ON THE RELATION BETWEEN ZONAL HEATING ASYMMETRIES
AND LARGE-SCALE ATMOSPHERIC FLUCTUATIONS IN SPACE AND TIME

by

JOHN ARTHUR YOUNG

B.A., Miami University
(1961)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February 1966

Signature of Author.

Department of Meteorology, February 1966

Certified by

Thesis Supervisor

Accepted by

Chairman, Departmental Committee on Graduate Students
ON THE RELATION BETWEEN ZONAL HEATING ASYMMETRIES
AND LARGE-SCALE ATMOSPHERIC FLUCTUATIONS IN SPACE AND TIME

Submitted to the Department of Meteorology on February 7, 1966 in
partial fulfillment of the requirements for the degree of
Doctor of Philosophy

ABSTRACT

The purpose of this thesis is to gain an understanding of how the
zonal heating asymmetries associated with the distribution of continents
and oceans produce a wave response in the atmosphere. In particular, we
wish to study the manner in which the advection processes of both the
planetary and cyclonic scales influence the atmospheric climate as described
by geographical variations in both the mean and transient flow states. We
will see that the interactions between the various wave scales play an
important role in the ultimate adjustment of such heating.

To gain this end, a quasi-geostrophic two-level model is introduced,
and some of its general properties are discussed. In view of the disper-
sive dynamics, it is convenient to express the flow as a time-varying
linear combination of 19 separate harmonic functions of x and y, each
with a particular associated scale of variation. The governing equations
then take a form which allows the inter-wave interactions to be treated
separately from those involving the zonal flow. In particular, this
direct wave coupling may be systematically excluded.

In anticipation of the later results found numerically, the linear prop-
erties of the system are investigated. The steady responses of given
forced waves imbedded in a zonal flow of variable properties are studied.
Attention is concentrated upon the conditions for resonant response; it
is shown that only those waves whose free modes would be both stable and
stationary may resonate in this case. In general, it is found that there
are two wave scales fulfilling the resonance condition for a given zonal
state. The influence of dissipative processes on the character of the
steady states is also examined, after which some examples of particular
steady wave solutions for a sequence of zonal flow states are given.

A simpler analysis is given to suggest the character of the influence
of a fixed planetary wave pattern on both the transient and mean states
of the shorter waves. Finally, stability analyses of both zonally sym-
metric and asymmetric baroclinic states are performed.

Next, numerical results are obtained using a three-step computational
procedure at each time step. The properties of this method for simple
linear and non-linear advective systems are compared with those of other
methods.
For particular choices of parameters, the behavior of the numerical solutions fall into two categories - periodic and non-periodic. In the periodic case, one experiment is considered in particular detail. It is found that the solutions possess qualitative behavior similar to that suggested by the linear analyses. In particular, development and maintenance of a free wave due essentially to the wave interactions is shown to occur. In addition, this wave exhibits an important influence on the nature of both the mean flow distribution and that of the transient statistics, both of which differ significantly from the pattern of continentality.

Variations of this experiment are also performed. It is shown that the mean and fluctuating properties may be significantly altered when the wave interactions are excluded. Feedback of the shorter waves onto the planetary wave is seen to depend upon the smaller scale heating field. It is also shown that the quasi-steady planetary waves may distort themselves, producing changes in both the zonal and wave flows.

The non-periodic solution is analyzed using power spectral analysis on the amplitude fluctuations of the waves, and cross-spectra are used to obtain the direction and speed of phase propagation. In addition to the fluctuations of cyclone period, it is found that oscillations with periods longer than three weeks are excited. The energy transformations are studied and found to exhibit similar "long-period" behavior. In addition they are related to the phase of the longest wave, and to some extent the shorter ones as well. From these analyses, an idealized "index cycle" explicitly involving the planetary wave and zonal flow, and implicitly the cyclone waves, is constructed. This cycle depends in an essential way upon the non-linear coupling of the planetary wave to the cyclone waves.

Finally, the solutions with and without wave interactions and the steady linear solution are compared with each other and with the real atmosphere. In addition, results obtained in the absence of continentality are studied. It is then found that development of energy in the longest waves is almost negligible. In addition, the total wave energy is smaller than in the case with asymmetric heating.

