XVIII. DETECTION AND ESTIMATION THEORY*

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A. THESES COMPLETED

1. STATE VARIABLES, THE FREDHOLM THEORY AND OPTIMAL COMMUNICATIONS

   This study has been completed by A. B. Baggeroer. It was submitted as a thesis in partial fulfillment of the requirements for the Degree of Doctor of Science, Department of Electrical Engineering, M. I. T., January 1968.

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2. ASYMPTOTIC APPROXIMATIONS TO THE ERROR PROBABILITY FOR DETECTING GAUSSIAN SIGNALS

   This study has been completed by L. D. Collins. It was submitted as a thesis in partial fulfillment of the requirements for the Degree of Doctor of Science, Department of Electrical Engineering, M. I. T., June 1968.

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3. CLOSED-FORM ERROR EXPRESSIONS IN LINEAR FILTERING

   This study has been completed by M. Mohajeri. It was submitted as a thesis in partial fulfillment of the requirements for the Degree of Master of Science, Department of Electrical Engineering, M. I. T., June 1968.

   H. L. Van Trees

B. PERFORMANCE OF THE OPTIMAL SMOOTHER

   The purpose of this report is to correct some errors in the analysis of the performance of the optimal smoother, which have appeared in recent publications.1, 2

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Let us assume that the generation of a state vector $x(t)$ and its observation in the presence of noise may be described by the following linear state representation and covariance matrices:

$$\frac{dx(t)}{dt} = F(t) x(t) + G(t) u(t), \quad T_0 \leq t;$$  

(1)

$$p(t) = C(t) x(t) + w(t), \quad T_0 \leq t \leq T_f;$$  

(2)

$$E[u(t)u^T(\tau)] = Q_s(t-\tau), \quad T_0 \leq t, \tau;$$  

(3)

$$E[x(T_0)x^T(T_o)] = \Sigma(T_0 | T_o);$$  

(4)

$$E[w(t)w^T(\tau)] = R(t) \delta(t-\tau), \quad T_0 \leq t, \tau.$$  

(5)

(Zero means for $u(t)$, $x(T_0)$, and $w(t)$ have been assumed for simplicity.)

The optimal smoother estimates the state vector $x(t)$ over the interval $[T_0, T_f]$, where the received signal $r(t)$ is observed over the same interval. The equations specifying the structure of the smoother may be found by assuming Gaussian statistics and using variational means,\(^1\,^2\) or by using a structured approach and solving the resulting Wiener-Hopf equation.\(^3\) This structure is

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} F(t) & G(t)QG^T(t) \\ C^T(t)R^{-1}(t)C(t) & -F^T(t) \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} - \begin{bmatrix} 0 \\ C^T(t)R^{-1}(t)r(t) \end{bmatrix}, \quad T_0 \leq t \leq T_f'$$  

(6)

with the boundary conditions

$$\hat{x}(T_0) = \Sigma(T_0 | T_o)p(T_o)$$  

(7)

$$p(T_f') = 0.$$  

(8)

One can also relate the optimal smoother estimate $\hat{x}(t)$ to the realizable filter estimate $\hat{x}_r(t)$ by\(^2\,^3\)

$$\hat{x}(t) - \hat{x}_r(t) = \Sigma(t | t)p(t), \quad T_0 \leq t \leq T_f,'$$  

(9)

where $\Sigma(t | t)$ is the realizable filter covariance of error.

Let us now consider some aspects of the performance of the smoother. First, we have the result

$$E[\xi(t)p^T(\tau)] = 0, \quad T_0 \leq t, \tau \leq T_f;'$$  

(10)
that is, the smoother error \( \xi(t) = \hat{x}(t) - x(t) \) is uncorrelated with the costate \( p(t) \) for all time within the observation interval.

**Proof:** The costate function \( p(t) \) is the result of a linear operation upon the received signal \( r(t) \). The error \( \xi(t) \) is uncorrelated with the received signal by the orthogonal projection lemma; see, for example, Van Trees\(^4\) or Kalman and Bucy.\(^5\) Equation 10 follows from this lemma and the linearity of the operation for finding the costate.

Our next result relates the smoother error covariance \( \Sigma(t|T_f) \), the realizable filter covariance \( \Sigma(t|t) \), and the costate covariance \( \Pi(t|T_f) \) by the formula

\[
\Sigma(t|T_f) + \Sigma(t|t) \Pi(t|T_f) \Sigma(t|t) = \Sigma(t|t), \quad T_o \leq t \leq T_f. \tag{11}
\]

**Proof:** Let us rewrite Eq. 9 in the form

\[
\xi(t) - \xi_F(t) = \Sigma(t|t) p(t), \quad T_o \leq t \leq T_f. \tag{12}
\]

where we have subtracted and added \( x(t) \) to the left side of Eq. 9, and \( \xi_F(t) \) is the realizable filter error. We can in turn rewrite Eq. 12 in the form

\[
\xi(t) - \Sigma(t|t) p(t) = \xi_F(t), \quad T_o \leq t \leq T_f. \tag{13}
\]

Equation 11 follows by multiplying (13) by its transpose, taking the expectation of the result, and using (10) to eliminate the cross-product terms between \( \xi(t) \) and \( p(t) \).

Our third result states that the covariance of the costate satisfies the matrix differential equation

\[
\frac{d}{dt} \Pi(t|T_f) = -\left(F(t) - \Sigma(t|t) C^T(t) R^{-1}(t) C(t)\right)^T \Pi(t|T_f)
\]

\[
-\Pi(t|T_f) \left(F(t) - \Sigma(t|t) C^T(t) R^{-1}(t) C(t)\right)
\]

\[
-C^T(t) R^{-1}(t) C(t), \quad T_o \leq t \leq T_f. \tag{14}
\]

with the boundary condition \( \Pi(T_f|T_f) = 0 \), since \( p(T_f) \) vanishes identically.

**Proof:** We simply differentiate Eq. 11 with respect to the variable \( t \) and substitute the differential equations for \( \Sigma(t|T_f) \) and \( \Sigma(t|t) \). The differential equation for \( \Sigma(t|T_f) \) can be found by several methods to be\(^2,3\)

\[
\frac{d}{dt} \Sigma(t|T_f) = (F(t) + G(t) QG^T(t) \Sigma^{-1}(t|t)) \Sigma(t|T_f)
\]

\[
\Sigma(t|T_f) \left(F(t) + G(t) QG^T(t) \Sigma^{-1}(t|t)\right)^T
\]

\[
-G(t) QG^T(t), \quad T_o \leq t \leq T_f. \tag{15}
\]
with the final boundary condition $\Sigma \left( T_f \mid T_f \right)$ provided by the realizable filter solution at $t = T_f$. The differential equation for $\Sigma (t \mid t)$ is well known.\(^5\)

$$\frac{d}{dt} \Sigma (t \mid t) = F(t) \Sigma (t \mid t) + \Sigma (t \mid t) F^T(t) + G(t) Q G^T(t)$$

$$-\Sigma (t \mid t) C^T(t) R^{-1}(t) C(t) \Sigma (t \mid t), \quad T_o \leq t,$$  \hspace{1cm} (16)

with the initial condition given by $\Sigma (T_o \mid T_o)$. Making these substitutions and using the positive definiteness of $\Sigma (t \mid t)$ yields the desired result.

