OFFSET VERTICAL SEISMIC PROFILING:
TWO-DIMENSIONAL FORWARD MODELING WITH ASYMPTOTIC RAY THEORY

by

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ABSTRACT

Offset vertical seismic profiling (VSP) is rapidly becoming a viable technique for delineating, in detail, subsurface geologic structure. In an effort to understand and use the information available in VSP data, a modeling study using asymptotic ray theory (ART) is undertaken. The basic theory and practical considerations for implementing ART in two-dimensional fully elastic media are discussed. A number of synthetic examples for simple models are presented to exhibit the effects of offset sources and laterally varying structures. Traveltime, amplitude, and phase effects are observed. The models show the significance of mode converted shear waves and the importance of full elastic modeling capabilities. Features in the data helpful in locating buried structures are investigated. A favorable comparison of ART synthetics to an actual VSP dataset acquired near a reef structure in northern Michigan is made. The agreement of both traveltime and amplitude effects shows the utility of the modeling technique in both field program planning and interpretive modeling.

Thesis Supervisor: M.N. Toksoz
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To my parents for their continual support of my endeavors. They are and always have been my closest friends.
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I. INTRODUCTION

Vertical Seismic Profiling (VSP) is a technique used to investigate in detail, the near surface of the earth (typically less than ~6km). Procedurely, VSP data is acquired using an artificial seismic source on the earth's surface in the vicinity of a deep borehole (figure 1). In the borehole, a receiver sonde, containing one or more geophones (vertically and/or horizontally oriented), is clamped in place. Seismic waves are generated from a stationary source position a number of times with the receiver at various depths in the borehole. The seismic wavefield is recorded digitally in time at each receiver position. The individual records are then displayed side-by-side to show the variations of the in situ wavefield with depth. This composite record is called a "Vertical Seismic Profile".

The VSP is an enhancement of previous borehole seismic techniques such as the checkshot/velocity survey. (Stewart 1983, Hardage 1983, and Gal'perin 1974 each give detailed reviews of the developmental history of the VSP technique.) This enhanced data contains much more information since the wavefield is typically recorded for several seconds (compared to just the first arrival times as in the checkshot survey) at a rather dense spatial sampling (typically 10m as compared to 150m for the checkshot survey). Both the transmitted (downgoing) and the reflected (upgoing) portions of the full elastic wavefield can be observed in situ. These added features of the VSP allow a detailed examination of the dynamics of the entire seismic wavefield. Gal'perin (1974) and Keho et al. (1984) have discussed such investigations with actual data recorded in the field.

For investigating large areas of the subsurface, the VSP is subordinate to the surface reflection seismic technique. This is because only the subsurface
region from the borehole to a distance of about one-half of the source offset and only that region in the direction of the source can be imaged. Surface reflection profiles, on the other hand, can be acquired over large areas independent of any borehole locations.

There are many advantages to acquiring VSP data, however. The amplitudes and the frequency band of the recorded signals are generally much higher than that found in surface reflection data. This is due to shorter propagation paths that reduce the effects of geometrical spreading and intrinsic attenuation as well as the larger amplitudes of transmitted waves as compared to reflected waves. Reflections need only propagate from the reflector up to a deeply positioned receiver rather than back to the surface as in the surface seismic data. Also, the wavefield need only pass once through the generally complex weathered zone (near surface low velocity zone), greatly simplifying the observed wavefield character. An additional feature is the abundance of many different wave types in the data. In addition to the recorded compressional waves of surface reflection profiles, transmitted, converted, and sub- super- and transcritically reflected waves are observed. These wave types, when present, are intentionally suppressed in surface reflection data by the standard acquisition and processing techniques. Finally, since the wavefield is recorded at various depths, the depth of origination of the different reflected/converted modes can be determined accurately without any preconceived assumptions about the local velocity structure, as is necessary for time-depth conversion of surface reflection data. Constraining certain seismic events in depth at the borehole provides a means of directly tying available well log information to surface seismic data. The added information available via these observations in VSP data allows the recovery of much more detailed information on the subsurface region around the borehole.
A major area of current interest is the use of the VSP in delineating geologic structure and stratigraphy in the vicinity of the borehole. By using a number of sources at various offsets and azimuths we can investigate a fairly large volume of earth in the vicinity of the borehole.

Much has been written in the recent past on the analysis and modeling of VSPs. Most of what has been published deals with the case of one-dimensional (vertically varying) acoustic media with seismic sources immediately adjacent to the borehole (zero-offset). Wyatt (1981) has presented a method for modeling the complete acoustic wavefield (all multiples included) for zero-offset VSP geometries. Stewart (1983) has discussed methods to extract the elastic and anelastic properties of horizontally layered media. Kennett et al. (1980) examined how VSP derived one-dimensional parameters can be used to extract more information from standard surface reflection data. Dietrich et al. (1984) used the discrete wavenumber method to synthesize longer offset source VSPs in one-dimensional acoustic and elastic media.

Very little work has been presented, however, that deals with the analysis and modeling of VSPs in two and three-dimensional (laterally heterogeneous) elastic media. Wyatt and Wyatt (1981), and Kennett and Ireson (1981) discussed the effects of structural dip and simple fault structures on VSP data. Chun et al. (1983) and Lines et al. (1984) each presented techniques to invert offset VSP traveltime data to determine local structural dip in two and three dimensions. Hardage (1983, chapter 6) presented a number of ray-traced synthetic VSPs (no amplitude or phase considerations) for offset sources in two-dimensional acoustic media. Their investigations show the beginning of an application of VSP to two-dimensional problems but again only for acoustic media and thus not taking advantage of a great deal of information available. Cassell and
Lasocki (1983) have discussed the use of zeroth-order asymptotic ray theory to model offset VSP in one dimensional elastic media. Cormier and Mellen (1984) discussed the basic theory and practical applications of dynamic ray tracing to modeling VSPs in two-dimensional media.

In order to take full advantage of the vast amount of information available in acquired multi-offset VSP data, modeling and inversion techniques capable of handling two and three-dimensional fully elastic media must be investigated. Presented here is a discussion of Asymptotic Ray Theory (ART) and its application to Vertical Seismic Profiling. An algorithm for computing synthetic surface reflection seismograms using ART in two-dimensional laterally heterogeneous elastic media (Červený and Pšenčík, 1981) has been modified to accept the more general source-receiver geometry used in VSP. Using this modeling technique a number of synthetic examples have been calculated to exhibit how various geologic situations may be observed on VSP data. Simplified models were chosen to illustrate general features common to typical observations rather than to illustrate how complicated synthetic seismograms can be made. As an example of how this modeling technique can be used as an aid in field experiment design and data interpretation, a short case history is presented through a comparison of ART synthetics with field data acquired during the MIT-CGG Experimental VSP Group Shoot (fall 1983).
II. ASYMPTOTIC RAY THEORY (ART)

Of the few techniques that have been developed to model seismic wave propagation in two and three-dimensional media, conceptually and computationally the simplest method invokes ray theory. The principles of tracing rays were developed from geometrical optics and have been known for some time. The classical technique has been used successfully as the basis of seismic traveltime and propagation path studies in complex three-dimensional media (Hubral and Krey, 1980; Aki et al., 1977, Lines et al., 1984). By incorporating energy considerations, the amplitude of the signal envelope can be computed. However, the actual signal shape (phase) cannot be determined this way. In order to observe waveform changes and compute synthetic seismograms we must resort to the more general asymptotic ray series approach.

A review of the principles of the asymptotic ray series technique, its application, and its relation to other forward modeling techniques appropriate to two and three-dimensional media, is given by Cormier and Mellen (1984, excerpts in Appendix A). A qualitative review of the aspects of asymptotic ray theory (ART) necessary to this study will be given here with no derivation. For more detail, the reader is referred to Appendix A and to detailed reviews of the technique by Červený et al. (1977) and Červený and Hron (1980).

The basic premise of ART is the assumption that the wavefield can be approximated by summing up the contributions from each of all possible raypaths from the source to the receiver. To obtain these raypaths along with amplitude and phase information we begin with the elastodynamic wave equation for perfectly elastic, isotropic, inhomogeneous media:
\[
\rho \frac{\partial^2 \mathbf{U}}{\partial t^2} = ((\lambda+\mu)\mathbf{V} \cdot \nabla \mathbf{U}) + \mu \nabla^2 \mathbf{U} + \nabla \lambda \nabla (\mathbf{V} \cdot \mathbf{U}) + \nabla \mu \times (\nabla \times \mathbf{U}) + 2(\nabla \mu \cdot \mathbf{V}) \mathbf{U} \quad (1)
\]

Assuming an asymptotic solution (particular for high frequency harmonic signals):

\[
\mathbf{U}(x_i, t) = e^{-i \omega (t - \tau(x_i))} \sum_{k=0}^{\infty} \mathbf{U}^{(k)}(x_i)(-i \omega)^{-k} \quad (2)
\]

an infinite series in inverse frequency (\(\omega\)) is obtained. \(\mathbf{U}(x_i, t)\) is the displacement vector, \(\rho\) is the medium density, and \(\lambda\) and \(\mu\) are medium Lame parameters. Equating terms of equal order in frequency results in an infinite number of equations in \(\mathbf{U}^{(k)}\). From the zeroth-order equation we obtain a system of differential equations (equations A8, appendix A) which specify the kinematic properties of the wavefield (wave front trajectories and traveltimes). Given the initial position and direction of wavefront propagation this solution specifies the ray paths throughout the medium. These results are identical to that of "standard" raytracing techniques with the exception that heterogeneous media are permitted with ART.

