FAIR-WEATHER ELECTRIC CHARGE TRANSFER BY CONVECTION
IN AN UNSTABLE PLANETARY BOUNDARY LAYER

BY

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ABSTRACT

A second-order-closure model for the upward turbulent transport of electric charge in an unstable, atmospheric, boundary layer is developed. The model identifies the surface fluxes of heat and momentum and the thickness of the "mixed layer" as the primary meteorological parameters determining the strength of this convection current. The planetary boundary layer is shown to function as a local generator, mechanically producing an electromotive force (EMF) which opposes the normal downward charge in fair weather. Numerical solutions of the model show that this EMF can apparently become as large as 130 kilovolts in cases of extreme instability, reducing the net downward current density locally by about 44%.

Two charge sources near the surface are considered for this model. The Blanchard bubbling effect applies only over the oceans, where it is shown to be less important than the electrode effect as a source for convection currents. The largest convection currents are predicted over land, where very unstable mixed layers can form due to strong solar heating during the daytime. The intense convection in these layers generates substantial EMF's with the charge provided by the electrode-effect source.

A simplified version of the turbulent electrode effect is developed for use as a charge source in the convection-current model. Asymptotic analysis is applied to the conservation equations for small ions to show that the conductivity profile in the electrode layer becomes independent of the conduction process in the limit of strong turbulent mixing. In this limit, which is found to be applicable to the atmospheric boundary layer under most circumstances, the electrode effect can be described simply by a charge-conservation equation with an externally determined, mean conductivity profile decreasing to zero at the surface.

The convection currents predicted by the second-order-closure model, together with information about the columnar resistance, should explain the hitherto poorly understood diurnal variations of electric field and current density observed at land stations. Since there are no suitable data available to test this hypothesis, experiments are proposed to fill this need.

Thesis Supervisor: Henry G. Houghton
Title: Professor of Meteorology

DEDICATION

To James H. Kraakevik,
whose pioneering work
inspired this research.

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1 INTRODUCTION

1.1 Atmospheric Electrical Background

Most of the classical theory of atmospheric electricity, embodied in the Theory of the Leaky Spherical Capacitor and the identification of World-Wide Thunderstorm Activity with the current source to charge that capacitor, was produced in the early Twentieth Century. This work is well summarized by the two excellent reviews of Chalmers (1967) and Israë1 (1970). The essential features of this theory are that there are strong generators operating in thunderstorms, which effectively pump positive charge upward into the highly conducting layers of the atmosphere, leaving the earth negatively charged, and that the rest of the atmosphere away from these storms acts simply as a passive return circuit, carrying a conduction current independent of height vertically downward from the upper atmosphere to the earth. Since the conductivity of the free atmosphere is observed to increase about exponentially with height above the surface, the height independence of the conduction current implies that the potential of the atmosphere relative to the earth increases as $(1 - e^{-z/H_\lambda})$. Here, H_λ is the conductivity scale height. Thus most of the positive charge, thought of as the upper plate of the spherical capacitor, actually resides in the lowest few kilometers of the atmosphere rather than in the ionosphere.

The part of the theory to focus on here is the description of the passive return path provided by the fair-weather atmosphere. Even after it became recognized that in the lowest few kilometers the conductivity is not a simple function of height, the idea of a vertical conduction current density independent of height was almost religiously held to. The theory was modified to allow atmospheric particles, convected upward from the surface

in the "Austausch layer", to reduce the conductivity in an irregular and variable manner, more strongly over land than over oceans, and especially over industrial or densely populated regions. But a compensating build-up of space charge was always presumed to exist in regions of vertical conductivity gradient, increasing the electric field in such a way as to maintain the uniformity of the conduction current.

There is still good reason to assume the uniformity and vertical direction of the current density $\vec{J}$ in the lower atmosphere, at least after time averaging over periods of several minutes. $\vec{J}$ remains nearly vertical because the fair weather space charge density ρ is small, so that the atmospheric electric field $\vec{E}$ sees only the surface charge density on the earth and the net space charge in fairly thick layers of atmosphere, which tend to look horizontally uniform. The near uniformity of $\vec{J}$ is due to the existence of an approximate steady state, $0 \approx \frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}$, making it divergenceless. Even if the assumed steady state breaks down instantaneously, it certainly holds for reasonable time averages, since the electrical relaxation time of the atmosphere is always less than an hour, and usually much less.

However, although it may be correct to assume the uniformity and vertical direction of the total current density, it is not correct to assume that this is all conduction current. In the presence of the substantial vertical gradients of space charge density that can exist in the Austausch layer, and especially near the surface, there is ample opportunity for turbulence and convection to set up significant mechanical transfer currents. The first recognition of this important fact seems to be due to Kraakevik and Clark, who, in a series of papers [Clark (1957), Kraakevik (1958a), Clark (1958), Kraakevik (1958b), Kraakevik and Clark (1958), and

Kraakevik (1961)], reported on measurements from an airplane of vertical profiles of conductivity λ and vertical electric field E up to 6 km elevation. Profiles of λ , E , and ρ (derived from $\rho = \epsilon_0 \frac{dE}{dz}$) for flights over the Greenland ice cap and over the Chesapeake Bay are presented in figure 1.1.

The vertical conduction current is defined as $J_\lambda \equiv \lambda E$. From the near-exponential behavior of λ and E in the Greenland profile, which was taken on a calm, clear, pre-dawn morning under extremely stable conditions, it is clear that J_λ is very nearly independent of height. However, the conduction current density derived from the Chesapeake Bay sounding, taken on a sunny, summer afternoon with convection in progress in a highly polluted area, and presented in figure 1.2, is clearly not independent of height.

If we make the steady-state assumption that the total vertical current density J is independent of height, we can define the difference between it and the conduction current density as the mechanical-transfer current density $J_m \equiv J - J_\lambda$. This can be evaluated from the J_λ profile if we take for J the value of J_λ above the Austausch layer in the region where J_λ is nearly independent of height. Thus we see that the convection current is carrying positive charge upward and opposing the conduction current.* In order for J to remain uniform, J_λ must become more negative in the Austausch layer. Kraakevik (1961) indicates that J_m is generally positive over most land, and even most ocean, surfaces. He estimates that at 15 m height J_m averages about -54% of J over the earth.

It was suggested earlier that these vertical convection currents might be due to transport of space charge by turbulent or convective

* The sign convention throughout this thesis will be positive upward, contrary to the tradition in atmospheric electricity, where the downward, fair-weather direction of J and E is considered positive.

Fig.1.1 Conductivity, potential gradient, and space charge density profiles for the Greenland and the Chesapeake Bay soundings, from Clark (1958).

Fig.1.2 Potential gradient and conduction current density profiles for the Chesapeake Bay sounding, from Kraakevik and Clark (1958).

mixing. Returning for a minute to the old theory, we see that a uniform conduction current flowing down a conductivity gradient implies a positive "conduction space charge":

$$\rho_c = \epsilon_0 \frac{dE}{dz} = \epsilon_0 J_\lambda \frac{d}{dz} \left(\frac{1}{\lambda} \right) = -\frac{\epsilon_0}{\lambda^2} J_\lambda \frac{d\lambda}{dz}$$

A comparison of ρ and λ in figure 1.1 for the Chesapeake Bay profile shows that the regions of strong $\frac{d\lambda}{dz}$ are in fact accompanied by strong positive ρ , which might be convected down its gradient to cause J_m . But comparison of ρ from figure 1.1 with J_λ from figure 1.2 suggests that at some levels the convection current may be acting to accumulate charge rather than to spread it out. There is a conspicuous lack of negative convection current anywhere in the profile. There is also an excess of positive charge in the lowest kilometer, unexplained by the conductivity gradient. All these observations suggest that there is more going on than simple down-gradient diffusion of the conduction space charge. Since there is presumably a source of positive charge at the surface due to the electrode effect (and the bubbling effect also [Blanchard (1966)] over the ocean), some of the observed charge density may be due to mechanical transport of this charge upward.

Evidently, the situation in the Austausch layer is complex. When a field exists in a region of non-uniform conductivity, the current flow approaches a steady state by building up space charge distributions tending to make the conduction current uniform. But in the presence of air motion, these space charges are moved about, setting up convection currents that tend to destroy the steady state, and demanding that the conduction currents become non-uniform again. Even the conductivity gradients themselves

depend on the field of motion, since the cause of the deviations of λ from the theoretical exponential profile is primarily the mixing of condensation nuclei upward from the surface to act as recombination sites for small ions.

The modification of the conductivity by nuclei in the surface layers was first investigated by Calahan (1951), Sagalyn and Faucher (1954), and Sagalyn and Faucher (1956), who measured the numerous profiles of λ , charged nucleus count, temperature, and humidity between 700 and 15,000 ft elevation at all times of day and during all seasons over southern New Hampshire. They identified the meteorological "mixing layer", defined by a near-adiabatic lapse rate with a sharp decrease in temperature lapse rate at the top, uniform specific humidity, and frictional or convective turbulent motions, with the electrical Austausch layer, defined by lower than normal conductivity deviating from the exponential profile and a high charged nucleus count. They established a typical diurnal variation of charged nucleus count N_+ , Austausch layer height H , and atmospheric conductivity that could be causally related to turbulent mixing.

Sagalyn and Faucher showed that turbulent transport was responsible for carrying the nuclei up into the atmosphere to establish an Austausch layer of the same height as the mixing layer, that the nuclei caused the observed decrease in conductivity according to a simple model presented in Appendix 1, and that, after the mixing layer had subsided, the strength and height of the decaying Austausch layer were reduced by coagulation of nuclei and by subsidence. All this leads to a typical diurnal variation of columnar resistance $R_z \equiv \int_0^z \frac{dz}{\lambda}$ in synchronism with the variations of N_+ , H , and λ , as shown in figure 1.3. A typical 24-hour series of profiles is shown in figure 1.4.

The importance of the electrical phenomena discussed above on the

Fig.1.3 Diurnal variations of the charged-nucleus concentration, conductivity, and average resistance of a 1 cm^2 column from the surface to 15,000 ft, from Sagalyn and Faucher (1956).

Fig.1.4 Time variations of the profiles of negatively charged nucleus concentration, conductivity, and temperature over southern New Hampshire, from Sagalyn and Faucher (1956).

local atmospheric electric circuit is considerable. Simplest to consider is the fact, stated by Sagalyn and Faucher, that in southern New Hampshire the Austausch layer alone is found to contribute 40% to 73% of the total columnar resistance R_{∞} of the atmosphere. Since that layer undergoes considerable diurnal variations, it is not surprising that the total columnar resistance executes a diurnal variation of about 30%. Within the somewhat modified framework of spherical capacitor theory, Pierce (1958) has evaluated the effect of such a diurnal variation with local time over all land areas of the globe on the total effective resistance of the global circuit return path. He was able to show that it probably amounts to a perturbation of only about 4%. Of course, a 30% variation in columnar resistance is large enough to entirely wipe out the local effect of the presumed diurnal variation in supply current, as Kraakevik (1961) claims to have observed in his soundings over Florida. Thus, it could cause a considerable variation in the distribution of J over the globe, though it should not significantly influence the "ionospheric potential" $V_{\infty} \equiv -\lim_{z \rightarrow \infty} \int_0^z E \, dz'$.

Much more important and interesting is the fact, entirely overlooked by Pierce (1958) and the spherical capacitor theory, that the existence of convection currents in the Austausch layer implies a local generator just as real as the thunderstorm generator. Let us consider the effect of some distribution of vertical convection current $J_m(z)$ and conductivity $\lambda(z)$. Assuming a steady state, this implies $J_{\lambda}(z) = J - J_m(z)$, where J is independent of height. From the corresponding ionospheric potential and total columnar resistance, we can define a uniform "passive conduction current" $J_p \equiv -\frac{V_{\infty}}{R_{\infty}}$, which is the current that would flow in the absence of convection. Subtracting J_p from J and using Ohm's Law, we find the "convective generator current":

$$J_g \equiv J - J_p = \lim_{z \rightarrow \infty} \left[\int_0^z \frac{J_m(z')}{\lambda(z')} dz' \right] R_\infty^{-1} \quad (1.1)$$

Clearly, if $J_m(z)$ is positive, as in the Chesapeake Bay sounding, J_g can be viewed as a uniform positive current flowing upwards from the earth to the upper atmosphere, driven by the Austausch generator! Here, the electro-

motive force (EMF) of this Austausch generator is $(EMF) = \lim_{z \rightarrow \infty} \int_0^z \frac{J_m(z')}{\lambda(z')} dz'$.

It is interesting to note that, for a given magnitude of J_m , J_g will be larger in proportion to the fraction of R_∞ spanned by the convection current. Since we have already seen that the Austausch layer frequently spans a large fraction of R_∞ and that J_m can be comparable to $|J|$ in that layer, we expect that J_g can often be of the same order of magnitude as J . This means that the Austausch generator has the potential of being a major contributor to the local atmospheric electric circuit.

Dr. Robert V. Anderson of the Naval Research Lab has kindly made available the processed data from the aircraft soundings of Kraakevik and Clark. From these, one example has been selected to illustrate a shallow but actively convecting mixed layer. This sounding was made over Chincoteague Island, Virginia, between 10:00 and 11:00 am, EST, on May 17, 1956. Figure 1.5 shows the measured profiles of conductivity λ and vertical electric field E on a semi-logarithmic scale. Notice the nearly uniform conductivity in the lowest 600m topped by a sudden increase in conductivity above. The temperature profile in figure 1.6 confirms the implication that there is a growing unstable layer there capped by an inversion, typical of fair-weather morning conditions in the summer over land.

The conduction current $J_\lambda \equiv \lambda E$ is also shown in figure 1.6. Notice

Fig.1.5 Electric field and conductivity profiles from the Chincoteague sounding of Kraakevik and Clark.

Fig.1.6 Temperature and conduction current density profiles for the Chincoteague sounding of Kraakevik and Clark. The total current density estimated from the profiles above 3 km is also indicated.

that it is substantially more negative in the mixing layer than above. If we take the total current J as about $-1.5 \times 10^{-12} \frac{\text{amps}}{\text{m}^2}$, the fairly uniform value of J_λ above 3 km, J_λ at 15 m altitude (the lowest measurement) is about $-4.5 \times 10^{-12} \frac{\text{amps}}{\text{m}^2}$, or three times J . This implies, with the assumption of a quasi-steady state, that the convection current is upward at about $3.0 \times 10^{-12} \frac{\text{amps}}{\text{m}^2}$ there, or twice $|J|$. By the top of the active layer at 600 m, this convection current has fallen to only about $0.4 \times 10^{-12} \frac{\text{amps}}{\text{m}^2}$. [However, it is not easy to explain why it does not seem to fall to zero until almost 3 km in this sounding.]

A profile of charge density can be calculated from the profile of E using Gauss's law. Although differentiation of E greatly magnifies the fractional error in the observations, it is easy to see from the profile in figure 1.5 that there is considerable positive charge throughout the lowest 1.2 km. Although the charge between 0.6 km and 0.9 km can be explained by the rapid increase in conductivity there, the only explanation for that between the surface and 600 m is convergence of the convection current. Furthermore, it appears that the convection current in this layer may not even be down the gradient, since there is no definite charge density gradient between 75 m and 600 m.

This profile was selected because it seemed to be the most clear-cut example of a mixing layer. Although it is only about 600 m thick, integration of the implied convection current through this layer in the presence of the measured uniform conductivity gives a convection-induced

EMF of about 65 kV. This is to be compared with an ionospheric potential V_{∞} of around 300 kV.

As a result of his observational work on convection currents in the lower atmosphere, Kraakevik (1958b) concluded that in at least one case there was a large convection current throughout the mixing layer which could not be explained by an eddy-diffusion model. He went as far as suggesting that there are "thermal chimneys of convection which carry charge upward from a region rich in positive charge near the surface and distribute it horizontally at the top of the secondary exchange layer, from which it is conducted downward."

In spite of this auspicious beginning to the study of convection currents in the lower atmosphere, little further observational or theoretical work seems to have been done on the subject. The reason for this is probably that there has been a lack of understanding of the nature of turbulent transport in an unstable layer of the atmosphere until fairly recently. Two experimental studies are worthy of note. Vonnegut, et al. (1962) showed that large concentrations of space charge generated artificially near the surface could be convected up into small, fair-weather cumulus clouds. Kohl (1968) found that the fluctuations of horizontal electric field in the mixing layer were much larger than those of the vertical component or than those of either component of the field above the mixing layer. The latter author concluded that his results were consistent with the presence of tall, narrow, vertical columns of positive space charge within the mixing layer. This is reminiscent of Kraakevik's earlier speculations.

On the theoretical side, the only studies attempted have been based on eddy-diffusion models, which have already been suggested to be

inadequate. Hoppel (1969), Hoppel and Gathman (1971), and Latham and Poor (1972) have reported on two numerical models of the electrode effect, including turbulent mixing. Although there is some question about the lower boundary conditions used by the latter authors, both these models agree that the convection currents, modeled this way, do not extend above a couple of hundred meters under normal atmospheric conditions. Presumably, it is the eddy-diffusion representation of turbulent transport in these models that prevents them from giving the deep convection currents actually found in observations of unstable planetary boundary layers.*

It would seem that the next step in understanding convection currents is to replace the eddy-diffusion approximation in an electrode-effect model like Hoppel's with a more realistic model of turbulent transport. However, it is apparent even from Hoppel's results that it is not necessary to carry separate equations for the individual ion species in the interior of the mixing layer. The reason for this is that, away from the immediate influence of the ground, the charge density is so small compared to the conductivity that the concentrations of ions of opposite sign are nearly equal.

In Appendix 1, the electrical equations for ionic conduction in a gas are given in rather general form and then simplified for use in the atmosphere. An argument is given there justifying the use of a single charge-conservation equation (A1.4) everywhere except in the immediate vicinity of the ground. This equation is

$$\frac{D\rho}{Dt} = -\nabla \cdot \lambda \underline{E} + Q \quad (1.2)$$

* The electrode effect as a charge source for these convection currents is discussed in considerable detail in Chapter 5.

where the substantial derivative $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{U} \cdot \nabla$ * includes any convection current, the conduction current is $\vec{J}_{\lambda} \equiv \lambda \vec{E}$, and Q is the net volume source strength of charge due, for example, to the Blanchard bubbling effect discussed in Chapter 4. Here, the distribution of λ must be known externally and is not modified significantly by the charge distribution or the conduction process.

In Appendix 2 a very simple model is developed, inspired by the speculations of Kraakevik and of Kohl and based on equation (1.2), to show crudely what kind of convection currents can be obtained from vertical columns of rising air. Before this "convective-column model" is discussed, however, we will digress to a general description of convection in the planetary boundary layer and the meteorological parameters used to describe it.

1.2 Convection in the Mixed Layer

The planetary boundary layer, interpreted as the layer of the atmosphere adjacent to the ground that is directly influenced by the surface fluxes of heat, moisture, and momentum, can assume very different forms depending on its stability. The stability of this layer, in turn, is controlled by these surface fluxes. Although the vertical moisture flux is important in driving convection in some circumstances, especially over the tropical oceans, we will refer here only to the vertical heat flux. It will be shown in Chapter 2 that the effects of moisture can be accounted for simply by substituting virtual temperatures for temperatures wherever they appear in the profiles or fluxes.

* $\vec{U}$ is assumed divergenceless ($\nabla \cdot \vec{U} \equiv 0$) throughout this thesis.

If the heat flux is downward, the resulting stable boundary layer tends to be thin, and the mixing processes are inhibited. If the heat flux is small enough relative to the momentum flux, the boundary layer has neutral stability and is characterized by an Ekman-like, spiral velocity profile. Neutral boundary layers are usually described by some form of mixing-length theory. They will be considered in this thesis only as the limiting form of an unstable boundary layer as the heat flux vanishes.

Perhaps the simplest form of the planetary boundary layer, and the form which will receive almost all the attention in this thesis, is the unstable boundary layer resulting from a strong upward heat flux. Such a layer has recently become known as a "mixed layer", and will be referred to as such henceforth. Mixed layers occur in the atmosphere when the ground is strongly heated by solar radiation, when cool air moves over a relatively warm surface, or when there is strong evaporation at the surface, as over the tropical oceans. Above a thin "surface layer", in which there are strong vertical gradients of temperature and momentum, mixed layers are characterized by a region of nearly uniform potential temperature and wind velocity which may be as much as three kilometers thick in extreme cases. This strongly turbulent region is usually capped by a fairly sharp temperature inversion, separating it from the relatively stable and quiescent air above.

Mixed layers over land generally form in the early morning as solar radiation begins to heat the ground. Turbulence is produced primarily by buoyancy in the superadiabatic surface layer and in the near-adiabatic interior of the mixed layer and is exported upward. This turbulence appears to take the form of plume-like structures of warm air rising

through an environment which is also turbulent but shows little temperature fluctuation. Many of these buoyant elements, which may be organized into larger structures such as roll vortices, probably make it all the way to the top of the adiabatic layer, entraining considerable environmental air on the way.

When these buoyant elements collide with the stable air capping the adiabatic region, they have enough momentum to penetrate a short distance and fall back. This "penetrative convection" is accompanied by some entrainment of the warmer air from above, which is carried back down with the buoyant elements into the mixed layer. Remembering that turbulent kinetic energy exported from the lower part of the mixed layer is converging on the boundary with the stable air above, one should not be surprised that some of this kinetic energy is converted into potential (and internal) energy, driving a downward heat flux at the top of the mixed layer.

During the morning and early afternoon of a summer day over land, the mixed layer eats into the stable atmosphere by this entrainment process, growing deeper at a rate that depends on the stability of the overlying air and the strength of the surface heat flux. Although the layer is being warmed continually by the upward heat flux from the surface and the smaller downward heat flux from above, the temperature of the air immediately above it is also increasing as the top of the layer rises. These competing processes develop and maintain a sharp inversion capping the mixed layer.

The mixed layer continues to grow deeper until either the surface heat flux turns downward, shutting off the buoyant production of turbulent kinetic energy, or the top of the layer approaches the lifting condensation

level. In the latter case, some of the buoyant elements penetrating the inversion overshoot high enough to condense. If the air above the inversion is conditionally unstable, some of these condensing elements will penetrate high enough of their own momentum to become buoyant again and rise some distance above the mixed layer. Since these runaways do not fall back into the mixed layer, the clouds represent a net mass flux upward through the inversion, which must be accompanied by subsidence of the environment. In this situation, a balance will develop between the growth rate of the mixed layer and the subsidence around the clouds. The subsidence will hold the inversion far enough below the lifting condensation level that the upward mass flux in the penetrating elements that became clouds just balances the downward mass flux in the environment. This appears to be the situation that obtains in the fair-weather trade-wind regime over the tropical oceans.

In this thesis, attention will be limited to dry convection as described above. Situations where large cumulus or cumulonimbus clouds form above the lifting condensation level will not be considered. The structure of the cloud layer which may be present above the mixed layer inversion in fair weather will also be ignored. The qualitative description of mixed layers given above is derived from numerous papers in the recent literature, of which a few are listed here: Warner and Telford (1967), Clarke (1970), Lenschow (1970, 1974), Wyngaard and Coté (1971), Wyngaard, Coté, and Izumi (1971), LeMone (1972), Wyngaard (1973), Augstein, et al., (1974), Deardorff (1974a, 1974b), Willis and Deardorff (1974), and Sarachik (1974).

In order to deal quantitatively with mixed layers, we must introduce some terminology. First, attention will be focused on the so-called "surface layer", defined as that layer of the atmosphere immediately above

the surface through which the fluxes of heat, moisture, and momentum change by only a negligible fraction of their surface values. In a quasi-steady state over flat terrain this layer may be 10 to 100 meters thick.

Following the very clear treatment by Wyngaard (1973), we observe that the surface layer should be characterized by four parameters: u_* , T_* , $\frac{g}{T_0}$, and z . Here $u_* \equiv (-\overline{u'w'}|_0)^{1/2}$ is the friction velocity, defined in terms of the turbulent momentum flux at the surface, $T_* \equiv -\frac{1}{u_*} \overline{w'\theta'}|_0$ is a temperature fluctuation scale, defined in terms of the surface heat flux, $\frac{g}{T_0}$ is the so-called buoyancy parameter, expressing how the acceleration is related to the temperature perturbation, and z is the height above the ground. Other parameters, such as z_0 , the surface roughness length, H , the thickness of the entire planetary boundary layer, ν , the kinematic viscosity of air, and κ , the thermometric conductivity of air, do not enter into this group of four because of various limiting processes. For example, z_0 does not enter because in the bulk of the surface layer $z_0 \ll z$.

Following what has become known as the Monin-Obukhov similarity hypothesis, we predict that any fifth physical variable can be expressed in the surface layer as a function of only the four parameters listed above. Dimensional analysis tells us that only two independent, dimensionless groups can be formed from such a collection of five variables, and that the two must be functionally related. Looking at this from another point of view, we observe that only four independent scales can be formed from the collection u_* , T_* , $\frac{g}{T_0}$, and z . Conventionally, these scales are chosen as u_* , T_* , z , and $L \equiv \frac{u_*^2 T_0}{kg T_*}$ — one velocity scale, one temperature scale, and two length scales. Here, k is a universal constant, called the von Karman constant, equal to approximately 0.35 and included in the definition of L by convention. So far, L is only a combination

of three of the four governing parameters that has the dimensions of length. It is called the Monin-Obukhov length, and it will be given a physical interpretation shortly.

Since any fifth physical variable can now be non-dimensionalized with u_* , T_* , and z , its dimensionless form must be a function of the only remaining independent dimensionless group, chosen as z/L by convention. This extremely powerful prediction has been verified by careful observations in the surface layer of the atmosphere over a fairly wide range of z/L . Businger (1973) presents considerable data and determines some of these universal functions of z/L . For example, the dimensionless profile of wind shear in the surface layer is given as

$$\frac{kz}{u_*} \frac{\partial \bar{u}}{\partial z} = \begin{cases} (1 - 15 \frac{z}{L})^{-1/4}, & \frac{z}{L} < 0 \\ 1 + 4.7 \frac{z}{L}, & \frac{z}{L} > 0 \end{cases} \quad (1.3)$$

where $\bar{u}$ is the mean wind speed.

Businger (1973) also presents a simple physical interpretation of the Monin-Obukhov length. Consider the flux Richardson number

$$R_f \equiv \frac{g}{T} \frac{\overline{w'\theta'}}{\overline{u'w'} \frac{\partial \bar{u}}{\partial z}} \quad (1.4)$$

defined as the ratio of buoyant production to shear production of turbulent kinetic energy. If the overbars are interpreted as horizontal averages, then this ratio applies to a particular height. Since $\overline{w'\theta'} = -u_* T_*$ and $\overline{u'w'} = -u_*^2$ are both uniform in the surface layer, the only problem in (1.4) is the mean wind shear $\frac{\partial \bar{u}}{\partial z}$. Reference to (1.3) suggests that $\frac{\partial \bar{u}}{\partial z}$ can be replaced by $\frac{u_*}{kz}$ for small enough z . With this substitution, the modified flux Richardson number is recognized as z/L :

$$R_f \approx \frac{g}{T_0} \frac{kz T_*}{u_*^2} = \frac{z}{L} \quad (1.5)$$

Businger goes on to show that (1.5) is a good approximation in unstable conditions, when $\frac{z}{L} < 0$, but a poor one in stable conditions, when $0 < \frac{z}{L}$. Thus, we can assume in this thesis that $\frac{z}{L} \approx R_f$.

Equation (1.5) gives us a physical interpretation of the parameter L . It is the height in the unstable surface layer at which shear production and buoyant production of turbulent kinetic energy are approximately equal. (Actually, as Businger (1973) points out, this equality occurs closer to $z = -0.57 L$.) Since buoyant production remains roughly constant with height, while shear production decreases rapidly with height, the latter dominates below $z \approx -L$ and the former, above. The minus sign arises because L is negative in unstable stratification.

Thus we see that L is not really a length scale characteristic of the turbulence, as is the height z . Instead it is simply a parameter characterizing the budget of turbulent kinetic energy. It turns out that most of the wind shear and vertical (superadiabatic) temperature gradient in the unstable surface layer is confined within a few $|L|$ of the surface.

In a strongly unstable surface layer $-L$ may become as small as 1 m, or even smaller. If we are interested in the mixed layer confined below an inversion at height H , we may therefore have $-L \ll H$. This suggests a limiting process in which $\frac{z}{H}$ replaces $\frac{z}{L}$ as the argument of the universal, non-dimensional profile-functions for the interior of the mixed layer. (This is analogous to the limiting process that made z_0 irrelevant in the interior of the surface layer, though it is certainly important very near the ground where $z \approx z_0$.) However, we cannot throw L away entirely, because it still tells us something about the relative importance of shear and buoyancy in the production of turbulent kinetic

energy. The introduction of the new mixed-layer parameter H will thus result in the replacement of $\frac{z}{L}$ with $\frac{z}{H}$ and the addition of $\frac{H}{L}$ as a new non-dimensional parameter.

By consideration of the budget equation for turbulent kinetic energy averaged vertically over a mixed layer, excluding the surface layer, Tennekes (1970) has shown that the magnitude of the vertical velocity fluctuations should be given by

$$\frac{\langle \overline{w'^2} \rangle}{u_*^2} \approx 0.4 \frac{w_*^2}{u_*^2} \equiv 0.4 \left(-\frac{H}{kL} \right)^{2/3} \quad (1.6)$$

where the angle brackets represent this vertical average as in Appendix 2. w_* , defined here, is a new velocity fluctuation scale, which can evidently become large compared to u_* when $-L \ll H$. This means that in a strongly unstable mixed layer the magnitude of the turbulent kinetic energy in the interior is large compared to the magnitude of the stress. It is this high value of turbulent energy produced by buoyancy which is responsible for the near-uniformity of potential temperature, momentum, and indeed, any conservative property, throughout the interior that gives the mixed layer its name. Equation (1.6) has been verified by measurements in the real atmosphere and by the detailed numerical calculations of Deardorff (1974b). The latter also show that the shape of the profile of $\overline{w'^2}$ does seem to depend only on $\frac{z}{H}$.

The introduction of the new parameter H for the mixed layer raises the question of why the Ekman depth $\frac{u_*}{f}$ has not also been included, where f is the Coriolis parameter. In a neutral planetary boundary layer, the characteristic depth is $\frac{u_*}{f}$, which is frequently on the order of kilometers in mid-latitudes. Since this is of the same order as typical values of H , one might expect that $\frac{u_*}{f}$ would be another important parameter

in the mixed layer. In fact, Tennekes (1970) takes the depth of the unstable "Ekman layer" to be proportional to $\frac{u_*}{f}$, a mistake which he later corrects.

It turns out that $\frac{u_*}{f}$ is not a relevant parameter in the mixed layer [Deardorff (1974a)] because of the large turbulence levels indicated by (1.6). $\frac{u_*}{f}$ is only a relevant length scale insofar as f^{-1} is a relevant time scale in the planetary boundary layer. In the mixed layer, however, the largest turbulence time scales are of the order of $\frac{H}{w_*}$, the turn-over time for the largest buoyant elements. Forming a ratio of these two time scales, we find

$$\frac{f^{-1}}{H/w_*} = \frac{u_*/f}{H} \frac{w_*}{u_*} \sim \left(-\frac{H}{kL}\right)^{1/3} \quad (1.7)$$

Here we have assumed that $\frac{u_*}{f}$ and H are of the same order. In a typical mixed layer, $125 \lesssim \frac{-H}{kL}$ so that $5 \lesssim \frac{w_*}{u_*}$. Evidently, the Coriolis acceleration ceases to be important as $\frac{-H}{kL}$ increases.

Because of the simplification discussed above due to the large turbulence intensity expressed in equation (1.6), it is possible to construct a very simple model of the growth of a mixed layer and the inversion capping it during the day. The original analysis of the growth of a mixed layer was performed by Ball (1960). He made the following assumptions based on observations of growing mixed layers: 1) the temperature lapse rate in the mixed layer is dry-adiabatic, 2) the mechanical generation of turbulent kinetic energy is negligible compared to the buoyant generation, 3) radiative heating and cooling are negligible within the layer, 4) the temperature inversion capping the mixed layer is a perfect discontinuity of strength $\delta\theta$ (this is a considerable idealization but is qualitatively correct), and 5) viscous dissipation of turbulence within the layer is

negligible compared to conversion of kinetic to potential energy by downward turbulent buoyancy transport at the inversion. This last assumption appears to be considerably in error in the light of more recent work, but again is qualitatively correct. With these assumptions, Ball was able to derive relationships between the surface heat flux, the temperature lapse rate in the stable layer just above the inversion, the inversion strength $\delta\theta$, and the rate of deepening of the mixed layer. He showed that these relationships compared well with the observations then available to him.

Recently this type of model, called a "jump model" because of the discontinuity at the inversion, has been carried further by Tennekes (1973). For assumption 5) above, he substituted the following argument. At the inversion, the budget equation for turbulent kinetic energy, where the local time derivative, shear production, and molecular dissipation are neglected according to assumptions made clearer in Zilitinkevich (1974) and Tennekes (1974), can be written as

$$\frac{g}{T_0} \overline{\theta'w'} \Big|_i \approx \frac{\partial}{\partial z} \left(\frac{1}{2} \overline{q'^2 w'} + \frac{1}{\rho_a} \overline{p'w'} \right) \Big|_i \quad (1.8)$$

The flux convergence term on the right hand side of (1.8) can be estimated as

$$\frac{\partial}{\partial z} \left(\frac{1}{2} \overline{q'^2 w'} + \frac{1}{\rho_a} \overline{p'w'} \right) \Big|_i \sim - \frac{(\overline{w'^2})^{3/2}}{H} \quad (1.9)$$

Using (1.6) and (1.9) in (1.8) gives

$$\overline{\theta'w'} \Big|_i \sim - (0.4)^{3/2} \frac{u_*^3 T_0}{g H} \left(-\frac{H}{kL} \right) \approx -0.2 \overline{\theta'w'} \Big|_0$$

Accordingly, Tennekes replaces assumption 5) with the relation

$$\overline{\theta'w'} \Big|_i = -0.2 \overline{\theta'w'} \Big|_0 \quad (1.10)$$

The choice of the factor -0.2 in equation (1.10) has been the subject of considerable debate. However, numerical models by Deardorff (1974a) and by Wyngaard and Coté (1974), laboratory experiments such as those by Willis and Deardorff (1974), and aircraft observations in the real atmosphere such as those of Lenschow (1970, 1974) all give small negative factors of the same order. It should be noted that Ball's assumption 5) implies a factor of -1.0 instead of -0.2 . Although this may not seem to be a large change, it corresponds to a major change in the turbulence energy budget. Ball assumed that all the energy produced by buoyancy goes into potential (and internal) energy by downward heat transport at the inversion. Tennekes' assumption in equation (1.10), on the other hand, implies that only about four percent of the energy is so converted, the rest being dissipated in the interior of the mixed layer.

Using (1.10), Tennekes (1973) produced equations predicting the time evolution of the inversion height H and strength $\delta\theta$ (and, therefore, the potential temperature of the mixed layer as well) from only the time history of the surface heat flux and the profile of temperature in the morning before the mixed layer appeared (or the time history of the temperature lapse rate just above the inversion). Equations like these have since been used successfully by Deardorff (1974a) to predict the growth of the mixed layer in his numerical model.

It was suggested early in this section that the mixed layer might turn out to be very simple, in spite of the relatively unknown complexities of the turbulence structure in buoyant convection. The fact that the evolution of the layer can apparently be completely described by a jump model confirms this remarkable simplicity, which has been shown to be due to the powerful mixing associated with strong buoyant production

of turbulent kinetic energy. Evidently, the only non-dimensional parameter necessary to characterize the structure of the mixed layer is $\frac{H}{L}$. The profiles of mean variables and turbulence statistics in the layer, non-dimensionalized with u_* , T_* , and H , become functions only of $\frac{H}{L}$ and $\frac{z}{H}$. This conclusion will be echoed in Chapter 2, when second-order-closure models are discussed and it is shown that the only parameters appearing in the dimensionless equation for charge flux are $R \equiv \frac{u_* \tau_\lambda}{H}$ and $\frac{H}{L}$. R is defined here as the ratio of an electrical relaxation time $\tau_\lambda \equiv \frac{\epsilon_0}{[\lambda]}$, where $[\lambda]$ is a scale for the conductivity, to a turbulence time scale $\frac{H}{u_*}$.

One other point is worth emphasizing here. The heat flux falls off linearly throughout the whole depth of the mixed layer, although the vertical potential temperature gradient changes, over a vertical distance of just a few $|L|$, from strongly superadiabatic in the surface layer to essentially zero in the interior. Clearly, the heat flux is not proportional, or even closely related, to the local temperature gradient in the bulk of an unstable planetary boundary layer. Similar conclusions apply to the momentum flux, as discussed by Wyngaard, Arya, and Coté (1974). Therefore, it is clear that a gradient diffusion hypothesis cannot be used to model vertical transports in such a layer.

1.3 Modeling the Convection Current

Since an eddy-diffusion model of the convection current cannot be used in the unstable planetary boundary layer, it is necessary to find a more complicated model which is adequate. The model eventually adopted for this thesis is the second-order-closure model described in Chapter 2. Before developing the relatively complex closure model, however, we should make some estimates of the magnitude and behavior of the electrical effects to be expected from a model of charge convection in the mixed layer. To

do this, we return to the convective-column model mentioned at the end of section 1.1 and developed in detail in Appendix 2. This is considered to be the simplest possible model of convective transport which is non-local and exhibits some of the important features of the structure of the mixed layer.

The convective-column model supposes that there are vertical columns of rising air extending through a gradually subsiding environment all the way from just above the surface to a rigid lid at height H . These columns rise at a uniform velocity and do not interact with their environment until they suddenly detrain their contents at height H to become the subsiding environment. A crude argument about the time scale for the dissipation of turbulent kinetic energy is used to connect the upward velocity of the columns and the fractional horizontal area covered by them to the stability parameter $\frac{H}{L}$ for a real mixed layer. The layer is assumed to have uniform conductivity λ_0 , giving rise to a second parameter $R \equiv \frac{u_* \tau_\lambda}{H}$, where $\tau_\lambda \equiv \epsilon_0 / \lambda_0$. Equations (A2.15) and (A2.16) summarize the non-dimensional results of the convective column model, showing that they can be expressed entirely in terms of the two dimensionless parameters R and $\frac{H}{L}$.

The noteworthy result of Appendix 2 is a graph in figure A2.1 of the dimensionless electromotive force, generated by the convection current, as a function of $\frac{H}{H_c} = 0.8 R^{-1} \left(\frac{-H}{kL} \right)^{-1/3}$. Here, the dimensionless EMF is defined as

$$\frac{\lambda_0}{\rho_* u_* H} (EMF) \equiv \int_0^1 \frac{\overline{\rho' w'}(\xi)}{\rho_* u_*} d\xi \quad (1.11)$$

where (EMF) is dimensional, $\rho_* \equiv \frac{1}{u_*} \overline{\rho' w'}|_0$ is a charge scale defined in terms of the surface charge flux in analogy with T_* , $\overline{\rho' w'}$ is the

turbulent charge flux or convection current, and $\zeta \equiv \frac{z}{H}$ is the dimensionless height. H_c is a scale height for the charge density in the model. The (EMF) has the same meaning here as in section 1.1. Its maximum dimensionless value of 0.570 occurs for $\frac{H}{H_c} = 0.573$. Before we can interpret this value dimensionally, however, we must choose values for the various scale factors, including ρ_* .

Choosing ρ_* brings up the question of the sources of charge for convection currents. In order to use the results of the simple, convective-column model, we must know the convection current supplied to the model at the bottom boundary, which is taken to be close to the ground. There are two sources of charge available near the surface in the fair-weather atmosphere: the Blanchard effect and the electrode effect. The former operates only over the ocean, where the bursting of small bubbles produced by whitecapping in sea water has been shown to eject small, positively-charged droplets into the air by Blanchard (1963). This effect as a charge source for the convection current is considered in more detail in Chapter 4. For the present, we will use the value $\overline{\rho'w'}|_0 = 1.07 \times 10^{-12} \frac{\text{coul}}{\text{m}^2\text{sec}}$, which is the maximum figure from the longitudinally and seasonally averaged charge fluxes, appropriate to 50°S latitude for June-August, given by Blanchard (1963). This is regarded as a reasonable value for the surface charge flux due to the Blanchard effect over the open ocean under conditions of moderate wind.

The second available charge source, the electrode effect, is a result of ionic conduction in a gaseous medium near a conducting boundary. The normal fair-weather electric field is directed perpendicularly downward toward the earth's surface and causes positive ions to drift downward and negative ions to drift upward. Since the latter cannot escape from the

surface, the layer of air nearest the ground is depleted of negative ions and develops positive space charge. It is not easy, however, to estimate the resulting convection current as this space charge is mixed upward by turbulence, except to say that it cannot be larger than the downward conduction current, which ultimately provides the positive charge for the electrode effect.

A scheme has been contrived for obtaining an estimate of $\overline{\rho'w'}|_0$ for our convective-column model from the results of Hoppel and Gathman's (1971) numerical calculations of the electrode effect with eddy diffusion. This scheme rests on the fact that, even in a very unstable boundary layer, turbulent transport can apparently be modeled by eddy diffusion in the lower part of the surface layer. The assumption made here is that at a height of $-0.57 L$, where buoyant production of turbulence is supposed to take over from shear production, it is appropriate to substitute Hoppel and Gathman's value of convection current into our convective-column model as $\overline{\rho'w'}|_0$.

We can calculate an effective eddy-diffusion coefficient for heat, from the unstable surface-layer temperature profile of Businger (1973).

$$K_h \equiv - \frac{\overline{\theta'w'}|_0}{\partial\theta/\partial z} = \frac{k z u_*}{0.74} \left(1.0 - 9.0 \frac{z}{L}\right)^{1/2} \quad (1.12)$$

If we ignore the radical in (1.12) for small enough z , we can identify χ in Hoppel and Gathman's eddy-diffusion formula approximately as

$$\chi \approx 47 u_* \quad (1.13)$$

Taking $L = -10$ m as a typical case of strong instability, we pick our convection current from the graphs in Hoppel and Gathman (1971) at a height of 5.7 m. For $\chi = 3, 6$, and $12 \frac{\text{m}^2}{\text{sec}}$, corresponding to $u_* = 0.063$,

0.13, and $0.25 \frac{\text{m}}{\text{sec}}$ according to (1.13), we find that the ratio of upward convection current to downward conduction current at $z = 5.7 \text{ m}$ is

$$J_m/J_\lambda|_{5.7\text{m}} = -0.25, -0.32, \text{ and } -0.36, \text{ respectively. Taking}$$

$J_\lambda(z=151\text{m}) = 1.83 \times 10^{-12} \frac{\text{coul}}{\text{m}^2\text{sec}}$, from the bottom point of the Chesapeake Bay sounding in figure 1.2, as a reasonable estimate for the downward conduction current near the surface, we finally have

$$\overline{\rho'w'}|_0 = 0.46, 0.59, \text{ and } 0.66 \times 10^{-12} \frac{\text{coul}}{\text{m}^2\text{sec}} \text{ in these three cases.}$$

Evidently, this is a very cumbersome and unsatisfying way of finding the lower boundary condition on $\overline{\rho'w'}$ for a convection current model with the electrode effect as a source. However, a model based on the charge conservation equation (1.2) does not seem to be able to produce an electrode effect at all, apparently forcing us to depend on a separate electrode-effect model near the ground and on some artificial scheme for coupling the two models, such as that in the previous paragraphs. For this reason, the electrode effect is studied in detail in Chapter 5, where a solution is found for this problem in a special case of atmospheric interest.

Proceeding with our estimates, let us take $L = -10 \text{ m}$, $H = 1000 \text{ m}$, $u_* = 0.25 \frac{\text{m}}{\text{sec}}$, and $\lambda_o = 1.0 \times 10^{-14} \frac{\text{mho}}{\text{m}}$. These values are reasonable over land, though they certainly represent an exaggeration of the instability in the mixed layer, and of the convectively generated EMF, over the ocean. Using these values, we find $R = 0.22$ and $\frac{H}{L} = -100$, implying $\frac{H}{H_c} = 0.55$ and $\frac{\lambda_o}{\rho_* u_* H} (\text{EMF}) = 0.57$ from Appendix 2. Making use of the estimates of $\overline{\rho'w'}|_0 = 1.07$ and $0.66 \times 10^{-12} \frac{\text{coul}}{\text{m}^2\text{sec}}$ for the Blanchard-effect and electrode-effect sources, respectively, we finally have the dimensional EMF generated by the convection current in these two cases as $(\text{EMF}) = 61 \text{ kV}$ and 38 kV .

These numbers, though probably only accurate to within an order of magnitude due to the crude nature of the model, compare favorably with the estimate of 65 kV made in section 1.1 from data taken by Kraakevik and Clark. The Blanchard effect result of 61 kV is undoubtedly an overestimate, as suggested above; whereas, the electrode effect result of 38 kV may be an underestimate, due to the crude modeling of the source. It is felt that these results justify further study of the convection current through the development of a better model.

1.4 Outline of the Thesis Research

The chapters that follow describe research on the convection currents in the fair-weather, unstable, planetary, boundary layer with the aid of a mathematical model of turbulent transport. The aim of this research was to develop relationships between the meteorological properties of such a layer and its effects on the local behavior of the atmospheric electrical properties. Convection in the mixed layer has already been identified in section 1.1 as a local generator, mechanically transporting positive charge upward. This generator will have the effect of enhancing the electric field and downward conduction current within the boundary layer, while at the same time locally reducing the total downward electric current from the upper atmosphere to the earth's surface.

Since the meteorological characteristics of the planetary boundary layer undergo diurnal variations, we expect these electrical characteristics to exhibit diurnal variations also. By developing a model to predict the electrical behavior of the layer from its meteorological properties, we aim to make possible the understanding of the electrical diurnal variations in terms of local and global causes. It is well known

that the global variations, caused by changes in the total current supplied to the upper atmosphere primarily by thunderstorms, have a peak-to-peak amplitude of some 30% of the mean. It is also well known that variations in local columnar resistance due to the upward transport of Aitkin nuclei in the "Austausch layer" can cause diurnal variations of electrical parameters of about 30% in some locations. The contention of this thesis is that local variations in convection currents can cause diurnal variations of at least this magnitude and, hence, can augment the effect of variations in columnar resistance.

Following a brief introduction to atmospheric electricity in section 1.1 and a review of the meteorological knowledge about mixed layers in section 1.2, a simple model of the convection current in such a layer was described in section 1.3. Having decided to proceed with the research, in Chapter 2 we develop a more realistic model of charge transport in the mixed layer from a known charge source near the surface. Chapter 3 is devoted to the numerical solution, testing, and justification of the model, ending up with a chart giving, in dimensionless form, the EMF generated by convection as a function of the two parameters R and $\frac{H}{L}$. In Chapter 4, a discussion of the Blanchard effect as a source is followed by a first attempt at calculating some effects of convection over the ocean.

Since we have already run into problems in using the electrode effect as a charge source in section 1.3, we devote Chapter 5 to an analysis of the electrode effect in the presence of turbulent mixing. The problems are solved in the special case of strong enough turbulence, and the model of Chapter 2 is modified in Chapter 6 to include the electrode effect in its new form. At the end of Chapter 6, another chart is presented of the dimensionless voltage drop across the mixed layer, due to convection

of electrode-effect charge, as a function of R and $\frac{H}{L}$. In Chapter 7, the results of Chapter 6 are used in a first try at computing the diurnal variation of electrical parameters over land due to the convection current. Also in Chapter 7 is a reconsideration of the Blanchard effect source in the light of the new model of Chapter 6.

Finally, Chapter 8 concludes the thesis with a discussion of future work to be done in this area. The various predictions of the theory and models developed here which can be readily tested are pointed out. Experiments are proposed to test these predictions. Possible future modifications to the model to take into account other physical effects, such as entrainment at the inversion capping a growing mixed layer, are discussed. Finally, the possibility of using electrical observations in the mixed layer to study the turbulence structure there is considered.

2 THE SECOND-ORDER-CLOSURE MODEL FOR CONVECTION CURRENT

2.1 Selection of the Model

In section 1.3 and Appendix 2 was developed a very simple physical model of the vertical transport process driven by convection in the mixed layer. Considering the crudeness of this "convective-column-model", one is amazed at the results it yielded. Therefore, it is tempting to pursue this line of attack on the convection current problem by constructing a more detailed and realistic physical model. The term "physical" is used here for a model which describes the air motions in detail, such as the assumption of convective columns covering a fractional area A_c and rising with uniform velocity v_c , as distinguished from some kind of statistical transport model. We shall use the term "mathematical" to describe models which make no direct reference to the details of the air motions but only to their statistical effects in transporting various properties of the air. The eddy-diffusion model, which approximates the vertical charge flux, for example, as $\overline{\rho'w'} \approx -K_h \frac{\partial \bar{\rho}}{\partial z}$, where K_h is some externally determined function of height, is the simplest possible mathematical model.

The obvious way to proceed in constructing a more valid physical model, based on a plume structure for the buoyant elements, would be to allow entrainment in the rising columns and to develop some way of relating their population per unit area and their intensity to the overall meteorological parameters of surface heat and momentum flux and mixed layer thickness. Such attempts have been made, for example by Priestly (1953), who imagined a statistical ensemble of plumes of various strengths drifting with the wind field. We feel that it would be possible to model

the transport by plumes in the interior of the mixed layer with reasonable accuracy, based on recent work such as that of Baines and Turner (1969).

Unfortunately, this approach encounters serious problems at the bottom boundary. How can the population density of plumes, or the spacing between them, be related to the meteorological parameters? How can the flux of the property to be transported be determined in the air entering the bottom of a plume? It seems clear that the answer to the former question is related to the growth rate of some kind of finite-amplitude instability in the surface layers of a fluid over a rigid, heated surface at very high Rayleigh number. This problem was structured by a brilliant suggestion of Howard (1964) and has been attacked by Foster (1965a, 1965b) and by Sparrow, Husar, and Goldstein (1970). However, the problem of plume spacing in the real atmosphere is still fraught with difficulties and does not seem close to a solution.

The answer to the latter question is even further from realization, because the flux of any quantity into the bottom of a buoyant plume is clearly dependent on the detailed structure of the plume base. A full understanding of this structure must wait until the instability problem stated above has been completely solved. Therefore, it was decided to abandon the search for a physical transport model and to fall back on the less appealing mathematical approach.

