

XVII. ELECTRODYNAMICS OF MEDIA *

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RESEARCH OBJECTIVES

1. Work is proceeding on the relativistic theory of quadrupolar media, and it is expected that this will come to a satisfactory conclusion during 1969. Research will continue to gain an understanding of power, energy, stress, and momentum in dispersive media, with the aim of providing a useful way of using frequency-domain information to prove conservation theorems.

2. The study of nonlinear interactions in CO₂ lasers is concerned on the one hand with gain saturation, as determined by relaxation among the vibrational-rotational levels and by spatial diffusion. The equivalent number of levels participating in cw laser action will be determined experimentally in a system in which spatial diffusion effects are minimized. The experimental and theoretical study of the nonlinear response of a laser amplifier to pulses of duration comparable to the inverse linewidth will be continued. A cavity-dumping scheme has been successfully implemented for this purpose.

On the new cw H₂O laser, which was built for NASA and is now located at the Research Laboratory of Electronics, studies on the relaxation rates and saturation mechanisms of the cw lines will be made. Frequency stability measurements will be carried out.

3. Numerical, theoretical, and analog analyses of coaxial and waveguide junctions are being made. The objectives are to determine a versatile computational method and to provide a meaningful display of the numerical results. Numerical and theoretical analyses are being made of the excitation of modes in hollow dielectric circular waveguide by a localized source. The use of the reciprocity theorem in designing antennas with parasitic reflectors is being studied.

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A. STEP RESPONSE OF INHOMOGENEOUSLY BROADENED LASER MEDIUM

In the bandwidth measurement of CO₂ at 10.6 μ it was necessary to match the observed laser amplifier output, attributable to a step input, to a theoretically predicted output.^{1,2} The theoretical predictions of the small-signal step response were based on

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equations developed for a homogeneously broadened medium. It was argued that the step response of a high-gain amplifier is determined mainly by the frequency components of the signal near the maximum of the gain vs frequency curve; but in this region the gain curves of a homogeneously broadened medium on one hand, and one that is both homogeneously and inhomogeneously broadened on the other hand, are hardly distinguishable. Hence one would expect that the step responses of two such media of high gain should be practically indistinguishable.

A computer program has been set up which evaluated the step response of a medium with both homogeneous and inhomogeneous broadening. Figure XVII-1 shows some results. The different curves labelled A-E are evaluated for different gain constants $\alpha(\text{m}^{-1})$. The length of the system is 5 m. The output is normalized to its steady-state value in each case.

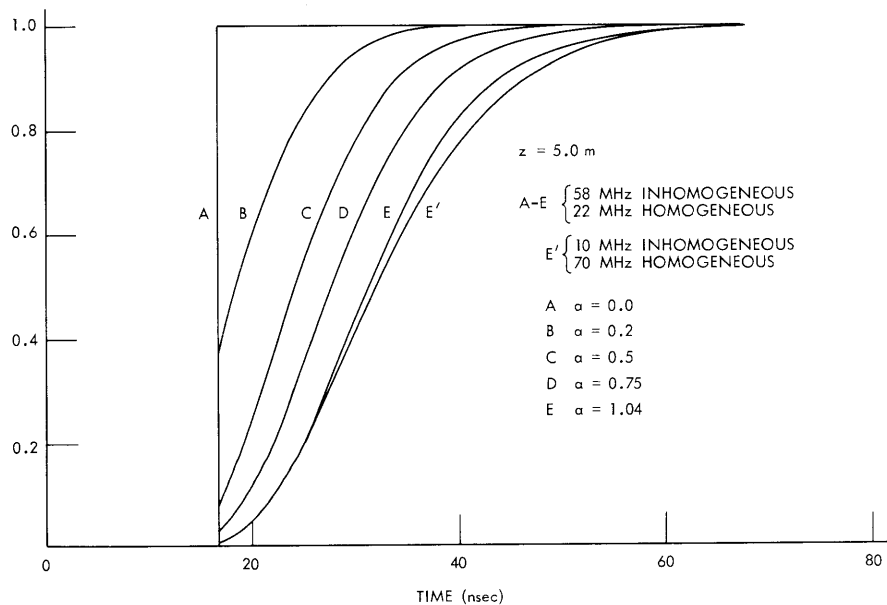


Fig. XVII-1. Normalized step response for inhomogeneously broadened medium.

The bandwidths of the homogeneous and inhomogeneous broadening are defined as the widths of the Lorentzian and Gaussian curves, respectively, at the points at which they drop to half of their value at the center. The curve labelled E' is evaluated for the same sum of bandwidths of 80 MHz, the same steady-state gain as that of E, but with different apportionment of the inhomogeneous vs homogeneous broadening. One can see that the difference in the output is relatively small.

H. A. Haus, C. M. Watson

References

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2. T. J. Bridges, H. A. Haus, and P. W. Hoff, "Small-Signal Step Response of Laser Amplifiers and Measurement of CO₂ Laser Linewidth," IEEE J. Quantum Electronics, Vol. QE-4, November 1968.

B. THEORY OF EXCITATION OF ELECTROMAGNETIC MODES IN HOLLOW CIRCULAR DIELECTRIC WAVEGUIDES

This report is concerned with the mathematical analysis of the electromagnetic waves excited in a hollow dielectric waveguide of circular cross section by a sinusoidally time-variant electric or magnetic point dipole having arbitrary orientation and location inside the waveguide. The waveguide shown in Fig. XVII-2 consists of a slightly lossy dielectric surrounding a vacuum hole; the hollow dielectric waveguide has been under consideration for the guided transmission of optical energy.

The problem of determining the allowed modes for the hollow dielectric waveguide has been formulated exactly and solved numerically for both the circular and the plane parallel cases.^{1,2} The excitation of the hollow dielectric waveguide by any source of electric or magnetic current has also been formulated and solved asymptotically for the plane parallel case.^{1,3} The excitation of the solid circular cylindrical dielectric waveguide by a longitudinal point electric dipole located on the axis of the cylinder has been formulated and solved.⁴

The exact analysis of the fields generated by the electric or the magnetic point dipole of arbitrary orientation and location inside a hollow circular cylindrical dielectric waveguide is accomplished in two steps. In the first step it is shown that the current distribution of an electric point dipole or a magnetic point dipole is equivalent to a Fourier superposition of both electric and magnetic surface currents on a circular cylindrical tube concentric with the waveguide axis. In the second step the exact fields generated by each of the Fourier components of the equivalent electric and magnetic surface currents on the tube are determined. The fields generated by the electric or magnetic point dipoles are then equal to the Fourier superposition of the fields generated by the Fourier components of the equivalent electric and magnetic surface currents on the tube.

1. Equivalence of Dipoles and current Tubes

We shall show that an electric point dipole and a magnetic point dipole, both of which have arbitrary orientation, are equivalent to a Fourier superposition of electric and

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magnetic surface currents on a circular cylindrical tube.