With or without continentality, it is shown that the efficiency of the cyclone waves in drawing energy from the zonal baroclinicity is reduced by the interwave interactions. It is also suggested that these interactions produce preferred directions of energy flow between scales in an enhanced manner when the influence of continentality is present.

Thesis Supervisor: Edward N. Lorenz
Title: Professor
DEDICATION

TO MY WIFE, JULIE, WHOSE PATIENT ENCOURAGEMENT AND MANIFEST SELFLESSNESS MADE THIS POSSIBLE.
ACKNOWLEDGEMENTS

I wish to express my appreciation to Professor E. N. Lorenz, whose advice proved very helpful, especially in obtaining the results of Chapter 5. Professor N. A. Phillips served as advisor during the first two years of work during which the approach to this thesis was finalized. This guidance, along with the careful attention paid to the final text, is greatly appreciated.

I am deeply grateful to my friend, Dr. R. Terry Williams, who contributed substantially to the broadening of my point of view. His active interest, reflected in untold numbers of discussions, was a welcome factor at all times.

Professor T. Madden's comments on time series analysis were helpful, and he generously contributed the use of his spectral subroutines. Much of the computational time was provided by the M.I.T. Computation Center. Financial support of the author during the first year was provided by a Ford Foundation fellowship. The remainder of the work was supported by Professor J. G. Charney's National Science Foundation Project.

The task of producing the final copy was masterfully directed by Mrs. Jane McNabb, who also labored many hours on the typewriter with Miss Penny Freeman. They were ably assisted by Miss Marie Louise Guillot.

Miss Isabelle Kole's deft touch is evident in the figures. Mr. M. K. Mak kindly contributed his time to help with the final details.
# TABLE OF CONTENTS

INTRODUCTION 1

1. QUASI-GEOSTROPHIC MOTIONS 7
   1.1 Basic Description 7
   1.2 Conservative Properties 9
   1.3 Heating and Friction 13

2. THE ATMOSPHERIC MODEL 16
   2.1 The Two-Level Model 16
   2.2 Choice of a Spectral Representation 23
   2.3 Some Characteristics of the Interactions 38
   2.4 Heating and Friction Influence 45

3. LINEAR DYNAMICS OF THE MODEL 53
   3.1 Introductory Remarks 53
   3.2 Free Wave Instability of the Zonal Flow 60
   3.3 Resonance in Stationary Forced Waves 65
   3.4 Examples of Stationary Wave Solutions 84
   3.5 Barotropic Waves on a Basic Wavy Flow 91
   3.6 Baroclinic Instability of a Wave Flow 97
   3.7 Final Remarks 102

4. SOLUTIONS WITH PERIODIC ELEMENTS 105
   4.1 Introduction 105
   4.2 Choice of Parameters 110
   4.3 Experiment I 116
   4.4 The Climate of Experiment I 137
   4.5 Experiments II, III and IV 145
   4.6 Experiment V: Planetary Scale Interactions 162

5. NON-PERIODIC FLOW 169
   5.1 Introduction 169
   5.2 Experiment VI 173
   5.3 Low Frequency Fluctuations 183
   5.4 Planetary Fluctuations and Energy Exchange 191
   5.5 Comparative Solutions for Three Representations of Advection 200
   5.6 Solutions Without Continentality 221
6. SUMMARY

6.1 Summary of relationships 226
6.2 Further Remarks 233
6.3 Final Remarks and Suggestions 236

APPENDIX A. DISCUSSION OF SIMPLIFIED HEATING LAWS 239
APPENDIX B. COMPARATIVE PROPERTIES OF SOME COMPUTATIONAL SCHEMES 246
APPENDIX C. SPECTRAL TECHNIQUES 257

BIBLIOGRAPHY 261
BIOGRAPHY 266
INTRODUCTION

This work is concerned with the examination of how the longitudinally variable heating associated with the patterns of continents and oceans influences the geostrophic flow of the atmospheric for periods longer than one day. In particular, we wish to gain an understanding of the dynamics of the resulting advections through which the flow evolves to produce the mean and transient characteristics of a longitudinally varying climate.

Earlier in the evolution of meteorological science much of the variability of atmospheric flows was thought to arise directly from the surface inhomogeneities. However, Jeffreys' (1926) suggestion that the atmospheric eddies contained an active dynamics of their own carried with it the implication that the flow behavior was not so simply related to the energy sources and sinks. Charney (1947) later demonstrated that the cyclone-scale eddies could develop as free instabilities on the baroclinic zonal flow. Hence, longitudinally variable heating did not seem necessary to explain the flow systems of cyclone scale.

