III. ELECTRODYNAMICS OF MEDIA

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A. MODEL OF MODE LOCKING WITH SATURABLE ABSORBER

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Of the many studies of mode locking using saturable absorbers, the work of Kuizenga and Siegman\(^1\) on forced mode locking has the greatest analytic simplicity. In this report our objective is to obtain simple expressions for the intensity and the pulse width of passively mode-locked pulses and to choose a model simple enough to make closed-form analysis possible. We assume, with Kuizenga and Siegman,\(^1\) that the gain of the laser medium changes negligibly through passage of a single pulse. We further assume that the response time of the saturable absorber is fast enough so that the population difference between the lower and upper levels in the saturable absorber can follow the pulse envelope instantaneously. Finally, we assume that the population perturbation in the saturable absorber is small enough that its rate equation may be linearized.

Our analysis is analogous to Kuizenga and Siegman's and, in particular, we require that a Gaussian pulse reproduce in magnitude and width after passage through the amplifier and the absorber and after reflection from the output mirror.

Adopting the notation of Kuizenga and Siegman and following their analysis of the passage of a Gaussian pulse through the amplifying medium, we obtain the electric field at the amplifier exit.

\[
E = \frac{E_0 G}{4\sqrt{\gamma A}} \exp\left[-(t-B)^2/4A\right].
\]  

When this pulse passes through the saturable absorber, it affects the population difference between the lower and upper levels of the absorber \(n\) according to the rate equation
All symbols in this equation refer to the parameters of the saturable absorber. Here $n^e$ is the steady-state value of $n$.

Because the main pulse-shaping occurs near the maximum of an optical pulse, we expand the E-field of Eq. 1 around the instant at which $E$ reaches a maximum and the absorption reaches a minimum. In this way, we obtain for the population difference in the absorber

$$n \approx n^e \left\{ 1 - \frac{2 t}{2} \frac{T_0^2}{16 \gamma A^2} \left[ 1 - 2(t-A)^2/4A \right] \right\}.$$ \hspace{1cm} \text{(3)}

The signal passing through the absorber is multiplied by

$$\exp(-\alpha \xi) = \exp(-\alpha n),$$ \hspace{1cm} \text{(4)}

where $S$ is a parameter characteristic of the relaxation time and matrix elements of the absorber. At this point our analysis connects with the Kuizenga-Siegman analysis, since the modeling of the absorber makes it perform a function identical to the function performed by the modulator used in forced mode locking. The parameter $\delta \omega_n$ of Kuizenga and Siegman is replaced by

$$\frac{1}{I_s} \frac{G^2}{16 \gamma A^2} \frac{S n^e}{4},$$ \hspace{1cm} \text{(5)}

where $I_s$ is now the saturation intensity in the absorber. Using the results of Kuizenga and Siegman, from the self-consistency requirement on the pulse width we find

$$\gamma = \frac{1}{4A} + \frac{1}{I_s} \frac{G^2}{16 \gamma A^2}.$$ \hspace{1cm} \text{(6)}

The self-consistency requirement on the pulse amplitude gives

$$\frac{R G}{2 \sqrt{\gamma A}} \exp(-\alpha n^e) = 1.$$ \hspace{1cm} \text{(7)}

The parameter $R$ is the reflectance of the output mirror. Solving these two equations for $\gamma$ and the intensity $I$, and using $A = 1/(4\gamma) + 4g/\Delta \omega^2$, we obtain

$$\gamma \approx (L^2 G^2 - 1) \frac{\Delta \omega^2}{16 g}.$$ \hspace{1cm} \text{(8)}
\[
\frac{I}{I_s} = 4 \ln \left(\frac{R}{L}\right)(L^2G^2-1)L^2, \tag{9}
\]

where we have defined a loss parameter

\[
L = R \exp(-S_n e). \tag{10}
\]

The pulse-width parameter of Eq. 10 can be rewritten as a pulse duration time \(\tau_p\), which becomes

\[
\tau_p = 8\sqrt{2} \ln 2 \sqrt{\frac{I_s}{I}} \sqrt{\frac{L}{\Delta \omega}} \sqrt{\ln(R/L)}. \tag{11}
\]

The parameters in Eq. 11 are identical to those used by Kuizenga and Siegman, with the exception of the loss parameter \(L\), which now involves both the attenuation of the saturable absorber and the reflectance of the mirror, and \(I_s\) which is the saturation intensity of the absorber.

These results may be compared with published experiments on a TEA CO\(_2\) laser that is mode-locked via a saturable absorber consisting of a heated CO\(_2\) cell at a pressure higher than the amplifier pressure.\(^2\) The assumptions here are generally consistent with these experimental conditions. The observed pulse length was 4 ns. If we introduce parameters that seem to match experimental results, we predict a pulse length of \(\sim 1.5\) ns. We believe that better agreement with experiment can be obtained if we generalize the analysis presented here to transient mode locking, by using an analysis analogous to Feldman's of transient forced mode locking.\(^3\)

References


B. INVERSION GROWTH IN AN HF LASER AMPLIFIER II

Joint Services Electronics Program (Contract DAAB07-71-C-0300)

J. R. Gersh, E. L. Frohring

1. System Description and Operation

In a previous report, preliminary results of the measurement of population inversion growth in an HF chemical laser amplifier were presented. In this work the amplifier gain was probed with a pulse from a similar HF laser oscillator. Further work on this system has corrected and extended those results.