Taking the first-order equation, we obtain the so-called transport equation (equation A9 of appendix A). From the solution to the transport equation, the dynamic properties (amplitude and phase) of high frequency body waves (compressional and shear independently) at one point in the medium can be related to those at a previous point along the same ray (the geometrical spreading factor). Using this relationship, the body waves can be "propagated" through an unbounded continuously inhomogeneous medium from point to point along the previously determined ray path.

Higher order equations can be used to determine the dynamic properties of higher order "modes" (head waves, surface waves, etc.). These modes,
however, are not considered in the present study.

When the ray strikes a curved interface where velocities (P and S) and/or density are discontinuous, the curved wavefront and the curved interface are replaced locally by planes tangent to each at the point of incidence. This approximation is valid where the radius of curvature of the interface is large with respect to the seismic wavelength. Standard reflection/transmission coefficients for elastic plane waves (Zoeppritz/Knott equations) are applied at the point of incidence and signal phases are matched using Snell's law. Hence, the direction of the ray, and the amplitude and phase of the signal immediately beyond the interface are determined. These properties are used as initial conditions in the procedure described above to "propagate" the signal through the new inhomogeneous medium. This procedure is continued from interface to interface until the ray reaches the receiver. Now the signal at the receiver can be recursively related to that at the source. Thus by specifying the conditions at the source (source radiation pattern specifying initial amplitude and phase) we obtain the amplitude and phase of the final signal at the receiver.

In several of the examples to be presented, the effects of intrinsic attenuation were also considered. By specifying a constant quality factor ($Q_n$) for each layer, the amplitude attenuation effect for a particular layer ($n$), can be approximated by:

$$A_n = \exp \left[ -\pi f_c \frac{t_n}{Q_n} \right]$$

(3)

where $f_c$ is the center frequency of the source and $t_n$ is the computed traveltime in that particular layer. This ignores the dispersive effects of intrinsic anelasticity, but is adequate for a narrow band source.
After all of the desired ray types have been computed for a particular receiver we can compute an approximate high frequency body wave seismogram. Incorporating all of the calculations for each ray and summing up the effects of all of the rays we have:

\[
\tilde{U} = \text{Re} \left\{ \sum_{i=1}^{NR} \sum_{n=1}^{NL(i)} S_i G_{in} T_{in} A_{in} \right\} \tag{4}
\]

where:

- \( NR \) = number of rays at the receiver of interest
- \( S_i \) = source radiation factor for the \( i^{\text{th}} \) ray
- \( NL(i) \) = number of layers passed through by the \( i^{\text{th}} \) ray
- \( G_{in} \) = geometrical spreading factor for the \( i^{\text{th}} \) ray in the \( n^{\text{th}} \) layer
- \( T_{in} \) = reflection/transmission coefficient for the \( i^{\text{th}} \) ray at the \( n^{\text{th}} \) interface
- \( A_{in} \) = attenuation factor for the \( i^{\text{th}} \) ray in the \( n^{\text{th}} \) layer

(\( S_i, G_{in}, \) and \( T_{in} \) are complex valued. \( A_{in} \) is pure real since dispersion is not considered.) The result of this summation is a "spike" synthetic. An arbitrary source function can be convolved with this to produce a synthetic seismogram that can be compared directly with observed data.

Practical considerations for implementation of this technique are discussed in detail by Červený and Hron (1980), Červený and Pšeničk (1981), and Cormier and Mellen (1984). One of the advantages of this technique over numerical solutions to the wave equation (e.g. finite-element, finite-difference) is that computation time is often an order of magnitude (or more) faster for even relatively simple models. An additional feature of this technique is that the wavefield can be decomposed into individual modes. In this way specific reflected/converted modes of interest can be studied in detail. This can be an
extremely important use of ART modeling since once the attributes of a particular mode are known they can be used to enhance or suppress that mode’s effect in both VSP and then surface reflection data.

II.1 Limitations

The quality of the amplitude and phase calculations depends on whether or not the high frequency assumption is valid. Kravtsov and Orlov (1980, 1981) discuss in detail the conditions under which asymptotic ray theory breaks down. Most generally stated, the medium properties should vary slowly across the cross section of the Fresnel volume of the ray. More specifically, the medium should adhere to the following conditions:

1. Any characteristic scale length of the medium should be much larger than the wavelength of the signal. Examples of such scale lengths include radii of curvature of interfaces and measures of heterogeneity such as $v/|\nabla v|$ and $\rho/|\nabla \rho|$, where $v = \text{velocity}$ and $\rho = \text{density}$.

2. The distance of the ray from surfaces of irregularity of the ray field should be much larger than the wavelength of the signal. Surfaces of irregularity consist of regions where signal amplitude variations are extremely rapid such as: caustic surfaces, critical regions, and shadow zones.

3. The distance the ray traverses between the source and receiver ($L$) should be much less than the ratio of the square of the previously mentioned scale lengths ($l^2$) and the wavelength of the signal ($\lambda$) (i.e. $\sqrt{L/\lambda} \ll l$).

The specific point at which these conditions break down depends on the particular model being considered. In real earth models, which tend to strain the conditions of validity, useful information can many times still be gained.
Special care must be taken with these models, however, as the degradation of results can be quite rapid.
III. SYNTHETIC EXAMPLES

The examples to be shown were computed using a program based on "SEIS81" (Červený and Pšenčík, 1981) which has been modified to accept VSP acquisition configurations. The program performs two-point ray tracing and computes synthetic body wave displacement seismograms at specified receiver locations. The model may contain discrete layers that are two-dimensionally inhomogeneous. Interfaces are approximated by a set of cubic splines and may be curved and/or discontinuous. Discontinuities and corners in the interfaces may violate the assumptions of ray theory and pose serious problems with the validity of the synthetic seismograms in that region (see Limitations of ART). Within each layer a grid of discrete velocity points is set up and a continuous velocity function is approximated using a series of bicubic splines. In the examples presented in this section, each model consists of a number of layers separated by curved and/or discontinuous interfaces. In each case the layers simulate homogeneous (no lateral velocity variations), isotropic, perfectly elastic solids. Inhomogeneous media are not considered so that the geometrical effects of the structures can be emphasized.

Ray path modeling is performed using the shooting method. The desired ray types are chosen by specifying the proper "ray code". The code describes the number and type of reflections, transmissions, and conversions which are required to take place at specified interfaces. The ray code is specified using a convenient notation developed by Červený et al. (1977). For each element of the ray (that portion of the ray between two interfaces) the ray code is a signed number whose magnitude defines the particular layer in which that element is to be situated (layers are numbered top to bottom). The sign determines whether the ray is to propagate as a P wave (code > 0) or as an S wave (code <
0). To specify an entire ray from source to receiver, one simply lists the code for each element in the order they actually propagate. Thus a ray which is to travel through the first two layers of the model as a P wave then reflect back up through them as an S wave would be specified by:

```
 1 2 -2 -1
```

This versatile coding technique allows arbitrary multiply reflected and converted rays to be calculated. In the examples presented here, we have limited the ray-traced events to include direct arrivals from the source, primary reflections, and only certain specific mode conversions (Table 1). For each event only one change in vertical direction is allowed but all mode conversions are considered at each interface. The synthetic VSPs do not, however, include the effects of multiple reflections, head waves, surface waves, borehole tube waves, or diffractions. The absence of these other wave types may be rather obvious in some situations as will be pointed out later. In later examples, one of the principal advantages of using ray theory, that of being able to choose specific events to "build" a seismogram, will be exhibited.

In the program, arbitrary source radiation patterns may be specified. Variations in initial amplitude and phase with ray takeoff angle are specified independently for compressional and shear wave sources. These radiation patterns correspond to far field approximations of the source. In all but one of the examples to be shown, the source is specified to be a spherically symmetric point compressional source which spreads geometrically in three dimensions. The source amplitude and phase do not vary with takeoff angle of the ray. The one exceptional source has a similar radiation pattern but generates only shear waves. These are not physical representations of actual seismic sources but their use reduces the number of variables in the computations and emphasizes structural effects in the synthetics. Also, as the source offset and receiver
depth increase these simple radiation patterns more closely approximate realistic source patterns.

