Corrections to *"Geometric Properties of Gradient Projection Anti-windup Compensated Systems"*

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***Abstract*—In a conference paper titled "Geometric Properties of Gradient Projection Anti-windup Compensated Systems," two main results were presented. The first is the controller state-output consistency property of *gradient projection anti-windup* (GPAW) compensated controllers. The second is a geometric bounding condition relating the vector fields of the uncompensated and GPAW compensated closed-loop systems with respect to a *star domain*. While the controller state-output consistency property stands without modifications, the proof of the geometric bounding condition depends on two lemmas, the proofs of which were found to be faulty. In this report, we present a new proof of the geometric bounding condition using concepts from convex analysis, together with minor miscellaneous corrections.**

***Index Terms*—gradient projection anti-windup, geometric properties, corrections, combinatorial optimization, quadratic program, convex analysis, projection on polyhedral cone.**

I. Introduction

In the conference paper titled "Geometric Properties of Gradient Projection Anti-windup Compensated Systems" [1], two main results were presented. The first is the controller state-output consistency property of *gradient projection anti-windup* (GPAW) compensated controllers, presented as Theorem 1 in [1]. The second is a geometric bounding condition relating the vector fields of the uncompensated and GPAW compensated closed-loop systems with respect to a *star domain* [1, Definition 2], which is presented as Theorem 2 in [1]. While the controller state-output consistency property [1, Theorem 1] stands without modifications, the proof of the geometric bounding condition [1, Theorem 2] depends on two lemmas [1, Lemmas 1 and 2], the proofs of which were found to be faulty. In this report, we present a new proof of the geometric bounding condition [1, Theorem 2] using concepts from convex analysis. The new proof is based on geometry and does not need [1, Lemmas 1 and 2], which is algebraic in nature.

This report is meant to be read *together* with [1]. As such, we will not repeat extraneous material, and refer the reader to [1]. We highlight the faults in the existing proofs in the following section. For consistency, we present some minor corrections to [1] in Section III. This is followed by the new proof of [1, Theorem 2] in Section IV. Equation, theorem, lemma, and proposition numbers in the present report have been prefixed by "C" for clarity.

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II. Faults with Existing Proofs

In [1], Theorem 2 depends on Lemma 2, which in turn depends on Lemma 1. It was found that both the proofs of Lemmas 1 and 2 in [1] are faulty, which invalidates Theorem 2. Note that Lemma 1 being faulty alone is sufficient to invalidate Theorem 2.

In the last part of the proof of Lemma 1 in [1], it was stated that when the columns of $N_{\mathcal{I}_{sat}}$ are linearly independent, then no columns of $N_{\mathcal{I}_{sat}\setminus\mathcal{I}^*}$ ($\mathcal{I}^*$ being an optimal solution to the combinatorial optimization subproblem (7) in [1]) can be in the span of the columns of $N_{\mathcal{I}^*}$. It was then *claimed* that this implies all columns of $\Phi N_{\mathcal{I}_{sat}\setminus\mathcal{I}^*}$ must lie *entirely* in the span of the columns of M, where M is chosen such that its columns, together with the columns of $\Phi N_{\mathcal{I}^*}$ forms a basis, and satisfy $N_{\mathcal{I}^*}^{\mathrm{T}}\Phi^{\mathrm{T}}M = 0$. The last assertion is wrong, as seen by the following counterexample.

Example 1: Set $\Phi N_{\mathcal{I}_{sat}} = \left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$ and $\Phi N_{\mathcal{I}^*} = [1,0]^{\mathrm{T}}$. Then $M = [0,\alpha]^{\mathrm{T}}$ for some $\alpha \neq 0$. Clearly, $\Phi N_{\mathcal{I}_{sat}\setminus\mathcal{I}^*} = [1,1]^{\mathrm{T}}$ must have a non-zero component lying in the span of the columns of $\Phi N_{\mathcal{I}^*}$, ie. it is not contained *entirely* in the span of M, which is the faulty claim. □

The preceding counterexample is sufficient to invalidate Lemma 2 in [1]. However, it was found that even if Lemma 1 holds, the proof of Lemma 2 is also faulty. In the latter part of the proof of Lemma 2 in [1], it was claimed that the cardinality of solutions $\phi(\hat{\mathcal{I}}_1)$ will be *increased* by the described construction. This is not true in general and invalidates Lemma 2, even if Lemma 1 holds.

