1} E(Y_1|S_{y_2} = t)f_{S_{y_2}}(t)dt + \int_{1}^{+\infty} E(Y_1|S_{y_2} = t)f_{S_{y_2}}(t)dt
\]

\[
= \int_{t=0}^{1} (\lambda a_1(p)t + \lambda a_1^{\text{R}}(p)(1 - t))f_{S_{y_2}}(t)dt + \int_{1}^{+\infty} \lambda a_1(p)f_{S_{y_2}}(t)dt
\]

\[
= \lambda a_1^{\text{R}}(p) - \lambda(a_1^{\text{R}}(p) - a_1(p)) \cdot \left( \int_{t=0}^{1} tf_{S_{y_2}}(t)dt + \int_{1}^{+\infty} f_{S_{y_2}}(t)dt \right)
\]

\[
= \lambda a_1^{\text{R}}(p) - \lambda(a_1^{\text{R}}(p) - a_1(p)) \cdot \left( P(S_{y_2} > 1) + \frac{y_2}{\lambda a_2(p)}P(S_{y_2 + 1} \leq 1) \right)
\]

\[
= \lambda a_1^{\text{R}}(p) - \frac{a_1^{\text{R}}(p) - a_1(p)}{a_2(p)}(\lambda a_2(p)P(N_2(1) \leq y_2 - 1) + y_2 P(N_2(1) \geq y_2 + 1))
\]

\[
= \lambda a_1^{\text{R}}(p) - \frac{a_1^{\text{R}}(p) - a_1(p)}{a_2(p)}E(\min\{pois(\lambda a_2(p)), y_2\})
\]

\[
= (\lambda - E(\min\{pois(\lambda a_2(p)), y_2\}))a_1^{\text{R}}(p),
\]

where the last equality follows from the fact that $\frac{a_1^{\text{R}}(p) - a_1(p)}{a_2(p)} = a_1^{\text{R}}(p)$. 58
If we identify conditions such that $\log \mu_1(p, y_2) = \log(\lambda - E(\min\{\text{pois}(\lambda a_2(p)), y_2\})) + \log a^\mathbb{R}_1(p)$ is concave in $p_1$, we can follow the same proof of Theorem 4.1.2 to show that there exists a pure strategy Nash equilibrium in the joint pricing and inventory game.

The following lemma provides sufficient conditions for the existence of a pure strategy Nash equilibrium in the joint pricing and inventory game. Define $C_i = \lambda a_i(p)P(N_i(1) = y_i - 1)$.

**Lemma 4.2.1.** If $a_i(p) \leq \min\{\frac{2}{\sqrt{4c_1+1+3}}, \frac{2}{\sqrt{4c_2+1+3}}\}$ and $a_i(p) + a^\mathbb{R}_i(p) \leq 1$, $\log \mu_i(p, y_j)$ is concave in $p_i$, for $i = 1, 2$ and $j \neq i$, and consequently, there exists a pure strategy Nash equilibrium in the joint pricing and inventory game.

**Proof.** It can be verified that $\frac{\partial a_i(p)}{\partial p_i} = -\lambda a_i(p)(1 - a_i(p))$ and $\frac{\partial a_i(p)}{\partial p_j} = \lambda_j a_i(p) a_j(p)$, for $i = 1, 2$ and $j \neq i$. These equations imply

$$\frac{\partial^2 a_i(p)}{\partial p_i^2} = -\lambda^2 a_i(p) a_j(p)(1 - 2a_j(p)), \quad (4.21)$$

and

$$\frac{\partial^2 a_i(p)}{\partial p_i^2} = \lambda^2 a_i(p)(1 - a_i(p))(1 - 2a_i(p)). \quad (4.22)$$

We also have

$$\frac{\partial E(\min\{N_2(1), y_2\})}{\partial p_1} = \lambda \frac{\partial a_2(p)}{\partial p_1} \left(\sum_{k=0}^{y_2-1} \frac{(\lambda a_2(p))^k}{k!} e^{-\lambda a_2(p)} - \sum_{k=0}^{y_2-1} \frac{(\lambda a_2(p))^{k+1}}{k!} e^{-\lambda a_2(p)}\right) + \frac{\partial^2 a_2(p)}{\partial p_1^2} P(N_2(1) \leq y_2 - 2) + P(N_2(1) \leq y_2 - 1) - \lambda a_2(p)$$

$$*P(N_2(1) \leq y_2 - 1) + y_2 P(N_2(1) \geq y_2) - y_2 P(N_2(1) \geq y_2 + 1))$$

$$= \lambda \frac{\partial a_2(p)}{\partial p_1} P(N_2(1) \leq y_2 - 1). \quad (4.23)$$

It implies that

$$\frac{\partial^2 E(\min\{N_2(1), y_2\})}{\partial p_1^2} = \lambda \frac{\partial^2 a_2(p)}{\partial p_1^2} P(N_2(1) \leq y_2 - 1) - \lambda^2 \left(\frac{\partial a_2(p)}{\partial p_1}\right)^2 P(N_2(1) = y_2 - 1).$$
Therefore,

\[
\frac{\partial^2 \log(\lambda - E(\min\{N_2(1), y_2\}))}{\partial p_1^2} = \frac{-\lambda^2 a_2(p) P(N_2(1) \leq y_2 - 1) + \lambda^2 (\partial a_2(P)/\partial p_1)^2 P(N_2(1) = y_2 - 1)}{\lambda - E(\min\{N_2(1), y_2\})} - (\lambda^2 a_2(P)/\partial p_1)^2 P(N_2(1) \leq y_2 - 1))^2 \\
= \frac{\lambda \lambda_1^2 a_2(p) a_1(p) (1 - 2a_1(p)) P(N_2(1) \leq y_2 - 1) + \lambda^2 \lambda_1^2 a_2(p)^2 a_1(p)^2 P(N_2(1) = y_2 - 1)}{\lambda - E(\min\{N_2(1), y_2\})} \\
- \frac{(\lambda \lambda_1 a_2(p) a_1(p) P(N_2(1) \leq y_2 - 1))^2}{(\lambda - E(\min\{N_2(1), y_2\}))^2}.
\]