All synthetic seismograms to be shown have been computed using an exponentially damped cosine wave source function (Gabor wavelet) given by:

\[ f(t) = e^{-\frac{\omega_c t}{\gamma}} \cos(\omega_c t + \nu - \psi) \]  

(5)

where:

\begin{align*}
\omega_c &= \text{source center frequency (50Hz unless otherwise specified)} \\
\gamma &= \text{source damping factor (3.5 approximates a Ricker wavlet)} \\
\nu &= \text{initial signal phase at the source (= 0 for all examples shown)} \\
\psi &= \text{cumulative phase changes due to propagation effects}
\end{align*}

In all cases the seismograms represent particle displacement shown in true amplitude variable area display with extremely large amplitude arrivals "clipped" for clarity. Trace gain is constant from trace to trace within an individual synthetic profile but may change from profile to profile.
III.1 Effect of Source Offset

In the first example we present the simplest case, a zero offset vertical seismic profile in a one-dimensional "layer cake" medium. The model (figure 2) consists of two 250m (820 ft) thick layers over a halfspace. The compressional wave velocity for each layer, from top to bottom, is 914 m/s (3000 ft/s), 1524 m/s (5000 ft/s), and 2286 m/s (7500 ft/s). Formation densities are 1.88, 2.00, and 2.16 g/m³ respectively. (Note: the shear velocities for this and all other models in this section are determined by \( V_s = V_p / \sqrt{3} \).) The source is located on the surface near the top of the borehole containing the receivers (actual offset = 20 m). Twenty-five vertical component receivers are located at a 30 m (98 ft) interval in a vertical borehole.

As can be seen from the synthetic seismograms in figure 2, even though the model is fully elastic and all of the ray codes in table 1 are used, only three events are present. Event A is the direct transmitted P-wave from the source. Events B and D are primary reflected P-waves from the first and second interfaces respectively. Since all waves strike the interfaces at normal incidence, no mode conversions take place. It would be sufficient to model this case with an acoustic wave propagation method. For future reference it should be noted that downward traveling events (e.g. A) increase in arrival time with depth (slope down and to the right in the figures) whereas upward traveling events (e.g. B and D) decrease in arrival time with depth (slope up and to the right). The depth at which an interface intersects the borehole can be located by the depth at which reflections originate (i.e. where upward traveling events intersect the first arrivals).

With a zero-offset source and a one-dimensional medium, interval compressional velocities can be measured directly from the moveout of the first
arrivals. (Use of a horizontally polarized shear wave source would allow determination of interval shear velocities.) After correcting for spherical divergence (note the rapid decrease in amplitude of the direct arrival with depth), direct arrival amplitudes can be used directly for estimating interval apparent attenuation. Stewart (1983) has devised an algorithm for least-squares inversion of the full VSP waveform to determine velocity and attenuation while considering the realities of real data.

The second example consists of the same earth model as above with the source now offset 500m (1640ft) from the borehole. As can be seen in figure 3, the synthetic VSP is much more complicated than the zero offset example, illustrating the full elastic nature of the model. Thus, even with the simplest of structures and an offset source, acoustic modeling is already insufficient.

One of the first items to be noted is that event moveout no longer directly indicates true interval velocities (compare event A in figure 2 versus figure 3). As can be illustrated geometrically (Wyatt and Wyatt, 1981), reflection moveouts are generally hyperbolic with an offset source. Also, the transmitted and reflected wave traveltime moveouts are no longer equal (in magnitude) and opposite in direction, as in the zero offset case, but are skew to each other (compare A/B moveout in figures 2 and 3). True interval velocities can be recovered through the use of standard traveltime inversion techniques. Shear wave velocities can also be recovered from compressional source data through the use of similar techniques on the mode converted arrivals. The discontinuities in first arrival times are due to the fact that these are "body-wave only" synthetics. Had the effects of interface waves been included, the first arrival curve would be continuous.
An additional important item to note is the change in relative amplitudes. Event B in the zero offset case is much smaller than the direct arrival (A) and event D is barely visible. In the offset source case, these compressional reflections can even be larger than the direct transmissions. A number of the mode converted shear waves, which are non-existent in the zero offset case, now have amplitudes comparable to the pure compressional events. The larger amplitudes (relative) of the reflected/converted modes are due mainly to three causes. First, the transmitted wave travel distance is much closer to that of the reflected wave with an offset source and thus geometrical spreading does not have as strong an effect (differentially). Secondly, the angle of incidence of the reflected waves, ~45° to 60° for event B of this source offset, is closer to the critical angle, ~37° for the first interface, and thus the reflection coefficients are more comparable to the transmission coefficients. Lastly, but equally important, is the angle at which the different arrivals impinge on the vertical component receiver. The direct compressional arrivals have a particle motion mainly in the horizontal direction (broadside arrivals) whereas the reflected modes impinge somewhat more steeply and the converted shear arrivals have a major vertical component (broadside arrival with transverse particle motion).

Changes in signal phases, limited to polarity reversals in the zero offset case, are now readily apparent. From the angles given above we see that event B for example, is a post-critical compressional reflection at all depths at which it is recorded. Phase changes with depth, due to complex reflection coefficients (which depend on both $V_p$ and $V_s$), are easily seen.

Over all, the far offset VSP data contains much more information on the elastic properties of the model. However, it requires more sophisticated analysis techniques to interpret than the zero offset case.
III.2 Effect of Dip

The next example consists of the same model as in the previous two examples with one modification. Interfaces now dip at 15°. The model is constructed to simulate a regional dip effect with no interface topography.

Figure 4 shows the results for a zero-offset source (actual offset = 20m). The direct P-wave arrivals (event A) have very similar times and amplitudes to those in the zero-offset no-dip situation, even below the interfaces. The ray paths for these direct events do not greatly deviate from the vertical. The amplitude of the compressional reflections has not changed greatly, however, the arrival times have moved up somewhat. This time advance is due to the reflection points "walking" updip from the borehole. One major change from the zero-offset no-dip situation (figure 2) is the appearance of mode converted shear waves. Their presence is easily noticed but they are not nearly as strong (relatively) as the shear arrivals in the long-offset no-dip case (figure 3).

Figure 5 shows similar results for a source offset 500m from the borehole in the updip direction. The results here are comparable to those in figure 3 for the long-offset no-dip case. Reflection curvature has increased slightly in the dipping layer case and the arrival times have moved up as they did in the zero-offset dipping layer case.

As previously mentioned, Lines et al. (1984) have devised a means for inverting transmitted and reflected (P-only) traveltimes for reflector dip. As can be seen from these examples, the problem could be quite difficult when only short source offsets are used. A zero-offset source (with only a vertical component receiver or an unoriented three component receiver) could conceivably indicate the amount of dip but in could not determine strike. At
least two offset source VSPs would be required to completely determine regional strike and dip. The use of an oriented three-component downhole geophone (orientation independent of wavefield measurements) could substitute for one of these sources.
III.3 Stratigraphic Effects

Figure 6 shows a simplified stratigraphic wedge. The model consists of a flat-lying surface layer which is 610m (2000ft) thick and overlies a wedge shaped layer which in turn overlies a halfspace. The base of the wedge dips at approximately $9^\circ$ away from the borehole (left edge of model). The wedge tip is 46m (150ft) from the borehole. The model attempts to simulate the response of dipping layers beneath an unconformity located at the base of the flat surface layer. The P wave velocities for the surface layer, wedge, and halfspace are 2130m/s (7000ft/s), 2740m/s (9000ft/s), and 3960m/s (13000ft/s) respectively. The layer densities are 2.1, 2.2, and $2.5\, g/cm^3$ respectively.

For this example two independent sources are considered (although they could have been used together). The first source is a point compressional source located on the surface and offset 823m (2700ft) from the borehole. The second source is a point vertically polarized shear source, located in the same position. In both cases the source radiation pattern is spherically symmetric. There are 70 two-component receivers (vertical and radial) located in a vertical borehole from the surface down to a depth of 1190m (3900ft) at a 17m (56ft) spacing. The source wavelet used in the calculation of the synthetic seismograms has a center frequency of 100Hz.

Figures 7 to 10 show the results of the dynamic ray tracing calculations. The seismograms are displayed in true amplitude (clipped) for each source-receiver component combination. Gain from one section to the next is not necessarily constant. Figures 7 and 8 are the vertical and horizontal component results for the P-wave source. As indicated in the figures, only nine of the fourteen specified ray types have immediately observable effects on the synthetics. Events which are converted from P to SV or SV to P more than once
are not easily observable.