Since we have seen that the eddy-diffusion model is inadequate for our purposes, we are forced to proceed to the next level of complexity in mathematical models of turbulence. Instead of approximating the turbulent fluxes such as $\overline{\rho'w'}$ directly in terms of mean variables like $\frac{\partial \bar{\rho}}{\partial t}$, we can derive budget equations for each of these fluxes in terms of mean variables, the other fluxes, and various other turbulence statistics which are

unknown. This brings out what is known as the closure problem of turbulence. Equations for the mean quantities such as $\bar{\rho}$ always contain second-order turbulence statistics like $\overline{\rho'w'}$. Approximations of these fluxes directly, as in eddy diffusion models, can be called first-order closure — the equations contain only mean variables. If we write equations for the second-order statistics like $\overline{\rho'w'}$, they always contain third-order statistics, in this case, $\overline{\rho'w'^2}$. Approximation of these higher-order quantities in terms of the mean variables and second-order quantities to obtain a closed system constitutes second-order closure. This process of developing higher and higher order sets of equations can be carried on indefinitely, but it will always be necessary to make closure assumptions. Practically, there is little point in going beyond second-order closure.

A detailed discussion of the development of a second-order-closure model for the dynamics of an atmospheric boundary layer has been given by Donaldson (1973). He first derives an approximate set of equations of motion appropriate to a thin horizontal layer of perfect gas in which the hydrostatic pressure change across the layer is small compared to the absolute pressure and the Mach number is small. These equations are essentially the Boussinesq set in which the gas behaves as though it were incompressible and buoyancy effects are completely described in terms of its potential temperature. The heat generated by frictional dissipation of kinetic energy is neglected in the heat equation, so that potential temperature is conserved except for heating by molecular conduction and perhaps by radiation.

Such equations have also been used by Deardorff (1973), who gives them as

$$\left. \begin{aligned} \frac{\partial u_i}{\partial t} &= -\frac{\partial}{\partial x_j} (u_i u_j) - \frac{1}{\rho_{ao}} \frac{\partial p}{\partial x_i} - 2 \epsilon_{ijk} \Omega_j u_k \\ &\quad + \delta_{ij} \left[\frac{g}{\theta_{vo}} (\theta_v - \theta_{vo}) - g \right] + \nu \nabla^2 u_i \\ \frac{\partial \theta_v}{\partial t} &= -\frac{\partial}{\partial x_j} (\theta_v u_j) + \kappa \nabla^2 \theta_v \end{aligned} \right\} \quad (2.1)$$

Here, we have neglected Deardorff's radiative heating term, and we will continue to neglect the effects of radiation throughout this thesis. The conventional Einstein subscript notation has been used, where u_i is the i^{th} component of air velocity, p is air pressure, ϵ_{ijk} is the alternating unit tensor (equal to 0 if any indices are duplicated, +1 if they are cyclic, and -1 if they are anti-cyclic), Ω_j is the j^{th} component of the earth's angular velocity, δ_{ij} is the delta function (equal to +1 if the indices are identical and 0 otherwise), g is the acceleration of gravity, θ_v is the virtual potential temperature, ν is the kinematic viscosity, and κ is the thermometric conductivity. ρ_{ao} and θ_{vo} are reference values of the air density and virtual potential temperature, assumed approximately uniform throughout the layer. The coordinate system is right-handed, chosen so that the 1- or x-axis is horizontal in the direction of the surface wind and the 3- or z-axis is vertically upward.

A word should be inserted here about the virtual potential temperature, which appears in equations (2.1). As mentioned in section 1.2, both the temperature and the water vapor mixing ratio are important in determining the buoyancy of a dry parcel relative to its environment. However, only the temperature itself enters the derivation of the Boussinesq equations; strictly speaking, only the potential temperature θ should appear in (2.1), not the virtual potential temperature θ_v .

It can be shown that the effect of water vapor with a mixing ratio r on the buoyancy of a parcel at temperature T can be accounted for

by assuming the parcel to be composed only of dry air having a temperature $T_v \equiv \left(\frac{1 + 1.609 r}{1 + r} \right) T \approx T + 0.61 T_o r$. T_v is called the virtual temperature, and the approximation follows for $r \ll 1$. If we define the virtual potential temperature in the same relation to the virtual temperature as potential temperature bears to temperature, we have

$$\theta_v \equiv \left(\frac{p_o}{p} \right)^{\frac{\bar{R}}{C_p}} T_v \approx \theta + 0.61 \theta_o r \quad (2.2)$$

where we continue to assume that $r \ll 1$ and that $(\theta - \theta_o)/\theta_o \ll 1$, and where θ_o is the reference value of θ . p is the pressure of the parcel under consideration and p_o is a reference value such as the surface pressure. $\bar{R}$ and C_p are the gas constant and specific heat at constant pressure for the mixture of gases under consideration. Although the potential temperature itself is conserved for adiabatic motions and is mass additive for mixing of dry parcels, this new virtual potential temperature will not be. However, it is shown in Appendix 4 that

$$|\theta_{v2} - \theta_{v1}| \approx 0.074 \theta_{v0} r \left| \ln \frac{p_1}{p_2} \right| < 0.33^\circ$$

for a parcel being moved adiabatically from pressure p_1 to pressure p_2 in the mixed layer and that

$$\left| \theta_{vf} - \frac{M_1 \theta_{v1} + M_2 \theta_{v2}}{M_1 + M_2} \right| \lesssim \frac{M_1 M_2}{(M_1 + M_2)^2} \left[0.87 + 0.61 \left(\frac{p_o}{p} \right)^{\frac{\bar{R}}{C_p}} \right] \left| (r_1 - r_2)(T_1 - T_2) \right| < 0.16^\circ$$

for mixing at pressure p of a mass M_1 of air at r_1 and T_1 with a mass M_2 of air at r_2 and T_2 in the mixed layer to give a final virtual potential temperature of θ_{vf} . Therefore, to a good approximation, the virtual potential temperature is conserved and mass additive just as θ was. Since water vapor is carried with the air just as is potential temperature, it is therefore legitimate to replace θ with θ_v in the Boussinesq equations

in order to account for the buoyancy effects of small amounts of water vapor in the mixed layer. From here on, we shall drop the subscript v on the virtual potential temperature, with the understanding that θ is to be replaced by θ_v if significant water vapor is present.

Returning to Donaldson's (1973) discussion of second-order-closure models, we observe that the next step is to apply the technique of Reynolds averaging to the equations of motion. Each quantity is decomposed into a mean value, represented by an overbar, and a fluctuation around that mean, denoted by a prime. For example, $\theta \equiv \bar{\theta} + \theta'$, where $0 \equiv \overline{(\theta')}$. In this thesis, the mean represented by the overbar will be taken as a horizontal average, since we will be interested only in the special case of horizontal uniformity. Averaging the equations of motion then gives the mean equations in terms of turbulent fluxes such as $\overline{\theta'w'}$ in the $\bar{\theta}$ equation.

Equations for the fluxes can be produced as follows. First form the equations for the fluctuations by subtracting each mean equation from its corresponding un-averaged equation of motion. To form the equation for $\overline{\theta'w'}$, for example, multiply the equation for θ' by w' and the equation for w' by θ' , add the two resulting equations, and average by applying the overbar. The equations for the other second-order moments can be obtained in a similar manner.

After carrying out this straightforward, if somewhat messy, procedure, we are left with a set of mean equations containing turbulent fluxes and a set of equations for these turbulent fluxes containing higher order turbulence moments like $\overline{\theta'w'^2}$, pressure correlation terms like

$\overline{\theta' \frac{\partial p'}{\partial z}}$, and dissipation terms like $\kappa \overline{w' \frac{\partial^2 \theta'}{\partial x_j \partial x_j}}$. These remaining three

kinds of terms in the equations for the second-order moments must be approximated in order to close the set of equations. It is the closure assumptions made at this point that are crucial to the success of a second-order-closure model.

Some ad hoc principles for making these closure assumptions are discussed by Donaldson (1973). They involve replacing an unknown turbulence statistic by a sum of terms each made up of some combination of known turbulence moments and mean variables which has the right dimensions and each multiplied by an arbitrary numerical coefficient. These coefficients must be determined from experimental data or by "tuning" the model to give reasonable results in cases where the solution is known. A more formal procedure for determining the closure assumptions has been described by Lumley and Khajeh-Nouri (1974). However, their method leads to even more terms than Donaldson's, leaving us with an unmanageable number of undetermined coefficients.

2.2 The Model of Wyngaard, et al.

Wyngaard, Coté, and Rao (1974) have developed a second-order-closure model for the dynamics of a horizontally uniform planetary boundary layer with the specific intention of studying the unstable mixed layer. These authors have adhered to the principle of parsimony in choosing their closure assumptions and have thereby reduced the number of arbitrary constants to the extent that they can be almost completely determined from surface layer data. This model has been quite successful in reproducing the behavior of the mixed layer, as further described by Wyngaard, Arya, and Coté (1974), with results comparing favorably to those of Deardorff (1974a, 1974b). The model of convection current developed

in this thesis will be based on the work of Wyngaard, et al., which will therefore be described briefly in this section.

The mean equations used by Wyngaard, Coté, and Rao (1974), based on (2.1), are

$$\left. \begin{aligned} \frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial z} \overline{u'w'} &= f(\bar{v} - v_g) \\ \frac{\partial \bar{v}}{\partial t} + \frac{\partial}{\partial z} \overline{v'w'} &= -f(\bar{u} - u_g) \\ \frac{\partial \bar{\theta}}{\partial t} + \frac{\partial}{\partial z} \overline{\theta'w'} &= 0 \end{aligned} \right\} \quad (2.3)$$

Here, $\bar{w} \equiv 0$, and $u_g \equiv -\frac{1}{f} \frac{\partial \bar{p}}{\partial y}$ and $v_g \equiv \frac{1}{f} \frac{\partial \bar{p}}{\partial x}$ are the only remnants of horizontal variability. Wyngaard et al. take the x-axis toward the east, so that the components of the earth's angular velocity in this coordinate system are $\Omega = (0, |\Omega| \cos \phi, |\Omega| \sin \phi)$, and the surface wind is from the west. The Coriolis parameter is $f \equiv 2|\Omega| \sin \phi$, where ϕ is the latitude angle. The equations for the second-order turbulence moments become

$$\begin{aligned} \frac{\partial}{\partial t} \overline{u'_i u'_j} &= -\overline{u'_j w'} \frac{\partial \bar{u}_i}{\partial z} - \overline{u'_i w'} \frac{\partial \bar{u}_j}{\partial z} - \frac{\partial}{\partial z} \overline{u'_i u'_j w'} \\ &\quad - \left(\overline{u'_j \frac{\partial p'}{\partial x_i}} + \overline{u'_i \frac{\partial p'}{\partial x_j}} \right) + \frac{g}{\theta_0} \left(\delta_{i3} \overline{u'_j \theta'} + \delta_{j3} \overline{u'_i \theta'} \right) \\ &\quad - \frac{2}{3} \delta_{ij} \bar{\epsilon} - 2 \left(\epsilon_{ikm} \Omega_k \overline{u'_m u'_j} + \epsilon_{jkm} \Omega_k \overline{u'_m u'_i} \right) \end{aligned} \quad (2.4)$$

$$\begin{aligned} \frac{\partial}{\partial t} \overline{u'_i \theta'} &= -\overline{w' \theta'} \frac{\partial \bar{u}_i}{\partial z} - \overline{u'_i w'} \frac{\partial \bar{\theta}}{\partial z} - \frac{\partial}{\partial z} \overline{u'_i w' \theta'} \\ &\quad - \overline{\theta' \frac{\partial p'}{\partial x_i}} + \frac{g}{\theta_0} \delta_{i3} \overline{\theta'^2} - 2 \epsilon_{ijk} \Omega_j \overline{u'_k \theta'} \\ \frac{\partial}{\partial t} \overline{\theta'^2} &= -2 \overline{w' \theta'} \frac{\partial \bar{\theta}}{\partial z} - \frac{\partial}{\partial z} \overline{\theta'^2 w'} - 2 \bar{\epsilon}_\theta \end{aligned}$$

In this somewhat mixed notation $(x_1, x_2, x_3) \equiv (x, y, z)$ are the space coordinates and $(u_1, u_2, u_3) \equiv (u, v, w)$ are the corresponding components

of the velocity. $\bar{\epsilon}$ is the molecular dissipation rate for turbulent kinetic energy, and $\bar{\epsilon}_\theta$ is the dissipation rate for turbulent temperature variance, related later to $\bar{\epsilon}$. Notice that equations have been included in (2.4) for $\overline{u'v'}$, $\overline{u'^2}$, $\overline{v'^2}$, $\overline{w'^2}$, $\overline{u'\theta'}$, and $\overline{v'\theta'}$, in spite of the fact that the only turbulence moments appearing in the mean equations are $\overline{u'w'}$, $\overline{v'w'}$, and $\overline{w'\theta'}$. This is because the budget equations for the latter three fluxes contain the others, so that equations are needed for all nine.

In writing equations (2.3) and (2.4) it has been assumed that the Reynolds number for the turbulence is so large that molecular diffusion may be neglected in comparison with turbulent transport and that molecular dissipation is only significant on very small scales where the turbulence is locally isotropic. Therefore, the only molecular terms appearing are $\bar{\epsilon}$ and $\bar{\epsilon}_\theta$ in the equations for $\overline{u'^2}$, $\overline{v'^2}$, $\overline{w'^2}$, and $\overline{\theta'^2}$. In the other covariance equations it is assumed that the destruction is not by molecular processes but by the tendency of the pressure correlations to de-correlate u'_i and u'_j ($i \neq j$) or u'_i and θ' at large scales. In order that molecular dissipation be modeled reasonably well, a separate equation is carried for $\bar{\epsilon}$:

$$\frac{\partial}{\partial t} \bar{\epsilon} = - \frac{\partial}{\partial z} \overline{w'\epsilon'} - 2\nu \left(\overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_j}{\partial x_k}} + \nu \overline{\frac{\partial^2 u'_i}{\partial x_k \partial x_j} \frac{\partial^2 u'_i}{\partial x_k \partial x_j}} \right) \quad (2.5)$$

In equations (2.4) and (2.5) there are a number of terms which must be approximated in order to close the system of equations. Wyngaard, Coté, and Rao (1974) based their closure assumptions in part on a simplified version of the functional expansion results of Lumley and Khajeh-Nouri (1974). Their closure assumptions will be listed here with a minimum of comment, since the ultimate justification is not their derivation but

their practical success. The pressure correlations are modeled in the spirit of their presumed role in de-correlating the anisotropic second-order moments:

$$\left. \begin{aligned} -\left(\overline{u'_j \frac{\partial p'}{\partial x_i}} + \overline{u'_i \frac{\partial p'}{\partial x_j}}\right) + \frac{2}{3} \delta_{ij} \frac{\partial}{\partial z} \overline{w' p'} &= -\left(\overline{u'_i u'_j} - \frac{1}{3} \delta_{ij} \overline{q'^2}\right) \frac{c_{ij}}{\tau} \\ -\overline{\theta' \frac{\partial p'}{\partial x_i}} &= -\frac{d_i}{\tau} \overline{\theta' u'_i} \end{aligned} \right\} \quad (2.6)$$

Here, $\overline{q'^2} \equiv \overline{u'_i u'_i}$ is twice the turbulent kinetic energy, and $\tau \equiv \overline{q'^2}/\epsilon$ is a turbulence dissipation time scale. The second term on the left-hand side of the first equation in (2.6) subtracts out the pressure transport terms, which will be added back in again in (2.9). The last two terms in equations (2.5), the molecular dissipation terms, are modeled according to the simplified functional expansion result of Lumley and Khajeh-Nouri (1974):

$$-2\nu \left(\overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_k}} + \nu \overline{\frac{\partial^2 u'_i}{\partial x_k \partial x_j} \frac{\partial^2 u'_i}{\partial x_k \partial x_j}} \right) = -4 \frac{\overline{\epsilon}^2}{\overline{q'^2}} + 3 \frac{\overline{\epsilon} \overline{P}}{\overline{q'^2}} \quad (2.7)$$

where $\overline{P}$ is the production rate of turbulent kinetic energy. Instead of carrying a separate budget equation for molecular dissipation of temperature variance, Wyngaard et al. model that simply as

$$-2\overline{\epsilon}_\theta = -\frac{2d_\theta}{\tau} \overline{\theta'^2} \quad (2.8)$$

Finally, the turbulent-transport or third-order-moment terms are modeled by an ad hoc gradient-diffusion model, since Lumley and Khajeh-Nouri's results for these terms are too complicated.

$$-\frac{\partial}{\partial z} \left(\overline{u'_i u'_j w'} + \frac{2}{3} \delta_{ij} \overline{p' w'} \right) = + a_t \frac{\partial}{\partial z} \left(\tau \overline{w'^2} \frac{\partial}{\partial z} \overline{u'_i u'_j} \right) \quad (2.9)$$

$$\left. \begin{aligned} -\frac{\partial}{\partial z} \overline{u'_i \theta' w'} &= +a_{te} \frac{\partial}{\partial z} \left(\bar{\epsilon} \bar{w'^2} \frac{\partial}{\partial z} \overline{u'_i \theta'} \right) \\ -\frac{\partial}{\partial z} \overline{w' \epsilon'} &= +a_{te} \frac{\partial}{\partial z} \left(\bar{\epsilon} \bar{w'^2} \frac{\partial \bar{\epsilon}}{\partial z} \right) \end{aligned} \right\} \quad \begin{array}{l} (2.9) \\ \text{cont.} \end{array}$$

It is hoped that these crude gradient-diffusion approximations will not cause too much trouble either because the turbulent transports of second-order moments are roughly proportional to their gradients or because such transports are not too important in the budgets for these moments. There is little evidence about the behavior of these transport terms — or indeed, about any of the closure-modeled terms — in the real atmosphere, but Wyngaard and Coté (1971) have found at least one contradiction to (2.9) from measurements in the surface layer. Their results indicate that $\overline{w'^2}$ is exported upward out of the surface layer under very unstable conditions. This up-gradient transport of $\overline{w'^2}$ is thought to be the reason that the second-order-closure model described here does not reproduce the surface-layer profiles perfectly. Nevertheless, we have at least succeeded in removing eddy-diffusion approximations from the mean equations.

Specializing the equations (2.4) and (2.5) to the surface layer, Wyngaard, Coté, and Rao (1974) were able to use surface-layer data to specify many of the closure constants:

$$\left. \begin{aligned} c_{11} = c_{22} = c_{33} &= 6.7 & d_3 &= 9.7 \\ c_{13} &= 13.2 & d_4 &= 1.4 \\ d_1 &= 4.4 & a_{te} &= 0.083 \end{aligned} \right\} \quad (2.10)$$

Because $\overline{u'v'}$, $\overline{v'w'}$, and $\overline{v'\theta'}$ vanish in the surface layer, they arbitrarily assumed $c_{12} \equiv c_{13} \equiv c_{23}$ and $d_2 \equiv \frac{d_1 + d_3}{2}$. The remaining transport coefficients were set to $a_t \equiv 0.15$. The method of obtaining

these numbers will be illustrated in section 2.4, where we do the same for the coefficients in our convection-current model.

In non-dimensional form, equations (2.3), (2.4), and (2.5) contain only two dimensionless parameters, $\frac{H}{kL}$ and $\frac{Hf}{u_*}$. Wyngaard, et al. integrated in time from arbitrary initial conditions to obtain the steady-state solutions for various degrees of instability ranging from $\frac{H}{L} = 0$ (neutral) to $\frac{H}{L} = -1000$. For all the unstable runs they took $\frac{Hf}{u_*} = 1.0$. This parameter has an effect on the velocity defect profiles in the mixed layer, as discussed by Wyngaard, Arya, and Côté (1974), but there is no reason to expect that it affects the turbulence structure in the unstable cases because of the time-scale argument given in section 1.2. The lower boundary conditions used required the solutions to match the surface-layer data. The upper boundary conditions were that all the turbulence moments, the mean shears, and the mean temperature gradient vanish at the rigid lid.

Only one of the interesting results obtained from this model will be mentioned here. Wyngaard, Arya, and Côté (1974) found that for $-\frac{H}{L} \gtrsim 10$, the stress profiles became asymptotically independent of $\frac{H}{L}$, as predicted in section 1.2. In the case of no geostrophic wind shear, $\overline{u'w'}$ became linear in $\frac{z}{H}$ and $\overline{v'w'}$ became essentially zero. Even when geostrophic wind shear was present, the mean wind shear remained essentially zero above the surface layer, with the curvature of the stress profile adjusting to balance the changing pressure gradient force. This vanishing of the mean wind shear in the mixed layer was traced to a feedback mechanism between stress and mean shear. Although the rotational production terms are dominant in the stress equations in (2.4), trying to drive the stress to large magnitudes of the order of $u_* w_*$, the coupling between stress profile curvature and mean shear in the mean equations (2.3) prevents

this. If the stress and its curvature try to become large, this coupling produces a large shear production term in the stress equations, tending to cancel the rotational production term. Thus, in the unstable mixed layer the stress equations seem to degenerate to a balance between shear and rotational production which tends to hold the mean shear down near zero.

2.3 The Model Equations for Charge Convection in a Layer of Uniform Conductivity

Having discussed Wyngaard's second-order-closure model by way of illustration, we are ready to develop such a model for the convection current. In this and the next two chapters we will be concerned only with the special case of uniform conductivity. Although this a good approximation in the interior of the mixed layer, as illustrated by the conductivity profiles shown in section 1.1, because of the strong mixing processes discussed in section 1.2, this special case rules out the electrode effect as a charge source near the ground. However, we can treat the convection current due to the Blanchard effect source with this case, as well as getting a good feel for the behavior of our model. In Chapters 5, 6, and 7, the restriction to uniform conductivity will be relaxed and the electrode effect will be treated.

In Appendix 1, it is argued that the single charge-conservation equation (A1.4) is adequate to describe electrical effects in the fair-weather planetary boundary layer away from the immediate proximity of the ground. If we are to neglect the electrode effect, we can use this equation right down to the ground. Restricting (A1.4) to the case of uniform conductivity $\lambda \equiv \lambda_0$, and using Gauss's law from (A1.3), we have a very simple charge conservation equation:

$$\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x_i} \rho u_i - \frac{\lambda_o}{\epsilon_o} \rho + Q + D \nabla^2 \rho \quad (2.11)$$

Here we have assumed $\frac{\partial u_i}{\partial x_i} = 0$, in accord with the Boussinesq approximation discussed in section 2.1, and the molecular diffusion term $D \nabla^2 \rho$ has been put back in for reasons that will appear later. This equation is similar in form to the equations of motion (2.1). Together with them it forms the set of basic equations from which the closure model will be derived.

Now the method of Reynolds averaging will be applied, as described in section 2.1, to produce a mean charge equation and budget equations for the necessary second-order moments. Define $\rho \equiv \bar{\rho} + \rho'$ and $u_i \equiv \bar{u}_i + u'_i$, where $\bar{\rho}' \equiv 0 \equiv \bar{u}'_i$ and the overbar represents a horizontal average. Inserting these definition into (2.11) and averaging gives the mean charge equation.

$$\frac{\partial \bar{\rho}}{\partial t} = - \frac{\partial}{\partial z} \bar{\rho' w'} - \frac{\lambda_o}{\epsilon_o} \bar{\rho} + \bar{Q} + D \frac{\partial^2}{\partial z^2} \bar{\rho} \quad (2.12)$$

Again we have assumed $\bar{w} \equiv 0$. (2.12) corresponds to the mean dynamical equations (2.3).

As an illustration of the technique used to derive the second-moment equations, we will derive here the equation for $\overline{\rho'^2}$. First, an equation for ρ' is formed by subtracting (2.12) from (2.11).

$$\frac{\partial \rho'}{\partial t} = - w' \frac{\partial \bar{\rho}}{\partial z} - \bar{u} \frac{\partial \rho'}{\partial x} - \bar{v} \frac{\partial \rho'}{\partial y} - \frac{\partial}{\partial x_i} \rho' u'_i + \frac{\partial}{\partial z} \bar{\rho' w'} - \frac{\lambda_o}{\epsilon_o} \rho' + D \nabla^2 \rho' \quad (2.13)$$

Here we have assumed that $Q' \equiv 0$. Multiplying this equation by ρ' and averaging gives

$$\frac{1}{2} \frac{\partial}{\partial t} \overline{\rho'^2} = - \frac{\lambda_o}{\epsilon_o} \overline{\rho'^2} - \overline{\rho' w'} \frac{\partial \bar{\rho}}{\partial z} - \frac{1}{2} \frac{\partial}{\partial z} \overline{\rho'^2 w'} + D \overline{\rho' \frac{\partial^2 \rho'}{\partial x_i \partial x_i}} \quad (2.14)$$

In a similar manner, the equations for w' and θ' obtained from equations (2.1) can be combined with (2.13) to form the equations for $\overline{\rho'w'}$ and $\overline{\rho'\theta'}$.

They are

$$\left. \begin{aligned} \frac{\partial}{\partial t} \overline{\rho'w'} &= -\frac{\lambda_o}{\epsilon_o} \overline{\rho'w'} - \overline{w'^2} \frac{\partial \bar{\rho}}{\partial z} + \frac{g}{\Theta_o} \overline{\rho'\theta'} + 2|\underline{\Omega}| \cos \phi \overline{\rho'u'} \\ &\quad - \frac{\partial}{\partial z} \overline{\rho'w'^2} - \overline{\rho' \frac{\partial \rho'}{\partial z}} + D \overline{w' \frac{\partial^2 \rho'}{\partial x_i \partial x_i}} + \nu \overline{\rho' \frac{\partial^2 w'}{\partial x_i \partial x_i}} \\ \frac{\partial}{\partial t} \overline{\rho'\theta'} &= -\frac{\lambda_o}{\epsilon_o} \overline{\rho'\theta'} - \overline{\rho'w'} \frac{\partial \bar{\theta}}{\partial z} - \overline{\theta'w'} \frac{\partial \bar{\rho}}{\partial z} \\ &\quad - \frac{\partial}{\partial z} \overline{\rho'w'\theta'} + D \overline{\theta' \frac{\partial^2 \rho'}{\partial x_i \partial x_i}} + K \overline{\rho' \frac{\partial^2 \theta'}{\partial x_i \partial x_i}} \end{aligned} \right\} \quad (2.15)$$

(2.14) and (2.15) correspond to the dynamical second-moment equations in (2.4). Clearly, the $\overline{\rho'w'}$ equation is needed for the mean equation (2.12). The $\overline{\rho'\theta'}$ equation is also needed because that turbulence moment appears in the equation for $\overline{\rho'w'}$. The equation for $\overline{\rho'^2}$ is not needed, since the charge variance does not appear in any of the other equations. However, it will be carried along anyhow, since it might be useful to have profiles of $\overline{\rho'^2}$ to compare with observations.

The term in each second-moment equation containing $\frac{\lambda_o}{\epsilon_o}$ represents dissipation of that turbulence moment by conduction. Following that term, there are two production terms in the second and third equations and one in the first. For example, the term $+\frac{g}{T_o} \overline{\rho'\theta'}$ represents the generation of $\overline{\rho'w'}$ by gravity acting on a positive $\overline{\rho'\theta'}$ correlation, and the $-\overline{\rho'w'} \frac{\partial \bar{\theta}}{\partial z}$ term represents the generation of $\overline{\rho'\theta'}$ by upward charge flux in the presence of a downward gradient of mean potential temperature. This particular pair of generation terms, acting together, forms a positive feedback loop whereby positive $\overline{\rho'\theta'}$ generates positive $\overline{\rho'w'}$ which, in the

presence of an unstable mean temperature gradient, generates more positive $\overline{\rho'\theta'}$. This loop would continue to increase $\overline{\rho'w'}$ and $\overline{\rho'\theta'}$ in the surface layer until other terms become large enough to compensate.

The $\overline{\rho'w'}$ equation also contains a correlation with the fluctuating pressure p' , and all the equations contain triple-moment and molecular-diffusion terms. None of these terms can be computed from the equations available here. They illustrate the closure problem in turbulence and must be approximated by some form of closure assumption before the equations can be solved. The Coriolis term containing $\overline{\rho'u'}$ is not strictly a closure problem because we could easily write an equation for $\overline{\rho'u'}$ in analogy with the one for $\overline{\rho'w'}$. We have not done this, however, because $|\Omega|$ is so small that the Coriolis term may be ignored compared to the conduction term as long as $\frac{\varepsilon_0}{\lambda_0}$ does not become too large.

Before taking up the closure assumptions, we will re-write the molecular-diffusion terms as follows:

$$\begin{aligned} D \overline{\rho' \frac{\partial^2 \rho'}{\partial x_i \partial x_i}} &= \cancel{\frac{D}{2} \frac{\partial^2}{\partial x_i \partial x_i} \overline{\rho'^2}} - D \overline{\frac{\partial \rho'}{\partial x_i} \frac{\partial \rho'}{\partial x_i}} \\ D \overline{w' \frac{\partial^2 \rho'}{\partial x_i \partial x_i}} + \nu \overline{\rho' \frac{\partial^2 w'}{\partial x_i \partial x_i}} &= \cancel{\nu \frac{\partial^2}{\partial x_i \partial x_i} \overline{\rho' w'}} + (D - \nu) \overline{w' \frac{\partial^2 \rho'}{\partial x_i \partial x_i}} - 2\nu \overline{\frac{\partial \rho'}{\partial x_i} \frac{\partial w'}{\partial x_i}} \\ D \overline{\theta' \frac{\partial^2 \rho'}{\partial x_i \partial x_i}} + K \overline{\rho' \frac{\partial^2 \theta'}{\partial x_i \partial x_i}} &= \cancel{K \frac{\partial^2}{\partial x_i \partial x_i} \overline{\rho' \theta'}} + (D - K) \overline{\theta' \frac{\partial^2 \rho'}{\partial x_i \partial x_i}} - 2K \overline{\frac{\partial \rho'}{\partial x_i} \frac{\partial \theta'}{\partial x_i}} \end{aligned}$$

The first term in each expansion represents the divergence of molecular-diffusive flux, which can be assumed negligible compared to the turbulent diffusion expressed by the triple moment terms. Assuming a very large Reynolds number for the turbulence implies local isotropy in the dissipation scales, so that we can also ignore the $\nu \frac{\partial \rho'}{\partial x_j} \frac{\partial w'}{\partial x_j}$ term in the $\overline{\rho'w'}$.

equation. This does not apply to the similar terms in the $\overline{\rho'\theta'}$ and $\overline{\rho'^2}$ equations, however, which represent important loss terms. If we were to assume $D = \nu = \kappa$, the second terms would vanish in the second and third equations, leaving only the dissipation terms $-2\kappa \frac{\partial \rho'}{\partial x_j} \frac{\partial \theta'}{\partial x_j}$ and $-D \frac{\partial \rho'}{\partial x_j} \frac{\partial \rho'}{\partial x_j}$ in the first and third equations. Although this is not true in our case, D being much smaller than $\nu \approx \kappa$, we will nevertheless make this assumption. We are in effect demanding that molecular effects vanish from the $\overline{\rho'w'}$ equation and assuming that in the $\overline{\rho'\theta'}$ equation they may be adequately modeled by the last term. This agrees with Wyngaard's assumptions in deriving his second-moment equations in (2.4).

In making the closure assumptions for the remaining pressure-correlation, dissipation, and triple-moment terms, we will follow Wyngaard's models given in equations (2.6), (2.8), and (2.9).

$$\left. \begin{aligned} -\frac{1}{2} \frac{\partial}{\partial z} \overline{w' \rho'^2} &= \frac{a_t}{2} \frac{\partial}{\partial z} \left(\overline{\tau} \overline{w'^2} \frac{\partial}{\partial z} \overline{\rho'^2} \right) \\ -D \frac{\partial \rho'}{\partial x_i} \frac{\partial \rho'}{\partial x_i} &= -\frac{d_4 D}{\kappa \overline{\tau}} \overline{\rho'^2} \\ -\overline{\rho' \frac{\partial \rho'}{\partial z}} &= -\frac{d_3}{\overline{\tau}} \overline{\rho' w'} \\ -\frac{\partial}{\partial z} \overline{\rho' w'^2} &= a_t \frac{\partial}{\partial z} \left(\overline{\tau} \overline{w'^2} \frac{\partial}{\partial z} \overline{\rho' w'} \right) \\ -\frac{\partial}{\partial z} \overline{\rho' \theta' w'} &= a_t \frac{\partial}{\partial z} \left(\overline{\tau} \overline{w'^2} \frac{\partial}{\partial z} \overline{\rho' \theta'} \right) \\ -2\kappa \frac{\partial \rho'}{\partial x_i} \frac{\partial \theta'}{\partial x_i} &= -\frac{d_4}{\overline{\tau}} \overline{\rho' \theta'} \end{aligned} \right\} \quad (2.16)$$

Strictly speaking, the right-hand side of the last equation should be $-\frac{2d_4}{\overline{\tau}} \overline{\rho' \theta'}$, but we have dropped the 2 because we expect about half the dissipation that would obtain if $D = \kappa$. Similar reasoning leads to the

ratio $\frac{D}{\kappa}$ appearing in the second equation. Putting (2.16) and the assumptions in the previous two paragraphs into (2.14) and (2.15) gives

$$\left. \begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \bar{\rho}^{1/2} &= \left(-\frac{\lambda_0}{\epsilon_0} - \frac{d_4 D}{\kappa \bar{\tau}} + \frac{a_t}{2} \frac{\partial}{\partial z} \bar{\tau} \bar{\omega}^{1/2} \frac{\partial}{\partial z} \right) \bar{\rho}^{1/2} \\ &\quad - \bar{\rho}^{1/2} \frac{\partial \bar{\rho}}{\partial z} \\ \frac{\partial}{\partial z} \bar{\rho}' \omega' &= \left(-\frac{\lambda_0}{\epsilon_0} - \frac{d_3}{\bar{\tau}} + a_t \frac{\partial}{\partial z} \bar{\tau} \bar{\omega}^{1/2} \frac{\partial}{\partial z} \right) \bar{\rho}' \omega' \\ &\quad - \bar{\omega}^{1/2} \frac{\partial \bar{\rho}}{\partial z} + \frac{g}{\Theta_0} \bar{\rho}' \Theta' \\ \frac{\partial}{\partial t} \bar{\rho}' \Theta' &= \left(-\frac{\lambda_0}{\epsilon_0} - \frac{d_4}{\bar{\tau}} + a_t \frac{\partial}{\partial z} \bar{\tau} \bar{\omega}^{1/2} \frac{\partial}{\partial z} \right) \bar{\rho}' \Theta' \\ &\quad - \bar{\Theta}' \omega' \frac{\partial \bar{\rho}}{\partial z} - \bar{\rho}' \omega' \frac{\partial \bar{\Theta}}{\partial z} \end{aligned} \right\} \quad (2.17)$$

Dropping the molecular diffusion term from the mean charge equation (2.12) gives

$$\frac{\partial \bar{\rho}}{\partial t} = -\frac{\lambda_0}{\epsilon_0} \bar{\rho} - \frac{\partial}{\partial z} \bar{\rho}' \omega' + \bar{Q} \quad (2.18)$$

Together, (2.17) and (2.18) form our second-order-closure model for the convection current in a layer of uniform conductivity.

It remains to non-dimensionalize equations (2.17) and (2.18) with the scales u_* , T_* , ρ_* , τ_λ , and H , defined in sections 1.2 and 1.3. Define dimensionless variables in terms of the dimensional ones as follows:

$$\left. \begin{aligned} t' &\equiv \frac{t}{\tau_\lambda}, \quad \mathcal{H} \equiv \frac{z}{H}, \quad \hat{Q} \equiv \frac{\tau_\lambda}{\rho_*} \bar{Q}, \quad \bar{\Theta}' \omega' \equiv \frac{\bar{\Theta}' \omega'}{T_* u_*} \\ \hat{\rho} &\equiv \frac{\bar{\rho}}{\rho_*}, \quad \tau \equiv \frac{\bar{\tau}}{\tau_\lambda}, \quad \hat{\Theta} \equiv \frac{\bar{\Theta}}{T_*}, \quad \bar{\omega}^2 \equiv \frac{\bar{\omega}^2}{u_*^2} \end{aligned} \right\} \quad (2.19)$$

$$\overline{\rho w} \equiv \frac{\overline{\rho' w'}}{\rho_* u_*}, \quad \overline{\rho \theta} \equiv \frac{\overline{\rho' \theta'}}{\rho_* T_*}, \quad \overline{\rho^2} \equiv \frac{\overline{\rho'^2}}{\rho_*^2} \quad (2.19)$$

cont.

Putting (2.19) into (2.18) and (2.17), using $R \equiv \frac{u_* \tau_\lambda}{H}$ and $\frac{H}{kL} \equiv \frac{g T_* H}{\theta_* u_*^2}$ as before, we finally have

$$\left. \begin{aligned} \frac{\partial \hat{\rho}}{\partial t'} &= -\hat{\rho} - R \frac{\partial}{\partial \zeta} \overline{\rho w} + \hat{Q} \\ \frac{\partial}{\partial t'} \overline{\rho w} &= \left(-1 - \frac{d_3}{\tau} + a_t R^2 \frac{\partial}{\partial \zeta} \tau \overline{w^2} \frac{\partial}{\partial \zeta} \right) \overline{\rho w} \\ &\quad - R \overline{w^2} \frac{\partial \hat{\rho}}{\partial \zeta} + R \frac{H}{kL} \overline{\rho \theta} \\ \frac{\partial}{\partial t'} \overline{\rho \theta} &= \left(-1 - \frac{d_4}{\tau} + a_t R^2 \frac{\partial}{\partial \zeta} \tau \overline{w^2} \frac{\partial}{\partial \zeta} \right) \overline{\rho \theta} \\ &\quad - R \overline{\theta w} \frac{\partial \hat{\rho}}{\partial \zeta} - R \overline{\rho w} \frac{\partial \hat{\theta}}{\partial \zeta} \\ \frac{1}{2} \frac{\partial}{\partial t'} \overline{\rho^2} &= \left(-1 - \frac{d_4 D}{k \tau} + a_t R^2 \frac{\partial}{\partial \zeta} \tau \overline{w^2} \frac{\partial}{\partial \zeta} \right) \overline{\rho^2} \\ &\quad - R \overline{\rho w} \frac{\partial \hat{\rho}}{\partial \zeta} \end{aligned} \right\} \quad (2.20)$$

The first three equations in (2.20) comprise a set of coupled, partial-differential equations in the dependent variables $\hat{\rho}$, $\overline{\rho w}$, and $\overline{\rho \theta}$. They are linear in these three variables as long as $\overline{w^2}$, $\overline{\theta w}$, $\frac{\partial \hat{\theta}}{\partial \zeta}$, and τ are known externally as functions of ζ and t' . These latter four turbulence statistics could be supplied, for example, by the independent solution of a dynamical model like Wyngaard's, described in section 2.2. The last equation in (2.20) for the fourth dependent variable $\overline{\rho^2}$ is uncoupled from the first three. It is therefore also linear and can be solved independently after solutions for $\overline{\rho w}$ and $\hat{\rho}$ have been obtained as functions of ζ and t' . The four equations above contain only two

dimensionless parameters, R and $\frac{H}{kL}$, and four closure parameters, a_t , d_3 , d_4 , and $\frac{d_4 D}{k}$, which will be specified in the next section.

We will be interested in the steady-state solutions to equations (2.20), which can readily be found directly because the equations are linear. Steady-state solutions will be adequate for our purposes because the turbulence time scales and electrical relaxation times in the mixed layer are usually small compared to the time scales for changes in the parameters R and $\frac{H}{kL}$ or in the scale factors u_* , T_* , ρ_* , and H . The method of obtaining a numerical solution to the steady-state problem will be discussed in Chapter 3.

2.4 Specification of the Closure Constants

Having decided on the closure models, we must now specify the four arbitrary constants appearing in them. This is done in the same manner in which Wyngaard, Coté, and Rao (1974) specified most of their closure constants, depending on the semi-empirical surface-layer results. We notice that, as $\frac{z}{L} \rightarrow 0$ in the surface layer, many of the dimensionless mean gradients and turbulence statistics approach constant values. Thus, the possibility arises of taking the limit of equations (2.20) as $\frac{z}{L} \rightarrow 0$ to obtain algebraic equations from which the closure constants can be determined.

The surface-layer forms of the relevant quantities are as follows:

$$\left. \begin{aligned} \overline{\theta w} &\equiv -1.0^1 \\ k \zeta \frac{\partial \bar{\theta}}{\partial y} &\equiv \phi_h = 0.74 \left(1 - 9 \frac{z}{L}\right)^{-1/2} \rightarrow 0.74^2 \\ \overline{w^2} &= 1.75 + 2 \left(-\frac{z}{L}\right)^{2/3} \rightarrow 1.75^3 \end{aligned} \right\} \quad (2.21)$$

$$\left. \begin{aligned}
 \overline{u^2} &= 4.0^4 \\
 \overline{v^2} &= 1.75^4 \\
 \frac{k_z}{u_*^3} \overline{\epsilon} &= \left[1.0 + 0.5 \left(-\frac{z}{L} \right)^{2/3} \right]^{3/2} \rightarrow 1.0^5
 \end{aligned} \right\} \quad (2.21) \text{ cont.}$$

The dimensionless forms $\overline{u^2}$ and $\overline{v^2}$ have been defined as was $\overline{w^2}$ in (2.19). The sources of these surface-layer formulae are referenced below. The limits approached as $\frac{z}{L} \rightarrow 0$ have been given where appropriate. From (2.21) and the definition of $\overline{\tau}$ given by Wyngaard, Coté, and Rao (1974), we find

$$\tau \equiv \frac{\overline{q'^2}}{\tau_* \overline{\epsilon}} = \frac{k_s}{R} \frac{7.5 + 2 \left(-\frac{z}{L} \right)^{2/3}}{\left[1 + 0.5 \left(-\frac{z}{L} \right)^{2/3} \right]^{3/2}} \rightarrow \frac{7.5 k_s}{R} \quad (2.22)$$

Define $\phi_p \equiv k_s \frac{\partial \hat{p}}{\partial \zeta}$ in analogy with ϕ_h , since we expect $\frac{\partial \hat{p}}{\partial \zeta}$ to become large in the surface layer as does $\frac{\partial \hat{\theta}}{\partial \zeta}$. However, we also expect $\overline{pw}$ (≈ 1.0), $\overline{\rho\theta}$, and $\overline{\rho^2}$ to be slowly varying through the surface layer in analogy with the traditional assumption of a constant flux layer in which $\overline{\theta'w'}$, $\overline{u'w'}$, and $\overline{\theta'^2}$ are essentially uniform. Putting these assumptions, along with (2.21) and (2.22), into the steady-state form of the three second-order-moment equations in (2.20) gives

$$0 = - \left[1 + \frac{d_s R}{7.5 k_s} \right] + 13.1 \frac{k_s}{R} \frac{\partial}{\partial \zeta} \left(\frac{\partial}{\partial \zeta} \overline{pw} \right) - \frac{1.75 R}{k_s} \left. \frac{\partial \phi_p}{\partial \zeta} \right|_0 + R \frac{H}{k_L} \overline{\rho\theta} \quad (2.23)$$

¹By definition.

²Businger (1973).

³Wyngaard and Coté (1974)

⁴Wyngaard, Coté, and Rao (1974)

⁵Wyngaard and Coté (1971)

$$\left. \begin{aligned} 0 &= -\left[1 + \frac{d_4 R}{7.5 k \zeta}\right] \overline{\rho \theta} \Big|_0 + 13.1 k a_t R \frac{\partial}{\partial \zeta} \zeta \frac{\partial}{\partial \zeta} \overline{\rho \theta} + \frac{R}{k \zeta} \phi_\rho \Big|_0 - \frac{0.74 R}{k \zeta} \\ 0 &= -\left[1 + \frac{d_4 D R}{7.5 k \zeta}\right] \overline{\rho^2} \Big|_0 + 13.1 k a_t R \frac{\partial}{\partial \zeta} \zeta \frac{\partial}{\partial \zeta} \overline{\rho^2} - \frac{R}{k \zeta} \phi_\rho \Big|_0 \end{aligned} \right\} \begin{array}{l} (2.23) \\ \text{cont.} \end{array}$$

In these equations, valid only in the surface layer limit as $\frac{z}{L} \rightarrow 0$, $\overline{\rho \theta} \Big|_0$, $\overline{\rho^2} \Big|_0$, and $\phi_\rho \Big|_0$ are constants. The second term in each equation is the turbulent transport term, containing the ζ -derivative of $\overline{\rho w}$, $\overline{\rho \theta}$, or $\overline{\rho^2}$. Since we expect these moments to be slowly varying, we can make the estimate $\frac{\partial}{\partial \zeta} \sim 1$, giving these terms the form $13.1 \cdot k a_t R \overline{\rho \theta} \Big|_0$. Thus, the conduction and turbulent transport terms (and the buoyant production term in the first equation) remain constant while the other terms blow up as $\zeta \rightarrow 0$. The $\zeta \rightarrow 0$ limit of (2.23) becomes

$$\begin{aligned} 0 &= -\frac{d_3}{7.5} - 1.75 \phi_\rho \Big|_0 \\ 0 &= -\frac{d_4}{7.5} \overline{\rho \theta} \Big|_0 + \phi_\rho \Big|_0 - 0.74 \\ 0 &= -\frac{d_4 D}{7.5 k} \overline{\rho^2} \Big|_0 - \phi_\rho \Big|_0 \end{aligned}$$

These three equations can readily be solved for three of the closure constants:

$$\left. \begin{aligned} d_3 &= -13 \phi_\rho \Big|_0 \\ d_4 &= \frac{7.5(\phi_\rho \Big|_0 - 0.74)}{\overline{\rho \theta} \Big|_0} \\ \frac{d_4 D}{k} &= -(7.5 \phi_\rho \Big|_0) / \overline{\rho^2} \Big|_0 \end{aligned} \right\} (2.24)$$

But we see that a_t has dropped out of the problem. In order to determine it, we must make recourse to comparing the implied eddy-diffusion coefficient

in the model, $K_t \equiv a_t \overline{w'^2}$, with the observed eddy diffusion coefficient for heat in the surface layer, $K_h \equiv -\frac{\overline{w'\theta'}}{\partial\theta/\partial z}$. Setting the two equal in the limit of $\frac{z}{L} \rightarrow 0$ gives

$$\left. \begin{aligned} a_t 7.5 \frac{kz}{u_*} 1.75 u_*^2 &= \frac{u_* T_* kz}{0.74 T_*} \\ \Rightarrow a_t &= 0.10 \end{aligned} \right\} \quad (2.25)$$

In order to evaluate d_3 , d_4 , and $\frac{d_4 D}{\kappa}$ we need the limiting values as $\frac{z}{L} \rightarrow 0$ of ϕ_ρ , $\overline{\rho\theta}$, and $\overline{\rho^2}$. These are the direct analogs of surface-layer profiles like ϕ_h and $\frac{1}{T_*^2} \overline{\theta'^2}$, which have been determined from extensive measurements on meteorological towers. Unfortunately, however, electrical turbulence statistics have not, to the author's knowledge, been measured in the surface layer. For this reason we must make educated guesses about their limiting values.

The easiest is $\phi_\rho|_0$, because it should behave (above any thin charge-source layer) just like any other passive scalar whose source is at the surface. This is because near the surface the turbulence time scale $\overline{\tau}$ will be so much shorter than the electrical relaxation time τ_λ that conduction may be neglected. But in the limit as $\frac{z}{L} \rightarrow 0$, temperature also behaves as a passive scalar; so we take $\phi_\rho|_0 = -\phi_h|_0 = -0.74$. The minus sign comes from the scaling, which takes $T_* < 0$ in unstable stratification so that $\phi_h \geq 0$.

If θ and ρ both behave like passive scalars in the $\frac{z}{L} \rightarrow 0$ limit, then $\overline{\rho\theta}$ ought to behave like $\overline{\theta^2}$. The limiting forms of their balance equations should look the same. But the molecular dissipation rate of $\overline{\theta^2}$ is twice as large as that of $\overline{\rho\theta}$ if we can neglect D compared to κ as in section 2.3. Since there is no vertical transport, production balances dissipation in these limiting, steady-state equations. Thus we conclude that $\overline{\rho\theta}|_0 = -2 \overline{\theta^2}|_0 = -2(4.0) = -8.0$, from Wyngaard, Coté, and Rao (1974).

Finally, we come to $\overline{\rho^2}|_0$. The situation here is the same as for $\overline{\theta^2}$ except that the dissipation rate should be smaller by the factor $\frac{D}{\kappa}$. We have estimated this factor to be $\frac{D}{\kappa} \approx 0.18$, taking $D = 3.75 \times 10^{-6} \frac{\text{m}^2}{\text{sec}}$ as the diffusion coefficient for small ions* and $\kappa = 2.08 \times 10^{-5} \frac{\text{m}^2}{\text{sec}}$ as the thermometric molecular conductivity for air. This implies that $\overline{\rho^2}|_0 = 5.6 \overline{\theta^2}|_0 = 22$.

Together, these estimated limiting values of the electrical profiles imply the following values for the closure constants:

$$\left. \begin{array}{l} \phi_p|_0 = -0.74 \\ \overline{\rho\theta}|_0 = -8.0 \\ \overline{\rho^2}|_0 = 22. \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} d_3 = 9.7 \\ d_4 = 1.4 \\ \frac{d_4 D}{\kappa} = 0.25 \end{array} \right\} \quad (2.26)$$

Comparing these results with (2.10) we see that d_3 and d_4 are the same as the corresponding closure constants used by Wyngaard, Coté, and Rao (1974) in their equations for $\overline{\theta w}$ and $\overline{\theta^2}$. Since these constants seem to work for them, we can hope that they will also be satisfactory here.

2.5 Choice of Input Profiles

The next task before the model can be run for an arbitrary unstable case is to choose profiles of the non-dimensional input functions $\overline{\theta w}$, $\frac{\partial \hat{\theta}}{\partial z}$, $\overline{w^2}$, and τ in terms of the two parameters R and $\frac{H}{kL}$ and non-dimensional height $\zeta \equiv \frac{z}{H}$. Since the results of the Wyngaard model did not properly reproduce the surface layer, due probably to a problem with the vertical transport of $\overline{w^2}$ near the surface, it was decided not to use the output of that model directly. Instead, empirical formulae are developed below from

* This value of D was calculated from the formula given in Appendix 1, $T = 299^\circ\text{K}$ and $k = 1.5 \times 10^{-4} \text{ m}^2/\text{volt sec}$.

a variety of sources. These are regarded to well-approximate the actual behavior of the relevant profiles in an unstable planetary boundary layer capped by a stable layer, provided that it is growing slowly enough to be in a quasi-steady state.