To simplify the above expression, we define \( C_i = \lambda a_i(p) P(N_i(1) = y_i - 1), i = 1, 2, \) and \( r = \min\{\frac{2}{\sqrt{4a_1+4+1}}, \frac{2}{\sqrt{4a_2+4+1}}\}. \) If \( \frac{a_i(p)}{1-a_i(p)} \leq r, \) i.e., \( a_i(p) \leq \min\{\frac{2}{\sqrt{4a_1+4+3}}, \frac{2}{\sqrt{4a_2+4+3}}\}, i = 1, 2, \) we have

\[
\frac{\partial^2 \log(\lambda - E(\min\{N_2(1), y_2\}))}{\partial p_1^2} = \frac{\lambda \lambda_1^2 a_2(p) a_1(p) (1 - 2a_1(p)) P(N_2(1) \leq y_2 - 1) + \lambda^2 \lambda_1^2 a_2(p)^2 a_1(p)^2 P(N_2(1) = y_2 - 1)}{\lambda - E(\min\{N_2(1), y_2\})} \\
- \frac{(\lambda \lambda_1 a_2(p) a_1(p) P(N_2(1) \leq y_2 - 1))^2}{(\lambda - E(\min\{N_2(1), y_2\}))^2} \\
\leq \frac{\lambda_1^2 a_2(p) a_1(p) (1 - 2a_1(p)) (1 - a_2(p))}{1 - a_2(p)} + \frac{C_2 \lambda^2 a_2(p) a_1(p)^2}{1 - a_2(p)} \\
\leq \frac{\lambda_1^2 a_1(p) (1 - a_1(p))(1 - a_1(p))}{1 - a_1(p)} (r + C_2 r^2) \leq \lambda_1^2 a_1(p) (1 - a_1(p)), \quad (4.24)
\]

where the first inequality follows from the facts that \( E(\min\{N_2(1), y_2\}) \leq E(N_2(1)) = \lambda a_2(p) \) and the last term is negative, the second inequality follows from \( P(N_2(1) \leq y_2 - 1) \leq 1 \) and definition of \( C_2 \), the third inequality follows from \( \frac{1 - 2a_1(p)}{1 - a_1(p)} \leq 1 \) and the assumption that \( \frac{a_i(p)}{1-a_i(p)} \leq r, \) and the last inequality follows from the definition of \( r. \) Similar inequality like
It can be verified that \( \frac{\log a_i^R(p)}{p_i^2} = -\lambda_i^2 a_i^R(p)(1 - a_i^R(p)) \). From (4.20), we know

\[
\frac{\partial^2 \log \mu_i(p, y_j)}{\partial p_i^2} = \frac{\partial^2 \log (\lambda - E(min\{N_i(1), y_j\}))}{\partial p_i^2} + \frac{\log a_i^R(p)}{\partial p_i^2}
\leq \lambda_i^2 a_i(p)(1 - a_i(p)) - \lambda_i^2 a_i^R(p)(1 - a_i^R(p))
= \lambda_i^2 (a_i(p) - a_i^R(p))(1 - (a_i(p) + a_i^R(p))) \leq 0,
\]

where the first inequality follows from (4.24), and the last inequality follows from \( a_i(p) \leq a_i^R(p) \) and the assumption that \( a_i(p) + a_i^R(p) \leq 1 \).

The assumption of Lemma 4.2.1 depends on the value of \( C_i \). To better understand this value, we define service level \( s_i = P(N_i(1) \leq y_i) \) and expected demand \( n_i = E(N_i(1)) = \lambda a_i(p) \) for retailer \( i, i = 1, 2 \), which is the retailer’s own demand without taking into account the additional demand due to stockout demand substitution. The following table shows how \( C_i \) depends on \( s_i \) and \( n_i \).

<table>
<thead>
<tr>
<th>( n_i )</th>
<th>( s_i=0.5 )</th>
<th>( s_i=0.6 )</th>
<th>( s_i=0.7 )</th>
<th>( s_i=0.8 )</th>
<th>( s_i=0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.25</td>
<td>1.25</td>
<td>1.1</td>
<td>0.9</td>
<td>0.73</td>
</tr>
<tr>
<td>50</td>
<td>2.8</td>
<td>2.75</td>
<td>2.5</td>
<td>2.1</td>
<td>1.4</td>
</tr>
<tr>
<td>100</td>
<td>4</td>
<td>3.95</td>
<td>3.6</td>
<td>3</td>
<td>1.9</td>
</tr>
<tr>
<td>500</td>
<td>8.9</td>
<td>8.7</td>
<td>7.8</td>
<td>6.4</td>
<td>4</td>
</tr>
<tr>
<td>1000</td>
<td>12.6</td>
<td>12.3</td>
<td>11.2</td>
<td>8.9</td>
<td>5.6</td>
</tr>
</tbody>
</table>

As we can see from above table, \( C_i \) increases with \( n_i \) and decreases with \( s_i \). For \( C_i = 2 \), \( \frac{2}{\sqrt{4c_i+1}+3} = \frac{1}{3} \), and for \( C_i = 12 \), \( \frac{2}{\sqrt{4c_i+1}+3} = \frac{1}{5} \).

Lemma 4.2.1 implies the following theorem.

**Theorem 4.2.2.** For a joint pricing and inventory game, there exists \( a_0^* \) such that if the no buy option satisfies \( a_0 \geq a_0^* \), a pure strategy Nash equilibrium exists.

**Proof.** Increasing the no buy option \( a_0 \) will decrease the probability \( a_i(p) \) and \( a_i^R(p) \), and also decrease the expected demand \( \lambda a_i(p) \), which decreases the parameters \( C_i \). Therefore,
the conditions of Lemma 4.2.1 can be satisfied by increasing $a_0$. \hfill \Box

Now, we give some insight on the joint pricing and inventory game. Let’s go back to Example 4.2.1 to see why retailer 1 can have two optimal price points that maximize $\pi_1(p, y_2)$, and how the assumptions in Lemma 4.2.1 can exclude this case. Although they lead to the same payoff, the two optimal price points represent two different strategies. The low price point represents a strategy of competing with retailer 2 on price. The high price point represents a strategy that pushes customer to retailer 2, and hence retailer 2 stocks out, after which retailer 1 becomes a monopolist.