The so-called "dishpan" experiments seemed to confirm Charney's hypothesis, and offered the further conclusion that the atmospheric wave motions could be unsteady in the absence of variable heating (Hide, 1953). More recently, Lorenz (1963a) has demonstrated that such non-periodic behavior is a characteristic of simple dynamical systems describing hydrodynamical flow like that of the atmosphere.
With this background, longitudinally variable heating seemed necessary only to explain the atmospheric disturbances of planetary scale. An important attempt to describe the dynamics of these motions had been made by Rossby, et. al., (1939), who pointed out the importance of the earth's sphericity (the so-called "beta effect") in this case. However, Smagorinsky's (1953) steady state solutions represented the first complete analysis of the problem since they included a linear representation of the interaction of the wave with the baroclinic zonal flow. It was found that the highly dispersive properties of these flows allowed the occurrence of resonance. Gilchrist (1953) further discussed the dependence of resonance on the zonal flow for a simpler two-level model.

Both of these studies were based on the assumption of steady-state, linear behavior of the planetary waves. In the real atmosphere, these waves are observed to interact non-linearly with other waves on a transient basis (Saltzman and Fleisher, 1960). Saltzman (1965) has therefore suggested that the effects of these interactions be included as known forcing functions for the steady waves. However, such an analysis would by necessity neglect any mean-state non-linear interactions.

Each of these approaches was based upon knowledge of a fixed heating field, taken to be independent of the flow itself. Doos: (1962) has presented solutions similar to Smagorinsky's and Saltzman's in which the heating was assumed to depend simply upon the steady flow
and a prescribed surface temperature field. One might realistically expect the transient interactions also to depend upon the mean flow, in which case the proper relation would be obtained by studying the full non-linear solutions in detail.

One object of the present work is to demonstrate the types of such relations that might exist, for both the time mean and slowly varying planetary flows, to gain an understanding in how the mean state develops. More importantly, we hope to learn more of the mechanisms which produce longitudinal variations in the transient flow statistics, for which little is presently known. In essence, this will involve the study of how the regular heating distributions associated with continentality are altered by the naturally occurring irregular flow. We will see that this question is closely connected with the direct exchange of energy between the various wave scales.

To obtain these results, the quasi-geostrophic model discussed in Chapters 1 and 2 will be used. The main results are presented and discussed in Chapters 3-5. Chapter 6 contains summarizing comments on the results, to which the reader is now referred.

In Chapter 1 the appropriateness of the quasi-geostrophic, beta plane dynamics for this problem will be discussed. It will be found that the equations may consistently describe flows of longitudinal planetary scale if the latitudinal scale is not too large. Both conservative and non-conservative properties of these equations will be reviewed to provide background for understanding the later results.
Chapter 2 contains the description of the two-level model employed as well as the final prediction equations. Representation of the flow in terms of time-varying combinations of fixed spatial functions will be utilized for two reasons. First, since these patterns have a unique measure of scale associated with them, the over-all dispersive behavior of the system becomes easier to study. An examination of the simple influence of heating and friction will be made in this manner in section 2.4. Secondly, the governing equations for the flow evolution have the property that interactions between these patterns may be consistently excluded at will. Some implications of the procedure will be discussed.

The purposes of Chapter 3 are two-fold. First, we wish to study the role of advection processes in determining the response of steady-state waves to heating. This will involve examining the relationships noted by Rossby and Gilchrist in more detail. We will find that consideration of the resonance condition alone will give much insight into these relations. Two important conclusions will be that only stable, stationary waves may resonate, and that two possible resonant scales of motion may exist for some zonal flow states.

The remaining part of Chapter 3 will be devoted to an attempt to predict ahead of time some qualitative features found in the later numerical solutions. To this end, simple linear systems describing the coupling of waves to the zonal flow and a fixed wave will be
studied. It will be seen that several effects involving the joint behavior of two wave scales for both transient and mean state flows can be anticipated. The possibility of a thermal wave field acting as a source of available potential energy for transient waves will be demonstrated.