The following modifications have been made in experimental apparatus described in our previous report: An NaCl flat was placed between the oscillator and amplifier to act as a beam splitter. A second Ge: Au photoconductive detector was added to monitor the signal reflected from this beam splitter. Thus both the input and output of the amplifier can be seen together on the Tektronix 556 dual-beam oscilloscope for any shot. The detectors were calibrated at the 3 \( \mu \) band of HF wavelengths; thus, direct measurements of the energy gained by the pulse in passing through the amplifier can now be made. A second gas-handling system was also added; hence, the oscillator and amplifier tubes are separately supplied with the flowing mixture containing 193 Torr He, 5.7 Torr SF\(_6\), 1.3 Torr He. A transverse discharge from pin electrodes dissociates fluorine from SF\(_6\) and this reacts with H\(_2\) according to

\[
F + H_2 \rightarrow HF + H
\]  

(1)
to produce the active medium of the laser.

Figure III-1 shows the oscillator output with the amplifier discharge turned off. The upper trace shows the signal from the amplifier input and the lower trace that from the output. This photograph was used to calibrate the system; equivalent signals on the
oscilloscope were given for identical signals at input and output of the amplifier.

The pulse has a characteristic triple-peaked shape. We previously assumed,\(^1\) as the result of preliminary spectroscopic measurements, that these three peaks represent separate and distinct vibrational transitions. More detailed measurements\(^2\) have shown that this is incorrect.

![Spectroscopic analysis of oscillator pulse](image)

**Fig. III-2.** Spectroscopic analysis of oscillator pulse.

Figure III-2 shows the timing and relative intensities of the vibrational-rotational transitions that make up the oscillator pulse according to these measurements.

The characteristic triple-peaked shape of the pulse seen in Fig. III-1 is thus explained by the sequence of separate transitions indicated in Fig. III-2. Of particular interest to the present work are the following observations:

a. The first peak is made up of \(P_2(3)\) and \(P_2(4)\) in almost equal amounts.

b. The second peak is for the most part \(P_2(5)\).

c. Transitions from the \(v = 3\) level form only a small fraction of the total pulse.

d. The third peak is made up of \(P_1(7)\), \(P_2(6)\), \(P_1(8)\), and \(P_2(7)\), with only a small amount of additional transitions.

It should also be noted that the third peak is made up of pairs of communicating transitions. The lower level of \(P_2(6)\) is the upper level of \(P_1(7)\), and similarly with \(P_2(7)\) and \(P_1(8)\). This information can now be exploited when we calculate how the populations in the \(v = 1\) and \(v = 2\) vibrational levels of the HF molecule grow with time after
the amplifier current discharge. This growth can then be compared with that predicted from reported values for the chemical rate coefficients involved in reaction (1).

Figure III-3 gives four examples of the amplification of pulses passing through the amplifier. The time delay between the amplifier discharge and the oscillator pulse is progressively longer in each example. The oscilloscope sweep is triggered by the amplifier current, and the energy gain of each section of the pulse as a function of this time is used to infer the population growth.

2. Predicted Growth

Growth of the population inversion in the amplifier can best be described by the overall rate at which the chemical reaction proceeds, and the fraction of the total amount of HF that goes into each of the vibrational levels.

Cohen\textsuperscript{3} has compiled the rate coefficients for all important reactions in the H\textsubscript{2}-F\textsubscript{2} laser system. Since chemical reaction (1) itself is the same in all situations, these
data are applicable to the present system, once the fluorine atoms are considered to have been liberated from the SF$_6$. Guesses will have to be made concerning the amount of fluorine produced; cross sections describing the process of SF$_6$ dissociation by electrons are not known.

Cohen quotes Homann's value for the overall rate coefficient for reaction (1), which is the best available, as $k = 1.01 \times 10^{13} \text{ cm}^3 \text{ mole}^{-1} \text{ s}^{-1}$ at 300°K. In the present system the initial concentration of H$_2$ is 1.3 Torr, that of F is not known. Since we are looking primarily for internal consistency in the analysis, we may make an assumption about the F concentration, to be justified by the results. The largest amount of HF that can be made is 2.6 Torr, if all of the available hydrogen is consumed. If we assume that this is not the case, that there is some excess of hydrogen, we may place limits on the time it will take for the reaction to go to completion. For a large excess of H$_2$, the reaction rate will be proportional to the amount of fluorine that remains and to this relatively constant H$_2$ concentration. The rate will then go as $\exp(-t/\tau)$, where $1/\tau = \{\text{H}_2\}k$. (1.3 Torr is equivalent to $7.5 \times 10^{-8} \text{ mole cm}^{-3}$, which gives $\tau = 130 \text{ ns}$.) The reaction will then be 90% complete at $t = 2.3\tau$ or $t = 300 \text{ ns}$.