Notice the probability that a customer chooses retailer 1 after retailer 2 stocks out,

$$a_1^R(p_1, a_0) = \frac{k_1 e^{-\lambda_1 p_1}}{a_0 + k_1 e^{-\lambda_1 p_1}},$$

has the property that $\log a_1^R(p_1, a_0)$ has decreasing differences in $(p_1, a_0)$, i.e., if $p'_1 \geq p_1$ and $a'_0 \geq a_0$, we have

$$\log a_1^R(p'_1, a'_0) - \log a_1^R(p_1, a_0) \leq \log a_1^R(p'_1, a'_0) - \log a_1^R(p_1, a_0),$$

and hence

$$\frac{a_1^R(p_1, a_0) - a_1^R(p'_1, a'_0)}{a_1^R(p_1, a_0)} \leq \frac{a_1^R(p_1, a_0) - a_1^R(p'_1, a'_0)}{a_1^R(p_1, a_0)},$$

which implies that demand sensitivity for retailer 1 is increasing with the no-buy option $a_0$.

Therefore, it is easy to understand that the high price strategy in Example 4.2.1 is based on the fact that the no buy option, $a_0 = 0.0001$, is relatively small such that the demand for retailer 1 is insensitive to her price, which makes the high price strategy as profitable as the low price strategy. Lemma 4.2.1 shows that if $a_i(p)$ and $a_i^R(p)$ are bounded in some way, the high price strategy of retailer 1 has to be excluded, which makes the best response correspondence be unique. Especially, as the no-buy option $a_0$ increases, demand becomes more sensitive to its price. Then, the high price strategy is excluded. We can conclude that the existence of a pure strategy Nash equilibrium in a joint pricing and inventory game depends on the sensitivity of the retailer’s demand to her price.

Lemma 4.2.1 also implies that if both retailers adopt high service level strategy, a joint pricing and inventory game will more likely to have a pure strategy Nash Equilibrium. This is because from the newsvendor model, we know high service level corresponds to high price $p_i$, which decreases $a_i(p)$ and $a_i^R(p)$. At the same time, as shown in the table, $C_i$ decreases with service level. Therefore, a high service level is more likely to imply that the assumptions
of Lemma 4.2.1 are satisfied.

4.3 Summary

In this chapter, I show the log-concavity of profit function under mild conditions. This result and the price-dependent demand substitution rule are applied to study a retail competition game. In this model, multiple retailers compete on price and inventory and shortage of one product affects the demand of other products. I identify conditions under which a pure strategy Nash equilibrium exists.
Chapter 5

Adaptive Safe Price

In this chapter, we use the structure developed in Almgren and Chriss [2] to model how security’s price is affected by the trader’s order and its evolution over periods. This model is also used in Huberman and Stanzl [13]. In particular, in each period, the trader’s order has temporary and permanent impact on the security’s price. The initial price of the security at time $i$, $p_i$, is observed by the trader. Given this price, the trader faces the transaction price $\hat{p}_i = p_i + \lambda_{1i} q_i$ to buy the quantity $q_i$, where $\lambda_{1i}$ is a positive constant that measures the temporary price impact of the trader’s order. The new initial price for the next period evolves according to $p_{i+1} = p_i + \lambda_{2i} q_i + \epsilon_i$, where $\lambda_{2i}$ is a positive constant that measures the permanent price impact of the trader’s order and $\epsilon_i$ is a random variable. This price motion law is illustrated in Figure 5-1.

Given this law of motion for price $p_i$ and the above analysis, the optimal policy of the trader is given as the optimal solution of the following optimization problem:

$$\max q_i \quad E\{\Sigma_{i=0}^N q_i\}$$

subject to

$$\hat{p}_i = p_i + \lambda_{1i} q_i$$

$$p_{i+1} = p_i + \lambda_{2i} q_i + \epsilon_i$$

$$\hat{p}_i \leq \bar{p}_i, \quad i = 0, \ldots, N.$$  

We point out that $\bar{p}_i$, $i = 0, 1, \ldots, N$, is the trader’s estimation of safety prices rather
than decision variables. At the beginning of each period $i$, the trader only has an estimate of $\bar{P}_i$ and $\bar{P}_j$ is unknown for $j > i$. The formulation above is general enough to incorporate changes in the safe price to adapt to market dynamics, which allows us to specify the order size as a function of the safe price estimation at that period.

The objective of this chapter is to characterize the optimal trading strategy based on the above formulation. It is easy to see that if $p_i \geq \bar{P}_i$ in period $i$, the trader will not purchase anything in this period. On the other hand, it is not clear how much should the trader purchase when $p_i < \bar{P}_i$. It is tempting to conclude that since the trader’s goal is to maximize the number of units of security, the trader should purchase to increase price up to $\bar{P}_i$, i.e., purchase $q_i^* = \frac{\bar{P}_i - p_i}{\lambda_{ii}}$. We refer to this policy as the greedy policy. Unfortunately, we show in Section 5.1, using a counter example, that the greedy policy is not always optimal. In Section 5.2, we show that under some reasonable conditions, the greedy policy is indeed optimal. Remarkably, we need only impose conditions on the price impact parameters $\lambda_{ki}$, $k = 1, 2$ and $i = 0, \ldots, N$. We do not need to make any assumption on the random variable $\epsilon_i$, price $p_i$ or safe price $\bar{P}_i$. In Section 5.3, we study the performance of the greedy policy when it is not optimal. We derive a lower bound on the ratio of the value returned by the greedy policy to the optimal value of (5.1)-(5.4). We also give a discussion on the structure

Figure 5-1: The law of price motion.
of the optimal policy in general.

5.1 Greedy Policy is Not Always Optimal

In this section, we give an example where the greedy policy is not optimal. First, we need the following lemma.

**Lemma 5.1.1.** If a function \( f(x) \) is convex over \([a, b]\), then \( \max_{x \in [a, b]} f(x) = \max\{f(a), f(b)\} \).

**Proof.** For any \( x \in [a, b] \), there exists some \( \lambda \in [0, 1] \) such that \( x = \lambda a + (1 - \lambda) b \). Then,

\[
\begin{align*}
f(x) &= f(\lambda a + (1 - \lambda) b) \\
&\leq \lambda f(a) + (1 - \lambda) f(b) \\
&\leq \lambda \max\{f(a), f(b)\} + (1 - \lambda) \max\{f(a), f(b)\} \\
&= \max\{f(a), f(b)\},
\end{align*}
\]

which completes the proof. \(\square\)

This lemma tells us that the maximal value of a univariate convex function is achieved at one of the endpoints.