One can determine a second set of differential equations for the smoother covariance $\Sigma (t \mid T_f)$. Let us define two $(2n \times 2n)$ matrices

$$P(t \mid T_f) = \begin{bmatrix} \Sigma (t \mid T_f) & -\Sigma (t \mid t) \Pi(t \mid T_f) \\ -\Pi(t \mid T_f) \Sigma (t \mid t) & -\Pi(t \mid T_f) \end{bmatrix}, \quad T_o \leq t \leq T_f, \quad (17a)$$

and

$$W(t) = \begin{bmatrix} F(t) & G(t) Q G^T(t) \\ C^T(t) R^{-1}(t) C(t) & -F^T(t) \end{bmatrix}, \quad T_o \leq t \leq T_f. \quad (17b)$$

By using Eqs. 11, 14, 15, and 16, it is a straightforward exercise to show that $P(t \mid T_f)$ satisfies the following matrix differential equation

$$\frac{d}{dt} P(t \mid T_f) = W(t) P(t \mid T_f) + P(t \mid T_f) W^T(t) + \begin{bmatrix} G(t) Q G^T(t) & 0 \\ 0 & C^T(t) R^{-1}(t) C(t) \end{bmatrix}, \quad T_o \leq t \leq T_f \quad (18a)$$

$$P(T_f \mid T_f) = \begin{bmatrix} \Sigma (T_f \mid T_f) & 0 \\ 0 & 0 \end{bmatrix}. \quad (18b)$$

Let us contrast these results with some that have been published. Equation 18a is identical in form to those given by Bryson and Frazier.\(^1\) Our interpretation of the partitions of the matrix $P(t \mid T_f)$, however, is quite different. We have demonstrated that $P(t \mid T_f)$ is not the covariance matrix of the augmented vector of $\xi(t)$ and $\mu(t)$; that is,

$$P(t \mid T_f) \neq E \left\{ \begin{bmatrix} \xi(t) \\ \mu(t) \end{bmatrix} \begin{bmatrix} \xi(t) \\ \mu(t) \end{bmatrix}^T \right\}. \quad (19)$$
If this is true, we can easily demonstrate a contradiction by using Eq. 18:

$$\frac{d}{dt} \Pi(t|T_f)_{t=T_f} = C^T(T_f) R^{-1}(T_f) C(T_f).$$

(20)

This in turn implies

$$\Pi(T_f - \Delta t|T_f) = -C^T(T_f) R^{-1}(T_f) C(T_f) \Delta t,$$

(21)

which is clearly impossible, since \(\Pi(t|T_f)\) is a covariance matrix. We have also shown in Eq. 10 that the diagonal terms in the expectation indicated in Eq. 19 are zero.

In Rauch, Tung, and Striebel's paper on this topic, they also assert that the smoothing error \(\xi(t)\) and the costate \(\lambda(t)\) are correlated. As indicated above, they are not. Also, their relation between \(\Sigma(t|T_f)\) and \(\Pi(t|T_f)\) differs from Eq. 11 with respect to sign, which will imply the same negative covariance as discussed above.

While the basic result concerning the equation for \(\Sigma(t|T_f)\) is correct in both of these papers, these differences and contradictions leave some of the derivation involved rather suspect.

A. B. Baggeroer

References


C. ASYMPTOTIC APPROXIMATIONS TO THE ERROR PROBABILITY FOR SQUARE-LAW DETECTION OF GAUSSIAN SIGNALS

In Quarterly Progress Report No. 85 (pages 253-265) and No. 88 (pages 263-276), we discussed the application of tilted probability distributions to the problem of evaluating the performance of optimum detectors for Gaussian signals received in additive Gaussian noise. In many situations it is convenient for either mathematical or physical reasons to use a suboptimum receiver. For example, instead of building a time-variant...
linear estimator followed by a correlator, we may choose to use a time-invariant filter followed by an envelope detector. In this report we develop the necessary modifications to allow us to approximate the error probabilities for such suboptimum receivers. We first derive an asymptotic expansion for the error probabilities. Since the test statistic is not the logarithm of the likelihood ratio, we must calculate the semi-invariant moment-generating function separately for each hypothesis. Second, we calculate these functions for the class of suboptimum receivers consisting of a linear filter followed by a squarer and an integrator. This is an accurate model for many of the receivers that are used in practice. We shall concentrate on the case in which the random processes and the receiver filter can be modeled via state variables. This includes as a subclass all stationary processes with rational spectra and all lumped RLC filters. In Section XVIII-E some numerical results will be presented which were obtained by using the techniques that we shall develop here.

The problem that we are considering is the zero-mean binary Gaussian problem.

\[
H_1: \quad r(t) = s_1(t) + w(t) \quad T_1 \leq t \leq T_f, \\
H_0: \quad r(t) = s_0(t) + w(t)
\] (1)

\(s_1(t)\) and \(s_0(t)\) are sample functions from zero-mean Gaussian random processes with known covariance functions \(K_1(t, \tau)\) and \(K_0(t, \tau)\), and \(w(t)\) is a sample function of white Gaussian noise with spectral density \(N_0/2\).

1. Approximations to the Error Probabilities

In this section, we develop bounds on and approximations to the error probabilities for suboptimum receivers. The development for hypothesis \(H_0\) parallels that for the optimum detector which was given in the previous report. We add a subscript to the semi-invariant moment-generating function to denote the hypothesis

\[
\mu_0(s) = \ln \mathbb{E}[e^{sL}|H_0],
\]

where \(L\) denotes the test statistic on which the decision is based. For our purposes it will suffice to consider \(s\) real. Furthermore, Eq. 2 is valid only over some range of \(s\), say \(a < s < b\), which is the familiar "region of convergence" associated with a Laplace transform.

Now we define a tilted random variable \(L_0s\) to have the probability density

\[
p_{L_0s}(L) = e^{sL - \mu_0(s)} p_{L|H_0}(L).
\] (3)
Then
\[ \Pr [e | H_0] = \int_\gamma^{\infty} p_{\ell_0}(L) \exp[\mu_0(s)-sL] \, dL, \] (4)

where \( \gamma \) denotes the threshold level. Just as before, we expand \( p_{\ell_0}(L) \) in an Edgeworth expansion. Therefore, we shall need the semi-invariants of the tilted random variable \( \ell_0 \), which are the coefficients in the power series expansion of the semi-invariant moment-generating function for \( \ell_0 \).

\[
\ln M_{\ell_0}(t) = \ln E \left[ e^{t \ell_0} \right] \\
= \ln \int_{-\infty}^{\infty} e^{(s+t)\ell_0 - \mu_0(s)} p_{\ell | H_0}(L) \, dL \\
= \mu_0(s+t) - \mu_0(s). \] (5)

Therefore
\[
\frac{d^k}{ds^k} \mu_0(s) = k^{th} \text{ semi-invariant of } \ell_0. \] (6)

and the coefficients in the Edgeworth expansion of \( p_{\ell_0}(L) \) are obtained from the derivatives of \( \mu_0(s) \) just as before.