If there are equal concentrations of H$_2$ and F, the rate will be proportional to the square of this reactant concentration. The amount remaining will then go as $(1+t/\tau)^{-1}$ and the reaction will be 90% complete at $t = 9\tau$ or $t = 1200 \text{ ns}$.

Thus it is expected that, depending on the precise value of the initial fluorine concentration, and under the assumption of no excess of fluorine, the reaction will be complete in 300-1200 ns after it is initiated. Since the fluorine is created over a finite time by the current, of course, this initiation time is not well defined.

Many values have been reported for the ratio of HF production in $v = 2$ to that in $v = 1$. (For the present analysis, it is assumed that no HF is produced in $v = 3$ or $v = 0$. This is only a fair approximation in the former case, but is substantially correct in the latter.) The earliest experimental result, by Parker and Pimentel$^5$ was a reported ratio of 5.5 at 594°K, determined by observations of the relative gain for different transitions from the same vibrational level as a function of temperature in an HF laser. Shortly thereafter, Polanyi and Tardy,$^6$ using a chemiluminescence technique, reported a value of 3.26 near room temperature. Theoretical studies have also been made. For example, Jaffe and Anderson,$^7$ using a classical trajectory analysis, calculated a ratio of 1.9 at 300°K. Wilkins$^4$ in a recent study calculated a ratio of 5.4 at 300°K, commenting that this seemed high compared with the result of Polanyi and Tardy.

We may expect, then, that the ratio of population produced in the $v = 2$ level to that produced in $v = 1$ should be between 2 and 5, possibly close to 3.

Relaxation of the excited vibrational populations back to thermal equilibrium takes place primarily by means of HF-HF collisions.$^8$ The energy-transfer mechanism may be either V-T or V-V. In the former case, the decay time has been measured.
as 0.014 μs atm, or 11 μs at 1 Torr HF pressure. On the other hand, the combined V-V, V-T transfer time has recently been determined in an elegant experiment by Osgood, Javan, and Sackett. They found the decay time to be 1.5 μs Torr. V-V transfer is therefore the dominant relaxation process, and its effects should become noticeable fairly quickly.

3. Energy-Gain Model

In order to infer the populations as a function of time after the amplifier current starts, a model must be developed to describe the passage of the pulse through the amplifier. The pulse will be analyzed in terms of its characteristic triple-peak shape; as noted above, the first two peaks are made up of $2 \rightarrow 1$ transitions, while the third is made up of both $2 \rightarrow 1$ and $1 \rightarrow 0$ transitions.

It has been shown in detail elsewhere, with the use of simplified rate equations, that the intensity of the radiation entering the amplifier is sufficient to saturate each of these transitions. This means that the energy added to each section of the input pulse as it transits the amplifier is related to the population inversion present on its arrival by

$$N + R\tau = \frac{2}{\kappa \hbar \nu} (E_o - E_i).$$

Here $N$ indicates the appropriate population inversion density (in cm$^{-3}$) present in the amplifier on the arrival of a particular transition in the input pulse. (The lack of subscript denotes a population difference.) $R$ is the net rate at which the chemical reaction pumps this inversion. The model used here approximates the intensity of any transition as a rectangle of length $\tau$, and assumes that $R$ stays constant during the pulse, although it is a function of the time the pulse arrives.

The effective length of the active medium is denoted by $\ell$. The width of the active region around each pin has been measured for a similar system to be 0.25, which gives $\ell = 38$ cm for 60 pins. $E_o - E_i$ is the energy per unit area of beam cross section, in J/cm$^2$, added to the pulse in passing through the amplifier.

The meaning of Eq. 2 is simply that the pulse, in passing through the amplifier, completely destroys the initially present inversion and also destroys any inversion added by chemical pumping during the duration of the pulse.

What we wish to evaluate is $N(t)$, the inversion that the pulse initially sees, as a function of $t$, the time at which the pulse enters the amplifier. Note that (2) is, in reality, a differential equation. Since $R$ is the rate at which the chemical reaction increases the inversion, it is also the rate of change of $N$ with respect to $t$. Therefore (2) becomes
We measure $E_o - E_i$ as a function of $t$ from experimental data, examples of which are shown in Fig. III-3. Separate measurements of the energy gain are made for each of the three sections of the input pulse. These are treated as individual pulses, with lengths of 40, 80, and 360 ns, respectively.