For problem (5.1)-(5.4), we define \( J_i(p_i) \) to be the optimal value to go from period \( i \) at price \( p_i \). In the following example, we show that the greedy policy is not optimal.

**Example 5.1.1.** We set \( N = 1 \) and assume \( \bar{p}_0 = \bar{p}_1 = \bar{p} \). Since period 1 is the last period, it is easy to see that

\[
J_1(p_1) = \begin{cases} 
\frac{\bar{p} - p_1}{\lambda_{11}} & \text{if } p_1 < \bar{p}, \\
0 & \text{if } p_1 \geq \bar{p}.
\end{cases}
\]  

(5.6)

\( J_1(p_1) \) is a convex function over \( R \). Therefore, for \( p_0 < \bar{p} \),

\[
J_0(p_0) = \max_{0 \leq q_0 \leq \frac{\bar{p} - p_0}{\lambda_{10}}} q_0 + E(J_1(p_0 + \lambda_{20} q_0 + \epsilon_0))
\]

\[
= \max\{E(J_1(p_0 + \epsilon_0)) + \frac{\bar{p} - p_0}{\lambda_{10}} + E(J_1(p_0 + \lambda_{20} \frac{\bar{p} - p_0}{\lambda_{10}} + \epsilon_0))\},
\]

where the second equality follows from Lemma 5.1.1. Note that if \( J_0(p_0) \) is achieved at
Figure 5-2: Greedy policy fails.

\[ E(J_1(p_0 + \epsilon_0)) \] the trader should not order anything in period 0; otherwise he should use the greedy policy in that period.

We define \( s = \bar{p} - p_0, \ a = 1 - \frac{\lambda_20}{\lambda_{10}} \) and

\[
\Delta = E(J_1(p_0 + \epsilon_0)) - E(J_1(p_0 + \lambda_{20} \frac{\bar{p} - p_0}{\lambda_{10}} + \epsilon_0)) - \frac{\bar{p} - p_0}{\lambda_{10}} \\
= \int_{-\infty}^{\bar{p} - p_0} \frac{\bar{p} - p_0 - t}{\lambda_{11}} \phi(t) dt - \int_{-\infty}^{\bar{p} - p_0 - \lambda_{20} \frac{\bar{p} - p_0}{\lambda_{10}}} \frac{\bar{p} - p_0 - \lambda_{20} \frac{\bar{p} - p_0}{\lambda_{10}} - t}{\lambda_{11}} \phi(t) dt - \frac{\bar{p} - p_0}{\lambda_{10}} \\
= \frac{1}{\lambda_{11}} \left( \int_{-\infty}^{s} (s - t) \phi(t) dt - \int_{-\infty}^{as - t} (as - t) \phi(t) dt \right) - \frac{s}{\lambda_{10}} = \frac{1}{\lambda_{11}} \int_{as}^{s} \Phi(t) dt - \frac{s}{\lambda_{10}}, \ (5.7)
\]

where \( \phi(t) \) and \( \Phi(t) \) are pdf and cdf of random variable \( \epsilon_0 \). The last equality follows from integration by parts.

We set \( \lambda_{10} = \lambda_{11} = 0.001, \ \lambda_{20} = 0.003, \) and assume that \( \epsilon_0 \) follows standard normal distribution, i.e., \( N(0, 1) \). In Figure 5-2, x-axis represents the value of \( s \) and the y-axis is \( \Delta \). As we can see, \( \Delta \geq 0 \) for all values of \( s \) and hence the optimal policy at period 0 is not to order at all instead of ordering up to \( \bar{p} \). Therefore, the greedy policy is not optimal in this example.
5.2 When is Greedy Policy Optimal?

In this section, we establish conditions under which the greedy policy is an optimal solution to (5.1)-(5.4). For this purpose, we first show some properties of \( J_i(p) \). For notational convenience, in this section we omit the time index of price \( p \). The following lemma shows a monotonicity property of \( J_i(p) \).

**Lemma 5.2.1.** For any \( i = 0, 1, \ldots, N \), \( J_i(p) \) decreases with \( p \).

**Proof.** We prove the lemma by applying backward induction. For \( i = N \), this property easily follows from the fact that

\[
J_N(p) = \begin{cases} \frac{p - \bar{p}_N}{\lambda_{1N}} & \text{if } p < \bar{p}_N, \\ 0 & \text{if } p \geq \bar{p}_N. \end{cases}
\] (5.8)

Assume \( J_{i+1}(p) \) decreases with \( p \). Without loss of generality, let \( p_1 \leq p_2 \). We consider three cases:

**Case 1:** For any \( p_1 \leq p_2 \leq \bar{p}_i \),

\[
J_i(p_1) = \max_{0 \leq q_i \leq \frac{p_1 - p}{\lambda_{1i}}} q_i + E(J_{i+1}(p_1 + \lambda_2 q_i + \epsilon_i)) \geq \max_{0 \leq q_i \leq \frac{p_2 - p}{\lambda_{1i}}} q_i + E(J_{i+1}(p_2 + \lambda_2 q_i + \epsilon_i)) \\
\geq \max_{0 \leq q_i \leq \frac{p - p}{\lambda_{1i}}} q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i)) \\
= J_i(p_2),
\] (5.9)

where the first inequality follows from the induction assumption and the second inequality follows from the facts that \( \frac{p_1 - p}{\lambda_{1i}} \geq \frac{p_2 - p}{\lambda_{1i}} \) and this is a maximization problem.

**Case 2:** For any \( p_2 \geq p_1 \geq \bar{p}_i \), we have

\[
J_i(p_1) = E(J_{i+1}(p_1 + \epsilon_i)) \geq E(J_{i+1}(p_2 + \epsilon_i)) = J_i(p_2).
\] (5.10)

**Case 3:** For any \( p_1 \leq \bar{p}_i \leq p_2 \), it follows from (5.9) and (5.10) that \( J_i(p_1) \geq J_i(\bar{p}_i) \geq J_i(p_2) \). □
We now provide an upper bound on the amount lost if the starting price in period $i$ is increased by $h$, i.e., an upper bound on $J_i(p) - J_i(p + h)$ for any $h > 0$ and $i = 0, \ldots, N$.