The dip of the base of the wedge is easily detected by both the reflected P-wave events (B and D) and the direct converted events (L and N). By measuring the difference in reflection curvature (moveout) between event B (P reflection at the unconformity) and event D (P reflection at the base of the wedge), one can determine the dip of the base of the wedge. A slightly different calculation could also be made using the difference in moveout between the direct event which is converted to shear at the unconformity (event N) and the event converted at the base of the wedge (event L). A similar effect occurs with the shear converted reflections, however, in this particular example the events C and G diverge more slowly making moveout measurement more difficult.

Figures 9 and 10 are the vertical and horizontal component results for the SV source. Again, the ray codes listed in table 1 were used with the exception that all events leave the source as S-waves (event labels in the figures are primed to indicate an initial S-wave ray element). In contrast to the P-wave source synthetics, almost all of the individual ray types specified are observable. A number of more complicated converted events have become more apparent (relatively) than those in the P-wave source sections. Five different converted modes which eventually propagate as upgoing P-waves in the surface layer (events B’,D’,F’,H’, and J’) have become more important. Two modes which eventually propagate as downgoing P-waves appear as precursory events to the much larger direct shear wave. These events are converted to P at one of the two interfaces and eventually arrive significantly before the direct shear waves.

The best indicator of the dip on the base of the wedge is given by the shear wave reflections (especially events C’ and K’). As in the P wave source case the divergence of these reflections is a measure of the dip.
In both the P and S wave source cases, the divergence of certain reflections can give information on the dip of the wedge. The most apparent indication of dip in each case is from the simplest (no conversion) reflections. Information on the specific location of the pinchout can be obtained from both the point of divergence of these reflections in each synthetic section.
III.4 STRUCTURAL EFFECTS

III.4.1 Simple Reef Model

Figure 11 shows a model of a grossly simplified reef structure. The model consists of a 1525 m (5000 ft) layer over a halfspace. At the contact between the layer and the halfspace is a high velocity anomaly (the reef). The reef is 640 m (2100 ft) across (equivalent to a 100 acre reef), has a maximum thickness of 140 m (460 ft) and is offset 90 m (300 ft) from the borehole. The P-wave velocities for the surface layer and the underlying halfspace are 3050 m/s (10000 ft/s) and 5200 m/s (17000 ft/s) respectively while the reef velocity is 4600 m/s (15000 ft/s). The densities of the surface layer, reef, and halfspace are 2.3, 2.6, 2.7 g/cm$^3$ respectively. For this particular example the source is a point compressional source located on the surface and offset 900 m (3000 ft) from the borehole. There are seventy vertical component geophones located in a vertical borehole at 30 m (98 ft) spacing. The source wavelet used in the calculation of the synthetic seismograms has a center frequency of 50 Hz.

The left side of figure 11 shows the layout of the model with no vertical exaggeration. A ray path model is overlain on top of the earth model. Only the direct compressional ray paths are shown for clarity. Just above the layer-halfspace contact there appear to be some reflected ray paths. These are actually directly transmitted rays that refract through the edge of the reef (there are no interface waves in these synthetics).

One of the advantages of the ray method over other modeling techniques is apparent in this example. The ray method allows us to look at the effects of each individual reflection and conversion event. Looking at the synthetics and the amplitudes of individual events, only six events (of 14 calculated) are
immediately apparent (marked A-D,G,and N in figure 11). These events include
the direct arrivals (A), the P-wave reflections from each interface (B and D), the
P-SV modes which are converted to shear upon reflection at each interface (C
and G), and event N which propagates as a P-wave until it converts to SV at the
reef. Modes that are converted from P to SV or SV to P more than once appear
as very small "side lobes" to the larger events.

In this particular example, it is possible to see some of the features of the
VSP technique that could be used to help locate and describe the reef in real
earth. Events B and D are P-wave reflections from the top and bottom of the
reef respectively. Using the difference in arrival times between these two events
in the upper part of the VSP section (above 1000m depth, away from the effects
of the edge of the reef) and knowledge of the velocity of the reef, one can
obtain an estimate of the reef's thickness. Using events C and G and shear wave
velocities for the reef, one can obtain additional constraints on the reef
thickness.

In the middle portion of the synthetic section, at approximately 1500m
(4900ft) depth, information on the shape of the reef and its location with
respect to the well may be obtained. From approximately 1000m (3400ft) to
1500m (4900ft) depth, events B and C become concave downward. This is due to
the reflections occurring at progressively greater depths on the side of the reef.
The shape of these events gives an indication of the slope of the side of the
reef. Where event D meets the direct arrivals (event A) gives a depth location
for the base of the reef. The depth and time of the intersection of event B and
event D can give information on the offset of the reef from the well (similarly for
C and G).
It should also be noted that the effects of the far edge of the reef are not observed in the synthetics. Consideration for the depth of the reef must be taken into account when choosing source offsets. In this case the source was not offset far enough from the borehole to observe the far side of the reef.
III.4.2 Anticline Model

The final synthetic example is shown in figure 12. The model is a simplified symmetric normal non-plunging anticline. The limbs of the anticline dip at \(\sim 30^\circ\). The crest of the anticline is 500m (1640ft) from the borehole at a depth of 760m (2500ft). The top and bottom interfaces of the folded formation intersect the borehole at 1000m (3280ft) and 1200m (3940ft) respectively. The compressional velocities, from top to bottom, are 2740m/s (9000ft/s), 4270m/s (14000ft/s), and 3050m/s (10000ft/s). Formation densities are 2.25, 2.55, and 2.31gm/cm\(^3\) respectively. There are sixty, two-component (vertical and radial) receivers at a 30m (98ft) spacing in a vertical borehole.

The problem was formulated as if a borehole had been drilled with an objective at the crest of the anticline. Since the crest was missed the intended purpose of the VSP is to locate it. The direction to the crest of the anticline is easily found by determining the direction of local dip at the borehole (as in the previous dipping layer case). The crest will be in the updip direction. The problem now is to determine the depth of and distance to the crest. To investigate the usefulness of the VSP in locating the crest, synthetic VSPs were computed using the model described above and four different source offsets.

Figures 13 to 16 show the results of computations for sources at offsets of 0, 500m (1640ft), 1000m (3281ft), and 1500m (4921ft) from the borehole respectively. In each figure, the vertical component of displacement is displayed on top with the radial component below (note that each section has its own and different fixed gain). Figures 17 and 18 show ray path diagrams for each offset. Figure 17 shows the direct transmitted arrivals whereas figure 18 shows P-to-P reflections from the upper interface of the anticline. It should be noted in figure 17 that the upgoing events in the longer offset cases are rays
which are refracted (not critically) through the convex part of the anticline and are not head waves. The presence of the rays indicates that head wave arrivals would be present in the synthetics had we the capability to model them.

In the zero-offset case, figure 13, we see results which are virtually identical to those expected if the model were simply two dipping layers, without an anticlinal crest. Figures 17a and 18a show clearly that the direct and primary reflected rays do not detect the presence of the crest. The direct arrivals, best seen on the vertical component synthetics (figure 13a), show distinct changes in velocity at depths of 1000m (3281ft) and 1200m (3937ft). The origination of reflection events at these depths confirms them as the depths at which the interfaces intersect the borehole.

Figures 14, 17b, and 18b show the results for a source offset of 500m (1640ft) (source immediately above the crest). As can be seen in figure 18b, this source offset does interrogate part of the crestal region but not all the way to its peak. In the synthetic VSP there should be some reflection traveltime delay because of the "rollover" of the interface but this might be difficult to detect. A comparison of these synthetics with those from a simple dipping layer model might make the delays more apparent.

Figures 15, 17c, and 18c and figures 16, 17d, and 18d show the results of source offsets of 1000m (3281ft) and 1500m (4921ft) respectively. The ray path diagrams show that both of these offsets interrogate the anticline all the way to its peak in both transmitted and reflected arrivals. The peak of the anticline acts somewhat like a point diffractor. All of the reflection points are grouped close together. The traveltimes of the reflected waves are almost as if the source was located at the peak but initiated at a later time (corresponding to the traveltime from the actual source to the peak). The shortest traveltime
from the crest to the borehole is along a horizontal path (figure 12). Therefore
the depth of the crest is the depth at which this reflection has a minimum
traveltime (see arrow figure 15b). Since the ray is traveling horizontally the
depth of the minimum time is easier to see on the radial component synthetics.

The traveltime calculations here are correct, however, care should be
taken when interpreting reflection amplitudes from this region. The apex of
the anticline is reasonably sharp (radius of curvature ~300m) with respect to
the seismic wavelength (~55m), the ratio being ~5:1. Thus the peak’s curvature
strains the conditions of ray theory validity.