Chapter 4 contains the results of numerical integrations of the full non-linear governing equations for the flow which have the property that the solutions are periodic in time. It will be shown that this convenient property may be obtained by making the heating rates unrealistically large. Several features of the simple cycles will be examined, with special emphasis placed upon the time-varying energy exchanges between waves. The solution will be compared with that obtained when these wave interactions are systematically neglected. Climatic maps for both solutions will be compared to show the importance of these interactions on the longitudinally varying climate. Further solutions will be given to demonstrate the sensitivity of the flow to the shape of the heating distribution and the beta effect, respectively. Finally, the behavior of a system of interacting forced waves of planetary scale will be studied to demonstrate the possibility of quasi-steady coupling.

One purpose of Chapter 5 is to demonstrate the fact that the relations found in Chapter 4 still hold in a statistical sense when the time variations are non-periodic. This will be accomplished by allowing the heating rates to assume more reasonable values. The
resulting climatic characteristics will be compared with the real atmosphere.

A second purpose is to show that the quasi-geostrophic motions are capable of undergoing long-period climatic oscillations, even in the case of constant forcing. A third purpose is to compare the solutions which develop under three separate approximations to the advection processes; the steady linear solution will be compared with the solutions which exclude and include the wave interactions, respectively.

Finally, Chapter 5 will be concluded with a discussion of the flow behavior in the absence of continentality. It will be seen that the development of planetary scale motions in the real atmosphere is evidently dependent upon external energy sources on that scale. Further differences in both the wave and zonal flows will be noted.
CHAPTER 1. QUASI-GEOSTROPHIC MOTIONS

1.1 Basic Description

The purpose of this chapter is to present a dynamical system capable of describing the long-term flow of a variably heated atmosphere when the longitudinal heating variations occur on the planetary scale. A secondary purpose is to discuss some of the properties of the chosen system, both conservative and otherwise.

The proper set of dynamics might be thought to be that given by Burger (1958) in his analysis of planetary scale motions. However, as Phillips (1963) has pointed out, the resulting equations are difficult to deal with analytically. More importantly, the observed interactions between the planetary and cyclone scales (Saltzman and Fleisher, 1960) preclude the significance of such an approach.

That study was based upon a zonal harmonic representation of the flow field, and thus it ignored explicit reference to the meridional wave scale. Ogura (1957), in a study on the harmonic distribution of the geostrophic velocity fields, found that the longest waves were quite anisotropic; the meridional scale noticeably was shorter than the zonal one. Haney (1961) found a typical "half-width" of these motion systems was typically less than 2,000 km. It thus appears that a realistic problem is to consider the dynamics of atmospheric wave motions possessing both planetary and cyclonic longitudinal scales which are somehow latitudinally confined. No attempt will be made to explain the restricting mechanism, however.
By considering only the relatively slow driving processes on large scales, it is clear that the immediate response of the atmosphere will be quasi-static in nature. In fact, studies (Rossby, 1938) of the adjustment of initially unbalanced fields of motion toward a final state of geostrophic balance indicate that the critical time scale separating primarily balanced responses from unbalanced ones is simply the inertia period $\frac{2\pi}{f}$, where $f = 2\Omega \sin \Theta$, $\Omega = 7.29 \times 10^{-5} \text{sec}^{-1}$ and $\Theta$ is latitude. Thus a Rossby number $R_o$ may be defined as the maximum of either $\frac{2\pi}{f} C$ or $\frac{f}{f}$ where $C$ is the time scale for the growth of hydrostatic pressure changes caused by the heating, and $f$ is the vertical vorticity component of the corresponding geostrophic flow field. We demand that $R_o$ be small compared to one, so that the effect of the earth's rotation on the horizontal motions is dominant.

In addition, we also demand that the heating at no time disrupt the state of positive static stability observed in the atmosphere. In fact, a Richardson number $R_i = \frac{g}{\Theta \frac{\partial \Theta}{\partial z}} \left| \frac{\partial \vec{v}_H}{\partial z} \right|^2$ may be defined, where $g$ is the local value of gravity, $\Theta$ is the potential temperature, and $\vec{v}_H$ is the horizontal velocity. We assume that the $\vec{v}_H$ field is given geostrophically and the typical depth scale for the motion is taken to be equal to the scale height $H = \frac{RT}{g}$, where $T$ is a typical temperature and $R$ is the gas constant for air. In this case, the effect of the static stability relative to the vertical wind shear as measured by $R_i$ is seen to be large, generally of the order of 100. The significance of this stratification
will be discussed later in a dynamical context.