To obtain the inversion history from the measured values of $(E_o - E_i)(t)$, the general solution to (3) is used.

$$N(t) = e^{-t/\tau} \int_0^t e^{t'/\tau} \frac{2}{\hbar \nu} (E_o - E_i)(t') \, dt'.$$

A simple computer program in the APL language was used to evaluate this expression for each $t$ at which a measurement was available, and the integral was calculated by using a trapezoid approximation. The two curves of Fig. III-4 and the upper curve of Fig. III-5 show the results of these calculations for each of the three sections of the pulse. The error bars represent the error in calculated inversion caused solely by the estimated error in measuring the area under the oscilloscope traces in the photographs.

The question now arises of exactly what these initial inversions represent. The rotational thermalization time of this system is quite short, as evidenced by the 1-GHz pressure-broadened linewidth. So, while this time must be long enough to allow the various transitions to lase sequentially, it is assumed that over the total duration of the first two peaks there is sufficient time for complete communication between rotational levels. The input pulse can then completely destroy the inversion between $v = 2$ and $v = 1$ over the total duration of these two peaks. In other
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words, the initial inversions as seen by the first two peaks at any time $t$ are taken together to be the total $2 \rightarrow 1$ inversion for that particular $t$. Thus the curve on Fig. III-5 labeled $N_2-N_1$ is the sum of the two smooth curves drawn by eye through the calculated points of Fig. III-4.

![Graph](image)

**Fig. III-5.** Sum and difference of populations.

Furthermore, since the $2 \rightarrow 1$ inversion is completely saturated by these first two peaks, it is assumed that at the start of the third peak the $v = 2$ and $v = 1$ populations are the same, each equal to $\frac{N_2(t) + N_1(t)}{2}$ for the $t$ at the beginning of the third peak.

It is important to notice at this point that the third peak is made up of both $2 \rightarrow 1$ and $1 \rightarrow 0$ transitions going simultaneously. These transitions share a common rotational level in the $v = 1$ band, and both the $2 \rightarrow 1$ and $1 \rightarrow 0$ transitions are saturated. Thus the passage of the third peak through the amplifier equalizes the populations of all three vibrational levels, 2, 1, and 0, to $\frac{N_2 + N_1 + (R_2 + R_1)\tau}{3}$. We assume the population in the ground state at the arrival of the third peak to be negligible; the third peak is also much longer than the first two.

In a three-level case like this the total energy extracted from the amplifier by a saturating pulse is independent of the initial population in, or the pumping rate into, the middle level. In general, for initial populations $N_2$, $N_1$, $N_0$, and final population in all levels $(N_2+N_1+N_0)/3$, the following is the case.

The energy emitted in the $2 \rightarrow 1$ transition is proportional to the initial population
in \( v = 2 \) minus the final population

\[
E_{2-1} = \frac{\hbar c}{2} \left\{ N_2 - \frac{(N_2 + N_1 + N_0)}{3} \right\}, \quad (5)
\]

or

\[
E_{2-1} = \hbar c \frac{(2N_2 - N_1 - N_0)}{3}. \quad (6)
\]

The energy emitted in the \( 1 \rightarrow 0 \) transition is proportional to the final population in \( v = 0 \) minus the initial population.

\[
E_{1-0} = \frac{\hbar c}{3} \left\{ \frac{(N_2 + N_1 + N_0)}{3} - N_0 \right\}, \quad (7)
\]

or

\[
E_{1-0} = \hbar c \frac{(N_2 + N_1 - 2N_0)}{3}. \quad (8)
\]

The total emitted in both transitions is the sum of (6) and (8).

\[
E = \hbar c (N_2 - N_0). \quad (9)
\]

If there is a pump at work during the saturation process, its effect may be added in a manner exactly analogous to the results of Eqs. 5 through 9, which gives

\[
E = \hbar c \left\{ (N_2 - N_0) + (R_2 - R_0) \right\}. \quad (10)
\]

Thus the total energy emitted is dependent only on the difference in population between \( v = 2 \) and \( v = 0 \). For the present case, where \( N_0 = 0 \) initially, the energy gain of the third peak is simply \( \hbar c \) times the initial \( v = 2 \) population. As noted above, this is just \( \frac{N_2(t) + N_1(t)}{2} \). This is why the top curve of Fig. III-5 is labeled \( N_2 + N_1 \); it represents the evaluation of (4) for the energy gain of the third peak.

An additional procedure had to be performed in order to make this calculation for the third peak properly. Using (10) we may rewrite (4) for the third peak as

\[
\frac{N_2(t) + N_1(t)}{2} + \tau R_2(t) = \frac{1}{\hbar c} \left( E_0 - E_1 \right)(t), \quad (11)
\]

or

\[
N_2(t) + N_1(t) + 2\tau R_2(t) = \frac{2}{\hbar c} \left( E_0 - E_1 \right)(t). \quad (12)
\]
(Recall that \( R_0 \) is assumed to be zero.) The problem here is that because a three-level saturation process is involved, only \( R_2 \) adds to the energy gain, while \( R_2 + R_1 \) is the rate of change with \( t \) of the initial inversion \( N_2 + N_1 \).

