A. THEORETICAL SMALL-SIGNAL ANALYSIS OF AN AVALANCHE DIODE IN THE s-PLANE

1. Theoretical-Impedance Calculation

In the avalanche diode structure of Fig. IV-1, a constant current $J_o$ is present because of carrier generation by impact ionization in a high electric field region. The small-signal impedance of this avalanche region is determined by solving a set of simultaneous linear differential equations derived from Maxwell's equations.

Assuming a time dependence of $e^{s t}$, we can write these equations as

$$\frac{dV}{dx} = [A(x)+sB(x)] \bar{V} +JC(x),$$

where $A(x)$ and $B(x)$ are $2 \times 2$ matrices, $C(x)$ is a $1 \times 2$ matrix, $s$ is the complex

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frequency variable $\sigma + j\omega$, and $\hat{J}$ is the amplitude of the total small-signal current in the diode. The vector $\hat{V}(x)$ comprises the small-signal electric field amplitude and the small-signal hole current amplitude. That is,

$$\hat{V}(x) = \begin{bmatrix} \hat{E}(x) \\ \hat{J}_p(x) \end{bmatrix}.$$  \hfill (2)

The elements of the matrices $A$, $B$, and $C$ depend on $x$ through the static electric field and hole and electron concentrations. These static values are found by solving the nonlinear static form of Maxwell's equations for the diode, given the doping $N(x)$ and the DC current $J_0$, in a special computer subroutine that has been written by Professor Paul Penfield. In his analysis, the hole and electron ionization coefficients and carrier velocities have their standard experimentally determined form.

Equation 1 must be solved under the boundary conditions that at the left edge of the avalanche region ($x=0$),

$$\hat{J}_p(0) = 0,$$  \hfill (3)

and at the right edge of the region ($x=w$),

$$\hat{J}_n(w) = 0,$$  \hfill (4)

where the electron current $\hat{J}_n(x)$ is given by

$$\hat{J}_n(x) = \hat{J} - \hat{J}_p(x) - s \epsilon \hat{E}(x).$$  \hfill (5)

The width $w$ of the avalanche region is determined by the doping profile $N(x)$ and the direct avalanche current $J_0$.

The boundary conditions can be satisfied by considering the solution of (1) as

$$\hat{V} = \lambda \hat{V}_h + \hat{V}_i,$$  \hfill (6)

where $\hat{V}_h$ is the homogeneous solution of (1), $(\hat{J}=0)$, with

$$\hat{V}_h(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$  \hfill (7)

$\hat{V}_i$ is the inhomogeneous solution of (1), $(\hat{J}=1)$, with

$$\hat{V}_i(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  \hfill (8)
and $\lambda$ is a complex constant to be determined. Boundary condition (3) is thus satisfied for any choice of $\lambda$.

By using the initial conditions (7) and (8), the solutions for $V_h(x)$ and $V_i(x)$ can be generated by iteration across the avalanche region at equally spaced points $\Delta x$ apart. The value of $\Delta x$ can be made as small as desired, but was usually chosen as $w/100$. The multiplier $\lambda$ is determined from the values of $V_h(w)$ and $V_i(w)$ combined with the boundary condition (4). From (5), we have

$$J_{nh}(w) = -J_{ph}(w) - se E_h(w) = -[1 se] V_h(w)$$

and

$$J_{ni}(w) = 1 - [1 se] V_i(w),$$

since $J = 1$. Then (4) requires

$$J_n(w) = \lambda J_{nh}(w) + J_{ni}(w) = 0$$

or

$$\lambda = \frac{1 - [1 se] V_i(w)}{[1 se] V_h(w)}.$$  \hspace{1cm} (11)

Equation 11 fails if $J_{nh}(w) = 0$, but this only happens if both boundary conditions are satisfied with $J = 0$, or when the value of $s$ used in the calculation is an open-circuit natural frequency.

The small-signal voltage amplitude $V$ is obtained from integration of the electric field $E(x)$ from $x = 0$ to $x = w$, and the impedance is then given as

$$Z(s) = V(s),$$  \hspace{1cm} (12)

since $\lambda = 1$. By following the procedure above for many values of $s$, we can generate the impedance throughout the $s$-plane. In the rest of this report, the diode admittance will be analyzed because this makes the data both easier to handle and readily comparable to the admittance of a circuit model.

The particular diode used in this study was a one-dimensional silicon diffused type with a background doping of $10^{21}/m^3$. Its admittance had the pole-zero pattern of Fig. IV-2 at each of the values of bias current tested, being unstable for both short-circuit and open-circuit conditions at the terminals of the avalanche region.