Now we proceed with an analogous development for hypothesis \( H_1 \).

\[
\mu_1(s) = \ln M_{\ell | H_1}(s). \] (7)

For the optimum detector discussed previously,
\[
\ell = \ln \Lambda(r(t)), \] (8)

where \( \Lambda(r(t)) \) is the likelihood ratio, and
\[
\mu_1(s) = \mu_0(s+1), \] (9)

so that it was sufficient for that case to consider only one of the two conditional moment-generating functions.

Returning to the suboptimum receiver, we must define a second tilted random variable \( \ell_1 \),
\[
p_{\ell_1}(L) = e^{sL-\mu_1(s)} p_{\ell | H_1}(s), \]
Then,
\[ \Pr \left[ \epsilon \left| \mathcal{H}_1 \right. \right] = \int_{-\infty}^{\gamma} e^{p_1(s)-sL} \cdot p_{f_{1s}}(L) \, dL. \]

Just as above
\[ \ln M_{f_{1s}}(t) = \mu_1(t+s) - \mu_1(s), \]
and
\[ \frac{d^k}{ds^k} \mu_1(s) = k^{\text{th}} \text{ semi-invariant of } f_{1s}. \]

so that the coefficients in the Edgeworth expansion of \( p_{f_{1s}}(L) \) are obtained from the derivatives of \( \mu_1(s) \).

Our bounds and approximations follow immediately from Eq. 4 and 7. The Chernoff bounds are
\[ \Pr \left[ \epsilon \left| \mathcal{H}_0 \right. \right] \leq \exp[\mu_0(s_0)-s_0\gamma] \quad \text{for } s_0 > 0 \tag{10} \]
\[ \Pr \left[ \epsilon \left| \mathcal{H}_1 \right. \right] \leq \exp[\mu_1(s_1)-s_1\gamma] \quad \text{for } s_1 < 0, \tag{11} \]

where
\[ \mu_0(s_0) = \gamma \tag{12} \]
and
\[ \mu_1(s_1) = \gamma. \tag{13} \]

Similarly, the first-order asymptotic approximations are
\[ \Pr \left[ \epsilon \left| \mathcal{H}_0 \right. \right] \approx \Phi\left(-s_0 \sqrt{\mu_0(s_0)}\right) \exp \left[ \mu_0(s_0)-s_0\mu_0(s_0) + \frac{s_0^2}{2} \mu_0(s_0) \right] \tag{14} \]
\[ \Pr \left[ \epsilon \left| \mathcal{H}_1 \right. \right] \approx \Phi\left(+s_1 \sqrt{\mu_1(s_1)}\right) \exp \left[ \mu_1(s_1)-s_1\mu_1(s_1) + \frac{s_1^2}{2} \mu_1(s_1) \right], \tag{15} \]

where \( s_0 \) and \( s_1 \) are given by Eqs. 12 and 13, and \( \Phi(X) \) is the Gaussian error function
\[ \Phi(X) = \int_{-\infty}^{X} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{x^2}{2} \right] \, dx. \tag{16} \]

Since
\[ \hat{\mu}_0(s) = \text{Var}(\ell_{0s}) \geq 0 \]  
\[ \hat{\mu}_1(s) = \text{Var}(\ell_{1s}) \geq 0. \]  

\[ \hat{\mu}_0(s) \text{ and } \hat{\mu}_1(s) \text{ are monotone increasing functions of } s. \text{ Then Eq. 12 has a unique solution for } s_0 > 0 \text{ if} \]
\[ \gamma > \hat{\mu}_0(0) = \mathbb{E}[\ell | H_0]. \]  
Equation 13 has a unique solution for \( s_1 \leq 0 \) if
\[ \gamma < \hat{\mu}_1(0) = \mathbb{E}[\ell | H_1]. \]  
Thus, just as for the optimum receiver, we require
\[ \mathbb{E}[\ell | H_0] \leq \gamma \leq \mathbb{E}[\ell | H_1] \]  
in order to be able to solve Eqs. 12 and 13. This restriction is in addition to that implied in the definition of the function \( \hat{\mu}_0(s) \text{ and } \hat{\mu}_1(s). \)

We have developed bounds on and approximations to the error probabilities for a rather general binary detection problem. The semi-invariant moment-generating functions \( \mu_j(s) \) played a central role in all of our results. These results were obtained without making any assumptions on the conditional statistics of the received signals. We shall now evaluate \( \mu_j(s) \) for a specific class of Gaussian random processes and suboptimum receivers.

2. Semi-Invariant Moment-Generating Functions for Square-Law Detectors

The class of suboptimum receivers that we shall consider is indicated in Fig. XVIII-1. For the class of problems that we are considering, \( r(t) \text{ and } y(t) \) are Gaussian random processes. Expanding \( y(t) \) in its Karhunen-Loève expansion
\[ y(t) = \sum_{i=1}^{\infty} y_i \phi_i(t), \text{ } T_i \leq t \leq T_f, \]  
we have
\[ \ell = \int_0^T y^2(t) \, dt \]
\[ = \sum_{i=1}^{\infty} y_i^2. \]  

Hence, \( \mu_i(s) = \ln E \left[ e^{s \ell} | H_j \right] \)

\[
\mu_i(s) = \ln E \exp \left[ s \sum_{i=1}^{\infty} Y_i^2 | H_j \right] \\
= \ln E \left[ \exp \left( sY_i^2 \right) | H_j \right] \\
= \sum_{i=1}^{\infty} \ln E \left[ \exp \left( sY_i^2 \right) | H_j \right] \\
= -\frac{1}{2} \sum_{i=1}^{\infty} \ln \left( 1 - 2s \lambda_{ij} \right) \quad \text{for } s < \frac{1}{2\lambda_{ij}}, \; j = 0, 1, \quad (24)
\]

where \( \{\lambda_{ij}\} \) denotes the eigenvalues of \( y(t) \), \( T_i < t < T_f \), conditioned on \( H_j \), for \( j = 0, 1 \); and we have used Eq. 7.67 of Wozencraft and Jacobs. ²

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The problem of computing the output probability distribution of a nonlinear detector, such as that shown in Fig. XVIII-1, has been studied for more than twenty years. ³ The previous approaches to this problem started with an expression for the characteristic function analogous to Eq. 24. Only recently has a satisfactory technique for finding the significant eigenvalues become available. ⁴ We thus can approximate the characteristic function by using the most significant eigenvalues. We are then faced, however, with the computational problem of evaluating (numerically) an inverse Fourier transform. Although highly efficient algorithms exist, ⁵ the constraint of computer memory size makes it difficult to obtain sufficient accuracy on the tail of the probability density.