In a steady state, the profile of $\overline{\theta w}$ must be linear. We use the rigid-lid approximation, assuming $\overline{\theta w}$ to vanish at the top of the mixed layer. Thus, our heat-transport profile becomes

$$\overline{\theta w} \equiv \frac{\overline{\theta' w'}}{u_* T_*} = - (1.0 - \xi) \quad (2.27)$$

It agrees fairly well with the form advocated by Lenschow (1974) and with model calculations.

The potential temperature gradient in the interior of a mixed layer is known to be small. Since the unstable surface-layer temperature profile as given by Businger (1973) becomes small for large $\frac{z}{L}$ at a rate which agrees well with the results of the Wyngaard model, we will use it here:

$$\frac{d\hat{\theta}}{d\xi} \equiv \frac{H}{T_*} \frac{d\bar{\theta}}{dz} = \frac{0.74}{k\xi} \left[1.0 - 9.0 \frac{z}{L} \right]^{-1/2} \quad (2.28)$$

This must be used with the understanding that it may not correctly represent the weak temperature gradient in the interior of the layer. There is speculation that the stratification may be slightly stable there. In the next chapter it will be shown that a change in sign of the interior potential-temperature gradient does not significantly change the results.

Somewhat more complicated is the behavior of $\overline{w^2}$ in the mixed layer. The surface-layer profile of $\overline{w^2}$ has already been given in (2.21) as

$$\overline{w^2} \equiv \frac{\overline{w'^2}}{u_*^2} = 1.75 + 2.0 \left(-\frac{z}{L} \right)^{2/3}, \quad -1 \lesssim \frac{z}{L} \lesssim 0$$

The correct form of $\overline{w^2}$ in the region of local free convection is given by Wyngaard and Coté, and Izumi (1971) as

$$\overline{w^2} = 3.6 \left(-\frac{z}{L}\right)^{2/3}, \quad \frac{z}{L} \lesssim -1, \quad \xi \ll 1$$

In the interior of the mixed layer Tennekes (1970) has shown that the average magnitude of $\overline{w^2}$ should scale with $(-\frac{H}{L})^{2/3}$. From the model results of Wyngaard and of Deardorff, we have chosen the constant and have used

$$\overline{w^2} \approx 0.88 \left(-\frac{H}{L}\right)^{2/3}$$

Finally we must demand that $\overline{w^2}$ fall off to zero at the rigid lid. Again the form of the roll-off is taken from the model results of Wyngaard and of Deardorff. It is approximated by the multiplicative factor

$$(1.0 - \xi^3)$$

Now we need a way of changing smoothly from one formula to another at the appropriate transition height. If f_1 and f_2 are two formulae, we choose the transition function

$$\frac{f_1 + (\xi/\xi_0)^3 f_2}{1.0 + (\xi/\xi_0)^3}$$

where ξ_0 is the transition height. Making use of this transition function twice to couple the three separate $\overline{w^2}$ profiles and multiplying by the roll-off factor finally gives

$$\overline{w^2} = \left\{ 1.75 + 2.0 \left(-\frac{z}{L}\right)^{2/3} + \left(-\frac{z}{L}\right)^3 \left[\frac{3.6 \left(-\frac{z}{L}\right)^{2/3} + 0.88 \left(-\frac{H}{L}\right)^{2/3} \left(\frac{\xi}{0.12}\right)^3}{1.0 + \left(\frac{\xi}{0.12}\right)^3} \right] \right\} \left\{ \frac{1.0 - \xi^3}{1.0 + \left(-\frac{z}{L}\right)^3} \right\} \quad (2.29)$$

The complicated formula appears to give the correct behavior of $\overline{w^2}$ for $\frac{H}{L} \gtrsim 10$. Its behavior is compared with the surface-layer, free-convection,

Fig.2.1 Comparison of the approximate formula (2.29) for $\overline{w^2}$ (plotted circles) with the surface-layer and free-convection profiles, the estimate of Tennekes (1970) for the interior of the mixed layer, and the profile predicted by the Wyngaard model, for $H/L = -5$.

Fig.2.2 Same as figure 2.1, for $H/L = -10$.

Fig.2.3 Same as figure 2.1, for $H/L = -50$.

Fig.2.4 Same as figure 2.3, but with a stretched ζ -coordinate.

Fig.2.5 Same as figure 2.1, for $H/L = -1000$.

and mixed layer forms given above, and with the Wyngaard-model results, for various values of $\frac{H}{L}$ in figures 2.1 through 2.5 .

The last input function we must specify is $\tau \equiv \frac{\overline{u^2} + \overline{v^2} + \overline{w^2}}{\epsilon}$, for which we need profiles of $\epsilon \equiv \frac{H}{u_*^3} \bar{\epsilon}$ and $\overline{u^2} + \overline{v^2}$. Wyngaard and Coté (1971) give the surface-layer form of $\bar{\epsilon}$ as

$$\frac{k z \bar{\epsilon}}{u_*^3} = \left[1.0 + 0.5 \left(-\frac{z}{L} \right)^{2/3} \right]^{3/2}$$

Lenschow (1974) indicates that in the upper part of the mixed layer the dissipation approaches the constant value of $\frac{H}{u_*^3} \bar{\epsilon} = 1.43 \left(-\frac{H}{L} \right)$. The surface layer form can be made to approach this limit by adjusting the 0.5 to 0.6 . Thus we will adopt the form

$$\epsilon \equiv \frac{H}{u_*^3} \bar{\epsilon} = \frac{1}{k \zeta} \left[1.0 + 0.6 \left(-\frac{z}{L} \right)^{2/3} \right]^{3/2} \quad (2.30)$$

The least satisfying input function of all is $\overline{u^2} + \overline{v^2}$. Here, the empirical data is not conclusive, and we have had to make some guesses. $\overline{u^2}$ and $\overline{v^2}$ do not obey Monin-Obukhov similarity in the unstable surface layer. This matter is discussed with regard to the Kansas data by Wyngaard and Izumi (1973), with the tentative conclusion that $\overline{v^2}$ shows evidence of w_* scaling.* The reason for this seems to be that the relatively large values of $\overline{w^2}$ occurring in the unstable planetary boundary layer, which do scale with $\left(-\frac{H}{KL} \right)^{2/3}$, are associated with some form of buoyant plumes extending down almost to the surface. These buoyancy-driven vertical motions must be accompanied at the surface by equally strong horizontal divergences and convergences in order to preserve continuity. Thus, below a strongly

* So-called w_* scaling occurs when a turbulence statistic in the appropriate non-dimensional form becomes proportional to $\frac{w_*}{u_*} \equiv \left(-\frac{H}{KL} \right)^{1/3}$.

convective layer $\overline{u^2}$ and $\overline{v^2}$ in the surface layer should also tend to scale with $\frac{w_*^2}{u_*^2}$.

The results of the numerical model presented by Deardorff (1974a) support this speculation, indicating that $\overline{u^2}$ and $\overline{v^2}$ have pronounced maxima in the surface layer which scale with $\frac{w_*^2}{u_*^2}$. Of course, these statistics also exhibit w_* scaling on the interior of the unstable layer, because there $\overline{u^2}$ and $\overline{v^2}$ are produced primarily by the cascade process from $\overline{w^2}$. As one might expect from a simple-minded consideration of the production of vortex lines by buoyancy, $\frac{\overline{u^2}}{\overline{w^2}} \sim \frac{\overline{v^2}}{\overline{w^2}} \sim \frac{1}{2}$ in the interior. Taking into account both shear and buoyant production, we have drawn our best estimate of the unstable profile for $\overline{u^2} + \overline{w^2}$ from Deardorff's curves:

$$\overline{u^2} + \overline{v^2} \equiv \frac{\overline{u^2}}{u_*^2} + \frac{\overline{v^2}}{u_*^2} = \left[0.25 + 0.75(1.0 - \xi)^4 \right] \frac{w_*^2}{u_*^2} + 5.75(1.0 - \xi)^2 \quad (2.31)$$

However, the alert reader will notice that this profile disagrees with the $\frac{z}{L} \rightarrow 0$ limit presented in equations (2.21). We have not actually used the above profile of $\overline{u^2} + \overline{v^2}$ for the very reason that it includes some $\frac{w_*^2}{u_*^2}$ down to the surface. This w_* scaling at the surface is undesirable for two reasons. Since w_* would then appear in τ , which appears in the formulae determining the closure constants (2.23), all these "constants" would depend on $\frac{H}{L}$. This situation would not only be unsatisfying from the modeling point of view, but would cause d_3 and d_4 to become inordinately large, and a_t much too small, at large $\frac{H}{L}$. The forms of the closure constants that result from using (2.31) are

$$\left. \begin{aligned} d_3 &= 9.7 + 1.3 \frac{w_*^2}{u_*^2} &= 45. \\ d_4 &= 0.14 d_3 &= 6.5 \\ a_t &= 1/d_3 &= 0.022 \end{aligned} \right\} \text{ for } \frac{H}{L} = -50$$

These new values of the closure constants cause a tremendous reduction in the height to which significant charge convection can occur in the model, making its predictions entirely unrealistic. Therefore, the $\overline{u^2} + \overline{v^2}$ profile given in (2.31) must be discarded.

The w_* scaling must be removed from $\overline{u^2} + \overline{v^2}$ in its surface-layer limit. A weak rationale can be produced for this as follows. τ is supposed to be the dissipation time scale relevant to the de-correlating of ρ' and w' by the cascade process. This de-correlating effect ought to be controlled by turbulent motions on the same scale as $\overline{\rho w}$, which scale should depend strongly on the height above the ground. In the surface layer most of $\overline{\rho w}$ will be in relatively small scales of motion or high frequencies. However, the part of $\overline{u^2}$ or $\overline{v^2}$ which exhibits w_* scaling should be in considerably larger scales or lower frequencies, because it is produced by vertical motions over the whole interior of the mixed layer. Therefore, it can be argued that this part of $\overline{u^2} + \overline{v^2}$ is irrelevant in calculating τ in the surface layer.

There seems to be no theoretical basis for making such a separation of the part of $\overline{u^2} + \overline{v^2}$ due to the convergence or divergence at the surface from the remainder generated by shear production and cascade from $\overline{w^2}$ in the interior. We have simply made the crude assumption that only shear production is relevant at the surface, and that higher up the horizontal energy is proportional to the buoyancy-generated $\overline{w^2}$. This agrees fairly well with the results of the Wyngaard model using our earlier-mentioned bottom boundary conditions (personal communication), and is considered to be the best we can do at present.

$$\overline{u^2} + \overline{v^2} = 0.80 \overline{w^2} + 4.35(1.0 - \xi)^2 \quad (2.32)$$

Since $\overline{w^2} = 1.75$ at $\zeta = 0$, it satisfies the requirement that $\overline{u^2} + \overline{v^2} = 5.75$ at the bottom boundary. Though the profile used here is quite uncertain, it is used only in combination with $\overline{w^2}$ to form τ , which is therefore not too sensitive to possible errors.

This completes the development of our convection-current model in the uniform-conductivity case. From the simplicity of the closure assumptions, the lack of data for determining the closure constants, and the uncertainty of some of the input profiles, it can be seen that the model results will be preliminary in the same sense that the Wyngaard, Coté, and Rao (1974) results are preliminary. They should give a good picture of the behavior of the convection current in various meteorological conditions and adequate estimates of the magnitude of the effects. However, they should be regarded as a guide for future measurements and modeling rather than as a final picture of charge convection in the mixed layer. In the next chapter the model developed here will be solved numerically and its sensitivity will be tested to variation of the closure constants and some of the input profiles.

3 SOLUTIONS OF THE UNIFORM-CONDUCTIVITY MODEL

3.1 Numerical Solution Methods

In this section we will discuss the method of obtaining numerical solutions to the dimensionless model equations (2.20). These equations contain only two dimensionless parameters, R and $\frac{H}{kL}$. They also contain the four input functions $\overline{\theta w}$, $\frac{d\hat{\theta}}{d\zeta}$, $\overline{w^2}$, and $\tau \equiv \frac{\overline{u^2} + \overline{v^2} + \overline{w^2}}{\epsilon}$ defined by equations (2.27), (2.28), (2.29), (2.30), and (2.32), and the four closure constants a_t , d_3 , d_4 , and $\frac{d_4 D}{\kappa}$ defined by (2.25) and (2.26). As discussed at the end of section 2.3, we will be concerned only with steady-state solutions and will therefore be able to separate the $\overline{\rho^2}$ equation from the other three.

The first step is to eliminate $\frac{d\hat{\rho}}{d\zeta}$ from the $\overline{\rho w}$ and $\overline{\rho \theta}$ equations by substitution from the $\hat{\rho}$ equation. In the steady-state we have from (2.20)

$$\hat{\rho} = -R \frac{d}{d\zeta} \overline{\rho w} + \hat{Q} \quad (3.1)$$

Differentiation by ζ and substitution of this into the steady-state $\overline{\rho w}$ and $\overline{\rho \theta}$ equations gives

$$\left. \begin{aligned} & \left[-\left(1 + \frac{d_3}{\tau}\right) + a_t R^2 \left(\frac{d}{d\zeta} \tau \overline{w^2}\right) \frac{d}{d\zeta} + R^2 \overline{w^2} (a_t \tau + 1) \frac{d^2}{d\zeta^2} \right] \overline{\rho w} \\ & + R \frac{H}{kL} \overline{\rho \theta} = R \overline{w^2} \frac{d\hat{Q}}{d\zeta} \\ & \left[-\left(1 + \frac{d_4}{\tau}\right) + a_t R^2 \left(\frac{d}{d\zeta} \tau \overline{w^2}\right) \frac{d}{d\zeta} + a_t R^2 \tau \overline{w^2} \frac{d^2}{d\zeta^2} \right] \overline{\rho \theta} \\ & + R \left(-\frac{d\hat{\theta}}{d\zeta} + R \overline{\theta w} \frac{d^2}{d\zeta^2} \right) \overline{\rho w} = R \overline{\theta w} \frac{d\hat{Q}}{d\zeta} \end{aligned} \right\} \quad (3.2)$$

Here we have two coupled, second-order, ordinary, differential equations in $\overline{\rho w}$ and $\overline{\rho \theta}$. If $\frac{d\hat{Q}}{d\zeta} \neq 0$, they are inhomogeneous. They can readily be

solved numerically by inversion of a block-tridiagonal matrix with 2×2 blocks. After obtaining the solution to (3.2) and using it in (3.1) to obtain $\hat{\rho}$, we can proceed to solve the $\bar{\rho}^2$ equation

$$\left[-\left(1 + \frac{d_4 D}{K \tau}\right) + \alpha_t R^2 \left(\frac{d}{d\zeta} \tau \bar{w}^2\right) \frac{d}{d\zeta} + \alpha_t R^2 \tau \bar{w}^2 \frac{d^2}{d\zeta^2} \right] \bar{\rho}^2 = R \bar{\rho} w \frac{d\hat{\rho}}{d\zeta} \quad (3.3)$$

This is another second-order, inhomogeneous, ordinary, differential equation which can be solved numerically by inversion of a tridiagonal matrix.

Before attempting to solve these equations, we will make a change of variable to stretch the ζ coordinate near the surface. The reason for this is that $\hat{\rho}$ and $\frac{d\hat{\rho}}{d\zeta}$, as well as $\frac{d\hat{\theta}}{d\zeta}$ and τ , will be rapidly changing near the bottom boundary. Since $\frac{d\hat{\theta}}{d\zeta}$ and $\frac{1}{\tau}$ go as ζ^{-1} in the surface layer, and we anticipate that $\frac{d\hat{\rho}}{d\zeta}$ will do the same, the appropriate new coordinate to use is $\ln \zeta$. We therefore make the following change of variable:

$$\frac{\zeta}{\zeta_0} \equiv e^{\Delta n} \iff n \equiv \frac{1}{\Delta} (\ln \zeta - \ln \zeta_0) \quad (3.4)$$

In our numerical solution we will take n as an integer in the range $[1, N]$, specifying the point in the finite-difference scheme. The bottom boundary will be ζ_0 at $n=1$ and the top boundary will be $\zeta = 1.0$ at $n=N$.

These definitions and equations (3.4) determine Δ and ζ_0 as

$$\zeta_0 = \zeta_{\infty}^{\frac{N}{N-1}} \quad \text{and} \quad \Delta = - \frac{\ln \zeta_{\infty}}{N-1} \quad (3.5)$$

according to the number of points and the bottom boundary height. From numerical experimentation, it was found that $N=100$ yielded enough accuracy for most purposes.

We use centered, finite-difference approximations to the first and second derivatives.

$$\left. \begin{aligned} \left(\frac{df}{dn} \right)_n &= \frac{1}{2} (f_{n+1} - f_{n-1}) \\ \left(\frac{d^2f}{dn^2} \right)_n &= f_{n+1} + f_{n-1} - 2f_n \end{aligned} \right\} \quad (3.6)$$

These formulae are of second-order accuracy, meaning in this case that their error is of the order of $-\frac{1}{6} \left(\frac{d^3f}{dn^3} \right)_n$ for the first derivative and $-\frac{1}{12} \left(\frac{d^4f}{dn^4} \right)_n$ for the second derivative. The factors of δn which would ordinarily appear in these formulae are missing because we have taken $\delta n \equiv 1$.

However, this does not mean that the errors remain finite as the number of grid points increases, because $\frac{d}{dn} = \frac{dz}{dn} \frac{d}{dz}$, where $\frac{dz}{dn} = \Delta \zeta_o e^{\Delta n}$. As can be seen from (3.5), the scale factor $\frac{dz}{dn}$ becomes small as N increases because Δ decreases as $\frac{1}{N-1}$. Therefore, we expect $\left(\frac{d^{(j)}f}{dn^{(j)}} \right)_n$ to decrease roughly as N^{-j} with increasing N and j ; formulae (3.6) are still of second-order accuracy, in spite of the missing factors of $(\delta n)^2$ in the error estimates.

Substitutions of (3.4) and (3.6) into equations (3.1), (3.2), and (3.3) gives the final finite-difference form of the model equations:

$$\hat{p}_n = -\frac{R}{2\Delta \zeta_o e^{\Delta n}} (\bar{\rho} \bar{w}_{n+1} - \bar{\rho} \bar{w}_{n-1}) + \hat{Q}_n \quad (3.7)$$

$$\left. \begin{aligned} & \frac{R^2}{\Delta \zeta_o^2 e^{2\Delta n}} \left[\frac{a_t}{4\Delta} (\tau_{n+1} \bar{w}_{n+1}^2 - \tau_{n-1} \bar{w}_{n-1}^2) + \left(\frac{1}{\Delta} - \frac{1}{2} \right) (a_t \tau_n + 1) \bar{w}_n^2 \right] \bar{\rho} \bar{w}_{n+1} \\ & - \left(1 + \frac{d_3}{\tau_n} + \frac{2R^2(a_t \tau_n + 1) \bar{w}_n^2}{\Delta^2 \zeta_o^2 e^{2\Delta n}} \right) \bar{\rho} \bar{w}_n \\ & + \frac{R^2}{\Delta \zeta_o^2 e^{2\Delta n}} \left[-\frac{a_t}{4\Delta} (\tau_{n+1} \bar{w}_{n+1}^2 - \tau_{n-1} \bar{w}_{n-1}^2) + \left(\frac{1}{\Delta} + \frac{1}{2} \right) (a_t \tau_n + 1) \bar{w}_n^2 \right] \bar{\rho} \bar{w}_{n-1} \\ & + R \frac{H}{\kappa L} \bar{\rho} \bar{\theta}_n = R \bar{w}_n^2 \left(\frac{\partial \hat{Q}}{\partial \zeta} \right)_n \end{aligned} \right\} \quad (3.8)$$

$$\left. \begin{aligned}
& \frac{a_t R^2}{\Delta g_o^2 e^{2\Delta n}} \left[\frac{1}{4\Delta} (\tau_{n+1} \bar{w}_{n+1}^2 - \tau_{n-1} \bar{w}_{n-1}^2) + \left(\frac{1}{\Delta} - \frac{1}{2} \right) \tau_n \bar{w}_n^2 \right] \bar{\rho} \bar{\theta}_{n+1} \\
& - \left(1 + \frac{d_4}{\tau_n} + \frac{2a_t R^2 \tau_n \bar{w}_n^2}{\Delta^2 g_o^2 e^{2\Delta n}} \right) \bar{\rho} \bar{\theta}_n \\
& + \frac{a_t R^2}{\Delta g_o^2 e^{2\Delta n}} \left[-\frac{1}{4\Delta} (\tau_{n+1} \bar{w}_{n+1}^2 - \tau_{n-1} \bar{w}_{n-1}^2) + \left(\frac{1}{\Delta} + \frac{1}{2} \right) \tau_n \bar{w}_n^2 \right] \bar{\rho} \bar{\theta}_{n-1} \\
& + \frac{R^2 \bar{\theta}_n}{\Delta g_o^2 e^{2\Delta n}} \left(\frac{1}{\Delta} - \frac{1}{2} \right) \bar{\rho} \bar{w}_{n+1} - R \left(\left(\frac{d\hat{\theta}}{dg} \right)_n + \frac{2R \bar{\theta}_n}{\Delta^2 g_o^2 e^{2\Delta n}} \right) \bar{\rho} \bar{w}_n \\
& + \frac{R^2 \bar{\theta}_n}{\Delta g_o^2 e^{2\Delta n}} \left(\frac{1}{\Delta} + \frac{1}{2} \right) \bar{\rho} \bar{w}_{n-1} = R \bar{\theta}_n \left(\frac{d\hat{Q}}{dg} \right)_n
\end{aligned} \right\} \quad (3.8) \text{ cont.}$$

$$\left. \begin{aligned}
& \frac{a_t R^2}{\Delta g_o^2 e^{2\Delta n}} \left[\frac{1}{4\Delta} (\tau_{n+1} \bar{w}_{n+1}^2 - \tau_{n-1} \bar{w}_{n-1}^2) + \left(\frac{1}{\Delta} - \frac{1}{2} \right) \tau_n \bar{w}_n^2 \right] \bar{\rho}^2_{n+1} \\
& - \left(1 + \frac{d_4 D}{k \tau_n} + \frac{2a_t R^2 \tau_n \bar{w}_n^2}{\Delta^2 g_o^2 e^{2\Delta n}} \right) \bar{\rho}^2_n \\
& + \frac{a_t R^2}{\Delta g_o^2 e^{2\Delta n}} \left[-\frac{1}{4\Delta} (\tau_{n+1} \bar{w}_{n+1}^2 - \tau_{n-1} \bar{w}_{n-1}^2) + \left(\frac{1}{\Delta} + \frac{1}{2} \right) \tau_n \bar{w}_n^2 \right] \bar{\rho}^2_{n-1} \\
& = R \bar{\rho} \bar{w}_n \left(\frac{d\hat{\rho}}{dg} \right)_n
\end{aligned} \right\} \quad (3.9)$$

Appendix 3 describes the numerical methods of solution for a tridiagonal system, as in (3.9), and for a block-tridiagonal system, as in (3.8), given boundary conditions on $\bar{\rho} \bar{w}$, $\bar{\rho} \bar{\theta}$, and $\bar{\rho}^2$ at $n=1$ and $n=N$.

Three kinds of lower boundary conditions can be used on these equations, depending on whether or not $\hat{Q} \neq 0$. If $\hat{Q} \equiv 0$ everywhere within the region $[z_{00}, 1]$, any convection current must arise from sources below the bottom boundary of the region. In this case the appropriate lower boundary conditions are those derived in the end of section 2.4. From equation (2.26) and the definition of ρ_* , we have

$$\hat{Q} \equiv 0, \quad \overline{\rho w}_1 = 1.0, \quad \overline{\rho \Theta}_1 = -8.0, \quad \overline{\rho^2}_1 = 22. \quad (3.10)$$

These conditions are analogous to those used by Wyngaard, Coté, and Rao (1974) in solving their dynamical second-order-closure model described in section 2.2. They are essentially a matching to the presumed surface-layer profiles in the limit $\frac{z}{L} \rightarrow 0$, and they correspond to a situation in which the convection current forced into the bottom of the mixed layer by some source is known externally. This is the same assumption that was made in section 1.3 to get solutions for the simple, convective-column model. However, it does not correspond exactly to any useful physical situation, as will be seen later.

In the numerical solutions to the second-order-closure model for convection current described above, not only the solutions for $\overline{\rho w}$, $\overline{\rho \theta}$, $\hat{\rho}$, and $\overline{\rho^2}$, but also the value of each term on the right-hand side of the $\overline{\rho w}$ and $\overline{\rho \theta}$ equations in (2.20), were printed out at each grid point. In addition to providing a check on the numerical scheme if these terms summed to zero for each equation at each grid point, printing out this additional information allowed the budgets for the two second moments $\overline{\rho w}$ and $\overline{\rho \theta}$ to be examined in detail. It was possible to determine, for example, whether buoyant production $(R \frac{H}{kL} \overline{\rho \theta})$ or gradient production $(-R \overline{w^2} \frac{d\hat{\rho}}{dz})$ was mainly responsible for the large value of $\overline{\rho w}$ at some level, and whether this

dominant production term was balanced primarily by conductive dissipation ($-\overline{\rho w}$), pressure-scrambling ($-\frac{d_3}{\tau} \overline{\rho w}$), or divergence of turbulent transport ($a_t R^2 \frac{d}{d\zeta} \tau \overline{w^2} \frac{d}{d\zeta} \overline{\rho w}$) there. The main result of running the model with the inhomogeneous boundary conditions (3.10) was that turbulent transport was indeed found to be negligible in the budgets of $\overline{\rho w}$ and $\overline{\rho \theta}$ in the surface layer, as argued in section 2.4.

A way was contrived of modifying these boundary conditions in order to further test the validity of the values $\overline{\rho \theta}|_0 = -8.0$ and $\overline{\rho^2}|_0 = 22$ developed in (2.26). Homogeneous lower boundary conditions were combined with inhomogeneous equations in which $\hat{Q} \neq 0$ in a thin layer just above the surface, so that $\overline{\rho \theta}$ and $\overline{\rho^2}$ could be self-determining near the bottom boundary.

$$\hat{Q} \neq 0, \quad \overline{\rho w}_1 = 0, \quad \overline{\rho \theta}_1 = 0, \quad \overline{\rho^2}_1 = 0 \quad (3.11)$$

In this case, ρ_* is no longer determined directly from the surface charge flux $\overline{\rho'w'}|_0$, as before, but from the normalization of $\hat{Q}$. Integration of the steady-state, mean charge equation (3.1) from the bottom boundary at ζ_{oo} to the level ζ_q above which $\hat{Q} \equiv 0$ gives

$$\int_{\zeta_{oo}}^{\zeta_q} \hat{\rho} d\zeta = -R \left(\overline{\rho w}|_{\zeta_q} - \overline{\rho w}|_{\zeta_{oo}} \right) + \int_{\zeta_{oo}}^{\zeta_q} \hat{Q} d\zeta \quad (3.12)$$

Reference to (2.18) reminds us that the left-hand side of this equation represents the decay by conduction of the mean charge density within the layer (ζ_{oo}, ζ_q) . If this layer is thin enough, the left-hand side will be negligible compared to the turbulent transport of charge out of the layer. Therefore ignoring this conduction term and demanding $\overline{\rho w}|_{\zeta_q} \equiv 1.0$ (similar to the earlier scaling, where ρ_* was determined by specifying $\overline{\rho w}|_0 \equiv 1.0$) gives

$$\int_{\xi_{oo}}^{\xi_r} \hat{Q} d\xi \equiv R \quad (3.13)$$

Further comment about the boundary conditions (3.11) and (3.13) is in order. These are intended to simulate the Blanchard-effect source, where the distribution of $\bar{Q}$ is known as a function of z and u_* (see Chapter 4) and where z_o , corresponding to the bottom boundary of the model at ξ_{oo} , is the surface roughness length. However, since we have used the boundary condition $\overline{\rho w}_1 = 0$, we are really still specifying $\overline{\rho w}(\xi_q)$ near the surface via equation (3.12). In the real Blanchard effect, on the other hand, only part of the charge source specified by $\int_{z_o}^{z_q} \bar{Q}(z) dz$ is convected upward and part is convected downward to the surface. Therefore, we should have $\overline{\rho w}_1 < 0$ self-determined by the model. The correct lower boundary condition in this case should be

$$0 \equiv \hat{\rho}(\xi_{oo}) = -R \left. \frac{d}{d\xi} \overline{\rho w} \right|_{\xi_{oo}} + \hat{Q}(\xi_{oo}) \quad (3.14)$$

since ions are annihilated by contact with the conducting surface. More will be said about this later.

Evidently, the modified boundary conditions given by (3.11) are really quite similar to the previous inhomogeneous boundary conditions (3.10). The results are quite similar also. The numerical solutions show that the profiles of $\overline{\rho w}$ and $\overline{\rho \theta}$ above ξ_q are essentially the same in the two cases. The turbulent transport terms in the $\overline{\rho w}$ and $\overline{\rho \theta}$ equations are still negligible immediately above the thin source layer. Furthermore, neither changing ξ_q by an order of magnitude in either direction, nor changing the shape of the $\hat{Q}(\xi)$ distribution, has an appreciable effect on the results. Thus it appears that the two sets of boundary

conditions described here are essentially equivalent as far as profiles of $\overline{\rho w}$, $\overline{\rho \theta}$, and $\hat{\rho}$ are concerned.

The story is different, however, in the case of the $\overline{\rho^2}$ profiles. Changing from the inhomogeneous to the homogeneous boundary conditions has a considerable effect on the magnitude of $\overline{\rho^2}$ throughout the mixed layer, which is also sensitive to changes in ζ_q . Although the individual terms in the budget equation for $\overline{\rho^2}$ have not been calculated, it appears that the reason for this lies in an increased importance of the turbulent-transport term throughout the mixed layer in this equation. However, little importance is attached to this matter because the $\overline{\rho^2}$ results are considered to be the least reliable of the model and because, in any case, they do not feed back to influence the other model results.

Finally, the third kind of boundary conditions used for the uniform-conductivity model is the most physically realistic. We take the bottom boundary to be at the surface roughness height z_0 . By surface-layer convention, this is the level at which the mean wind speed vanishes, and we will take it also as the level at which the conducting surface annihilates ions. Therefore, we demand that all charge fluctuations, and the mean charge density itself, vanish there. However, we allow $\overline{\rho'w'}(z_0) < 0$, because we expect some of the charge from a source layer near the surface to be carried down to the surface by convection and molecular diffusion. The missing boundary condition on $\overline{\rho w}$ is replaced by one on $\hat{\rho}$ given by (3.14).

$$\hat{Q} \neq 0, \quad \left. \frac{d}{dz} \overline{\rho w} \right|_1 = \frac{1}{R} \hat{Q}, \quad \overline{\rho \theta}_1 = 0, \quad \overline{\rho^2}_1 = 0 \quad (3.15)$$

Again, $\hat{Q}$ is normalized by equation (3.13) so that $\rho_* u_* \equiv \int_{z_0}^{z_q} \overline{Q}(z) dz$.

Results obtained with these boundary conditions will be described in section 3.5.

In all the model runs presented in this thesis, the upper boundary conditions are homogeneous.

$$\overline{\rho w}_N = 0, \quad \overline{\rho \theta}_N = 0, \quad \overline{\rho^2}_N = 0 \quad (3.16)$$

These are the same upper boundary conditions as those used by Wyngaard, Coté, and Rao (1974) in their dynamical second-order-closure model, corresponding to a rigid lid at $z = H$. It would be interesting to relax this restriction in a future study in order to investigate the convection of charge from the sharp conductivity gradient usually associated with the inversion capping a mixed layer.

3.2 Comparison with Simple Models

In this section, the results of the second-order-closure model for convection current will be examined in two special cases in an effort to lend credibility to its results. First, the results in the case of neutral stability will be compared with those of an eddy-diffusion model. And second, the results in two unstable cases will be compared with those of the simple convective-column model of Appendix 2.

Unfortunately, the only simple case in which we can determine the correct result with any degree of certainty is the neutral case. In a neutral boundary layer it is fairly well accepted that the transport of conservative properties can be represented by an eddy-diffusion model. The only trick is to choose the proper profile of the eddy-diffusion coefficient. Drs. Wyngaard and Coté (personal communication) have kindly made available the results of one of their preliminary neutral runs with the model described

in section 2.2. These results include profiles of $\frac{H}{u_*} \frac{d\bar{u}}{dz}$, $\frac{H}{u_*} \frac{d\bar{v}}{dz}$, $\frac{1}{u_*^2} \overline{u'w'}$, $\frac{1}{u_*^2} \overline{v'w'}$, $\frac{\overline{u'^2}}{u_*^2}$, $\frac{\overline{v'^2}}{u_*^2}$, $\frac{\overline{w'^2}}{u_*^2}$, and $\frac{H}{u_*^3} \bar{\epsilon}$ from the surface layer (the lower boundary was at $z/H = 0.00271$) up to $z/H = 1.85$, where a rigid lid provided the upper boundary conditions. The height scale for the neutral, barotropic, boundary layer was taken as $H \equiv \frac{u_*}{f}$. The run we used was appropriate to 60°N latitude, although the runs for 45°N and 30°N were essentially the same.

It was decided to use the results of the Wyngaard model in the neutral case, instead of observational results, because the model provided all the turbulence statistics throughout the entire Ekman layer, not just the eddy-diffusion coefficient. This allowed us to make a comparison between our closure model of the convection current and a simple eddy-diffusion model. From the results of the Wyngaard-model run we calculated the profiles of the effective eddy-diffusion coefficients for momentum, $K_{mu} \equiv -\frac{d\bar{u}/dz}{\overline{u'w'}}$ and $K_{mv} \equiv \frac{-d\bar{v}/dz}{\overline{v'w'}}$, and of the eddy-diffusion coefficient used in the closure assumptions for the turbulent-transport terms (2.9), $K_t \equiv a_t \overline{\tau w'^2}$.* These two profiles are presented in dimensionless form in figure 3.1, along with the surface-layer form of $\frac{f}{u_*^2} K_h$. Below $\frac{zf}{u_*} \approx 0.15$, K_{mu} and K_{mv} agreed quite well, but above that level there was considerable scatter, especially in K_{mu} near $\frac{zf}{u_*} = 0.31$, where $\frac{d\bar{u}}{dz}$ changed sign. Therefore, the curve for $\frac{f}{u_*^2} K_m$ shown in the figure is a smoothed composite of $\frac{f}{u_*^2} K_{mu}$ and $\frac{f}{u_*^2} K_{mv}$ above $\frac{zf}{u_*} \approx 0.15$.

Although these curves of dimensionless K_m and K_t cannot be taken as gospel, they are certainly of the right general behavior and

* In the latter we used our value of $a_t = 0.10$, instead of Wyngaard's value of 0.15, because it gave better agreement with the surface-layer form of K_h in (1.12).

Fig.3.1 Dimensionless profiles of the eddy-diffusion coefficient for momentum, K_m , and that used in the turbulent-transport terms, K_t , from the neutral run of the Wyngaard model are compared with the surface-layer profile and our approximate formula (plotted circles) based on (3.17).

magnitude. From Lettau's (1950) reexamination of the famous Leipzig wind profile, we calculate a maximum of $\frac{f}{u_*^2} K_m \approx 0.0048$, though this maximum occurs at the much lower height of $zf/u_* \approx 0.05$, appearing to disagree with the surface layer profile mentioned earlier. From two profiles of $\frac{K_m}{ku_*}$ given by Blackadar (1962) for widely differing values of surface roughness, we calculate maxima of $\frac{f}{u_*^2} K_m \approx 0.0035$ and 0.0050 at heights $\frac{zf}{u_*} \approx 0.041$ and 0.056 , respectively. Finally, from the neutral run of Deardorff's (1972) giant numerical model with the rigid lid at $\frac{zf}{u_*} = 0.45$, which was probably too low, we find a maximum of $\frac{f}{u_*^2} K_m \approx 0.02$ at $\frac{zf}{u_*} \approx 0.2$. We place more confidence in the height of the maximum calculated by Deardorff, though all of the maximum values are of the same order as those shown in figure 3.1.

In figure 3.2 the profile of $f\bar{\tau} \equiv f \frac{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}{\varepsilon}$, computed from the Wyngaard neutral run, is presented along with the surface-layer form from (2.22). Figure 3.3 shows the corresponding profile of $\frac{\overline{w'^2}}{u_*^2}$. For computational purposes, these profiles have been fitted crudely with simple analytic expressions.

$$\left. \begin{aligned} f\bar{\tau} &= (2.63 - 1.25 \xi^{1/2}) \xi \\ \frac{\overline{w'^2}}{u_*^2} &= 1.75 e^{\frac{-\xi}{0.240}} \end{aligned} \right\} \quad (3.17)$$

These approximate profiles have been used with $a_t = 0.10$ to calculate an approximate analytic form of $\frac{f}{u_*^2} K_t$, and the three analytic profiles have been indicated by the circles in figure 3.1, 3.2, and 3.3. The expression for $\frac{\overline{w'^2}}{u_*^2}$ in (3.17) evidently yields values which are too large above

Fig.3.2 The approximate formula (3.17) for the dimensionless dissipation time scale (plotted circles) is compared with the surface-layer profile and the results of the neutral run of Wyngaard's model (solid curve).

Fig.3.3 The approximate formula (3.17) for $\overline{w^2}$ (plotted circles) is compared with the results of the neutral run of Wyngaard's model (solid curve).

$\frac{zf}{u_*} = 0.5$, leading to somewhat exaggerated values of $\frac{f}{u_*^2} K_t$; but this minor discrepancy has been ignored, since the Wyngaard profiles of $\bar{\tau}$ and K_t increase in uncertainty above this level anyhow.

A steady-state, one-dimensional, eddy-diffusion model for the convection current is easily constructed from the mean charge equation (2.18). Differentiating by z and substituting $\overline{\rho'w'} \equiv J_m \equiv -K_t \frac{d\bar{\rho}}{dz}$ gives

$$0 = \frac{\lambda_o}{\epsilon_o K_t} J_m - \frac{d^2 J_m}{dz^2} + \frac{d\bar{Q}}{dz}$$

Non-dimensionalization by τ_λ , u_* , ρ_* , and H , as before, then gives

$$\frac{d^2 \hat{J}_m}{d\zeta^2} - \frac{1}{R \hat{K}_t} \hat{J}_m = \frac{1}{R} \frac{d\hat{Q}}{d\zeta} \quad (3.18)$$

Here, the new dimensionless variables are defined by $J_m \equiv \rho_* u_* \hat{J}_m$ and $K_t \equiv u_* H \hat{K}_t$. For our present purposes, we will set $\frac{d\hat{Q}}{d\zeta} \equiv 0$ and take the boundary conditions on (3.18) as $\hat{J}_m(\zeta_{oo}) = 1$ and $\hat{J}_m(1) = 0$. This amounts to imposing a rigid lid at $z = H$, which will be shown not to interfere significantly with the solutions.

Equation (3.18) can readily be solved in finite difference form, as a tridiagonal system similar to (3.9), after the substitution of the coordinate stretching (3.4) and finite differences (3.6). The solution has been carried out, using the profile of $\hat{K}_t \equiv a_t f\bar{\tau} \frac{\overline{w'^2}}{u_*^2}$ obtained from the analytic formulae in (3.17), for the case $R = 0.1$. For comparison purposes, we also carried out a solution to the closure model based on the same turbulence statistics, namely, $\overline{\theta w} \equiv 0 \equiv \frac{d\hat{\theta}}{d\zeta}$, $\overline{w^2} \equiv \frac{\overline{w'^2}}{u_*^2}$, and $\tau \equiv \frac{1}{R} (f\bar{\tau})$, with $\frac{\overline{w'^2}}{u_*^2}$ and $f\bar{\tau}$ specified by (3.17). The factor of $\frac{1}{R}$ in the expression for τ appears because in the closure model $\bar{\tau}$ is non-dimensionalized by τ_λ ,

whereas in (3.17) it is non-dimensionalized by $f^{-1} = \frac{H}{u_*}$. The resulting two profiles of $\hat{J}_m$ appear in figure 3.4.

Two conclusions are inevitable from an examination of figure 3.4. First, there is negligible difference between the results of the two models. Evidently, the closure model developed in Chapter 2 is just as good as the eddy-diffusion model in describing the convection current in a neutral, planetary boundary layer. Second, the convection current falls off very rapidly with height in the neutral case. It is down by a factor of 10 at $\zeta = 0.075$ and by a factor of 100 at $\zeta = 0.23$. This makes it clear that the uncertain behavior of the eddy-diffusion coefficient in the upper part of the Ekman layer is irrelevant. Because $\tau_\lambda \ll f^{-1}$, there is no significant convection current in the bulk of the layer. No wonder the turbulent electrode-effect models discussed in the Introduction were unable to predict the observed convection currents in the unstable boundary layer.

Neither of these results is particularly surprising. In the neutral case, the important buoyant-production term drops out of the $\overline{\rho w}$ equation. If the turbulent-transport term can also be neglected, and it can in the lower part of the boundary layer, we can write a steady-state version of the $\overline{\rho w}$ budget equation from (2.20) as

$$\overline{\rho w} = - \frac{R \overline{w^2}}{(1 + d_3/\tau)} \frac{d\hat{\rho}}{d\zeta}$$

This is immediately recognized as a gradient-diffusion model with an effective eddy-diffusion coefficient given by $\frac{R \overline{w^2}}{1 + d_3/\tau}$. In the surface layer, where $d_3/\tau \gg 1$, this expression reduces to $\frac{1}{d_3} (f\tau) \frac{\overline{w'^2}}{u_*^2}$. With $d_3 = 9.7$, it is essentially the same as $\frac{f}{u_*^2} K_t$. Examination of the numerical solution shows that the conductive-dissipation and turbulent-transport terms can

Fig.3.4 Dimensionless convection-current density profiles predicted by the second-order-closure model and by the eddy-diffusion model are compared for the neutral case.

indeed be neglected in the $R = 0.1$ case below $\zeta = 0.043$.

Next, we will compare the results of the second-order-closure model with those of the simple convective-column model developed in section 1.3 and Appendix 2. As in the previous paragraphs, we will use the turbulence statistics τ , $\overline{w^2}$, $\overline{\theta w}$, and $\frac{d\hat{\theta}}{d\zeta}$ corresponding to the simple model as input to the closure model. However, in this case the procedure is not so straightforward as in the neutral case, because the convective-column model is only a caricature of a real physical situation and, as such, fails to satisfy the equations of motion.

All four turbulence statistics have been derived for the convective-column model in equations (A2.15). Notice that $\overline{w^2}$, τ , and $\frac{d\hat{\theta}}{d\zeta}$ are independent of height in this model; only $\overline{\theta w}$ depends on ζ . Before using these statistics in the closure model, however, we must set $d_3 = 0 = d_4$. This amounts to neglecting the pressure-scrambling loss term in the $\overline{\rho w}$ equation and the molecular-dissipation term in the $\overline{\rho \theta}$ equation for the purpose of making the two models more comparable. These are the dominant loss terms in the closure model in most cases, but there is no corresponding loss process in the convective-column model. There, the only process destroying $\overline{\rho w}$ is conduction, so we have kept only the conductive-destruction terms in the closure model for this comparison. The bottom boundary conditions used in the closure model are also taken from the convective-column model results (A2.15): $\overline{\rho w}_1 = 1.0$, $\overline{\rho \theta}_1 = -2.50 \left(-\frac{H}{KL} \right)^{-2/3}$, and $\overline{\rho^2}_1 = 2.50 \left(-\frac{H}{KL} \right)^{-2/3}$.

The profile of $\overline{\rho w}$ for the convective-column model is given in equations (A2.15). This result is compared with the closure-model result (using the convective-column model statistics and boundary conditions and $d_3 = 0 = d_4$) for a typical unstable case, $\frac{H}{L} = -50$ and $R = 0.1$, in

Fig.3.5 Dimensionless convection-current density profiles predicted by the second-order-closure model and the simple convective-column model are compared for $R = 0.1$, $H/L = -50$.

Fig.3.6 Same as figure 3.5, for $H/L = -1000$.

figure 3.5. Also included in the figure is the $\overline{\rho w}$ profile from an unmodified closure-model run, using the closure constants, boundary conditions, and turbulence statistics developed in sections 2.4 and 2.5. Figure 3.6 shows the same three profiles for a case of extreme instability, $\frac{H}{L} = -1000$ and $R = 0.1$.

It is apparent from the figures that the convective-column model exaggerates the convection current compared to that predicted by the unmodified closure model. This is not surprising, considering that the former is based on uniform rising columns with none of the dissipative mechanisms inherent in real turbulent transport. The only physical constraints obeyed by the simple convective-column model are continuity of charge and of mass. For these reasons, it would be disastrous for the closure model if it did not predict smaller convection currents than the convective-column model.

In spite of this exaggeration, however, agreement between the two models is well within an order of magnitude throughout the mixed layer in both of the cases shown. Agreement is considerably improved in the $\frac{H}{L} = -50$ case by using the convective-column model statistics, as discussed above. These results, though certainly no proof of the validity of the second-order-closure model developed in Chapter 2, are very encouraging and show that our model is capable of giving large convection currents similar to those predicted by the simple model of Appendix 2.

3.3 Behavior of the Model

In this section solutions of the model, using lower boundary conditions of the type (3.11) with an arbitrary profile of $\hat{Q}(\zeta)$ normalized by (3.13), will be presented for the whole range of the parameters R and $\frac{H}{L}$ of physical interest. The behavior of the model will be characterized

primarily by the dimensionless electromotive force generated by the convection current, defined as in (A2.16) and identified in section 1.1 as an electrical parameter of primary importance.

$$(emf) \equiv \frac{\lambda_o}{\rho_* u_* H} (EMF) \equiv \int_{\xi_o}^1 \overline{\rho w} d\xi \quad (3.19)$$

These solutions will then be discussed in terms of the relative importance of the various terms in the budget equations for $\overline{\rho w}$ and $\overline{\rho \theta}$ given in (2.20). Finally, the closure constants will be varied in one case to indicate the sensitivity of the model to the somewhat uncertain numerical values chosen in section 2.4. Throughout this section, the results for $\overline{\rho^2}$ will be suppressed, since it has been shown that they are sensitive to the exact form of the lower boundary conditions.

Since we will be considering only neutral and unstable boundary layers, we will take the range of stability as $-\frac{H}{L} \in [0, 1000]$. This parameter depends primarily on the thickness of the mixed layer H and the strength of the surface heat flux $\rho_a C_p \overline{\theta'w'}|_0$, and secondarily on the surface stress $-\rho_a u_*^2$. Wyngaard (1973) has given a curve of the diurnal variation of $\overline{\theta'w'}|_0$ during the "Kansas experiment" in his figure 3.15. During the daytime it ranged between small negative values in the early morning and late evening and a maximum of about $+0.25 \frac{\text{deg m}}{\text{sec}}$ in the early afternoon. During the most unstable run of the Kansas experiment, run #46, the Monin-Obukhov length was calculated to be $L \equiv -\frac{T_o}{g} \frac{u_*^3}{k \overline{\theta'w'}|_0} = -10.3 \text{ m}$, from measured values of $\overline{\theta'w'}|_0 = 0.246 \frac{\text{deg m}}{\text{sec}}$, $u_* = 0.31 \frac{\text{m}}{\text{sec}}$, and $T = 292^\circ\text{K}$, according to Izumi (1971). Although H was not measured, using a value of $H = 2000 \text{ m}$ gives $\frac{H}{L} = -194$. This value is relatively moderate because of

the fairly strong winds and correspondingly large surface stresses during the Kansas experiment. The upper limit of $-\frac{H}{L} = 1000$ is close to the extreme value of this stability parameter reached by the model boundary layer of Deardorff (1974a, 1974b), discussed further in Chapter 7.

The range of the other dimensionless parameter is taken as $R \equiv \frac{u_* \tau_\lambda}{H} \in [0.005, 2.0]$, designed to cover the ranges of $u_* \in [0.1, 0.5] \frac{m}{sec}$, $\tau_\lambda \in [160, 1200] sec$, and $H \in [300, 3000] m$, considered to be typical of the unstable planetary boundary layer. In the neutral case, we take $H \equiv \frac{u_*}{f}$, implying $R = \tau_\lambda f$. Since $f \sim 10^{-4} sec^{-1}$, except near the equator, this suggests $R \in [0.02, 0.1]$ for the neutral planetary boundary layer.

Figure 3.7 shows the behavior of the dimensionless EMF, (emf), predicted by our model as a function of R and $\frac{H}{L}$. For comparison purposes, the same graph is presented for the simple convective-column model in figure 3.8. Of course, the slopes of all the constant-(emf) contours on the latter figure are the same, as shown by equations (A2.15) and (A2.16); and the values of (emf) are consistently higher than on the former figure, as discussed in section 3.2. However, it is noteworthy that the slopes of these contours agree quite well in the lower half of the two graphs, suggesting that there is more than a superficial resemblance between the two models.

Vertical profiles of $\overline{\rho w}$ for all of the closure-model runs plotted in figure 3.7 are presented in figures 3.10 through 3.13, each graph being a family of curves for fixed $\frac{H}{L}$ and variable R . The results of the eddy-diffusion model for the neutral case, described in section 3.2, are presented in figure 3.9. To show more clearly the behavior of the model as $\frac{H}{L}$ is varied, some of these profiles have been re-plotted in

Fig.3.7 Contours of constant dimensionless EMF in parameter space for the uniform- λ model with the $\overline{\rho w}(z_{00}) = 0$ lower boundary conditions. The plotted circles are the model runs on which the contours were based.

Fig.3.8 Same as figure 3.7, but for the simple convective-column model.

Fig.3.9 Profiles of the dimensionless convection-current density predicted by the eddy-diffusion model for the neutral case. The values of the dimensionless EMF corresponding to each curve have also been tabulated.

Fig.3.10 Profiles of the dimensionless convection-current density predicted by the uniform- λ model for $H/L = -10$. The values of the dimensionless EMF corresponding to each curve have also been tabulated.

Fig.3.11 Same as figure 3.10, for $H/L = -50$.

Fig.3.12 Same as figure 3.10, for $H/L = -200$.