III. Minor Corrections

For consistency, the following changes to [1] are needed.

(i) Following the proof of Proposition 1, the material stating Lemmas 1 and 2 is to be removed. The paragraph preceding equation (9) should be changed to read: "At each fixed time (so that $(x_g, y, \tilde{r}(t))$ is fixed), let $\mathcal{I}^*$ be a solution to subproblem (7). The GPAW compensated controller derived from (2) is then given by (4) with ..."

(ii) The Appendix containing the (faulty) proofs of Lemmas 1 and 2 is to be removed.

IV. Correction to Proof of Geometric Bounding Condition

Here, we present a new proof of Theorem 2 in [1]. The new proof is based on the underlying geometry, in contrast with the existing attempt in [1], which is algebraic in nature. We first show that a vector defined by a solution to the *combinatorial* optimization subproblem (7) in [1] must be the unique solution to a convex optimization problem. Then geometric properties of the solution to the convex optimization problem is used to prove Theorem 2 in [1].

Observe that the GPAW parameter $\Gamma \in \mathbb{R}^{q\times q}$ is symmetric positive definite, so that it can always be decomposed as $\Gamma = \Phi^{\mathrm{T}}\Phi$ for some *nonsingular* matrix $\Phi \in \mathbb{R}^{q\times q}$ [2, Theorem 7.2.7, pp. 406]. For any $\mathcal{I}$ such that $|\mathcal{I}| = 0$, or such that $N_{\mathcal{I}}(x_g)$ is full rank, the matrix $R_{\mathcal{I}}(x_g)$ is well-defined and given by (see its definition following (6) in [1])

$$R_{\mathcal{I}}(x_g) = \begin{cases} I - \Gamma N_{\mathcal{I}}(N_{\mathcal{I}}^{\mathrm{T}}\Gamma N_{\mathcal{I}})^{-1}N_{\mathcal{I}}^{\mathrm{T}}(x_g), & \text{if } |\mathcal{I}| > 0, \\ I, & \text{otherwise.} \end{cases}$$

For any well-defined $R_{\mathcal{I}}(x_g)$, let the *principal projection matrix* $\tilde{R}_{\mathcal{I}}(x_g)$ and *complementary projection matrix* $S_{\mathcal{I}}(x_g)$ induced by $R_{\mathcal{I}}(x_g)$ be

$$\tilde{R}_{\mathcal{I}}(x_g) := \Phi^{-\mathrm{T}}R_{\mathcal{I}}(x_g)\Phi^{\mathrm{T}}, \quad S_{\mathcal{I}}(x_g) := I - \tilde{R}_{\mathcal{I}}(x_g), \tag{C1}$$

so that

$$R_{\mathcal{I}}(x_g) = \Phi^{\mathrm{T}}\tilde{R}_{\mathcal{I}}(x_g)\Phi^{-\mathrm{T}} = I - \Phi^{\mathrm{T}}S_{\mathcal{I}}(x_g)\Phi^{-\mathrm{T}}. \tag{C2}$$

It can be verified that $\tilde{R}_{\mathcal{I}}(x_g)$ and $S_{\mathcal{I}}(x_g)$ take the explicit forms

$$\tilde{R}_{\mathcal{I}}(x_g) = \begin{cases} I - \tilde{N}_{\mathcal{I}}(\tilde{N}_{\mathcal{I}}^{\mathrm{T}}\tilde{N}_{\mathcal{I}})^{-1}\tilde{N}_{\mathcal{I}}^{\mathrm{T}}(x_g), & \text{if } |\mathcal{I}| > 0, \\ I, & \text{otherwise,} \end{cases}$$
$$S_{\mathcal{I}}(x_g) = \begin{cases} \tilde{N}_{\mathcal{I}}(\tilde{N}_{\mathcal{I}}^{\mathrm{T}}\tilde{N}_{\mathcal{I}})^{-1}\tilde{N}_{\mathcal{I}}^{\mathrm{T}}(x_g), & \text{if } |\mathcal{I}| > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{C3}$$

where $\tilde{N}_{\mathcal{I}}(x_g) = \Phi N_{\mathcal{I}}(x_g)$. From [3, Lemma 1, Theorem 1], it can be seen that both are projection matrices. Using (C2) and (C3), the objective function of subproblem (7) in [1] can be written as