Instead, we use the moment-generating function (with real values for its argument) to obtain bounds and approximations as discussed above. All that remains is to obtain...
closed-form expressions for $\mu_0(s)$ and $\mu_1(s)$. Recall that the Fredholm determinant for a random process $y(t)$, $T_{i_1} \leq t \leq T_{i_2}$, is defined as

$$D_\mathcal{F}(z) = \prod_{i=1}^{\infty} (1 + z \lambda_i),$$  \hspace{1cm} (25)$$

where $\{\lambda_i\}$ are the eigenvalues of $y(t)$. Then from Eq. 23,

$$\mu_j(s) = -\frac{1}{2} \ln D_\mathcal{F} |H_j(-2s)| \text{ for } s < \frac{1}{2\lambda_{ij}}, \ j = 0, 1. \hspace{1cm} (26)$$

The restriction on $s$ below Eq. 26 is the "region of convergence" discussed above. Combining this with the ranges on $s$ for which Eqs. 12 and 13 have a solution, we have

$$0 < s_0 < \frac{1}{2\lambda_{i0}}. \hspace{1cm} (27a)$$

and

$$s_1 < 0. \hspace{1cm} (27b)$$

The techniques discussed in the previous report for evaluating Fredholm determinants are applicable here, too. Here the pertinent random process is the input to the square-law device in the receiver, conditioned on the two hypotheses. In the case in which the random-process generation and the receiver filter can be modeled via state variables, we can readily evaluate the Fredholm determinant. The model, conditioned on one of the hypotheses, is shown in Fig. XVIII-2. For simplicity in the discussion that follows, we drop the subscript denoting the hypothesis.

The equations specifying the message generation are

$$\dot{x}_1(t) = F_1(t) x_1(t) + G_1(t) u_1(t) \hspace{1cm} (28a)$$

and

$$r(t) = C_1(t) x_1(t) + w(t) \hspace{1cm} (28b)$$

Fig. XVIII-2. Suboptimum receiver: State-variable model.
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and those specifying the receiver are

\[ \dot{x}_2(t) = F_2(t) x_2(t) + G_2(t) r(t) \]  
\[ y(t) = C_2(t) x_2(t). \]  

(29a)  

(29b)

The initial conditions are zero-mean Gaussian random variables.

\[ E\left[ x_1(T_1) x_1^T(T_1) \right] = P_0 \]  
\[ E\left[ x_2(T_1) x_2^T(T_1) \right] = 0. \]  

(30a)  

(30b)

The entire system can be rewritten in canonical state-variable form.

\[ \frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} F_1(t) & 0 \\ G_2(t) C_1(t) & F_2(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} G_1(t) & 0 \\ 0 & G_2(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ w(t) \end{bmatrix} \]  
\[ E\left[ \begin{bmatrix} x_1(T_1) \\ x_2(T_1) \end{bmatrix} \begin{bmatrix} x_1^T(T_1) \\ x_2^T(T_1) \end{bmatrix} \right] = \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix} \]  

(31a)  

(31b)

Define

\[ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \]  

(32a)

\[ F(t) = \begin{bmatrix} F_1(t) & 0 \\ G_2(t) C_1(t) & F_2(t) \end{bmatrix} \]  

(32b)

\[ G(t) = \begin{bmatrix} G_1(t) & 0 \\ 0 & G_2(t) \end{bmatrix} \]  

(32c)

\[ Q(t) = \begin{bmatrix} Q_1 & 0 \\ 0 & N_0 \end{bmatrix} \delta(t-t_0) \]  

(32d)

\[ C(t) = [0, C_2(t)] \]  

(32e)
$u(t) = \begin{bmatrix} u_1(t) \\ w(t) \end{bmatrix}$

$\Sigma_0 = \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix}$. 

Then

$\dot{x}(t) = F(t) x(t) + G(t) u(t)$

$y(t) = C(t) x(t)$

$E[x(t) x^T(t)] = \Sigma_0$, 

and the evaluation of the Fredholm determinants is carried out exactly as before, using

$$\ln D_\mathcal{F}(z) = \ln \det \phi_2(T_f) + \int_{T_i}^{T_f} \operatorname{tr} [F(t)] \, dt,$$

where

$$\frac{d}{dt} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \begin{bmatrix} F(t) & G(t) Q(t) Q^T(t) \\ C^T(t) z C(t) & -F^T(t) \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}$$

$$\begin{bmatrix} \phi_1(T_i) \\ \phi_2(T_i) \end{bmatrix} = \begin{bmatrix} \Sigma_0 \\ 1 \end{bmatrix}.$$ 

3. Application to Stationary Bandpass Random Processes

In many applications, the random processes of interest are narrow-band around some carrier frequency $\omega_c$. That is,

$$s(t) = \sqrt{2} A(t) \cos (\omega_c t + \theta(t)),$$

where the envelope and phase waveforms, $A(t)$ and $\theta(t)$, have negligible energy at frequencies comparable to $\omega_c$. Equivalently, we can write $s(t)$ in terms of its quadrature components.

$$s(t) = \sqrt{2} s_c(t) \cos \omega_c t + \sqrt{2} s_q(t) \sin \omega_c t.$$
In some of our applications we shall consider stationary bandpass random processes that can be modeled as in Eq. 37 over some interval \( T_i \leq t \leq T_f \), where \( s_c(t) \) and \( s_s(t) \) are statistically independent stationary random processes with identical statistics. For this case, the eigenvalues of the bandpass random process \( s(t) \) are equal to the eigenvalues of the quadrature components. Each eigenvalue of \( s_c(t) \) and \( s_s(t) \) of multiplicity \( N \) is an eigenvalue of \( s(t) \) with multiplicity \( 2N \). It follows immediately that

\[
\mu_{BP} \left( \frac{2E}{N_0} \right) = 2\mu_{LP} \left( \frac{E}{N_0} \right),
\]

where the subscripts BP and LP denote "bandpass" and "lowpass," and we have explicitly indicated the signal-to-noise ratio in each term.

We comment that the results in Eq. 38 are not the most general that can be obtained for bandpass processes, but they suffice for the examples that we shall consider. A more detailed discussion of properties and representation for bandpass signals would take us too far afield. There are two appropriate references. 7, 8

4. Summary

In this report we have discussed the necessary modifications of our asymptotic approximations to error probabilities to allow us to analyze suboptimum receivers. The results were expressed in terms of two semi-invariant moment-generating functions. For the problem of square-law detection of Gaussian signals, these functions can be expressed in terms of the Fredholm determinant. For the important case of processes and systems that can be modeled via state variables, there is a straightforward technique available for computing their Fredholm determinants.

In Section XVIII-E numerical results obtained by using the techniques developed in this report will be presented. A second problem in which these results have been successfully applied is the random phase detection problem. 9 For this problem the received signals are not Gaussian, but since we can compute \( \mu_j(s) \), the approximations presented here are still useful.

L. D. Collins, R. R. Kurth

References

3. Davenport and Root, op. cit., Chap. 9.
D. CHANNEL CAPACITY FOR AN RMS BANDWIDTH CONSTRAINT

The usual definition of the channel capacity of a bandlimited additive noise channel implies that the channel is strictly bandlimited. In some applications a strictly bandlimited assumption cannot be realistically imposed on the transmitted signal and/or channel. For example, a transmitted signal of finite duration is obviously not strictly bandlimited. Comparison of the performance of such an approximately bandlimited system with the theoretical performance implied by the strictly bandlimited channel capacity can lead to contradictions (such as system performance better than the "theoretical" ultimate performance). In this report the strictly bandlimited assumption of channel capacity is replaced by a mean-square bandwidth (rms) constraint and the resulting channel capacity is computed.