This problem may be dealt with by assuming a value for the ratio \( x = R_2 / R_1 \). (This is just the ratio of chemical pumping into \( v = 2 \) to that into \( v = 1 \), estimated to be between 2 and 5.) We may now express

\[
R_2 = \frac{R_2 + R_1}{1 + 1/x}.
\]

This gives, for (12)

\[
\{N_2(t)+N_1(t)\} + \frac{2\tau}{1 + 1/x} \{R_2(t)+R_1(t)\} = \frac{2}{\hbar \nu} (E_o-E_1(t)). \tag{14}
\]

Assuming a value for \( x \), we may then use the program mentioned above to integrate this equation, using the value \( 2\tau/(1+1/x) \) instead of \( \tau \). A consistency check can then be made for the choice of \( x \). Since the final result of this computation leads to values for \( N_2(t) \) and \( N_1(t) \), we may compare \( x \) with \( N_2(t)/N_1(t) \). This was done for the time of peak population and for all values of \( x \) from 1 to 10 in steps of 0.1. The result was found to be consistent for \( x = 2.7 \), and the top curve of Fig. III-5 represents the integration of (14) using this value.

4. Conclusion

Figure III-5 gives \( N_2 + N_1 \) and \( N_2 - N_1 \) as functions of \( t \). Figure III-6 is a plot of half the sum and half the difference of these two curves, and therefore plots the populations of the \( v = 2 \) and \( v = 1 \) levels as functions of the time after the start of amplifier current. These curves must be checked for consistency with the predictions made above of a reaction going to completion in 300-1200 ns, pumping \( v = 2 \) from 2 to 5 times as strongly as \( v = 1 \), with V-V relaxation finally setting in.

First, we must notice that the ratio of \( N_2 \) to \( N_1 \) in Fig. III-6 is by no means constant. The initial portion of the plot, up to 400 ns, is the least reliable, and there the assumption that \( R \) is approximately constant during the separate portions of the pulse (which is basic to this analysis) is least valid. This gives an inflated value to \( N_2 + N_1 \), and thereby makes \( N_2/N_1 \) smaller than it should be.

Once a fairly stable value for \( N_2/N_1 \) is reached, the ratio, 2.7, agrees well with the prediction of 2 to 5 with a likely value around 3.

The total amount of HF produced in these two levels is \( 4.9 \times 10^{16} \) cm\(^{-3} \) or 1.4 Torr. This is less than the maximum possible amount, 2.6 Torr, although of the same order of magnitude. Since less than the maximum amount was produced, the
Fig. III-6. \( v = 2 \) and \( v = 1 \) populations.

The maximum total population occurs 1200 ns after the current starts, although there is little increase after 900 ns. If we consider that the reaction is initiated over a finite span of time, approximately 400 ns,\(^1\) while the current through the tube is at a high level, this time is not inconsistent with the 300-1200 ns range for time to completion of the reaction.

The total amount of HF produced is larger than that noted in our previous report.\(^1\) We believe that the closed-tube experiment reported there was incorrectly interpreted, by assuming that all of the hydrogen in the HF generated in one pulse was permanently lost. Our present results suggest that hydrogen is liberated from the HF building up in the closed tube, thereby prolonging the intensity decay.

After 900 ns the population in \( v = 2 \) decreases, while that in \( v = 1 \) increases, after being almost constant for a time. This effect is consistent with a process of one-step vibrational relaxation populating \( v = 1 \) by decay from \( v = 2 \). The characteristic time cannot be determined accurately from the plot, but it does not seem to be inconsistent with the time of 1.5 \( \mu \text{s} \) Torr noted above.

Thus the available data concerning the kinetics of reaction (1) and describing HF relaxation can be seen to provide an adequate and useful qualitative picture for explaining how energy is extracted from an HF laser amplifier.

References

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C. ELECTROMAGNETIC WAVES IN LAYERED BIAXIAL MEDIA

Joint Services Electronics Program (Contract DAAB07-71-C-0300)

J. A. Kong

Propagation of electromagnetic waves in layered isotropic media, which has application to wave optics,¹ has been well documented. Studies have also been made of wave propagation in uniaxial thin-film optical waveguides,²³ in view of applications to modulation and mode conversion. In remote sensing and geophysical probing, there has been increasing attention to anisotropic behavior of geological features. For instance, because of brine content, the anisotropy of sea ice has been measured.⁴ In this report we consider layered biaxial media looking toward application to thin-film optics, integrated optics, and remote sensing. We use propagation matrix formulation to find, first, reflection coefficients attributable to the layered media. Waves in each layer can then be determined from the reflection coefficients and the propagation matrices.