The electric or magnetic point dipole is assumed to be located at the point x_0, y_0, z_0 in a Cartesian coordinate system and to have an orientation such that its current moment

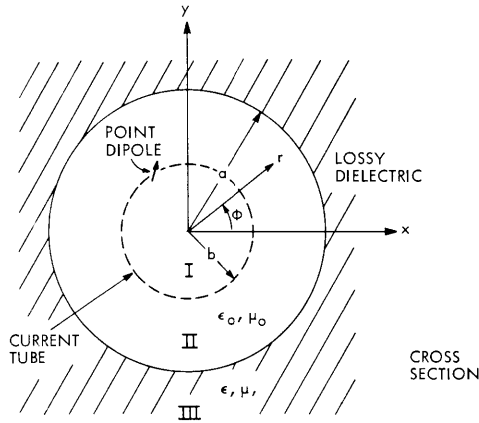
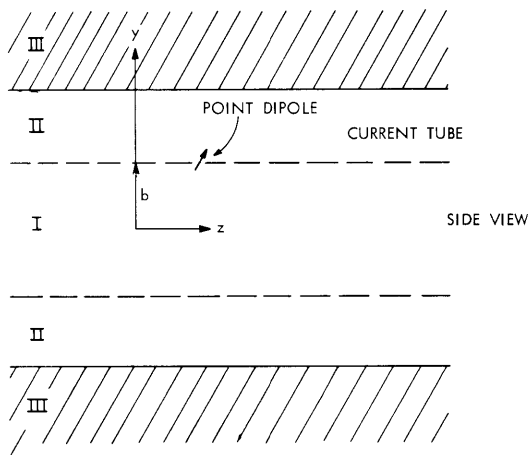


Fig. XVII-2. Cylindrical coordinate system.



has r , ϕ , and z components in the cylindrical coordinate system shown in Fig. XVII-2. The electric and magnetic current moments are given in MKS units as

$$I\bar{d} = I(\bar{i}_r d_r + \bar{i}_\phi d_\phi + \bar{i}_z d_z) \quad \text{A/m} \quad (1)$$

$$I^* \bar{d}^{**} = I^* (\bar{i}_r d_r^* + \bar{i}_\phi d_\phi^* + \bar{i}_z d_z^*) \quad \text{V/m}, \quad (2)$$

respectively, and the current densities are given by

$$\bar{J} = I\bar{d} \delta(x-x_0) \delta(y-y_0) \delta(z-z_0) \quad \text{A/m}^2 \quad (3)$$

$$\bar{\mathbf{J}}^* = I^* d^* \delta(x-x_0) \delta(y-y_0) \delta(z-z_0) \quad \text{V/m}^2, \quad (4)$$

where δ denotes a Dirac delta function.

The ϕ and z components of the current densities can be expressed as a Fourier superposition of ϕ and z directed surface currents on a circular cylindrical tube located at $r = b$, where

$$b = \sqrt{x_0^2 + y_0^2}. \quad (5)$$

The current density on the tube can be obtained by expressing the delta functions in cylindrical coordinates

$$\delta(x-x_0) \delta(y-y_0) \delta(z-z_0) = \frac{1}{b} \delta(r-b) \delta(\phi-\phi_0) \delta(z-z_0), \quad (6)$$

where ϕ_0 is given by

$$\phi_0 = \tan^{-1} \frac{y_0}{x_0}. \quad (7)$$

The ϕ and z directed surface current densities on the tube are then given by

$$\bar{\mathbf{K}} = \frac{I}{b} (\bar{i}_\phi d_\phi + \bar{i}_z d_z) \delta(\phi-\phi_0) \delta(z-z_0) \quad \text{A/m} \quad (8)$$

$$\mathbf{K}^* = \frac{I^*}{b} (\bar{i}_\phi d_\phi^* + \bar{i}_z d_z^*) \delta(\phi-\phi_0) \delta(z-z_0) \quad \text{A/m} \quad (9)$$

Equations 8 and 9 can be expressed in terms of a Fourier integral to obtain the required z dependence and a Fourier sum in order to obtain the required ϕ dependence.

$$\mathbf{K} = \int_{-\infty}^{\infty} \frac{dh}{2\pi} \sum_{n=-\infty}^{\infty} \bar{\mathbf{K}}_n(h) e^{jn\phi - jhz} \quad (10)$$

$$\bar{\mathbf{K}}^* = \int_{-\infty}^{\infty} \frac{dh}{2\pi} \sum_{n=-\infty}^{\infty} \bar{\mathbf{K}}_n^*(h) e^{jn\phi - jhz}, \quad (11)$$

where n is an integer, h is a real number, and $\bar{\mathbf{K}}_n$ and $\bar{\mathbf{K}}_n^*$ are vectors given by

$$\bar{\mathbf{K}}_n = \bar{i}_\phi K_{n\phi} + \bar{i}_z K_{nz} \quad (12)$$

$$\bar{\mathbf{K}}_n^* = \bar{i}_\phi K_{n\phi}^* + \bar{i}_z K_{nz}^*. \quad (13)$$

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The Fourier coefficients are given by

$$\bar{K}_n = \frac{I}{b} (\bar{i}_r d_r + \bar{i}_\phi d_\phi) e^{-jn\phi_0 + jhz_0} \quad (14)$$

$$\bar{K}_n^* = \frac{I^*}{b} (\bar{i}_r d_r^* + \bar{i}_\phi d_\phi^*) e^{-jn\phi_0 + jhz_0}. \quad (15)$$

\bar{K}_n and \bar{K}_n^* can be expressed in terms of \bar{K} and \bar{K}^* by the inverse transform relations.

$$\bar{K}_n = \int_{-\infty}^{\infty} \int_0^{2\pi} \bar{K} e^{-jn\phi + jhz} \frac{d\phi dz}{2\pi} \quad (16)$$

$$\bar{K}_n^* = \int_{-\infty}^{\infty} \int_0^{2\pi} \bar{K}^* e^{-jn\phi + jhz} \frac{d\phi dz}{2\pi}. \quad (17)$$

Equations 10 and 11 and 16 and 17 can be considered as an expansion of any ϕ - and z -directed currents on a tube of radius b into traveling waves in the ϕ and z directions.

The r -directed electric-point dipole current density can be expressed in terms of ϕ - and z -directed magnetic surface currents and the r -directed magnetic-point dipole current density in terms of ϕ - and z -directed electric surface currents by using the principle of equivalence. The principle of equivalence states that an electric-point dipole generates the same fields as an infinitesimal magnetic current loop, and that a magnetic dipole generates the same fields as an infinitesimal current loop. The equivalence principle can be expressed by the differential relations

$$\begin{array}{l} \text{electric} \\ \text{dipole} \end{array} \quad \bar{J}_{\text{eq.}}^* = - \frac{\nabla \times \bar{J}^*}{j\omega\epsilon_0} \quad (18)$$

$$\begin{array}{l} \text{magnetic} \\ \text{dipole} \end{array} \quad \bar{J}_{\text{eq.}} = \frac{\nabla \times \bar{J}}{j\omega\mu_0}, \quad (19)$$

where the subscript eq. stands for equivalent, ω is the angular frequency of oscillation, ϵ_0, μ_0 are the permittivity and permeability of free space. The equivalent currents are obtained by applying Eqs. 18 and 19 to the r -directed currents.

$$\bar{J}_{\text{eq.}}^* = - \frac{1}{j\omega\epsilon_0} \frac{I d_r}{b} \left(\bar{i}_\phi \delta(\phi - \phi_0) \delta'(z - z_0) - i_z \frac{\delta'(\phi - \phi_0) \delta(z - z_0)}{b} \right) \delta(r - b) \quad (20)$$

$$\bar{J}_{\text{eq.}} = + \frac{1}{j\omega\mu_0} \frac{I^* d_r^*}{b} \left(\bar{i}_\phi \delta(\phi - \phi_0) \delta'(z - z_0) - i_z \frac{\delta'(\phi - \phi_0) \delta(z - z_0)}{b} \right) \delta(r - b), \quad (21)$$

where the prime denotes differentiation. The Fourier coefficients of the equivalent surface currents can then be obtained from Eqs. 16 and 17.

$$\bar{K}_{n_{\text{eq}}}^* = \frac{1}{\omega \epsilon_0} \frac{I d_r}{b} \left(\bar{i}_\phi h + \bar{i}_z \frac{n}{b} \right) e^{-jn\phi_0 + jhz_0} \quad (22)$$

$$\bar{K}_{n_{\text{eq}}} = -\frac{1}{\omega \mu_0} \frac{I^* d_r^*}{b} \left(\bar{i}_\phi h + \bar{i}_z \frac{n}{b} \right) e^{-jn\phi_0 + jhz_0} \quad (23)$$

3. Fields Produced by Fourier Components of Tubular Currents

We shall now obtain the longitudinal fields produced by ϕ - and z -directed currents on a tube of radius b of the form $e^{jn\phi - jhz}$. It will be shown that the allowed modes for the waveguide appear as poles in the coefficients for the field components for complex h .