1. Derivation of rms Bandlimited Channel Capacity

As is well known, the channel capacity of an additive white noise channel (spectral density $N_0/2$) for which the transmitter spectrum is $S(f)$ is

$$C = \frac{1}{2} \int_{-\infty}^{\infty} df \ln \left( 1 + \frac{2S(f)}{N_0} \right). \quad (1)$$

It is convenient to define a normalized spectrum, $\sigma(f)$

$$S(f) = P\sigma(f), \quad (2)$$

where $P$ is the average transmitted power which is assumed finite. Equation 1 becomes

$$C = \frac{1}{2} \int_{-\infty}^{\infty} df \ln \left( 1 + \frac{2P}{N_0} \sigma(f) \right) \text{nats/sec.} \quad (3)$$

The remaining part of the solution for $C$ is to maximize Eq. 3 subject to any transmitter
or channel constraints. Here an infinite bandwidth channel is assumed with power and bandwidth constraints on the transmitter.

For example, if a strictly bandlimited constraint is made at the transmitter

\[ \sigma(f) = 0 \quad |f| > W \]  

(4)

and the optimal choice of \( \sigma(f) \) is

\[ \sigma(f) = \frac{1}{2W} \quad |f| \leq W \]  

(5)

with the resulting well-known capacity formula from Eq. 3:

\[ C = W \ln \left( 1 + \frac{P}{N_0} \right). \]  

(6)

Defining a signal-to-noise ratio \( \lambda \) in the transmitter bandwidth

\[ \lambda = \frac{P}{N_0 W} \]  

(7)

implies that the channel capacity increases logarithmically with increasing signal-to-noise ratio.

For an rms bandwidth \( B \) constraint at the transmitter,

\[ B^2 = \int_{-\infty}^{\infty} df f^2 \sigma(f), \]  

(8)

which represents a constraint on \( \sigma(f) \). The other implied constraints are

\[ \sigma(f) \geq 0 \]  

(9)

\[ \int_{-\infty}^{\infty} df \sigma(f) = 1. \]  

(10)

In order to maximize Eq. 3 subject to the three constraints on \( \sigma(f) \) (Eqs. 8-10), define

\[ J = \frac{1}{2} \int_{-\infty}^{\infty} df \ln \left( 1 + \frac{2P}{N_0} \sigma(f) \right) + a \int_{-\infty}^{\infty} df \sigma(f) + \gamma \int_{-\infty}^{\infty} f^2 \sigma(f) df, \]  

(11)

where \( a \) and \( \gamma \) are Lagrange multipliers. Perturbation of \( J \) with respect to \( \sigma(f) \) yields
\( \sigma(f) = \max \left[ 0, -\frac{1}{2} \left( \frac{1}{\sigma + \gamma f^2} + \frac{N_0}{P} \right) \right] \) \hspace{1cm} (12)

The maximum operation is necessary to satisfy \( \sigma(f) \geq 0 \). Clearly, if \( \sigma, \gamma \) are positive, \( \sigma(f) = 0 \), which does not satisfy the constraint Eqs. 8 and 10. Similarly, if the two multipliers are of different sign, the constraints cannot be satisfied; hence, \( \sigma \) and \( \gamma \) are both negative. Define two new positive multipliers \( Q \) and \( f_c \) such that

\[ \sigma(f) = \max \left[ 0, \frac{1}{2B\lambda} \left( \frac{Q^2 - \left( \frac{Qf^2}{f_c} \right)}{1 + \left( \frac{Qf^2}{f_c} \right)} \right) \right] \]

where the signal-to-noise ratio in the rms bandwidth

\[ \lambda = \frac{P}{N_0 B} \]

has been introduced. The transmitter spectrum is that of a one-pole process shifted down to cutoff at \( f = f_c \); the spectrum shape is plotted in Fig. XVIII-3 for large and small \( Q \).

For \( \sigma(f) \) as given in Eq. 13, direct evaluation of the constraints Eqs. 8 and 10 yields

\[ \lambda B^3 = f_c^3 \left\{ \frac{2}{3} + \frac{1}{Q^2} - \left( \frac{1}{Q} + \frac{1}{Q^3} \right) \tan^{-1} Q \right\} \]

and

\[ QPR \text{ No. 90} \quad 203 \]
\( \lambda B = f_c \left\{ \left( \frac{1}{Q} + Q \right) \tan^{-1} Q - 1 \right\}. \) \hspace{1cm} (16)

These two equations determine the unknowns \( f_c \) and \( Q \). Given \( f_c \) and \( Q \) as the solution of these equations, the channel capacity from Eq. 3 is

\[ C = 2 f_c \left\{ 1 - \frac{1}{Q} \tan^{-1} Q \right\}. \]

It can be shown from the equations above that \( C \) can also be written

\[ C = B g(\lambda), \]

where \( g(\lambda) \) is a complicated implicit function. The important observation is that channel capacity for rms bandwidth is of the same functional form as the strictly bandlimited form (Eq. 6), providing signal-to-noise ratios in the transmitter bandwidth are defined. Unfortunately \( g(\lambda) \) is implicit and cannot be determined analytically.

Fig. XVIII-4. Channel capacity per unit rms bandwidth.
2. Large Signal-to-Noise Ratio Approximations

The equations can be solved approximately for $\lambda \gg 1$. For large $Q$ the channel capacity in Eq. 17 is

$$C \approx 2f_c.$$  \hspace{1cm} \text{(19)}

Similarly, for large $Q$, Eq. 15 implies

$$\frac{2}{3} f_c^3 \approx \lambda B^3$$  \hspace{1cm} \text{(20)}

or combining

$$C \approx B(12)^{1/3} \lambda^{1/3} \quad (\lambda \gg 1)$$  \hspace{1cm} \text{(21)}

which implies that channel capacity increases as the cube root of $\lambda$ for an rms constraint, but only logarithmically for a strict bandwidth constraint. Thus, using the strict bandwidth capacity formula for channels that are actually rms bandlimited yields a capacity much lower than the true capacity.

$g(\lambda)$ is plotted in Fig. XVIII-4 along with its asymptote (Eq. 21).

T. J. Cruise

E. PERFORMANCE OF A CLASS OF RECEIVERS FOR DOPPLER-SPREAD CHANNELS

This report considers a class of generally suboptimum receivers for the binary detection of signals transmitted over a Doppler-spread channel and received in additive white Gaussian noise. Bounds on the error probabilities for these receivers are given. The performance of a suboptimum system is compared with the optimum one in several examples.

In the binary detection problem one of two narrow-band signals

$$f_i(t) = \sqrt{2} \Re \left[ i \pi c \right], \quad 0 \leq t \leq T, \quad i = 0, 1,$$  \hspace{1cm} \text{(1)}

is transmitted over a Doppler-spread channel and received in additive white Gaussian noise. The complex envelope of the received signal is

$$\tilde{r}(t) = \tilde{b}_i \tilde{f}_i(t) + \tilde{w}(t) = \tilde{s}_i(t) + \tilde{w}(t), \quad 0 \leq t \leq T, \quad i = 0, 1,$$  \hspace{1cm} \text{(2)}
where \( \tilde{b}_i(t) \) and \( \tilde{w}(t) \) are zero-mean, independent, stationary, complex Gaussian processes. Details of the representation of complex random processes may be found in Van Trees.1

The binary detection problem thus becomes one of deciding between the two hypotheses

\[
H_1: \quad \tilde{r}(t) = \tilde{s}_1(t) + \tilde{w}(t) \quad 0 \leq t \leq T.
H_0: \quad \tilde{r}(t) = \tilde{s}_0(t) + \tilde{w}(t)
\]

The known complex covariance functions for \( \tilde{s}_i(t) \) and \( \tilde{w}(t) \) are \( \tilde{K}_i(t, u) \) and \( \tilde{N}_0 \delta(t-u) \).