Consider an n-layer plane stratified medium composed of biaxial material, oriented with principal axes parallel to the coordinate axes. Let the plane of incidence be the x-z plane, and consider the TE waves. The solution in region k is a wave having a component in the positive x direction and a wave having a component in the negative x direction,
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\[ E_{fy} = \left[ A_{\ell} \exp(ik_{\ell}x) + B_{\ell} \exp(-ik_{\ell}x) \right] \exp(i k z) \]  \hspace{1cm} (1a)

\[ H_{fx} = \frac{k_{\ell} k_z}{\mu_{\ell} \ell_x} \left[ A_{\ell} \exp(ik_{\ell}x) + B_{\ell} \exp(-ik_{\ell}x) \right] \exp(i k z) \]  \hspace{1cm} (1b)

\[ H_{fz} = \frac{k_{\ell} k_x}{\omega \ell_x} \left[ A_{\ell} \exp(ik_{\ell}x) - B_{\ell} \exp(-ik_{\ell}x) \right] \exp(i k z). \]  \hspace{1cm} (1c)

We omit subscript \( f \) for \( k \) because, from boundary conditions, the \( k_z \) must be the same in all regions. The wave number \( k_{\ell}x \) satisfies the dispersion relation in the \( \ell \)'th region.

\[ \frac{\mu_y}{\mu_z} k_{\ell x}^2 + \frac{\mu_y}{\mu_z} k_{\ell z}^2 = \omega^2 \mu_y n_{\ell} \ell_x. \]  \hspace{1cm} (2)

The wave amplitudes \( A_{\ell} \) and \( B_{\ell} \) are related to wave amplitudes in neighboring regions by the boundary conditions.

The boundary condition at \( x = -d_{\ell} \) requires that \( E_y \) and \( H_z \) be continuous across the boundary. We can express \( A_{\ell} \) and \( B_{\ell} \) in terms of \( A_{\ell+1} \) and \( B_{\ell+1} \). The result is

\[ \begin{bmatrix} A_{\ell} \\ B_{\ell} \end{bmatrix} = U_{\ell}(\ell+1) \begin{bmatrix} A_{\ell+1} \\ B_{\ell+1} \end{bmatrix} \]  \hspace{1cm} (3)

\( U_{\ell}(\ell+1) \) is the upward propagation matrix.

\[ U_{\ell}(\ell+1) = \frac{1}{2} \left( 1 + \frac{\mu_z k_{\ell}^2(\ell+1)x}{\mu_{\ell} k_{\ell} x} \right) \]  \hspace{1cm} (4)

\[ \times \begin{bmatrix} \exp[i(k_{\ell}x-k_{\ell}(\ell+1)x)d_{\ell}] & R_{\ell}(\ell+1) \exp[i(k_{\ell}x+k_{\ell}(\ell+1)x)d_{\ell}] \\ R_{\ell}(\ell+1) \exp[-i(k_{\ell}x+k_{\ell}(\ell+1)x)d_{\ell}] & \exp[-i(k_{\ell}x-k_{\ell}(\ell+1)x)d_{\ell}] \end{bmatrix} \]

where

\[ R_{\ell}(\ell+1) = \frac{1 - \frac{\mu_z k_{\ell}^2(\ell+1)x}{\mu_{\ell} k_{\ell} x}}{1 + \frac{\mu_z k_{\ell}^2(\ell+1)x}{\mu_{\ell} k_{\ell} x}} \]  \hspace{1cm} (5)

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is the reflection coefficient from $k$ to $(k+1)$ for the TE waves.

Similarly, we can express $A_{k+1}$ and $B_{k+1}$ in terms of $A_{k}$ and $B_{k}$ and define a downward propagation matrix such that

$$
\begin{bmatrix}
A_{k+1} \\
B_{k+1}
\end{bmatrix} = D_{(k+1)k} \begin{bmatrix}
A_{k} \\
B_{k}
\end{bmatrix}
$$

(6)

$$
D_{(k+1)k} = \frac{1}{2} \left[ 1 + \frac{\mu_{(k+1)} k_{x}}{\mu_{k} k_{(k+1)x}} \right] \left[ \begin{array}{cc}
\exp[i(k_{(k+1)x} - k_{(k)x} + k_{x} d_{k})] & R_{(k+1)k} \exp[i(k_{(k+1)x} + k_{x} d_{k})] \\
R_{(k+1)k} \exp[-i(k_{(k+1)x} + k_{x} d_{k})] & \exp[-i(k_{(k+1)x} - k_{x} d_{k})]
\end{array} \right]
$$

(7)

and

$$
R_{(k+1)k} = -R_{k(k+1)}^{-1}
$$

(8)

It can be shown that the product of the downward and upward propagation matrices at a boundary surface yields the identity matrix

$$
U_{k(k+1)} D_{(k+1)k} = D_{(k+1)k} U_{k(k+1)} = I.
$$

(9)

Note that the results above are derived for TE waves. For simplicity, we have omitted the superscript TE on the propagation matrices and reflection coefficients. A parallel analysis can be carried out for TM waves, simply by replacing $E$ by $H$, $H$ by $-E$, $\mu$ by $\epsilon$, and $\epsilon$ by $\mu$.

