Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Problem Set 8
Fall 2004

Issued: Wednesday, October 27, 2004 **Due:** Monday, November 8, 2004

Reading: For entanglement and measures of entanglement:

- L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995), Sect. 12.14.
- D. Bouwmeester, A. Ekert, and A. Zeilinger, *The Physics of Quantum Information* (Springer Verlag, Berlin, 2000), Sects. 3.4 and 3.5.

For qubit teleportation:

- D. Bouwmeester, A. Ekert, and A. Zeilinger, *The Physics of Quantum Information* (Springer Verlag, Berlin, 2000), Sects. 3.3 and 3.7.

For quadrature teleportation:

- D. Bouwmeester, A. Ekert, and A. Zeilinger, *The Physics of Quantum Information* (Springer Verlag, Berlin, 2000), Sect. 3.9.

For optimum binary hypothesis testing:

- C.W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976) Sects. 4.2 and 6.1.

Problem 8.1

Here we shall begin a treatment of optimum binary hypothesis testing. Suppose that a quantum system is known to be in either state $|\psi_{-1}\rangle$ or $|\psi_1\rangle$, where $|\psi_{-1}\rangle \neq |\psi_1\rangle$. Let hypothesis H_{-1} denote "state $= |\psi_{-1}\rangle$" and hypothesis H_1 denote "state $= |\psi_1\rangle$." Assume that these two hypotheses are equally likely, i.e., before we make any measurement on the quantum system, it has probability 1/2 of being in state $|\psi_{-1}\rangle$ and probability 1/2 of being in state $|\psi_1\rangle$. Our task is to make a measurement on this system to determine—with the lowest probability of being wrong—whether the system's state was $|\psi_{-1}\rangle$ or $|\psi_1\rangle$ before we make our measurement. (The projection postulate implies that the system's state will likely be changed by our having made a measurement.)

Because we know the system can only be in $|\psi_{-1}\rangle$ or $|\psi_1\rangle$ we can—and we will—limit all our analysis in the reduced Hilbert space,

$$\mathcal{H} \equiv \mathrm{span}(|\psi_{-1}\rangle, |\psi_1\rangle),$$

i.e., to the Hilbert space of kets of the form

$$|\psi\rangle = \alpha|\psi_{-1}\rangle + \beta|\psi_1\rangle,$$

where α and β are complex numbers.

Define a decision operator,

$$\hat{D} \equiv |d_1\rangle\langle d_1| - |d_{-1}\rangle\langle d_{-1}|,$$

where $\{|d_{-1}\rangle, |d_1\rangle\}$ are a pair of *orthonormal* kets on the reduced Hilbert space $\mathcal{H}$. Clearly, $\hat{D}$ is an observable on $\mathcal{H}$. Suppose that we measure $\hat{D}$ on the quantum system under study. If the outcome of this measurement is -1, we will say that the state before the measurement was $|\psi_{-1}\rangle$. If the outcome of this measurement in 1, we will say that the state before the measurement was $|\psi_1\rangle$.

(a) Find the conditional probabilities,

$$\begin{aligned}
\Pr(\text{ say "state was } |\psi_{-1}\rangle\text{" } \mid \text{state was } |\psi_1\rangle\,) &= \Pr(\hat{D} = -1 \mid |\psi_1\rangle), \\
\Pr(\text{ say "state was } |\psi_{1}\rangle\text{" } \mid \text{state was } |\psi_{-1}\rangle\,) &= \Pr(\hat{D} = 1 \mid |\psi_{-1}\rangle).
\end{aligned}$$

and the unconditional error probability,

$$\begin{aligned}
\Pr(e) \equiv\; & \Pr(\text{state was } |\psi_{-1}\rangle) \Pr(\hat{D} = 1 \mid |\psi_{-1}\rangle) \\
+\; & \Pr(\text{state was } |\psi_1\rangle) \Pr(\hat{D} = -1 \mid |\psi_1\rangle).
\end{aligned}$$

(b) Suppose that $\langle\psi_{-1}|\psi_1\rangle = 0$, so that $\{|\psi_{-1}\rangle, |\psi_1\rangle\}$ is an orthonormal basis for $\mathcal{H}$. Find the measurement eigenkets $\{|d_{-1}\rangle, |d_1\rangle\}$ that minimize your error probability expression from (a). [The error probability of your optimum decision operator for this case shows why orthonormal kets are said to be "distinguishable."]