Consider first the last period $N$. For any $p$ and $h > 0$, we must have

$$J_N(p) - J_N(p + h) \leq \frac{h}{\lambda_{1N}}. \quad (5.11)$$

This is easy to see by inspecting equation (5.8) for $p, p + h < \bar{p}_N$ or $p, p + h \geq \bar{p}_N$. If $p < \bar{p}_N$ and $p + h \geq \bar{p}_N$, then the same equation suggests that $J_n(p) - J_n(p + h) = \frac{\bar{p}_N - p}{\lambda_{1N}} \leq \frac{h}{\lambda_{1N}}$ because $p + h \geq \bar{p}_N$. The following theorem extends this observation by providing an upper bound for any $i$.

**Theorem 5.2.1.** Define $a_N = \frac{1}{\lambda_{1N}}$ and $a_i = \max\{a_{i+1}, \frac{1}{\lambda_{ii}}, a_{i+1} + \frac{(1-\lambda_{ii}a_{i+1})}{\lambda_{ii}}\}$ for $i = 0, 1, \ldots, N - 1$. Then for any price $p$ and $h > 0$, we have

$$J_i(p) - J_i(p + h) \leq a_i h, \quad (5.12)$$

for any $i$.

**Proof.** We prove the result using backward induction. We have shown that (5.12) holds for $i = N$. Assume (5.12) holds for $i + 1$, for any $p$ with $p + h < \bar{p}_i$, we know

$$J_i(p) = \max_{0 \leq q_i \leq \frac{p - h}{\lambda_{ii}}} q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i)) = \max_{0 \leq q_i \leq \frac{p - h}{\lambda_{ii}}} \max_{\frac{p - h}{\lambda_{ii}} \leq q_i \leq \frac{p}{\lambda_{ii}}} q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i)), \max_{\frac{p - h}{\lambda_{ii}} \leq q_i \leq \frac{p}{\lambda_{ii}}} q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i)) \}.$$}

We start by focusing on the first component of the right hand side of the above equation.

$$\max_{0 \leq q_i \leq \frac{p - h}{\lambda_{ii}}} q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i)) \leq \max_{0 \leq q_i \leq \frac{p - h}{\lambda_{ii}}} \{q_i + E(J_{i+1}(p + h + \lambda_2 q_i + \epsilon_i))\} + a_{i+1} h = J_i(p + h) + a_{i+1} h \leq J_i(p + h) + a_i h,$$

where the first inequality follows from induction assumption, the equality follows from definition of $J_i(p + h)$, and the last inequality follows from the definition of $a_i$.

For the second component, observe that the definition of $J_i(p+h)$ implies that $J_i(p+h) \geq \ldots$
\[
\bar{E}_{\lambda_i}(p - h) + E(J_{i+1}(p + h + \lambda_2 \bar{E}_{\lambda_i}(p - h) + \epsilon_i)). \]

It follows that
\[
\max_{\bar{E}_{\lambda_i}(p - h) \leq q_i \leq \bar{E}_{\lambda_i}(p)} q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i))
\leq \max_{\bar{E}_{\lambda_i}(p - h) \leq q_i \leq \bar{E}_{\lambda_i}(p)} \{q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i)) - \bar{p}_i - p - h \over \lambda_i \}
- E(J_{i+1}(p + h + \lambda_2 {\bar{p}_i - p - h \over \lambda_i} + \epsilon_i)) + J_i(p + h).
\]

(5.13)

If \(\lambda_2 q_i \geq h + \lambda_2 {\bar{p}_i - p - h \over \lambda_i} \), Lemma 5.2.1 and inequality (5.13) imply that
\[
\max_{\bar{E}_{\lambda_i}(p - h) \leq q_i \leq \bar{E}_{\lambda_i}(p)} q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i))
\leq \max_{\bar{E}_{\lambda_i}(p - h) \leq q_i \leq \bar{E}_{\lambda_i}(p)} \{q_i - \bar{p}_i - p - h \over \lambda_i \} + J_i(p + h)
= h \over \lambda_i + J_i(p + h) \leq a_i h + J_i(p + h).
\]

If \(\lambda_2 q_i < h + \lambda_2 {\bar{p}_i - p - h \over \lambda_i} \), the induction assumption and inequality (5.13) imply that
\[
\max_{\bar{E}_{\lambda_i}(p - h) \leq q_i \leq \bar{E}_{\lambda_i}(p)} q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i))
\leq \max_{\bar{E}_{\lambda_i}(p - h) \leq q_i \leq \bar{E}_{\lambda_i}(p)} \{q_i - \bar{p}_i - p - h \over \lambda_i \}
+ a_{i+1} \{h + \lambda_2 {\bar{p}_i - p - h \over \lambda_i} - \lambda_2 q_i \} + J_i(p + h)
= \max_{\bar{E}_{\lambda_i}(p - h) \leq q_i \leq \bar{E}_{\lambda_i}(p)} \{(1 - \lambda_2 a_{i+1}) \{q_i - \bar{p}_i - p - h \over \lambda_i \} \}
+ a_{i+1} h + J_i(p + h)
\leq \max \{(1 - \lambda_2 a_{i+1}) \{q_i - \bar{p}_i - p - h \over \lambda_i \} + a_{i+1} \} h + J_i(p + h)
\leq a_i h + J_i(p + h),
\]

(5.14)

where the second inequality follows from considering the two cases that \(1 - \lambda_2 a_{i+1} \leq 0 \) and \(1 - \lambda_2 a_{i+1} \geq 0 \).

In summary, for any \(p, p + h < \bar{p}_i \), (5.12) holds.

We next consider any \(p \) with \(p, p + h \geq \bar{p}_i \),
\[
J_i(p) - J_i(p + h) = E(J_{i+1}(p + \epsilon_i)) - E(J_{i+1}(p + h + \epsilon_i)) \leq a_{i+1} h \leq a_i h,
\]

(5.15)
where the first inequality follows from induction assumption and the second inequality follows from definition of $a_i$.

The last possible case is $p < \bar{p}_i$ and $p + h \geq \bar{p}_i$. We have

$$J_i(p) - J_i(p + h) = \max_{0 \leq q_i \leq \frac{h}{\lambda_i}} q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i)) - E(J_{i+1}(p + h + \epsilon_i))$$

$$\leq \max_{0 \leq q_i \leq \frac{h}{\lambda_i}} q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i)) - E(J_{i+1}(p + h + \epsilon_i)), \quad (5.16)$$

where the last inequality follows from $\bar{p}_i - p \leq h$.