$$\begin{aligned} J(\mathcal{I}) &= f_c^{\mathrm{T}}\Gamma^{-1}R_{\mathcal{I}}f_c = f_c^{\mathrm{T}}\Phi^{-1}\Phi^{-\mathrm{T}}(I - \Phi^{\mathrm{T}}S_{\mathcal{I}}\Phi^{-\mathrm{T}})f_c, \\ &= \tilde{f}_c^{\mathrm{T}}\tilde{f}_c - \tilde{f}_c^{\mathrm{T}}S_{\mathcal{I}}\tilde{f}_c = \|\tilde{f}_c\|^2 - \tilde{f}_c^{\mathrm{T}}S_{\mathcal{I}}S_{\mathcal{I}}\tilde{f}_c, \\ &= \|\tilde{f}_c\|^2 - \tilde{f}_c^{\mathrm{T}}S_{\mathcal{I}}^{\mathrm{T}}S_{\mathcal{I}}\tilde{f}_c = \|\tilde{f}_c\|^2 - \|S_{\mathcal{I}}\tilde{f}_c\|^2, \end{aligned}$$

where $\tilde{f}_c := \Phi^{-\mathrm{T}}f_c = \Phi^{-\mathrm{T}}f_c(x_g, y, \tilde{r}(t))$. Observe that for any $\mathcal{I}$ such that $R_{\mathcal{I}}(x_g)$ is well-defined, we have $N_{\mathcal{I}}^{\mathrm{T}}(x_g)f_{\mathcal{I}}(x_g, y, \tilde{r}(t)) = 0$ (≤ 0), so that the solutions to subproblem (7) in [1] remains unchanged by addition of these *redundant* constraints. Defining $\tilde{N}_{\mathcal{I}}(x_g) := \Phi N_{\mathcal{I}}(x_g)$, it can be seen that subproblem (7) in [1] can be rewritten as a *minimization* problem with $|\mathcal{I}_{sat}|$ inequality constraints,

$$\min_{\mathcal{I}\in\mathcal{J}} \|(I - \tilde{R}_{\mathcal{I}})\tilde{f}_c\|^2, \qquad \text{subject to } \operatorname{rank}(\tilde{N}_{\mathcal{I}}) = |\mathcal{I}|, \quad \tilde{N}_{\mathcal{I}_{sat}}^{\mathrm{T}}\tilde{R}_{\mathcal{I}}\tilde{f}_c \leq 0, \tag{C4}$$

where all function arguments have been dropped, and $\mathcal{J}$ remains unaltered as the set of all subsets of $\mathcal{I}_{sat}$ with cardinality less than or equal to q.

Now, consider the quadratic programming problem

$$\min_{x\in\mathbb{R}^q} \|\tilde{f}_c - x\|^2, \qquad \text{subject to } \tilde{N}_{\mathcal{I}_{sat}}^{\mathrm{T}}x \leq 0, \tag{C5}$$

which is a convex optimization problem with a unique solution [4, pp. 214]. It can be seen that problem (C5) seeks the unique projection of $\tilde{f}_c$ onto the (polyhedral) polar cone $\mathcal{K}^\circ$ [4, pp. 513] induced by the (finitely generated) cone $\mathcal{K}$ generated by the columns of $\tilde{N}_{\mathcal{I}_{sat}}$ [4, pp. 143]. Together with the dual cone $\mathcal{K}^*$ [4, pp. 149] (dual to $\mathcal{K}$), these cones are defined by

$$\begin{aligned} \mathcal{K} &= \{x \in \mathbb{R}^q \mid x = \tilde{N}_{\mathcal{I}_{sat}}z, \forall z \geq 0 \in \mathbb{R}^{|\mathcal{I}_{sat}|}\}, \\ \mathcal{K}^\circ &= \{x \in \mathbb{R}^q \mid \tilde{N}_{\mathcal{I}_{sat}}^{\mathrm{T}}x \leq 0 \in \mathbb{R}^{|\mathcal{I}_{sat}|}\}, \\ \mathcal{K}^* &= \{x \in \mathbb{R}^q \mid \tilde{N}_{\mathcal{I}_{sat}}^{\mathrm{T}}x \geq 0 \in \mathbb{R}^{|\mathcal{I}_{sat}|}\}. \end{aligned} \tag{C6}$$