In order to determine the distance of the peak from the borehole we note
the geometry outlined in figure 12. \( d \) is the travel distance from the source to
the crest. Using the Pythagorean theorem we can write:

\[
d = \sqrt{h_p^2 + (z_s - x_p)^2}
\]  \hspace{1cm} (6)

where:

\( h_p \) = depth of anticline peak
\( z_p \) = distance from borehole to peak
\( x_s \) = source offset

\( z_s \) is known and \( h_p \) can be determined from the depth at which the reflection
time is minimized. Therefore we can write the traveltime along this ray as:

\[
t = \frac{d + x_p}{v_1}
\]  \hspace{1cm} (7)

where \( v_1 \) is the compressional velocity of the upper medium (can be measured
from direct arrival traveltimes at the shallow receivers). This can be rewritten
as:
Since all other are known we can solve for \( x_p \). Thus, we have located the anticline crest's depth and distance from the borehole. This simple method depends entirely on our ability to pick the depth at which the reflection traveltine is minimized. When the crestal region is broad and/or the dip of the anticline limbs are shallow, the determination of the required time and depth will be difficult.

It is not expected that a situation this simple will occur in the real earth, however, we have shown some of the features which may help interpret VSP data in more complex structures. In a more realistic situation, the offset VSP data could be combined together for use with several more quantitative techniques. Simultaneous migration/inversion of all offsets may show detail of the crestal region rather than locate a particular point at its peak.
IV. A COMPARISON WITH FIELD DATA

IV.1 The Field Experiment

The MIT-CGG VSP Groupshoot was conducted during the fall of 1983 to study the feasibility of using the VSP technique to image the subsurface away from a borehole in real earth. In this particular experiment the objective was to image a reef structure (the "Springdale Reef") adjacent to the St. Burch 1-20 well located in Manistee Co., Michigan. The field operations and results of the experiment are presented elsewhere. Here we present results of ray theoretical modeling as applied to field program planning and an example of its use in interpretative modeling of actual data.

Figure 19 shows an interpreted cross section of the reef area as it was perceived before the field work began. The stratigraphy has been simplified from a previous model of the area based on conventional well log data from the three indicated wells and surface seismic data in the region (courtesy of West Bay Exploration Co., Traverse City, Michigan). The compressional wave velocities are based on sonic logs recorded in the St. Burch 1-20 well (from 2900ft. to 6300ft.) and available information from the surrounding region.

In choosing the data acquisition parameters, ray theoretical modeling is most useful as an aid in locating optimum source and receiver positions. In this particular experiment, receiver locations were constrained to positions within St. Burch 1-20 (left edge of figure 19). An additional constraint was made by this well's casing program. When the borehole was drilled a steel casing was set in place from the surface to a depth of 2900ft. From previous VSP studies (Hardage 1983, chapter 3) it is known that good quality data can be recorded within the casing only in areas where the casing is well cemented to the earth.
When the geophone is positioned in regions of poor cementation, the casing begins to resonate at the onset of the first seismic arrival. This resonate motion dominates the rest of the recorded signal obscuring later seismic arrivals. In St. Burch 1-20 the casing was cemented only in the region from ~2400ft. to its bottom at 2900ft. From these facts it seems likely that good quality data can only be recorded at depths from 2400ft. to the bottom of the well at 6300ft.

Since the receiver locations are constrained by non-seismic conditions we need to position the sources such that we gain the most information about the reef. Previous surface seismic data recorded in the vicinity of St. Burch 1-20 indicate the strongest P-wave reflection originates from the top of the A2-carbonate formation just above the reef formation (the Brown Niagarian). We are therefore most concerned with placing the sources so that the recorded direct arrivals and these reflections interrogate the reef region.

A number of ray path models, using different source offsets, were computed using the computer algorithm described in the previous section on synthetic examples and the velocity structure shown in figure 19. Figures 20 to 22 show some of the results for source offsets of 2000, 4000, and 6000ft. respectively. In each case the upper figure shows the direct transmitted arrivals and the lower one shows P-wave reflections from the top of the A2-carbonate.

The diagrams show that as the source offset increases both the transmitted and reflected wavefields interrogate more of the reef region. Figure 21 shows that reflections from the 4000ft. source offset completely cover the top of the reef. We can see that the reflections from the far side of the reef will be recorded well above the bottom of the steel casing in an area of anticipated poor data quality. It is not until the source is offset 5000ft. from
the well that these reflections would be recorded below the casing (figure 22). Thus, source offset should be extended to at least 6000ft. in order to "illuminate" the entire reef in the specified receiver "aperture". Longer offsets would enable us to record the reflected arrivals farther down the borehole but were rejected because of additional information suggesting that recorded arrivals would be weak due to spherical divergence and intrinsic attenuation. Logistical reasons also restrict the use of longer offsets.

Figure 23 shows the locations of 11 source positions selected for the actual VSP experiment. Six source positions, labeled B, C, D, F, H, and J in the figure, were selected on the "reef side" (east-southeast) of the St. Burch 1-20 well based on the ray tracing calculations and logistical considerations. Two sources positions at each of 2000, 4000, and 6000ft. offsets were selected both for redundancy of information and to observe some possible three-dimensional features of the reef. Three other source positions, labeled E, I, and L, were selected on the "off-reef side" for differential comparisons of data acquired "on" and "off" of the reef. Offset A was selected to acquire conventional zero-offset VSP data.
IV.2 The Springdale Reef Model

From the data acquired during the MIT-CGG VSP Groupshoot one 6000ft. offset, J in figure 23, was selected to illustrate the utility of asymptotic ray theory in interpretative modeling. The vertical component geophone data for offset J are shown in figure 24. For completeness, the acquisition and processing parameters are given in table 2.

In addition to VSP data, a number of ancillary well logs were acquired during the course of the experiment. A Borehole Gravimeter and/or Gamma-Gamma Density Log were run throughout the entire depth of the well to provide constraints on formation densities. A full waveform acoustic log (EVA) was run in the lower part of the well, below the casing shoe, to provide information on the compressional and shear wave velocities in the vicinity of the borehole. Shear wave velocity measurements were available only in certain regions. These data were simplified and used to formulate a new model for the reef area. Note that \( V_p/V_s \) is not constant for all layers as was the case in previous models. The data are tabulated in table 3 and the new model is presented in figure 25. Using the new model a vertical component ray theoretical VSP was computed. The source offset and receiver locations were matched to those of the actual data. As in the previous synthetic examples, a point compressional source with a spherical radiation pattern was used. The ray path diagrams in figures 20 to 22 show that the high velocity contrast at the base of the glacial till creates a very narrow range of ray take-off angles from the source. Since the angles are similar a complicated source radiation pattern is unnecessary. The measured peak frequency of the field data was \(~50\text{Hz.}\), therefore 50Hz. was chosen for the synthetic source wavelet. Ray codes were chosen so that at each interface transmitted and reflected compressional and shear waves were
generated. No multiply-reflected or multiply-converted rays were generated.

Figure 26 shows the results of the seismogram synthesis. In comparing the synthetics to the actual data a number of features are immediately apparent. First, the direct arrival times are almost identical. When the model was first constructed, the traveltime moveout of the first arrivals was correct but the absolute arrival time was off considerably. The acoustic logs gave us good compressional velocity control in the depth range of the data but not up nearer the surface. As the near surface velocities were inferred anyway, they were adjusted to bring the first arrivals into alignment. Only the upper two layer velocities could be changed without degrading the relative arrival time moveout. A second point to note is the change in relative amplitudes of the first arrivals. At ~2760ft., ~3300ft., and ~4800ft. there are significant decreases in amplitude. The synthetics show very similar features except at 4800ft. The upper two low amplitude regions can be identified in the synthetics as regions where the rays cross over into significantly higher velocity materials. The rays are refracted away from vertical and show a marked decrease in amplitude on the vertical component geophone. The low amplitude region at 4800ft. requires further study to explain.

The synthetics are also useful for interpreting the origin of specific events seen in the data. Figures 27 and 28 show interpreted versions of the real and synthetic data. From the ray tracing results we can identify event 1 as a mode converted shear wave originating from just above 2400ft. This is an inferred interface in the middle of the D.R. Salt formation which was not present in the original stratigraphic models. Events 2 and 3 are compressional and shear reflections from the top of the A2-carbonate. Event 4 is a downgoing converted shear wave from the reef region. Events 2-4 are expected to be the most useful
in determining the properties of the reef.

Event 5 is a P-wave reflection from the top of the PDC sandstone below the bottom of the well. By adjusting the depth of this interface such that the reflection times match the field data we obtained an estimate of its depth beneath the well. This interface was not included in the original ray trace model and shows the application of VSP to delineating features beneath the bottom of the well (referred to as "prediction ahead of the bit").