The current distribution on the tube will be taken as a single Fourier component, n, h . The fields generated by this current will consist of a sum of E and H cylindrical waves in the three regions shown in Fig. XVII-2. The E_z electric field component defines the E wave uniquely and the H_z magnetic field defines the H wave uniquely. The form of the complex amplitudes E_z in the three regions of space is the same.

$$\text{I. } E_z = (k^2 - h^2) a_{1n} J_n(\lambda r) e^{jn\phi - jhz} \quad \text{V/m} \quad (24)$$

$$H_z = (b_{1n}/a_{1n}) E_z \quad \text{A/m}$$

$$\text{II. } E_z = (k^2 - h^2)(a_{2n} J_n(\lambda r) + c_{2n} N_n(\lambda r)) e^{jn\phi - jhz} \quad (25)$$

$$H_z = (k^2 - h^2)(b_{2n} J_n(\lambda r) + d_{2n} N_n(\lambda r)) e^{jn\phi - jhz}$$

$$\text{III. } E_z = (k_3^2 - h^2) a_{3n} H_n^{(2)}(\lambda_3 r) e^{jn\phi - jhz} \quad (26)$$

$$H_z = (b_{3n}/a_{3n}) E_z,$$

where J_n denotes a Bessel function of the first kind, N_n denotes a Bessel function of the second kind, and $H_n^{(2)}$ represents a Hankel function of the second kind.⁵ The constants $a_{1n}, a_{2n}, c_{2n}, a_{3n}$ refer to the excited E waves, and the constants $b_{1n}, b_{2n}, d_{2n}, b_{3n}$ refer to the excited H waves. k and k_3 denote the propagation constants in the vacuum hole and lossy dielectric, respectively,

$$k^2 = \omega^2 \mu_0 \epsilon_0 \quad (\text{m})^{-2} \quad (27)$$

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$$k_3^2 = \omega^2 \mu_o \epsilon - j\omega\mu_o \sigma, \quad (28)$$

where ϵ and σ are the permittivity and conductivity of the lossy dielectric, λ and λ_3 denote the transverse propagation constants in the vacuum and lossy dielectric, respectively, for uniform plane waves having a longitudinal propagation constant h .

$$\lambda = \sqrt{k^2 - h^2} \quad (m) \quad (29)$$

$$\lambda_3 = \sqrt{k_3^2 - h^2} \quad (30)$$

All field components can be obtained from the E_z and H_z fields by differentiation.

The excitation coefficients of the E and H waves can be obtained by matching the boundary conditions on the tangential fields at $r = a$ and $r = b$. It is clear that the coefficients c_{2n} and d_{2n} which determine the N_n type E and H waves, respectively, in Region II are not affected by the presence of the lossy dielectric waveguide. They are given by

$$c_{2n} = \frac{1}{j\omega\epsilon_o} \left(\frac{\pi b}{2}\right) J_n(W) \left(K_{nz} + K_{n\phi} \frac{nhb}{W^2}\right) + \left(\frac{\pi b^2}{2}\right) \frac{J'_n(W)}{W} K_{n\phi}^* \quad V \cdot m \quad (31)$$

$$d_{2n} = -\frac{1}{j\omega\mu_o} \left(\frac{\pi b}{2}\right) J_n(W) \left(K_{nz}^* + K_{n\phi}^* \frac{nhb}{W^2}\right) + \left(\frac{\pi b^2}{2}\right) \frac{J'_n(W)}{W} K_{n\phi} \quad A \cdot m, \quad (32)$$

where the prime denotes differentiation, the dimensionless propagation constant W is defined as

$$W = \sqrt{k^2 - h^2} b,$$

and $K_{n\phi}$, $K_{n\phi}^*$, K_{nz} , K_{nz}^* denote the Fourier coefficients for the ϕ - and z -directed currents on the tube. The coefficients a_{2n} and b_{2n} are given by

$$a_{2n} = -\frac{\Delta(J_n, N_n)}{\Delta(J_n, J_n)} c_{2n} + \frac{j2\omega\mu_o nh}{\pi} \left(\frac{1}{u^2} - \frac{1}{v^2}\right) \frac{d_{2n}}{\Delta(J_n, J_n)} \quad (33)$$

$$b_{2n} = -\frac{j2\omega\epsilon_o nh}{\pi} \left(\frac{1}{u^2} - \frac{1}{v^2}\right) \frac{c_{2n}}{\Delta(J_n, J_n)} - \frac{\Delta(N_n, J_n)}{\Delta(J_n, J_n)} d_{2n}, \quad (34)$$

where the normalized propagation constants u and v are defined as

$$u = \sqrt{k^2 - h^2} a \quad (35)$$

$$v = \sqrt{k_3^2 - h^2} a. \quad (36)$$

$\Delta(A_n, B_n)$ is a complex function of two cylinder functions $A_n(u)$ and $B_n(u)$.

$$\Delta(A_n, B_n) = \frac{u^2}{a^2} \left[\left(A_n'(u) - \frac{H_n^{(2)'(v)}}{vH_n^{(2)}(v)} u A_n(u) \right) \left(k^2 B_n'(u) - k_3^2 \frac{H_n^{(2)'(v)}}{vH_n^{(2)}(v)} u B_n(u) \right) - \frac{n^2 h^2}{u^2} \left(\frac{1}{u^2} - \frac{1}{v^2} \right)^2 A_n(u) B_n(u) \right]. \quad (37)$$

$\Delta(J_n, N_n)$ determines the coupling between the N_n type E wave and the J_n type E wave, and $\Delta(N_n, J_n)$ determines the coupling between the N_n type H wave and the J_n type H wave in Region II. $\Delta(J_n, J_n)$ determines the coupling between the J_n type E and H waves, and reduces to the transcendental equation for the free modes of the hollow waveguide when

$$\Delta(J_n, J_n) = 0 \quad (38)$$

(see Stratton⁵).

The remaining coefficients are given by

$$a_{1n} = a_{2n} + \frac{N_n(W)}{J_n(W)} c_{2n} - \frac{K_{n\phi}^* b^2}{W^2 J_n(W)} \quad (39)$$

$$b_{1n} = b_{2n} + \frac{N_n(W)}{J_n(W)} d_{2n} + \frac{K_{n\phi} b^2}{W^2 J_n(W)} \quad (40)$$

$$a_{3n} = \frac{u^2}{v^2} \left[\frac{J_n(u)}{H_n^{(2)}(v)} a_{2n} + \frac{N_n(u)}{H_n^{(2)}(v)} c_{2n} \right] \quad (41)$$

$$b_{3n} = \frac{u^2}{v^2} \left[\frac{J_n(u)}{H_n^{(2)}(v)} b_{2n} + \frac{N_n(u)}{H_n^{(2)}(v)} d_{2n} \right]. \quad (42)$$

The mathematical results may be expressed physically; in the absence of the lossy dielectric the tubular currents generate E and H waves of the J_n type in Region I and

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of the $H_n^{(2)}$ type in Regions II and III. For this case,

$$a_{2n} = jc_{2n} = a_{3n} \quad (43)$$

$$b_{2n} = jd_{2n} = b_{3n} \quad (44)$$

The $H_n^{(2)}$ type waves are outgoing cylindrical waves produced by the tubular currents. In the presence of the lossy dielectric waveguide the E waves of the $H_n^{(2)}$ type are not only "reflected" and "transmitted" at $r = a$ but also produce "reflected" and "transmitted" H waves, because of ϕ -directed conduction and polarization currents induced in the lossy dielectric. "Reflected wave" refers to an $H_n^{(1)}$ type incoming cylindrical wave in Region II, and "transmitted wave" refers to an $H_n^{(2)}$ type outgoing cylindrical wave in Region III. The H waves of the $H_n^{(2)}$ type are also reflected and transmitted at $r = a$ and produce reflected and transmitted E waves.