States such as these possessing nearly complete geostrophic balance are an outstanding manifestation of stably stratified rotating fluid systems, and our present understanding of them has been summarized by Phillips (1963). There it is demonstrated that motions having these properties of \( R_0 << 1 \), \( R_0^2 R_i \leq 1 \) and \( L_y/\alpha << 1 \) satisfy the criteria defining "type 1" geostrophic motion. Here, \( L_y \) is the latitudinal length scale and \( \alpha \) is the earth's radius.

This is true despite the fact that the motion may be of planetary dimension in the longitudinal direction. The critical fact is simply that \( L_y \) be small enough to allow the use of the "beta plane" approximation. Accordingly, \( L_y \) will be taken to be of cyclone scale, or a few thousand kilometers.

In this case, one then has a tractable dynamical system described in Cartesian geometry. Further, the single set of dynamics consistently determines the behavior of both cyclone and planetary scale motions. It is therefore suitable for studying their interaction. This feature forms the basis for the results contained in later chapters.

1.2 Conservative Properties

Expressed in the \((x,y,p,t)\) system, the frictionless, adiabatic dynamics for this quasi-geostrophic system on the "beta plane" are given by the hydrostatic relation, the vorticity equation, and the first law of thermodynamics:

\[
\frac{\partial \phi}{\partial p} = -\alpha = -c_p \left[ \frac{\partial}{\partial p} \left( \frac{\Theta}{p_{00}} \right) \right] \Theta
\]

(1.2.1)
Here, we have defined

\[ p_0 = 1000 \text{ mb} \]
\[ \alpha = RT/p = \text{specific volume} \]
\[ T = \text{temperature} \]
\[ \Theta = T\left(\frac{p_0}{p}\right)^k = \text{potential temperature} \]
\[ \Theta_s = \Theta_s(p) = \text{a known "basic" potential temperature field} \]
\[ R = \text{gas constant for air} \]
\[ C_p = \text{specific heat of air at constant pressure} \]
\[ \kappa = R/C_p \]
\[ \Phi = g \Phi = \text{geopotential} \]
\[ f_o = f_0(\sin \Theta_0) = 2 \Omega \sin \Theta_0 \]
\[ \beta = \frac{1}{\kappa} \left. \frac{\partial f}{\partial \Theta} \right|_{\Theta = \Theta_0} \]
\[ \Theta_0 = \text{a constant mid-latitude} \]
\[ \Psi_h = (\|k \times \nabla \Phi\|)^{-1} f_o = \text{geostrophic wind} \]
\[ z = \|k \cdot \nabla \Psi_h\| = f_o^{-1} \|k \times \nabla \Phi\| \]
\[ k = \text{vertical unit vector} \]
\[ \omega = \frac{\partial p}{\partial t} = \text{"vertical" velocity relative to a pressure surface}. \]
The vertical boundary conditions are given by

\[ \omega(x, y, 0, t) = 0 \] (1.2.5)

and

\[ \omega(x, y, p, t) = \omega_g(x, y, t) \] (1.2.6)

where \( \omega_g(x, y, t) \) is the vertical "p-velocity" at the ground.

For the moment, the presence of lateral boundaries will be ignored.

Equation (1.2.2) contains the dynamic influence of the fixed stratification \( \frac{\partial}{\partial p} \Theta_s(p) \). Expressed non-dimensionally, Phillips (1963) has shown that it is given by a parameter \( B = R_0^2 R_i \sim 1 \) (1.2.7)

In this form, we see that the static effect of the stratification \( R_i \) is modulated by the dynamical factor \( \frac{R_0^2}{R_i} \frac{U^2}{f^2} \frac{1}{L^2} \) which measures the departure of the system from pure geostrophic balance. Here \( U \) is a typical value of \( |\Psi_H| \) and \( L \) is a characteristic length. It is seen that this rotational constraint changes the effective stability in a manner depending strongly upon the length scale. Clearly the circulation implied by \( \omega \) will also respond selectively.