Fig.3.13 Same as figure 3.10, for $H/L = -1000$.

Fig.3.14 Same as figures 3.9 and 3.10, for $R = 0.1$.

figure 3.14 as a family for fixed R and variable $\frac{H}{L}$. The values of (emf) for each curve are tabulated on the corresponding figures for easy comparison. Figures 3.15 and 3.16 give a cross section of the profiles of $-\overline{\rho\theta}$ for $\frac{H}{L} = -50$ and for $R = 0.1$, respectively. It is to be recalled when looking at these figures that the effective lower boundary condition is $\overline{\rho\theta}(\zeta_{o.o}) = -8.0$, so $-\overline{\rho\theta}$ is falling off very rapidly in the surface layer. Also, the surface-layer sign convention takes $T_* < 0$ in unstable conditions, so negative $\overline{\rho\theta}$ means a positive correlation between ρ' and θ' .

The most striking aspect of these figures is the dramatic increase of (emf) with increasing instability. The convection current in the neutral cases is insignificant compared to that occurring in the cases of strong instability. It is no wonder that the eddy-diffusion electrode-effect models referred to in the Introduction were unable to predict the strong convection currents observed by Kraakevik and Clark.

One might ask what physical processes, as represented in the model, bring about this dramatic increase in charge convection with increasing instability. To answer this question, we must examine the various terms in the budget equations for $\overline{\rho w}$ and $\overline{\rho\theta}$. These equations are re-written below from (2.20), and labels are attached to each term for easier reference.

$$\begin{aligned}
 0 \equiv \frac{\partial}{\partial t} \overline{\rho w} = & \quad \text{local rate of change of } \overline{\rho w} \\
 & - \overline{\rho w} \quad \text{conductive dissipation} \\
 & - \frac{d_3}{\tau} \overline{\rho w} \quad \text{pressure scrambling} \\
 & + a_t R^2 \frac{\partial}{\partial \zeta} \tau \overline{w^2} \frac{\partial}{\partial \zeta} \overline{\rho w} \quad \text{turbulent-transport convergence} \\
 & - R \overline{w^2} \frac{\partial \hat{\rho}}{\partial \zeta} \quad \text{gradient production} \\
 & + R \frac{H}{K_L} \overline{\rho\theta} \quad \text{buoyant production}
 \end{aligned} \tag{3.20}$$

Fig.3.15 Profiles of the dimensionless charge density-temperature covariance predicted by the uniform- λ model for $H/L = -50$.

Fig.3.16 Same as figure 3.15, for $R = 0.1$.

$$\begin{aligned}
0 \equiv \frac{\partial}{\partial t} \overline{\rho\theta} = & \quad \text{local rate of change of } \overline{\rho\theta} \\
& - \overline{\rho\theta} \quad \text{conductive dissipation} \\
& - \frac{d_u}{\tau} \overline{\rho\theta} \quad \text{molecular dissipation} \\
& + a_t R \frac{\partial}{\partial \zeta} \tau \overline{w^2} \frac{\partial}{\partial \zeta} \overline{\rho\theta} \quad \text{turbulent-transport convergence} \\
& - R \overline{\theta w} \frac{\partial \hat{\rho}}{\partial \zeta} \quad \text{gradient production} \\
& - R \overline{\rho w} \frac{\partial \hat{\theta}}{\partial \zeta} \quad \text{stratification production}
\end{aligned} \tag{3.21}$$

Except for the conductive-dissipation, pressure-scrumbling, and molecular-dissipation terms, which obviously are always loss terms in these equations, all the terms have been labeled as convergence or production terms, although any one of them can also act as a loss term.

The value of each term has been printed out at each grid point by the numerical solutions so that the balances among the various terms can be examined in detail. Figures 3.17 and 3.18 delineate these balances for the $\overline{\rho w}$ equation in some of the cases, giving the dominant gain and loss terms as a function of n and $\frac{H}{L}$ for $R = 0.1$ (figure 3.17) and as a function of n and R for $\frac{H}{L} = -50$ (figure 3.18). Figures 3.19 and 3.20 do the same for the $\overline{\rho\theta}$ equation. Recall that n is the gridpoint number, increasing from 1 at ζ_{oo} to 100 at $\zeta = 1.0$; n is proportional to $\ln \zeta$, as in (3.4). With the boundary conditions used, $\zeta_{oo} = 10^{-5}$ and $\zeta_q = 10^{-3}$, so that $\hat{Q} = 0$ above $n=39$. We have therefore considered only the part of the boundary layer above the source region, $n \in [40, 100]$, for which the results are essentially independent of the boundary conditions.

Apparently, the large convection current in a mixed layer of strong instability is due to the buoyant-production term, which is dominant in the interior of the layer and increases with increasing R and $|\frac{H}{L}|$. The $\overline{\rho\theta}$

Fig.3.17 The dominant gain and loss terms (in that order) in the $\bar{\rho_w}$ -budget equation (3.20), indicating the balance between the various terms in different regions of the mixed layer and of the range of H/L , for $R = 0.1$. n is the numerical vertical coordinate, proportional to $\ln \zeta$.

Fig.3.18 The dominant gain and loss terms (in that order) in the $\bar{\rho w}$ -budget equation (3.20), indicating the balance between the various terms in different regions of the mixed layer and of the range of R , for $H/L = -50$.

Fig.3.19 Same as figure 3.17, for the $\overline{\rho\theta}$ -budget equation (3.21).

Fig.3.20 Same as figure 3.18, for the $\overline{p\theta}$ -budget equation (3.21).

correlation on which this term depends is produced in the surface layer by stratification production and carried up into the interior by turbulent transport. Since there is no $\overline{\rho\theta}$ in the neutral case, this mechanism cannot operate, and $\overline{\rho w}$ is limited to that generated by the gradient-production term in the surface layer and carried into the interior by turbulent transport.

Calculation of the individual terms in equations (3.20) and (3.21) also makes it possible to examine the importance of the terms which have been approximated by closure models. The pressure-scrambling term in the $\overline{\rho w}$ equation and the molecular-dissipation term in the $\overline{\rho\theta}$ equation are the dominant loss terms in the lower part of the mixed layer in all of the cases of smaller R . Therefore, it is to be expected that the values of d_3 and d_4 are important in determining the model results.

The turbulent-transport term is negligible in the $\overline{\rho w}$ equation in all the cases except those for $R = 0.005$ and those for $R = 0.02$ with $\frac{H}{L} = -10, -50$, and -200 , in which cases it becomes the dominant gain term in the upper part of the mixed layer. However, this term is always the dominant gain term in the $\overline{\rho\theta}$ equation in the interior of the layer, so we expect the value of a_t to be important also in determining the model results. It will influence $\overline{\rho w}$ through the buoyant-production term by its effect on $\overline{\rho\theta}$.

These speculations on the importance of the closure constants are confirmed by a sensitivity study based on the $R = 0.1$, $\frac{H}{L} = -50$ case. Each of the three closure constants d_3 , d_4 , and a_t has been halved and doubled in turn and the resulting $\overline{\rho w}$ profiles are compared with that for the unmodified case in figure 3.21. As expected, d_3 makes the biggest difference, followed by d_4 and then by a_t . The influence of a_t is

Fig.3.21 Profiles of dimensionless convection-current density from a sensitivity study of the uniform- λ model in which each of the closure constants was halved and doubled in turn from the values of $a_t = 0.10$, $d_4 = 1.4$, and $d_3 = 9.7$ used in the standard case (heavy solid curve), for $R = 0.1$ and $H/L = -50$.

only comparable to that of d_3 and d_4 in the interior of the mixed layer, because turbulent transport of $\overline{\rho\theta}$, and consequent buoyant production of $\overline{\rho w}$, is only important above the surface layer. Therefore, the model does not seem to be as sensitive to a_t as to the other two constants.

While we are on the subject of sensitivity, we recall that there was some question in section 2.5 about the exact behavior of the potential temperature gradient in the upper part of the mixed layer. In order to test the importance of this interior temperature gradient, we made a run exactly like the $\frac{H}{L} = -50$, $R = 0.1$ case except for a reversal of the sign of $\frac{d\hat{\theta}}{d\zeta}$ in the region $\zeta \in [0.5, 1.0]$. This sign reversal made a negligible change in the $\overline{\rho w}$ profile and only about a 10% decrease in the magnitude of $\overline{\rho\theta}$ in the upper part of the mixed layer. The reason for the small effect of this change was traced to the dominance of turbulent transport over stratification production of $\overline{\rho\theta}$ in the upper part of the mixed layer, as shown in figures 3.19 and 3.20. Therefore, it was concluded that the profile of $\frac{d\hat{\theta}}{d\zeta}$ given in equation (2.28) is adequate to our purposes.

3.4 Theoretical Considerations

One of the most striking characteristics of the $\overline{\rho w}$ profiles in figures 3.10 through 3.14 is their approach to linearity with increasing R and increasing $-\frac{H}{L}$. We invoke an explanation for this phenomenon similar to that proposed by Wyngaard, Arya, and Coté (1974) to account for the linear stress profiles predicted by their model in the unstable cases. Differentiate the mean charge equation (3.1) and ignore $\hat{Q}$ in the interior of the mixed layer:

$$\frac{d\hat{f}}{d\zeta} = -R \frac{d^2}{d\zeta^2} \overline{\rho w} \quad (3.22)$$

As long as the curvature of the $\overline{\rho w}$ profile is positive $\left(\frac{d^2}{d\zeta^2} \overline{\rho w} > 0\right)$ in the interior, the charge gradient will therefore be negative. This means that the gradient-production term in the $\overline{\rho w}$ equation (3.20) will be positive, and that in the $\overline{\rho \theta}$ equation, negative (remember that $\overline{\theta w} < 0$ because $T_* < 0$ in unstable cases), both tending to increase $\overline{\rho w}$.

However, as $-R \frac{H}{kL}$ increases, the buoyant-production term becomes a very large gain term in the $\overline{\rho w}$ equation, tending to drive the curvature negative. By (3.22) this means a positive charge gradient which makes the gradient-production terms in both (3.20) and (3.21) change sign. The resulting destruction of $\overline{\rho w}$ tends to drive the curvature back toward zero, with the result that the $\overline{\rho w}$ profile stays near linear. This explanation receives support from the fact that the gradient-production terms change from gain terms to significant loss terms in both budget equations as $-R \frac{H}{kL}$ becomes very large.

Although the mechanism is different, a similar result obtains in the simple convective-column model, where the $\overline{\rho w}$ profile in equation (A2.15) approaches linearity as R and $-\frac{H}{L}$ are increased. In this case, it is simple charge continuity which does it, forcing the charge flux leaving the rising columns at $z = H$ to equal that entering the descending environment. Ultimately, however, the apparently differing mechanisms in the two models must amount to the same thing, because both are based directly on the same charge-conservation equation.

3.5 The Ion-Annihilation Bottom Boundary Condition

As pointed out in section 3.1, the boundary conditions used in section 3.3 to derive the contours of constant dimensionless EMF in parameter space, presented in figure 3.5, are not exactly appropriate to the solution of any realistic physical problems. The reason for this is that,

even with the Blanchard effect as the charge source, we don't know the magnitude of the upward convection current forced into the bottom of the model at some height z_q ; we know only the source strength $\bar{Q}(z)$. The only satisfactory way around this problem is to make use of the third kind of boundary conditions (3.15) and (3.13), based on $\hat{\rho}(\zeta_{oo}) = 0$, and to let the model itself determine the upward convection current at the top of the source layer z_q . Unfortunately, this set of boundary conditions makes the solutions more complicated by introducing two new dimensionless parameters, $\frac{z_o}{H}$ and $\frac{z_q}{H}$. Actually, these parameters already appeared in the boundary conditions (3.11) and (3.13), because there $\hat{Q} \neq 0$ for $\zeta \in [\zeta_{oo}, \zeta_q]$; but it was shown that the solutions above ζ_q were independent of them in that case.

Preliminary calculations have been made with this third type of boundary conditions. However, for the very small values of $\frac{z_o}{H}$ and $\frac{z_q}{H}$ which turn out to be of interest for the Blanchard effect over the tropical oceans, some problems were encountered in obtaining convergence of the numerical solutions. For this reason, and because of the large number of dimensionless parameters involved (four, instead of just the two appearing in section 3.3), no attempt has been made to do a comprehensive survey of parameter space. A general summary of the results will be given here, with the details deferred to Chapter 4, where the Blanchard effect is discussed.

The preliminary calculations indicate that on the order of 60% to 90% of the total current $\int_{\zeta_{oo}}^{\zeta_q} \hat{Q}(\zeta) d\zeta$ supplied by the source is convected down to the surface. Only the remaining 10% to 40% is convected upward past ζ_q . It is this smaller fraction of the total that corresponds to the current forced into the lower boundary of the model by the earlier types of boundary conditions. In fact, it is found that the (emf) and the profiles

of $\overline{\rho w}$, $\overline{\rho \theta}$, and $\hat{\rho}$ predicted by the model with the new boundary conditions agree perfectly with those predicted by the model as solved in section 3.3, provided the latter are scaled down by this fraction $\overline{\rho w}(\zeta_q)$.

The implication of this result is that we need to use the new boundary conditions (3.15) and (3.13) only to find the fraction $\overline{\rho w}(\zeta_q)$ of the source current that goes up. Once we know this fraction, we can use it with the results of section 3.3 to find all the other results except the profile of $\overline{\rho^2}$. Since $\zeta_q \ll |L|$ in all reasonable cases, one might expect that an eddy-diffusion model could be used to find $\overline{\rho w}(\zeta_q)$. This would at least limit the determination of this fraction to the three parameters R , $\frac{z_o}{H}$, and $\frac{z_q}{H}$. Unfortunately, the preliminary calculations indicate that $\overline{\rho w}(\zeta_q)$ does depend somewhat on $\frac{H}{L}$ as well as on the other three parameters. Nevertheless, an eddy-diffusion model using the $\hat{\rho}(\zeta_{oo}) = 0$ lower boundary condition has been developed, and some of its results will be reported in Chapter 4. In the light of the results presented there, it appears that this problem does not merit further study at present.

4 THE BUBBLING OF SEA WATER AS A CHARGE SOURCE FOR THE CONVECTION CURRENT

4.1 The Blanchard Effect

It has been mentioned in the Introduction that over the ocean the bubbling of air through a sea-water surface by whitecapping can produce positive charge near the surface. This charge source was investigated in detail by Blanchard (1963), and his work will be summarized briefly here. It consisted of an experimental phase, in which careful laboratory work established the behavior and charge of the "jet drops" produced by the bursting of small bubbles, and a more speculative phase in which an attempt was made to deduce the importance of the jet drop charging mechanism over the real oceans.

When small bubbles burst at an air-water interface, a jet rises from the bottom of the collapsing bubble cavity at high velocity, breaking up into several tiny drops which are ejected into the air above. If they are small enough, these "jet drops" evaporate before they fall back into the water, forming the airborne salt particles which are so important as condensation nuclei over the oceans. By laboratory experimentation, Blanchard (1963) was able to determine the size of the first one or two jet droplets, and the height to which they were ejected, as a function of bubble size. He went on to measure the charge on the jet drops as a function of drop size and bubble age. For example, in sea water at room temperature a bubble of diameter 0.45 mm, aged 8 seconds, fires a top jet drop of about 20 μm radius to a height of about 3.8 cm with a positive charge of about 6.7×10^{-16} coul. Smaller bubbles produce smaller jet drops with lower ejection heights and less charge. One interesting and unexpected result was that the charge on the drops increased with bubble age up to a maximum

achieved for ages between 20 and 100 sec. Temperature effects on charging have not been fully studied, but it appears that at 4°C the jet drops may carry an even higher charge than at room temperature. In every case, the charging of sea water jet drops was positive.

Blanchard (1963) was unable to obtain a direct relation between the spectra of bursting bubbles and the wind speed over the ocean. Therefore, in order to estimate the global production of charge by the bubbling mechanism, he was forced to follow a rather indirect approach. Woodcock (1953) had measured the spectra of salt nuclei near cloud base (about 600 m over the ocean) in the Hawaiian area at various wind speeds. Blanchard assumed that these spectra could be applied to the top of the thin "laminar boundary layer" over the water surface and computed sedimentation spectra from the gravitational fall velocities of the nuclei. Assuming that these sedimentation spectra were balanced in the steady state by production spectra due to bubbling, he then had the drop production spectra.

Assuming further that these drops were all top jet drops of age 8 sec, and ignoring other factors, Blanchard deduced the charge production rate over the ocean as a function of wind speed. This relation is given in his figure 29. It is to be noted that this is not the total charge production rate, since there are other loss processes for nuclei than sedimentation. Nevertheless, using global maps of the average wind speed over the oceans along with this charge production rate, Blanchard computed the partial charge production over the whole globe to be 47 amps, with a negligible seasonal variation.

The other major loss processes for salt nuclei, in addition to gravitational sedimentation over the oceans, are washout by rain over the oceans and advection over the continents. The former was estimated by

obtaining the ratio of sea salt brought down by rain to that brought down by sedimentation near Hawaii and generalizing it to the world ocean. Assuming that this ratio applies also to charge production, Blanchard (1963) estimated an additional 89 amps corresponding to rain washout over the oceans. The salt advected over the continents was assumed to return to the oceans in river run-off. This led to a further estimated 14 amps, bringing the grand total to 150 amps.

The relation between partial charge production and wind speed given in Blanchard's (1963) figure 29 has been multiplied by the factor $\frac{150}{47} = 3.2$ to account for the charging corresponding to the two additional nucleus-loss processes. This leads to one of the curves in our figure 4.2, representing the vertically integrated charge source $\int_{z_0}^{z_q} \bar{Q}(z) dz$ as a function of wind speed. However, due to the rather indirect method by which Blanchard arrived at his results, and in view of the many assumptions and approximations therein, it would be desirable to arrive at the charge production rate by another method.

4.2 Alternate Estimates of the Charge Source

Blanchard (1966) and Gathman and Hoppel (1970) both made measurements of the electric field and charge density down wind of a surf zone. From these measurements they separately deduced values for charge production in the surf zone of about $4 \times 10^{-11} \frac{\text{coul}}{\text{m}^2 \text{sec}}$. Blanchard (1966) speculated that this value might be representative of the charge production by whitecaps on the open sea.

In order to use this estimate of the charge source in whitecaps, we must have a relation between the fractional area of the sea covered with whitecaps and the wind speed. Blanchard (1963) presented such a relation in his figure 31, derived from a series of five U.S. Navy Weather Squadron

aerial photographs of the sea surface at various winds speeds. A more thorough study of this matter was undertaken by Monahan (1971), who gave his best estimate of the whitecap area coverage on a fully developed sea as

$$W(\%) = \begin{cases} < 0.1 & \bar{u} < 4 \frac{\text{m}}{\text{sec}} \\ 0.00135 (\bar{u})^{3.4} & 4 \frac{\text{m}}{\text{sec}} < \bar{u} < 10 \frac{\text{m}}{\text{sec}} \end{cases}$$

Here, $\bar{u}$ is the mean wind speed 10 m above the mean water surface. Both authors agree with the suggestion by Gathman and Trent (1968) that whitecaps do not appear until the wind speed exceeds about $3.1 \frac{\text{m}}{\text{sec}}$.

In figure 4.1, Blanchard's (1963) figure 31 has been replotted together with Monahan's (1971) relation. The latter gives considerably smaller fractional coverage over its range of validity, but intersects with the Blanchard curve if extrapolated to $12 \frac{\text{m}}{\text{sec}}$. Monahan's relation has been multiplied by the value of $4 \times 10^{-11} \frac{\text{coul}}{\text{m}^2 \text{sec}}$ for whitecap charge production to arrive at the "Monahan-Gathman-Hoppel" curve of charge source as a function of wind speed in figure 4.2, for $\bar{u} < 10 \frac{\text{m}}{\text{sec}}$. If Monahan's relation is used for $\bar{u} < 10 \frac{\text{m}}{\text{sec}}$ and Blanchard's for $\bar{u} > 10 \frac{\text{m}}{\text{sec}}$, and if a single smooth curve is drawn through these values for $3.1 \frac{\text{m}}{\text{sec}} < \bar{u} < 20 \frac{\text{m}}{\text{sec}}$, multiplication by $4 \times 10^{-11} \frac{\text{coul}}{\text{m}^2 \text{sec}}$ leads to the "Whitecap Composite" curve in figure 4.2. This curve probably represents the best data available at present on charge production by whitecapping on the ocean.

Ideally, we want not only the vertically integrated charge source $\int_{z_0}^{z_q} \bar{Q}(z) dz$ due to whitecapping but also the profile of $\bar{Q}(z)$ to insert in the charge conservation equation of our model (2.18). We could obtain this profile from Blanchard's (1963) data on the charge and ejection height of

Fig.4.1 Fractional area of the sea surface covered by whitecaps as a function of the mean wind speed at 10 m, from Blanchard (1963) and Monahan (1971).

Fig.4.2 Vertically integrated charge-source strength due to the Blanchard effect as a function of the mean wind speed at 10 m. See text for an explanation.

jet drops as a function of bubble size if we had the bubble spectrum produced by whitecaps. Unfortunately, it is very difficult to obtain a spectrum of the bubbles bursting in a whitecap on the open sea. Blanchard and Woodcock (1957) measured the bubble spectrum in sea water a few seconds after the passage of a breaking wave, separating the bubbles into four size classes centered on 75 μm , 150 μm , 250 μm , and 350 μm diameter. We have used their data to estimate $\bar{Q}(z)$ in a whitecap in Table 4.1. The values in the last column have been plotted as a bar graph in figure 4.3. Integration over height to obtain the scale factor $\rho_* u_* \equiv \int_{z_0}^{z_q} \bar{Q}(z) dz$ gives $6.0 \times 10^{-11} \frac{\text{coul}}{\text{m}^2 \text{sec}}$ in startling agreement with the measured value of $4 \times 10^{-11} \frac{\text{coul}}{\text{m}^2 \text{sec}}$ quoted earlier.

This confirmation encourages us to use the whitecapping charge production curves given in figure 4.2 as our best estimate of the Blanchard-effect charge source on the open sea. We shall use the "Monahan-Cathman-Hoppel" curve for $\bar{u} \leq 10 \frac{\text{m}}{\text{sec}}$ and the "Whitecap Composite" curve for $10 \frac{\text{m}}{\text{sec}} < \bar{u} < 20 \frac{\text{m}}{\text{sec}}$. It is suspected that the Blanchard (1963) result, also shown in figure 4.2, underestimates the charge production because the assumption that nucleus spectra measured at cloud base represent the spectra at the top of the thin "laminar boundary layer" underestimates the jet drop production. It must be recognized, however, that there is considerable uncertainty in these estimates of the charge source due to bubbling. They may be good only to within an order of magnitude.

4.3 The Friction Velocity and Surface Roughness

In order to obtain solutions for the Blanchard effect source over the oceans, we must know u_* and z_0 , both of which depend primarily on wind speed. Sheppard, et al. (1972) have recently made a comprehensive study of mean profiles and turbulence statistics over a large fresh-water

Table 4.1

The Charge Source Profile from Blanchard and Woodcock (1957)

Bubble Diameter (μm)	Bubble Production per 100 μm diameter interval ($\text{cm}^{-2}\text{sec}^{-1}$ per 100 μm)	Top Jet Drop Diameter (μm)	Top Jet Drop Charge (coul.)	Ejection Height (cm)	Charge Production per 100 μm diameter interval (coul $\text{cm}^{-2}\text{sec}^{-1}$ per 100 μm)	Charge Production per Unit Volume (coul $\text{cm}^{-3}\text{sec}^{-1}$)
75	30.	9	1.7×10^{-16}	0.5	5.0×10^{-15}	8.3×10^{-15}
150	3.24	15	2.5×10^{-16}	0.9	8.0×10^{-16}	1.3×10^{-15}
250	0.47	24	3.3×10^{-16}	1.7	1.6×10^{-16}	1.8×10^{-16}
350	0.15	32	5.0×10^{-16}	2.5	7.5×10^{-17}	6.8×10^{-17}

Fig.4.3 Estimated profile of the charge-source function due to the Blanchard effect, based on Blanchard and Woodcock (1957) and Blanchard (1963).

lake. By fitting the well-known logarithmic wind formula

$$\frac{\bar{u}}{u_*} = \frac{1}{k} \ln \frac{z}{z_0}$$

to a large number of wind profiles observed under conditions of near-neutral stability, they deduced relationships between the mean wind speed at 10 m and the drag coefficient and between u_* and z_0 . These data have been replotted here as graphs of u_* versus $\bar{u}$ (10 m) in figure 4.4 and of z_0 versus $\bar{u}$ (10 m) in figure 4.5.

The lake where these data were taken was of roughly uniform depth of about 15 m and had an upwind fetch of 8 to 20 km. For these reasons, and because the air-water temperature difference over the open ocean is usually quite small, Sheppard, et al. (1972) suggest that the relationships shown in our figures 4.4 and 4.5 are probably applicable to the open sea. They agree broadly with the data presented by Roll (1965). We will therefore use them for all further computations over the ocean.

4.4 Results for the Blanchard Effect Source in the Tropics

Except in special circumstances of rather limited horizontal extent, the air-ocean temperature difference is not large. Therefore, we expect to find mixed layers of much weaker instability over the ocean than over land. As a typical unstable case, we will examine the fair-weather trade-wind regime in the tropics, elucidated by BOMEX. Holland and Rasmusen (1973) worked out a preliminary analysis of the budgets of heat and momentum during the five-day undisturbed period of the Barbados Oceanographic and Meteorological EXperiment. They give an evaporation rate of $6.0 \frac{\text{mm}}{\text{day}}$, a surface stress of $0.065 \frac{\text{nt}}{\text{m}^2}$, an estimated surface flux of sensible heat of $3.0 \times 10^5 \frac{\text{cal}}{\text{m}^2 \text{day}}$, and an average height of the mixed-layer inversion of

Fig.4.4 Friction velocity over water as a function of mean wind speed at 10 m, from Sheppard, et al. (1972).

Fig.4.5 Surface roughness length over water as a function of mean wind speed at 10 m, from Sheppard et al. (1972).

600 m . The typical surface wind (measured at masthead height of 20 m or 27 m) was about $8 \frac{\text{m}}{\text{sec}}$. These are the values used by Sarachik (1974) in his analysis of the tropical boundary layer, and we will use them also.

The following conversion factors are useful:

$$\left[\text{Surface Stress} \left(\frac{\text{nt}}{\text{m}^2} \right) \times \frac{1}{\rho_a} \left(\frac{\text{m}^3}{\text{kg}} \right) \right]^{\frac{1}{2}} = \text{Friction Velocity} \left(\frac{\text{m}}{\text{sec}} \right)$$

$$\text{Sensible Heat Flux} \left(\frac{\text{cal}}{\text{m}^2 \text{day}} \right) \times 4.84 \times 10^{-5} \left(\frac{\text{joule day}}{\text{cal sec}} \right) =$$

$$\text{Sensible Heat Flux} \left(\frac{\text{watts}}{\text{m}^2} \right)$$

$$\text{Sensible Heat Flux} \left(\frac{\text{watts}}{\text{m}^2} \right) \times \frac{1}{\rho_a C_p} \frac{\text{m}^3 \text{day}}{\text{joule}} = \text{Temperature Flux} \left(\frac{\text{deg m}}{\text{sec}} \right)$$

$$\text{Latent Heat Flux} \left(\frac{\text{cal}}{\text{m}^2 \text{day}} \right) \times 1.98 \times 10^{-11} \left(\frac{\text{kg day}}{\text{cal sec}} \right) =$$

$$\text{Water Vapor Flux} \left(\frac{\text{kg}}{\text{m}^2 \text{sec}} \right)$$

$$\text{Evaporation Rate} \left(\frac{\text{mm}}{\text{day}} \right) \times 1.16 \times 10^{-5} \left(\frac{\text{kg day}}{\text{mm m}^2 \text{sec}} \right) =$$

$$\text{Water Vapor Flux} \left(\frac{\text{kg}}{\text{m}^2 \text{sec}} \right)$$

$$\text{Water Vapor Flux} \left(\frac{\text{kg}}{\text{m}^2 \text{sec}} \right) \times \frac{1}{\rho_a} \left(\frac{\text{m}^3}{\text{kg}} \right) = \text{Mixing Ratio Flux} \left(\frac{\text{m}}{\text{sec}} \right)$$

Making use of the above along with the average BOMEX surface temperature of 26°C , surface pressure of 1013 mb, and surface mixing ratio of 15×10^{-3} , corresponding to a surface air density of $\rho_a = 1.17 \frac{\text{kg}}{\text{m}^3}$ and a specific heat of $C_p = 1.017 \times 10^3 \frac{\text{joule}}{\text{kg deg}}$, we find $u_* = 0.24 \frac{\text{m}}{\text{sec}}$, $\overline{\theta'w'} = 1.2 \times 10^{-2} \frac{\text{deg m}}{\text{sec}}$, and $\overline{r'w'} = 5.9 \times 10^{-5} \frac{\text{m}}{\text{sec}}$. Consultation of figure 4.4 shows that the value of u_* given here corresponds to a mean wind speed of better than $7 \frac{\text{m}}{\text{sec}}$, in good agreement with the measured value.

In section 2.1 it was pointed out that the water vapor flux could make a significant contribution to the buoyancy flux in situations where

the surface is moist. This is especially true over the tropical oceans, where about half of the buoyancy flux is typically due to evaporation at the sea surface. In such circumstances we must use the virtual potential temperature flux $\overline{\theta'_v w'}$ instead of the potential temperature flux $\overline{\theta' w'}$ in our equations. In Appendix 4 the following approximate formula is derived, valid for $r \ll 1$:

$$\overline{\theta'_v w'} \approx \overline{\theta' w'} + \left(0.609 + 0.0743 \ln \frac{p_o}{p}\right) \theta_o \overline{r' w'} \quad (4.1)$$

When the BOMEX values are entered in this formula, we find

$$\overline{\theta'_v w'} \approx 1.2 \times 10^{-2} + 1.1 \times 10^{-2} = 2.3 \times 10^{-2} \frac{\text{deg m}}{\text{sec}}$$

Combined with the values of friction velocity and surface temperature given above, this virtual potential temperature flux implies a Monin-Obukhov length of $L \approx -50 \text{ m}$, so that $\frac{H}{L} \approx -12$.

A typical value for the conductivity within the tropical, oceanic, mixed layer is $2.0 \times 10^{-14} \frac{\text{mho}}{\text{m}}$ (Anderson, personal communication), implying $\tau_\lambda \approx 440 \text{ sec}$. Together with the above values deduced from BOMEX, this implies $R \approx 0.17$. Entering figures 4.2 and 4.5 with $\bar{u} = 8 \frac{\text{m}}{\text{sec}}$ gives the Blanchard-effect charge source as $\rho_* u_* = 6.4 \times 10^{-13} \frac{\text{coul}}{\text{m}^2 \text{sec}}$ and the roughness length as $z_o = 6.0 \times 10^{-5} \text{ m}$. Taking $z_q \approx 3.0 \text{ cm}$, as suggested by figure 4.3, we finally have $\zeta_{oo} = 1.0 \times 10^{-7}$ and $\zeta_q = 5.0 \times 10^{-5}$. This completes specification of all the parameters needed for a model solution of the tropical, trade-wind case.

A numerical solution was obtained for this case, using an arbitrary profile of $\bar{Q}(z)$ resembling that of figure 4.3, which appears to be accurate to about 10% in spite of the convergence problems mentioned in section 3.5. This solution indicates that about 85% of the current from the source layer is convected down to the sea surface. The remainder is convected upward, generating a dimensionless EMF of about 0.025. Making use of the values of λ_o , $\rho_* u_*$, and H quoted above, we find that this implies a dimensional EMF generated by convection of Blanchard-effect charge over the fair-weather tropical ocean of only about 480 volts, according to equation (3.19). This is entirely negligible compared to an ionospheric potential of roughly 300 kilovolts, showing that in this case convection plays no significant role in the electrical budget of the mixed layer.

There are several reasons why this number is so much smaller than our original estimate in section 1.3. First, the degree of instability, over the ocean, as measured by $|\frac{H}{L}|$, is small compared to that over sun-heated land, as will be seen further in Chapter 7. Second, the source layer is so thin that most of the charge produced there is convected downward, generating no EMF. And third, the Blanchard effect is fairly small at the low wind speeds necessary to give even the weak instability level of $\frac{H}{L} = -12$; there is not enough whitecapping. Of course, this EMF could be considerably larger if we have underestimated the thickness of the source layer or used an inappropriate profile shape for $\hat{Q}(\zeta)$. However, because of the considerable uncertainties involved, and the negligible result obtained here, we have decided that neither a survey of parameter space with the $\hat{\rho}(\zeta_o) = 0$ lower boundary condition, nor even a more careful computation of this tropical trade-wind case, is justified at present.

4.5 The Blanchard Effect in a Neutral Case

Although $\left|\frac{H}{L}\right|$ decreases as u_* increases relative to $\overline{\theta'w'}$, the whitecapping area and the turbulence intensity both increase with increasing mean wind speed. Therefore it seems worthwhile to investigate the convection of the Blanchard-effect charge under conditions of strong wind. To do this, we developed another eddy-diffusion model, using the $\hat{\rho}(\zeta_{oo}) = 0$ boundary condition and a profile of $\hat{Q}(\zeta)$ similar to that used in the previous section. The model and its results will be described briefly here.

Starting again from the charge conservation equation (2.18), we use the eddy-diffusion model for $\overline{\theta'w'}$ as in section 3.2. Leaving the equation in terms of $\bar{\rho}$ and re-arranging, we have in the steady state

$$K_t \frac{d^2 \bar{\rho}}{dz^2} + \frac{dK_t}{dz} \frac{d\bar{\rho}}{dz} - \frac{1}{\tau_\lambda} \bar{\rho} = -\bar{Q}$$

Non-dimensionalization, as before, gives

$$R \left(\hat{K}_t \frac{d^2 \hat{\rho}}{d\zeta^2} + \frac{d\hat{K}_t}{d\zeta} \frac{d\hat{\rho}}{d\zeta} \right) - \hat{\rho} = -\hat{Q} \quad (4.2)$$

This equation has been derived in terms of $\hat{\rho}$ instead of $\hat{J}_m$, as in (3.18), because this time the lower boundary condition is $\hat{\rho}(\zeta_{oo}) = 0$ instead of $\hat{J}_m(\zeta_{oo}) = 1$. For an upper boundary condition, we will use $\hat{\rho}(1) = 0$. This will not interfere with the solution because of the rapid fall-off of $\hat{\rho}$ with height above the source layer.

Equation (4.2) was solved as a tridiagonal system after substitution of the same coordinate stretching and finite-difference scheme as used for equations (3.18). The same profile of $\hat{K}_t$ was also used, as

obtained from (3.17). The first case worked out was intended to correspond to the tropical trade-wind case of section 4.4, except that here the upward buoyancy flux was ignored: neutral stability was assumed. We used $f = 3.8 \times 10^{-5} \text{sec}^{-1}$, corresponding to the latitude of 15°N at the center of the BOMEX array, and $\tau_\lambda = 440 \text{ sec}$, giving $R \equiv \tau_\lambda f = 0.017$. $u_* = 0.24 \frac{\text{m}}{\text{sec}}$ implies $H \equiv \frac{u_*}{f} = 6.3 \times 10^3 \text{ m}$, so that $\zeta_{oo} = 9.6 \times 10^{-9}$ and $\zeta_q = 4.8 \times 10^{-6}$. With these parameters, the numerical solution gave $(\text{emf}) = 0.0020$ and $\overline{\rho w}(\zeta_{oo}) = -0.72$.

Using $\rho_* u_* = 6.4 \times 10^{-13} \frac{\text{coul}}{\text{m}^2 \text{sec}}$ again, we find the dimensional EMF generated in this neutral run to be 400 volts. This compares well with the value of 480 volts obtained in the previous section. The larger fraction of the source current that is convected upward in this case (28% as compared to about 15% in section 4.4) because of the slightly different $\hat{Q}(\zeta)$ profile, is more than balanced by the weaker convection current predicted by the eddy-diffusion model for neutral stability. The dimensional height at which $\overline{\rho w}$ falls off to 10% of its maximum upward value is only 120 m in this neutral run, as compared to 290 m in the unstable run of section 4.4. The net result is that the EMF predicted here is somewhat smaller than that in the previous section, as expected.

In a situation of strong wind, we expect $|\frac{H}{L}|$ to be small enough that the boundary layer will be essentially neutral. As an example of such a strong wind case, we have taken $\bar{u} = 16 \frac{\text{m}}{\text{sec}}$, corresponding to $u_* = 0.71 \frac{\text{m}}{\text{sec}}$, $z_o = 1.2 \times 10^{-3} \text{ m}$, and $\rho_* u_* = 4.5 \times 10^{-12} \frac{\text{coul}}{\text{m}^2 \text{sec}}$ from figures 4.4, 4.5, and 4.2. If the latitude and conductivity remain unchanged, $R = 0.017$ as before, and $H = 1.9 \times 10^4 \text{ m}$. This makes $\zeta_{oo} = 6.4 \times 10^{-8}$ and $\zeta_q = 1.6 \times 10^{-6}$, if we continue to assume $z_q = 3.0 \text{ cm}$. With these parameters, the new eddy-diffusion model gives

(emf) = 0.00088 and $\overline{\rho w}(\zeta_{oo}) = -0.88$. The dimensional EMF becomes 3.7 kilovolts and the height of 10% $\overline{\rho w}$ is 350 m.

Evidently, the effect of convection of Blanchard-effect charge is considerably larger for a neutral layer with a $16 \frac{\text{m}}{\text{sec}}$ mean wind than for the tropical trade-wind case with $\overline{u} = 8 \frac{\text{m}}{\text{sec}}$. However, 3.7 kV is still small compared to an ionospheric potential of some 300 kV. Therefore, we feel justified in concluding that, except in special circumstances of limited horizontal extent, convection of the Blanchard-effect charge over the ocean has negligible effect on the electrical structure of the planetary boundary layer. It seems that in order to find large convection currents we must look over land, where there is no Blanchard effect. Consequently, we turn our attention to a consideration of the electrode effect as a charge source.

It should be pointed out, however, that our findings here in no way contradict the results of Blanchard (1963). The charge source due to bubbling undoubtedly exists and is important in influencing the charge density in the lowest few meters over ocean surface. Our results simply indicate that the salt nuclei produced by bubbling are convected upward so slowly that they lose most of their charge before reaching significant altitudes. Therefore, the upward transport of these nuclei does not usually result in significant convection currents above a few tens of meters and cannot generate large EMF's. Although Blanchard's prediction that bubbling transfers some 160 amps from the World Ocean to the atmosphere may well be correct, this current is certainly not carried as high as cloud base.

5 THE ELECTRODE EFFECT IN A SPECIAL CASE

5.1 The Electrode Effect

As described in the equations of Appendix 1, the conduction current in the atmosphere is carried by ions of both signs, produced throughout the volume of air by cosmic and terrestrial radiation, drifting in an electric field. Although this current is Ohmic (proportional to the field) in the interior of a volume of air, as long as the field is not too strong, nonlinearities arise near a conducting boundary. Imagine the horizontally uniform, vertically downward, electric field in a layer of air produced by a uniform, negative, surface charge on flat ground. This is an idealization of the situation in the real atmospheric boundary layer. If there is no turbulent mixing, the positive ions will drift downward and be neutralized at the surface, while the negative ions drift upward away from the surface.

After a period of time a layer of air adjacent to the ground will be depleted of negative ions. The positive ions produced throughout the atmosphere continue to drift downward into the layer. However, no negative ions can enter the layer from below because of the ground, and those that are produced there by the volume ionization drift upward out of the layer. Therefore, all the current is being carried by positive ions at the surface.

Since the air in the interior of the atmosphere contains approximately equal numbers of positive and negative ions of nearly equal mobility, the conductivity must be reduced near the surface to about half its interior value. Consequently, to maintain a vertically uniform conduction current in the steady state, the electric field must increase in magnitude near the surface to roughly twice its interior value. This

increase of the electric field corresponds, through Gauss's law, to a layer of positive space charge lying next to the ground where the negative ions have been depleted. It is the existence of this layer of positive space charge with reduced conductivity and increased electric field that is referred to as the electrode effect.

The theory of the electrode effect has been expounded in considerable detail by numerous authors in the atmospheric electricity literature. However, before attempting to incorporate the electrode effect as a source in our charge-convection model, we will take another look at it in hopes of finding a simplification. As mentioned in the introduction, the only successful models of the electrode effect in the presence of the turbulent mixing which is nearly always important in the lower atmosphere have been fairly recent. Our work here is based on that of Hoppel (1969) and of Hoppel and Gathman (1971), since their treatment of the lower boundary conditions is more careful, and in our opinion, more valid, than that of Latham and Poor (1972).

Work toward a simplified treatment of the electrode effect in the special case of strong turbulent mixing was motivated by a study of the results of Hoppel (1969). He used a gradient-diffusion model for the turbulent-transport term in each of his budget equations for positive and negative small ions. Nuclei and large ions were not considered. The eddy-diffusion coefficient increased linearly with height near the ground. Qualitatively, his key results were as follows: 1) As the strength of the turbulent mixing was increased, the depth of the electrode layer increased, while the gradient of the vertical electric field (proportional to the local charge density) near the surface decreased. 2) Increasing the turbulence did not, however, decrease the overall

"strength" of the electrode effect, since the fractional change in electric field (proportional to the total integrated charge in the layer) across the whole electrode layer was unchanged. 3) As the turbulence was increased, the profiles of the individual ion densities n_1 and n_2 became more and more similar, and relative to these, $n_1 - n_2$ (proportional to the charge density) became small throughout the layer. 4) Although the profiles of electric field and charge density were shown to be quite sensitive to the profile of ionization near the ground (which varies due to surface radiation and, more importantly, due to trapping of radon-220 gas near the surface) in the case of no turbulent mixing, this did not seem to be a problem in the case of strong turbulence due to the resulting dispersal of the radon throughout the boundary layer.

Point 3) above suggested that something special is going on in the limit of strong turbulent mixing. It appeared that the "true electrode effect", interpreted as the depletion of ions of one sign near a conducting surface, was being replaced by simple turbulent diffusion of ions of both signs to the surface, where they were annihilated. If this was the case, perhaps the electrode effect in the limit of strong turbulent mixing could be modeled simply by a conductivity profile decreasing to zero at the surface.

5.2 Asymptotic Analysis

To test this hypothesis, we will study the electrode effect, using a gradient-diffusion model like Hoppel's for simplicity, in the limit of strong turbulent mixing. We start with the steady-state budget equations for positive and negative small ions of concentration n_1 and n_2 ($\frac{1}{m^3}$), respectively, with identical uniform mobilities $\pm k$ ($\frac{m^3}{\text{volt sec}}$) and recombination coefficient α ($\frac{m^3}{\text{sec}}$) in a region of uniform ionization rate

$q \left(\frac{1}{\text{m}^3 \text{sec}} \right)$. The vertical electric field is $E \left(\frac{\text{volt}}{\text{m}} \right)$; the molecular-diffusion coefficient is $D \left(\frac{\text{m}^2}{\text{sec}} \right)$, and the eddy-diffusion coefficient is taken as Kz with $K \left(\frac{\text{m}}{\text{sec}} \right)$ a constant. These equations, a special case of (A1.5), are similar to those of Hoppel and Gathman (1971).

$$\left. \begin{aligned} -K \frac{d}{dz} z \frac{d}{dz} n_1 - D \frac{d^2}{dz^2} n_1 + k \frac{d}{dz} n_1 E &= q - \alpha n_1 n_2 \\ -K \frac{d}{dz} z \frac{d}{dz} n_2 - D \frac{d^2}{dz^2} n_2 - k \frac{d}{dz} n_2 E &= q - \alpha n_1 n_2 \\ \frac{dE}{dz} &= \frac{e}{\epsilon_0} (n_1 - n_2) \end{aligned} \right\} \quad (5.1)$$

Here, $e \equiv 1.602 \times 10^{-19}$ coul is the unit electron charge assumed to reside on each ion and $\epsilon_0 \equiv 8.85 \times 10^{-12} \frac{\text{farad}}{\text{m}}$ is the permittivity of free space. The proper boundary conditions on these equations are $n_1(0) \equiv 0 \equiv n_2(0)$, $n_1(z \rightarrow \infty) \equiv \sqrt{\frac{q}{\alpha}} \equiv n_2(z \rightarrow \infty)$, and $E(z \rightarrow \infty) \equiv E_\infty$, as discussed by Hoppel (1969) and Hoppel and Gathman (1971). We now define

$$y \equiv n_1 + n_2, \quad \frac{\rho}{e} \equiv n_1 - n_2 \quad (5.2)$$

and re-write equations (5.1) in these new variables:

$$\left. \begin{aligned} -K \frac{d}{dz} z \frac{d}{dz} y - D \frac{d^2}{dz^2} y + \frac{k}{e} \frac{d}{dz} E \rho &= 2q - \frac{\alpha}{2} \left(y^2 - \frac{\rho^2}{e^2} \right) \\ -\frac{K}{e} \frac{d}{dz} z \frac{d}{dz} \rho - \frac{D}{e} \frac{d^2}{dz^2} \rho + k \frac{d}{dz} E y &= 0 \\ \frac{dE}{dz} &= \frac{\rho}{\epsilon_0} \end{aligned} \right\} \quad (5.3)$$

Here $y \equiv \frac{\lambda}{ek}$ is proportional to the conductivity. The new boundary conditions are $\rho(0) = 0 = y(0)$, $\rho(z \rightarrow \infty) = 0$, $y(z \rightarrow \infty) = 2\sqrt{\frac{q}{\alpha}}$, and $E(z \rightarrow \infty) \equiv E_\infty$. Next, equations (5.3) are scaled according to deviations from the asymptotic solutions approached as $z \rightarrow \infty$.

$$\left. \begin{aligned} y &\equiv \bar{y} + y' \equiv \sqrt{\frac{q}{\alpha}} (2 + y_I) \\ \rho &\equiv \bar{\rho} \rho' \equiv \frac{\epsilon_0 |E_\infty|}{L'} \rho_I \\ E &\equiv E' - |E_\infty| \equiv |E_\infty| (E_I - 1)^* \\ z &\equiv L' z_I \end{aligned} \right\} \quad (5.4)$$

The prime on the height scale is used to distinguish it from the Monin-Obukhov length L . Substitution of (5.4) into (5.3) and some rearrangement gives the non-dimensional equations:

$$\left. \begin{aligned} -\left(z_I \frac{d^2 y_I}{dz_I^2} + \frac{dy_I}{dz_I}\right) - \left[\frac{D}{KL'}\right] \frac{d^2 y_I}{dz_I^2} + \left[\frac{k|E_\infty|}{K}\right] \left[\frac{2\bar{\rho}}{e\bar{y}}\right] \left(\frac{d}{dz_I} E_I \rho_I - \frac{d\rho_I}{dz_I}\right) \\ = -\left[\frac{L'}{K} \sqrt{q\alpha}\right] \left(2y_I + \frac{1}{2}y_I^2\right) + \frac{1}{2} \left[\frac{L'}{K} \sqrt{q\alpha}\right] \left[\frac{2\bar{\rho}}{e\bar{y}}\right]^2 \rho_I^2 \\ -\left(z_I \frac{d^2 \rho_I}{dz_I^2} + \frac{d\rho_I}{dz_I}\right) - \left[\frac{D}{KL'}\right] \frac{d^2 \rho_I}{dz_I^2} + \left[\frac{k|E_\infty|}{K}\right] \left[\frac{2\bar{\rho}}{e\bar{y}}\right]^{-1} \left(2\frac{dE_I}{dz_I} - \frac{dy_I}{dz_I} + \frac{d}{dz_I} E_I y_I\right) = 0 \\ \frac{dE_I}{dz_I} = \rho_I \end{aligned} \right\} \quad (5.5)$$

* This definition of E_I is valid only for $E_\infty < 0$. The absolute value signs on E_∞ have been employed to give E_I the same sign as E .

Finally, we assume that the molecular-diffusion term is negligible compared to the eddy-diffusion term in the interior of the electrode layer, or that $[\frac{D}{KL}] \ll 1$. If this is so, then the first and third terms in the ρ_I equation must balance, implying

$$\frac{\epsilon_0 |E_\infty|}{L' e} \sqrt{\frac{\kappa}{q}} \equiv \left[\frac{2\bar{\rho}}{e\bar{y}} \right] = \frac{k |E_\infty|}{K} \Rightarrow L' = \frac{\epsilon_0 K}{k e} \sqrt{\frac{\kappa}{q}} \quad (5.6)$$

Using (5.6) to eliminate $[\frac{2\bar{\rho}}{e\bar{y}}]$ and to define the height scale L' gives

$$\left. \begin{aligned} & - \left(z_I \frac{d^2 y_I}{dz_I^2} + \frac{dy_I}{dz_I} \right) + \delta \left(2y_I + \frac{1}{2} y_I^2 \right) = \beta \left\{ \gamma \left[\frac{d\rho_I}{dz_I} - \frac{d}{dz_I} (E_I \rho_I) \right] \right. \\ & \quad \left. + \frac{\delta}{2} \rho_I^2 \right\} + \frac{d^2 y_I}{dz_I^2} \\ & - \left(z_I \frac{d^2 \rho_I}{dz_I^2} + \frac{d\rho_I}{dz_I} \right) + 2 \frac{dE_I}{dz_I} - \frac{dy_I}{dz_I} + \frac{d}{dz_I} E_I y_I = \beta \frac{d^2 \rho_I}{dz_I^2} \\ & \frac{dE_I}{dz_I} - \rho_I = 0 \end{aligned} \right\} \quad (5.7)$$

where we have defined three non-dimensional parameters:

$$\left. \begin{aligned} \beta & \equiv \frac{D k e}{\epsilon_0 K^2} \sqrt{\frac{q}{\kappa}} \\ \gamma & \equiv \frac{k |E_\infty|^2 \epsilon_0}{D e} \sqrt{\frac{\kappa}{q}} \approx 6.6 \times 10^3 \\ \delta & \equiv \frac{\epsilon_0 K}{k e} \approx 0.59 \end{aligned} \right\} \quad (5.8)$$

The approximate values given above come from the following typical values

of the parameters for clean continental surface air;

$$\left. \begin{aligned} |E_{\infty}| &= 100 \frac{\text{volt}}{\text{m}} \\ q &= 10^7 \frac{1}{\text{m}^3 \text{sec}} \\ D &= 5 \times 10^{-6} \frac{\text{m}^2}{\text{sec}} \\ k &= 1.5 \times 10^{-4} \frac{\text{m}^2}{\text{volt sec}} \\ \alpha &= 1.6 \times 10^{-12} \frac{\text{m}^3}{\text{sec}} \end{aligned} \right\} \quad (5.9)$$

The boundary conditions have now become $y_I(0) = -2$, $\rho_I(0) = 0$, $y_I(z_I \rightarrow \infty) = 0$, $\rho_I(z_I \rightarrow \infty) = 0$, and $E_I(z_I \rightarrow \infty) = 0$. If we take the further rough estimate of $K = 0.1 \frac{\text{m}}{\text{sec}}$, implying that the eddy-diffusion coefficient at 10 m is $1 \frac{\text{m}^2}{\text{sec}}$, we have $\beta \approx 3.4 \times 10^{-6}$ and $L' \approx 15 \text{ m}$. This scaling implies that in the interior of the electrode layer ρ_I drops out of the equation for y_I in (5.7) because of the smallness of β . However, this need not be true close to the surface, where there may be a boundary layer. We emphasize here that no approximations have been made in obtaining equations (5.7) from equations (5.1). The approximations will come when we ignore the right-hand sides.