If $\lambda_2 q_i > h$, Lemma 5.2.1 and inequality (5.16) imply that

$$J_i(p) - J_i(p + h) \leq \max_{0 \leq q_i \leq \frac{h}{\lambda_i}} q_i = \frac{h}{\lambda_i} \leq a_i h \quad (5.17)$$

If $\lambda_2 q_i \leq h$, the induction assumption and inequality (5.16) imply that

$$J_i(p) - J_i(p + h) \leq \max_{0 \leq q_i \leq \frac{h}{\lambda_i}} q_i + a_{i+1}(h - \lambda_2 q_i) = \max_{0 \leq q_i \leq \frac{h}{\lambda_i}} (1 - \lambda_2 a_{i+1}) q_i + a_{i+1}h$$

$$\leq \max \{a_{i+1}, \frac{(1 - \lambda_2 a_{i+1})}{\lambda_i} + a_{i+1}\} h \leq a_i h,$$

where the first inequality follows from considering the two cases: $1 - \lambda_2 a_{i+1} \leq 0$ and $1 - \lambda_2 a_{i+1} \geq 0$.

In summary, for any $p < \bar{p}_i$ and $p + h \geq \bar{p}_i$, (5.12) holds. \qed

Theorem 5.2.1 tells us that the difference between $J_i(p)$ and $J_i(p + h)$ is bounded by a linear function of $h$ with slope $a_i$, which depends on the price impact parameters $\lambda_{ki}$ but not on the price $p$, noise $\epsilon_j$, $j = i, \ldots, N - 1$ and how the trader sets the safe prices $\bar{p}_j$ in the later periods $j = i, \ldots, N$. This theorem motivates the following sufficient condition under which the greedy policy is optimal.

**Theorem 5.2.2.** At period $i$, if the temporary and permanent price impact parameters satisfy

$$\lambda_2 a_{i+1} \leq 1, \quad (5.18)$$
then greedy policy is optimal to (5.1)-(5.4), i.e., if $p < \bar{p}_i$, $q_i^* = \frac{\bar{p}_i}{\lambda_{ii}}$ is the optimal quantity.

**Proof.** For any $0 \leq q_i \leq q_i^*$, we have

$$q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i)) - (q_i^* + E(J_{i+1}(p + \lambda_2 q_i^* + \epsilon_i))) \leq q_i - q_i^* + a_{i+1} \lambda_2 (q_i^* - q_i)$$

$$= (q_i - q_i^*)(1 - a_{i+1} \lambda_2)$$

$$\leq 0,$$  \hspace{1cm} (5.19)

where the first inequality follows from Theorem 5.2.1.

Thus, since $J_i(p) = \max_{0 \leq q_i \leq \frac{\bar{p}_i}{\lambda_{ii}}} q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i))$, we have that $q_i^* = \frac{\bar{p}_i}{\lambda_{ii}}$ is the optimal quantity. $\square$

We now explain the intuition behind Theorem 5.2.2. Consider purchasing one additional unit in period $i$. This unit will increase the price in period $i + 1$ by $\lambda_2 i$. Theorem 5.2.1 tells us that starting from period $i + 1$, we will lose at most $\lambda_2 i a_{i+1}$ due to this price increase. Therefore, if Eq. (5.18) is satisfied, greedy policy is optimal since what we can lose in subsequent periods is less than what we gain in the current period.

Next we identify cases where the condition $\lambda_2 i a_{i+1} \leq 1$ is satisfied, that is, we identify cases where the greedy policy is optimal. The first case is when the temporary price impact is greater than or equal to the permanent price impact, i.e., $\lambda_{ii} \geq \lambda_2 i$, and the permanent price impact is nondecreasing with time, i.e., $\lambda_{2(i+1)} \leq \lambda_2 i$. This is shown in the following proposition.

**Proposition 5.2.1.** If for each $i$, $\lambda_{ii} \geq \lambda_2 i$ and $\lambda_2 i \leq \lambda_{2(i+1)}$, then the greedy policy is optimal.

**Proof.** We show $\lambda_2 i a_{i+1} \leq 1$ by backward induction. For $i = N-1$, $\lambda_{2(N-1)} a_N = \frac{\lambda_{2(N-1)}}{\lambda_{1N}} \leq \frac{\lambda_{2N}}{\lambda_{1N}} \leq 1$. Assume $\lambda_2 i a_{i+1} \leq 1$, then $a_{i+1} + \frac{(1-\lambda_2 i a_{i+1})}{\lambda_{ii}} \geq a_{i+1}$. Also, $a_{i+1} + \frac{(1-\lambda_2 i a_{i+1})}{\lambda_{ii}} = (1 - \frac{\lambda_2 i}{\lambda_{ii}}) a_{i+1} + \frac{1}{\lambda_{ii}} \geq \frac{1}{\lambda_{ii}}$, hence $a_i = (1 - \frac{\lambda_2 i}{\lambda_{ii}}) a_{i+1} + \frac{1}{\lambda_{ii}}$. Therefore, $\lambda_{2(i-1)} a_i \leq \lambda_2 a_i = (1 - \frac{\lambda_2 i}{\lambda_{ii}}) \lambda_2 i a_{i+1} + \frac{\lambda_2 i}{\lambda_{ii}} \leq (1 - \frac{\lambda_2 i}{\lambda_{ii}}) + \frac{\lambda_2 i}{\lambda_{ii}} \leq 1$. By Theorem 5.2.2, we know that the greedy policy is optimal. $\square$
**Practical Indication**: After the trader places an order for the security, he observes an immediate price increase. If after a short time the security price begins to drop down, then this indicates that temporary price impact is greater than permanent price impact. If, in addition, placing the same order in the next period results in a higher permanent price impact, and this is true for any consecutive periods, then there is a reason to believe that the conditions of Proposition 5.2.1 are satisfied. This proposition shows that under current market condition, the trader should greedily buy this security whenever the price falls below his safe price. This scenario is illustrated in Figure 5-3.

![Figure 5-3: First case that greedy policy is optimal.](image)

The second case is when a unit ordered in this period will increase next period price more than current period price, i.e., \( \lambda_{1i} \leq \lambda_{2i} \), but less than buying an additional unit next period, i.e., \( \lambda_{2i} \leq \lambda_{1(i+1)} \). This intuitively implies that as time progresses, the security is desirable by more and more people.

**Proposition 5.2.2.** If for each \( i \), \( \lambda_{1i} \leq \lambda_{2i} \) and \( \lambda_{2i} \leq \lambda_{1(i+1)} \), then the greedy policy is optimal.