4. Conclusion

It has been shown that the fields produced by an electric or magnetic dipole of arbitrary orientation and location inside the hollow circular dielectric waveguide can be determined exactly by a Fourier superposition of ϕ - and z -directed tubular surface currents. Future consideration will include a numerical computation for the interesting case of a small transverse dipole located on the waveguide axis. The dipole will be approximated by a current distribution on a tube of radius b of the form

$$\bar{K} = \bar{i}_\phi K_1 \cos \phi \delta(z-z_0) \quad (46)$$

in the limit as b goes to zero and K_1 goes to infinity so that bK_1 is constant.

The author wishes to acknowledge the assistance of Professor L. J. Chu who suggested the idea of the tubular currents.

J. I. Glaser

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C. MAXWELL'S EQUATIONS FOR DISCRETE SPACE

1. Introduction

This report is the first of a series that is concerned with numerical analysis techniques for waveguide scattering problems. A waveguide scattering problem is defined as the determination of the scattered fields produced when a specified waveguide mode impinges on a highly conducting body, junction, or other discontinuity located inside the waveguide. Numerical analysis techniques have been applied to electromagnetic scattering problems in free space.¹ The technique involves formulating the problem mathematically with the aid of Maxwell's equations and the boundary conditions at the conducting surface to obtain an exact integral equation relating the incident electric field distribution tangent to the surface to the unknown induced surface current distribution, and another exact integral equation relating the scattered electric or magnetic field distribution to the induced surface current distribution. The first integral equation is then approximated by a set of N algebraic equations that relate the induced surface currents at N distinct points on the surface to the incident electric field tangent to the surface at N distinct points. Solution of these N equations yields the induced surface currents at N points. The second integral equation is then approximated as an algebraic equation relating the scattered field at any point to the previously computed induced surface currents at the N points.

The numerical analysis technique described for the free-space scattering problem can be applied directly to the waveguide scattering problem. The integral equation, however, will involve a superposition of all of the waveguide modes, which are infinite in number. This infinite sum can be approximated by a finite sum that can then be evaluated numerically for waveguides whose modes are known analytically, e. g., coaxial, circular, elliptical, rectangular or parallel-plate waveguides. On the other hand, if the waveguide modes are not available analytically, they must be computed numerically by using available techniques.² These techniques involve the point-by-point solution of the Helmholtz wave equation in two dimensions with Dirichlet or Neumann boundary conditions on a curved boundary.

It is clear that the problem of determining the scattered fields produced when a specified waveguide mode in a waveguide of arbitrary shape impinges on a highly conducting body, junction or other obstacle located inside the waveguide will involve a large amount of numerical analysis and approximation. Implicit in the analysis will be the evaluation of the fields at a large number of discrete points of space and of the currents at a large number of discrete points on the surfaces of the conductors. As the scattering problems become more complex, the numerical analysis techniques, for all practical purposes, amount to the solution of Maxwell's equations on a point-by-point basis!

For these previous reasons, it is of interest to determine a simple set of difference

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equations relating the electric and magnetic field components at discrete points in space which approaches Maxwell's equations in the limit that the number of points per unit volume goes to infinity. The discrete points and the difference equations will be a "medium" and a "mode" of propagation, respectively, which are amenable to digital computer analysis. Furthermore, by studying the simple solutions for the discrete case, the fundamental limitations imposed on field solutions obtained by computer will be derived.

In this report Maxwell's equations will be approximated as a set of difference equations relating the electric and magnetic fields evaluated at discrete points in space. These difference equations apply to specially constructed cubic grids, and approach the exact Maxwell's equations as the grid spacing goes to zero. It will be shown that the grid comprises a medium of wave propagation and the simplest allowed waves will be

obtained and discussed. The treatment for discrete space closely follows the treatment for continuous space given by Adler, Chu, and Fano.³ The "TE" and "TM" waves derived for discrete space reduce to the TE and TM waves presented by these authors in the limit that the grid spacing goes to zero.

2. Discrete Space

We shall assume that space is filled with a simple cubic grid or lattice. A unit cell is a cube of volume a^3 . The lattice points are assumed to lie on a rectangular coordinate system x, y, z as shown in Fig. XVII-3. The coordinates of any lattice point can be written (pa, qa, ra) , where $p, q,$ and r are positive or negative integers. The distance vector to all points in the lattice can be written

$$\vec{D} = \bar{i}_x pa + \bar{i}_y qa + \bar{i}_z ra. \quad (1)$$

The lattice will now be used as a basis to construct two sublattices, an E lattice and an H lattice, which are displaced replicas of each other. The E lattice will be taken as the midpoints of all sides of the cubic lattice, and the H lattice will be taken as the center of all faces of the cubic lattice. Both the E and H lattices are

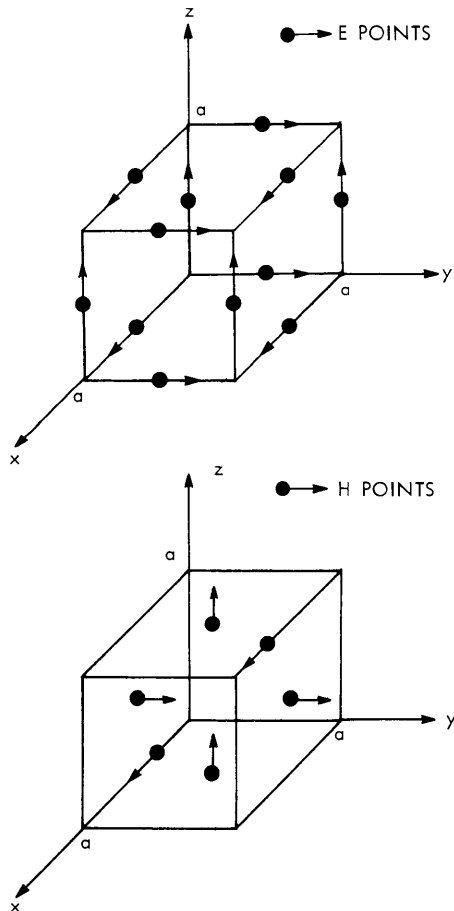


Fig. XVII-3.

E and H points for a unit cell at the origin.

face-centered cubic lattices, and the H lattice points can be considered to be midpoints of the sides of a cubic lattice identical to the original lattice but displaced by $a/2$ in each direction.

According to the previous definitions, the coordinates of the E and H lattices are given by

$$\text{E lattice} \quad \left(p + \frac{1}{2}, q, r\right) \left(p, q + \frac{1}{2}, r\right) \left(p, q, r + \frac{1}{2}\right) \quad (2)$$

$$\text{H lattice} \quad \left(p + \frac{1}{2}, q + \frac{1}{2}, r\right) \left(p + \frac{1}{2}, q, r + \frac{1}{2}\right) \left(p, q + \frac{1}{2}, r + \frac{1}{2}\right). \quad (3)$$

The H lattice then consists of the midpoints of the sides of a simple cubic lattice defined by the distance vector

$$\bar{D} = \bar{i}_x \left(p + \frac{1}{2}\right) a + \bar{i}_y \left(q + \frac{1}{2}\right) a + \bar{i}_z \left(r + \frac{1}{2}\right) a. \quad (4)$$

Both E and H lattices are shown in Fig. XVII-3.

3. Discretization of Maxwell's Equations

Maxwell's equations in integral form can be approximated by difference equations relating the electric field on the E lattice to the magnetic field on the H lattice. In complex notation Maxwell's equations (ME) can be written in MKS units for source-free space as

$$\oint_c \bar{E} \cdot d\bar{s} = -j\omega\mu_0 \oint_s \bar{H} \cdot \bar{n} da \quad (5)$$

$$\oint_c \bar{H} \cdot d\bar{s} = +j\omega\epsilon_0 \oint_s \bar{E} \cdot \bar{n} da \quad (6)$$

where c is any contour, and s is any surface bounded by the contour c . Since Eqs. 5 and 6 are of identical form, except for a similarity transformation, the approximation technique for one will apply in the same way to the other.