(c) Suppose that $|\psi_{-1}\rangle$ and $|\psi_1\rangle$ are normalized (unit length), but *not* orthogonal. In particular, let $\{|x\rangle, |y\rangle\}$ be an orthonormal basis for $\mathcal{H}$, and assume that,

$$|\psi_{-1}\rangle = \cos(\theta)|x\rangle - \sin(\theta)|y\rangle \quad \text{and} \quad |\psi_1\rangle = \cos(\theta)|x\rangle + \sin(\theta)|y\rangle,$$

where $0 < \theta < \pi/4$. Using the expansions,

$$|d_{-1}\rangle = \cos(\phi)|x\rangle - \sin(\phi)|y\rangle \quad \text{and} \quad |d_1\rangle = \sin(\phi)|x\rangle + \cos(\phi)|y\rangle,$$

where $0 \le \phi < 2\pi$, and your $\Pr(e)$ result from (a) find the ϕ value—hence the $\{|d_{-1}\rangle, |d_1\rangle\}$—that minimizes the error probability for this case.

[Hint: By assiduous use of trig identities, you should be able to reduce the error probability expression to the following form:

$$\Pr(e) = \frac{1}{2}[1 - \sin(2\phi)\sin(2\theta)],$$

which is easily minimized over ϕ.]

Problem 8.2

Here we shall continue our treatment of optimum binary hypothesis testing. Suppose that the quantum system considered in Problem 8.1 is a single-mode optical field with annihilation operator $\hat{a}$.

(a) Let $|\psi_{-1}\rangle = |n_{-1}\rangle$ and $|\psi_1\rangle = |n_1\rangle$ be photon number states with $n_{-1} \neq n_1$. Show that making the number operator measurement, $\hat{N} \equiv \hat{a}^\dagger \hat{a}$, on the single-mode field allows a zero-error-probability decision to be made as to whether the state before the measurement was $|n_{-1}\rangle$ or $|n_1\rangle$.

(b) Let $|\psi_{-1}\rangle = |\alpha_{-1}\rangle$ and $|\psi_1\rangle = |\alpha_1\rangle$ be coherent states with $\langle\alpha_{-1}|\alpha_1\rangle = \cos(2\theta)$ for a θ value satisfying $0 < \theta < \pi/4$. Find the error probability achieved by the minimum-error-probability decision operator for deciding whether the state before the measurement was $|\alpha_{-1}\rangle$ or $|\alpha_1\rangle$.

(c) Evaluate your error probability from (b) when on-off keying (OOK) is used: $|\alpha_{-1}\rangle = |0\rangle$ and $|\alpha_1\rangle = |\sqrt{N}\rangle$, i.e., when the two coherent states we are trying to distinguish are the vacuum state, and a coherent state with average photon number N. Compare this error probability with what is achieved when we make the $\hat{N}$ measurement and say "state was $|0\rangle$" when this measurement yields outcome 0 and say "state was $|1\rangle$" when this measurement yields a non-zero outcome.

[Hint: First find the conditional error probabilities,

$$\Pr(\text{ say "state was } |0\rangle\text{"} \mid \text{state was } |\sqrt{N}\rangle\ ),$$

and

$$\Pr(\text{ say "state was } |\sqrt{N}\rangle\text{"} \mid \text{state was } |\sqrt{0}\rangle\ ).$$

and then find the unconditional error probability using these intermediate results.]

(d) Evaluate your error probability from (b) when binary phase-shift keying (BPSK) is used: $|\alpha_{-1}\rangle = |-\sqrt{N}\rangle$ and $|\alpha_1\rangle = |\sqrt{N}\rangle$. Compare this error probability with what is achieved when we make the $\hat{a}_1 = \mathrm{Re}(\hat{a})$ measurement and say "state was $|-\sqrt{N}\rangle$" when this measurement yields a negative outcome and say "state was $|\sqrt{N}\rangle$" when this measurement yields a non-negative outcome. Express your answer in terms of,

$$Q(x) \equiv \int_x^\infty dt\, \frac{e^{-t^2/2}}{\sqrt{2\pi}},$$

i.e., the probability that a zero-mean, unity-variance Gaussian random variable exceeds x.

[Hint: First find the conditional error probabilities,

$$\Pr(\text{ say "state was } |-\sqrt{N}\rangle\text{"} \mid \text{state was } |\sqrt{N}\rangle\,),$$

and

$$\Pr(\text{ say "state was } |\sqrt{N}\rangle\text{"} \mid \text{state was } |-\sqrt{N}\rangle\,).$$

and then find the unconditional error probability using these intermediate results.]