One area where the synthetics deviate significantly from the field data is event 6. This is a mode converted shear wave from near the top of the F-unit. This event may be present in the field data but obviously does not have as strong an amplitude as predicted by the synthetics. P-wave velocities and formation densities are well constrained in this region, however, shear wave velocities are based on crude inferences. Adjusting the shear wave velocities can significantly reduce the amplitude of this event without effecting the traveltime moveout of the other identified events. The introduction of intrinsic attenuation effects would also help somewhat.

Overall, the comparison of the first ~250msec. of data beyond the first arrivals is extremely favorable both in traveltime moveout and amplitude effects. Beyond this time, multiply-reflected/converted events, that were not modeled, appear to have a significant effect on the data. There are no theoretical or computational restrictions that prevent these later arrivals from being modeled, so that a synthetic seismogram can be "built" to compare, in as much detail as desired, with actual field data.
V. CONCLUSIONS

The basic theory of seismogram synthesis in three-dimensional heterogeneous media using Asymptotic Ray Theory (ART) has been reviewed along with techniques for practical application of this method to two-dimensional models. A number of synthetic offset vertical seismic profiles in two-dimensional media were computed and discussed.

Although admittedly over simplified from that expected to occur in the real earth, the examples presented show the utility of ART in VSP investigations of complex geologic structures. From these simple models it is apparent that offset VSPs cannot easily be interpreted directly in complex areas. However, it is assured that a great deal of detailed information is available in multi-offset/multi-azimuth/multi-component VSPs. This wealth of information can only be completely taken advantage of through the use of multi-dimensional full elastic modeling and inversion techniques which retain amplitude and phase information. In every example, except for the zero-offset no-dip case, mode converted shear waves played a significant part in the synthetic VSPs. If "acoustic-only" modeling/inversion techniques are used, a great deal of this information will be lost.

An additional result of this modeling study is an illustration of the importance of using a three-component downhole geophone. In a number of the examples presented, a significant portion of the seismic energy for both compressional and shear waves would be recorded on the horizontal components of the downhole geophone. The maximum amount of information could be gained through the use of such a receiver particularly if the horizontal components could be oriented in space independent of the seismic wavefield. Observations of particle motion directions in three-dimensions can
assist in identifying various phases within the VSP data and the direction to their point of origin in the earth.

To date very few sophisticated processing techniques have been developed specifically for VSP data. The application of new processing techniques to true amplitude synthetics could provide a worthwhile understanding of the technique before the complexities of real data are considered.

A modeling study of ART synthetics to an actual VSP acquired near a reef structure in Manistee Co., Michigan gave favorable results. Using sparse information available prior to the experiment a number of models were computed as an aid in choosing optimum source/receiver placements. After the field data were acquired additional models were computed as interpretive aids. Traveltime and amplitude effects of both compressional and shear wave transmissions and reflections were matched quite well. Depth and means of origination of a number of seismic events could be related directly to geologic models of the area via the synthetics.

In all, Asymptotic Ray Theory was shown to be a viable technique for modeling offset Vertical Seismic Profiles. In many cases the synthetic sections can be "built" event by event until a comparison with real data is as close as desired. The extension of this technique to include head and surface waves appears to be the most logical and important next step in its development.
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APPENDIX A


Representation of Body Waves

In an inhomogeneous, perfectly elastic and isotropic medium, ART seeks to solve the elastodynamic wave equation,

$$\rho \frac{\partial^2 \ddot{U}}{\partial t^2} = (\lambda + \mu) \nabla (\nabla \cdot \ddot{U}) + \mu \nabla^2 \ddot{U} + \nabla \lambda (\nabla \cdot \ddot{U}) + \nabla \mu (\nabla \times \ddot{U}) + 2(\nabla \mu \cdot \nabla) \ddot{U} \quad (A1)$$

with an asymptotic series of the form

$$O(z_i, t) = \sum_{k=0}^{\infty} \tilde{U}_k(z_i) F_k(t - \tau(z_i)) \quad (A2)$$

where $\tilde{U}(z_i, t)$ is the displacement vector. For a harmonic source this becomes

$$\tilde{U}(z_i, t) = e^{-i\omega(t - \tau(z_i))} \sum_{k=0}^{\infty} \ddot{U}^{(k)}(z_i)(-i\omega)^{-k} \quad (A3)$$

The solution for a source having an arbitrary time dependence can be obtained from a superposition of harmonic components using equation (A3), the source spectrum, and the inverse Fourier transform.

Restrictions and Limitations of ART

Application of ART must be restricted to models in which the layer velocities and topography vary slowly with respect to wavelength and to ray
paths in which the cumulative errors of propagating an approximate solution through a heterogeneous medium are small. These restrictions are discussed rigorously by Kravtsov and Orlov (1980). A simplified summary of the restrictions is given by the inequalities

\[ \lambda \ll l \quad l = \frac{v}{|\nabla v|} \quad (A4a) \]
\[ \lambda \ll n \quad (A4b) \]
\[ \sqrt{L/\lambda} \ll l \quad (A4c) \]

where

- \( \lambda \) = wavelength
- \( n \) = distance of the ray from a surface of irregularity
- \( L \) = distance from source to receiver
- \( l \) = a characteristic scale length of heterogeneities.
- \( v \) = velocity of the medium

The breakdown in the asymptotic expansion (equation A2) can be best illustrated by performing the expansion in powers of \((\lambda/l)\) or \((\lambda/n)\) rather than powers of \((1/\omega)\). When inequalities (A4a-c) are violated, the methods of finite difference and finite element can be used to synthesize the wavefield to any desired accuracy. These numerical methods are complementary to ray methods in the sense that they are best used in different domains of frequency, range, and traveltime. Although the numerical methods can provide exact solutions, their computational expense generally limits their application to models or regions of models that have dimensions less than several wavelengths. Provided that material properties vary slowly over a wavelength \((A4a,b)\) and cumulative errors are small \((A4c)\), ray methods can be used to propagate the wavefield through large volumes of the model.
Another restriction on the use of ART occurs because its simplest formulation requires that the amplitude observed at a station be the result of a single ray. In a plane layered medium, in which ray solutions can be evaluated by transform methods, this approximation originates from a stationary phase or saddle point evaluation of a wavenumber integral (e.g. Ch. 9 of Aki and Richards, 1980). It is an approximation strictly valid only at infinite frequency, and it leads to singularities in the amplitude of rays evaluated on a caustic surface, where the geometric spreading function vanishes and the ray tube collapses. In these situations, ART can be extended by any strategy that does not force all of the amplitude contribution to be concentrated at a single ray. Examples of such extensions are given by Maslov's asymptotic theory (Chapman and Drummond, 1982), the Gaussian beam method (Červený et al., 1982; Červený, 1983), and the Kirchhoff integral method (e.g. Haddon and Buchen, 1981; Scott and Helmberger, 1983). These extensions are also what are required to describe the frequency dependence of body waves in the vicinity of caustics and cusps in traveltime curves.

Solution of Amplitude Coefficients

The unknown coefficients $U^{(k)}$ are obtained by substituting (A3) into (A1) and equating terms of equal order in frequency. The lowest-order equation becomes

$$\mathbf{N}[\mathbf{U}^{(0)}] = 0 \quad (A5a)$$

where $\mathbf{N}$ is a vector operator such that

$$\mathbf{N}[\mathbf{U}^{(0)}] = \rho \mathbf{U}^{(0)} + (\lambda + \mu) \nabla \tau \left[ \mathbf{U}^{(0)} \nabla \tau \right] + \mu (\nabla \tau)^2 \mathbf{U}^{(0)} \quad (A5b)$$

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In order for \((A5a)\) to have a non-trivial solution, the determinants of the coefficients \(U^{(0)}_i\), \(i=1,2,3\) must be zero. This gives

\[
\left( (\nabla \tau)^2 - \frac{1}{v_p^2} \right) \left( (\nabla \tau)^2 - \frac{1}{v_s^2} \right) = 0 \tag{A6}
\]

Solutions of \((A6)\) giving propagating elastic waves thus exist when

\[
(\nabla \tau)^2 = \frac{1}{v_p^2} \tag{A7a}
\]

or

\[
(\nabla \tau)^2 = \frac{1}{v_s^2} \tag{A7b}
\]

These are the eikonal equations. They show, that to zero order in inverse frequency, P and S waves travel independently in an inhomogeneous, isotropic medium. This is an approximation that breaks down at low frequencies when the scale length of variations in the medium approaches the wavelength of the body wave. It should be noted that the breakdown of the asymptotic series (A2 or A3) in the presence of strong gradients has not been precisely defined. In regions where the inequalities given by equations (A4a-c) begin to be violated, it is rarely worthwhile to calculate higher order terms in the asymptotic series. This is because the asymptotic series cannot properly account for back-reflected and converted radiation by strong gradients. In regions of strong gradient, Chapman (1981) has proposed that an iterative approach be used to solve the wave equation rather than an asymptotic series.