The contours applied to Eq. 5 will be taken as the edges of the square faces of the cubic grid defined by Eq. 1, and the surfaces of interest will be taken as the faces of the same cubic grid. The contours and surfaces to be applied to Eq. 6 are to be taken as the edges and faces of the cubic grid defined by Eq. 4.

The contribution to the line integral around a given square face from a single edge will be approximated by the value of the integrand times the length of the edge. The contribution to the surface integral over a given face will be approximated by the value of the integrand at the center of the face times the area of the face

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$$\int_{\text{edge}} \bar{\mathbf{E}} \cdot d\bar{\mathbf{s}} \cong \bar{\mathbf{E}}_c \cdot \bar{\mathbf{i}}_s a \quad (7)$$

$$\iint_{\text{square}} \bar{\mathbf{H}} \cdot \bar{\mathbf{n}} da \cong \bar{\mathbf{H}}_c \cdot \bar{\mathbf{n}} a^2. \quad (8)$$

It is evident that the H points taken for the surface integrals in Eq. 5 are identical to the H points taken for the line integral in Eq. 6, and vice versa for the E points. Therefore, at each point only one field component is required, and the field component can be identified by knowing the coordinates of the point of evaluation. These points are now listed for classification purposes.

$$\begin{aligned} E_x & \left(p + \frac{1}{2}, q, r \right) & H_x & \left(p, q + \frac{1}{2}, r + \frac{1}{2} \right) \\ E_y & \left(p, q + \frac{1}{2}, r \right) & H_y & \left(p + \frac{1}{2}, q, r + \frac{1}{2} \right) \\ E_z & \left(p, q, r + \frac{1}{2} \right) & H_z & \left(p + \frac{1}{2}, q + \frac{1}{2}, r \right). \end{aligned} \quad (9)$$

The discrete form of Maxwell's equations can now be obtained by using the approximations indicated by Eqs. 7 and 8. In order to simplify the notation, each field component will be labeled by an argument of the form (p, q, r) . The point in space where the field is being evaluated can be determined by adding the appropriate half-increments to the arguments.

When Eq. 5 is applied to the three faces of the unit cube centered at the point $\left(p + \frac{1}{2}, q + \frac{1}{2}, r + \frac{1}{2} \right)$, three equations are obtained:

$$\begin{aligned} a(E_z(q+1) - E_z(q) - E_y(r+1) + E_y(r)) &= -j\omega\mu_o a^2 H_x(p, q, r) \\ a(E_x(r+1) - E_x(r) - E_z(p+1) + E_z(p)) &= -j\omega\mu_o a^2 H_y(p, q, r) \\ a(E_y(p+1) - E_y(p) - E_x(q+1) + E_x(q)) &= -j\omega\mu_o a^2 H_z(p, q, r). \end{aligned} \quad (10)$$

When Eq. 6 is applied to the three faces of the unit cube centered at the point (p, q, r) , three equations are obtained:

$$\begin{aligned} a(H_z(q) - H_z(q-1) - H_y(r) + H_y(r-1)) &= +j\omega\epsilon_o a^2 E_x(p, q, r) \\ a(H_x(r) - H_x(r-1) - H_z(p) + H_z(p-1)) &= +j\omega\epsilon_o a^2 E_y(p, q, r) \\ a(H_y(p) - H_y(p-1) - H_x(q) + H_x(q-1)) &= +j\omega\epsilon_o a^2 E_z(p, q, r). \end{aligned} \quad (11)$$

All possible equations can be obtained by allowing the variables p , q , r to be all possible positive and negative integers including zero, for Eqs. 10 and 11. The "plane wave" solutions of Eqs. 10 and 11 will now be investigated.

4. Uniform and Nonuniform TE and TM Plane Waves in Discrete Space

TE and TM plane waves that satisfy Maxwell's equations for discrete space will be investigated. A plane wave is defined as a solution of the complex Maxwell's equations for the electric and magnetic field that varies as

$$\bar{E} = \bar{E}_0 e^{-pu-qv-rw} \quad (12)$$

$$\bar{H} = \bar{H}_0 e^{-pu-qv-rw}, \quad (13)$$

where p , q , r are positive and negative integers including zero, \bar{E}_0 , \bar{H}_0 are complex vectors having three complex components, and u , v , w are taken as the x , y , and z components of a propagation vector $\bar{\gamma}$, where

$$\bar{\gamma} = u\bar{i}_x + v\bar{i}_y + w\bar{i}_z, \quad (14)$$

with u , v , and w complex quantities. The convention for the labelling of E and H is the same as that discussed above.

A reduced set of Maxwell's equations for the complex amplitudes \bar{E}_0 and \bar{H}_0 is obtained by substituting Eqs. 12 and 13 in Eqs. 10 and 11. Under the assumption that the amplitudes are independent of p , q , r , just as in real space the plane wave amplitudes are independent of x , y , z , the following equations are obtained:

$$\bar{\Gamma}_1 \times \bar{E}_0 = +j\omega\mu_0 a\bar{H}_0 \quad (15)$$

$$\bar{\Gamma}_2 \times \bar{H}_0 = -j\omega\epsilon_0 a\bar{E}_0, \quad (16)$$

where $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are given by

$$\bar{\Gamma}_1 = (1-e^{-u})\bar{i}_x + (1-e^{-v})\bar{i}_y + (1-e^{-w})\bar{i}_z \quad (17)$$

$$\bar{\Gamma}_2 = (e^u-1)\bar{i}_x + (e^v-1)\bar{i}_y + (e^w-1)\bar{i}_z. \quad (18)$$

In the limit that u , v , w are small, $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ both approach the vector $\bar{\gamma}$.

In order for Eqs. 15 and 16 to be consistent, three conditions must be satisfied.

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$$\bar{\Gamma}_1 \cdot \bar{\Gamma}_2 = -\omega^2 \mu_o \epsilon_o a^2 = -(ka)^2 \quad (19)$$

$$\bar{\Gamma}_1 \cdot \bar{H}_o = 0 \quad (20)$$

$$\bar{\Gamma}_2 \cdot \bar{E}_o = 0. \quad (21)$$

Equation 19 leads to a dispersion equation for u, v, w given by

$$4(\sinh^2 (u/2) + \sinh^2 (v/2) + \sinh^2 (w/2)) = -(ka)^2. \quad (22)$$

For small u, v, w, Eq. 22 reduces to

$$\bar{\gamma} \cdot \bar{\gamma} = -(ka)^2 \quad (23)$$

which is identical to the dispersion equation for continuous space if the following normalizations are made:

$$\gamma'_x = (u/a) \quad \gamma'_y = (v/a) \quad \gamma'_z = (w/a) \quad (24)$$

$$\bar{\gamma}' = \gamma'_x \bar{i}_x + \gamma'_y \bar{i}_y + \gamma'_z \bar{i}_z = (\bar{\gamma}'/a). \quad (25)$$

Equations 20 and 21 indicate that there are two types of solutions to Eqs. 15 and 16 which are independent of each other. The first is obtained by picking \bar{E}_o to lie perpendicular to the plane in which $\bar{\Gamma}_2$ lies and by using Eq. 15 to solve for \bar{H}_o . The second is obtained by picking H_o to lie perpendicular to the plane in which $\bar{\Gamma}_1$ lies and by using Eq. 16 to solve for \bar{E}_o . These two solutions are the transverse electric (TE) waves and the transverse magnetic (TM) waves for discrete space. The pertinent relations for the two possibilities are given by

$$\text{TE} \quad \bar{E}_o \perp \bar{\Gamma}_2, \bar{H}_o = (\bar{\Gamma}_1 \times \bar{E}_o)/(j\omega\mu_o a) \quad (26)$$

$$\text{TM} \quad \bar{H}_o \perp \bar{\Gamma}_1, \bar{E}_o = -(\bar{\Gamma}_2 \times \bar{H}_o)/(j\omega\epsilon_o a). \quad (27)$$

It will now be assumed that the $\bar{\Gamma}$ vectors can only lie in three different planes xy, zy, and xz. If $\bar{\Gamma}_1$ lies in either the xy, yz or xz plane, then $\bar{\Gamma}_2$ and $\bar{\gamma}$ also lie in the same plane. Under these restrictions, it is clear that the plane of propagation of the TE and TM waves can be defined as the plane in which $\bar{\gamma}$, $\bar{\Gamma}_1$, and $\bar{\Gamma}_2$ lie, and that the magnetic field in the TE case and the electric field in the TM case both lie in the plane of propagation. These results are shown in Fig. XVII-4.