The projection of $\tilde{f}_c$ onto $\mathcal{K}^\circ$, together with the cone $\mathcal{K}$ and its *dual* $\mathcal{K}^*$, are illustrated in Fig. 1. Observe that the polar of $\mathcal{K}^\circ$, ie. $\mathcal{K}^{\circ\circ}$, satisfy $\mathcal{K}^{\circ\circ} = \mathcal{K}$ for every finitely generated

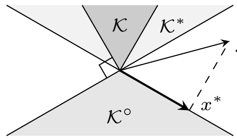

Fig. 1. The projection x^* of $\tilde{f}_c$ onto the polyhedral cone $\mathcal{K}^\circ$, together with $\mathcal{K}$ $(= \mathcal{K}^{\circ\circ})$ and its dual $\mathcal{K}^*$ $(= -\mathcal{K}^\circ)$.

cone $\mathcal{K}$ [5, Lemma 2.7.9, pp. 54]. Moreover, a theorem due to Minkowski states that every polyhedral cone (eg. $\mathcal{K}^\circ$ in (C6)) is finitely generated [5, Theorem 2.8.6, pp. 55], and a theorem due to Weyl states that every finitely generated cone (eg. $\mathcal{K}$ in (C6)) is a polyhedral cone [5, Theorem 2.8.8, pp. 56]. These allow results applicable to finitely generated cones to be applied to polyhedral cones, and vice versa.

We will need the following result from [6].

Proposition C1 ([6, Proposition 2]): Let x^* be the projection of a vector y of $\mathbb{R}^n$ into a convex polyhedral cone $\mathcal{K} = \mathcal{K}(S)$ that is generated by a set $S = \{s_1, \dots, s_k\}$. Let R be the set of vectors s_i of S orthogonal to $y - x$. Then the vector x^* is equal to the projection of y into the subspace $L(R)$ generated by the vectors of R.

Proposition C1 applied to the convex polyhedral cone $\mathcal{K}^\circ$ shows that the unique solution x^* to problem (C5) is equal to the projection of $\tilde{f}_c$ onto a subspace containing a face of $\mathcal{K}^\circ$ that x^* resides in (including possibly the *face* $\mathcal{K}^\circ$). This leads to the next result, which is crucial to the new proof of Theorem 2 in [1].

Lemma C1: The unique solution x^* to the convex optimization problem (C5) satisfies $x^* = \tilde{R}_{\mathcal{I}^*}\tilde{f}_c$ for any solution $\mathcal{I}^*$ to the combinatorial optimization problem (C4).

Proof: When $\tilde{f}_c \in \mathcal{K}^\circ$, the unique optimal solution x^* of problem (C5) must be $x^* = \tilde{f}_c$. In this case, $\emptyset$ is a feasible solution to problem (C4), so that the objective function satisfies $\|\tilde{f}_c - \tilde{R}_{\mathcal{I}^*}\tilde{f}_c\|^2 = \tilde{J}(\mathcal{I}^*) \leq \tilde{J}(\emptyset) = 0$ for any optimal solution $\mathcal{I}^*$. This implies $\tilde{R}_{\mathcal{I}^*}\tilde{f}_c = \tilde{f}_c$, and $x^* = \tilde{R}_{\mathcal{I}^*}\tilde{f}_c$.