*Proof.* We show \( \lambda_{2i}a_{i+1} \leq 1 \) using backward induction. For \( i = N - 1 \), \( \lambda_{2(N-1)}a_N = \frac{\lambda_{2(N-1)}}{\lambda_{1(N-1)}} \leq 1 \). Assume \( \lambda_{2i}a_{i+1} \leq 1 \), then \( a_{i+1} + \frac{(1-\lambda_{2i}a_{i+1})}{\lambda_{1i}} \geq a_{i+1} \), and \( a_{i+1} + \frac{(1-\lambda_{2i}a_{i+1})}{\lambda_{1i}} \leq 1 \).
(1 - \frac{\lambda_{2i}}{\lambda_{ii}})a_{i+1} + \frac{1}{\lambda_{ii}} \leq \frac{1}{\lambda_{ii}}. \text{ Hence, using definition of } a_i, \text{ we have } a_i = \frac{1}{\lambda_{ii}}. \text{ Therefore, } \\
\lambda_{2(i-1)}a_i = \frac{\lambda_{2(i-1)}}{\lambda_{ii}} \leq 1. \text{ Theorem 5.2.2 implies that the greedy policy is optimal. } \square \\

**Practical Indication**: Figure 5-4 illustrates the expected motion of prices that satisfies the conditions of Proposition 5.2.2. After the trader places an order for the security, he observes an immediate price increase. If he doesn’t observe a price drop after a short time, then we conclude that the temporary price impact is no more than the permanent price impact. In addition, placing the same order in the next period results in a higher temporary price impact than the permanent price impact, and this is true for any consecutive periods.

![Figure 5-4: Second case that greedy policy is optimal.](image)

**5.3 Lower Bound for Greedy Policy**

In Section 5.2, we identified conditions under which the greedy policy is optimal. Unfortunately, as we saw in Section 5.1, the greedy policy is not always optimal. In these cases, it is important to identify the effectiveness of this policy.

For this purpose we start by providing insight on the structure of the optimal policy. Evidently, the greedy policy is optimal in the last period since future prices do not matter.
Consider the second to last period, period $N - 1$, since $J_N(p)$ is a convex function, Lemma 5.1.1 implies that for a given starting price in period $N - 1$, the optimal policy in period $N - 1$ is either the greedy policy or a “no-buy policy”, a policy in which the decision maker does not buy any security. The following example shows that indeed in period $N - 1$, the optimal policy may be either greedy or “no-buy” depending on the starting price in period $N - 1$.

**Example 5.3.1.** We consider a two-period model, $N=1$. We use the same notation as we defined in Example 5.1.1 except that we set $\lambda_{11} = 0.001$, $\lambda_{10} = 0.00105$ $\lambda_{20} = 0.00315$, and assume that $\epsilon_0$ follows normal distribution $N(0.15, 1)$. In Figure 5-5, the $x$-axis is $s$, the difference between the current price and the safety price, and the $y$-axis is $\Delta$, the difference between the objective function for ”no-buy” policy and ”greedy” policy. As you can see, there are two separate regions where the no-buy policy is optimal; otherwise the greedy is optimal.

![Figure 5-5: Optimal regions for No-buy and greedy policy.](image)

In general, for periods other than the last two periods, we do not expect optimal policy to be well-structured because the function $J_i(p)$ need not be convex for $i \leq N - 1$. To see this, let $\epsilon_{N-1}$ be uniformly distributed. Then $J_{N-1}(p)$ is a piecewise linear or quadratic function.
that is not necessarily a convex function.

Since the structure of the optimal policy may be complex in general, it is interesting to characterize the effectiveness of the greedy policy. For this purpose, we derive a lower bound on the ratio of the value returned by greedy policy to the optimal value of (5.1)-(5.4).

**Theorem 5.3.2.** For any \( p \), let \( J_0(p) \) be the optimal value of (5.1)-(5.4) and \( J_0^*(p) \) be the value returned by greedy policy. Denote \( \mathcal{I} = \{i \in \{0, 1, \ldots, N - 1\} | \lambda_2 a_{i+1} \geq 1\} \), then

\[
\frac{J_0^*(p)}{J_0(p)} \geq \prod_{i \in \mathcal{I}} \frac{1}{\lambda_2 a_{i+1}}.
\]  

\[ (5.20) \]

**Proof.** For any \( p < \bar{p} \), let \( q_i^* = \frac{E_i - p}{\lambda_{ti}} \) and \( \bar{q}_i = \arg\max_{q_i \in \frac{[E_i - p]}{\lambda_{ti}}} \{q_i + E(J_{i+1}(p + \lambda_2 q_i + \epsilon_i))\} \). Then for any \( i \) such that \( \lambda_2 a_{i+1} \geq 1 \), we have

\[
\frac{q_i^* + E(J_{i+1}(p + \lambda_2 q_i^* + \epsilon_i))}{\bar{q}_i + E(J_{i+1}(p + \lambda_2 \bar{q}_i + \epsilon_i))} \geq \frac{q_i^* + E(J_{i+1}(p + \lambda_2 q_i^* + \epsilon_i))}{\bar{q}_i + E(J_{i+1}(p + \lambda_2 q_i^* + \epsilon_i)) + a_{i+1}\lambda_2 (q_i^* - \bar{q}_i)} \geq \frac{q_i^* + E(J_{i+1}(p + \lambda_2 q_i^* + \epsilon_i))}{a_{i+1}\lambda_2 (q_i^* + E(J_{i+1}(p + \lambda_2 q_i^* + \epsilon_i)))} = \frac{1}{a_{i+1}\lambda_2} \]  

\[ (5.21) \]

where the first inequality follows from (5.12) and second inequality follows from the assumption that \( \lambda_2 a_{i+1} \geq 1 \). Now we prove (5.20) by induction on the number of periods. Assume

\[
\frac{J_0^*(p)}{J_1(p)} \geq \prod_{i \in \mathcal{I}\setminus\{0\}} \frac{1}{\lambda_2 a_{i+1}},
\]