The reflection and transmission coefficients of the stratified medium can be obtained by using the propagation matrix between the incident region 0 and the transmitted region $t$, which is a product of $n$ individual propagation matrices. If we are primarily interested in the reflection coefficient $R$, we can derive a formula for $R$ directly by using the upward propagation matrix between $k$ and $(k+1)$ as expressed in Eq. 7 to find the ratio for $A_{k}/B_{k}$, written in the form of continuous fractions.

$$
\frac{A_{k}}{B_{k}} = e^{\frac{i2k_{x}d_{k}}{R_{k(k+1)}}} \frac{1 - \left( \frac{R_{k(k+1)}}{R_{k(k+1)}} \right)^{-1}}{\exp[i(k_{(k+1)x} + k_{x} d_{k})]} + \frac{A_{k+1}}{B_{k+1}}.
$$

(10)
In the transmitted region \( t \), there is no reflected wave, \( A_t/B_t = 0 \). In the incident region \( 0 \), \( A_0/B_0 = R \). Thus we have

\[
R = \frac{1}{R_{01}} + \frac{1}{R_{12}} \left( 1 - \frac{1}{R_{12}} \right) \frac{\exp(i2k_{1x}d_1)}{R_{12}} + \frac{\exp(i2k_{2x}d_2)}{R_{23}} + \frac{\exp(i2k_{3x}d_3)}{R_{23}} + \cdots
\]

This is a closed-form solution for the reflection coefficient which is true for both TE and TM fields; for the TE field, we use \( R_{TE}^\ell \), and for the TM field, \( R_{TM}^\ell \).

As an example, consider a three-layer stratified medium. The reflection coefficient, which can be calculated from (11), is

\[
R = \frac{R_{01} + R_{1t} \exp(i2k_{1x}d_1)}{1 + R_{01}R_{1t} \exp(i2k_{1x}d_1)}. \tag{12}
\]

In order to calculate the transmission coefficient, we use the upward propagation matrices for TE waves,

\[
U_{01} = \frac{1}{4} \begin{bmatrix}
1 + \frac{\mu_{k 1x}}{k} & \frac{\mu_{1z k 2x}}{k} \\
1 + \frac{\mu_{1z k 2x}}{k} & \frac{\mu_{2z k 1x}}{k}
\end{bmatrix} \begin{bmatrix}
1 & R_{01} \\
R_{01} & 1
\end{bmatrix}
\times \begin{bmatrix}
\exp[i(k_{1x} - k_{2x})d_1] & R_{12} \exp[i(k_{1x} + k_{2x})d_1] \\
R_{12} \exp[-i(k_{1x} + k_{2x})d_1] & \exp[-i(k_{1x} - k_{2x})d_1]
\end{bmatrix}. \tag{13}
\]
After multiplying out the two matrices, the transmission coefficient is equal to the reciprocal of the element at the first column and first row. Thus

\[ T = \frac{4e^{\text{i}k_2x_1d}}{\left(1 + \frac{\mu_z^1k_{1x}^2}{\mu_z^2k_{1x}^1}\right)\left(1 + \frac{\mu_z^1k_{1x}^2}{\mu_z^2k_{1x}^1}\right)} \]

Note that similarly the transmission coefficient can be calculated by using R and downward propagation matrices.

It is important to observe that with the arrangement above, the TE and TM waves are decoupled. There is no mode conversion upon reflection and transmission. Mode conversion will occur when the medium is bianisotropic; it also occurs when the principal axes are rotated an angle with respect to the z axis or the x axis. A study of mode conversion caused by material anisotropy and bianisotropy will be discussed in a subsequent report.

References


D. RADIATION PATTERNS OF ELECTRIC DIPOLES OVER ANISOTROPIC HALF-SPACE

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In geophysical subsurface probing with dipole antennas, it is of fundamental importance that the radiation patterns of the dipole antennas be clearly understood. In this report, we study radiation patterns arising from an electric dipole, either vertical or horizontal, in the presence of an anisotropic half-space medium. Let the dipole be at height \( d_0 \) above the interface of the half-space. Let the dipole moment of the dipole be \( M \). All field components are expressed as integrals.\(^1\) Using the saddle-point method, we find that the field vectors in the radiation zone are the following.
Above the interface

\[ \bar{E} = \hat{\alpha} \left( -i \frac{\mu_0}{4\pi} \right) k^2 \sin \alpha \left( e^{\frac{i\pi z_0}{d_0}} + R^{TM} e^{i k z_0} \right) \frac{e^{i k R}}{R} \]  
(1a)

\[ \bar{H} = \hat{\alpha} \left( -i \frac{\mu_0}{4\pi} \right) k \sin \alpha \left( e^{\frac{i\pi z_0}{d_0}} + R^{TM} e^{i k z_0} \right) \frac{e^{i k R}}{R} \]  
(1b)

\[ R^{TM} = \frac{\epsilon_{1z} k_z - \epsilon^{(e)}_{1z}}{\epsilon_{1z} k_z + \epsilon^{(e)}_{1z}} \]  
(1c)

\[ R = \sqrt{(z + d_0)^2 + \rho^2} \]  
(1d)

\[ \alpha = \tan^{-1} \frac{\rho}{z + d_0} \]  
(1e)