When $\tilde{f}_c \notin \mathcal{K}^\circ$, Proposition C1 shows that x^* is equal to the projection of $\tilde{f}_c$ onto some (possibly non-unique) subspace L containing a face of $\mathcal{K}^\circ$ that x^* resides in. This subspace L must be orthogonal to the span of some collection of column vectors of $\tilde{N}_{\mathcal{I}_{sat}}$. The unique projection of $\tilde{f}_c$ onto the subspace orthogonal to the subspace spanned by the columns of $\tilde{N}_{\mathcal{I}}$ for some $\mathcal{I} \subset \mathcal{I}_{sat}$ (such that the columns of $\tilde{N}_{\mathcal{I}}$ are linearly independent) is given by $\tilde{R}_{\mathcal{I}}\tilde{f}_c$ [3, Theorem 1]. Since the definition of the set of candidate solutions $\mathcal{J}$ is exhaustive, it must include a set of indices $\mathcal{I}_L$ such that the columns of $\tilde{N}_{\mathcal{I}_L}$ are linearly independent, and L is orthogonal to the subspace spanned by the columns of $\tilde{N}_{\mathcal{I}_L}$. Hence $x^* = \tilde{R}_{\mathcal{I}_L}\tilde{f}_c$ by Proposition C1 and $\tilde{J}(\mathcal{I}_L) = \|\tilde{f}_c - \tilde{R}_{\mathcal{I}_L}\tilde{f}_c\|^2 = \|\tilde{f}_c - x^*\|^2$. The minimization in problem (C4) yields $\|\tilde{f}_c - \tilde{R}_{\mathcal{I}^*}\tilde{f}_c\|^2 = \tilde{J}(\mathcal{I}^*) \leq \tilde{J}(\mathcal{I}_L) = \|\tilde{f}_c - x^*\|^2$. Uniqueness of the solution x^* of problem (C5) [4, pp. 214] ensures that $\|\tilde{f}_c - x^*\|^2 < \|\tilde{f}_c - y\|^2$ for all $y \neq x^*$. This, together with $\|\tilde{f}_c - \tilde{R}_{\mathcal{I}^*}\tilde{f}_c\|^2 \leq \|\tilde{f}_c - x^*\|^2$, shows that $x^* = \tilde{R}_{\mathcal{I}^*}\tilde{f}_c$, as desired. ∎

Remark 1: Observe that $\emptyset \in \mathcal{J}$ by the definition of $\mathcal{J}$, and that the constraint $\mathrm{rank}(\tilde{N}_{\mathcal{I}}) = |\mathcal{I}|$ in (C4) is to ensure linear independence of the columns of $\tilde{N}_{\mathcal{I}}$, or that $\mathcal{I} = \emptyset$. Note also that the second part of the proof of Lemma C1 includes the case $\tilde{f}_c \in \mathcal{K}^\circ$, but is presented as such for clarity. □

Remark 2: Lemma C1 shows that even if problem (C4) has no unique solutions, the projection of $\tilde{f}_c$ defined by any solution $\mathcal{I}^*$, namely $\tilde{R}_{\mathcal{I}^*}\tilde{f}_c$, *is unique*. Since Φ is nonsingular, it implies uniqueness of $R_{\mathcal{I}^*}f_c = \Phi^{\mathrm{T}}\tilde{R}_{\mathcal{I}^*}\tilde{f}_c$ as well. □

Remark 3: An implication of Lemma C1 is that the GPAW compensated controller can be defined by (compare with (4) and (6) in [1])

$$\dot{x}_g = \Phi^{\mathrm{T}}x^*, \qquad x_g(0) = x_{c0}, \qquad u = g_c(x_g),$$

where x^* is obtained as a solution to the quadratic program (C5) at each fixed time. This realization will be useful if the quadratic program proves to be more computationally attractive than the posed combinatorial optimization problems. However, observe that most of the inherent structure of the GPAW compensated controller is then concealed by this representation, which renders it ill suited for further analysis. □

Remark 4: Much work has been devoted to developing algorithms for projection onto polyhedral cones (see [6], [7] and the references therein). This suggests that the quadratic program formulation may be computationally inefficient. □

For the next result, observe that any solution $\mathcal{I}^*$ of subproblem (7) in [1] satisfies $\mathrm{rank}(N_{\mathcal{I}^*}(x_g)) = |\mathcal{I}^*|$, so that either $\mathcal{I}^* = \emptyset$ or $N_{\mathcal{I}^*}(x_g)$ is full rank. This ensures that $R_{\mathcal{I}^*}(x_g, y, \tilde{r}(t))$, $\tilde{R}_{\mathcal{I}^*}(x_g, y, \tilde{r}(t))$, and $S_{\mathcal{I}^*}(x_g, y, \tilde{r}(t))$ are well-defined. Recall the definition of the unsaturated region in [1] (preceding [1, Remark 9])

$$K = \{x \in \mathbb{R}^q \mid h_i(x) \leq 0, \forall i \in \mathcal{I}_{2m}\} \subset \mathbb{R}^q,$$

where h_i are the saturation constraint functions in (5) of [1].