Next, we derive the boundary layer forms of equations (5.7). We begin by assuming that the surface is aerodynamically rough and that it is effectively located at some small height $z_0 > 0$. This allows us to ignore the molecular-diffusion sublayer because at $z = z_0$ the eddy-diffusion coefficient is already much larger than the molecular-diffusion coefficient. This is analogous to standard practice in surface-layer, semi-empirical theory where, for example, the profile of mean horizontal wind in a neutrally-stratified surface layer is given by Businger (1973) as

$$\frac{\bar{u}}{u_*} = \frac{1}{k} \ln \frac{z}{z_0}$$

We will apply all lower boundary conditions at $z_I = z_{I0} \equiv \frac{z_0}{L'}$.

Because y_I and ρ_I are both forced to assume certain values at the lower boundary, where ions are annihilated, we expect boundary layers in these variables. Since $\rho_I = \frac{dE_I}{dz_I}$ vanishes at z_{I0} , we do not expect a boundary layer in E_I . Therefore we make the following re-scaling of the boundary-layer variables:

$$\left. \begin{aligned} z_I &\equiv \beta^n z_B \\ y_I &\equiv y_B \\ \rho_I &\equiv \rho_B \\ E_I &\equiv E_0 + \beta^r E_B \end{aligned} \right\} \quad (5.10)$$

Here E_0 is a constant of order unity from the point of view of the boundary-layer independent variable z_B . The forms assumed in (5.10) incorporate the above assumptions that there is no boundary layer in E_I (to lowest order) while there are strong boundary layers in y_I and ρ_I of dimensional thickness proportional to $\beta^n L'$. The bottom boundary conditions on these new boundary-layer variables become $y_B(z_{B0}) = -2$,

$\rho_B(z_{B0}) = 0$, and $E_B(z_{B0}) = 0$, where $z_{B0} \equiv \beta^{-n} z_{I0}$. The upper boundary condition is a matching with the interior solution: $y_B(z_B \rightarrow \infty) = y_I(z_I \rightarrow 0)$, $\rho_B(z_B \rightarrow \infty) = \rho_I(z_I \rightarrow 0)$, and (to lowest order) $E_0 = E_I(z \rightarrow 0)$. Putting (5.10) into (5.7) and multiplying through by β^n in the first two equations gives

$$\begin{aligned}
& -\left(z_B \frac{d^2 y_B}{dz_B^2} + \frac{dy_B}{dz_B}\right) = \beta^{1-n} \frac{d^2 y_B}{dz_B^2} - \beta^n \delta \left(2y_B + \frac{1}{2}y_B^2\right) + \beta \gamma (1-E_0) \frac{dp_B}{dz_B} \\
& \quad - \beta^{1+r} \gamma \frac{d}{dz_B} (E_0 \rho_B) + \beta^{1+n} \gamma \frac{\delta}{2} \rho_B^2 \\
& -\left(z_B \frac{d^2 \rho_B}{dz_B^2} + \frac{d\rho_B}{dz_B}\right) - (1-E_0) \frac{dy_B}{dz_B} = \beta^{1-n} \frac{d^2 \rho_B}{dz_B^2} \\
& \quad - \beta^r \left[2 \frac{dE_0}{dz_B} + \frac{d}{dz_B} (E_0 y_B)\right]
\end{aligned}
\tag{5.11}$$

The last equation above suggests $r = n$. We can use this equality with the understanding that we have now defined what we mean by "zero order". Hereafter, we are no longer allowed to assume that two or more terms in each of the other equations must balance to zero order. Any further such assumption is only legitimate insofar as it does not lead to a contradiction.

With $r = n$ above we have the following powers of β : β^0 , β^{1-n} , β^n , $\beta^1 \gamma$, and $\beta^{1+n} \gamma$. Since δ is of order unity, we have ignored it; but γ is a large parameter which causes trouble and hence has been included. We are hoping to find a situation where conduction remains negligible in the y_B equation in the boundary layer as well as in the interior, so we want $\beta^{1+n} \gamma \ll 1$. Since it is reasonable to expect the $\beta \gamma (1-E_0) \frac{dp_B}{dz_B}$ conduction term to be the largest, because of the vanishing of $\frac{dE_I}{dz_I}|_{z_{I0}}$ at the surface, we expect $0 < n$. This is also required in order to have a thin boundary layer. Furthermore, we have already required molecular diffusion to be negligible above our rough surface, so we must have $n < 1$.

We will try taking $n = \frac{1}{2}$, which gives the simplest ordering of the terms subject to these constraints.

The simplest way to handle γ is to write

$$\gamma \equiv A \beta^m, \quad A \sim \mathcal{O}(1) \quad (5.12)$$

where A is the new parameter of order unity, and the magnitude of γ is taken care of by the β^m factor. For the range of parameters of interest to us, and in order to maintain the simple ordering of powers of β , we take $m = \frac{1}{2}$. With these assumptions, (5.11) becomes

$$\left. \begin{aligned} -\left(z_B \frac{d^2 y_B}{dz_B^2} + \frac{dy_B}{dz_B}\right) &= \beta^{1/2} \left[A(1-E_0) \frac{d\rho_B}{dz_B} + \frac{d^2 y_B}{dz_B^2} - \delta \left(2y_B + \frac{1}{2} y_B^2 \right) \right] \\ &+ \beta' A \left[\frac{\delta}{2} \rho_B^2 - \frac{d}{dz_B} (E_0 \rho_B) \right] \\ -\left(z_B \frac{d^2 \rho_B}{dz_B^2} + \frac{d\rho_B}{dz_B}\right) - (1-E_0) \frac{dy_B}{dz_B} &= \beta^{1/2} \left[\frac{d^2 \rho_B}{dz_B^2} - 2 \frac{dE_0}{dz_B} - \frac{d}{dz_B} (E_0 y_B) \right] \\ \frac{dE_0}{dz_B} - \rho_B &= 0 \end{aligned} \right\} \quad (5.13)$$

Since the first term in the y_B equation above has nothing to balance it, before proceeding further we must expand y_B , ρ_B , and E_B in powers of $\beta^{1/2}$:

$$\left. \begin{aligned} y_B &\equiv {}^{(0)}y_B + \beta^{1/2} {}^{(1)}y_B + \dots \\ \rho_B &\equiv {}^{(0)}\rho_B + \beta^{1/2} {}^{(1)}\rho_B + \dots \end{aligned} \right\} \quad (5.14)$$

$$E_B \equiv {}^{(0)}E_B + \beta^{1/2} {}^{(1)}E_B + \dots \quad (5.14)$$

cont.

Putting (5.14) into (5.13), we collect the zero-order equations:

$$\left. \begin{aligned} -\left(z_B \frac{d^2 {}^{(0)}y_B}{dz_B^2} + \frac{d {}^{(0)}y_B}{dz_B}\right) &= 0 \\ -\left(z_B \frac{d^2 {}^{(0)}\rho_B}{dz_B^2} + \frac{d {}^{(0)}\rho_B}{dz_B}\right) - (1-E_0) \frac{d {}^{(0)}y_B}{dz_B} &= 0 \\ \frac{d {}^{(0)}E_B}{dz_B} - {}^{(0)}\rho_B &= 0 \end{aligned} \right\} \quad (5.15)$$

and the first-order equations:

$$\left. \begin{aligned} -\left(z_B \frac{d^2 {}^{(1)}y_B}{dz_B^2} + \frac{d {}^{(1)}y_B}{dz_B}\right) &= A(1-E_0) \frac{d {}^{(0)}\rho_B}{dz_B} + \frac{d^2 {}^{(0)}y_B}{dz_B^2} - \delta\left(2 {}^{(0)}y_B + \frac{1}{2} {}^{(0)}y_B^2\right) \\ -\left(z_B \frac{d^2 {}^{(1)}\rho_B}{dz_B^2} + \frac{d {}^{(1)}\rho_B}{dz_B}\right) - (1-E_0) \frac{d {}^{(1)}y_B}{dz_B} &= \frac{d^2 {}^{(0)}\rho_B}{dz_B^2} - \frac{d}{dz_B} \left({}^{(0)}E_B {}^{(0)}y_B \right) - 2 \frac{d {}^{(0)}E_B}{dz_B} \\ \frac{d {}^{(1)}E_B}{dz_B} - {}^{(1)}\rho_B &= 0 \end{aligned} \right\} \quad (5.16)$$

From (5.7), the zero-order interior equations are:

$$-\left(z_I \frac{d^2 {}^{(0)}y_I}{dz_I^2} + \frac{d {}^{(0)}y_I}{dz_I}\right) + \delta\left(2 {}^{(0)}y_I + \frac{1}{2} {}^{(0)}y_I^2\right) = 0 \quad (5.17a)$$

$$-\left(z_I \frac{d^2 {}^{(0)}\rho_I}{dz_I^2} + \frac{d {}^{(0)}\rho_I}{dz_I}\right) + 2 \frac{d {}^{(0)}E_I}{dz_I} - \frac{d {}^{(0)}y_I}{dz_I} + \frac{d}{dz_I} \left({}^{(0)}E_I {}^{(0)}y_I \right) = 0 \quad (5.17b)$$

$$\frac{d^{(0)}E_I}{dz_I} - {}^{(0)}\rho_I = 0 \quad (5.17c)$$

Notice that (5.15) is really a special case of (5.17) where some of the terms become negligible but no new terms are added. In order to verify the validity of this asymptotic analysis, we must actually solve (5.17) and (5.15) and then substitute these zero-order solutions back into the higher-order terms on the right hand sides of (5.7) and (5.13) to show that these terms are in fact negligible. This has been done numerically, by solving only the interior equations (5.17) right down to the bottom boundary at z_{I0} . The numerical solution confirms in detail the results of the above asymptotic analysis within its range of validity.

The limits of validity of the zero-order equations derived above may be estimated as follows. First, we must have $\beta \ll 1$; this condition is easily satisfied. But we must also have $A \equiv \gamma\beta^{1/2} \lesssim 0(1)$. When A approaches $0(\beta^{-1/2})$, it moves all the terms containing γ an order higher in equations (5.7) and (5.13), and some of the conduction terms start appearing in the zero-order equations for y_I and y_B . Suppose we guess that the condition on A is satisfied as long as $A \leq \beta^{-1/4}$. This implies

$$\gamma\beta^{1/2} \leq \beta^{-1/4} \Rightarrow \beta \leq \gamma^{-4/3} \quad (5.18a)$$

For the values of the parameters stated in (5.9) above, we have $\beta \leq 8.1 \times 10^{-6}$, corresponding to $K \geq 0.065 \frac{\text{m}}{\text{sec}}$ and $L' \geq 9.6 \text{ m}$. Thus, our previously suggested value of $\beta = 3.4 \times 10^{-6}$ should be within the range of validity of this analysis.

A further restriction is that $z_{I0} \gtrsim 0(\beta^{1/2})$ since we have scaled z_B by $\beta^{1/2}$ in the boundary layer. If z_{I0} approaches $0(\beta)$, it will

move the molecular-diffusion terms up to zero order in equations (5.13).

Proceeding in the same manner, we assume this restriction to be satisfied as long as

$$\beta^{3/4} \leq z_{10} \quad (5.18b)$$

Thus, the maximum allowed value for β also gives the largest lower bound on $z_{10} \geq 1.5 \times 10^{-4}$. With the aid of the scale relation $z = L'z_I$ and equation (5.6), (5.18b) can be translated into a lower bound on the dimensional roughness length z_0 :

$$z_0 \geq L' \beta^{3/4} = \left(\frac{\epsilon_0 D^3}{k e K^2 \sqrt{\frac{\alpha}{q}}} \right)^{1/4} \quad (5.18c)$$

This tells us that increasing K (and, hence, decreasing β) also decreases the lower bound on z_0 , so its largest lower bound can be given as $z_0 \geq 1.4 \text{ mm}$. This is certainly not large enough to restrict us much in the real atmosphere, at least over land.

5.3 Numerical Treatment

Before attempting a numerical solution of equations (5.17), we made the following coordinate stretching to take account of the fact that there is a boundary layer near the surface:

$$z_I \equiv z_{\infty} e^{\Delta n} ; \quad z_{\infty} \equiv z_{10} e^{-\Delta} , \quad \Delta \equiv \frac{\ln(z_{10}/z_{10})}{N-1} \quad (5.19)$$

This is essentially the same stretching function used in the second-order-closure model, arranged so that the bottom boundary z_{I0} is at $n=1$ and the upper limit of integration $z_{I\infty}$ is at $n=N$. This form was chosen because $y_B \propto \ln z_B \propto \ln z_I \propto \Delta n$ is the form of the solution in the boundary layer.

It is clear that the y_I equation (5.17a) can be solved independently of the others, giving the conductivity profile $\lambda = ek y = ek \left(\frac{g}{\alpha}\right)^{1/2} (2+y_I)$. This equation was solved by a shooting method. The second initial condition $\left.\frac{dy_I}{dz_I}\right|_{z_{I0}}$ was guessed, and the equation was integrated as an initial-value problem from z_{I0} upward until either y_I became positive or $\left.\frac{dy_I}{dz_I}\right|_{z_{I0}}$ became negative. Then $\left.\frac{dy_I}{dz_I}\right|_{z_{I0}}$ was either decreased or increased, respectively, and the procedure was repeated until the upper limit of integration $z_{I\infty}$ was reached without meeting either test condition. At this point the upper asymptotic boundary condition $y_I(z_I \rightarrow \infty) = 0$ was considered satisfied.

In order to solve the equation (5.17b) for ρ_I , we used equation (5.17c) to eliminate $\frac{dE_I}{dz_I}$ in the two places it occurred. The remaining E_I was eliminated by using the integral of (5.17b), rearranged as follows:

$$E_I = \frac{z_I \frac{d\rho_I}{dz_I} + y_I}{2 + y_I} \quad (5.20)$$

The resulting equation

$$-\left(z_I \frac{d^2 \rho_I}{dz_I^2} + \frac{d\rho_I}{dz_I}\right) + \frac{z_I \frac{dy_I}{dz_I}}{2 + y_I} \frac{d\rho_I}{dz_I} + (2 + y_I) \rho_I = \frac{2 \frac{dy_I}{dz_I}}{2 + y_I} \quad (5.21)$$

was solved as a boundary-value problem, using the previous solution for y_I in the coefficients and on the right-hand side. Finally, equation (5.20) was solved diagnostically for E_I .

Having obtained these zero-order solutions for y_I , ρ_I , and E_I , we substituted them into the right-hand sides of the full interior equations (5.7) to evaluate the validity of the approximations. It is easy to iterate for the exact solutions to (5.7) by using the non-zero right-hand sides calculated in this manner. This gives us a way to determine precisely how good the zero-order equations are as well as which of the ignored terms are most important.

As examples, two cases have been worked out. Case 1 is for $\beta = 3.4 \times 10^{-6}$ and $z_{I0} = 1.23 \times 10^{-3}$, corresponding to $K = 0.1 \frac{m}{sec}$, $L' \approx 15 m$, and $z_0 \approx 1.8 cm$. Case 2 is for $\beta = 1.4 \times 10^{-5}$ and $z_{I0} = 1.23 \times 10^{-4}$, corresponding to $K = 0.05 \frac{m}{sec}$, $L' \approx 7.4 m$, and $z_0 \approx 0.91 mm$. In both we have taken γ and δ as in (5.8). The results are shown in figures 5.1 to 5.6, where both the zero-order (approximate) solutions and the exact (corrected) solutions appear. It is evident that the zero-order solutions for Case 1 are quite good, whereas those for Case 2 are not acceptable.

In Case 1, an examination of the terms in the budget equations (5.7) yields the following results. Molecular diffusion is indeed negligible throughout, as assumed; in the ρ_I equation this term is two orders of magnitude smaller than the largest term at the bottom boundary. Thus equation (5.17b) is validated. Below $z_I \approx 0.025$ the conduction terms involving $\frac{dE_I}{dz_I}$ drop out of the ρ_I equation also, validating (5.15b).

At first glance there appears to be trouble in the y_I equation, because the sum of the neglected terms is larger than the largest term in

Fig.5.1 Approximate and corrected profiles of dimensionless conductivity in the electrode layer on a semi-logarithmic scale for Cases 1 and 2, calculated from equations (5.17) and (5.7) as described in the text.

Fig.5.2 Same as figure 5.1, for dimensionless charge density.

Fig.5.3 Approximate and corrected profiles of dimensionless conductivity in the electrode layer on a linear scale for Case 1, calculated from equations (5.17) and (5.7) as described in the text.

Fig.5.4 Same as figure 5.3, for dimensionless charge density.

Fig.5.5 Same as figure 5.3, for dimensionless electric field.

Fig.5.6 Same as figure 5.3, for dimensionless convection-current density.

the zero-order equation up to $z_I \approx 0.011$ and it does not become negligible until above $z_I \approx 1.39$. However, it is observed that the exact solution for y_I , including these terms, differs very little from the zero-order solution. This can be taken as evidence that (5.15a) is valid in the lower part. Further evidence for this is the near-linearity of the graph of $\hat{\lambda}$ on semi-log paper in figure 5.1. The balance we are observing in the numerical solution is really the first-order balance as expressed by the first equation in (5.16).

Looking now at Case 2, we find that the zero-order equations are no longer obeyed. Although the molecular diffusion terms are not quite so large as the dominant included terms at the bottom boundary, this term is not negligible in the ρ_I equation until above $z_I \approx 0.00047$. In the y_I equation the sum of the neglected terms is now larger than the largest term in the zero-order equation up to $z_I \approx 0.72$, and it is evident from a comparison of the "approximate" and "corrected" curves for $\hat{\lambda}$ in figure 5.1 that these terms have now made it up to zero order. From these calculations it appears that the range of validity of the zero-order equations for the given value of γ is $\beta \lesssim 3.4 \times 10^{-6}$ and $z_0 \gtrsim 1 \text{ mm}$. These results agree quite well with the previous estimates of $\beta \leq 8.1 \times 10^{-6}$ and $z_0 \geq 1.4 \text{ mm}$. Therefore, we feel justified in accepting equations (5.18).

5.4 Results on the Electrode Effect

We have succeeded in identifying a range in parameter space, corresponding roughly to the range of moderate to strong turbulent mixing of relevance to our problem, in which the electrode effect degenerates to diffusion of ions to the surface. In this range, the conduction process does not affect the conductivity profile, which can be described by a single ordinary differential equation containing the one parameter δ , subject to

boundary conditions containing another parameter z_{I0} . While there seems to be no observational evidence available at present with which to compare our theoretical predictions concerning the conductivity profiles near the ground, these profiles appear to agree with those of Hoppel (1969). Let us briefly review the simplifications and approximations that make this result possible.

The major simplifications made in the analysis above were the neglect of large ions and nuclei and the assumption of an eddy-diffusion model for the turbulent transport of small ions. The former simplification is the most serious. Neglect of nuclei and large ions is never justified over land, where they considerably reduce the conductivity and the lifetime of small ions by acting as recombination sites. In extremely polluted areas, such as near large cities, they may even carry a large fraction of the space charge. Over an ocean surface nearly free of whitecaps, however, the results of Hoppel and Gathman (1971) indicate that the density of nuclei is small enough to have a relatively small effect on the electrical parameters. And over an ice surface, such as in the arctic, the effect is probably negligible.

It is hard to see just what effect the presence of nuclei will have on our analysis above. Results of earlier studies on the non-turbulent electrode effect, such as Hoppel (1966), have shown that nuclei decrease the thickness of the electrode layer by reducing the lifetime of small ions. However, Hoppel and Gathman (1971) found that the introduction of nuclei in the turbulent case increases the thickness of the electrode layer. It is not anticipated that the introduction of a moderate concentration of nuclei will qualitatively change our results on the behavior of the strong-turbulence limit, as long as the nuclei do not become the

dominant charge carriers, but this matter merits further study in the future. We will assume for the purposes of this thesis that even over land the conductivity profile becomes independent of the conduction process during moderate to strong turbulent mixing.

The assumption of an eddy-diffusion model for the turbulent transport of ions in our analysis was made to give a simple form to that term in the equations. This is expected to be a fairly good approximation in this case, because the dominant process is diffusion to the surface, and its effects are concentrated in a relatively shallow layer due to the short lifetime of the ions. This statement requires some clarification, since we have already shown that the eddy-diffusion model is inadequate for describing charge transport in the unstable boundary layer.

There is reason for this apparent turnabout. The conductivity profile is determined by diffusion of ions of both signs, produced and annihilated throughout the atmosphere by ionization and recombination, to a surface where they are destroyed. This process is necessarily controlled very close to the surface, where the greatest resistance to turbulent diffusion is concentrated. The convection current, on the other hand, is carried upward into the interior of the mixed layer from a charge source region distributed over the lowest several meters. This charge is produced and destroyed by processes quite different from those controlling the ion densities themselves. The lifetime of an ion is based on the production and annihilation rates and can be expressed as

$$\tau_n \equiv \frac{n_{1,2}}{q} \approx \frac{\lambda}{2keq} \quad (5.22)$$

whereas, the decay of charge density is controlled by the electrical

relaxation time $\tau_\lambda \equiv \frac{\epsilon_0}{\lambda}$. In many parts of the lower atmosphere, $\tau_n \ll \tau_\lambda$. This means that the charge produced by the electrode effect can be carried by turbulent transport to much higher levels than those to which the surface-produced depletion of conductivity extends. Thus, the entire conductivity profile, and perhaps the decrease of charge density from its maximum to zero at the surface, can be adequately modeled with an eddy-diffusion model; whereas, the profiles of charge density above the maximum, and of convection current, cannot.

Table 5.1
Time Scales in the Electrode Layer

Value:	$\lambda \left(10^{-14} \frac{\text{mbo}}{\text{m}} \right)$	$q \left(10^6 \frac{1}{\text{m}^3 \text{sec}} \right)$	$\tau_\lambda \text{ (sec)}$	$\tau_n \text{ (sec)}$
Arctic	4.5	1.8	200	520
Oceanic	2.0	1.5	440	280
Continental	1.0	10.1	880	20
Polluted Continental	0.4	10.1	2200	8.2

Table 5.1 gives typical values of τ_n and τ_λ in surface air over various terrains, using values of λ and q drawn from Israel (1970). Comparison of τ_n and τ_λ in this table indicates that the above justification for the use of an eddy-diffusion model to obtain λ profiles is valid over land, but perhaps not over oceans, and certainly not in the arctic. However, it is rare to find strong instability over the ocean or in the arctic, so eddy-diffusion may be appropriate there after all. For the purposes of this thesis we will use λ profiles derived from equation (5.17a) for all models involving the electrode effect.

5.5 Development of an Approximate λ Profile

Referring to the graph of $\hat{\lambda}$ versus $\ln z_I$ in figure 5.1, we observe that the solution is very nearly linear in the lower part, in agreement with the boundary layer equation (5.15a), and rolls off asymptotically toward 1.0 in the vicinity of $z_I = 1.0$. Based on this behavior, an approximate formula has been developed for $\hat{\lambda}$. The linear behavior in the lower part

$$y_I = -(2 + C \ln z_{I0}) + C \ln z_I, \quad z_I \ll 1$$

can be matched asymptotically to 1.0 as follows:

$$y_I' \equiv -(2 + C \ln z_{I0}) - \frac{1}{A} \ln \left[\frac{1}{z_I^{AC}} + e^{-A(2 + C \ln z_{I0})} \right] \quad (5.23)$$

Here, $C = \left. \frac{dy_I}{d(\ln z_I)} \right|_{z_{I0}}$ is determined by the numerical solution, and A is an arbitrary constant controlling how fast the roll-off occurs. By proper choice of A, y_I' can be matched well to y_I from the numerical solutions of (5.17a). For z_{I0} in the range $[1.23 \times 10^{-6}, 1.23 \times 10^{-3}]$ it has been found that $AC = 1.175$ gives a good match. For z_{I0} larger than 1.23×10^{-3} , however, AC must be increased, and errors start appearing in the lower boundary condition because the second term in brackets in (5.23) is no longer negligible at the surface. Subject to these limitations, this formula can be used in place of interpolation of the actual numerical solution, saving considerable numerical effort.

Equation (5.23) can be put into closure-model, non-dimensional variables with the aid of the following formulae:

$$\left. \begin{aligned}
 \lambda &\equiv 1 + \frac{y_2}{2} \\
 z_1 &= \frac{H}{L'} \zeta \\
 \frac{H}{L'} &= \frac{H k e}{\epsilon_0 K} \sqrt{\frac{g}{\alpha}} = \frac{H \lambda_\infty}{2 \epsilon_0 K} = \frac{H}{2 \tau_\lambda K} = 1.06 \frac{H}{u_* \tau_\lambda} = \frac{1.06}{R}
 \end{aligned} \right\} \quad (5.24)$$

In the last line above, we have used previous definitions of L' , λ_∞ , and R , as well as the surface-layer form of the eddy-diffusion coefficient. In the surface layer, Businger (1973) gives the unstable temperature profile as

$$\frac{k z}{T_*} \frac{d\bar{\theta}}{dz} = 0.74 \left(1 - 9 \frac{z}{L}\right)^{-1/2}$$

Using this in the limit as $\frac{z}{L} \rightarrow 0$, we have

$$K_z = K_h \equiv \frac{u_* T_*}{d\bar{\theta}/dz} = \frac{k z u_*}{0.74} = 0.47 u_* z \quad (5.24a)$$

Putting (5.24) into (5.23) gives

$$\left. \begin{aligned}
 \lambda \approx -\frac{C}{2} \left\{ \ln \frac{1.06 \lambda_\infty}{R} + \frac{1}{AC} \ln \left[\left(\frac{R}{1.06 \zeta} \right)^{AC} \right. \right. \\
 \left. \left. + e^{-AC \left(\frac{2}{C} + \ln \frac{1.06 \lambda_\infty}{R} \right)} \right] \right\}
 \end{aligned} \right\} \quad (5.25)$$

The results of (5.25), shown in figure 5.7, compare well with the actual numerical solutions of (5.17a), shown in figure 5.8 for several values of

Fig.5.7 Profiles of dimensionless conductivity predicted by the approximate formula (5.25) with $AC = 1.175$, except as otherwise indicated on the dashed curves, and with C chosen from figure 5.9.

Fig.5.8 Profiles of dimensionless conductivity obtained by numerical solution of equation (5.17a) for $\delta = 0.585$.

Fig.5.9 The constant C in the approximate formula (5.25) for dimensionless conductivity as a function of z_{I0} .

z_{I0} . The variation of $C = \frac{dy_I}{d(\ln z_I)} \Big|_{z_{I0}}$ with z_{I0} in the numerical solutions is plotted in figure 5.9. All these results are for $\delta = 0.585$.

6 MODIFICATIONS TO THE SECOND-ORDER-CLOSURE MODEL

FOR NON-UNIFORM CONDUCTIVITY

6.1 The Problem of the Electrode-Effect Source

The second-order-closure model described in Chapter 2 was specialized to the case of uniform conductivity, since this appeared to be a reasonable approximation within the thermally mixed boundary layer and yielded considerable simplifications. This uniform- λ model was adequate for describing the effects of a convection current forced into the layer at the lower boundary. It was used to analyze the Blanchard effect as a source. However, there is no satisfactory way of incorporating the electrode effect as a source into this model.

At first glance, it appears that a second-order-closure model to describe the electrode effect would require budget equations for at least $\overline{n_1'w'}$, $\overline{n_1'\theta'}$, $\overline{n_2'w'}$, $\overline{n_2'\theta'}$, and $\overline{n_1'n_2'}$, instead of just $\overline{\rho'w'}$ and $\overline{\rho'\theta'}$. This would make the problem complex and unwieldy and, therefore, is to be avoided. It would also be unsatisfying from an aesthetic point of view, since we know that everywhere in the interior of the mixed layer $\frac{\rho}{e} \ll n_1 \approx n_2$. Thus, all the extra computation to get n_1 and n_2 individually would be wasted and, indeed, would be a very inaccurate way of calculating $\rho = e(n_1 - n_2)$ as the small difference of large numbers.

For the above reasons, it was considered important to find a simplified way of handling the electrode effect as a source of convection current — a way which would allow an accurate treatment of the physical processes involving the two ion species, but which would not unduly complicate the model. The results of Chapter 5 on the electrode effect in the strong-turbulence limit indicate that this is easily done. As long as the

turbulent mixing is strong enough, the conductivity profile is unaffected by the conduction process. Therefore, we may continue to use the simple charge-conservation equation on which the uniform- λ model was based. The new closure-model equations for $\overline{\rho'w'}$ and $\overline{\rho'\theta'}$, derived below, are essentially the same as in the uniform- λ case. The only real difference is the appearance of another term in the equation for mean charge density, expressing the effect of conduction down a conductivity gradient.

6.2 Derivation of the Non-Uniform- λ Equations

The charge-conservation equations, written in terms of the potential $-\nabla V \equiv \underset{\sim}{E}$ with the substantial derivative expanded and $\nabla \cdot \underset{\sim}{U} \equiv 0$, is

$$\frac{\partial}{\partial t} \nabla^2 V = -\nabla \cdot \underset{\sim}{U} \nabla^2 V - \nabla \cdot \frac{\lambda}{\epsilon_0} \nabla V - \frac{Q}{\epsilon_0} \quad (6.1)$$

Here, $\rho = -\epsilon_0 \nabla^2 V$. Making a Reynolds expansion, as before, with

$V \equiv \bar{V} + V'$, $\underset{\sim}{U} \equiv \bar{\underset{\sim}{U}} + \underset{\sim}{U}'$, and $\lambda \equiv \bar{\lambda} + \lambda'$, and averaging horizontally with $\bar{w} \equiv 0$, we obtain the new mean charge equation:

$$\frac{\partial \bar{\rho}}{\partial t} = -\bar{\lambda} \bar{\rho} - \bar{E}_z \frac{\partial \bar{\lambda}}{\partial z} - \frac{\partial}{\partial z} \overline{\lambda' E'_z} - \frac{\partial}{\partial z} \overline{\rho' w'} + \bar{Q} \quad (6.2)$$

Here, the subscript z has been retained on E_z to emphasize that we are talking about the vertical component. Equations for $\overline{\rho'w'}$, $\overline{\rho'\theta'}$, and $\overline{\rho'^2}$ may also be obtained as before. They are

(equations on following page)

$$\left.
\begin{aligned}
\frac{\partial}{\partial t} \overline{\rho' w'} &= -\frac{\bar{\lambda}}{\epsilon_0} \overline{\rho' w'} - \overline{w'^2} \frac{\partial \bar{\rho}}{\partial z} - \frac{\partial}{\partial z} \overline{\rho' w'^2} - \overline{\rho' \frac{\partial p'}{\partial z}} + \frac{g}{T_0} \overline{\rho' \theta'} \\
&\quad - \frac{\partial \bar{\lambda}}{\partial z} \overline{w' E'_z} - \overline{w' \frac{\partial}{\partial z} (\lambda' E'_z)} - \overline{w' \frac{\partial}{\partial x_j} \lambda' E'_j} \\
\frac{\partial}{\partial t} \overline{\rho' \theta'} &= -\frac{\bar{\lambda}}{\epsilon_0} \overline{\rho' \theta'} - \overline{\theta' w'} \frac{\partial \bar{\rho}}{\partial z} - \overline{\rho' w'} \frac{\partial \bar{\theta}}{\partial z} - \frac{\partial}{\partial z} \overline{w' \theta' \rho'} \\
&\quad - \overline{\theta' E'_z} \frac{\partial \bar{\lambda}}{\partial z} - \overline{\theta' \frac{\partial}{\partial z} (\lambda' E'_z)} - \overline{\theta' \frac{\partial}{\partial x_j} \lambda' E'_j} \\
\frac{\partial}{\partial t} \frac{1}{2} \overline{\rho'^2} &= -\frac{\bar{\lambda}}{\epsilon_0} \overline{\rho'^2} - \overline{\rho' w'} \frac{\partial \bar{\rho}}{\partial z} - \frac{1}{2} \frac{\partial}{\partial z} \overline{\rho'^2 w'} \\
&\quad - \overline{\rho' E'_z} \frac{\partial \bar{\lambda}}{\partial z} - \overline{\rho' \frac{\partial}{\partial z} (\lambda' E'_z)} - \overline{\rho' \frac{\partial}{\partial x_j} \lambda' E'_j}
\end{aligned}
\right\} (6.3)$$

In these equations the molecular-diffusion and Coriolis terms have been omitted for simplicity. As before, the former will be important in the $\overline{\rho' \theta'}$ and $\overline{\rho'^2}$ equations, while the latter will be neglected. In the last terms in each of equations (6.3) the subscript j has been used to indicate summation over all three coordinates.

Equations (6.2) and (6.3) are similar to the corresponding equations (2.12), (2.15), and (2.14) derived earlier for the uniform- λ case.

In the mean charge equation (6.2), note the appearance of two new terms, $-\bar{E}_z \frac{\partial \bar{\lambda}}{\partial z} - \frac{\partial}{\partial z} \overline{\lambda' E'_z}$, representing the effect of conduction down a conductivity gradient in accumulating space charge. In (6.3), three new terms appear in the last line of each equation. Of this total of eleven new terms, only one can be evaluated directly: the $-\bar{E}_z \frac{d\bar{\lambda}}{dz}$ term in the mean charge

equation. All the others must be either ignored or approximated by closure models of some sort.

The physical significance of the new terms may be brought out by the following analogy. Re-writing (6.2) in the steady state as

$$\bar{\rho} = \frac{\epsilon_0}{\lambda} \left[\bar{Q} - \bar{E}_z \frac{\partial \bar{\lambda}}{\partial z} - \frac{\partial}{\partial z} \overline{\lambda' E'_z} - \frac{\partial}{\partial z} \overline{\rho' w'} \right]$$

we can consider the second term in the brackets as an additional source term due to conduction. Make the following definitions to extend this analogy:

$$\bar{Q}_{\lambda \bar{E}} \equiv -\bar{E}_z \frac{\partial \bar{\lambda}}{\partial z}$$

$$Q'_{\lambda' \bar{E}} \equiv -\bar{E}_z \frac{\partial \lambda'}{\partial z}$$

$$Q'_{\lambda' E'} \equiv -E'_z \frac{\partial \lambda'}{\partial z}$$

$$Q''_{\lambda' E'} \equiv -E'_j \frac{\partial \lambda'}{\partial x_j}$$

Using these definitions, we can re-write the new terms in each equation as a combination of source and conductive-dissipation terms.

$$\frac{\partial \bar{\rho}}{\partial t} : \quad + \bar{Q}_{\lambda \bar{E}} + \overline{Q''_{\lambda' E'}} - \frac{1}{\epsilon_0} \overline{\lambda' \rho'}^* \quad (6.2a)$$

$$\left. \begin{aligned} \frac{\partial}{\partial t} \overline{\rho' w'} : \quad & + \overline{w' Q'_{\lambda' E}} + \overline{w' Q'_{\lambda' \bar{E}}}^* + \overline{w' Q''_{\lambda' E'}} \\ & - \frac{1}{\epsilon_0} \left(\overline{\lambda' w' \rho}^* + \overline{\lambda' w' \rho'} \right) \end{aligned} \right\} \quad (6.3a)$$

$$\left. \begin{aligned}
 \frac{\partial}{\partial t} \overline{\rho' \theta'} : & \quad + \overline{\theta' Q'_{\lambda E}} + \overline{\theta' Q'_{\lambda E}}^* + \overline{\theta' Q''_{\lambda E}} \\
 & \quad - \frac{1}{\epsilon_0} (\overline{\lambda' \theta' \rho}^* + \overline{\lambda' \theta' \rho'}) \\
 \frac{\partial}{\partial t} \frac{1}{2} \overline{\rho'^2} : & \quad + \overline{\rho' Q'_{\lambda E}} + \overline{\rho' Q'_{\lambda E}}^* + \overline{\rho' Q''_{\lambda E}} \\
 & \quad - \frac{1}{\epsilon_0} (\overline{\lambda' \rho' \rho}^* + \overline{\lambda' \rho'^2})
 \end{aligned} \right\} \begin{array}{l} (6.3a) \\ \text{cont.} \end{array}$$

The starred terms above each contain one of the three correlations, $\overline{w' \lambda'}$, $\overline{\theta' \lambda'}$, or $\overline{\rho' \lambda'}$. We could write equations for these correlations, extending the closure model to include convection of λ as well as of ρ . However, this would still leave ten terms to be approximated by closure models, and it would introduce other troublesome terms as well in the three new equations. Such a procedure could be justified pragmatically only if the starred terms turned out to be by far the most important of the new terms (except for the mean-conduction term, $\overline{Q_{\lambda E}}$, which is bound to be important).

Instead of resorting to the unsatisfying possibility of introducing three new coupled equations, we will attempt to show that all of the new terms except $\overline{Q_{\lambda E}}$ are negligible. In the argument below the magnitude of each new term in each equation is compared, via crude scaling arguments, to the magnitude of a reference term in that equation whose importance is known. If we can show that a new term is much smaller than an important term or of the same order as a negligible term, we are justified in ignoring it.

6.3 Scale Analysis of the New Terms

Scale analysis will be carried out on the $\overline{\rho'\theta'}$ equation in (6.3). Because of the similar form of the three equations, the scaling will apply equally well to the $\overline{\rho'w'}$ and $\frac{1}{2}\overline{\rho'^2}$ equations, except where otherwise indicated. The mean-conduction term $-\frac{\bar{\lambda}}{\epsilon_0}\overline{\rho'\theta'}$ will be chosen as the reference term for convenience of comparison. This term is important in some cases in the interior of the mixed layer, but it is always negligible in the surface layer because of the dominance of the molecular-dissipation term (or the pressure-correlation term, in the $\overline{\rho'w'}$ equation) near the surface. The terms to be scaled are listed in the first column of table 6.1, taken from the second equation in (6.3).

Some definitions are in order. We define a fluctuation magnitude of each variable based on the variance.

$$\rho_*^2 \equiv \overline{\rho'^2}, \quad \theta_*^2 \equiv \overline{\theta'^2}, \quad \lambda_*^2 \equiv \overline{\lambda'^2} \quad (6.4)$$

(Note that ρ_* has been given a somewhat different definition from that used in the uniform- λ closure-model scaling.) Based on the fluctuation magnitudes, we define an integral scale, or mixing length, which has been assumed the same for each variable because the same turbulence is mixing them all.

$$\frac{\rho_*}{\ell} \sim \left| \frac{d\bar{\rho}}{dz} \right|, \quad \frac{\lambda_*}{\ell} \sim \left| \frac{d\bar{\lambda}}{dz} \right| \quad (6.5)$$

This ℓ is the length scale for the large-scale derivatives or the scale of the largest eddies, which have fluctuations of magnitude ρ_* , θ_* , and λ_* . For the small-scale derivatives, we define Taylor microscales:

$$2\left(\frac{\rho_*}{\lambda_\rho}\right)^2 \equiv \overline{\left(\frac{\partial \rho'}{\partial z}\right)^2}, \quad 2\left(\frac{\lambda_*}{\lambda_\lambda}\right)^2 \equiv \overline{\left(\frac{\partial \lambda'}{\partial z}\right)^2} \quad (6.6)$$

λ_ρ and λ_λ are somewhat larger than the molecular-dissipation scales of these quantities because they are based on the large-scale fluctuation magnitudes ρ_* and λ_* , and they are not necessarily identical. However, for the large-Reynolds-number atmospheric turbulence of interest here, it is certain that the microscales are much smaller than the mixing length. [See Tennekes and Lumley (1970).]

$$\lambda_\rho, \lambda_\lambda \ll \ell \quad (6.7)$$

Before proceeding, we need a way to scale E in terms of the scales for ρ . From the averaged form of Gauss's law, we have

$$\frac{d\bar{E}_z}{dz} = \frac{\bar{\rho}}{\epsilon_0} \Rightarrow \bar{E}_z = \bar{E}_{z_0} + \frac{1}{\epsilon_0} \int_0^z \bar{\rho}(z') dz' \quad (6.8)$$

Since we know from the electrode-effect results that the surface electric field tends to be about twice that in the interior, we can make use of (6.8) and the electrode layer depth h to write the following estimate:

$$|\bar{E}_z| \sim \left| \frac{1}{2} \bar{E}_{z_0} \right| \sim \frac{\bar{\rho} h}{\epsilon_0} \quad (6.9)$$

From the perturbation form of Gauss's law, we can write

$$\rho' = \epsilon_0 \frac{\partial E_i'}{\partial x_i} \Rightarrow \overline{\rho'^2} = \epsilon_0^2 \overline{\frac{\partial E_i'}{\partial x_i} \frac{\partial E_j'}{\partial x_j}} \sim 3 \epsilon_0^2 \overline{\left(\frac{\partial E_z'}{\partial z} \right)^2} \quad (6.10)$$

The last approximate equality in (6.10) follows exactly if the individual components of the field are uncorrelated and isotropic. Based on (6.10) and (6.4) we can make the estimate

$$\overline{\left(\frac{\partial E_z'}{\partial z} \right)^2} \sim \frac{\rho_*^2}{3 \epsilon_0} \quad (6.11)$$

Comparison of this perturbation derivative with the mean derivative in (6.8) gives a surprise:

$$\sqrt{\overline{\left(\frac{\partial E_z'}{\partial z} \right)^2}} / \left| \frac{\partial \bar{E}_z}{\partial z} \right| \sim \frac{\rho_*}{\bar{\rho}} \quad (6.12)$$

This is in marked disagreement with the behavior of the derivatives of λ from (6.5) and (6.6):

$$\sqrt{\overline{\left(\frac{\partial \lambda'}{\partial z} \right)^2}} / \left| \frac{\partial \bar{\lambda}}{\partial z} \right| \sim \frac{\ell}{\lambda_\lambda} \gg 1 \quad (6.13)$$

This paradox is resolved by the difference in behavior of λ and E_z as we go to smaller scales. The fluctuations of λ remain of order λ_* (and the fluctuations of ρ , of order ρ_*) down to the small scales where molecular dissipation begins to work on them. Thus, the derivatives of λ' (and of ρ') are largest at the smallest scales, and an estimates of their magnitude is given by $\frac{\lambda_*}{\lambda_\lambda}$ (or $\frac{\rho_*}{\lambda_\rho}$). The fluctuations of E_z , on the other hand, are produced by integrating those of ρ . This integration introduces

a length factor which also becomes small at small scales. Therefore, we expect the derivatives of E'_z to be of the order given by (6.11) at large and at small scales. This means that the fluctuations of E_z themselves must be much smaller at small scales than at large scales, where they are of order

$$\overline{E_z^2} \sim \frac{\rho_*^2 \ell^2}{3 \epsilon_0^2} \quad (6.14)$$

Using (6.4), (6.5), (6.6), (6.9), (6.11), and (6.14), we have made first estimates of the magnitudes of the terms in the first column of table 6.1. These estimates have been entered in the second column with approximate equalities ($\sim$) where the correlation coefficients can be assumed of order unity. Inequalities have been indicated where there is reason to suspect that the correlation coefficients may be small.

The table shows that the fourth and sixth terms, marked by an asterisk, may be large compared to the reference term in the first row, because they contain the large length ratios $\frac{h}{\lambda_\lambda}$ and $\frac{\ell}{\lambda_\lambda}$. However, the inequalities indicated in these rows suggest that we should examine the correlation coefficients involved. Looking first at the term in the last row, $\left| \overline{\theta' E'_j \frac{\partial \lambda'}{\partial x_j}} \right|$, we see that we have used a scale for $E'_j \sim \frac{\rho_* \ell}{\epsilon_0}$ appropriate only to large eddies with a scale for $\frac{\partial \lambda'}{\partial x_j} \sim \frac{\lambda_*}{\lambda_\lambda}$ appropriate only to small eddies. Since these eddies are of widely separated wavelengths, the correlation should be very poor. A better estimate is obtained either by using the E'_j scale appropriate for small eddies or the $\frac{\partial \lambda'}{\partial x_j}$ scale appropriate for large eddies. These two alternatives both yield the same result.

Table 6.1

Magnitude Estimates for the New Terms in the $\overline{\rho'\theta'}$ Equation

<u>Term</u>	<u>First Estimate</u>	<u>Second Estimate</u>
$\left \frac{\bar{\lambda}}{\epsilon_0} \overline{\rho'\theta'} \right $	$\sim \frac{\bar{\lambda} \rho_* \theta_*}{\epsilon_0} = \left(\frac{\bar{\lambda} \rho_* \theta_*}{\epsilon_0} \right)^\dagger$	
$\left \frac{d\bar{\lambda}}{dz} \overline{\theta'E'_z} \right $	$\sim \frac{\lambda_* \theta_* \rho_*}{\epsilon_0} = \frac{\lambda_*}{\bar{\lambda}} \left(\frac{\bar{\lambda} \rho_* \theta_*}{\epsilon_0} \right)^\dagger$	
$\left \frac{d\bar{E}_z}{dz} \overline{\theta'\lambda'} \right $	$\sim \frac{\bar{\rho} \theta_* \lambda_*}{\epsilon_0} = \frac{\bar{\rho} \lambda_*}{\rho_* \bar{\lambda}} \left(\frac{\bar{\lambda} \rho_* \theta_*}{\epsilon_0} \right)$	
$\left \overline{E_z} \overline{\theta' \frac{d\lambda'}{dz}} \right $	$\sim \frac{\bar{\rho} h \theta_* \lambda_*}{\epsilon_0 \lambda \bar{\lambda}} = \frac{h}{\lambda \bar{\lambda}} \frac{\bar{\rho} \lambda_*}{\rho_* \bar{\lambda}} \left(\frac{\bar{\lambda} \rho_* \theta_*}{\epsilon_0} \right)^* \dagger$	$\sim \frac{h}{\ell} \frac{\bar{\rho} \lambda_*}{\rho_* \bar{\lambda}} \left(\frac{\bar{\lambda} \rho_* \theta_*}{\epsilon_0} \right)$
$\left \frac{1}{\epsilon_0} \overline{\theta'\lambda'\rho'} \right $	$\sim \frac{\theta_* \lambda_* \rho_*}{\epsilon_0} = \frac{\lambda_*}{\bar{\lambda}} \left(\frac{\bar{\lambda} \rho_* \theta_*}{\epsilon_0} \right)$	
$\left \overline{\theta'E'_j} \frac{\partial \lambda'}{\partial x_j} \right $	$\sim \frac{\theta_* \rho_* \ell \lambda_*}{\epsilon_0 \lambda \bar{\lambda}} = \frac{\ell}{\lambda \bar{\lambda}} \frac{\lambda_*}{\bar{\lambda}} \left(\frac{\bar{\lambda} \rho_* \theta_*}{\epsilon_0} \right)^* \dagger$	$\sim \frac{\lambda_*}{\bar{\lambda}} \left(\frac{\bar{\lambda} \rho_* \theta_*}{\epsilon_0} \right)^\dagger$

* These two terms appear to be large compared to the standard term in the first line.
† These three terms contain correlation coefficients which may be small. They require closer examination.
‡ This is the reference term to which the others are being compared.

$$\left| \overline{\theta E_j' \frac{\partial \lambda'}{\partial x_j}} \right| \lesssim \frac{\theta_* \rho_* \lambda_*}{\epsilon_0} = \frac{\lambda_*}{\lambda} \left(\frac{\gamma \rho_* \theta_*}{\epsilon_0} \right) \quad \dagger \quad (6.15)$$

Turning our attention to the term in the fourth row of the table, we find the correlation $\overline{\theta' \frac{d\lambda'}{dz}}$. θ' and λ' (or ρ' and λ' , in the corresponding term in the $\frac{1}{2} \rho'^2$ equation) are produced together when air parcels with average properties come into contact with the surface and are heated and purged of ions.* Therefore we expect good correlations between θ' and λ' (or between ρ' and λ'). We can make use of these good correlations to substitute $\frac{\lambda'}{\lambda_*}$ for $\frac{\theta'}{\theta_*}$ (or for $\frac{\rho'}{\rho_*}$) in this term, giving

$$\overline{\theta' \frac{d\lambda'}{dz}} \sim \frac{\theta_*}{\lambda_*} \overline{\lambda' \frac{d\lambda'}{dz}} = \frac{\theta_*}{2\lambda_*} \frac{d}{dz} \overline{\lambda'^2} \sim \frac{\theta_* \lambda_*}{2\ell} \quad (6.16)$$

Thus, the term in the fourth row becomes

$$\left| \overline{E_z \theta' \frac{d\lambda'}{dz}} \right| \sim \frac{\bar{\rho} h \theta_* \lambda_*}{2 \epsilon_0 \ell} = \frac{h}{\ell} \frac{\bar{\rho} \lambda_*}{\rho_* \gamma} \left(\frac{\gamma \rho_* \theta_*}{\epsilon_0} \right) \quad (6.17)$$

† Here we have retained the inequality because we still do not know how well E_j' is correlated with θ' or $\partial \lambda' / \partial x_j$.