Once the plane of propagation has been defined as xy, yz, or xz, $\bar{\gamma}$ will be a function of two parameters that satisfy the dispersion equation. For example, if the plane of propagation is taken as the xz plane, u, and w determine $\bar{\gamma}$, $\bar{\Gamma}_1$, and $\bar{\Gamma}_2$. The

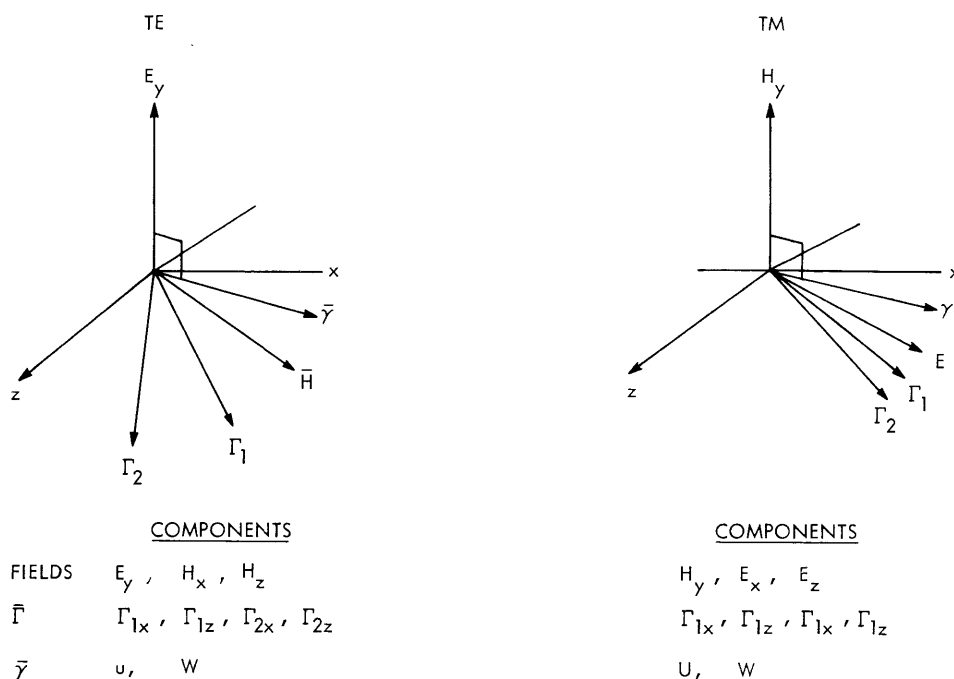


Fig. XVII-4. Transverse electric (TE) waves and transverse magnetic (TM) waves for discrete space.

electric field is taken in the direction for a TE wave, and the magnetic field in the z direction for a TM wave. It is clear from Eq. 22 that u and w cannot both be real numbers. For $v = 0$, Eq. 22 becomes

$$4(\sinh^2(u/2) + \sinh^2(w/2)) = -(ka)^2. \quad (29)$$

For example, if u is imaginary, then w can be either real or imaginary, depending on how large u is. If u is imaginary, then the wave will propagate from point to point in the x direction without attenuation but with a constant phase shift from cell to cell given by

$$\phi_x = \text{Im}(u) \text{ rad.} \quad (30)$$

If u is real, then the wave will propagate from point to point in the x direction with no phase shift but with a constant attenuation from cell to cell given by

$$a_x = \text{Re}(u) \text{ nepers.} \quad (31)$$

In order to simplify the notation, it will be assumed that u, v, w are either real or imaginary numbers but not complex numbers. This assumption will not affect the discussion but will restrict the generality of these results. The uniform and nonuniform

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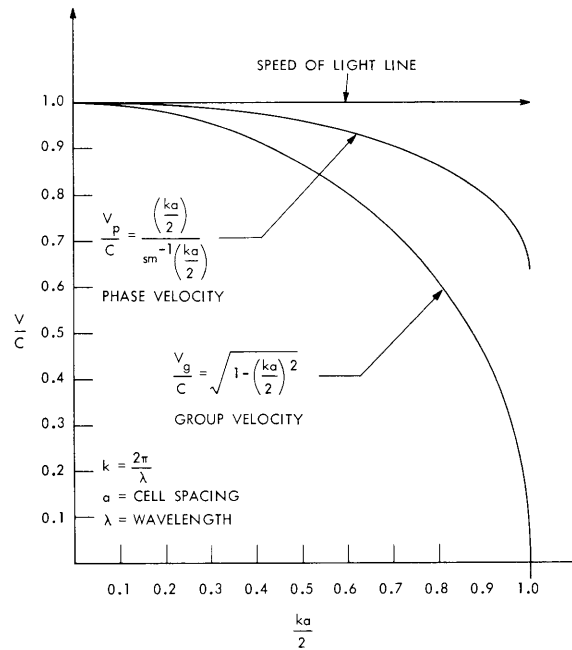


Fig. XVII-5. Dispersion introduced in phase and group velocity by discretization.

Definition: A uniform plane wave in discrete space is a TE or TM wave whose plane of propagation is the xy , yz , or xz plane and whose propagation constants uv , vw , uw , respectively, are imaginary numbers. If either propagation constant is real, the wave will be denoted as a nonuniform plane wave.

The properties of the uniform and nonuniform plane waves will be illustrated with the aid of an example. Consider a nonuniform TE plane wave propagating in the x - z plane and described by the dispersion equation (22). Assume that u is imaginary and that w is obtained by solving Eq. 22. The fields associated with this wave are given by

$$\bar{E}_0 = A e^{-j\beta p - wr} \bar{i}_y \quad (32)$$

$$H_0 = -(A/j\omega\mu_0 a) e^{-j\beta p - wr} (1 - e^{-w})\bar{i}_x - (1 - e^{-j\beta})\bar{i}_z, \quad (33)$$

where

$$u = j\beta \quad (34)$$

$$w = \pm 2 \sinh^{-1} \left(\sqrt{\frac{-(ka)^2}{4} + \sin^2(\beta/2)} \right). \quad (35)$$

It is clear from Eq. 35 that β affects the propagation of the wave in the z (r) direction in the following way

$$\text{propagation } -2 \sin^{-1} (ka/2) < \beta < 2 \sin^{-1} (ka/2) \quad (36)$$

$$\text{attenuation } \beta > 2 \sin^{-1} (ka/2) \quad \text{or} \quad \beta < -2 \sin^{-1} (ka/2). \quad (37)$$

The dispersion introduced by the discretization can be observed by letting $\beta = 0$; this case corresponds to a uniform plane wave propagating in the z direction. w is given by

$$w = j 2 \sin^{-1} (ka/2). \quad (38)$$

The apparent phase velocity is then given by

$$V_p = c(ka)/\text{Im}(w), \quad (39)$$

whereas the group velocity is given by

$$V_g = c/(d(\text{Im}(w))/d(ka)). \quad (40)$$

The curves are plotted in Fig. XVII-5. It is clear that for small ka the phase and group velocities approach the value of c very closely. Of course, the phase error increases linearly with the number of cells taken.

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D. RELATIVISTIC TREATMENT OF MULTIPOLAR MEDIA

In a previous report¹ and a paper² the theory of quadrupolar media was developed as an extension of Chu's theory of electrodynamics of dipolar media. The thermodynamic aspects of the theory were treated nonrelativistically. It was not clear at the time whether the kinematics, and the field equations, were consistent with relativity. We report here the relativistic formulation of "multipolar" media; that is, we give a treatment of a macroscopically neutral medium containing two oppositely charged clouds of charge. An expansion in terms of the displacement of the (+) charge with respect

to the (-) charge is carried out.