Below the interface

\[ \bar{E} = \hat{\beta} \left( -i \frac{\mu_0}{4\pi \epsilon_0} \right) a k^2 \sin \beta \left( T^{TM} e^{i k z_0} \right) \frac{i \sqrt{a} k R_1}{R_1} \]  
(2a)

\[ \bar{H} = \hat{\beta} \left( -i \frac{\mu_0}{4\pi} \right) \frac{\epsilon_{1z}}{\epsilon} a k \sin \beta \left( T^{TM} e^{i k z_0} \right) \frac{i \sqrt{a} k R_1}{R_1} \]  
(2b)

\[ T^{TM} = \frac{2 \epsilon k^{(e)}_{1z}}{\epsilon^{(e)}_{1z} + \epsilon_{1z} k_z} \]  
(2c)

\[ R_1 = \sqrt{(z + d_0)^2 + \rho^2} \]  
(2d)

\[ \beta = \tan^{-1} \frac{a \rho}{z + d_0} \]  
(2e)

In (1a) and (1b), the \( \mp \) sign in the exponential term applies to regions above the antenna (minus) and below the antenna (plus). At the plane containing the antenna and parallel to the interface, the fields are continuous because \( k = 0 \). The angle \( \beta \) is not the physical
observation angle; it is the angle of power flow in the uniaxial medium. Only the extraor-
dinary waves are excited in the anisotropic medium.

In plotting the radiation patterns, we normalize with respect to the free-space
Poynting's power $P_f = 10(kl)^2$. The gain factor, defined as

$$g(\theta) = \frac{2\pi R^2 |E_\theta H_\phi^*|}{10(kl)^2},$$

is plotted. In Fig. III-7a, we show a pattern with a vertical dipole at different heights
above the same medium. When the antenna is higher, more power radiates into upper
space. In Fig. III-7b, we compare an anisotropic medium with an isotropic medium. The
anisotropic medium is negative uniaxial, thus less power is coupled into the lower
medium.

In the case of a horizontal electric dipole, the broadside field components are purely
TE and the anisotropic nature of the medium has no effect on the radiation patterns.
The end-fire field components are primarily TM and are calculated as follows.

Above the interface

$$\vec{E}^{TM} = \hat{\alpha} \left( i \frac{ie}{4\pi\omega} \right) k^2 \cos \phi \left( e^{\frac{i k z_o}{R} - R} e^{\frac{i k d_o}{R}} \right) e^{i k R}$$

$$\vec{H}^{TM} = \hat{\phi} \left( i \frac{ie}{4\pi\omega} \right) k \cos \phi \left( e^{\frac{i k z_o}{R} - R} e^{\frac{i k d_o}{R}} \right) e^{i k R}$$

Fig. III-7. (a) Radiation patterns for a half-space anisotropic medium with
$\varepsilon_1 = 2\varepsilon_o$ and $a = 0.5$, and antenna heights $d_o = 0$, $d_o = \lambda/2$.
(b) Radiation patterns for a half-space medium with $\varepsilon_1 = 2\varepsilon_o$ and
$a = 1.0, 0.8$, and antenna height $d_o = \lambda/2$. 

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Below the interface

\[
\begin{align*}
\mathbf{E}^{TM} &= \alpha \left( \frac{16}{4\pi \omega \varepsilon_0 \varepsilon_1} \right) e^{ik_1 \cos \beta \ e} \ \text{e}^{-}\ |z_d| \ \text{e}^{i\sqrt{\alpha} k_1 R_1} \\
\mathbf{H}^{TM} &= \alpha \left(-i \frac{16}{4\pi} \right) e^{ik_1 \cos \beta \ e} \ \text{e}^{-}\ |z_d| \ \text{e}^{i\sqrt{\alpha} k_1 R_1} 
\end{align*}
\]

The radiation patterns are plotted in Fig. III-8. In Fig. III-8a, we show radiation patterns for the same anisotropic medium with different antenna heights. It is seen that for the antenna on the surface, there is a side lobe in the pattern as compared with

Fig. III-8. (a) Radiation pattern for a half-space anisotropic medium with \(\varepsilon_1 = 2\varepsilon_0\) and \(a = 2\), and antenna heights \(d_0 = 0\), \(d_0 = \lambda/2\). (b) Radiation pattern for a half-space medium with \(\varepsilon_1 = 2\varepsilon_0\) and \(a = 1.0, 0.8, 0.6\), and antenna height \(d_0 = 0\).

the pattern when \(d_0 = \lambda/2\). In the upper half-space, the higher the antenna, the more the radiated power. In Fig. III-8b we compare radiation patterns for an antenna on the surface of an anisotropic medium with that of an isotropic medium. Since the anisotropic medium as shown is negative uniaxial, and the fields are extraordinary waves, less power is coupled into the lower medium. Note also the different angles for the side lobes.

References