The characteristics of the eikonal equation form a system of trajectories orthogonal to surfaces \(\tau(x_i) = \text{constant}\) and can be shown to represent extremals of Fermat’s integral. Thus the ray’s trajectories can be found by
solving the equations for the characteristics of the eikonal equation. These form a system of six ordinary differential equations of first order:

\[ \frac{dz_i}{d\tau} = v^2 p_i, \quad \frac{dp_i}{d\tau} = -\frac{d\ln v}{dz_i} \]  \hspace{1cm} (A6)

where a slowness vector has been introduced by the relation \( \vec{p} = (p_1, p_2, p_3) = \vec{v}\tau \).

By noting that the slowness vector is normal to the wavefront at any point along the ray, equation (A6) can also be used to determine the direction of \( \vec{U} \). This is achieved by taking the scalar product \( \vec{N}(\vec{U}_0) \cdot \vec{v}\tau \) and cross product \( \vec{N}(\vec{U}_0) \times \vec{v}\tau \), showing that the P wave displacement is parallel and the S wave displacement is perpendicular to the ray direction.

The eikonal equation itself \( p_1^2 + p_2^2 + p_3^2 = 1/v^2 \) can be used to reduce (A6) to a system of five differential equations. They are solved by specifying a set of five initial conditions at the source, which amount to specifying a source location and two take off angles. The ray connecting a specified receiver and source is a two point boundary value problem that can be solved by the method of shooting, in which (A6) are numerically integrated and the ray connecting a specified receiver and source found by iteration. The method of bending may often be computationally more efficient than shooting and iteration. Both methods are reviewed by Julian and Gubbins (1977). The method of bending may often be a computationally more efficient technique. The iteration step in the shooting method can be optimized by by exploiting the stationarity of the phase of plane waves arriving near the geometrical arrival time (Buland and Chapman, 1983). Once the ray trajectories have been calculated they may be plotted. Already at this stage, the ray plots can profitably be used to interpret zones of focusing, defocusing, and interference.
To calculate \( \dot{U} \), the next higher order equation in frequency must be used. These equations are the so-called transport equations. The transport equation for P waves is given by:

\[
\frac{dU_p^{(0)}}{ds} + \frac{1}{2} U_p^{(0)} \left[ v_p \nabla^2 \tau + \frac{d}{ds} \ln(\rho u^2) \right] = 0 \tag{A9}
\]

The P wave solution is independent of that for S waves, but in order for the transport equations for SV and SH polarized S waves to decouple, the solution must be sought in a new coordinate frame (Popov and Psenõík, 1976, 1978). The new coordinate frame is an orthogonal curvilinear frame that moves along the ray (figure 29), having a vector basis \((\vec{t}, \vec{e}_1, \vec{e}_2)\), where \(\vec{t} = v_{t}^2\) is tangent to the ray. The vectors \(\vec{e}_1\) and \(\vec{e}_2\) are initially chosen to coincide with the vectors \(\vec{n}_{SH}\) and \(\vec{n}_{SV}\) respectively. In three-dimensions \(\vec{t}_1\) and \(\vec{t}_2\) can rotate about the central ray by an angle \(\theta\). \(\theta\) can be obtained by integrating the equation

\[
\frac{d\theta}{d\tau} = \frac{p_3 \left[ p_2 \frac{\partial u_z}{\partial x_1} - p_1 \frac{\partial u_z}{\partial x_2} \right]}{p_1^2 + p_2^2} \tag{A10}
\]

along the ray. This procedure does not require calculation of the ray torsion as in the Frenet frame. Chin et al. (1983) have given an alternative derivation of the transport equation, in which the ray-centered coordinate system follows from applying the theory of operators in linear spaces.

The solution of the transport equation is given by

\[
U(s) = U(s_0) \left[ \frac{J(s_0) \rho(s_0) u(s_0)}{J(s) \rho(s) u(s)} \right]^k \tag{A11}
\]

where \(s_0\) is the source location and where \(J\) is the Jacobian of the transformation from the fixed Cartesian coordinate system to a coordinate
system that moves along the ray. For P waves this can be taken to be the
system \((s, \gamma_1, \gamma_2)\), where \(s\) is the distance from the source measured along the
ray and \(\gamma_1, \gamma_2\) are the ray parameters specifying the ray. For decoupled SV and
SH waves the coordinate system must be the ray-centered coordinate system
\((s, q_1, q_2)\) where \(q_1\) is taken along \(\vec{e}_1\) and \(q_2\) along \(\vec{e}_2\). Using the ray-centered
coordinate system, Červený and Pšeničk (1979) derived a system of linear
equations from which \(J\) can be calculated. These \textit{dynamic ray-tracing} systems
require evaluation of second derivatives of velocity with respect to ray-
centered coordinates. Evaluation of these derivatives is described in Popov and
Pšeničk (1976, 1978). The spreading functions calculated from dynamic ray-
tracing can also be used to estimate the travel time of the ray arriving at the
receiver. This paraxial approximation eliminates two-point ray tracing if a fan
of shot rays covers an array of receivers sufficiently densely (Klimeš and
Červený, 1983). Since the Jacobian measures the differential area of a ray tube
(figure 30) an alternative, although possibly less precise, method of determining
\(J\) is simply to shoot three closely space rays near a receiver and compute

\[
J \approx \left| \frac{\Delta \sigma}{\Delta \gamma_1 \Delta \gamma_2} \right| \quad \text{(A12)}
\]

(e.g. Cerveny et al., 1977), where \(\Delta \sigma\) denotes the cross sectional area of a
surface that includes the receiver in the borehole.

Reflection, refraction, and transmission of rays at arbitrarily curved
interfaces has been described by Červený and Ravindra (1971) and Hubral
(1979). The calculation proceeds by specifying a local coordinate system in the
plane of incidence defined by the slowness vector at the point of incidence and
the normal to the interface. Phases are matched using Snell's \textit{law} in the local
coordinate system. Details of the procedure are given in Popov and Pšeničk
and Cervený and Hron (1980). SH and SV reflection and transmission is calculated by resolving the incident S polarization into components in (SV) and perpendicular (SH) to the plane of incidence. For a zeroth-order approximation and a smooth, first-order interface (one for which the elastic constants are discontinuous), only the local slope of the interface in the plane of incidence affects the problem provided that the principle radii of curvature of the interface are large compared to the wavelength.

The complete calculation of amplitudes must include conservation of energy through ray tubes, which requires a relation between the Jacobians of incident, and reflected or converted waves. The Jacobian $J'$ of a reflected, transmitted, or converted wave must be related to the Jacobian $J$ of the incident wave by

$$\frac{J'(\theta_j)}{J(\theta_j)} = \frac{\cos \delta_j'}{\cos \delta_j}$$

(A13)

where $\delta_j$ is the angle of incidence and $\delta_j'$ is the angle of reflection, transmission, or conversion at $\theta_j$. Compact formulae for the amplitude of P waves, which include all of these effects, are given in Cervený et al. (1977). In the case of a 3-D medium described by homogeneous layers separated by arbitrarily curved boundaries, simple geometric formulae can be given for the geometric spreading of all P multiples (e.g. May and Hron, 1978). In many applications, models having layers in which velocities are constant can illustrate many of most important effects of 3-D structure because the most dramatic effects will be given by layer topography. This also simplifies the calculation of ray trajectories by eliminating the numerical integration of the ray tracing equations. The Jacobians needed for amplitude calculations can be
determined by analytic expressions for curvature of the wavefront (Červený and Ravindra, 1971).
Table 1
Ray traced events included in the synthetic VSPs.