The charge density and current density are expressed up to second order in the displacement, thereby giving both dipolar and quadrupolar terms. This expression is formulated relativistically by introducing in the laboratory frame a four-vector of displacement having the three-vector displacement for its space part and zero in its time part. A four-vector of this kind is necessarily frame-bound. If transformed into another laboratory frame, moving with respect to the first one at a constant velocity, \bar{u} , its three-space part does not give the vector distance of the charge displacement as measured by an observer in the new frame and its four-part is not zero. Hence, in the new frame, one needs to redefine the expression for the current vector. The formulation is frame-bound.

Next, we develop the transformation of the length four-vectors defined in two different frames. Then, we use these transformation laws to compare the expressions for the four-current density obtained by two observers. When one current density is transformed by the relativistic transformation corresponding to the relative velocity of the two frames, one expects to get the same result. This is found to be the case.

Consider a negative-charge cloud described by the density $-\rho(\bar{x})$. We assume that another, positive-charge cloud, is displaced with respect to the negative one by the vector distance, $\bar{\xi}(\bar{x})$. We would like to compute the net charge density $\tilde{\rho}$ to second order in $\bar{\xi}$. This problem has been solved by Sturrock.³ Since his solution is a geometric one, carried out in one particular frame, it is also valid relativistically. The result is

$$\begin{aligned} \tilde{\rho}(\bar{x}) = & -\frac{\partial}{\partial x_r} \{ \rho(\bar{x}) \xi_r(\bar{x}) \} \\ & + \frac{1}{2} \frac{\partial^2}{\partial x_r \partial x_s} \{ \rho(\bar{x}) \xi_r(\bar{x}) \xi_s(\bar{x}) \}. \end{aligned} \quad (1)$$

The first term is the dipolar term, the second is the quadrupolar term. Suppose that the negative charge density $-\rho(\bar{x})$ moves with a space-dependent velocity v_r . The current density \bar{j}_r arising from the motion of the two clouds of charge is³

$$\begin{aligned} \tilde{j}_r(\bar{x}) = & \rho \left(\frac{\partial \xi_r}{\partial t} + v_s \frac{\partial \xi_r}{\partial x_s} \right) \\ & - \frac{\partial}{\partial x_u} \left\{ \rho \left(v_r + \frac{\partial \xi_r}{\partial t} + v_s \frac{\partial \xi_r}{\partial x_s} \right) \xi_u \right\} + \frac{1}{2} \frac{\partial^2}{\partial x_s \partial x_u} \{ \rho v_r \xi_s \xi_u \}. \end{aligned} \quad (2)$$

The current density and charge density may be grouped into a four-vector

$$\tilde{j}_a = [\tilde{j}_r, ic\tilde{\rho}] \quad (3)$$

which enters as a source into Maxwell's equation in four-notation.

$$\frac{\partial G_{\alpha\beta}}{\partial x_{\alpha}} = j_{\beta}. \quad (4)$$

Equations 1 and 2 can be cast into four-notation by introducing a distance four-vector defined in the laboratory frame in which the current and charge densities are viewed:

$$\xi_{\alpha} = [\xi_r, 0]. \quad (5)$$

The three-space part of the four-vector is the vector displacement between the two charge clouds, the four-part of the four-vector is equal to zero. With this notation one can write the four-vector current in the form

$$\tilde{j}_{\alpha} = -\frac{\partial}{\partial x_{\beta}} qn_{\alpha} (v_{\alpha} \xi_{\beta} - v_{\beta} \xi_{\alpha}) - \frac{\partial}{\partial x_{\beta}} \left[qn_{\alpha} v_{\gamma} \xi_{\beta} \frac{\partial \xi_{\alpha}}{\partial x_{\gamma}} \right] + \frac{1}{2} \frac{\partial^2}{\partial x_{\beta} \partial x_{\gamma}} \{ qn_{\alpha} v_{\alpha} \xi_{\beta} \xi_{\gamma} \}, \quad (6)$$

where we have used the four-vector velocity

$$v_{\alpha} = \gamma [v_r, ic]. \quad (7)$$

In expression (6) appears a four-vector ξ_{α} which has a simple physical meaning only in one inertial frame, the particular laboratory frame chosen for its definition. If one considers an observer in a different inertial frame, he would use a differently defined four-vector and hence end up with a four-current expression that is not the transform of (6), but which would contain a four-vector defined in its frame according to the definition (5). This in itself would not be particularly disturbing. The way in which the current and charge densities were evaluated, by its nature, is frame-bound. The detailed expressions may differ in different frames, provided a relativistic transformation of the current four-vector as a whole from one frame to the other leads to the same net four-vector.

In order to compare the current four vectors derived above in two different frames, we want to study the transformation law of the four-vector (5). The analysis below follows closely the derivation given elsewhere,⁴ with the modification that the length is allowed to vary with time. Consider a length element, with end points \bar{r} and \bar{p} .

The end point \bar{r} will be associated later with the negative charge, \bar{p} with the positive charge. Suppose that the length vector is observed in the unprimed frame at the time $t = 0$, and that the position of the end point \bar{r} at this time is \bar{k} . An observer in a frame (primed frame) moving with respect to the frame under consideration (unprimed frame) is assumed to set his clock with respect to the clock of the first frame at position \bar{r} . Events at positions \bar{r} and \bar{p} that are simultaneous in one frame are not simultaneous in another frame. We can evaluate the time shift by expressing the position \bar{p} as a function of time, taking into account velocity and acceleration.

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$$\bar{\mathbf{r}} = \bar{\mathbf{k}}$$

$$\rho = \bar{\kappa} + \bar{\nu}\tau + \frac{1}{2}\bar{\alpha}\tau^2 + \dots, \quad (8)$$

where $\bar{\mathbf{v}}$ is the velocity, and $\bar{\mathbf{a}}$ the acceleration. The expansion is broken off at a term of second order in τ . In this way, we shall obtain a law of transformation that is correct to second order in τ which, in measurements of length, is correct to second order in $|\bar{\mathbf{d}}| = |\bar{\rho} - \bar{\mathbf{r}}|$. Inclusion of derivatives of the acceleration would be necessary only when octopolar terms are to be retained.

The relativistic transformations for length and time can be used to express the distance $\bar{\mathbf{d}}' = \bar{\rho}' - \bar{\mathbf{k}}$ in the primed frame in terms of the unprimed quantities and τ , with $\tau' = t' = t = 0$. When this is done and that solution of the quadratic equation for τ is picked that reduces to the proper solution for $\bar{\mathbf{a}} \rightarrow 0$, and finally an expansion is made to first order in $\bar{\mathbf{a}}$ (and thus second order in $\bar{\alpha}$), one obtains (compare Penfield and Haus⁵)

$$\begin{aligned} \bar{\mathbf{d}}' = \frac{1}{1 - \frac{\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}}{c^2}} & \left[\bar{\mathbf{d}}_{\perp} + \frac{\bar{\mathbf{d}}_{\parallel}}{\gamma} - \frac{\bar{\mathbf{u}} \times (\bar{\mathbf{d}} \times \bar{\mathbf{v}})}{c^2} \right. \\ & \left. + \frac{1}{2} \left(\frac{\bar{\mathbf{u}} \cdot \bar{\mathbf{d}}}{c} \right)^2 \frac{1}{\left(1 - \frac{\bar{\mathbf{u}} \cdot \bar{\mathbf{v}}}{c^2} \right)^2} \left(\bar{\mathbf{a}}_{\perp} + \frac{\bar{\mathbf{a}}_{\parallel}}{\gamma} - \frac{\bar{\mathbf{u}} \times (\bar{\mathbf{a}} \times \bar{\mathbf{v}})}{c^2} \right) \right]. \end{aligned} \quad (9)$$