* Actually, this is strictly true only for λ' . θ' may also be produced by long wave radiation from the heated surface being absorbed by air parcels in the lowest few meters, but this process is still speculative and will not be considered here. ρ' is also produced by conduction down gradients of conductivity, a process which would lead to a good correlation between ρ' and $d\lambda'/dz$ and therefore would tend to make the term in question large in the $\frac{1}{2} \rho'^2$ equation. However, the relaxation time for conduction has already been shown in Chapter 5 to be large in many cases compared with the ion lifetimes which control the persistence of a feature in the conductivity distribution. And furthermore, such a feature imbedded in a small eddy would frequently turn over and be exposed to an electric field of opposite sign, tending to cancel out the buildup of appreciable ρ' . For these reasons, we will ignore this mechanism of producing $\rho' (d\lambda'/dz)$ correlations.

In the $\overline{\rho'w'}$ equation in (6.3), this term in the fourth row of table 6.1 contains the correlation $\overline{w' \frac{d\lambda'}{dz}}$. Again we may expect w' to be correlated fairly well with λ' , because upward moving parcels near the surface tend to carry surface air deficient in λ , and because the good $\overline{\theta'\lambda'}$ correlation mentioned in the previous paragraph tends to produce $\overline{w'\lambda'}$ higher up by buoyancy. However, correlations between w' and any conservative property tend to disappear at small scales because of the scrambling effect of the pressure fluctuations. For these reasons, we expect the estimate analogous to (6.17) to be an upper limit for the $|\overline{E_z w' \frac{d\lambda'}{dz}}|$ term as well.

The above scaling analysis of the new terms in the $\overline{\rho'w'}$, $\overline{\rho'\theta'}$, and $\frac{1}{2} \overline{\rho'^2}$ equations, summarized in table 6.1, applies primarily to the electrode layer, where there is appreciable variation of $\overline{\lambda}$ with height and, hence, appreciable λ_* . (In the interior of the mixed layer $\frac{d\overline{\lambda}}{dz} \approx 0$ and $\overline{\lambda'^2} \approx 0$, so we can certainly ignore all the new terms there.) This is the same region where the mean conduction terms used for reference are negligible. Therefore, we are justified in ignoring these terms as long as $\frac{\lambda_*}{\overline{\lambda}}$, $\frac{\overline{\rho\lambda_*}}{\rho_* \overline{\lambda}}$, and $\frac{h}{\ell}$ are of order unity or less. These assumptions appear to be justified except very close to the surface, where ℓ becomes small compared to h , suggesting that the term in the fourth row of the table is probably the most important. However, the larger $\frac{h}{\ell}$ becomes, the more negligible the reference term becomes. For these reasons, we have ignored all the new terms added to the $\overline{\rho'w'}$, $\overline{\rho'\theta'}$, and $\frac{1}{2} \overline{\rho'^2}$ equations (6.3) by allowing variable conductivity.

Finally we turn attention to the mean charge equation (6.2), where we hope to show that one of the two new terms, $-\frac{d}{dz} \overline{\lambda' E_z'}$, can also be ignored. We will scale the first three terms on the right-hand side in

the electrode layer. Based on (6.4), (6.5), (6.9), and (6.14), the estimates appear in table 6.2 below.

Table 6.2
Magnitude Estimates of the New Terms
in the $\bar{\rho}$ Equation

<u>Term</u>	<u>Estimate</u>
$\left \frac{\bar{\lambda}}{\epsilon_0} \bar{\rho} \right $	$= \frac{\bar{\lambda} \bar{\rho}}{\epsilon_0} = \left(\frac{\bar{\lambda} \bar{\rho}}{\epsilon_0} \right)^{\dagger}$
$\left \bar{E}_z \frac{d\bar{\lambda}}{dz} \right $	$\sim \frac{\bar{\rho} h \lambda_*}{\epsilon_0 \ell} = \frac{h}{\ell} \frac{\lambda_*}{\bar{\lambda}} \left(\frac{\bar{\lambda} \bar{\rho}}{\epsilon_0} \right) = \left(\frac{h \lambda_* \bar{\rho}}{\epsilon_0 \ell} \right)$
$\left \frac{d}{dz} \frac{\bar{\lambda}' E'_z}{\bar{\lambda}} \right $	$\lesssim \frac{\lambda_* \rho_*}{\epsilon_0} = \frac{\lambda_* \rho_*}{\bar{\lambda} \bar{\rho}} \left(\frac{\bar{\lambda} \bar{\rho}}{\epsilon_0} \right)^{\dagger} = \frac{\ell}{h} \frac{\rho_*}{\bar{\rho}} \left(\frac{h \lambda_* \bar{\rho}}{\epsilon_0 \ell} \right)$

$\dagger$, $\dagger$ See footnotes on table 6.1

In this case, the reference term in the first row of the table is important everywhere. The scaling results indicate that the mean-conduction term appearing in the second row is important near the surface, where $\ell < h$. The term in the third row may be ignored either if $\frac{\lambda_* \rho_*}{\bar{\lambda} \bar{\rho}} \ll 1$ or if $\frac{\ell}{h} \frac{\rho_*}{\bar{\rho}} \ll 1$, in which case one of the other two terms is dominant. In the interior of the electrode layer we expect the first condition to be met. This is analogous to the interior of the surface layer, where $u_* \ll \bar{u}$. And near the surface, where the first condition may break down, the second condition should be met as ℓ becomes small compared to h . Therefore, we feel justified in ignoring the $-\frac{d}{dz} \frac{\bar{\lambda}' E'_z}{\bar{\lambda}}$ term in equation (6.2).

Summarizing, we collect the remaining terms in the approximated versions of (6.2) and (6.3) below:

$$\frac{\partial \bar{\rho}}{\partial t} = -\frac{\bar{\lambda}}{\epsilon_0} \bar{\rho} - \bar{E}_z \frac{\partial \bar{\lambda}}{\partial z} - \frac{\partial}{\partial z} \bar{\rho}' \bar{w}' + \bar{Q} \quad (6.18)$$

$$\left. \begin{aligned} \frac{\partial}{\partial t} \bar{\rho}' \bar{w}' &= -\frac{\bar{\lambda}}{\epsilon_0} \bar{\rho}' \bar{w}' - \bar{w}'^2 \frac{\partial \bar{\rho}}{\partial z} - \frac{\partial}{\partial z} \bar{\rho}' \bar{w}'^2 - \bar{\rho}' \frac{\partial \bar{w}'}{\partial z} + \frac{q}{T_0} \bar{\rho}' \bar{\theta}' \\ \frac{\partial}{\partial t} \bar{\rho}' \bar{\theta}' &= -\frac{\bar{\lambda}}{\epsilon_0} \bar{\rho}' \bar{\theta}' - \bar{\theta}' \bar{w}' \frac{\partial \bar{\rho}}{\partial z} - \bar{\rho}' \bar{w}' \frac{\partial \bar{\theta}}{\partial z} - \frac{\partial}{\partial z} \bar{w}' \bar{\theta}' \bar{\rho}' \\ \frac{\partial}{\partial t} \frac{1}{2} \bar{\rho}'^2 &= -\frac{\bar{\lambda}}{\epsilon_0} \bar{\rho}'^2 - \bar{\rho}' \bar{w}' \frac{\partial \bar{\rho}}{\partial z} - \frac{1}{2} \frac{\partial}{\partial z} \bar{\rho}'^2 \bar{w}' \end{aligned} \right\} \quad (6.19)$$

These equations (6.18) and (6.19) comprise the new version of the second-order-closure model for non-uniform conductivity.

6.4 The Non-Uniform- λ Model

It remains to make the appropriate closure assumptions and to scale equations (6.18) and (6.19) in order to obtain the final, non-dimensional model equations. The closure assumptions made here are identical to those discussed in section 2.3 for the uniform- λ model. The scaling is also identical, with the following minor exceptions. A scale for $\bar{\lambda}$ must be introduced, since it is no longer simply equal to λ_0 . This scale is taken as $[\lambda] \equiv \bar{\lambda}(z \rightarrow \infty)$, the undisturbed asymptotic value of $\bar{\lambda}$. $[\lambda]$ replaces λ_0 in the definition of τ_λ .

$$\bar{\lambda} \equiv [\lambda] \hat{\lambda}, \quad \tau_\lambda \equiv \frac{\epsilon_0}{[\lambda]} \quad (6.20)$$

Current density is scaled by $\rho_* u_*$, as before, and the natural scale for $\bar{E}_z$ is therefore $\frac{\rho_* u_*}{[\lambda]}$.

$$J_m \equiv \rho_m u_x \hat{J}_m, \quad \bar{E}_2 \equiv \frac{\rho_+ u_x}{[\lambda]} \hat{E} \quad (6.21)$$

With these scales, the new equations are remarkably similar to their uniform- λ counterparts in (2.20).

$$\frac{\partial \hat{p}}{\partial t'} = -\lambda' \hat{p} - R \frac{\partial}{\partial \xi} \bar{\rho w} - R \hat{E} \frac{\partial \lambda'}{\partial \xi} + \hat{Q} \quad (6.22)$$

$$\left. \begin{aligned} \frac{\partial}{\partial t'} \bar{\rho w} &= \left(-\lambda' - \frac{d_2}{\tau} + a_t R^2 \frac{\partial}{\partial \xi} \tau \bar{w}^2 \frac{\partial}{\partial \xi} \right) \bar{\rho w} \\ &\quad - R \left(\bar{w}^2 \frac{\partial}{\partial \xi} \hat{p} - \frac{H}{kL} \bar{\rho \theta} \right) \\ \frac{\partial}{\partial t'} \bar{\rho \theta} &= \left(-\lambda' - \frac{d_4}{\tau} + a_t R^2 \frac{\partial}{\partial \xi} \tau \bar{w}^2 \frac{\partial}{\partial \xi} \right) \bar{\rho \theta} \\ &\quad - R \left(\bar{\theta w} \frac{\partial \hat{p}}{\partial \xi} + \frac{\partial \hat{\theta}}{\partial \xi} \bar{\rho w} \right) \\ \frac{\partial}{\partial t'} \frac{1}{2} \bar{\rho^2} &= \left(-\lambda' - \frac{d_4 D}{k\tau} + a_t R^2 \frac{\partial}{\partial \xi} \tau \bar{w}^2 \frac{\partial}{\partial \xi} \right) \bar{\rho^2} \\ &\quad - R \bar{\rho w} \frac{\partial \hat{p}}{\partial \xi} \end{aligned} \right\} \quad (6.23)$$

The only differences are the appearance of λ' in the conductive dissipation term in each of the four equations and the addition of the new

mean-conduction term $-R \hat{E} \frac{d\hat{\lambda}}{d\zeta}$ to the mean charge equation (6.22). The new variable $\hat{E}$ is taken care of by a non-dimensional form of Gauss's law:

$$\hat{\rho} = R \frac{\partial \hat{E}}{\partial \zeta} \quad (6.24)$$

In the steady state, the left-hand sides of equations (6.22) and (6.23) vanish. In this case, (6.22) can be integrated to produce an important integration constant:

$$\hat{J} \equiv \frac{1}{R} \int_{\zeta}^1 \hat{Q}(\zeta') d\zeta' + \overline{\rho w} + \hat{\lambda} \hat{E} \quad (6.25)$$

Here, $\hat{J}$ is the total dimensionless current flowing through the system. This is the forcing which drives the electrode effect. Together, equations (6.22) through (6.25) complete the non-uniform- λ model.

Problems arise when we try to solve this steady-state model, as before, as a coupled pair of equations in $\overline{\rho w}$ and $\overline{\rho \theta}$. It will be recalled that the uniform- λ model was put into this form in Chapter 3 by solving the mean charge equation for $\hat{\rho}$ and differentiating. This expression was then used in the equations for $\overline{\rho w}$ and $\overline{\rho \theta}$ to eliminate $\frac{d\hat{\rho}}{d\zeta}$. This same procedure can be carried out for equations (6.22) and (6.23) above, elimination of $\hat{E}$ being accomplished with the use of (6.25). Aside from making the equations much more involved, however, this procedure causes a major problem. In order to solve (6.22) for $\hat{\rho}$, we must divide by $\hat{\lambda}$, with the eventual result that the pair of equations for $\overline{\rho w}$ and $\overline{\rho \theta}$ has a singular point at the bottom boundary, where $\hat{\lambda} = 0$.

This singular point might perhaps be avoided numerically by a procedure of applying the boundary conditions slightly above ζ_0 . Truncated Taylor series for the lowest few grid points can be written in terms

of the first few derivatives of $\overline{\rho w}$ and $\overline{\rho \theta}$, evaluated at ζ_{oo} . These can then be combined algebraically to obtain boundary conditions applicable at a new lower boundary, one grid point above ζ_{oo} . This procedure is not entirely satisfactory, however, because of the nature of the original lower boundary condition.

Because of the electrode effect, as discussed in the previous chapter, the proper lower boundary conditions are $\overline{\rho \theta}(\zeta_{oo}) = 0$ and $\hat{\rho}(\zeta_{oo}) = 0$. In terms of $\overline{\rho w}$, the lowest order derivative in which the $\hat{\rho}(\zeta_{oo}) = 0$ boundary condition can be used is $\frac{d^2}{d\zeta^2} \overline{\rho w}|_{\zeta_{oo}}$, as can be seen from equations (6.22) and (6.25). This makes the Taylor series procedure outlined here rather inaccurate.

It was found that all these difficulties could be avoided by eliminating $\overline{\rho w}$ in favor of $\hat{E}$ as a dependent variable. This works because it involves no division by $\hat{\lambda}$ and yields equations which are not singular. The lower boundary condition is easily introduced, because of equation (6.24). First, we write expressions for $\overline{\rho w}$ and its derivatives from equations (6.22), (6.24), and (6.25).

$$\left. \begin{aligned} \overline{\rho w} &= -\hat{\lambda} \hat{E} - \frac{1}{R} \int_{\zeta}^{\zeta'} \hat{Q} d\zeta' + \hat{J} \\ \frac{d}{d\zeta} \overline{\rho w} &= -\hat{\lambda} \frac{d\hat{E}}{d\zeta} - \hat{E} \frac{d\hat{\lambda}}{d\zeta} + \frac{\hat{Q}}{R} \\ \frac{d^2}{d\zeta^2} \overline{\rho w} &= -\hat{\lambda} \frac{d^2 \hat{E}}{d\zeta^2} - 2 \frac{d\hat{\lambda}}{d\zeta} \frac{d\hat{E}}{d\zeta} - \hat{E} \frac{d^2 \hat{\lambda}}{d\zeta^2} + \frac{1}{R} \frac{d\hat{Q}}{d\zeta} \end{aligned} \right\} \quad (6.26)$$

From (6.24) we have

$$\frac{d\hat{\rho}}{d\zeta} = R \frac{d^2\hat{E}}{d\zeta^2} \quad (6.27)$$

Substitution of (6.26) and (6.27) into the steady-state forms of the $\overline{\rho w}$ and $\overline{\rho \theta}$ equations in (6.23) yields the following pair of coupled, second-order, ordinary differential equations.

$$\begin{aligned} 0 = & -R^2 \overline{w^2} (a_t \tau \hat{\lambda} + 1) \frac{d^2 \hat{E}}{d\zeta^2} \\ & - a_t R^2 \left[\hat{\lambda} \left(\frac{d}{d\zeta} \tau \overline{w^2} \right) + 2 \tau \overline{w^2} \frac{d\hat{\lambda}}{d\zeta} \right] \frac{d\hat{E}}{d\zeta} \\ & + \left\{ \hat{\lambda} \left(\hat{\lambda} + \frac{d_3}{\tau} \right) - a_t R^2 \left[\left(\frac{d}{d\zeta} \tau \overline{w^2} \right) \frac{d\hat{\lambda}}{d\zeta} + \tau \overline{w^2} \frac{d^2 \hat{\lambda}}{d\zeta^2} \right] \right\} \hat{E} \\ & + R \frac{H}{kL} \overline{\rho \theta} - \left(\hat{\lambda} + \frac{d_3}{\tau} \right) \left(\hat{J} - \frac{1}{R} \int_{\zeta}^1 \hat{Q} d\zeta' \right) \\ & + a_t R \left[\left(\frac{d}{d\zeta} \tau \overline{w^2} \right) \hat{Q} + \tau \overline{w^2} \frac{d\hat{Q}}{d\zeta} \right] \end{aligned} \quad (6.28)$$

$$\begin{aligned} 0 = & a_t R^2 \left[\tau \overline{w^2} \frac{d^2 \overline{\rho \theta}}{d\zeta^2} + \left(\frac{d}{d\zeta} \tau \overline{w^2} \right) \frac{d}{d\zeta} \overline{\rho \theta} \right] \\ & - \left(\hat{\lambda} + \frac{d_4}{\tau} \right) \overline{\rho \theta} \\ & - R^2 \overline{\theta w} \frac{d^2 \hat{E}}{d\zeta^2} + R \frac{d\hat{\theta}}{d\zeta} \hat{\lambda} \hat{E} \\ & - R \frac{d\hat{\theta}}{d\zeta} \left(\hat{J} - \frac{1}{R} \int_{\zeta}^1 \hat{Q} d\zeta' \right) \end{aligned}$$

The boundary conditions on these equations are $\overline{\rho\theta}(\zeta_{oo}) = 0 = \overline{\rho\theta}(1)$, $\frac{d\hat{E}}{d\zeta}|_{\zeta_{oo}} = 0$, and $\hat{E}(1) = \frac{\hat{J}}{\hat{\lambda}(1)}$. This model is to be solved with the aid of the input profiles for $\overline{w^2}$, τ , $\overline{\theta w}$, and $\frac{d\hat{\theta}}{d\zeta}$ derived in section 2.5 and the $\hat{\lambda}$ profile derived in Chapter 5.

6.5 Solutions for the Electrode-Effect Source

The numerical solution to the non-uniform- λ model is carried out in the same manner as that of the uniform- λ model described in section 3.1. The stretching function (3.4) and the finite-difference representations (3.6) are substituted into (6.28) to give the finite-difference equations:

$$\begin{aligned}
 & \left\{ \frac{R^2}{\Delta \zeta_n} \left[\frac{\overline{w_n^2}}{\zeta_n} \left(1 + a_t \tau_n \lambda_n' \right) \left(\frac{1}{2} - \frac{1}{\Delta} \right) - \frac{a_t}{2} \left(\lambda_n' \left(\frac{d}{d\zeta} \tau \overline{w} \right)_n + 2 \tau_n \overline{w_n^2} \left(\frac{d\lambda_n'}{d\zeta} \right) \right) \right] \right\} \hat{E}_{n+1} \\
 & + \left\{ \frac{2R^2 \overline{w_n^2}}{\Delta^2 \zeta_n^2} \left(1 + a_t \tau_n \lambda_n' \right) + \lambda_n' \left(\lambda_n - \frac{d_3}{\tau_n} \right) - a_t R^2 \left[\left(\frac{d\lambda_n'}{d\zeta} \right)_n \left(\frac{d}{d\zeta} \tau \overline{w} \right)_n + \left(\frac{d^2 \lambda_n'}{d\zeta^2} \right)_n \tau_n \overline{w_n^2} \right] \right\} \hat{E}_n \\
 & + \left\{ \frac{R^2}{\Delta \zeta_n} \left[\frac{\overline{w_n^2}}{\zeta_n} \left(1 + a_t \tau_n \lambda_n' \right) \left(-\frac{1}{2} - \frac{1}{\Delta} \right) + \frac{a_t}{2} \left(\lambda_n' \left(\frac{d}{d\zeta} \tau \overline{w} \right)_n + 2 \tau_n \overline{w_n^2} \left(\frac{d\lambda_n'}{d\zeta} \right) \right) \right] \right\} \hat{E}_{n-1} \\
 & + R \frac{H}{kL} \overline{\rho\theta}_n = \left(\lambda_n' + \frac{d_3}{\tau_n} \right) \left(\hat{J} - \frac{1}{R} \int_{\zeta_n}^1 \hat{Q} d\zeta' \right) - a_t R \left[\hat{Q}_n \left(\frac{d}{d\zeta} \tau \overline{w} \right)_n + \left(\frac{d\hat{Q}}{d\zeta} \right)_n \tau_n \overline{w_n^2} \right] \\
 & \left[\frac{R^2 \overline{\theta w_n}}{\Delta \zeta_n^2} \left(\frac{1}{2} - \frac{1}{\Delta} \right) \right] \hat{E}_{n+1} + \left[\frac{2R^2 \overline{\theta w_n}}{\Delta^2 \zeta_n^2} + R \left(\frac{d\hat{\theta}}{d\zeta} \right)_n \lambda_n' \right] \hat{E}_n + \left[\frac{R^2 \overline{\theta w_n}}{\Delta \zeta_n^2} \left(-\frac{1}{2} - \frac{1}{\Delta} \right) \right] \hat{E}_{n-1} \\
 & + \left\{ \frac{a_t R^2}{\Delta \zeta_n} \left[\frac{\tau_n \overline{w_n^2}}{\zeta_n} \left(\frac{1}{2} - \frac{1}{\Delta} \right) + \frac{1}{2} \left(\frac{d}{d\zeta} \tau \overline{w} \right)_n \right] \right\} \overline{\rho\theta}_{n+1} \\
 & + \left[-\frac{2a_t R^2 \tau_n \overline{w_n^2}}{\Delta^2 \zeta_n^2} - \left(\lambda_n' + \frac{d_4}{\tau_n} \right) \right] \overline{\rho\theta}_n \\
 & + \left\{ \frac{a_t R^2}{\Delta \zeta_n} \left[\frac{\tau_n \overline{w_n^2}}{\zeta_n} \left(\frac{1}{2} + \frac{1}{\Delta} \right) - \frac{1}{2} \left(\frac{d}{d\zeta} \tau \overline{w} \right)_n \right] \right\} \overline{\rho\theta}_{n-1} = R \left(\frac{d\hat{\theta}}{d\zeta} \right)_n \left(\hat{J} - \frac{1}{R} \int_{\zeta_n}^1 \hat{Q} d\zeta' \right)
 \end{aligned} \tag{6.29}$$

These two coupled equations are solved as a block-tridiagonal system just as equations (3.8) were. In finite-difference form, the derivative condition on $\hat{E}$ at the lower boundary becomes

$$\hat{E}_1 = \frac{1}{3}(4\hat{E}_2 - \hat{E}_3) \quad (6.30)$$

The other boundary conditions are

$$\overline{\rho\theta}_1 = 0 = \overline{\rho\theta}_N, \quad \hat{E}_N = \frac{\hat{J}}{\hat{\lambda}_N} \quad (6.31)$$

After a solution for $\hat{E}$ and $\overline{\rho\theta}$ is obtained, $\overline{\rho w}$ and $\hat{\rho}$ are calculated diagnostically from equations (6.25) and (6.24) as

$$\left. \begin{aligned} \overline{\rho w}_n &= -\hat{\lambda}_n \hat{E}_n - \frac{1}{R} \int_{\xi_n}^1 \hat{Q} d\xi' + \hat{J} \\ \hat{\rho}_n &= \frac{R}{\Delta \xi_n} (\hat{E}_{n+1} - \hat{E}_{n-1}) \end{aligned} \right\} \quad (6.32)$$

Finally, these solutions are used in the finite-difference form of the $\overline{\rho^2}$ equations from (6.23)

$$\left. \begin{aligned} &\left\{ \frac{a_* R^2}{\Delta \xi_n} \left[\frac{\tau_n \bar{w}_n^2}{\xi_n} \left(\frac{1}{\Delta} - \frac{1}{2} \right) + \frac{1}{2} \left(\frac{d}{d\xi} \tau \bar{w} \right)_n \right] \right\} \overline{\rho^2}_{n+1} + \left(-\frac{2a_* R^2 \tau_n \bar{w}_n^2}{\Delta^2 \xi_n^2} - \hat{\lambda}_n - \frac{D d_*}{K \tau_n} \right) \overline{\rho^2}_n \\ &+ \left\{ \frac{a_* R^2}{\Delta \xi_n} \left[\frac{\tau_n \bar{w}_n^2}{\xi_n} \left(\frac{1}{\Delta} + \frac{1}{2} \right) - \frac{1}{2} \left(\frac{d}{d\xi} \tau \bar{w} \right)_n \right] \right\} \overline{\rho^2}_{n-1} = R \overline{\rho w}_n \left(\frac{d\hat{\rho}}{d\xi} \right)_n \end{aligned} \right\} \quad (6.33)$$

to obtain the solution for $\overline{\rho^2}$. This equation is solved as a tridiagonal system just as (3.9) was. Its boundary conditions are

$$\overline{\rho_1^2} = 0 = \overline{\rho_N^2} \quad (6.34)$$

A survey of parameter space for the non-uniform- λ model will be presented here as in section 3.3. Again we take $R \in [0.005, 2.0]$ and $-\frac{H}{L} \in [0, 1000]$. For the neutral case, we have not constructed another eddy-diffusion model like (3.18). Instead, we have simply used the neutral profiles of $\tau \equiv \frac{f\tau}{R}$ and $\overline{w^2} \equiv \frac{\overline{w'^2}}{u_*^2}$ from (3.17), along with $\frac{d\hat{\theta}}{d\zeta} \equiv 0 \equiv \overline{\theta w}$, in the non-uniform- λ version of the second-order-closure model for $\frac{H}{L} = 0$ and $R \in [0.02, 0.1]$. In every case the profiles of $\hat{\lambda}$ and its derivatives are taken from the approximate formula (5.25) with $AC = 1.175$, unless otherwise indicated, and with C taken from figure 5.9 for the appropriate value of $z_{I0} = \frac{1.06}{R} \zeta_{oo}$.

However, there are two problems here that did not arise in section 3.3. First, we cannot use a generalization of (3.19) to define a dimensionless EMF generated by convection in the model. Such a generalization would look like

$$(emf)' \equiv \int_{\zeta_{oo}}^1 \frac{\overline{\rho w}}{\lambda} d\zeta$$

which is not defined because $\hat{\lambda}(\zeta_{oo}) = 0$. Instead, we must characterize the solutions by the entire potential drop across the layer:

$$\overline{V}(H) - \overline{V}(\zeta_o) = - \int_{\zeta_o}^H \overline{E_z} dz$$

Therefore, we define the dimensionless potential drop as

$$\hat{V} \equiv -\frac{[\lambda]}{\rho_* u_* H} \int_{z_0}^H \overline{E}_2 dz = -\int_{\xi_0}^1 \hat{E} d\xi \quad (6.35)$$

We cannot even compare the value of $\hat{V}$ computed by the model with that which would obtain in the absence of convection,

$$\hat{V}_0 \equiv -\int_{\xi_0}^1 \frac{d\xi}{\lambda}$$

because the latter integral is also undefined. And in any case, $\hat{V}_0$ would be meaningless because the profile of $\hat{\lambda}$ depends on the existence of turbulent mixing. The best we can do is to compare the value of $\hat{V}$ in a given unstable case with the values obtained in less unstable or neutral cases. It will be seen shortly that $\hat{V}$ approaches a minimum value of 1.0 as the strength of the turbulent mixing decreases. More will be said about this problem in section 7.2.

The second problem is that we have introduced the new parameter $\frac{z_0}{H} \equiv \zeta_{00}$ in addition to the two parameters R and $\frac{H}{L}$ discussed in section 3.3. ζ_{00} becomes important in the non-uniform- λ model not only because it is the level at which charge is annihilated, as discussed in section 3.5 with regard to the uniform- λ model, but also because it affects the conductivity profile through equation (5.25) and figure 5.9. Fortunately, it turns out that the competing results of these two effects tend to cancel, so that changes in ζ_{00} over several orders of magnitude for given values of R and $\frac{H}{L}$ cause rather small changes in $\hat{V}$. We will take the range of this new parameter as $\zeta_{00} \in [5 \times 10^{-7}, 1 \times 10^{-3}]$, corresponding to a range of roughness length $z_0 \in [1.5 \text{ mm}, 30 \text{ cm}]$ and the range of mixed layer depth

$H \in [300, 3000]m$ given in section 3.3. The minimum value of $z_o = 1.5$ mm given here is close to the largest lower bound on z_o derived in (5.18c).

For the moment we will ignore the effects of changing ζ_{oo} .

Corresponding to figure 3.7 for the uniform- λ model, figure 6.1 shows the behavior of $\hat{V}$ predicted by the non-uniform- λ model as a function of R and $\frac{H}{L}$, for $\zeta_{oo} = 1.0 \times 10^{-5}$. These two figures show similar overall behavior of the two models, indicating generally increasing effects of the convection current with increasing $|\frac{H}{L}|$ and increasing R . For more about the model behavior, see Appendix 5. Corresponding to figures 3.9 through 3.13, figures 6.2 through 6.6 show profiles of $\overline{\rho w}$ for all the cases displayed in figure 6.1, plus the neutral cases. In figure 6.7, some of these profiles have been replotted as a family of curves for $R = 0.1$ and variable $\frac{H}{L}$, as in figure 3.14. The value of $\hat{V}$ for each curve is tabulated on the figures for easy comparison. For these and all other runs discussed in this section, we have taken $\hat{Q}(\zeta) \equiv 0$ and $\hat{J} = -1.0$. From the scaling for current density given in equation (6.21), we see that the scale factor ρ_* is therefore determined by the value of the total current J flowing through the layer as

$$\rho_* u_* = -J \quad (6.36)$$

In table 6.3, the values of $\hat{V}$, $\hat{E}(\zeta_{oo})$, and $\hat{E}(1)$ predicted in these model runs are tabulated as functions of R and $\frac{H}{L}$. $R[\hat{E}(1) - \hat{E}(\zeta_{oo})]$ gives the vertically integrated, dimensionless charge density in the mixed layer. Since the charge density is positive throughout the layer, $|\hat{E}|$ reaches its maximum value at the bottom boundary. $\hat{E}(1)$ deviates from -1.0 because $\hat{\lambda}$ does not quite reach its asymptotic value of 1.0 at the

Fig.6.1 Contours of constant dimensionless potential drop across the mixed layer in parameter space for the non-uniform- λ model with $\zeta_{\infty} = 1.0 \times 10^{-5}$ and the electrode-effect source. The plotted circles are the model runs on which the contours were based.

Fig.6.2 Profiles of the dimensionless convection-current density predicted by the non-uniform- λ model for $H/L = 0$ and $\zeta_{00} = 1.0 \times 10^{-5}$. The values of the dimensionless potential drop across the mixed layer corresponding to these curves have also been tabulated.

Fig.6.3 Same as figure 6.2, for $H/L = -10$.

Fig.6.4 Same as figure 6.2, for $H/L = -50$.

Fig.6.5 Same as figure 6.2, for $H/L = -200$.

Fig.6.6 Same as figure 6.2, for $H/L = -1000$.

Fig.6.7 Same as figure 6.2, for $R = 0.1$.

Table 6.3
 $\hat{V}$, $\hat{E}(\zeta_{oo})$, and $\hat{E}(1)$ versus R and $\frac{H}{L}$ for the
 Non-Uniform- λ Model with $\zeta_{oo} = 1.0 \times 10^{-5}$

H/L	R=0.005	R=0.02	R=0.1	R=0.5	R=2.0
0	X	1.013 -2.068 -1.000	1.047 -2.047 -1.002	X	X
-10	1.0045 -2.109 -1.000	1.019 -2.118 -1.000	1.122 -2.199 -1.002	1.404 -2.329 -1.009	1.558 -2.265 -1.033
-50	1.0058 -2.147 -1.000	1.041 -2.218 -1.000	1.323 -2.460 -1.002	1.734 -2.740 -1.009	1.744 -2.526 -1.033
-200	1.012 -2.238 -1.000	1.124 -2.439 -1.000	1.678 -2.941 -1.002	2.056 -3.239 -1.009	1.837 -2.672 -1.033
-1000	1.047 -2.510 -1.000	1.407 -3.047 -1.000	2.272 -3.901 -1.002	2.391 -3.840 -1.009	1.808 -2.598 -1.033
In each box, the values are: $\begin{pmatrix} \hat{V} \\ \hat{E}(\zeta_{oo}) \\ \hat{E}(1) \end{pmatrix}$					

upper boundary of the model in the cases of large R . However, this deviation appears to be small enough that it is not worth correcting for. An indication of the behavior of the other model results is given in figures 6.8 through 6.13, where families of profiles of $\overline{\rho\theta}$, $\hat{\rho}$, and $\overline{\rho^2}$ are given for $R = 0.1$ and variable $\frac{H}{L}$ or for $\frac{H}{L} = -50$ and variable R .

The effects of varying ζ_{oo} are illustrated in table 6.4, where $\hat{V}$, $\hat{E}(\zeta_{oo})$, and $\hat{E}(1)$ are tabulated as functions of ζ_{oo} for the five cases $\frac{H}{L} = 0$ and $R = 0.1$, $\frac{H}{L} = -50$ and $R = 0.005$, $\frac{H}{L} = -50$ and $R = 0.1$, $\frac{H}{L} = -50$ and $R = 2.0$, and $\frac{H}{L} = -1000$ and $R = 0.1$. As indicated earlier, the variations of these results within a given case due to changing ζ_{oo} are considerably smaller than the variations between cases. Therefore, it will often be possible to ignore changes in ζ_{oo} . A further illustration of this fact is provided in figure 6.14, where three $\overline{\rho w}$ profiles are given for the case $\frac{H}{L} = -50$ and $R = 0.1$ with $\zeta_{oo} = 5.0 \times 10^{-7}$, 1.0×10^{-5} , and 1.0×10^{-3} . Because of the relative unimportance of variations of ζ_{oo} , it was decided not to make an exhaustive survey of the third dimension of parameter space introduced by this new parameter. Table 6.5 gives the values of z_{I0} , C , and AC used in the formula (5.25) for $\hat{\lambda}$ in the above cases, as functions of R and ζ_{oo} .

In order to investigate the sensitivity of the model results to uncertainties in the profile of $\hat{\lambda}$, we made several runs varying the parameters in equation (5.25). In the case $\frac{H}{L} = -50$ and $R = 0.1$ with $\zeta_{oo} = 1.0 \times 10^{-5}$, table 6.5 indicates that $C = 0.250$ and $AC = 1.175$ give the best fit to the $\hat{\lambda}$ profile calculated numerically in Chapter 5. Holding C fixed, we changed AC to 1.0 and 1.50. We also made runs with $AC = 1.175$ but with C changed to 0.240 and 0.260. These four new $\hat{\lambda}$ profiles are compared with the original one and with the numerical

Fig.6.8 Profiles of dimensionless charge density-temperature covariance predicted by the non-uniform- λ model for $R = 0.1$ and $\zeta_{00} = 1.0 \times 10^{-5}$.

Fig.6.9 Profiles of dimensionless charge density-temperature covariance predicted by the non-uniform- λ model for $H/L = -50$ and $\zeta_{00} = 1.0 \times 10^{-5}$.

Fig.6.10 Same as figure 6.8, for the dimensionless mean charge density.

Fig.6:11 Same as figure 6.9, for the dimensionless mean charge density.

Fig.6.12 Same as figure 6.8, for the dimensionless charge-density variance.

Fig.6.13 Same as figure 6.9, for the dimensionless charge-density variance.

Table 6.4
 $\hat{V}$, $\hat{E}(\zeta_{oo})$, and $\hat{E}(1)$ versus ζ_{oo}
 for five cases of $\frac{H}{L}$ and R

ζ_{oo}	$\frac{H}{L}=0$ R = 0.1	$\frac{H}{L}=-50$ R = 0.001	$\frac{H}{L}=-50$ R = 0.1	$\frac{H}{L}=-50$ R = 2.0	$\frac{H}{L}=-1000$ R = 0.1
5×10^{-7}	1.043 -2.043 -1.001	1.0050 -2.124 -1.000	1.307 -2.384 -1.001	X	2.085 -3.418 -1.001
2×10^{-6}	1.045 -2.045 -1.001	1.0052 -2.123 -1.000	1.314 -2.419 -1.001	1.767 -2.571 -1.032	2.171 -3.632 -1.001
1×10^{-5}	1.047 -2.047 -1.002	1.0058 -2.147 -1.000	1.323 -2.460 -1.002	1.744 -2.526 -1.033	2.272 -3.901 -1.002
5×10^{-5}	1.051 -2.048 -1.002	1.0059 -2.127 -1.000	1.328 -2.484 -1.002	1.703 -2.436 -1.041	2.335 -4.111 -1.002
2×10^{-4}	1.057 -2.063 -1.003	1.0064 -2.113 -1.000	1.329 -2.492 -1.003	1.618 -2.254 -1.048	2.325 -4.186 -1.003
1×10^{-3}	1.061 -2.052 -1.002	X	1.305 -2.407 -1.002	1.475 -2.945 -1.064	1.985 -3.522 -1.002
In each box, the values are: $\begin{pmatrix} \hat{V} \\ \hat{E}(\zeta_{oo}) \\ \hat{E}(1) \end{pmatrix}$

Fig.6.14 Profiles of dimensionless convection-current density predicted by the non-uniform- λ model for $R = 0.1$ and $H/L = -50$, showing the effect of varying ζ_{00} .

Table 6.5
 z_{I0} , C, and AC for the $\hat{\lambda}$ formula
as functions of R and ζ_{00}

ζ_{00}	R = 0.005	R = 0.02	R = 0.1	R = 0.5	R = 2.0
5×10^{-7}	1.06×10^{-4} 0.250 1.175	2.65×10^{-5} 0.213 1.175	5.30×10^{-4} 0.183 1.175	1.06×10^{-6} 0.158 1.175	X
2×10^{-6}	4.24×10^{-4} 0.302 1.175	1.06×10^{-4} 0.250 1.175	2.12×10^{-5} 0.209 1.175	4.24×10^{-6} 0.179 1.175	1.06×10^{-6} 0.158 1.175
1×10^{-5}	2.12×10^{-3} 0.387 1.20	5.30×10^{-4} 0.311 1.175	1.06×10^{-4} 0.250 1.175	2.12×10^{-5} 0.209 1.175	5.30×10^{-6} 0.183 1.175
5×10^{-5}	1.06×10^{-2} 0.537 1.40	2.65×10^{-3} 0.403 1.20	5.30×10^{-4} 0.311 1.175	1.06×10^{-4} 0.250 1.175	2.65×10^{-5} 0.213 1.175
2×10^{-4}	4.24×10^{-2} 0.755 1.60	1.06×10^{-2} 0.537 1.40	2.12×10^{-3} 0.387 1.20	4.24×10^{-4} 0.302 1.175	1.06×10^{-4} 0.250 1.175
1×10^{-3}	X 	5.30×10^{-2} 0.800 1.60	1.06×10^{-2} 0.537 1.40	2.12×10^{-3} 0.387 1.20	5.30×10^{-4} 0.311 1.175
In each box, the values are: $\begin{cases} z_{I0} \\ C(z_{I0}) \\ AC \end{cases}$					

result in figure 6.15. The results for $\hat{V}$, $\hat{E}(\zeta_{oo})$, and $\hat{E}(1)$ in these five runs are compared in table 6.6.

Table 6.6

The Effects of Changing AC and C in the $\hat{\lambda}$ Profile
For the Case $\frac{H}{L} = -50$, $R = 0.1$, and $\zeta_{oo} = 1.0 \times 10^{-5}$

	C=0.250 AC=1.175	C=0.250 AC=1.0	C=0.250 AC=1.50	C=0.240 AC=1.175	C=0.260 AC=1.175
$\hat{V}$	1.323	1.330	1.317	1.344	1.305
$\hat{E}(\zeta_{oo})$	-2.460	-2.473	-2.448	-2.547	-2.381
$\hat{E}(1)$	-1.002	-1.004	-1.000	-1.002	-1.001

Evidently, the results are somewhat more sensitive to changes in C than to changes in AC, but not enough so to be important.

We also bodily stretched and shrank the $\hat{\lambda}$ profile in the vertical by replacing ζ with $B(\zeta - \zeta_{oo}) + \zeta_{oo}$ in (5.25). The results are shown in table 6.7.

Table 6.7

The Effects of Stretching or Shrinking the $\hat{\lambda}$ Profile
In the Case $\frac{H}{L} = -50$, $R = 0.1$, and $\zeta_{oo} = 1.0 \times 10^{-5}$

	B = 1.0	B = 2.0	B = 0.50
$\hat{V}$	1.323	1.242	1.433
$\hat{E}(\zeta_{oo})$	-2.460	-2.098	-2.935
$\hat{E}(1)$	-1.002	-1.001	-1.004

Shrinking the profile vertically reduces $\hat{V}$ and $|\hat{E}(\zeta_{oo})|$ in spite of the fact that it increases $\frac{d\hat{\lambda}}{d\zeta}$ in the surface layer. Stretching the profile, on the other hand, increases the effects of the convection current in this case. However, the fact that $\hat{V}$ decreases with increasing R for the

Fig.6.15 Profiles of dimensionless conductivity used in the sensitivity study of table 6.6, obtained by decreasing and increasing C and AC in turn in (5.25) from the values of $C = 0.250$ and $AC = 1.175$ used in standard profile (heavy solid curve) for $\zeta_{\infty} = 1.0 \times 10^{-5}$. For comparison, the result of a numerical solution of equation (5.17a) is also shown (fine solid curve).

largest values of $\left|\frac{H}{L}\right|$ and R , as shown in figure 6.1, indicates that this trend does not continue indefinitely. Eventually, the region of significant $\frac{d\hat{\lambda}}{d\epsilon}$ must get sufficiently stretched out that most of it is outside the surface layer and loses its effect on the production terms of $\overline{\rho w}$ and $\overline{\rho \theta}$.

Again, as in section 3.1, it was found that $N = 100$ gave sufficient numerical accuracy for our purposes in the non-uniform- λ model. The general behavior of the model in terms of sensitivity to the closure constants and balances of the various terms in the $\overline{\rho w}$ and $\overline{\rho \theta}$ equations is similar to that described in section 3.4 and will not be discussed further here. Having described the general behavior of the model results, we will next turn to the consideration of realistic atmospheric cases.

7 THE CONVECTION OF ELECTRODE-EFFECT CHARGE

7.1 The Diurnal Variations over Land

It has long been known that in fair weather the vertical electric field measured at the surface over land undergoes irregular diurnal variations which are not synchronized between stations in different parts of the world. When many 24-hour time series of the surface electric field at a particular land station are averaged together, a mean diurnal variation emerges which is characteristic of that station and which may change considerably from season to season. Israël (1970) has classified such mean diurnal variations into "single" and "double oscillations" and has presented several examples of the seasonal variations of these curves. In summer a typical double-oscillation type curve has a broad minimum in the early afternoon by local time, which may disappear in the winter, regardless of the longitude of the station. In general, the surface electric field, and even the total downward current density, at land stations seem to be controlled by local meteorological and aerological conditions and to vary with local time. There is little correlation between these curves at stations of widely differing longitude.

Over the ocean, however, the local conditions are not so strongly influenced by time of day. Although individual 24-hour time series of the surface electric field at sea are still quite irregular, the mean diurnal variations tend to be of the single-oscillation type with little seasonal variation. From statistical analysis of the data taken on the world cruises of the research vessel Carnegie in the beginning of this century, Mauchly (1923) discovered that these mean diurnal variations were synchronized over the oceans of the whole earth. His average diurnal variation of the surface electric field over the World Ocean is given in figure 7.1

in terms of percent of the mean as a function of Greenwich Mean Time (GMT). This has come to be known as the "Carnegie variation". Also shown in the figure is the mean diurnal variation computed from measurements taken aboard the Maud on the Arctic Ocean. The agreement is excellent.

For a long time, the cause of the fair-weather electric field of the atmosphere had been the central problem of atmospheric electricity. The discovery of the Carnegie variation provided an important key. It was soon found that the number of thunderstorms active on the earth at any particular time underwent a diurnal variation very similar to that of figure 7.1 [see Pierce (1958)]. As pointed out in the Introduction, it is now generally accepted that thunderstorms provide the main supply current for the global circuit. Since the resistance of the return path from the ionosphere to the earth's surface is fairly constant, variations in this supply current should cause similar variations of the ionospheric potential simultaneously over the whole earth. These variations in V_{∞} are generally believed to follow the Carnegie curve and to be responsible for Mauchley's results. Evidence supporting this theory has recently come from Clark (1958), Anderson (1969), and Markson (1974).

The question that concerns us here is why the presumed Carnegie variation of V_{∞} does not show up in the surface electric field or total downward current measured over land. The main reasons for this were both pointed out in the Introduction. It is generally accepted that diurnal variations of columnar resistance due to local meteorological and aerological influences can have a considerable effect on these variables. It is not generally accepted but it is the contention of this thesis that convection currents in the mixed layer act as a local generator which may have an even larger effect. It seems that the noise produced by these two disturbing

Fig.7.1 The mean diurnal variations of the electric field observed aboard the Carnegie on the World Ocean and the Maud on the Arctic Ocean, from Chalmers (1967).

influences obscures the Carnegie variation in measurements made over land. If this is true, an understanding of the convection currents should make it possible to explain the diurnal variations observed over land in terms of the Carnegie variation and the two disturbing influences.

7.2 Convection of Electrode-Effect Charge over Land

In section 4.5 we learned that the turbulent transport of Blanchard-effect charge over the oceans does not generally produce significant convection currents in the mixed layer. Therefore, any important electrical effects of convection in common occurrence and with significant horizontal extent must be due to the electrode effect and will probably be confined to the continents. Having developed a simplification of the electrode effect in Chapter 5 and incorporated it into our second-order-closure model for the convection current in Chapter 6, we are now in a position to investigate these electrical effects over land.

In order to use our non-uniform- λ model to predict the effects of convection of electrode-effect charge at a particular land station, we must have values of the three dimensionless parameters $\frac{H}{L}$, R , and ζ_{oo} . This requires a knowledge of the diurnal variations of the surface fluxes of heat and momentum and of the thickness and conductivity of the mixed layer, as well as the value of z_o . In addition, we must have some way of specifying the charge density scale ρ_* , either through the total current density J or through some surface value such as $\overline{E_z}(z_o)$. Unfortunately, there seem to be no data on these diurnal variations measured simultaneously at the same location. Drs. Paul Frenzen and G. Dale Hess of Argonne National Laboratory (personal communication) have an experiment under way at the time of writing to measure time series of the surface fluxes and mixed-layer thickness during the morning hours. These data should be

most interesting when they become available.

In the absence of suitable observational data, we have made use of the detailed numerical simulation of the growth of an unstable planetary boundary layer by Deardorff (1974a, 1974b), based on Day 33 of the Wangara data of Clarke, et al. (1971), taken at Hay in south-eastern Australia. Deardorff used a roughness length of $z_0 = 1.0$ cm, in the middle of the range of 1.2 mm to 2.0 cm found by Clarke. The numerical "data" on $\overline{w'u'}|_0$, $\overline{w'\theta'_v}|_0$, H , and $\bar{u}(10m)$ are tabulated in table 7.1, from Deardorff (1974a, 1974b). The values marked with an asterisk were interpolated by eye from his graphs and tables.

Table 7.1
Numerical "Data" from Deardorff's Model

Hour (Local)	$\overline{w'u'} _0 \left(\frac{m^2}{sec^2} \right)$	$\overline{w'\theta'_v} _0 \left(\frac{deg\ m}{sec} \right)$	H(m)	$\bar{u} \left(\frac{m}{sec} \right)$
09	-2.0×10^{-2} *	0.100 *	180	—
10	-2.2×10^{-2}	0.140	300	2.4
11	-1.3×10^{-2}	0.155	850	2.2
12	-2.0×10^{-2}	0.185	1050	2.0
13	-2.0×10^{-2} *	0.200 *	1200	—
14	-2.0×10^{-2}	0.185	1300	2.2
15	-3.0×10^{-2} *	0.155 *	1400	—
16	-3.3×10^{-2}	0.145	1500	3.0
17	-2.0×10^{-2} *	0.050 *	1500	—
18	-2.0×10^{-2} *	0.000	1400	—

Taking the rough values $\frac{T_0}{kg} = 87.5 \frac{deg\ sec^2}{m}$, $[\lambda] = 1.0 \times 10^{-14} \frac{mho}{m}$ (typical of clean continental air), and $z_0 = 1.0 \times 10^{-2}$ m, we have calculated L , $\frac{H}{L}$, R , and ζ_{00} in table 7.2 from the "data" in table 7.1. At hour 18, we used $H \equiv \frac{u_*}{f}$, with $f = 1.0 \times 10^{-4} sec^{-1}$.

Table 7.2

Dimensionless parameters $\frac{H}{L}$, R , and ζ_{oo}
and Potential Drop $\hat{V}$ for Deardorff's Simulation

Hour	L(m)	$\frac{H}{L}$	R	ζ_{oo}	$\hat{V}$
09	2.47	73	0.69	5.6×10^{-5}	1.84
10	2.04	150	0.44	3.3×10^{-5}	1.97
11	0.84	1000	0.12	1.2×10^{-5}	2.30
12	1.34	780	0.12	9.5×10^{-6}	2.22
13	1.24	970	0.10	8.3×10^{-6}	2.23
14	1.34	970	0.096	7.7×10^{-6}	2.22
15	2.93	480	0.11	7.1×10^{-6}	2.03
16	3.62	410	0.11	6.7×10^{-6}	1.88
17	4.95	310	0.082	6.6×10^{-6}	1.73
18	∞	0	0.088	7.1×10^{-6}	1.047

We observe that the instability parameter $\frac{H}{L}$ has much greater magnitude here than in the tropical trade-wind case of section 4.4. This is because of the considerably stronger surface heat flux found over land ($0.20 \frac{\text{deg m}}{\text{sec}}$ as compared to $0.012 \frac{\text{deg m}}{\text{sec}}$ in the tropical case) and the light surface winds leading to a lower friction velocity ($\sim 0.14 \frac{\text{m}}{\text{sec}}$ as compared to $0.24 \frac{\text{m}}{\text{sec}}$ in the tropical case), as well as the greater thickness of the mixed layer. $u_* = 0.14 \frac{\text{m}}{\text{sec}}$, corresponding to $\overline{w'u'} = -2.0 \times 10^{-2} \frac{\text{m}^2}{\text{sec}^2}$, is just within the limit of applicability of the strong-turbulence electrode-effect model given in (5.18a), and ζ_{oo} does not deviate far from 1.0×10^{-5} . Therefore, we have used figure 6.1 to obtain the values of $\hat{V}$ in the last column of table 7.2. For the neutral case, we have used the value corresponding to $R = 0.1$ and $\zeta_{oo} = 1.0 \times 10^{-5}$ from table 6.3.