As $J_0$ varied, the zero locations moved considerably, while the pole frequency remained essentially fixed on the real axis. A searching routine was written to
accurately determine these frequencies. Figure IV-3 shows their movement in the $s$-plane for values of bias $J_0$ between $10^4$ A/m$^2$ and $10^6$ A/m$^2$.

The computed admittance will now be compared with that of a simple circuit model that has the same pole-zero pattern. This is accomplished by comparing constant contours of the real and imaginary parts of the calculated admittance with those of the
circuit admittance in a normalized-frequency plane.

2. Circuit Model for the Avalanche Diode

a. Model Characteristics

The proposed circuit model for the diode is shown in Fig. IV-4. The admittance of this circuit is

\[ y_d(s) = \frac{1 - rg + s(rc - lg) + s^2 fc}{r + s \ell}, \tag{13} \]

and has the same pole-zero pattern as Fig. IV-2, with the zeros at

\[ s_o = \frac{1}{2} \left[ \frac{g}{c} - \frac{r}{\ell} \right] \pm j \sqrt{\frac{1}{\ell c} - \frac{1}{4} \left[ \frac{g}{c} + \frac{r}{\ell} \right]^2}, \tag{14} \]

and the pole at

\[ \sigma_p = -\frac{r}{\ell} \tag{15} \]

where in this case \( r \) will be negative. If a new frequency variable \( z \) is defined as

\[ z = x + jy = s \sqrt{\ell c} + \frac{r}{\ell} \sqrt{\ell c}, \tag{16} \]

where
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\[ x = (\sigma + r/\ell) \cdot \sqrt{f_c} \]  \hspace{1cm} (17)

and

\[ y = \omega \sqrt{f_c}, \]  \hspace{1cm} (18)

then (13) can be written

\[ \overline{y}_d(z) = \frac{y_d(z)}{Y_0} + \gamma = z + \frac{1}{z}, \]  \hspace{1cm} (19)

where

\[ Y_0 = \left( \frac{Q}{\ell} \right)^{1/2} \]  \hspace{1cm} (20)

and

\[ \gamma = g/Y_0 + rY_0. \]  \hspace{1cm} (21)

Normalized and shifted in this way, \( \overline{y}_d(z) \) is the universal impedance of the circuit, since it does not depend on any of the circuit-element values. In the z-plane, \( \overline{y}_d(z) \) can be modeled as a parallel combination of an inductance and a capacitance, each with value one, so the pole is now at \( z = 0 \) and the zeros at \( \pm j \, 1 \). This is shown in Fig. IV-5.

![Impedance transformation from s-plane to z-plane.](image)

\textbf{Fig. IV-5. Impedance transformation from s-plane to z-plane.}

Constant conductance and susceptance contours in the z-plane are determined by setting
Re \{\overline{y}_d\} = x \left[ 1 + \frac{1}{x^2 + y^2} \right] = n = \text{const.} \tag{22}

and

Im \{\overline{y}_d\} = y \left[ 1 - \frac{1}{x^2 + y^2} \right] = m = \text{const.} \tag{23}

Each choice of m and n defines a curve in the z-plane, with curves of constant \(\text{Re} \{\overline{y}_d\}\) orthogonal to curves of constant \(\text{Im} \{\overline{y}_d\}\). The first quadrant of such a plot is shown in Fig. IV-6, where the susceptance is odd about the x-axis and even about the y-axis, and the conductance is odd about the y-axis and even about the x-axis.

![Fig. IV-6. Constant Re \{\overline{y}_d\} and Im \{\overline{y}_d\} contours in the z-plane.]

b. Normalization of Calculated Admittance

Figure IV-6 serves as a reference with which the computed admittance will be compared. Operations equivalent to Eqs. 16-21 for the computed admittance are
determined at each value of $\sigma_0$ from the pole and zero frequencies $\sigma_p$ and $s_o$. These equivalent operations are

$$z' = x' + jy' = \frac{s}{|s_o' - \sigma'|} - \frac{\sigma_p'}{|s_o' - \sigma'|}$$  \hspace{1cm} (24)

$$x' = \frac{\sigma - \sigma_p}{|s_o' - \sigma'|}$$  \hspace{1cm} (25)

$$y' = \frac{\omega}{|s_o' - \sigma'|}$$  \hspace{1cm} (26)

$$Y_o' = |s_o' - \sigma'| c_d$$  \hspace{1cm} (27)

$$\gamma' = \frac{2(\sigma_p' - \sigma_o')}{|s_o' - \sigma'|}$$  \hspace{1cm} (28)

and

$$\bar{y}'_d(z') = \frac{\bar{y}'_d(z' |s_o' - \sigma'| + \sigma_p')}{Y_o'} + \gamma',$$  \hspace{1cm} (29)

where $c_d$ is considered to be a known quantity that is equal to the avalanche region capacitance per unit area, $\varepsilon/w$. The primes refer to quantities defined from the calculated data. Again, the pole is at $z' = 0$ and the zeros at $z' = \pm j1$, as shown in Fig. IV-7.