Consider now an observer in the unprimed frame who measures a vector distance $\bar{\mathbf{d}}$. He defines the four-vector (5). The observer in the primed frame moving at velocity $\bar{\mathbf{u}}$ with respect to the unprimed frame, defines a vector distance $\bar{\mathbf{d}}'$ and a four-vector ζ'

$$\zeta'_a = [\bar{\mathbf{d}}', 0]. \quad (10)$$

When ζ'_a is transformed by the conventional transformation law into the unprimed frame, we obtain the four-vector ζ_a which differs from the four-vector ξ_a . With some manipulation one obtains for $\xi_a - \zeta_a$ in four-notation

$$\begin{aligned} \delta_a \equiv \xi_a - \zeta_a = \frac{u_{\beta} \xi_{\beta}}{u_{\gamma} v_{\gamma}} v_a - \frac{1}{2} \left(\frac{u_{\beta} \xi_{\beta}}{u_{\gamma} v_{\gamma}} \right)^2 v_{\delta} \frac{\partial}{\partial x_{\delta}} v_a \\ + \frac{1}{2} \left(\frac{u_{\beta} \xi_{\beta}}{u_{\gamma} v_{\gamma}} \right)^2 (v_{\delta} - u_{\delta}) v_{\epsilon} \left(\frac{\partial}{\partial x_{\epsilon}} v_{\delta} \right) v_a. \end{aligned} \quad (11)$$

Here we have introduced the four-vector velocity

$$v_a = \frac{1}{\sqrt{1 - \frac{\gamma^2}{c^2}}} [\bar{v}, ic]. \quad (12)$$

This velocity differs from the velocity v_γ of the negative charge, to which it is assigned, on two counts: The three-velocities \bar{v} and \bar{v} differ by $d\bar{d}/dt$; and the γ -factors differ, $\gamma(\bar{v}) \neq \gamma(\bar{v})$. Hence, to second order in ξ , δ_γ can be written

$$\left(\frac{u_\beta \xi_\beta}{u_\gamma v_\gamma} + \text{second order scalar term} \right) v_a + \frac{u_\beta \xi_\beta}{u_\gamma v_\gamma} v_\delta \frac{\partial \xi_a}{\partial x_\delta} + \frac{1}{2} \left(\frac{u_\beta \xi_\beta}{u_\gamma v_\gamma} \right)^2 \left[v_\delta \frac{\partial}{\partial x_\delta} v_a \right]. \quad (13)$$

We can now compare the current densities obtained by the two observers.

The observer in the primed frame constructs a current four-vector analogous to (6)

$$\begin{aligned} \tilde{j}'_a &= - \frac{\partial}{\partial x'_\beta} qn_o (v'_a \xi'_\beta - v'_\beta \xi'_a) - \frac{\partial}{\partial x'_\beta} qn_o v'_\gamma \xi'_\beta \frac{\partial \xi'_a}{\partial x'_\gamma} \\ &+ \frac{1}{2} \frac{\partial^2}{\partial x'_\beta \partial x'_\gamma} \{ qn_o v'_a \xi'_\beta \xi'_\gamma \}. \end{aligned} \quad (14)$$

This current four-vector is transformed into the unprimed frame simply by replacing all primed four-vectors by their unprimed counterparts.

Now the current density $\tilde{j}'_a(\xi)$ is not necessarily equal to the current four-vector $\tilde{j}_a(\xi)$ although on physical grounds we should expect them to be equal. Let us take the difference between the two expressions for the current four-vector.

$$\begin{aligned} \tilde{j}'_a(\xi_\beta) - \tilde{j}_a(\xi_\beta) &= - \frac{\partial}{\partial x_\beta} qn_o (v_a \delta_\beta - v_\beta \delta_a) - \frac{\partial}{\partial x_\beta} qn_o v_\gamma \delta_\beta \frac{\partial \xi_a}{\partial x_\gamma} \\ &- \frac{\partial}{\partial x_\beta} qn_o v_\gamma \xi_\beta \frac{\partial \delta_a}{\partial x_\gamma} - \frac{\partial}{\partial x_\beta} qn_o v_\gamma \delta_\beta \frac{\partial \delta_a}{\partial x_\gamma} + \frac{\partial^2}{\partial x_\beta \partial x_\gamma} \{ qn_o v_a \xi_\beta \delta_\gamma \} \\ &+ \frac{1}{2} \frac{\partial^2}{\partial x_\beta \partial x_\gamma} \{ qn_o v_a \delta_\beta \delta_\gamma \}. \end{aligned} \quad (15)$$

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The first term in (15), the dipolar term, gives no contribution attributable to the term proportional to v_γ in (13). Since the remaining term in (13) is of second order in ξ , we see that, to first order, the two currents $j_a(\xi_\beta)$ and $j_a(\zeta_\beta)$ are identical, even though the "measurements" on the basis of which they were defined were performed in different frames. Next, consider terms of second order. For this purpose, one need consider in the dipolar term solely the contribution

$$\frac{u_\beta \xi_\beta}{u_\gamma v_\gamma} u_\delta \frac{\partial \xi_a}{\partial x_\delta} - \frac{1}{2} \left(\frac{u_\beta \xi_\beta}{u_\gamma v_\gamma} \right)^2 \left[v_\delta \frac{\partial}{\partial x_\delta} v_a \right],$$

whereas in the quadrupolar terms δ_γ can be approximated by $\delta_\gamma \approx \frac{u_a \xi_a}{u_\beta v_\beta} v_\gamma$ because everywhere δ_γ appears multiplied by a first-order term. When this is done, one finds after some manipulation,

$$\tilde{j}_a(\zeta_\beta) - \tilde{j}_a(\xi_\beta) = 0.$$

In carrying out the computation, one finds that quadrupolar terms defined as terms of second order in ζ on the one hand, ξ on the other hand, would not have cancelled by themselves. Cancellation is achieved in that the first-order terms in ζ of $\tilde{j}_a(\zeta_\beta)$ contribute second-order terms in ξ when expressed in terms of ξ . Physically, this means that effects identified as quadrupolar by one observer are identified as part of a dipolar contribution by another observer. The concept of a quadrupole is not frame-invariant. This can be checked by direct comparison of a quadrupole in two different frames when the quadrupole is time-variant. Because of nonsimultaneity in the two frames, a time-variant charge configuration with zero net dipole moment acquires a dipole moment in a frame moving with respect to the first frame.

The results obtained here agree with those from a (nonrelativistic) treatment in one frame.^{1, 2} The quadrupole tensor $\frac{1}{2} qn_o (\xi_a \xi_\beta + \xi_\beta \xi_a)$ is not general in the sense that its determinant vanishes. The analysis can be generalized to cover arbitrary quadrupolar density distributions by superposition of N pairs of charge distributions. One defines the quadrupole tensor

$$Q_{\alpha\beta} = \sum_{i=1}^N \frac{1}{2} qn_o^{(i)} \xi_a^{(i)} \xi_\beta^{(i)} \tag{16}$$

with components that are not now in any way restricted. In the same way, one may define the antisymmetric tensor

$$A_{\alpha\beta} = - \sum_{i=1}^N \frac{1}{2} qn_o^{(i)} v_\gamma \left(\frac{\partial \xi_a^{(i)}}{\partial x_\gamma} \xi_\beta - \xi_a \frac{\partial \xi_\beta^{(i)}}{\partial x_\gamma} \right). \tag{17}$$

Then,

$$\begin{aligned}
 \tilde{j}_a = & -\frac{\partial}{\partial x_\beta} (v_a P_\beta - v_\beta P_a) \frac{\partial^2}{\partial x_\beta \partial x_a} (v_a Q_{\beta\gamma}) \\
 & -\frac{\partial}{\partial x_\beta} \left[qn_o v_\gamma \frac{\partial}{\partial x_\gamma} \left(\frac{Q_{a\beta}}{n_o} \right) \right] \\
 & + \frac{\partial}{\partial x_\beta} A_{a\beta},
 \end{aligned} \tag{18}$$

where

$$P_\beta = \sum_{i=1}^N qn_o^{(i)} \xi_\beta^{(i)}.$$

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H. A. Haus, P. Penfield, Jr.

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