Lemma C2: Let $\mathcal{I}^*$ be a solution to subproblem (7) in [1]. If the unsaturated region $K \subset \mathbb{R}^q$ is a star domain, then

$$(x - x_{ker})^{\mathrm{T}}\Phi^{-1}S_{\mathcal{I}^*}(x, y, \tilde{r}(t))\Phi^{-\mathrm{T}}f_c(x, y, \tilde{r}(t)) \geq 0$$

holds for any boundary point $x \in \partial K$ and any $x_{ker} \in \ker(K)$.

Proof: For notational convenience, define

$$\begin{aligned} \tilde{N}_{\mathcal{I}} &:= \Phi N_{\mathcal{I}}(x), & \tilde{N}_{\mathcal{I}^*} &:= \Phi N_{\mathcal{I}^*}(x,y,\tilde{r}(t)), \\ R_{\mathcal{I}^*} &:= R_{\mathcal{I}^*}(x,y,\tilde{r}(t)), & \tilde{R}_{\mathcal{I}^*} &:= \tilde{R}_{\mathcal{I}^*}(x,y,\tilde{r}(t)), \\ S_{\mathcal{I}^*} &:= S_{\mathcal{I}^*}(x,y,\tilde{r}(t)), & \tilde{f}_c &:= \Phi^{-\mathrm{T}} f_c(x,y,\tilde{r}(t)), \end{aligned}$$

bearing in mind that $x \in \partial K$ and $\Gamma = \Phi^{\mathrm{T}}\Phi$ [2, Theorem 7.2.7, pp. 406]. With $\tilde{x} := \Phi^{-\mathrm{T}}(x - x_{ker})$, we need to show that

$$\tilde{x}^{\mathrm{T}} S_{\mathcal{I}^*} \tilde{f}_c \geq 0. \tag{C7}$$

Lemma C1 shows that $x^* = \tilde{R}_{\mathcal{I}^*}\tilde{f}_c$ where x^* is the unique optimal solution to problem (C5). Since x^* is the projection of $\tilde{f}_c$ onto $\mathcal{K}^\circ$, it satisfies $x^* - \tilde{f}_c \in \mathcal{K}^{\circ *}$ [4, Theorem E.9.2.0.1, pp. 726] where $\mathcal{K}^{\circ *}$ is the dual [4, pp. 149] to $\mathcal{K}^\circ$. Moreover, because the dual cone is the negative polar cone [4, footnote 2.53, pp. 149], and $\mathcal{K}^{\circ\circ} = \mathcal{K}$ [5, Lemma 2.7.9, pp. 54], we have $\mathcal{K}^{\circ *} = -\mathcal{K}$. Hence the relation $x^* - \tilde{f}_c \in \mathcal{K}^{\circ *}$ is equivalent to $\tilde{f}_c - x^* \in \mathcal{K}$. Using (C1) and $x^* = \tilde{R}_{\mathcal{I}^*}\tilde{f}_c$ (by Lemma C1), we have

$$\tilde{f}_c - x^* = \tilde{f}_c - \tilde{R}_{\mathcal{I}^*}\tilde{f}_c = S_{\mathcal{I}^*}\tilde{f}_c \in \mathcal{K}. \tag{C8}$$

From [1, Lemma 3], we have $\langle x - x_{ker}, \nabla h_i(x)\rangle \geq 0$ for all $i \in \mathcal{I}_{sat}$, where $\mathcal{I}_{sat}$ is the set of indices of active saturation constraints (see its definition preceding (7) in [1]). This means

$$N_{\mathcal{I}_{sat}}^{\mathrm{T}}(x - x_{ker}) = N_{\mathcal{I}_{sat}}^{\mathrm{T}}\Phi^{\mathrm{T}}\Phi^{-\mathrm{T}}(x - x_{ker}) = \tilde{N}_{\mathcal{I}_{sat}}^{\mathrm{T}}\tilde{x} \geq 0,$$

which implies that $\tilde{x}$ is in the dual of $\mathcal{K}$, ie. $\tilde{x} \in \mathcal{K}^*$ [4, pp. 149]. This, together with the definition of the dual cone [4, pp. 149] and (C8) shows that (C7) holds. ∎

The corrected proof of the geometric bounding condition, with a minor generalization, is presented next.