then

\[
J_0^*(p) = q_0^* + E(J_1^*(p + \lambda_2 q_0^* + \epsilon_0)) \geq q_0^* + (\prod_{i \in \mathcal{I}\setminus\{0\}} \frac{1}{\lambda_2 a_{i+1}})E(J_1(p + \lambda_2 q_0^* + \epsilon_0)) \geq (\prod_{i \in \mathcal{I}\setminus\{0\}} \frac{1}{\lambda_2 a_{i+1}})(q_0^* + E(J_1(p + \lambda_2 q_0^* + \epsilon_0))) \geq (\prod_{i \in \mathcal{I}\setminus\{0\}} \frac{1}{\lambda_2 a_{i+1}} \frac{1}{\lambda_2 a_1}) \frac{1}{\lambda_2 a_1} (\bar{q}_0 + E(J_1(p + \lambda_2 \bar{q}_0 + \epsilon_0))) = (\prod_{i \in \mathcal{I}} \frac{1}{\lambda_2 a_{i+1}}) J_0(p),
\]  

\[ (5.22) \]

where the second inequality follows from \( \lambda_2 a_{i+1} \geq 1 \) and third inequality follows from (5.21). \( \Box \)
Observe that the theorem is noise-independent and price-independent, i.e., we do not make any assumptions on the noise \( \epsilon_i \), price \( p \) and how the trader sets the safe prices \( \bar{p}_j \) in the later periods \( j = 1, \ldots, N \). As we can see, the performance of the greedy policy only depends on the temporary and permanent price impact parameters \( \lambda_{ki}, k = 1,2 \) and \( i = 0,1, \ldots, N - 1 \).

**Example 5.3.3.** Consider a three-period example with \( \lambda_{01} = 10^{-5}, \lambda_{11} = 1.2 \times 10^{-5}, \lambda_{21} = 9 \times 10^{-6} \) and \( \lambda_{02} = \lambda_{12} = \lambda_{22} = 10^{-5} \); these values are consistent with the examples provides in Huberman and Stanzl [13]. Theorem 5.3.2 tells us that \( \frac{J^*(p)}{J_0(p)} \geq 0.81 \), i.e., if a trader use greedy policy in every period, he can get at least 81% of the maximal number of shares he can possible get.

### 5.4 Summary

In this chapter, I apply a dynamic programming approach to study optimal trading with adaptive safe price. The analysis indicates that while the greedy is not always optimal, there are important cases where it is. To characterize the effectiveness of greedy policy, we also provide a lower bound on the ratio of the value returned by greedy policy to the optimal value of (5.1)-(5.4).
Chapter 6

Conclusion and Future Research

In Chapter 2, I apply an axiomatic approach to characterize how stock-out demand substitution depends on the prices of all the products. Without any assumption on the structure of the demand model for the remaining products, I show that there is a unique price-dependent demand substitution rule to determine the remaining products demand model. This demand substitution rule serves as a building block to study joint pricing and inventory coordination as well as retail competition. Also in this chapter, I use the demand substitution rule to study how demand sensitivity and system demand depend on the number of products.

In Chapter 3, I extend the demand substitution rule to a stochastic environment by using consumer choices theory. I apply this result to a joint pricing and inventory coordination model, and demonstrate the impact of the substitution rule on the optimal inventory and pricing strategy. Specifically, we compare two inventory-pricing models: one in which demand for a product disappear during stockout time and one where substitution occurs according to our demand substitution rule. The result illustrates that if the axiomatic approach is acceptable, the optimal strategy and corresponding expected profit are quite different than models that ignore stockout demand substitution. Also in this chapter, I show that more products don’t always lead retailers a higher profit and identify conditions under which two products can provide a higher profit than a single product.

Of course Chapters 2 and 3 focus on a a single-period model, which is appropriate for per-
ishable products and fashion items. For non-perishable products, if the retailer overstocks, inventory can be carried over to the next period. Evidently, in this case the retailer needs to pay the inventory holding cost. If some products are out of stock, customers substitute to other products according to the price-dependent demand substitution rule. At the same time, the retailer needs to coordinate the price of all products. This is a multi-period joint pricing and inventory coordination model for substitutable products, which we plan to explore in the future.

Notice that in Chapter 3, we apply an extensive search algorithm to find the optimal price and inventory of each product so as to maximize total expected profit. To be computationally efficient, it will be interesting to characterize the optimal structure of the optimal policy, or at least to obtain some bounds on the optimal solutions. In fact, the extensive search algorithm is not likely to be effective as the number of products increases. One possible way to address this challenge is to apply sample path methods. Heuristic algorithms, based perhaps on the characteristic of this optimal strategy, may also be effective. We plan to analyze both strategies in the future.

In practice, retailers don’t know the specific structure of customer demand as a function of all product prices. Indeed, they may have only a few data points on the relationship between price and demand, and hence this precludes the applications of inventory-pricing optimization models. An interesting future research therefore is an empirical work that includes (i) design of experiment; (ii) collecting data and (iii) developing a multi-product demand model for a set of specific products. Such a model must incorporate the demand substitution rule developed in this thesis. Otherwise, the model will be inconsistent; see Examples 2.1.1 and 2.1.2. If it’s difficult to build the exact demand model, robust optimization and data-driven optimization that takes into account stock-out demand substitution, are reasonable methods to solve this problem.

In Chapter 4, I show that the profit function is log-concave under a mild condition. This result generalizes the work on the newsvendor problem with price effect, and the existing
work on pricing games. By using this result and the price-dependent demand substitution rule developed in Chapters 2 and 3, I study the joint pricing and inventory game and obtain some interesting insights on the existence of Nash equilibrium. In the future research, it will be interesting to study the joint pricing and inventory game with more than two retailers.

In Chapter 5, I apply a dynamic programming approach to study the optimal trading with adaptive safe price. The model assumes that a trader has a “safe price” for the security, which is the highest price that the trader is willing to pay for this security in each time period. The analysis indicates that while the greedy algorithm is not always optimal, there are important cases where it is. I also provide bounds on the performance of the greedy policy relative to the performance of the optimal policy. Future work can consider nonlinear price impact and infinite time horizon models.

Another extension of the work in Chapter 5 is to incorporate a “target selling price” into the model. I define the target selling price as the price that the trader will start selling his shares. The price region between the buying (safe) price and the target selling price is the trader’s no trade region. The trader can simultaneously buy and sell the security with adaptive buying (safe) and selling (target) price to maximize his profit over multiple periods. It is interesting to identify the conditions under which the multi-period arbitrage opportunity exists if the trader has the power to affect market price by his trade.
Bibliography