For the time being we may eliminate explicit reference to that divergent part of the dynamics by elimination of \( \omega \) in (1.2.2) and (1.2.3) using (1.2.1). A single equation is obtained which expresses the fact that the potential vorticity

\[ \Phi = \beta y + f + \frac{\partial}{\partial p} \left[ \frac{f_0}{\Theta_s(p)} \Theta \right] \] (1.2.8)
is conserved following the geostrophic flow:

\[
\frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot \mathbf{v} = 0
\]  

(1.2.9)

If we restrict ourselves to the situation \( \frac{\partial \mathbf{q}}{\partial p} = \text{constant} \), then the last term in (1.2.8) becomes

\[
\frac{\mathbf{f}}{\partial \phi} \frac{\partial \theta}{\partial p}
\]  

(1.2.10)

This term is proportional to the static stability of the spatially varying field, which does not play a direct role in temperature changes for these type 1 quasi-geostrophic motions.

To the extent that these adiabatic dynamics are applicable to the real atmosphere, one finds that this sum of the absolute vertical component of vorticity and the "stability" term can never exceed the largest initial value \( g(x, y, p_0) \). Therefore, frontal regions, which possess large positive values of both \( \mathbf{f} \) and \( \frac{\partial \theta}{\partial p} \), can not develop from a smoothly varying initial field of \( \mathbf{q} \). We note, however, that these motions can develop concentrated regions of flow which resemble broad-scale fronts (Phillips, 1956). Presumably these regions are catalytic to the subsequent frontogenesis by ageostrophic processes (Williams, 1965).

This suggests that some measure of the three dimensional scale of the geostrophic field may never become small. In fact, Charney (1965, private communication) has demonstrated that a cascade of geostrophic energy to very short wavelengths is impossible, at least when the bottom boundary of the fluid is taken to be an
isentropic surface. His arguments represent a generalization of those appropriate for two dimensional, incompressible flow. Fjortoft (1953) derived several theorems showing that the spectral exchange of energy was limited by the initial state in that case. For example, he showed that the spectral energy density for scales shorter than those contained in the initial state was bounded by a known function of scale for all time.

Applied to the geostrophic motions, this would imply that initially broad flows could never produce a cascade of energy to a highly localized geostrophic state. The geostrophic dynamics, derived under the assumption of rather broad flow distributions, would thus be self-preserving for all time.

On this basis, it seems relevant to consider the behavior of a geostrophic system for lengths of time far beyond the advective time scale upon which the equations (1.2.1 - 1.2.4) were based. A simple two-level analogue to the continuous quasi-geostrophic system was first studied in this manner by Phillips (1956), in a general circulation experiment. The present work will also utilize such a model, but for the purpose of examining the interplay of longitudinal heating differences with the advective processes.

1.3 Heating and Friction

The geostrophic dynamics discussed in the last section describe only the adiabatic response of the fluid after the introduction of
energy by some means. We now briefly consider how heating and friction may operate to change the geostrophic flow field.

Firstly, the heating must have a distribution whose scale coincides with that of the motion. While it is known that local regions of intense heating occasionally exist in the atmosphere, we will confine our attention to large scale distributions alone. Some rough estimates of the average heating distribution for winter have been made (Clapp, 1961; Wiin-Nielsen and Brown, Jr., 1962); as expected, the resulting patterns were related to the distribution of continental and oceanic areas.

These results exhibited relatively small estimated rates of temperature change. For example, the time averaged values were always less than 10°C/day. As shown by Eliassen (1951) such "slow" heating produces a small divergent component of circulation which takes the altered pressure field into a new state of geostrophic balance. The ease with which this balance is reached depends upon the scale of the system, and the required divergent circulation reflects this dependence.

With regard to friction, it is safe to say that its precise mechanism in the atmosphere is poorly understood. One is more or less forced to assume that a small scale turbulent eddy viscosity can be assigned to the fluid, taken to be independent of the motion itself. In this case, the frictional processes operate vertically, in the direction of maximum flow variation. The strong tendency
for geostrophic balance then makes itself felt by limiting the
dissipation to a thin surface boundary layer. Away from this
Ekman boundary layer, the flow behaves as though it were nearly
frictionless. However, the small divergent circulation again
develops, this time in response to the vertical mass outflow
induced by horizontal variations of the surface flow. In the
event that internal friction were greatly increased, significant
dissipation might also occur in the interior, in which case a
further divergent motion would develop to achieve equilibrium.