Fig. IV-7. Impedance transformation from s-plane to z'-plane.
c. Comparison of Calculated Data with Reference Data

The contours of constant Re \( \frac{\vfi_{yd}}{j} \) and Im \( \frac{\vfi_{yd}}{j} \) were determined from computation of (29) after the pole and zero frequencies had been determined in a separate calculation. For a given contour value and a given \( x' \) value, the \( y' \) value was found by a searching routine. At one value of bias \((10^5 \text{ A/m}^2)\), the \( y' \) values were determined for several values of \( x' \) for all values of the reference contours specified in Fig. IV-6. In each case the \( y' \) value agreed with the reference \( y \) value to at least 3 decimal places. At the other values of \( J_o' \), only the contour intercepts along the \( x' \) and \( y' \) axes were checked and the agreement was also as good. These results show that the diode can be accurately modeled by the circuit of Fig. IV-4 if we choose the element values correctly. This choice was already made by stating equivalence of Eqs. 16-21 to Eqs. 24-29, which is equivalent to matching pole and zero frequencies of both admittances with \( c = c_d \). The model element values are thus given as

\[
g = (2\sigma_o' - \sigma_i') c_d \tag{30}
\]

\[
r = \frac{-\sigma_i'}{c_d \left| s_o' - \sigma_i' \right|^2} = \frac{-\sigma_i'}{c_d \left[ \omega_o^2 + (\omega_o' - \sigma_i')^2 \right]} \tag{31}
\]

\[
\ell = \frac{1}{c_d \left[ \omega_o^2 + (\omega_o' - \sigma_i')^2 \right]} \tag{32}
\]

and

\[
c = c_d. \tag{33}
\]

As further verification of the model's validity, Table IV-1 shows the actual admittance values vs the equivalent circuit admittance values evaluated along the \( j\omega \) axis. At a frequency of approximately 10 times the resonant frequency, the two sets of data begin to disagree, and the model is no longer valid.

d. Bias Dependence

To complete the modeling of the diode, we must include the bias dependence of the admittance along with its frequency dependence. Since the pole and zero frequencies can easily be found at several values of bias current \( J_o \), the dependence of the circuit element values on \( J_o \) can readily be determined from Eqs. 30-33. The movement in the \( s \)-plane of the natural frequencies vs bias was seen in Fig. IV-3, and Fig. IV-8 is
Table IV-1. Frequency response of calculated admittance and equivalent-circuit admittance.

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Fig. IV-8. Equivalent-circuit elements vs bias $J_o$.

Fig. IV-9. Diode equivalent circuit.

- $c = 1.605 \times 10^{-5} \text{ F/m}^2$
- $g = 1.34 J_o \times 10^{-2} \text{ mhos/m}^2$
- $\ell = (4.16 / J_o) \times 10^{-10} \text{ H/m}^2$
- $J_o$ = direct current density, $\text{A/m}^2$
- $10^4 \text{ A/m}^2 \leq J_o \leq 10^6 \text{ A/m}^2$
- $r = -0.983 / J_o \ \Omega \text{-m}^2$
a log-log plot of the corresponding element values $r$, $l$, and $g$ vs $J_o$. Since $\sigma_p = -r/l$ remains nearly constant, $r$ and $l$ vary together and appear to be inversely proportional to $J_o$. The conductance $g$ is proportional to $J_o$, while $c$ remains essentially constant over the test range in bias $10^4 \text{A/m}^2 \leq J_o \leq 10^6 \text{A/m}^2$. The complete circuit model is thus shown in Fig. IV-9, where the admittance is in mhos/m$^2$, since all calculations were made on a unit area basis.

3. Conclusions

The small-signal admittance of an avalanche diode calculated directly from Maxwell's equations can be modeled by a simple four-element circuit in a region of the $s$-plane that includes the natural frequencies. The bias dependence of the admittance is characterized by the dependence of the element values on the direct current $J_o$. This simple model can be easily handled by existing circuit-analysis computer programs and is useful in design of microwave active circuits.

One should recognize that the model only characterizes the region of the diode in which impact ionization is taking place and the electric field is high, and does not include effects occurring in the high-conductivity, low electric field inactive end regions.

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References