From Eq. 2,

\[
\tilde{K}_i(t, u) = f_i(t) \tilde{K}_i(t-u) f_i^*(u), \quad i = 0, 1.
\]

(The superscript star denotes complex conjugation.)

A special case of this hypothesis test occurs when the \( f_i(t) \) have identical complex envelopes but carrier frequencies that are separated enough so that the \( f_i(t) \) are orthogonal. Then

\[
H_1: \quad r(t) = \sqrt{2} \Re \left[ \tilde{s}_1(t) e^{j \omega_1 t} \right] + w(t)
H_0: \quad r(t) = \sqrt{2} \Re \left[ \tilde{s}_0(t) e^{j \omega_0 t} \right] + w(t), \quad (5a)
\]

where

\[
\tilde{s}(t) = \tilde{b}(t) \tilde{f}(t). \quad (5b)
\]

This model will be called the binary symmetric orthogonal communication case.

1. Optimum Receiver and Its Performance

The optimum receiver for the detection problem of Eq. 3 for a large class of criteria is well known1: the detector compares the likelihood ratio to a threshold. An equivalent test is

\[
\int_0^T \int_0^T \tilde{r}^*(t) \tilde{r}(u) \left[ \tilde{h}_1(t, u) - \tilde{h}_0(t, u) \right] du \, dt \begin{cases} \gamma \quad H_1 \\ \gamma \quad H_0 \end{cases} \quad (6)
\]

The \( \tilde{h}_i(t, u) \) are the complex envelopes of the bandpass filter impulse responses that satisfy the integral equations.
For the binary symmetric orthogonal communication case with equally likely hypotheses, the threshold \( \gamma \) is zero and the test of Eq. 6 reduces to choosing the larger of

\[
T \int_0^T \int_0^T \tilde{r}_i(t) \hat{h}(t,u) \tilde{r}_i(u) \, dt \, du, \quad i = 0, 1.
\] (8)

The subscript \( i \) indicates that the lowpass operation on the right-hand side of Eq. 8 is performed twice, once with \( \tilde{r}_1(t) \) obtained by heterodyning \( r(t) \) from \( \omega_1 \), and the other with \( \tilde{r}_0(t) \), the result of heterodyning \( r(t) \) from \( \omega_0 \). The filter \( h(t,u) \) satisfies Eq. 7 with the kernel \( \tilde{K}_{S_i}(t,u) \) of the signal in Eq. 5b.

Bounds on the performance of the optimum receiver are known. For the detector of Eq. 6

\[
P(\epsilon \mid H_0) \leq \exp[\mu(s) - s\gamma]
\]

\[
P(\epsilon \mid H_1) \leq \exp[\mu(s) + (1-s)\gamma]
\] (9)

where

\[
\mu(s) = \sum_{i=1}^{s} \left[ s \ln \left( 1 + \frac{\lambda_i}{N} \right) + (1-s) \ln \left( 1 + \frac{\lambda_{i+1}}{N} \right) - \ln \left( 1 + \frac{\lambda_{i+1}}{N} \right) \right]
\] (10)

The \( \lambda_i \) and \( \lambda_{i+1} \) are the eigenvalues of the complex random processes \( \tilde{s}_1(t), \tilde{s}_0(t) \) of Eq. 2, respectively, under \( H_0 \). The \( \tilde{\lambda}_{i,\text{comp}} \) are the eigenvalues of the composite process with covariance function

\[
\tilde{K}_{\text{comp}}(t,x) = (1-s) \tilde{K}_{\tilde{s}_1}(t,x) + s\tilde{K}_{\tilde{s}_0}(t,x).
\] (10a)

The value of \( s \) is chosen to minimize the bounds in Eq. 9. In the binary communication problem the processes \( s_1(t) \) and \( s_0(t) \) have essentially orthogonal eigenfunctions, and the probability of error for the detector of Eq. 8 is bounded by

\[
\frac{\exp(\mu_c(1/2))}{2 \left( 1 + \sqrt{\frac{\pi}{8} \cdot \mu_c(1/2)} \right)} \leq P(\epsilon) \leq \frac{\exp(\mu_c(1/2))}{2 \left( 1 + \sqrt{\frac{\pi}{8} \cdot \mu_c(1/2)} \right)}
\] (11)

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The subscript \(C\) in Eqs. 11 and 12 denotes the binary symmetric orthogonal communication problem. The \(\lambda_i\) are the eigenvalues of the process \(\tilde{s}(t)\) in Eq. 5b. The functions \(\mu_C(s)\) and \(\mu(s)\) can be expressed as integrals of minimum mean-square filtering errors and can be evaluated easily when the random processes \(\tilde{s}_1(t)\) have state-variable representations. Collins presents details on this and on other approximations to the error probabilities.

If the eigenvalues in Eq. 12 are chosen to minimize \(\mu_C(s)\), subject to a constraint on the average received energy \(E_r\), a bound on the probability of error for a binary orthogonal communication system results

\[
P(\epsilon) \leq \frac{1}{2} \exp \left[ -0.1488 \frac{E_r}{N_0} \right].
\]

Thus Eq. 13 gives a bound on the exponential performance of any signal and receiver for the communication problem of Eq. 5.

2. Suboptimum Receiver

One implementation of the optimum receiver follows directly from Eq. 6 if the filter \(h_1(t, u)\) is factored

\[
\tilde{h}_1(t, u) = \int_0^T \tilde{g}_1(x, t) \tilde{g}_1^*(x, u) \, dx, \quad 0 \leq t, u \leq T.
\]

Then the optimum receiver is realized as the filter-squarer configuration of Fig. XVIII-5. At bandpass these operations correspond to a narrow-band filter followed by a square-law envelope detector and integrator.

Solving Eqs. 7 and 14 for \(\tilde{g}_1(t, u)\) is difficult in general, but for several special cases the solution is possible and motivates the choice of the suboptimum receiver. The first case occurs when the observation interval \(T\) is large and the process \(\tilde{s}_1(t)\) is stationary. This implies that \(f(t)\) is a constant and the filter \(\tilde{g}_1(t, u)\) is time-invariant. Its Fourier transform is

\[
\tilde{G}_1(j\omega) = \left[ \frac{\tilde{s}_1(\omega)}{\tilde{s}_1(\omega) + N_0} \right]^+. \tag{15}
\]

The superscript + indicates the factor containing the left-half \(s\)-plane poles and zeros.
Fig. XVIII-5. Filter-squarer receiver for the binary detection problem.

Fig. XVIII-6. State variable model for a Doppler-spread channel and the $i^{th}$ branch of the filter-squarer receiver.

Fig. XVIII-7. Suboptimum receiver branch with signal multiplication and time-invariant filtering.
Another case in which the filter \( \tilde{g}_i(t, u) \) can be found is under a low-energy coherence, or "threshold" assumption. In this situation the impulse response \( \tilde{h}_i(t, u) \) becomes \( \tilde{K}_i(t, u) \). From Eqs. 4 and 14, this implies that for a large observation interval \( T \) the filter \( \tilde{g}_i(t, u) \) is a cascade of a time-variant gain, \( f_i(t) \), and a filter \( [\tilde{S}_b^*(\omega)]^\dagger \), where \( \tilde{S}_b(\omega) \) is the Fourier transform of \( \tilde{K}_b(t, u) \) given by Eq. 4.