We thus see that heating and frictional processes can be
incorporated into the quasi-geostrophic system in a consistent
manner, but that their effects may be altered by the immediate
influence of the Coriolis force. The resulting divergent cir-
culation thus acts as a scale-dependent filter for the influence
of heating and friction. Some of its characteristics will be
noted in Chapter 2.

While this initial tendency is of some interest, it is not
the primary concern of this work. Of more importance is to
arrive at an increased understanding of how the geostrophically
balanced flow evolves in time under the continuous influence of
heating and its own motion. Chapter 2 presents a model designed
to answer the question most straightforwardly.
2.1 The Two-Level Model

The complicated interplay of heating, advection, and dissipation suggests studying as simple a system as the basic processes permit. We will therefore limit consideration to a two-level representation of the vertical structure. The first such model was introduced by Phillips (1951), but a form similar to that used recently by Lorenz (1963) will be adopted here.

Figure 2.1 shows the assignment of variable names for each of the equally spaced hydrostatic pressure levels \( p_i \), \( i = 0, 1, 2, 3, 4 \). We define \( \Delta p = p_4 - p_2 = p_2 - p_0 \) as the thickness of the "upper" and "lower" layers. We see that \( \psi = \frac{\psi_1 + \psi_3}{2} \) and \( \zeta = \frac{\psi_1 - \psi_3}{2} \) represent the vertically averaged and vertical shear stream functions for the geostrophic flow, respectively. \( \Theta = \frac{\Theta_1 + \Theta_3}{2} \) is the vertically averaged potential temperature, while \( \Theta = \frac{\Theta_1 - \Theta_3}{2} \) is a measure of the static stability. In addition, \( \omega \), \( i = 0, 1, 2, 3, 4 \) are the vertical "p-velocities" at the levels \( p_i \).

These variables at each level are functions of \( \nu \), \( \mu \) and \( t \), where \( \nu \) is linear distance measured eastward, \( \mu \) is linear distance northward, and \( t \) is the time. The pressure levels are taken to be at \( p_0 = 200 \text{ mbar}, p_1 = 400 \text{ mbar}, p_2 = 600 \text{ mbar}, p_3 = 800 \text{ mbar}, \) and \( p_4 = 1000 \text{ mbar} \). \( \omega \) is thus the vertical velocity at the top of the troposphere, and will be assumed to be zero. This "rigid top" condition is equivalent to having an infinitely stable stratosphere lying above
Figure 2.1. Vertical information levels and primary dependent variables for the two-level model.
the level $p_0$. Thus, the model atmosphere can only interact across its lower boundary; two mechanisms of this sort exist in this model. First, if the earth's surface is not considered flat, then geostrophic flow over the topography can introduce vertical velocities $\omega_4$ at $p_4$. The influence of topography is not a primary objective of this study, but its effects will be briefly mentioned in Chapter III. Secondly, the system pictured in Figure 2.1 can be assumed to overlie an Ekman friction layer. In this case Charney and Eliassen (1949) showed that a vertical velocity $\omega_4 = -D \nabla^2 \psi_4$ (2.1.1) results. Here $D$ is a suitably defined coefficient with units of length, and $\nabla^2$ is the two-dimensional Laplacian operator.

Away from this friction layer, the state of geostrophic balance is expressed through the thermal wind relation

$$\nabla^2 \theta = \frac{f_0}{b^*} c_p \nabla^2 z$$

(2.1.2)

where $b^* = \frac{1}{2} \left[ (\frac{\psi_3}{\psi_4})^* - (\frac{\psi_1}{\psi_4})^* \right]$. The prediction equations for the geostrophic field are the vorticity equation and the First Law of Thermodynamics. The sum and difference of the vorticity equations evaluated at levels 1 and 3 take the forms (2.1.3) and (2.1.4). The vertical sum of the first law expressions of these levels yields (2.1.5). Here the approximation $\left( \frac{\omega_1 + \omega_2}{2} \right) = \frac{\omega_2}{2}$ was used; this merely states that the vertically averaged $\omega$ is about $1/2$ of the maximum $\omega \ (\sim \omega_2$ in the 2 level model).