In order to interpret these results in dimensional form, we must find some way to specify the scale factor ρ_* . In the absence of any

electrical measurements, consider the following simple model. Divide the atmosphere from the ground to the ionosphere into two layers with the mixed layer on the bottom and a quiescent layer sustaining no convection current on the top. In a steady state the total current density J is uniform throughout. In the upper layer the conduction current density is $J_\lambda = J$, and the partial columnar resistance will be called R_u , so that the dimensional potential drop from the ionosphere to the mixed layer inversion is $-JR_u$. In the lower layer the dimensional potential drop is $-\frac{JH}{[\lambda]} \hat{V}$, because $\rho_* u_* = -J$ from (6.36). These two potential drops must add up to the ionospheric potential:

$$V_\infty = -J \left(R_u + \frac{H}{[\lambda]} \hat{V} \right) \quad (7.1)$$

The remaining question is, how does R_u depend on H ? We are interested in the variations of J caused by the convection current, not those caused by variations in columnar resistance à la Sagalyn and Faucher (1956). Therefore, we will assume that the total columnar resistance R_∞ is fixed. This brings up a problem discussed in section 6.5; the partial columnar resistance in the lower layer $R_H \equiv \int_{z_0}^H \frac{dz}{\lambda}$ is undefined because $\bar{\lambda}(z_0) = 0$. Consideration of the real meaning of the columnar resistance will help us to sidestep this problem.

The columnar-resistance concept was introduced into atmospheric electricity in order to relate the conduction current density J_λ to the ionospheric potential V_∞ by a simple equivalent-circuit model. Defined in this way, R_∞ is simply the proportionality factor between J_λ and V_∞ . In a steady-state with no convection current, J_λ is independent of height, and we have $R_\infty = \lim_{z \rightarrow \infty} \int_0^z \frac{dz'}{\lambda}$, as in the Introduction. However,

there is a problem with this definition at the bottom boundary. Even in the absence of turbulent mixing, the current density there is not carried by conduction but by molecular diffusion of ions and by the coulomb attraction of their image charges in the conducting surface. Therefore, it does not matter that the conductivity vanishes at the surface; current still flows. For this reason, the singularity of the integrand in the above formula for R_∞ should be ignored. The formula for the columnar resistance should be changed to $R_\infty = \lim_{z \rightarrow \infty} \int_{z_d}^z \frac{dz'}{\lambda}$, where z_d is some small length related to the thickness of the molecular-diffusion sublayer.

Accordingly, we define the "effective columnar resistance" of the lower layer in our simple model as $R_\ell \equiv \frac{H}{[\lambda]}$ and require that $R_\infty = R_u + R_\ell$. This amounts to neglecting the deviations of $\bar{\lambda}$ from $[\lambda]$ in the electrode layer and demanding the dimensionless potential drop across the boundary layer in the absence of convection to be $\hat{V}_0 \equiv 1.0$. This is not an unreasonable approximation, since $\bar{\lambda}$ is only significantly smaller than $[\lambda]$ over a layer shallow compared to H , and since $\hat{V}$ in figure 6.1 seems to be approaching 1.0 for small R and $|\frac{H}{L}|$. Armed with this approximation, we can re-write equation (7.1) as

$$J = -V_\infty \left[R_\infty + \frac{H}{[\lambda]} (\hat{V} - 1) \right]^{-1} \quad (7.2)$$

Since the total current density in the absence of convection would be $J_p \equiv -V_\infty/R_\infty$, according to our approximation, we find the "current reduction factor" due to convection of the electrode-effect charge as

$$\frac{J}{J_p} = \frac{R_\infty}{R_\infty + \frac{H}{[\lambda]} (\hat{V} - 1)} \quad (7.3)$$

The more $\hat{V}$ exceeds 1.0, and the larger $\frac{H}{[\lambda]}$ becomes relative to R_∞ , the smaller this factor gets. Using $R_\infty = 2 \times 10^{17}$ ohm m² [see Kraakevik and Clark (1958)], $[\lambda] = 1.0 \times 10^{-14} \frac{\text{mho}}{\text{m}}$, $\hat{V}$ from table 7.2, and H from table 7.1, we find the time series of R_u/R_∞ and J/J_p given in table 7.3.

Table 7.3

The Fraction of R_∞ above the Mixed Layer and the "Current Reduction Factor" for Deardorff's Simulation

Hour	R_u/R_∞	J/J_p
09	0.91	0.93
10	0.85	0.87
11	0.58	0.64
12	0.48	0.61
13	0.40	0.58
14	0.35	0.56
15	0.30	0.58
16	0.25	0.60
17	0.25	0.65
18	0.30	0.97

The hypothetical time series worked out in table 7.3 is an extreme case in that it is based on fixed R_∞ and $[\lambda]$. As such, it somewhat exaggerates the variations of R_u relative to R_ℓ . From equation (7.1) we can see that this causes some exaggeration of the variations of J relative to J_p . Nevertheless, this is the best that can be done without more data. In figure 7.2 we have plotted $\overline{w'\theta'}_v|_0$, H , and the Carnegie variation against local time for Hay, Australia, where the data of Clarke, et al. (1971) were taken. Assuming that the Carnegie curve represents the time variation of the ionospheric potential over Hay, we have multiplied it by the current reduction factor of table 7.3 to obtain a modified relative

Fig.7.2 The surface virtual-potential-temperature flux and the depth of the mixed layer during the Dear-dorff simulation (lower frame), and the Carnegie variation expected in the total downward current density, with a modification due to convection-current effects (upper frame), are plotted against local time for Hay, Australia.

variation of the total downward current due to both the Carnegie variation of V_{∞} and the convection-current effects. In doing this, we have assumed that $J/J_p \approx 1.0$ for the hours before 0800 and after 1900 local time. If variations of R_{∞} were included, we would expect the deviations from the Carnegie variation to be even larger, since R_{∞} should be well correlated with H .

Table 7.3 and figure 7.2 show that the convection current can be at least as important as variations in V_{∞} in producing variations in J . The convection of electrode-effect charge in the mixed layer acts as a local generator which, in this case, reduces the total downward current by as much as 44%. It is not hard to see how different characteristic diurnal variations of J or of $\bar{E}_z(z_0)$, such as the single and double oscillations discussed by Israël (1970), could be produced at land stations of different longitudes by changing the phase difference between local time and GMT. We believe that the combined effects of the Carnegie variation, the columnar resistance, and the convection current will explain most of the diurnal variations observed at land stations. In the future, the important meteorological parameters $\overline{w'\theta'}|_0$, $\overline{w'u'}|_0$, and H must be observed when such electrical measurements are made.

7.3 The Electrode Effect over the Ocean

In sections 4.4 and 4.5 the uniform- λ model was used to calculate the convection of Blanchard-effect charge over the ocean in two cases. We would like to calculate the convection of electrode-effect charge in these same cases, in order to determine which is the more important source. Unfortunately, the tropical trade-wind case of section 4.4 is outside the range of validity of our approximate electrode-effect model because of the extremely small values of z_0 over the ocean in light to moderate winds.

In that case we had $u_* = 0.24 \frac{\text{m}}{\text{sec}}$, corresponding to a lower limit of validity of $z_0 \geq 1.1 \text{ mm}$ by (5.18c); but figure 4.5 gave us $z_0 = 0.060 \text{ mm}$ — more than an order of magnitude too small. This means that the molecular-diffusion term needs to be included in equation (5.17a) to obtain the $\hat{\lambda}$ profile.

Moving on to the strong-wind neutral case of section 4.5, we find from (5.18c) that $u_* = 0.71 \frac{\text{m}}{\text{sec}}$ corresponds to a lower limit of validity of $z_0 \geq 0.64 \text{ mm}$. Since figure 4.5 gave us $z_0 = 1.2 \text{ mm}$ in that case, our approximate $\hat{\lambda}$ profile should be valid. Therefore, the non-uniform- λ model was run with the neutral profiles of τ and $\overline{w^2}$, as in section 6.5, for the case $R = 0.017$ and $\zeta_{oo} = 6.4 \times 10^{-8}$, with $C = 0.178$ from figure 5.9 and $AC = 1.175$. With these parameters, the model predicts $\hat{V} = 1.0092$, $\hat{E}(\zeta_{oo}) = -2.065$, and $\hat{E}(1) = -1.000$. This time $\overline{\rho w}$ is down to 10% of its maximum upward value by $\zeta = 0.027$, corresponding to $z = 510 \text{ m}$ (as compared to 350 m in section 4.5) if $H = 1.9 \times 10^4 \text{ m}$.

In order to interpret these results, we again have recourse to equation (7.3), using $R_{\infty} = 1.0 \times 10^{17} \text{ ohm m}^2$, $[\lambda] = 2.0 \times 10^{-14} \frac{\text{mho}}{\text{m}}$, and $H = 1.9 \times 10^4 \text{ m}$. Notice that these values imply $R_{\ell} \equiv \frac{H}{[\lambda]} = 9.5 \times 10^{17} \text{ ohm m}^2 \gg R_{\infty}$, an obvious contradiction. This is because we cannot expect $\bar{\lambda}$ to remain uniform throughout a 19 km deep layer. However, these numbers are the appropriate ones for equation (7.3) because significant convection currents extend only to heights of 500 m or so, over which depth $\bar{\lambda}$ can be assumed to be roughly constant. Using them, we obtain a current reduction factor of only $J/J_p = 0.92$.

We can put this in the language of Chapter 4 by re-writing equation (7.2) as

$$V_{\infty} = -J R_{\infty} - J R_{\ell} (\hat{V} - 1) \quad (7.4)$$

Visualizing (7.4) in terms of a simple Thevenin equivalent circuit in which the ionospheric potential V_{∞} is applied across a series resistor and EMF, we identify the resistor as R_{∞} and the effective EMF generated by the convection current as $(EMF) = -J R_{\ell} (\hat{V} - 1)$. A typical value of the total current density over the ocean is $J = -2.7 \times 10^{-12} \frac{\text{amp}}{\text{m}^2}$, according to Kraakevik (1961). With $R_{\ell} = 9.5 \times 10^{17} \text{ ohm m}^2$ and $\hat{V} = 1.0092$, this gives $(EMF) \approx 24 \text{ kV}$. Comparing this with the 3.7 kV calculated in section 4.5, we conclude that the electrode effect is much more important than the Blanchard effect as a charge source for convection currents in this case. We expect this conclusion to hold true for the tropical trade-wind case as well, because the strength of the Blanchard-effect source falls off rapidly with decreasing wind speed, as shown in figure 4.2.

As in section 4.5, we regard the strong-wind neutral case worked out above as a reasonable upper limit for fair-weather convection currents over the ocean, excluding unusual situations of limited horizontal extent. Therefore, we feel justified in concluding that the Blanchard effect is generally unimportant compared to the electrode effect as a charge source for convection currents over the open sea.

Equation (7.4) can be re-written in yet another way:

$$(EMF) = V_{\infty} \left(1 - \frac{J}{J_p} \right) \quad (7.5)$$

From the solution for the Deardorff simulation, discussed in the previous section, we find a maximum of $(EMF) \approx 130 \text{ kV}$, using $V_{\infty} = 300 \text{ kV}$ and

$J/J_p = 0.56$ for hour 14 from table 7.3. Therefore, it appears that the effects of convection currents over the ocean are generally small compared to those which can occur over land, in spite of the larger results obtained here than in Chapter 4.

7.4 The Blanchard Effect Recapitulated

The reader will have noticed, after the discussion of the electrode effect in Chapter 5, that the Blanchard effect was not treated completely in Chapter 4. There we used the uniform- λ model with the $\hat{\rho}(\zeta_{oo}) = 0$ lower boundary conditions given in (3.15) and (3.13). However, an examination of equations (5.1) and (5.2) and the boundary conditions used on them in section 5.2 shows that the ion-annihilation boundary conditions discussed in section 3.5 are not strictly consistent with the uniform- λ model. $\hat{\rho}(\zeta_{oo})$ vanishes because the individual ion densities n_1 and n_2 vanish at the bottom boundary, implying that $\hat{\lambda}(\zeta_{oo})$ should vanish also. Therefore, the results for the Blanchard-effect source given in Chapter 4 are underestimates; they do not include the electrode effect due to the conduction current produced by convection of Blanchard-effect charge. We have chosen to classify this phenomenon under the electrode effect because it represents an augmentation of the convection current due to non-uniform conductivity.

The non-uniform- λ model can be used to compute the convection of Blanchard-effect charge by setting $\hat{J} = 0$ and using a profile of $\hat{Q}(\zeta)$ like that of figure 4.3. This has been done for the strong-wind neutral case of the previous section with the same profile of $\hat{Q}(\zeta)$ as in section 4.4 and with $\zeta_q = 1.6 \times 10^{-6}$. The results are $\hat{V} = 0.0014$ and $\hat{E}(\zeta_{oo}) = -0.194$. Dimensionally, this corresponds to $(EMF) = 6.0$ kV, almost twice the result obtained from the uniform- λ model in section 4.5.

Evidently, the electrode effect augments the convection current due to the Blanchard-effect charge. However, the result is still considerably smaller than that for the pure electrode effect obtained in the previous section. Since the model equations (6.28) are linear, these two results may be superimposed to obtain the combined effects of convection of electrode-effect and Blanchard-effect charge in the strong-wind neutral case: $\hat{V} = 1.0106$, $\hat{E}(\zeta_{oo}) = -2.259$, $\hat{E}(1) = -1.000$, and $(EMF) \approx 30$ kV . This last is still small compared to the maximum result of $(EMF) \approx 130$ kV for the Deardorff simulation. Therefore, the conclusion at the end of the last section still stands: the strongest manifestations of convection currents in the fair-weather planetary boundary layer will be found over land.

8 CONCLUSIONS

8.1 Summary of the Results

The main conclusion to be drawn from the research reported in this thesis is that the convection of space charge in the fair-weather, unstable, planetary boundary layer acts as a local generator which can have a large effect on the electrical structure of the atmosphere. In the example of strong instability over land worked out in section 7.1, convection of electrode-effect charge was found to generate an EMF of about 130 kV, reducing the total downward current density locally by some 44%. The convection-current effects generally seem to be much stronger over sun-heated land than over the ocean, in spite of the additional charge source available in the latter case. It also appears that the electrode effect is much more important than the Blanchard effect as a charge source for convection currents on the open sea. In the extreme example worked out in section 4.5, 7.2, and 7.3 for strong wind ($\bar{u}(10 \text{ m}) = 16 \frac{\text{m}}{\text{sec}}$) and neutral stability over the ocean, the convection current was found to generate about 30 kV, of which only some 4 kV could be ascribed purely to the Blanchard effect. The effects of convection were even weaker in the tropical trade-wind case worked out in section 4.4, thought to be an example of fairly strong instability for the open ocean. A key to obtaining further dimensional results from the dimensionless model solutions presented in this thesis has been provided in Appendix 5.

The implications of these results for the local behavior of atmospheric electric variables are considerable, at least over land. We have shown that a large part of the diurnal variations of these variables can be due to convection currents, as in figure 7.2. Therefore, future studies must take into account the meteorological parameters of the mixed

layer as well as the electrical variables. In particular, the surface fluxes of heat, moisture, and momentum and the height of the mixed-layer inversion must all be recorded in order to make time series of the total downward current or the conduction current at some level meaningful. In addition, of course, some estimate is needed of the variations in total columnar resistance. With such measurements the models developed in the previous chapters should lead to the explanation of some aspects of the electrical behavior of the boundary layer hitherto not fully understood.

8.2 Future Work: Experimental

The first priority for future work in this area is observational. As mentioned in section 7.1, simultaneous meteorological and atmospheric-electrical data adequate to verify the results of the model calculations in this thesis are non-existent. Further refinements of the models are pointless until there are some reliable data with which to compare them.

The most dramatic prediction of the convection-current model developed here is the large EMF generated by mixed-layer convection of electrode-effect charge under strongly unstable conditions over sun-heated land. Therefore, the most critical test of the model would be provided by an observational program designed to measure the diurnal variations over semi-arid land in the summertime. As an absolute minimum, such a program should measure time series of the surface fluxes of heat, moisture (unless the contribution of the moisture flux to the buoyancy flux could be shown to be negligible at the experimental site), and momentum, the height of the mixed-layer inversion, and the profiles of vertical electric field, conductivity, and temperature through the boundary layer. These measurements would make it possible to specify R and H/L and to calculate model predictions for the electrical structure of the boundary layer, which

could then be compared with the observations. ζ_{oo} could be estimated from estimates of z_o at the site.

The measured profiles of $\overline{E_z}$ and $\overline{\lambda}$ would make possible the calculation of profiles of conduction-current density J_λ . From the values of J_λ above the mixed layer, the diurnal variations of the total current density J could be estimated and the convection-current density could be deduced as $J_m = J - J_\lambda$. The profiles of J_m and $\overline{\lambda}$ could be used to estimate the diurnal variations of the EMF generated by convection. For comparison purposes, the diurnal variations of R_∞ could be estimated from the profiles of $\overline{\lambda}$ to determine whether the columnar resistance, the convection current, or the ionospheric potential, was the main cause of variations in J . The profiles of $\overline{E_z}$ could also be used with Gauss's law to calculate profiles of $\overline{\rho}$ for a further comparison with the model.

The main differences between the observations proposed here and those performed by Kraakevik and Clark, as described in section 1.1, are as follows. 1) We recommend horizontal averaging of E_z , λ , and T at each level in the profiles. Kraakevik and Clark were apparently unable to do this, at least with E_z , because of their measuring apparatus. 2) We have included the meteorological measurements relevant to the structure of the mixed layer. These measurements are essential for comparison with our model predictions.

There are several other measurements that would be desirable along with the minimum requirements listed above, if the necessary apparatus were available. The total current density J should be monitored directly at the surface with a collector plate. This might provide a better time series for J than deduction from the $\overline{E_z}$ and $\overline{\lambda}$ profiles above the inversion. The value of J is important for separating the current density

into conduction and convection current. It would also be desirable to measure the charge density ρ directly during the profiles so that $\bar{\rho}$, $\overline{\rho'^2}$, and $\overline{\rho'\theta'}$ could be calculated for comparison with the model predictions. If in addition the vertical air velocity w could be obtained during the taking of profiles, it would allow direct calculation of the convection current $\overline{\rho'w'}$. This has never been done before and would avoid all the assumptions inherent in $J_m(z) = J = J_\lambda(z)$. Some of these auxiliary measurements may require the development of new instrumentation for the direct measurement of charge-density fluctuations.

There is another important prediction in this thesis which needs to be tested. The results on convection currents over land described herein are all predicated on the special case of the electrode effect developed in Chapter 5. The central conclusion there was that the mean profile of conductivity becomes independent of the conduction process when the turbulent mixing becomes strong enough relative to the electric field. The resulting $\bar{\lambda}$ profile decreases from $[\lambda]$ in the interior of the boundary layer to zero at the surface over a vertical distance of the order of L' . This independence of the $\bar{\lambda}$ profile from the conduction process is equivalent to the positive and negative ion densities remaining nearly equal right down to the surface, or to the charge density being small compared to the conductivity throughout the electrode layer.

This prediction can be tested by a much simpler measurement program conducted entirely from the ground. In order to verify that the positive and negative ion densities are nearly equal and large compared to the net charge density, it should be sufficient to measure the two polar conductivities and the charge density close to the ground. To check the profile of $\bar{\lambda}$ predicted by equation (5.17a), it will be necessary to monitor

the conductivity at three or more heights above the ground along with the surface stress. These measurements should be made at a horizontally-homogeneous site under conditions of moderate winds. An estimate of z_0 will also be needed.

As far as other experimental work is concerned, there are two studies that might be proposed as a result of the work in this thesis. One is a measurement program to investigate the convection current over the ocean and to compare it with the predictions of our model. The other is a study of bubbling on the open sea to obtain a better estimate of the charge source profile $\bar{Q}(z)$ due to the Blanchard effect and its dependence on wind speed. However, both of these are of secondary importance to the two experiments proposed above because of the lesser importance of convection currents over the ocean.

8.3 Future Work: Theoretical

There is an important modification to the models developed here that might be contemplated in the future. This is the explicit inclusion of the inversion capping the mixed layer, with its strong stability and associated gradient of conductivity. This inversion should act as another charge source, giving rise to a downward convection current at the top of the mixed layer. Although it is not anticipated that this will have a large impact on the EMF generated by convection, it will certainly have some observable effects near the inversion that might be worthy of study.

More attention should also be given to the closure assumptions made in sections 2.3 and 2.4 and to the wholesale disposal of terms in the non-uniform- λ model discussed in section 6.3. However, it is our feeling that such refinements of the present models are not justified until the

experimental work suggested above has allowed their strengths and weaknesses to be identified.

There is one area of theoretical research in which further work is much needed. This is the area of "physical" models of convective transport, discussed briefly in section 2.1. If work like that of Foster (1965a, 1965b) could be extended and applied to the atmospheric boundary layer, it would be possible to model the vertical transports of heat, moisture, momentum, and charge with more confidence. This would lead to a much better understanding of these important exchange processes.

REFERENCES

- Anderson, R.V. (1969), "Universal diurnal variation in air-earth current density", J. Geophys. Res., 74, 1697-1700.
- Augstein, E., H. Schmidt, and F. Ostapoff (1974), "The vertical structure of the atmospheric planetary boundary layer in undisturbed trade winds over the Atlantic ocean", Boundary-Layer Met., 6, 129-150.
- Baines, W.D., and J.S. Turner (1969), "Turbulent buoyant convection from a source in a confined region", J. Fluid Mechanics, 37, 51-80.
- Ball, F.K. (1960), "Control of inversion height by surface heating", Q. J. Royal Met. Soc., 86, 483-494.
- Blackadar, A.K. (1962), "The vertical distribution of wind and turbulent exchange in a neutral atmosphere", J. Geophys Res., 67, 3095-3102.
- Blanchard, D.C. (1963), "The electrification of the atmosphere by particles from bubbles in the sea", in Progress in Oceanography, Vol.I, M. Sears (ed.), New York: The Macmillan Company, 383 pp.
- Blanchard, D.C. (1966), "Positive space charge from the sea", J. Atmos. Sci., 23, 507-515.
- Blanchard, D.C., and A.H. Woodcock (1957), "Bubble formation and modification in the sea and its meteorological significance", Tellus, 9, 145-158.
- Businger, J.A. (1973), "Turbulent transfer in the atmospheric surface layer", in Workshop on Micrometeorology, D. Haugen (ed.), Boston: American Meteorological Society, 392 pp.
- Callahan, R.C., S.C. Coroniti, A.J. Parziale, and R. Patten (1951), "Electrical conductivity of air in the troposphere", J. Geophys. Res., 56, 545-551.
- Chalmers, J.A. (1967), Atmospheric Electricity, Second Edition, Oxford: Pergamon Press, 515 pp.
- Clark, J.F. (1957), "Airborne measurement of atmospheric potential gradient", J. Geophys. Res., 62, 617-628.
- Clark, J.F. (1958), "The fair-weather atmospheric electric potential and its gradient", in Recent Advances in Atmospheric Electricity, L.G. Smith (ed.), New York: Pergamon Press, 631 pp.
- Clarke, R.H. (1970), "Observational studies in the atmospheric boundary layer", Q. J. Royal Met. Soc., 96, 91-114.
- Clarke, R.H., A.J. Dyer, R.R. Brook, D.G. Reid, and A.J. Troup (1971), "The Wangara experiment: boundary layer data", Technical Paper No.19, Commonwealth Scientific and Industrial Research Organization, Australia.

- Deardorff, J.W. (1972), "Numerical investigation of neutral and unstable planetary boundary layers", J. Atmos. Sci., 29, 91-115.
- Deardorff, J.W. (1973), "An explanation of anomalously large Reynolds stresses within the convective planetary boundary layer", J. Atmos. Sci., 30, 1070-1076.
- Deardorff, J.W. (1974a), "Three-dimensional numerical study of the height and mean structure of a heated planetary boundary layer", Boundary-Layer Met., 7, 81-106.
- Deardorff, J.W. (1974b), "Three-dimensional numerical study of turbulence in an entraining mixed layer", Boundary-Layer Met., 7, 199-226.
- Donaldson, C. duP. (1973), "Construction of a dynamic model of the production and the dispersal of atmospheric pollutants", in Workshop on Micrometeorology, D.A. Haugen (ed.), Boston: American Meteorological Society, 392 pp.
- Foster, T.D. (1965a), "Stability of a homogeneous fluid cooled uniformly from above", Physics of Fluids, 8, 1249-1257.
- Foster, T.D. (1965b), "Onset of convection in a layer of fluid cooled from above", Physics of Fluids, 8, 1770-1774.
- Gathman, S.R., and W.A. Hoppel (1970), "Surf electrification", J. Geophys. Res., 75, 4525-4529.
- Gathman, S.G., and E.M. Trent (1968), "Space charge over the open ocean", J. Atmos. Sci., 25, 1075-1079.
- Holland, J.A., and E.M. Rasmussen (1973), "Measurements of the atmospheric mass, energy, and momentum budgets over a 500-kilometer square of tropical ocean", Monthly Weather Rev., 101, 44-55.
- Hoppel, W.A. (1969), "Electrode effect: comparison of theory and measurement", in Planetary Electrodynamics, Vol. II, S.C. Coroniti and J. Hughes (eds.), New York: Gordon and Breach, 503 pp.
- Hoppel, W.A., and S.G. Gathman (1971), "Determination of eddy diffusion coefficients from atmospheric electrical measurements", J. Geophys. Res., 76, 1467-1477.
- Howard, L.N. (1964), "Convection at high Rayleigh number", Proc. 11th Int. Congress on Applied Mechanics (H. Görtler, ed.), Berlin: Springer-Verlag, 1109-1115.
- Israël, H. (1970), Atmospheric Electricity, Vols. I and II, authorized and revised translation from the second German edition, Jerusalem: Israel Program for Scientific Translations, 317 pp. and 796 pp.
- Izumi, Y. (1971), "Kansas 1968 field program data report", AFCRL-72-0041, Environmental Research Paper No. 379, Air Force Cambridge Research Lab., 79 pp.

- Kohl, D.A. (1968), "Aircraft measurements of fair weather electric field noise within the Austausch layer", J. Geophys. Res., 73, 1913-1919.
- Kraakevik, J.H. (1958a), "The airborne measurement of atmospheric conductivity", J. Geophys. Res., 63, 161-169.
- Kraakevik, J.H. (1958b), "Electrical conduction and convection currents in the troposphere", in Recent Advances in Atmospheric Electricity, L.G. Smith (ed.), New York: Pergamon Press, 631 pp.
- Kraakevik, J.H. (1961), "Measurements of current density in the fair weather atmosphere", J. Geophys. Res., 66, 3735-3748.
- Kraakevik, J.H., and J.F. Clark (1958), "Airborne measurements of atmospheric electricity", Trans. American Geophys. U., 39, 827-834.
- Latham, D.J., and H.W. Poor (1972), "A time-dependent model of the electrode effect", J. Geophys. Res., 77, 2669-2676.
- LeMone, M.A. (1973), "The structure and dynamics of horizontal roll vortices in the planetary boundary layer", J. Atmos. Sci., 30, 1077-1091.
- Lenschow, D.J. (1970), "Airplane measurements of planetary boundary layer structure", J. Applied Met., 9, 874-884.
- Lenschow, D.J. (1974), "Model of the height variation of the turbulent kinetic energy budget in an unstable planetary boundary layer", J. Atmos. Sci., 31, 465-474.
- Lettau, H. (1950), "A reexamination of the 'Leipzig wind profile' considering relations between wind and turbulence in the friction layer", Tellus, 2, 125-129.
- Lumley, J.L., and B. Khajeh-Nouri (1974), "Computational modeling of turbulent transport", in Advances in Geophysics, Vol.18A, H.E. Landsberg and J. von Mieggh (eds.), New York: Academic Press, 462 pp.
- Markson, R. (1974), "Investigation of the temporal variation of the ionospheric potential and electrical structure through the atmospheric exchange layer", Ph.D. Thesis, Department of Atmospheric Sciences, New York State University at Albany, 134 pp.
- Mauchly, S.J. (1923), "Diurnal variation of the potential gradient of atmospheric electricity", Terrestrial Magnetism and Atmos. Electricity, 28, 61-81.
- Monahan, E.C. (1971), "Oceanic Whitecaps", J. Physical Oceanog., 1, 139-144.
- Pierce, E.T. (1958), "Some topics in atmospheric electricity", in Recent Advances in Atmospheric Electricity, L.G. Smith (ed.), New York: Pergamon Press, 631 pp.
- Priestley, C.H.B. (1959), Turbulent Transfer in the Lower Atmosphere, Chicago: University of Chicago Press, 130 pp.

- Richtmyer, R.D., and K.W. Morton (1967), Difference Methods for Initial-Value Problems, second edition, New York: Interscience Publishers, 405 pp.
- Roll, H.U. (1965), Physics of the Marine Atmosphere, New York: Academic Press, 426 pp.
- Sagalyn, R.C., and G.F. Faucher (1954), "Aircraft investigation of the large ion content and conductivity of the atmosphere and their relation to meteorological factors", J. Atmos. and Terrestrial Physics, 5, 253-272.
- Sagalyn, R.C., and G.F. Faucher (1956), "Space and time variations of charged nuclei and electrical conductivity of the atmosphere", Q. J. Royal Met. Soc., 82, 428-443.
- Sarachik, E.S. (1974), "The tropical mixed layer and cumulus parameterization", J. Atmos. Sci., 31, 2225-2230.
- Sheppard, P.A., D.T. Tribble, and J.R. Garratt (1972), "Studies of turbulence in the surface layer over water (Lough Neagh). Part I. Instrumentation, programme, profiles", Q. J. Royal Met. Soc., 98, 627-641.
- Sparrow, E.M., R.B. Husar, and R.J. Goldstein (1970), "Observations and other characteristics of thermals", J. Fluid Mech., 41, 793-800.
- Tennekes, H. (1970), "Free convection in the turbulent Ekman layer of the atmosphere", J. Atmos. Sci., 27, 1027-1034.
- Tennekes, H. (1973), "A model for the dynamics of the inversion above a convective boundary layer", J. Atmos. Sci., 30, 558-567.
- Tennekes, H. (1975), "Reply", J. Atmos. Sci., 32, 992-995.
- Tennekes, H., and J.L. Lumley (1970), A First Course in Turbulence, Cambridge: The MIT Press, 300 pp.
- Vonnegut, B., C.B. Moore, R.G. Semonin, J.W. Bullock, D.W. Staggs, and W.E. Bradley (1962), "Effect of atmospheric space charge on initial electrification of cumulus clouds", J. Geophys Res., 67, 3909-3922.
- Warner, J., and J.W. Telford (1967), "Convection below cloud base", J. Atmos. Sci., 24, 374-382.
- Willis, G.E., and J.W. Deardorff (1974), "A laboratory model of the unstable planetary boundary layer", J. Atmos. Sci., 31, 1297-1307.
- Woodcock, A.H. (1953), "Salt nuclei in marine air as a function of altitude and wind force", J. Meteor., 10, 362-371.
- Wynngaard, J.C. (1973), "On surface-layer turbulence", in Workshop on Micrometeorology, D. Haugen (ed.), Boston: American Meteorological Society, 392 pp.

- Wynngaard, J.C., S.P.S. Arya, and O.R. Coté (1974), "Some aspects of the structure of convective planetary boundary layers", J. Atmos. Sci., 31, 747-754.
- Wynngaard, J.C., and O.R. Coté (1971), "The budgets of turbulent kinetic energy and temperature variance in the atmospheric surface layer", J. Atmos. Sci., 28, 190-201.
- Wynngaard, J.C. and O.R. Coté (1974), "The evolution of a convective planetary boundary layer — a higher-order-closure model study", Boundary-Layer Met., 7, 289-308.
- Wynngaard, J.C., and O.R. Coté, and Y. Izumi (1971), "Local free convection, similarity, and the budgets of shear stress and heat flux", J. Atmos. Sci., 28, 1171-1182.
- Wynngaard, J.C., O.R. Coté, and K.S. Rao (1974), "Modeling the atmospheric boundary layer", in Advances in Geophysics, Vol.18A, H.E. Landsberg and J. van Mieghm (eds.), New York: Academic Press, 462 pp.
- Wynngaard, J.C., and Y. Izumi (1973), "Deviations from Monin-Obukhov similarity in the unstable surface layer" (unpublished).
- Zilitinkevich, S.S. (1975), "Comments on 'A Model for the Dynamics of the Inversion Above a Convective Boundary Layer'", J. Atmos. Sci., 32, 991-992.

APPENDIX 1 The Electrical Equations

The electric equations for the lower atmosphere, where geomagnetic effects are negligible and where we ignore electromagnetic effects by considering only situations where the relevant velocity scales are small compared to the speed of light, are summarized below from Latham and Poor (1972).

$$\left. \begin{aligned} \frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \underline{U} + n_i k_i \underline{E} - D_i \nabla n_i) &= P_i \\ P_i &\equiv q_i + \sum_{i \neq j} \beta_{ij} n_i n_j \end{aligned} \right\} \quad (\text{A1.1})$$

These are the equations for the concentration of the various types of ions, where n_i , k_i , D_i , and q_i are respectively the volume concentration, mobility, diffusion coefficient, and volume source strength of the i^{th} ion species. $\underline{U}$ is the air velocity, P_i is called the production term, and β_{ij} is a combination coefficient between the i^{th} and j^{th} species. For simplicity, and usually without grave errors, these equations are immediately simplified to consider only small ions of both signs and one size of Aitken nuclei, uncharged and in the form of large ions of both signs: n_1 , n_2 , N_0 , N_1 , and N_2 , where subscript 1 denotes positive and 2 is negative. Furthermore, it is assumed that $\nabla \cdot \underline{U} \equiv 0$ and that D_i is small enough for diffusion to be neglected compared to the other transport processes. Since $D_i = \frac{\bar{K} k_i T}{e}$, deduced by Einstein in 1905, where $\bar{K} = 1.38 \times 10^{-23} \frac{\text{joule}}{\text{deg}}$ and $e = 1.602 \times 10^{-19}$ coul, for atmospheric temperatures $D \sim 2.5 \times 10^{-2} k$. Thus $|D \nabla n| \ll |n k E|$ if $2.5 \times 10^{-2} \frac{\text{joules}}{\text{coul}} \ll |E| \ell$, where ℓ is the length scale of ∇n ;

for normal fair-weather fields of ~ 100 V/m, ℓ must be greater than about 2.5 mm. With the final restriction that all ions are only singly charged, we rewrite (A1.1) as

$$\left. \begin{aligned}
 \frac{Dn_1}{Dt} + \nabla \cdot k_1 n_1 \tilde{E} &= q_1 - \alpha n_1 n_2 - \gamma_{10} n_1 N_0 - \gamma_{12} n_1 N_2 \\
 \frac{Dn_2}{Dt} + \nabla \cdot k_2 n_2 \tilde{E} &= q_2 - \alpha n_1 n_2 - \gamma_{20} n_2 N_0 - \gamma_{21} n_2 N_1 \\
 \frac{DN_1}{Dt} + \nabla \cdot K_1 N_1 \tilde{E} &= Q_1 + \gamma_{10} n_1 N_0 - \gamma_{21} n_2 N_1 - \nu N_1 N_2 - \gamma_{12} N_1 N_2 \\
 \frac{DN_2}{Dt} + \nabla \cdot K_2 N_2 \tilde{E} &= Q_2 + \gamma_{20} n_2 N_0 - \gamma_{12} n_1 N_2 - \nu N_1 N_2 - \gamma_{12} N_1 N_2 \\
 \frac{DN_0}{Dt} &= Q_0 + \gamma_{12} n_1 N_2 + \gamma_{21} n_2 N_1 + 2\nu N_1 N_2 - \gamma_{10} n_1 N_0 - \gamma_{20} n_2 N_0 \\
 &\quad - \gamma_{00} N_0^2
 \end{aligned} \right\} \quad (A1.2)$$

The γ terms here represent the coagulation pointed out by Sagalyn and Faucher (1956) as the important loss process for nuclei.

In addition to the above ion conservation equations, we also need the electrical equations

$$-\nabla V = \tilde{E} \quad , \quad \nabla \cdot \tilde{E} = \frac{\rho}{\epsilon_0} \quad (A1.3)$$

where $\rho \equiv e(n_1 - n_2 + N_1 - N_2)$. The missing familiar equations of charge conservation and Ohm's Law are contained in (A1.2). They become

$$\frac{D\rho}{Dt} = -\nabla \cdot \tilde{J}_\lambda + Q \quad , \quad \tilde{J}_\lambda \equiv \lambda \tilde{E} \quad (A1.4)$$

where $\lambda \equiv e(k_1 n_1 - k_2 n_2 + K_1 N_1 - K_2 N_2)^*$ and $Q \equiv e(q_1 - q_2 + Q_1 - Q_2)$.

As for the magnitudes of the various coefficients, we obtain the following approximate values from Israëli (1970) for near the surface of the earth:

$$q \equiv q_1 = q_2 \sim 10^7 \text{ m}^{-3} \text{ sec}^{-1}$$

$$k_1 = 1.37 \times 10^{-4} \frac{\text{m}^2}{\text{volt sec}}$$

$$k_2 = -1.87 \times 10^{-4} \frac{\text{m}^2}{\text{volt sec}}$$

$$\alpha \approx 1.6 \times 10^{-12} \frac{\text{m}^3}{\text{sec}}$$

$$\eta_1 \equiv \eta_{12} \approx \eta_{21} \approx (2.2, 14.9) \times 10^{-12} \frac{\text{m}^3}{\text{sec}}$$

$$\eta_0 \equiv \eta_{10} \approx \eta_{20} \approx (0.39, 12.3) \times 10^{-12} \frac{\text{m}^3}{\text{sec}}$$

$$\nu \approx (0.6, 17.0) \times 10^{-15} \frac{\text{m}^3}{\text{sec}}$$

$$K_1 \approx K_2 \approx (10^{-4}, 10^{-2}) \cdot k_{1,2}$$

Sagalyn and Faucher (1956) give a value for coagulation as

$$\gamma \sim 1.6 \times 10^{-15} \frac{\text{m}^3}{\text{sec}}$$

Because γ and ν are so small, because $Z \equiv N_0 + N_1 + N_2$ is usually regarded as a known uniform constant in solutions for the ion balance, and because $|K_{1,2}| \ll |k_{1,2}|$, the sources, coagulation, recombination, and conduction current of nuclei are usually ignored; and the most common form of equations (A1.2) is

* Note that the mobilities of negative ions are negative because they move in the opposite direction from the field.

$$\left. \begin{aligned}
 \frac{Dn_1}{Dt} + \nabla \cdot k_1 n_1 \tilde{E} &= q - \alpha n_1 n_2 - \eta_0 n_1 N_0 - \eta_1 n_1 N_2 \\
 \frac{Dn_2}{Dt} + \nabla \cdot k_2 n_2 \tilde{E} &= q - \alpha n_1 n_2 - \eta_0 n_2 N_0 - \eta_1 n_2 N_1 \\
 \frac{DN_1}{Dt} &= \eta_0 n_1 N_0 - \eta_1 n_2 N_1 \\
 \frac{DN_2}{Dt} &= \eta_0 n_2 N_0 - \eta_1 n_1 N_2 \\
 \frac{DZ}{Dt} &= 0
 \end{aligned} \right\} \quad (A1.5)$$

This requires the modification that $\lambda \equiv e(k_1 n_1 - k_2 n_2)$ in equations (A1.4)

Several simple analytical solutions to this set of equations have been found. One of interest here is obtained by Sagalyn and Faucher (1956) to relate conductivity to charged nucleus concentration. Making the approximations $n_1 = n_2$, $N_1 = N_2$, and $\frac{Dn}{Dt} = \frac{DN}{Dt} = 0$, and neglecting conduction, they find

$$n = \frac{-\beta N \pm (\beta^2 N^2 + 4\alpha q)^{1/2}}{2\alpha}, \quad \beta \equiv \eta_1 + \eta_0 \frac{N_0}{N} \quad (A1.6)$$

A graph of the experimental points plotted with equation (A1.6) is presented in figure A1.1, illustrating the decrease of n (hence λ) with increasing N (hence Z).

For the purposes of this thesis, the most important of the equations developed here are (A1.4). If we are interested only in the distribution, convection, and conduction of net charge, we need not consider the individual charge species n_1 , n_2 , N_1 , and N_2 except in so far as

Fig.A1.1 Small ion density n_+ versus charged nucleus concentration N_+ at 5,000 ft elevation, from Sagalyn and Faucher (1956).

they determine λ . And in most of the atmosphere, the charge density is so small relative to the absolute ion densities that these latter may be considered independent of the former. In the language of (A1.5),

$$\frac{\rho}{e} = n_1 - n_2 \ll \frac{\lambda}{ek} \approx n_1 + n_2$$

where we have taken $k \equiv k_1 \approx -k_2$. Therefore, in many cases we can use (A1.4) in solving for the charge distribution, avoiding the cumbersome complexity of sets of non-linear equations like (A1.2) or (A1.5).

In order to use (A1.4) alone, we must know the distribution of conductivity externally at all times. The main processes tending to modify the conductivity in a particular location are advection and conduction. Advection of air of different conductivity into the location is a separate problem, independent of the charge distribution, and will be ignored here. However, the charge distribution does control the electric field, which in turn determines the conduction current. Therefore, we must find out under what conditions conduction does not influence the conductivity. Obviously, the classical electrode effect is an example of the conduction process grossly modifying the conductivity near a conducting boundary; it cannot be treated by (A1.4) alone. However, we will postpone consideration of this effect until Chapter 5, and here consider only conduction in the interior of the gas volume.

An equation for λ can be formed by adding the first two equations in (A1.5). Making the further simplifications $k_1 \equiv k \equiv -k_2$, $\nabla k \equiv 0$, and $N_1 \equiv 0 \equiv N_2$, we have

$$\frac{D\lambda}{Dt} + k^2 \nabla \cdot \rho \underline{E} = 2ekq - \frac{k\alpha}{2e} \left[\left(\frac{\lambda}{k} \right)^2 - \rho^2 \right] \quad (\text{A1.7})$$

Ignoring advection in the steady state amounts to setting $\frac{D}{Dt} \lambda \equiv 0$. We re-write the equation so that the terms involving ρ appear on the right-hand side (RHS):

$$2ekq - \frac{\alpha}{2ek} \lambda^2 = k^2 \nabla \cdot \rho \underline{E} - \frac{k\alpha}{2e} \rho^2 \quad (\text{A1.8})$$

These RHS terms may be thought of as the influences of charge density on the conduction and recombination processes, both tending to modify the steady-state solution for λ which would otherwise exist. This steady-state solution is found by setting the RHS equal to zero:

$$\lambda_s \equiv 2ek \sqrt{\frac{q}{\alpha}} \quad (\text{A1.9})$$

In order that λ be independent of ρ and the conduction process, we find two conditions from (A1.8)

$$\left. \begin{aligned} |k \nabla \cdot \rho \underline{E}| &\ll 2eq \Rightarrow \frac{\rho_s E_s}{l} \ll \frac{2eq}{k} \\ \frac{\alpha}{2e} \rho^2 &\ll 2eq \Rightarrow \rho^2 \ll \frac{4e^2 q}{\alpha} = \frac{\lambda_s^2}{k^2} \end{aligned} \right\} \quad (\text{A1.10})$$

Here, we have written the magnitudes of ρ and $\underline{E}$ as ρ_s and E_s and the length scale for the derivative as l , and we have used (A1.9). From figure 1.1 in the Introduction and from the list of typical atmospheric values above, we have $E_s \lesssim 200 \frac{\text{V}}{\text{m}}$, $\rho_s \lesssim 2.0 \times 10^7 \frac{\text{e}}{\text{m}^3}$, $\lambda_s \gtrsim 1.0 \times 10^{-14} \frac{\text{mho}}{\text{m}}$, $q \approx 10^7 \frac{1}{\text{m}^3 \text{sec}}$, and $k \approx 1.5 \times 10^{-4} \frac{\text{m}^2}{\text{V sec}}$. Using these values in (A1.10) gives

$$\left. \begin{aligned} l &\gg \frac{k \rho_s E_s}{2 e q} \lesssim 3.0 \times 10^{-2} \text{ m} \\ 4.0 \times 10^{14} \text{ m}^{-6} &\geq \frac{\rho^2}{e^2} \ll \frac{\lambda_s^2}{k^2 e^2} \gtrsim 1.7 \times 10^{17} \text{ m}^{-6} \end{aligned} \right\} \quad (\text{A1.11})$$

Evidently, these conditions are very well satisfied in the interior of the atmosphere. In fact, the first equation in (A1.11) indicates that equations (A1.4) can be used down within a meter or so of the ground before the classical electrode effect becomes important in modifying the conductivity profile. More is said about this in Chapter 5.

APPENDIX 2 The Simple Convective- Column Model

Based on the speculations of Kraakevik (1958b), a very simple model of the convection current is developed here for the purposes of estimating its magnitude and its behavior in various meteorological conditions. This model supposes that convection extends through a layer from the surface to height H in the form of vertical columns of rising air of uniform diameter and vertical velocity v_c , occupying a fractional horizontal area A_c . It is supposed that all the air enters these convective columns near the ground and leaves them at the upper boundary of the layer, so that there is no entrainment into the columns along the way. Therefore, there must be a uniform subsidence velocity in the environment $v_s = \frac{-v_c A_c}{1-A_c}$ covering a fractional area $1-A_c$. As long as $A_c \ll 1$, this model can be expected to transport properties in a manner similar to a real unstable atmospheric boundary layer. Of course, the idealizations involved, particularly that of zero entrainment, mean that this model will exaggerate the charge transport and, therefore, will probably represent an upper bound on the convection current.

The charge transport by this "convective-column model" is based on equations (A1.4). With the further idealizations of uniform $\lambda \equiv \lambda_0$, $Q \equiv 0$, and a steady state, and with Gauss's law from (A1.3), these equations reduce to

$$\vec{u} \cdot \nabla \rho = w \frac{d\rho}{dz} = -\frac{\lambda_0}{\epsilon_0} \rho \quad (\text{A2.1})$$

From geometry and kinematics, we can write

$$\left. \begin{aligned}
 \bar{\rho} &= A_c \rho_c + (1-A_c) \rho_s \\
 \rho'_c &\equiv \rho_c - \bar{\rho} = (1-A_c)(\rho_c - \rho_s) \\
 \rho'_s &\equiv \rho_s - \bar{\rho} = -A_c(\rho_c - \rho_s) \\
 \overline{\rho'w'} &= v_c A_c (\rho_c - \rho_s) \\
 \bar{\rho}'^2 &= A_c \rho_c'^2 + (1-A_c) \rho_s'^2
 \end{aligned} \right\} \quad (A2.2)$$

Here, the subscripts c and s refer to the rising column and the subsiding environment, respectively, the overbars represent horizontal averages, and the primes represent deviations from those averages. Thus, $\overline{\rho'w'}$ is the convection current. Application of (A2.1) to the columns and the environment individually gives profile forms of ρ in each region.

$$\left. \begin{aligned}
 \rho_c &= \rho_{c0} e^{-z/H_c}, \quad H_c \equiv \frac{v_c \epsilon_0}{\lambda_0} \\
 \rho_s &= \rho_{s0} e^{+z/H_s}, \quad H_s \equiv \frac{H_c A_c}{1-A_c}
 \end{aligned} \right\} \quad (A2.3)$$

H_c and H_s defined here are the respective scale heights for the charge density variations. Since at $z = H$ the outflow from the rising columns becomes the subsiding environment, we can solve for ρ_{s0} :

$$\rho_{s0} = \rho_{c0} e^{-\frac{H}{A_c H_c}} \quad (A2.4)$$

Insertion of (A2.3) and (A2.4) into (A2.2) then gives

$$\left. \begin{aligned} \bar{\rho} &= \rho_{co} e^{-\frac{z}{H_c}} \left[A_c + (1-A_c) e^{-\frac{H}{A_c H_c} \left(1 - \frac{z}{H}\right)} \right] \\ \rho'_c &= \rho_{co} (1-A_c) e^{-\frac{z}{H_c}} \left[1 - e^{-\frac{H}{A_c H_c} \left(1 - \frac{z}{H}\right)} \right] \\ \rho'_s &= -\rho_{co} A_c e^{-\frac{z}{H_c}} \left[1 - e^{-\frac{H}{A_c H_c} \left(1 - \frac{z}{H}\right)} \right] \\ \overline{\rho'w'} &= \rho_{co} v_c A_c e^{-\frac{z}{H_c}} \left[1 - e^{-\frac{H}{A_c H_c} \left(1 - \frac{z}{H}\right)} \right] \\ \overline{\rho'^2} &= A_c (1-A_c) \rho_{co}^2 e^{-\frac{2z}{H_c}} \left[1 - e^{-\frac{H}{A_c H_c} \left(1 - \frac{z}{H}\right)} \right]^2 \end{aligned} \right\} \quad (A2.5)$$

Finally, defining the convection current at the lower boundary as

$\overline{\rho'w'}(z=0) \equiv \overline{\rho'w'}|_0$, we can solve for ρ_{co} .

$$\rho_{co} = \frac{\overline{\rho'w'}|_0}{v_c A_c} \left(1 - e^{-\frac{H}{A_c H_c}} \right)^{-1} \quad (A2.6)$$

Before going any further with (A2.5), we notice that there are two unknown constants in this analysis: v_c and A_c . In order to compare the results of this simple model to those of the second-order-closure model developed in this thesis or to calculate convection currents in particular meteorological conditions, we must find a way of specifying these constants. To do this, we will return to the geometry and kinematics of the convective-column model to derive some more profiles of turbulence statistics.