A suboptimum receiver structure for Doppler-spread signals is the same as that shown in Fig. XVIII-5, but with the filter \( \tilde{g}_i(t, u) \) chosen arbitrarily. An attractive candidate for the filter \( \tilde{g}_i(t, u) \) is a time-variant gain \( f_i(t) \) followed by a time-invariant filter with the same form as \( [\tilde{S}_b^*(\omega)]^\dagger \) but with different time constants. This particular choice is motivated by the similar optimum filter-squarer-integrator configurations in the two limiting cases outlined above.

Both the error probability bounds presented below for the suboptimum receiver and those for the optimum detector can be evaluated conveniently when the processes \( \tilde{s}_i(t) \) or equivalently \( \tilde{b}_i(t) \), have state-variable representations. If the filter in the suboptimum receiver also has a state-variable representation, then the suboptimum system can be represented as shown in Fig. XVIII-6. The fading process under the \( i \)th hypothesis is generated by

\[
\begin{align*}
\tilde{x}_i(t) &= \tilde{f}_i \tilde{x}_i(t) + \tilde{G}_i \tilde{u}_i(t) \\
\tilde{b}_i(t) &= \tilde{C}_i \tilde{x}_i(t) \\
E[\tilde{u}_i(t) \tilde{u}_i^*(\tau)] &= \tilde{Q}_i \delta(t-\tau) \\
E[\tilde{s}_i(0) \tilde{s}_i^*(0)] &= \tilde{P}_i
\end{align*}
\]  

(16)

and the receiver is specified by

\[
\begin{align*}
\tilde{r}(t) &= \tilde{E}_i(t) \tilde{f}_i(t) + \tilde{w}(t) \\
\tilde{y}_i(t) &= \tilde{A}_i(t) \tilde{y}_i(t) + \tilde{B}_i(t) \tilde{r}(t) \\
\tilde{f}_i(t) &= |\tilde{H}_i(t)\tilde{y}_i(t)|^2 \\
E[\tilde{y}_i(0)\tilde{y}_i^*(0)] &= 0 \\
E[\tilde{w}(t)\tilde{w}^*(\tau)] &= N_0 \delta(t-\tau).
\end{align*}
\]  

(17)

The superscript \( \dagger \) indicates a complex conjugate transpose operation. Complex
state-variable representations have been discussed in detail by Baggeroer, Collins, and Van Trees. Figure XVIII-7 shows the state-variable receiver when the filter gains are chosen so that \( r(t) \) is multiplied by \( \tilde{\gamma}_1(t) \) and passed through a time-invariant filter of the same form as the one that generates \( \tilde{b}_1(t) \). This was suggested above as a promising suboptimum configuration. The gains of Fig. XVIII-6 become

\[
\tilde{A}_1(t) = \tilde{F}_r
\]

\[
\tilde{B}_1(t) = \tilde{G}_r \tilde{f}_1(t)
\]

\[
\tilde{H}_1(t) = \tilde{C}_r.
\]

The constant matrices \( \tilde{F}_r, \tilde{G}_r, \) and \( \tilde{C}_r \) have the same form as \( F, G, \) and \( C \) of Fig. XVIII-6.

### 3. Performance Bounds for the Suboptimum Receiver

The receivers of Figs. XVIII-5, XVIII-6, and XVIII-7 fall into the class treated by Collins and Kurth in Section XVIII-C. Bounds on the error probabilities for the receiver of Fig. XVIII-5 are

\[
P(\epsilon | H_1) \leq \exp[\mu_1(s_1) - s_1], \quad s_1 < 0 \tag{18}
\]

\[
P(\epsilon | H_0) \leq \exp[\mu_0(s_0) - s_0], \quad s_0 > 0 \tag{19}
\]

\[
\mu_1(s_1) = \ln E \left[ e^{s_1 \ell} | H_1 \right] = - \sum_{i=1}^{\infty} \ln (1-s_1 \lambda_{i1}) \tag{20}
\]

\[
\mu_0(s_0) = \ln E \left[ e^{s_0 \ell} | H_0 \right] = - \sum_{i=1}^{\infty} \ln (1-s_0 \lambda_{i0}), \tag{21}
\]

where \( \ell \) is indicated in Fig. XVIII-5, and the \( \lambda_{ij} \) are the eigenvalues of the complex random process \( \tilde{\gamma}_1(t) - \tilde{\gamma}_0(t) \) under hypothesis \( j = 0, 1 \). The bounds above and those of Eq. 9 permit a comparison of the optimum and suboptimum detectors. Evaluation of the bounds is feasible where the representations of Figs. XVIII-6 or XVIII-7 are used. The procedure is outlined below for a special case.

For the equilikely binary orthogonal communication problem of Eq. 5, the threshold \( \gamma \) in Eqs. 18 and 19 is zero, and the bound on the probability of error becomes

\[
P(\epsilon) = P(\epsilon | H_1) \leq \exp[\mu_1(s_1)], \tag{22}
\]
where

\[
\mu_1(s_1) = \ln E \left[ e^{s_1 (T_1 - T_0)} \right]_{H_1} = \ln E \left[ e^{s_1 T_1} \right]_{H_1} E \left[ e^{-s_1 T_0} \right]_{H_1}
\]

\[
= - \sum_{i=1}^{\infty} \left[ \ln (1-s_1 \lambda_{i s}) + \ln (1+s_1 \lambda_{i n}) \right], \quad -[\max \lambda_{in}]^{-1} < s_1 \leq 0. \tag{23}
\]

The \( \lambda_{is} \) and \( \lambda_{in} \) are the eigenvalues of the processes \( \tilde{f}_1(t) \) and \( \tilde{f}_0(t) \) in Fig. XVIII-5, under \( H_1 \). Since \( f_1(t) \) and \( f_2(t) \) are orthogonal, \( \tilde{f}_1(t) \) and \( \tilde{f}_0(t) \) are independent.