Theorem C1 ([1, Theorem 2]): If $K \subset \mathbb{R}^q$ is a star domain, then for any $z \in (\mathbb{R}^n \times K)$ and any $z_{ker} \in (\mathbb{R}^n \times \ker(K))$,

$$\langle z - z_{ker}, \tilde{\Gamma}^{-1} f_p(t,z)\rangle \leq \langle z - z_{ker}, \tilde{\Gamma}^{-1} f_n(t,z)\rangle,$$

holds for all $t \in \mathbb{R}$, where $\tilde{\Gamma} = \left[\begin{smallmatrix}\Theta & 0 \\ 0 & \Gamma\end{smallmatrix}\right] \in \mathbb{R}^{(n+q)\times(n+q)}$ and $\Theta \in \mathbb{R}^{n\times n}$ is any nonsingular square $n \times n$ matrix.

Proof: Let $z = (e, x) \in (\mathbb{R}^n \times K)$, $z_{ker} = (e_\infty, x_\infty) \in (\mathbb{R}^n \times \ker(K))$, $\bar{e} = e - e_\infty \in \mathbb{R}^n$, and $\bar{x} = x - x_\infty \in \mathbb{R}^q$. With reference to (3) and (11) in [1], we need to show that

$$\bar{e}^{\mathrm{T}}\Theta^{-1}(f - \dot{r}(t)) + \bar{x}^{\mathrm{T}}\Gamma^{-1} f_{\mathcal{I}^*} \leq \bar{e}^{\mathrm{T}}\Theta^{-1}(f - \dot{r}(t)) + \bar{x}^{\mathrm{T}}\Gamma^{-1} f_c,$$

or equivalently, $\bar{x}^{\mathrm{T}}\Gamma^{-1} f_{\mathcal{I}^*} \leq \bar{x}^{\mathrm{T}}\Gamma^{-1} f_c$, where the function arguments have been dropped. Using the definition of $f_{\mathcal{I}^*}$ (see (6) in [1]), $\Gamma = \Phi^{\mathrm{T}}\Phi$ [2, Theorem 7.2.7, pp. 406], and (C2), the preceding can be reduced to

$$\begin{gathered} \bar{x}^{\mathrm{T}}\Gamma^{-1} f_c(x) - \bar{x}^{\mathrm{T}}\Gamma^{-1} R_{\mathcal{I}^*}(x) f_c(x) \geq 0, \\ \bar{x}^{\mathrm{T}}\Gamma^{-1} f_c(x) - \bar{x}^{\mathrm{T}}\Gamma^{-1}(I - \Phi^{\mathrm{T}} S_{\mathcal{I}^*}(x)\Phi^{-\mathrm{T}}) f_c(x) \geq 0, \\ \bar{x}^{\mathrm{T}}\Phi^{-1} S_{\mathcal{I}^*}(x,y,\tilde{r}(t))\Phi^{-\mathrm{T}} f_c(x,y,\tilde{r}(t)) \geq 0, \end{gathered} \tag{C9}$$

where by an abuse of notation, we mean $R_{\mathcal{I}^*}(x) := R_{\mathcal{I}^*}(x,y,\tilde{r}(t))$, $S_{\mathcal{I}^*}(x) := S_{\mathcal{I}^*}(x,y,\tilde{r}(t))$, and $f_c(x) := f_c(x,y,\tilde{r}(t))$.

If x is in the interior of K, then $h_i(x) < 0$ for all $i \in \mathcal{I}_{2m}$ and $\mathcal{I}_{sat} = \emptyset$. Since $\mathcal{I}^* \subset \mathcal{I}_{sat}$, we have $\mathcal{I}^* = \emptyset$. From the definition of $S_{\mathcal{I}^*}$ (C3) with $\mathcal{I}^* = \emptyset$, we see that (C9) holds with equality. If x is on the boundary of K, ie. $x \in \partial K$, Lemma C2 yields (C9), and hence the conclusion. ∎

V. Conclusions

Faults in the existing proof of the geometric bounding condition in [1] were revealed. Necessary changes and a new proof of a marginally generalized result was presented, using concepts from convex analysis. An intermediate result is that the GPAW compensated controller can be defined by the solution to a quadratic program with a unique optimal solution. The geometric bounding condition as originally stated stands with this new proof.

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