The steady-state assumption already made in connection with (A2.1)

requires that the shape of the mean potential-temperature profile* remain unchanged as the boundary layer is heated and convection proceeds. This means that the potential temperature flux must be linear with height and the surface flux must be constant with time.

$$\overline{\theta'w'}(z) = \overline{\theta'w'}|_0 \left(1 - \frac{z}{H}\right) \quad (\text{A2.7})$$

This situation can only obtain if the temperature of the air entering the bottom of the rising columns, and hence, the temperature everywhere in the layer, increases linearly with time. If $\theta_0(t)$ is the temperature of this air entering the columns, then

$$\left. \begin{aligned} \theta_c(z,t) &= \theta_0(t) - \frac{\overline{\theta'w'}|_0}{v_c} \frac{z}{H} \\ \theta_s(z,t) &= \theta_0(t) - \frac{\overline{\theta'w'}|_0}{v_c} \left[\frac{1}{A_c} + \frac{z}{H} \left(1 - \frac{1}{A_c}\right) \right] \\ \overline{\theta}(z,t) &= \theta_0(t) - \frac{\overline{\theta'w'}|_0}{v_c} \left[\frac{1}{A_c} - 1 + \frac{z}{H} \left(2 - \frac{1}{A_c}\right) \right] \\ \frac{\partial \overline{\theta}}{\partial z} &= \frac{\overline{\theta'w'}|_0}{H v_c} \left(\frac{1}{A_c} - 2 \right) \\ \theta'_c &= \frac{\overline{\theta'w'}|_0}{v_c} \frac{1}{A_c} (1 - A_c) \left(1 - \frac{z}{H}\right) \\ \theta'_s &= - \frac{\overline{\theta'w'}|_0}{v_c} \left(1 - \frac{z}{H}\right) \\ \rho' \theta' &= \frac{\rho_0 \overline{\theta'w'}|_0}{v_c} (1 - A_c) \left(1 - \frac{z}{H}\right) e^{-\frac{z}{H}} \left[1 - e^{-\frac{H}{A_c H} \left(1 - \frac{z}{H}\right)} \right] \end{aligned} \right\} \quad (\text{A2.8})$$

* Note, that, if there is significant moisture flux in the boundary layer, the potential temperature θ should be replaced throughout by the virtual potential temperature θ_v in profiles and fluxes.

Note that, by continuity, the air in the subsiding environment must become the air entering the rising columns when the former reaches the ground. However, this change of identity is imagined to occur within the surface layer, where properties of the air such as temperature and charge density are changed by intimate contact with the ground. Therefore, there is no inconsistency in specifying the charge density or potential temperature of the air entering these convective columns according to the surface fluxes $\overline{\rho'w'}|_0$ and $\overline{\theta'w'}|_0$.

Finally, we can write down the mean-squared vertical velocity throughout the layer simply as

$$\overline{w'^2} = \frac{A_c v_c^2}{1 - A_c} \quad (\text{A2.9})$$

This equation gives us one relation between v_c , A_c , and turbulence statistics. Another such relation will now be obtained using (A2.7), and the two will be used to specify v_c and A_c , as desired.

The buoyant-production rate of turbulent kinetic energy per unit area in our unstable boundary layer is given approximately by

$$H\langle P \rangle \equiv \frac{\rho_{a0} g}{T_0} \int_0^H \overline{\theta'w'} dz = \frac{\rho_{a0} g H}{2 T_0} \overline{\theta'w'}|_0$$

Here the angle brackets represent a vertical average, ρ_{a0} and T_0 are magnitudes of the air density and absolute temperature in the layer, and g is the gravitational acceleration. If we neglect shear production (justified in a strongly convective layer for which $L \ll H$, where L is the Monin-Obukhov length) and conversion of kinetic to potential energy at the inversion (justified for our purposes because apparently only a few percent

of the kinetic energy is used in this way in a real, inversion-capped, unstable layer), we can equate $\langle P \rangle$ to the dissipation rate by molecular viscosity, $\langle \bar{\epsilon} \rangle$. This dissipation rate, together with twice the total kinetic energy in the layer per unit area

$$H \langle \bar{q}^2 \rangle = \int_0^H \rho_{a0} \bar{w}^2 dz = \frac{\rho_{a0} H A_c v_c^2}{1 - A_c}$$

can be used to give a turbulence-dissipation time scale, defined as was Wyngaard's time scale described in Chapter 2.

$$\bar{\tau} = \frac{\langle \bar{q}^2 \rangle}{\langle \bar{\epsilon} \rangle} = \frac{2 T_0 A_c v_c^2}{g (1 - A_c) \overline{\theta' w'}|_0} \quad (\text{A2.10})$$

We can get another estimate of the turbulence time scale in our simple model from the time required for a parcel to ride up a convective column from bottom to top.

$$\bar{\tau} \sim \frac{H}{v_c} \quad (\text{A2.11})$$

Equating this estimate to the time scale defined in (A2.10) gives the second relation needed between v_c and A_c , allowing us to solve for them both.

$$\left. \begin{aligned} \frac{A_c}{1 - A_c} &= \frac{4 T_0^2 (\bar{w}^2)^3}{g^2 H^2 (\overline{\theta' w'}|_0)^2} \\ v_c &= \frac{g H \overline{\theta' w'}|_0}{2 T_0 \bar{w}^2} \end{aligned} \right\} \quad (\text{A.12})$$

The equating of (A2.10) and (A2.11) is admittedly a questionable operation, since the meaning of a turbulence-dissipation time scale is unclear in our simple model; but this procedure would be reasonable in a real unstable layer, where the dissipation time should be on the order of one turn-over time for the largest eddies. The real justification for (A2.12) must be that it gives physically reasonable values for v_c and A_c .

In order to find a numerical value for A_c , we make use of the result of Tennekes (1970) relating $\overline{w'^2}$ in an unstable mixed layer to the surface temperature flux. This has become known as " w_* scaling" because of the definition of w_* .

$$\left. \begin{aligned} \langle \overline{w'^2} \rangle &\approx 0.4 u_*^2 \left(-\frac{H}{kL} \right)^{2/3} = 0.4 w_*^2 \\ \frac{w_*^2}{u_*^2} &\equiv \left(-\frac{H}{kL} \right)^{1/3} \equiv \left(\frac{g H \overline{\theta' w'}|_0}{u_*^3 T_0} \right)^{1/3} \end{aligned} \right\} \quad (\text{A2.13})$$

Here u_* is the friction velocity, and k is the von Karman constant [$k \approx 0.35$ according to Businger (1973)]. Using (A2.13) in (A2.12) gives

$$\left. \begin{aligned} A_c &= 0.204 \\ \frac{v_c}{u_*} &= 1.25 \left(\frac{g H \overline{\theta' w'}|_0}{u_*^3 T_0} \right)^{1/3} = 1.25 \left(-\frac{H}{kL} \right)^{1/3} = 1.25 \frac{w_*}{u_*} \end{aligned} \right\} \quad (\text{A2.14})$$

This value of A_c seems reasonable in the light of the intuitive idea that the rising columns occupy a relatively small fraction of the horizontal area. The fraction should not be too small, since in the interior of a real mixed layer the rising plumes have expanded considerably by entrainment. An examination of the horizontal maps of vertical velocity and

virtual-temperature perturbation presented by Deardorff (1974b) suggests that A_C should be between 0.1 and 0.5 .

Finally, we collect all the results for future reference. Making them non-dimensional with u_* , $\tau_\lambda \equiv \epsilon_0/\lambda_0$, $T_* \equiv -\frac{1}{u_*} \overline{w'\theta'}|_0$, H , and $\rho_* \equiv \frac{1}{u_*} \overline{\rho'w'}|_0$, we have, from (A2.3), (A2.5), (A2.6), (A2.7), (A2.8), (A2.11), (A2.13), and (A2.14),

$$\frac{\overline{w'^2}}{u_*^2} = 0.4 \left(-\frac{H}{kL} \right)^{2/3}$$

$$\frac{\overline{\tau}}{\tau_\lambda} = 0.8 R^{-1} \left(-\frac{H}{kL} \right)^{-1/3}$$

$$\frac{\overline{\theta'w'}}{u_* T_*} = - \left(1 - \frac{z}{H} \right)$$

$$\frac{H}{T_*} \frac{d\overline{\theta}}{dz} = -2.32 \left(-\frac{H}{kL} \right)^{-1/3} \quad (A2.15)$$

$$\frac{\overline{\rho'^2}}{\rho_*^2} = 2.50 \left(-\frac{H}{kL} \right)^{-2/3} \left(1 - e^{-4.91 \frac{H}{H_c}} \right)^{-2} e^{-2 \frac{H}{H_c} \frac{z}{H}} \left[1 - e^{-4.91 \frac{H}{H_c} \left(1 - \frac{z}{H} \right)} \right]^2$$

$$\frac{\overline{\rho'\theta'}}{\rho_* T_*} = -2.50 \left(-\frac{H}{kL} \right)^{-2/3} \left(1 - e^{-4.91 \frac{H}{H_c}} \right)^{-1} \left(1 - \frac{z}{H} \right) e^{-\frac{H}{H_c} \frac{z}{H}} \left[1 - e^{-4.91 \frac{H}{H_c} \left(1 - \frac{z}{H} \right)} \right]$$

$$\frac{\overline{\rho'w'}}{\rho_* u_*} = \left(1 - e^{-4.91 \frac{H}{H_c}} \right)^{-1} e^{-\frac{H}{H_c} \frac{z}{H}} \left[1 - e^{-4.91 \frac{H}{H_c} \left(1 - \frac{z}{H} \right)} \right]$$

$$\frac{H}{H_c} = 0.8 R^{-1} \left(-\frac{H}{kL} \right)^{-1/3} = \frac{\overline{\tau}}{\tau_\lambda}$$

where we have defined $R \equiv \frac{u_* \tau_\lambda}{H}$. The first four equations here will be used elsewhere for a comparison with the second-order-closure model.

We now proceed to use the last two equations above to derive the behavior of the EMF generated by convection in our simple model. The non-dimensional EMF is defined as in the Introduction:

$$\left. \begin{aligned} \frac{\lambda_0}{\rho_* u_* H} (\text{EMF}) &\equiv \int_0^1 \frac{\overline{\rho' w'}}{\rho_* u_*} d\left(\frac{z}{H}\right) \\ &= \frac{H_c}{H} \left[1 - e^{-\frac{H}{A_c H_c}} \right]^{-1} \left[1 - \left(\frac{1}{1-A_c} \right) e^{-\frac{H}{H_c}} + \left(\frac{A_c}{1-A_c} \right) e^{-\frac{H}{A_c H_c}} \right] \end{aligned} \right\} \quad (\text{A2.16})$$

A graph of the value of this integral versus H/H_c is shown in figure A2.1, using $A_c = 0.204$. It is noteworthy that the integral has a maximum of $\frac{\lambda_0}{\rho_* u_* H} (\text{EMF}) = 0.570$ at $H/H_c = 0.573$, corresponding to $(-H/L)^{1/3} R = 0.983$. If $R = 0.1$, for example, this maximum occurs for $H/L = -951$. This brings out another important result. The non-dimensional EMF, and $\frac{\overline{\rho' w'}}{\rho_* u_*}$ itself, depend in this simple model only on the combination H/H_c . Therefore, the results are the same along straight lines in parameter space on log-log paper:

$$\log R + \frac{1}{3} \log \left(-\frac{H}{L} \right) = \text{constant} \quad (\text{A2.17})$$

All these results will be found to correspond closely to results from the second-order-closure model developed in Chapter 3.

There are two simple limiting cases. For large H/H_c , the charge going up in the rising columns does not get all the way to the upper boundary at $z = H$.

Fig.A2.1 Dimensionless EMF versus H/H_c predicted by the simple convective-column model, with asymptotic forms for large and small H/H_c .

$$\left. \begin{aligned} \frac{\overline{\rho'w'}}{\rho_* u_*} &\approx e^{-\frac{H}{H_c} \frac{z}{H}} \\ \frac{\lambda_o}{\rho_* u_* H} (EMF) &\approx \frac{H_c}{H} \end{aligned} \right\} \frac{H}{A_c H_c} \gg 1 \quad (A2.18)$$

For small H/H_c , the charge decays little over a full round trip up in the rising columns and down in the subsiding environment.

$$\left. \begin{aligned} \frac{\overline{\rho'w'}}{\rho_* u_*} &\approx 1 - \frac{z}{H} \\ \frac{\lambda_o}{\rho_* u_* H} (EMF) &\approx \frac{1}{2} \end{aligned} \right\} \frac{H}{A_c H_c} \ll 1 \quad (A2.19)$$

Reference to figure A2.1 shows that these limiting cases are accurate for $H/H_c \gtrsim 3$ and $H/H_c \lesssim 0.1$, respectively.

APPENDIX 3 Numerical Solution of Tridiagonal and Block-Tridiagonal Systems of Equations

A tridiagonal system of linear, algebraic equations is always obtained when second-order, centered differences are substituted into a first- or second-order, linear, ordinary differential equation with boundary values. This is the case, for example, with the $\overline{\rho^2}$ equation (3.3), which appears in tridiagonal, finite-difference form in (3.9). The general form of such a system of equations is

$$A \underset{\sim}{x} = \underset{\sim}{r} \quad (\text{A3.1})$$

where $\underset{\sim}{x}$ is the vector of unknowns, $\underset{\sim}{r}$ is the vector of constant terms, and A is a square, tridiagonal matrix of the form

$$A \equiv \begin{pmatrix} b_1 & c_1 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & & \\ 0 & a_3 & b_3 & c_3 & & \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & 0 & a_N & b_N \end{pmatrix} \quad (\text{A3.2})$$

In the form of (3.9) this system would be written

$$a_n x_{n-1} + b_n x_n + c_n x_{n+1} = r_n, \quad 1 \leq n \leq N \quad (\text{A3.3})$$

The boundary conditions x_0 and x_{N+1} have been used in (A3.3) to eliminate the a_1 and c_N terms and square off the corners of the matrix (A3.2). If these boundary conditions are not homogeneous, they will result

in the modification of one or more of b_1 , c_1 , r_1 , a_N , b_N , and r_N in going from (A3.3) to (A3.2).

An economical method of solution of this system of equations has been given by Richtmyer and Morton (1967), and is repeated below in somewhat modified form. It requires going through the matrix twice — from top to bottom, and back again. On the first pass, the following operations are performed:

$$\left. \begin{aligned} b_1 &= b_1^{-1} \\ r_1 &= b_1 r_1 \\ b_n &= (b_n - a_n c_{n-1} b_{n-1})^{-1} \\ c_{n-1} &= -c_{n-1} b_{n-1} \\ r_n &= b_n (r_n - a_n x_{n-1}) \end{aligned} \right\} \begin{array}{l} \\ \\ n=2, 3, \dots, N \\ \end{array} \quad (A3.4)$$

Note that the equals signs here and in (A3.5) represent numerical assignment operations (as in FORTRAN), not algebraic equalities. Evidently, this algorithm destroys the original constant vector $\tilde{r}$ and the matrix diagonals b_n and c_n in the interest of saving storage space. On the second pass, the final solution vector is calculated:

$$r_n = c_n r_{n+1} + r_n, \quad n = N-1, N-2, \dots, 1 \quad (A3.5)$$

This method represents a very fast way of solving the system (A3.1), provided $b_1 \neq 0$ and $b_n \neq a_n c_{n-1} b_{n-1}$ in (A3.4).

If it is a system of M coupled, linear, second-order, ordinary differential equations that is to be solved, substitution of second-order, centered-difference formulae results in a block-tridiagonal system with $M \times M$ blocks. The pair of equations for $\overline{\rho w}$ and $\overline{\rho \theta}$ given in (3.2) results in such a system (3.8), with $M=2$. The general form of such a system can still be written as (A3.1) and (A3.2), with each element x_n and r_n in the vectors itself being considered as an M -dimensional vector and each matrix element a_n , b_n , and c_n itself being considered as an $M \times M$ matrix. For example, $x_n = (\overline{\rho w_n}, \overline{\rho \theta_n})$ in our case.

A modification of the algorithm of (A3.4) and (A3.5) can be used on a block-tridiagonal system as well, the only difference being that the matrix elements a_n , b_n , and c_n , being matrices themselves, cannot be divided; they must be inverted and multiplied. This method is still quite economical, however, as long as M is small, since it only involves the inversion of N $M \times M$ matrices. An outline of the derivation is given below.

Define matrices L and U as

$$L \equiv \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_2 & 1 & 0 & & \\ 0 & l_3 & 1 & & \\ \vdots & & & \ddots & \\ 0 & \cdots & 0 & l_N & 1 \end{pmatrix} \quad (A3.6)$$

$$U \equiv \begin{pmatrix} d_1 & u_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & u_2 & 0 & & \\ 0 & 0 & d_3 & u_3 & & \\ \vdots & & & \ddots & & \\ 0 & & & & d_{N-1} & u_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & d_N \end{pmatrix} \quad \begin{matrix} \text{(A3.6)} \\ \text{cont.} \end{matrix}$$

so that

$$A \equiv LU \quad \text{(A3.7)}$$

where A is defined by (A3.2), as before. Multiplying out (A3.7) and solving for u_n , d_n , and l_n , being careful to use only operations valid for matrices, we find the following sequential algorithm:

$$\left. \begin{aligned} d_1 &= b_1, \quad u_1 = c_1 \\ u_i &= c_i \\ l_i &= a_i d_{i-1}^{-1} \\ d_i &= b_i - l_i c_{i-1} \end{aligned} \right\} \quad i = 2, 3, \dots, N \quad \text{(A3.8)}$$

Returning to (A3.1) and (A3.7), we can write the following equations:

$$\underline{L} \underline{g} = \underline{L} (\underline{U} \underline{x}) = \underline{r} \quad (\text{A3.9})$$

$$\underline{g} = \underline{U} \underline{x} = \underline{L}^{-1} \underline{r} \quad (\text{A3.10})$$

The elements of the newly defined vector $\underline{g}$ can be calculated sequentially from (A3.9).

$$\left. \begin{aligned} g_1 &= r_1 \\ g_n &= r_n - l_n g_{n-1}, \quad n = 2, 3, \dots, N \end{aligned} \right\} \quad (\text{A3.11})$$

Finally, the elements of $\underline{x}$ can be calculated from (A3.10) in reverse sequence.

$$\left. \begin{aligned} x_N &= d_N^{-1} g_N \\ x_n &= d_n^{-1} (g_n - c_n x_{n+1}), \quad n = N-1, N-2, \dots, 1 \end{aligned} \right\} \quad (\text{A3.12})$$

Collecting the operations in (A3.8), (A3.11), and (A3.12), we can write a two-pass numerical algorithm for the solution of the block-tridiagonal system. After simplifying it to save as much storage as possible, we have

$$\left. \begin{aligned} b_{n-1} &= b_{n-1}^{-1} \\ l &= a_n b_{n-1} \end{aligned} \right\} \quad (\text{A3.13})$$

$$\left. \begin{aligned} b_n &= b_n - \ell c_{n-1} \\ r_n &= r_n - \ell r_{n-1} \end{aligned} \right\} n = 2, 3, \dots, N \quad (A3.13) \text{ cont.}$$

and

$$\left. \begin{aligned} b_N &= b_N^{-1} \\ r_N &= b_N r_N \\ r_n &= b_n (r_n - c_n r_{n+1}), \quad n = N-1, N-2, \dots, 1 \end{aligned} \right\} \quad (A3.14)$$

Here again, the equals signs mean numerical assignments rather than algebraic equalities, and the solutions appear in the vector $\underline{r}$. It has been necessary to define one additional temporary storage element ℓ . Remember that a_n , b_n , c_n , and ℓ are $M \times M$ matrices and the r_n 's are M -dimensional vectors. Therefore, b_{n-1}^{-1} represents the inverse of b_{n-1} , $a_n b_{n-1}$ represents a matrix multiplication, and ℓr_{n-1} represents a matrix operating on a vector. In the $M=2$ case used in this thesis, all these operations are trivial.

The algorithm in (A3.13) and (A3.14) gives the solution to the block-tridiagonal system (A3.1) as long as all the inverses exist. (A3.4) and (A3.5) can be derived from them with a little simplification due to the validity of the division operation in the algebraic case.

APPENDIX 4 Virtual Potential Temperature

The virtual temperature T_v of a moist parcel of air with water-vapor mixing ratio r and temperature T at pressure p is defined as the temperature of a dry parcel with the same density at that pressure. If $\bar{R}_d$ is the gas constant for dry air, then the gas constant for the moist parcel is given by

$$\bar{R}_m = \frac{1 + 1.609r}{1+r} \bar{R}_d$$

because the ratio of molecular weights of dry air and water vapor is 1.609. Application of the perfect gas law then gives

$$\left. \begin{aligned} \bar{R}_m T &= \frac{p}{\rho_a} \equiv \bar{R}_d T_v \\ \Rightarrow T_v &= \frac{1 + 1.609r}{1+r} T \end{aligned} \right\} \quad (A4.1)$$

We define the virtual potential temperature of the parcel as

$$\Theta_v \equiv \left(\frac{p_0}{p} \right)^{\kappa_d} T_v \quad (A4.2)$$

where $\kappa_d \equiv \frac{\bar{R}_d}{c_{pd}}$, c_{pd} is the specific heat at constant pressure for dry air, and p_0 is a reference pressure. The actual potential temperature of the parcel is given by

$$\Theta = \left(\frac{p_0}{p} \right)^{\kappa_m} T$$

where $\kappa_m \equiv \frac{\bar{R}_m}{C_{pm}}$ and $C_{pm} = \frac{1 + 1.869r}{1 + r} C_p$ because C_{pv} for water vapor is 1.869 times C_{pd} at about 10°C. These definitions imply

$$\left. \begin{aligned} \theta_v &= \left(\frac{1 + 1.609r}{1 + r} \right) \left(\frac{P_0}{P} \right)^{K_d - K_m} \theta \\ \text{where } K_m &= \frac{1 + 1.609r}{1 + 1.869r} K_d \end{aligned} \right\} \quad (A4.3)$$

Making the following approximations for small $r \ll 1$,

$$\left. \begin{aligned} \left(\frac{1 + 1.609r}{1 + r} \right) &\approx 1 + 0.609r \\ \left(\frac{P_0}{P} \right)^{K_d - K_m} &\approx 1 + 0.0743r \ln \left(\frac{P_0}{P} \right) \end{aligned} \right\} \quad (A4.4)$$

we can write approximate forms of (A4.1) and (A4.3) as

$$\left. \begin{aligned} T_v &\approx T + 0.609 T_0 r \\ \theta_v &\approx \theta + (0.609 + 0.0743 \ln \frac{P_0}{P}) \theta_0 r \end{aligned} \right\} r \ll 1 \quad (A4.5)$$

where T_0 and θ_0 are reference temperatures.

From (A4.5) it is clear that θ_v will not be exactly conserved for adiabatic motions, nor will it be exactly mass-additive for mixing of parcels at constant pressure, in spite of the fact that potential temperature and specific humidity are both conserved and mass-additive. It must be demonstrated that θ_v behaves enough like θ that it can be used in

place of θ in the thermodynamic equation in (2.1). First we consider the conservation of θ_v for adiabatic changes in pressure. Suppose our parcel has virtual potential temperature θ_{v1} at pressure p_1 . Its potential temperature θ and its mixing ratio r will be conserved, and it will arrive at pressure p_2 with a new virtual potential temperature θ_{v2} given by

$$\begin{aligned} \left(\frac{1+r}{1+1.609r} \right) \left(\frac{p_1}{p_0} \right)^{K_d-K_m} \theta_{v1} &= \theta = \left(\frac{1+r}{1+1.609r} \right) \left(\frac{p_2}{p_0} \right)^{K_d-K_m} \theta_{v2} \\ \Rightarrow \theta_{v2} &= \left(\frac{p_1}{p_2} \right)^{K_d-K_m} \theta_{v1} \\ &\approx \theta_{v1} + 0.0743 \theta_0 r \ln \left(\frac{p_1}{p_2} \right), \quad r \ll 1 \end{aligned} \quad (A4.6)$$

from (A4.3) and (A4.4). Using $\theta_0 \leq 303^\circ\text{K}$, $r \leq 0.04$, and $p_2 \geq 0.691 p_1$ (corresponding to a change in elevation from zero to 3 km in the U.S. Standard Atmosphere) we find the deviation of θ_v from strict conservation to be

$$|\theta_{v2} - \theta_{v1}| \approx 0.0743 r \theta_0 \left| \ln \frac{p_1}{p_2} \right| \leq 0.33^\circ$$

Since the total adiabatic temperature change over a vertical displacement of 3 km is about 30 degrees, this amounts to a fractional error of only about 1%. Evidently, θ_v is conserved to a good approximation in the mixed layer.

Next we consider the mass-additiveness of θ_v for mixing at constant pressure. Using the identity $q' \equiv \frac{r}{1+r}$, where the specific

humidity q' is mass-additive, we can re-write equations (A4.1) and (A4.2) as

$$\Theta_v = (1 + 0.609 q') \left(\frac{p_o}{p} \right)^{K_d} T \quad (\text{A4.7})$$

This separates the problem into two parts. We can consider T alone, to show that it is approximately mass-additive in the mixed layer. We can consider separately the product of two mass-additive quantities to estimate its deviations from mass-additiveness.

Taking the second part of the problem first, suppose that we mix mass M_1 , with properties a_1 and b_1 , with mass M_2 , with properties a_2 and b_2 , to give mass $(M_1 + M_2)$ with properties a_f and b_f . Both a and b are mass-additive, but their product (ab) will not be. A little algebra shows that the deviations of (ab) from mass-additiveness are given by

$$(ab)_f - \frac{a_1 b_1 M_1 + a_2 b_2 M_2}{M_1 + M_2} = - \frac{M_1 M_2}{(M_1 + M_2)^2} (a_1 - a_2)(b_1 - b_2)$$

In (A4.7) the expression $\left[1 + 0.609 q' \right] \left(\frac{p_o}{p} \right)^{K_d}$ is mass-additive. If we assume for the moment that T is mass-additive also, we can take $a \equiv \left[1 + 0.609 q' \right] \left(\frac{p_o}{p} \right)^{K_d}$ and $b \equiv T$, to find

$$\Theta_{vf} - \frac{\Theta_{v1} M_1 + \Theta_{v2} M_2}{M_1 + M_2} \approx -0.609 \frac{M_1 M_2}{(M_1 + M_2)^2} \left(\frac{p_o}{p} \right)^{K_d} (r_1 - r_2)(T_1 - T_2) \quad (\text{A4.8})$$

Here $(q'_1 - q'_2)$ has been replaced by $(r_1 - r_2)$ for the limit of small r .

Returning to the first part of the problem, we must examine the mass-additiveness of T . We have

$$T_f = \frac{M_1 C_{p1} T_1 + M_2 C_{p2} T_2}{M_1 C_{p1} + M_2 C_{p2}}$$

because enthalpy and specific heat are mass-additive in a perfect gas.

After some algebra, we can write

$$T_f - \frac{M_1 T_1 + M_2 T_2}{M_1 + M_2} = (T_f - T_2) \left(\frac{M_2}{M_1 + M_2} \right) \left(\frac{C_{p1} - C_{p2}}{C_{p1}} \right)$$

Substitution of the estimate $T_f \approx \frac{T_1 M_1 + T_2 M_2}{M_1 + M_2}$ in the right-hand side of this equation gives

$$T_f - \frac{M_1 T_1 + M_2 T_2}{M_1 + M_2} \approx \frac{M_1 M_2}{(M_1 + M_2)^2} (T_1 - T_2) \left(\frac{C_{p1} - C_{p2}}{C_{p1}} \right)$$

With more algebra, we can derive

$$\frac{C_{p1} - C_{p2}}{C_{p1}} \approx 0.869 (r_1 - r_2), \quad r \ll 1$$

Putting this expression in the above equation finally gives

$$T_f - \frac{M_1 T_1 + M_2 T_2}{M_1 + M_2} \approx 0.869 \frac{M_1 M_2}{(M_1 + M_2)^2} (r_1 - r_2) (T_1 - T_2) \quad (\text{A4.9})$$

Our best estimate of the deviation from mass-additiveness of θ_v itself comes from combining the deviations given by equations (A4.8) and (A4.9). This suggests an upper limit of

$$\left| \theta_{vf} - \frac{\theta_{v1} M_1 + \theta_{v2} M_2}{M_1 + M_2} \right| \lesssim$$

$$\frac{M_1 M_2}{(M_1 + M_2)^2} \left[0.869 + 0.609 \left(\frac{p_0}{p} \right)^{k_d} \right] \left| (r_1 - r_2)(T_1 - T_2) \right| \lesssim 0.16^\circ$$

The numerical estimate here comes from $|T_1 - T_2| \leq 10^\circ$,

$|r_1 - r_2| \leq 0.04$, and $p \geq 0.691 p_0$, considered to be extreme values for the mixed layer. Again we have a fractional error on the order of 1%.

Evidently, θ_v is mass-additive to a good approximation in the mixed layer.

Finally, we derive the relation between the turbulent fluxes of ordinary and virtual potential temperature used in section 4.4. Starting with the second equation in (A4.5), we write

$$\begin{aligned} \overline{\theta_v w} &\approx \overline{\theta w} + \left(0.609 + 0.0743 \ln \frac{p_0}{p} \right) \theta_0 \overline{r w} \\ &= \overline{\theta w} + \overline{\theta' w'} + \left(0.609 + 0.0743 \ln \frac{p_0}{p} \right) \theta_0 (\overline{r w} + \overline{r' w'}) \end{aligned}$$

Subtracting $\overline{\theta_v w} \approx \overline{\theta w} + \left(0.609 + 0.0743 \ln \frac{p_0}{p} \right) \theta_0 \overline{r w}$ from this gives the result

$$\overline{\theta_v' w'} \approx \overline{\theta' w'} + \left(0.609 + 0.0743 \ln \frac{p_0}{p} \right) \theta_0 \overline{r' w'}, \quad r \ll 1 \quad (\text{A4.10})$$

APPENDIX 5 A Key to Obtaining Dimensional
Results from this Thesis

Three related models of the convection current have been developed in this thesis. We will summarize here the physical parameters which must be specified to determine a dimensional solution of each of these models, so that the interested reader can more easily make use of the dimensionless results presented in earlier chapters. The uniform- λ model of section 3.3 is based on a known flux of convection current entering through the bottom boundary, and is consequently the simplest. Two dimensionless parameters, $R \equiv \frac{u_* \tau_\lambda}{H}$ and $H/L \equiv \frac{kgT_*H}{T_o u_*^2}$, determine a dimensionless solution. Here $\tau_\lambda \equiv \epsilon_o / \lambda_o$, $u_*^2 \equiv -\overline{u'w'}|_0$, and $u_* T_* \equiv -\overline{w'\theta'_v}|_0$. There are five scale factors needed to re-dimensionalize the results: H , λ_o , u_* , T_* , and ρ_* . The physical variables which must be known to determine these scale factors, and the two dimensionless parameters as well, are the mixed-layer thickness, the conductivity of the interior of the layer (presumed uniform), the surface wind stress, the surface fluxes of heat and moisture, and the magnitude of the convection current forced into the bottom boundary, $\rho_* u_* \equiv \overline{\rho'w'}|_0$. Given R and H/L , one can look up the dimensionless EMF generated by convection in figure 3.7. Profiles of $\overline{\rho w}$ for many cases are found in figures 3.10 through 3.13. If the boundary layer in question has neutral stability, $H \equiv \frac{u_*}{f}$, and the Coriolis parameter must also be known. In that case, the solutions are found in figure 3.4.

The second model, discussed in section 3.5, is just the first model with a different lower boundary condition. This model is designed to compute the convection of Blanchard-effect charge from a source distribution given by $\hat{Q}(\zeta)$, which must be specified. It also requires an

additional dimensionless parameter, $\zeta_{oo} \equiv z_o/H$. In this case, the scale factor ρ_* is specified by the integrated source strength as

$\rho_* u_* \equiv \int_{z_o}^{z_q} \bar{Q}(z') dz'$. Therefore, two additional physical variables must be known, the surface roughness length and the charge-source profile.

Figures 4.2, 4.4, and 4.5 give estimates of $\rho_* u_*$, u_* , and z_o as functions of $\bar{u}$ (10 m) over the ocean. Two cases are worked out with this model in sections 4.4 and 4.5.

The non-uniform- λ model, discussed in section 6.5, is the third and most complicated one. It requires one additional dimensionless parameter beyond those of the second model: $\hat{J} \equiv \frac{J}{\rho_* u_*}$. This corresponds to the new physical variable J , the total downward current through the layer, which must be specified for the electorde effect. There are two cases. If only the electrode effect is considered, $\hat{J} = -1.0$, $\hat{Q}(\zeta) \equiv 0$, and $\rho_* u_* = -J$. If only the Blanchard effect is considered, $\hat{J} = 0$, $\hat{Q}(\zeta) \neq 0$, and $\rho_* u_*$ is specified as in the second model. The only other change is the substitution of the conductivity scale $[\lambda]$ for λ_o .

In the non-uniform- λ model the dimensionless conductivity profile $\hat{\lambda}(\zeta)$ is specified internally. However, two additional parameters must be provided to make this possible. C is obtained as a function of $z_{IO} \equiv \frac{1.06}{R} \zeta_{oo}$ from figure 5.9. AC is usually set equal to 1.175, but is tabulated along with z_{IO} and C for a number of cases in table 6.5. Given R and H/L for $\zeta_{oo} = 1.0 \times 10^{-5}$, one can look up the dimensionless potential drop across the layer in figure 6.1. An indication of the effect of changing ζ_{oo} is provided by table 6.4. Profiles of $\bar{\rho w}$ in many cases are given in figures 6.2 through 6.6. A diurnal variation due to convection of the electrode-effect charge over land has been worked out with this model in section 7.2.

A crucial step in the development of the non-uniform- λ model was the simplification of the electrode effect in a special case derived in Chapter 5. There a simple eddy-diffusion model was developed for calculating the conductivity profile from two dimensionless parameters, $z_{I0} \equiv \frac{z_o}{L'}$ and $\delta \equiv \frac{\epsilon_o \alpha}{ke}$, where $L' \equiv \frac{0.47 \epsilon_o u_*}{ke} (\alpha/q)^{1/2}$. Here, the new physical parameters needed are the mobility, recombination coefficient, and source strength of ions given in (5.9). Once these are fixed, the profile of $\bar{\lambda}$ depends only on z_o , u_* , and $[\lambda]$. An approximate profile of $\hat{\lambda}(\zeta)$ is given in equation (5.25), with the parameters ζ_{oo} , R , AC , and C discussed above.

In the previous chapters the behavior of the models has been discussed primarily in terms of dimensionless parameters and solutions. While such a treatment contains all the necessary information, it often obscures the physical meaning of the results. Therefore, we have undertaken here a brief discussion of the behavior of the convection current in terms of dimensional physical parameters. Attention will be confined to the dependence of the dimensional EMF, defined as $(EMF) = \frac{(\rho_* u_*) H}{\lambda} (emf)$ for the uniform- λ model in (3.19) or as $(EMF) = -J \frac{H}{[\lambda]} (\hat{V}-1)$ for the electrode-effect model in (7.4), on the friction velocity u_* , the surface temperature flux $F \equiv \overline{w'\theta'}_v|_0$, the inversion height H , and the conductivity scale λ_o or $[\lambda]$. The parameter J or $\overline{\rho'w'}|_0$ appears only as a multiplicative factor in the results and will be ignored. z_o and all other parameters will be held fixed.

To get a feeling for the behavior of the second-order-closure model itself, uncomplicated by the peculiarities of the charge source, we will first consider the uniform- λ model with the simple $\overline{\rho w}(\zeta_{oo}) = 1.0$ lower boundary conditions. Taking partial derivatives of the EMF with respect to the four independent variables chosen above, we have

$$\left. \begin{aligned}
 \frac{\partial}{\partial u_*} (EMF) &= \frac{(\rho_* u_*) H}{\lambda_0} \left[\frac{\partial(emf)}{\partial R} \frac{\partial R}{\partial u_*} + \frac{\partial(emf)}{\partial(H/L)} \frac{\partial(H/L)}{\partial u_*} \right] \\
 \frac{\partial}{\partial F} (EMF) &= \frac{(\rho_* u_*) H}{\lambda_0} \frac{\partial(emf)}{\partial(H/L)} \frac{\partial(H/L)}{\partial F} \\
 \frac{\partial}{\partial H} (EMF) &= \frac{(\rho_* u_*)}{\lambda_0} (emf) + \frac{(\rho_* u_*) H}{\lambda_0} \left[\frac{\partial(emf)}{\partial R} \frac{\partial R}{\partial H} + \frac{\partial(emf)}{\partial(H/L)} \frac{\partial(H/L)}{\partial H} \right] \\
 \frac{\partial}{\partial \lambda_0} (EMF) &= - \frac{(\rho_* u_*) H}{\lambda_0^2} (emf) + \frac{(\rho_* u_*) H}{\lambda_0} \frac{\partial(emf)}{\partial R} \frac{\partial R}{\partial \lambda_0}
 \end{aligned} \right\} \quad (A5.1)$$

because $(\rho_* u_*)$ is fixed, $\frac{\partial R}{\partial F} \equiv 0$, and $\frac{\partial(H/L)}{\partial \lambda_0} \equiv 0$. From the definitions of R and H/L , we find

$$\left. \begin{aligned}
 R &\equiv \frac{u_* \epsilon_0}{\lambda_0 H}, \quad \frac{H}{L} \equiv - \frac{k q H F}{T_0 u_*^3} \\
 \frac{\partial R}{\partial u_*} &= \frac{R}{u_*}, \quad \frac{\partial R}{\partial H} = - \frac{R}{H}, \quad \frac{\partial R}{\partial \lambda_0} = - \frac{R}{\lambda_0} \\
 \frac{\partial(H/L)}{\partial u_*} &= - \frac{3}{u_*} \left(\frac{H}{L} \right), \quad \frac{\partial(H/L)}{\partial F} = \frac{(H/L)}{F}, \quad \frac{\partial(H/L)}{\partial H} = \frac{H/L}{H}
 \end{aligned} \right\} \quad (A5.2)$$

Inserting (A5.2) into (A5.1), we have

$$\left. \begin{aligned}
 \frac{\partial}{\partial u_*} (EMF) &= \frac{(\rho_* u_*) H}{u_* \lambda_0} \left[R \frac{\partial(emf)}{\partial R} - 3 \left(\frac{H}{L} \right) \frac{\partial(emf)}{\partial(H/L)} \right] \\
 \frac{\partial}{\partial F} (EMF) &= \frac{(\rho_* u_*) H}{F \lambda_0} \left(\frac{H}{L} \right) \frac{\partial(emf)}{\partial(H/L)} \\
 \frac{\partial}{\partial H} (EMF) &= \frac{(\rho_* u_*)}{\lambda_0} \left[(emf) - R \frac{\partial(emf)}{\partial R} + \left(\frac{H}{L} \right) \frac{\partial(emf)}{\partial(H/L)} \right] \\
 \frac{\partial}{\partial \lambda_0} (EMF) &= - \frac{(\rho_* u_*) H}{\lambda_0^2} \left[(emf) + R \frac{\partial(emf)}{\partial R} \right]
 \end{aligned} \right\} \quad (A5.3)$$

In these equations, the value and derivatives of (emf) must be obtained from figure 3.7 for the appropriate values of R and H/L. We will work out only the one case $R = 0.1$ and $H/L = -50$, which appears to be qualitatively representative of most of parameter space. In this case $(emf) = 0.223$, and we estimate $\frac{\partial(emf)}{\partial R} \approx 1.9$ and $\frac{\partial(emf)}{\partial(H/L)} \approx -2.0 \times 10^{-3}$. Putting these values into (A5.3) along with the typical values $(\rho_* u_*) = 1.0 \times 10^{-12} \frac{\text{amp}}{\text{m}^2}$, $H = 1.0 \text{ km}$, $\lambda_o = 1.0 \times 10^{-14} \frac{\text{mho}}{\text{m}}$, $u_* = 0.2 \frac{\text{m}}{\text{sec}}$, and $F = 0.2 \frac{\text{deg m}}{\text{sec}}$, corresponding to $(EMF) = 22.3 \text{ kV}$, we have

$$\left. \begin{aligned} \frac{\partial}{\partial u_*}(EMF) &\approx -5.5 \times 10^4 \frac{\text{volt sec}}{\text{m}} \\ \frac{\partial}{\partial F}(EMF) &\approx +5.0 \times 10^4 \frac{\text{volt sec}}{\text{deg m}} \\ \frac{\partial}{\partial H}(EMF) &\approx +1.3 \times 10^1 \frac{\text{volt}}{\text{m}} \\ \frac{\partial}{\partial \lambda_o}(EMF) &\approx -4.1 \times 10^{18} \frac{\text{volt m}}{\text{mho}} \end{aligned} \right\} \quad (A5.4)$$

These results tell us that increasing the friction velocity or the conductivity decreases the EMF; whereas, increasing the heat flux or the inversion height increases it.

Similar equations can be worked out for the non-uniform- λ model with the electrode-effect source. In that case there is an additional dimensionless parameter:

$$\xi_{oo} \equiv \frac{z_o}{H}, \quad \frac{\partial \xi_{oo}}{\partial H} = -\frac{\xi_{oo}}{H} \quad (A5.5)$$

The partial derivatives of (EMF) corresponding to (A5.3) are

$$\left. \begin{aligned}
 \frac{\partial}{\partial u_*}(\text{EMF}) &= -\frac{JH}{u_*[\lambda]} \left[R \frac{\partial \hat{V}}{\partial R} - 3 \left(\frac{H}{L} \right) \frac{\partial \hat{V}}{\partial (H/L)} \right] \\
 \frac{\partial}{\partial F}(\text{EMF}) &= -\frac{JH}{F[\lambda]} \left(\frac{H}{L} \right) \frac{\partial \hat{V}}{\partial (H/L)} \\
 \frac{\partial}{\partial H}(\text{EMF}) &= -\frac{J}{[\lambda]} \left[(\hat{V}-1) - R \frac{\partial \hat{V}}{\partial R} + \left(\frac{H}{L} \right) \frac{\partial \hat{V}}{\partial (H/L)} - \zeta_{oo} \frac{\partial \hat{V}}{\partial \zeta_{oo}} \right] \\
 \frac{\partial}{\partial [\lambda]}(\text{EMF}) &= +\frac{JH}{[\lambda]^2} \left[(\hat{V}-1) + R \frac{\partial \hat{V}}{\partial R} \right]
 \end{aligned} \right\} \quad (\text{A5.6})$$

For the case $R = 0.1$, $H/L = -50$, and $\zeta_{oo} = 1.0 \times 10^{-5}$, $(\hat{V}-1) = 0.323$, and we estimate $\frac{\partial \hat{V}}{\partial R} \approx 3.1$ and $\frac{\partial \hat{V}}{\partial (H/L)} \approx -3.8 \times 10^{-3}$ from figure 6.1. From table 6.4 we estimate $\frac{\partial \hat{V}}{\partial \zeta_{oo}} \approx 4.9 \times 10^2$. Using these numbers, $-J = 1.0 \times 10^{-12} \frac{\text{amp}}{\text{m}^2}$, and $[\lambda] = 1.0 \times 10^{-14} \frac{\text{mho}}{\text{m}}$, corresponding to $(\text{EMF}) = 32.3 \text{ kV}$, in (A5.6) gives

$$\left. \begin{aligned}
 \frac{\partial}{\partial u_*}(\text{EMF}) &\approx -1.3 \times 10^5 \frac{\text{volt sec}}{\text{m}} \\
 \frac{\partial}{\partial F}(\text{EMF}) &\approx +9.5 \times 10^4 \frac{\text{volt sec}}{\text{deg m}} \\
 \frac{\partial}{\partial H}(\text{EMF}) &\approx +2.0 \times 10^1 \frac{\text{volt}}{\text{m}} \\
 \frac{\partial}{\partial [\lambda]}(\text{EMF}) &\approx -6.3 \times 10^{18} \frac{\text{volt m}}{\text{mho}}
 \end{aligned} \right\} \quad (\text{A5.7})$$

Although the signs are the same here as in (A5.4), the magnitudes are roughly twice as large, indicating that in this region of parameter

space the effective strength of the electrode-effect source is also decreasing with increasing u_* and $[\lambda]$ and increasing with increasing F and H . This result is partially due to the feedback process whereby increased convection current implies increased downward conduction current driving a stronger electrode effect. Note that the added term $-\zeta_{oo} \frac{\partial \hat{V}}{\partial \zeta_{oo}}$ in the equation for $\frac{\partial}{\partial H} (\text{EMF})$ has a negligible effect because $\frac{\partial \hat{V}}{\partial \zeta_{oo}}$ is so small.

LIST OF FREQUENTLY-USED SYMBOLS

a_t	closure constant in the turbulent-transport terms
A_c	fractional horizontal area of rising columns in the convective-column model
C_p	specific heat of air at constant pressure
d_3	closure constant in the pressure-scrambling term in the $\overline{\rho'w'}$ equation
d_4	closure constant in the molecular-dissipation term in the $\overline{\rho'\theta'}$ equation
D	molecular-diffusion coefficient for small ions
e	charge on an electron
E	vertical component of $\vec{E}$
$\vec{E}$	electric field vector
$\hat{E}$	dimensionless, horizontally averaged form of E
E_B	dimensionless, boundary-layer form of E for the electrode layer
E_I	dimensionless, interior form of E for the electrode layer
E_o	dimensionless surface value of E for the electrode layer
E_∞	value of E approached asymptotically at the top of the electrode layer
(EMF)	electromotive force generated in the mixed layer
(emf)	dimensionless form of (EMF)
f	Coriolis parameter
g	gravitational acceleration
H	thickness of the mixed layer or vertical scale of the model boundary layer
J	vertical component of $\vec{J}$
$\vec{J}$	total current-density vector
$\hat{J}$	dimensionless form of J
J_m	vertical mechanical-transfer- or convection-current density

J_λ	vertical component of $\vec{J}_\lambda$
$\vec{J}_\lambda$	conduction-current density vector
k	small ion mobility
$\frac{t}{K}$	synonym for k
K	eddy-diffusion constant for the electrode effect
K_h	eddy-diffusion coefficient for heat
K_t	closure-model eddy-diffusion coefficient
$\hat{K}_t$	dimensionless form of K_t
ℓ	length scale for mean vertical derivatives
L	Monin-Obukhov length
L'	vertical length scale for the electrode layer
n	numerical vertical coordinate — grid point number
n_1	positive small-ion number density
n_2	negative small-ion number density
N	total number of grid points in the numerical solution
$\mathcal{O}$	synonym for 0
p	air pressure
q	volume ionization rate
$\overline{q'^2}$	twice the total turbulent kinetic energy per unit mass
Q	volume source of charge
$\hat{Q}$	dimensionless, horizontally averaged form of Q
r	water-vapor mixing ratio
R	dimensionless ratio of conduction and turbulence time scales
R_z	partial columnar resistance to height z
R_∞	total columnar resistance
t	time
t'	dimensionless form of t

T	air temperature
T_*	temperature-fluctuation scale
T_o	reference temperature for the mixed layer
u	horizontal component of $\vec{U}$ in the direction of the mean surface wind
u_*	surface friction velocity
u_i	i^{th} component of $\vec{U}$
$\overline{u'w'} _0$	surface turbulent momentum flux per unit mass
$\vec{U}$	air velocity vector
v	horizontal component of $\vec{U}$ to the left of the mean surface wind
v_c	vertical velocity of the rising columns in the convective-column model
V	electrostatic potential function
$\hat{V}$	dimensionless potential drop across the mixed layer
V_∞	"ionospheric potential"
w	vertical component of $\vec{U}$
w_*	mixed-layer velocity-fluctuation scale
$\overline{w'^2}$	dimensionless form of $\overline{w'^2}$
$\overline{w'u'} _0$	synonym for $\overline{u'w'} _0$
$\overline{w\theta}$	dimensionless form of $\overline{w'\theta'}$
$\overline{w'\theta'} _0$	surface turbulent temperature flux
$\overline{w\rho}$	dimensionless form of $\overline{w'\rho'}$
$\overline{w'\rho'} _0$	surface turbulent charge flux
x	horizontal dimension in the direction of the mean surface wind
x_i	i^{th} component of the position vector
y	horizontal dimension to the left of the mean surface wind
y_B	dimensionless boundary-layer form of the total number density of positive and negative small ions for the electrode layer

y_I	dimensionless interior form of the total number density of positive and negative small ions for the electrode layer
z	vertical dimension
z_B	dimensionless, boundary-layer form of z for the electrode layer
z_I	dimensionless interior form of z for the electrode layer
z_o	surface roughness length
z_q	height of the top of the charge-source layer
<hr/>	
α	recombination coefficient for small ions
β	dimensionless expansion parameter for the electrode effect
γ	dimensionless parameter for the electrode effect
δ	dimensionless parameter for the electrode effect
Δ	coordinate-stretching parameter
$\bar{\epsilon}$	dissipation rate of turbulent kinetic energy
ϵ_o	dielectric permittivity of free space
ζ	dimensionless form of z
ζ_o	coordinate-stretching parameter
ζ_{oo}	dimensionless form of z_o
ζ_q	dimensionless form of z_q
θ	potential temperature
$\hat{\theta}$	dimensionless, horizontally averaged form of θ
θ_o	reference potential temperature for the mixed layer
θ_v	virtual potential temperature
$\overline{\theta w}$	dimensionless form of $\overline{\theta' w'}$
$\overline{\theta' w'} _0$	synonym for $\overline{w' \theta'} _0$
$\overline{\theta \rho}$	dimensionless form of $\overline{\theta' \rho'}$
$\frac{d\hat{\theta}}{d\zeta}$	dimensionless form of $\frac{d\theta}{dz}$
κ	thermometric conductivity of air

λ	electrical conductivity of air
$\hat{\lambda}$	dimensionless, horizontally averaged form of λ
$[\lambda]$	scale for $\bar{\lambda}$ in the mixed layer
λ_0	uniform value of λ assumed in the mixed layer for the uniform- λ model
ν	kinematic viscosity of air
ρ	charge density
$\hat{\rho}$	dimensionless, horizontally averaged form of ρ
ρ_*	charge-density fluctuation scale
ρ_a	air density
ρ_B	dimensionless, boundary-layer form of ρ for the electrode layer
ρ_I	dimensionless, interior form of ρ for the electrode layer
$\overline{\rho^2}$	dimensionless form of $\overline{\rho'^2}$
$\overline{\rho w}$	dimensionless form of $\overline{\rho'w'}$
$\overline{\rho'w'} _0$	synonym for $\overline{w'\rho'} _0$
$\overline{\rho\theta}$	synonym for $\overline{\theta\rho}$
τ	dimensionless form of $\bar{\tau}$
$\bar{\tau}$	turbulence-dissipation time scale
τ_λ	electrical relaxation time
Ω	the earth's vector angular velocity of rotation
k	von Karman constant
O	order-of-magnitude operator
$\mathfrak{z}$	mathematical symbol for "is in the range"
$\overline{(\)}$	horizontal average of $(\)$
$(\)'$	deviations of $(\)$ from $\overline{(\)}$

BIOGRAPHICAL NOTE

The author was born in Washington, D.C. on May 23, 1947. He attended secondary school at St. Albans in Washington, and he received his B.A. in Physics from Swarthmore College in Pennsylvania in 1970. He entered graduate studies at M.I.T. in the same year, where he was supported by graduate fellowships from the National Science Foundation and the Sloan Foundation.