The expression for \( \mu_1(s) \) in Eq. 23 can be evaluated when the processes \( \tilde{f}_1(t) \) and \( \tilde{f}_0(t) \) have state-variable representations, as in Figs. XVIII-6 and XVIII-7. The first term on the right side of Eq. 23 is related to a Fredholm determinant and can be expressed as

\[
\sum_{i=1}^{\infty} \ln (1-s_1 \lambda_{is}) = \ln \det \left[ \tilde{\theta}_{21}(0, T) \tilde{P}_c + \tilde{\theta}_{22}(0, T) + \int_0^T \text{Tr} \left[ \tilde{F}_c(t) \right] \, dt \right] \tag{24}
\]

where \( \tilde{\theta}_{21}(0, T) \) and \( \tilde{\theta}_{22}(0, T) \) are partitions of the transition matrix satisfying

\[
\frac{d\tilde{\theta}(t, \tau)}{dt} = \left[ \begin{array}{cc}
\tilde{F}_c(t) & \tilde{G}_c(t) \tilde{Q}_c \tilde{G}_c(t) \\
-s_1 \tilde{C}_c(t) \tilde{C}_c(t) & -\tilde{F}_c(t)
\end{array} \right] \tilde{\theta}(t, \tau). \tag{25a}
\]

\[
\tilde{\theta}(\tau, \tau) = I_c. \tag{25b}
\]

Under \( H_1 \), the input to the receiver branch containing \( \tilde{f}_1(t) \) is \( s(t) + \tilde{n}(t) \). Thus the composite matrices \( \tilde{F}_c(t) \), \( \tilde{G}_c(t) \), \( \tilde{Q}_c \), \( \tilde{C}_c(t) \), and \( \tilde{A}(t) \) are those for the system shown in Fig. XVIII-6, with \( i = 1 \). If the state vectors \( \tilde{x}_1(t) \) and \( \tilde{x}_1(t) \) are adjoined, these matrices are

\[
\tilde{F}_c(t) = \left[ \begin{array}{c|c}
\tilde{F} & 0 \\
\tilde{f}(t) & \tilde{B}(t) \tilde{C} & \tilde{A}(t)
\end{array} \right] \tag{26}
\]

\[
\tilde{G}_c(t) = \left[ \begin{array}{c|c}
\tilde{G} & 0 \\
0 & \tilde{B}(t)
\end{array} \right] \tag{27}
\]
\[ \tilde{C}_c(t) = \begin{bmatrix} 0 & \tilde{H}(t) \\ \tilde{H}(t) & 0 \end{bmatrix} \]  \hspace{1cm} (28)

\[ \tilde{Q}_c = \begin{bmatrix} \tilde{Q} & 0 \\ 0 & N_0 \end{bmatrix} \]  \hspace{1cm} (29)

\[ \tilde{P}_c = \begin{bmatrix} \tilde{P} & 0 \\ 0 & 0 \end{bmatrix} \]  \hspace{1cm} (30)

The subscripts in Fig. XVIII-6 have been ignored above.

The second term in Eq. 23 is computed in a similar manner. In Eqs. 24 and 25 the sign of \( s_1 \) is reversed, and the composite matrices of Eqs. 26-30 reduce to just those of the receiver portion of Fig. XVIII-6. This is because under \( H_1 \) the input to the 0th branch of the demodulator is just \( \tilde{w}(t) \).

The special case of Fig. XVIII-7 is included in the preceding discussion. The optimum and suboptimum performances now can be contrasted by minimizing \( \mu_1(s_1) \) over \( s_1 \) and comparing it with \( \mu_c(1/2) \) from Eq. 12.

4. Examples

In the following examples the performance of the suboptimum receiver of Fig. XVIII-7 for the binary symmetric communication problem of Eq. 5 is compared with the optimum performance. The exponent \( \mu_1(s_1) \) of Eq. 23 is minimized over \( s_1 \). The exponent for the optimum receiver is \( \mu_c(1/2) \) in Eq. 12.

The first case to be treated comprises a first-order Butterworth fading spectrum and a transmitted signal with a constant envelope. The spectrum is

\[ \tilde{S}_b(\omega) = \frac{2kP}{\omega^2 + k^2} \]

and

\[ \tilde{f}(t) = \sqrt{\frac{E_1}{T}}, \quad 0 \leq t \leq T. \]

The average received energy in the signal component is

\[ E_r = E_t P. \]

This spectrum implies that the receiver of Fig. XVIII-8 is of the form

\[ \tilde{y}(t) = -k \tilde{y}(t) + \tilde{r}(t) r(t). \]
Fig. XVIII-8. Probability of error exponent for optimum and suboptimum receivers.

Table XVIII-1. Performance comparison for pulse train with first-order Butterworth fading.

<table>
<thead>
<tr>
<th>$E_r/N_0$</th>
<th>N</th>
<th>d</th>
<th>$\mu_c(\frac{1}{2})$</th>
<th>$\min \mu_1(s_1)$</th>
<th>$k_r/k$</th>
</tr>
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<tbody>
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<td>10</td>
<td>3</td>
<td>.1</td>
<td>-1.42</td>
<td>-1.41</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>.5</td>
<td>-1.26</td>
<td>-1.25</td>
<td>2.5</td>
</tr>
<tr>
<td>30</td>
<td>8</td>
<td>.1</td>
<td>-4.21</td>
<td>-4.17</td>
<td>6</td>
</tr>
</tbody>
</table>
Figure XVIII-8 shows $\mu_c(1/2)$ and $\mu_1(s_1)$ versus $E_r/N_o$ for two choices of the product $kT$. First, $kT$ is chosen to minimize $\mu_c(1/2)$ at each $E_r/N_o$. Then $k_rT$ is picked to optimize $\mu_1(s_1)$. The second set of curves is for $kT$ equal to $1/10$ of the optimum value at each $E_r/N_o$. The curves of Fig. XVIII-8 show that the "optimized" suboptimum receiver of Fig. XVIII-7 achieves a performance within 0.5 dB of the optimum for $E_r/N_o$ greater than one. Although it is not indicated, the value of $k_rT$ which maximizes the performance of the suboptimum scheme for large $E_r/N_o$ (large $kT$) is close to that predicted by Eq. 15. With regard to the bound of Eq. 13, the receiver exponent $\mu_c(1/2)$ never exceeds 0.121 ($E_r/N_o$) in Fig. XVIII-8.

A somewhat better signal for the one-pole fading spectrum is the pulse train of Fig. XVIII-9. It consists of $N$ square pulses each with duty cycle $d$. The exponent $\mu_c(1/2)$ is minimized when $N$ is approximately equal to the optimum $kT$ at any given $E_r/N_o$, and $d$ is made as small as possible, subject to the transmitter peak-power constraint. The resulting magnitude of $\mu_c(1/2)$ is bounded by 0.142 $E_r/N_o$. Table XVIII-1 gives a comparison of $\mu_c(1/2)$ with $\mu_1(s_1)$ for some representative signal-to-noise ratios. Given $E_r/N_o$ and $d$, $N$ and $kT$ are chosen to approximately minimize $\mu_c(1/2)$, and then $k_rT$ is adjusted to optimize $\mu_1(s_1)$. The suboptimum receiver performs nearly as well as the optimum receiver with these parameter values. The value of the optimum-time constant, $k_rT$, is consistent with a diversity interpretation of the signal of Fig. XVIII-9. In such a case the optimum receiver correlates only over the duration of each component pulse and sums the squares of the correlator outputs. Hence $k_rT$ is of the order of $N \cdot kT$ and the summation is approximated by the integrator following the squarer in Fig. XVIII-7.

The performance comparison for a second-order Butterworth fading spectrum has been carried out for the two signals used above. The results are not presented here, but they are similar to those of Fig. XVIII-8 and Table XVIII-1.

5. Summary

A suboptimum receiver has been analyzed for the detection of known signals transmitted over a Gaussian, Doppler-spread channel and received in additive Gaussian white noise. The form of the receiver is suggested by the structure of the optimum filter-squarer detector in several limiting cases. Bounds on the error probabilities for the suboptimum receiver have been presented and compared with the performance of the optimum receiver in several examples. For the signals and spectra evaluated, the suboptimum receiver performance is close to optimum when the system parameters are appropriately chosen.

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References