Problem 8.3

Here we shall consider a different variant of the binary hypothesis testing problem. Suppose, as in Problem 8.1, that a quantum system is known to be in either state $|\psi_{-1}\rangle$ or $|\psi_1\rangle$, where $|\psi_{-1}\rangle \neq |\psi_1\rangle$. Let hypothesis H_{-1} denote "state $= |\psi_{-1}\rangle$" and hypothesis H_1 denote "state $= |\psi_1\rangle$." Assume that these two hypotheses are equally likely, i.e., before we make any measurement on the quantum system, it has probability 1/2 of being in state $|\psi_{-1}\rangle$ and probability 1/2 of being in state $|\psi_1\rangle$. Our task is still to make a measurement on this system to determine whether the system's state was $|\psi_{-1}\rangle$ or $|\psi_1\rangle$ before we make our measurement. Now, however, we do not want to make *any* mistakes, i.e., when we say "state was $|\psi_{-1}\rangle$" we must be correct, and when we say "state was $|\psi_1\rangle$" we must also be correct. This does *not* require that we limit ourselves to orthonormal states $|\psi_{-1}\rangle$ and $|\psi_1\rangle$, because we will also allow our measurement outcome to be "error," meaning it cannot reliably determine whether the state was $|\psi_{-1}\rangle$ or $|\psi_1\rangle$. In other words, we will require a measurement on the two-dimensional reduced Hilbert space $\mathcal{H}$ that has three possible outcomes: "state was $|\psi_{-1}\rangle$," "state was $|\psi_1\rangle$," and "error."

Assume that,

$$|\psi_{-1}\rangle = \cos(\theta)|x\rangle - \sin(\theta)|y\rangle \quad \text{and} \quad |\psi_1\rangle = \cos(\theta)|x\rangle + \sin(\theta)|y\rangle,$$

where $0 < \theta < \pi/4$, as in Problem 8.1(c), where $|x\rangle$ and $|y\rangle$ are an orthonormal basis for $\mathcal{H}$. Define a pair of kets,

$$|\xi_{-1}\rangle = -\sin(\theta)|x\rangle + \cos(\theta)|y\rangle \quad \text{and} \quad |\xi_1\rangle = -\sin(\theta)|x\rangle - \cos(\theta)|y\rangle$$

and a set of operators $\{\hat{\Pi}_{-1}, \hat{\Pi}_1, \hat{\Pi}_e\}$,

$$\begin{aligned}
\hat{\Pi}_{-1} &\equiv a|\xi_{-1}\rangle\langle\xi_{-1}|, \\
\hat{\Pi}_1 &\equiv a|\xi_1\rangle\langle\xi_1|, \\
\hat{\Pi}_e &\equiv b|x\rangle\langle x|,
\end{aligned}$$

where a and b are real-valued constants.

(a) Find a and b such that $\{\hat{\Pi}_{-1}, \hat{\Pi}_1, \hat{\Pi}_e\}$ is a probability operator valued measure (POVM) on the reduced Hilbert space $\mathcal{H}$, i.e., find the values of a and b for which

$$\hat{\Pi}_j^\dagger = \hat{\Pi}_j, \quad \text{for } j = -1, 1, e,$$

and

$$\hat{\Pi}_{-1} + \hat{\Pi}_1 + \hat{\Pi}_e = \hat{I}_2,$$

where $\hat{I}_2$ is the identity operator on $\mathcal{H}$.

(b) When we measure $\{\hat{\Pi}_{-1}, \hat{\Pi}_1, \hat{\Pi}_e\}$—with a and b as found in (a), so that these operators form a POVM and hence represent a measurement—and the state of the quantum system is $|\psi\rangle \in \mathcal{H}$, the outcome will be either -1, 1, or e, with the following probabilities:

$$\begin{aligned}
\Pr(\text{outcome} = -1) &= \langle\psi|\hat{\Pi}_{-1}|\psi\rangle, \\
\Pr(\text{outcome} = 1) &= \langle\psi|\hat{\Pi}_1|\psi\rangle, \\
\Pr(\text{outcome} = e) &= \langle\psi|\hat{\Pi}_e|\psi\rangle.
\end{aligned}$$

Suppose that we measure this POVM on our quantum system. If the measurement outcome is -1, we will say "state was $|\psi_{-1}\rangle$." If the measurement outcome is 1, we will say "state was $|\psi_1\rangle$." If the measurement outcome is e, we will say "error." Show that this decision procedure will never be incorrect when it says "state was $|\psi_{-1}\rangle$," or when it says "state was $|\psi_1\rangle$."

(c) For the POVM decision rule from (b), find the unconditional error probability, $\Pr(\text{outcome} = \text{"error"})$.

(d) Evaluate your error probability from (c) when $|\psi_{-1}\rangle = |-\sqrt{N}\rangle$ and $|\psi_1\rangle = |\sqrt{N}\rangle$, for $|\pm\sqrt{N}\rangle$ being coherent states.