<table>
<thead>
<tr>
<th>Event</th>
<th>Mode*</th>
<th>Ray Code</th>
<th>Conversions</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>p</td>
<td>1 2 3</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>pP</td>
<td>1 1</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>pS</td>
<td>1 -1</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>ppPP</td>
<td>1 2 2 1</td>
<td>0</td>
</tr>
<tr>
<td>E</td>
<td>ppPS</td>
<td>1 2 -2 -1</td>
<td>1</td>
</tr>
<tr>
<td>F</td>
<td>ppSP</td>
<td>1 2 -2 1</td>
<td>2</td>
</tr>
<tr>
<td>G</td>
<td>ppSS</td>
<td>1 -2 -2 -1</td>
<td>1</td>
</tr>
<tr>
<td>H</td>
<td>psPP</td>
<td>1 -2 2 1</td>
<td>2</td>
</tr>
<tr>
<td>I</td>
<td>psPS</td>
<td>1 -2 2 -1</td>
<td>3</td>
</tr>
<tr>
<td>J</td>
<td>psSP</td>
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<td>2</td>
</tr>
<tr>
<td>K</td>
<td>psSS</td>
<td>1 -2 -2 -1</td>
<td>1</td>
</tr>
<tr>
<td>L</td>
<td>ppS</td>
<td>1 2 -3</td>
<td>1</td>
</tr>
<tr>
<td>M</td>
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<td>1 -2 3</td>
<td>2</td>
</tr>
<tr>
<td>N</td>
<td>pss</td>
<td>1 -2 -3</td>
<td>1</td>
</tr>
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</table>

* (lowercase indicates downgoing event, uppercase indicates upgoing event)

Table 2
Offset J VSP Acquisition and Processing Parameters

<table>
<thead>
<tr>
<th>Source Parameters</th>
<th>Type: Litton 311 Vertical Vibrator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Offset:</td>
<td>6038ft. (1840m.)</td>
</tr>
<tr>
<td>Azimuth:</td>
<td>N106° 55′ E</td>
</tr>
<tr>
<td>Sweep:</td>
<td>14-112Hz log. 15dB boost</td>
</tr>
<tr>
<td>Sweep Length:</td>
<td>20% start amp, 10% break point</td>
</tr>
<tr>
<td></td>
<td>12sec</td>
</tr>
<tr>
<td>Downhole Tool</td>
<td>Type: IFP Geolock-H 3-Component</td>
</tr>
<tr>
<td></td>
<td>(2-arm hydraulic)</td>
</tr>
<tr>
<td>Geophone Type:</td>
<td>Sensor SM4U, 14Hz</td>
</tr>
<tr>
<td>Recording Parameters</td>
<td>Recording System: Sercel 338B</td>
</tr>
<tr>
<td></td>
<td>Sampling Rate: 2msec</td>
</tr>
<tr>
<td></td>
<td>Record Length: 12+2 sec</td>
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<tr>
<td></td>
<td>Recording Filters: High Cut 250Hz 72dB/oct</td>
</tr>
<tr>
<td></td>
<td>Low Cut 12.5Hz 12dB/oct</td>
</tr>
<tr>
<td>Processing Sequence</td>
<td>Vertical Stack: Σ9</td>
</tr>
<tr>
<td></td>
<td>Correlate</td>
</tr>
<tr>
<td></td>
<td>Demultiplex</td>
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<td></td>
<td>Trace Sort</td>
</tr>
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</table>
### Table 3

Parameters for the Springdale Reef Model

<table>
<thead>
<tr>
<th>Formation Name</th>
<th>Depth (feet)</th>
<th>(V_p) (feet/sec)</th>
<th>(V_s) (feet/sec)</th>
<th>Density ((gm/cm^3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glacial Till</td>
<td>0</td>
<td>6500↑</td>
<td>1860↑</td>
<td>2.04</td>
</tr>
<tr>
<td>Sunbury - Ellsworth Sh.</td>
<td>600</td>
<td>8000↑</td>
<td>4000↑</td>
<td>2.55</td>
</tr>
<tr>
<td>Antrim Sh.</td>
<td>930</td>
<td>12000↑</td>
<td>6000↑</td>
<td>2.45</td>
</tr>
<tr>
<td>Traverse Ls.</td>
<td>1210</td>
<td>18500↑</td>
<td>8250↑</td>
<td>2.62</td>
</tr>
<tr>
<td>Bell Sh.</td>
<td>1800</td>
<td>7000↑</td>
<td>2000↑</td>
<td>2.62</td>
</tr>
<tr>
<td>Dundee Ls. - D.R.Salt</td>
<td>1900</td>
<td>13000↑</td>
<td>6500↑</td>
<td>2.67</td>
</tr>
<tr>
<td>D.R.Salt - Amherstburg</td>
<td>2400</td>
<td>19200</td>
<td>9600</td>
<td>2.72</td>
</tr>
<tr>
<td>Bois Blanc</td>
<td>3050</td>
<td>14300</td>
<td>8550↑</td>
<td>2.72</td>
</tr>
<tr>
<td>Bass Island - Salina</td>
<td>3150</td>
<td>18900</td>
<td>11300↑</td>
<td>2.81</td>
</tr>
<tr>
<td>F, E, C, &amp; B Units</td>
<td>3500</td>
<td>14300</td>
<td>8550</td>
<td>2.37</td>
</tr>
<tr>
<td>A2 Carbonate</td>
<td>4580</td>
<td>20000</td>
<td>12000</td>
<td>2.79</td>
</tr>
<tr>
<td>A2 Evaporite</td>
<td>4690</td>
<td>15200</td>
<td>9100</td>
<td>2.29</td>
</tr>
<tr>
<td>A1C, Niagarian</td>
<td>4760</td>
<td>19200</td>
<td>11500</td>
<td>2.77</td>
</tr>
<tr>
<td>Cabot - Manitoulin</td>
<td>5530</td>
<td>13900</td>
<td>7900</td>
<td>2.76</td>
</tr>
<tr>
<td>Cincinnati</td>
<td>5770</td>
<td>18000</td>
<td>10800↑</td>
<td>2.73</td>
</tr>
<tr>
<td>Utica Sh.</td>
<td>5940</td>
<td>13500</td>
<td>7500</td>
<td>2.68</td>
</tr>
<tr>
<td>Trenton - Black Riv.</td>
<td>6130</td>
<td>20000</td>
<td>12000↑</td>
<td>2.72</td>
</tr>
<tr>
<td>PDC Ss.</td>
<td>6450</td>
<td>17000↑</td>
<td>10200↑</td>
<td>2.80↑</td>
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</tbody>
</table>

↑ inferred
Figure 1. Vertical Seismic Profile Acquisition Geometry.
Figure 2. Vertical component synthetic VSP for a zero-offset pure compressional wave source in the plane layer model shown.
Figure 3. Vertical component synthetic VSP for a long-offset pure compressional wave source in the plane layer model shown. Source offset = 500m.
Figure 4. Vertical component synthetic VSP for a zero-offset pure compression- 
al wave source in the dipping layer model shown. Interfaces dip at 15°.
Figure 5. Vertical component synthetic VSP for a long-offset pure compressional wave source in the dipping layer model shown. Source offset = 500m. Interfaces dip at 15°.
Figure 6. Unconformity model showing direct transmission ray paths.
Figure 7. Vertical component synthetic VSP for the Unconformity model with pure compressional wave source. Source offset = 823 m.
Figure 8. Radial component synthetic VSP for the Unconformity model with pure compressional wave source. Source offset = 823m.
Figure 9. Vertical component synthetic VSP for the Unconformity model with pure shear wave source. Source offset = 823m.
Figure 10. Radial component synthetic VSP for the Unconformity model with pure shear wave source. Source offset = 823m.
Figure 11. Simple Reef model showing direct transmission ray paths and vertical component synthetic VSP for a pure compressional wave source of offset 900m.
Figure 12. Anticline Model with geometry parameters.
Figure 13. Synthetic VSP sections for the Anticline model. Compressional wave source offset = 0. (Arrow indicates depth of anticline crest.)
Figure 14. Synthetic VSP sections for the Anticline model. Compressional wave source offset = 500. (Arrow indicates depth of anticline crest.)
Figure 15. Synthetic VSP sections for the Anticline model. Compressional wave source offset = 1000. (Arrow indicates depth of anticline crest.)
Figure 16. Synthetic VSP sections for the Anticline model. Compressional wave source offset = 1500. (Arrow indicates depth of anticline crest.)
Figure 17. Direct transmission ray paths from various source offsets for the Anticline Model.
Figure 18. P-to-P reflection ray paths from various source offsets for the Anticline Model.
Figure 19. The Springdale Reef Model prior to the MIT-CGG VSP Groupshoot.
Figure 20. Direct transmission and P-to-P reflection ray paths for the Springdale Reef Model with a source offset of 2000ft.
Figure 21. Direct transmission and P-to-P reflection ray paths for the Springdale Reef Model with a source offset of 4000ft.
Figure 22. Direct transmission and P-to-P reflection ray paths for the Springdale Reef Model with a source offset of 6000ft.
Figure 23. MIT-CGG VSP Groupshoot Field Acquisition Geometry.
Figure 24. Offset J vertical component VSP data (source offset = 6000 ft).
Figure 25. The Springdale Reef Model modified with full waveform sonic velocities and Borehole Gravimeter densities.
Figure 26. Vertical component synthetic VSP for a source offset of 6000ft in the improved Springdale Reef Model.
Figure 27. Interpreted version of Offset J vertical component VSP data.
Figure 28. Interpreted version of the vertical component synthetic VSP for a source offset of 6000ft in the improved Springdale Reef Model.
Figure 29. Ray centered coordinate system. $s$ is the distance from the source, $q_1$ is in the $\hat{e}_1$ direction and $q_2$ is in the $\hat{e}_2$ direction.
$\Delta \sigma = \sigma_2 - \sigma_1$

$\Delta \sigma \approx J \Delta \gamma_1 \Delta \gamma_2$

Figure 30. Ray tube showing differential area and calculation of the Jacobian.