In addition, the stability $\gamma$ can be changed by heating or by a vertical heat flux (Lorenz, 1960b).
Here we have defined the overbar $\overline{\left(\begin{array}{c}
\end{array}\right)}$ as the horizontal mean. Terms of the form $\overline{\left(\begin{array}{c}
\end{array}\right)}$ are the second order Jacobians, and are the mathematical representations of the horizontal advection processes. The "beta effect" is represented by the constant $\beta$ again. $\overline{\left(\begin{array}{c}
\end{array}\right)}$ is a constant equal to the Coriolis parameter evaluated at a typical mid-latitude. $Q_\varnothing$ and $Q_\varpi$ represent heating effects and will be given later. The constant $K'$ is seen to represent an internal friction coefficient relating the frictional drag between layers to the velocity difference between them.

The system (2.1.2 - 2.1.5), with $\overline{\left(\begin{array}{c}
\end{array}\right)}$ assuming a given constant value, nearly corresponds to the conventional quasi-geostrophic version of the two-level model. In this case, addition and subtraction of the frictionless, adiabatic forms of (2.1.3) and (2.1.4), using (2.1.5) to eliminate the $\omega \left(\begin{array}{c}
\end{array}\right)$ terms, yields

$$\frac{\partial \varpi}{\partial t} + \overline{\left(\begin{array}{c}
\end{array}\right)} = 0$$ (2.1.7)
\[ \frac{\partial}{\partial t} \mathcal{Q}_3 + \mathcal{I}(\psi_3, \mathcal{Q}_3) = 0 \]  

(2.1.8)

Here (2.1.2) was used to set \( \mathcal{I}(\mathcal{Q}, \Theta) = 0 \). (2.1.7) and (2.1.8) state that the potential vorticities\( \mathcal{Q}_i = \nabla^2 \psi_i + \beta y - \frac{f_0}{\Theta} \theta \)
and\( \mathcal{Q}_3 = \nabla^2 \psi_3 + \beta y + \frac{f_0}{\Theta} \theta \)
at each of the levels 1 and 3 are conserved following the motion at their individual level. This acts as a constraint on the motion in the manner discussed in Chapter I. Accordingly, we expect (2.1.7) and (2.1.8) to possess a spatially integrated form of energy which is constant when the flow over a closed region is considered.

This is obtained by multiplying (2.1.3), (2.1.4) and (2.1.5) by \( \psi, \mathcal{Q}, \Theta \) respectively, after setting \( \omega \psi, \Theta \)
and \( \kappa' \) all equal to zero. Integration of the resulting forms yields with the help of (2.1.2) the expressions

\[ \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} \]  

(2.1.9)

\[ \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} \]  

(2.1.10)

Here, we define the average kinetic energy of the fluid as

\[ K = \frac{\partial^2}{\partial \rho^2} \left[ \nabla \mathcal{Q} \cdot \nabla \mathcal{Q} + \nabla \mathcal{Q} \cdot \nabla \mathcal{Q} \right] \]  

(2.1.11)

and the average available potential energy (Lorenz, 1955) as
\[ A = \frac{2\alpha_p \kappa}{g} \frac{\theta'}{aT}, \quad \text{where} \quad \theta' = \theta - \overline{\theta} \]  

(2.1.12)

The terms involving the Jacobians are zero when the integrations are carried out and the system is considered closed laterally. Thus, they represent an internal redistribution of either \( K \) or \( A \) by the advective processes. The remaining terms in (2.1.9) and (2.1.10) are identical except for sign. They are proportional to the vertical heat flux, which develops when relatively warm or cold air parcels move in a particular vertical direction, on the average. Hence, such a process may be considered an internal conversion of available potential energy to kinetic energy.

From these remarks, addition of (2.1.9) and (2.1.10) yields

\[ \frac{\partial}{\partial t} (\dot{A} + K) = 0 \]  

(2.1.13)

That is, the quantity \( (\dot{A} + K) \) represents the conserved total energy of the geostrophic system, when \( \dot{A} \) and \( K \) are defined as above. While the total energy is conserved, the individual forms \( \dot{A} \) or \( K \) are not, due to energy conversion. In addition, even when either \( \dot{A} \) or \( K \) are nearly constant, their spatial distributions may not be. That is, the motions may evolve to mix initial flow states into different forms. In general, the spatial spectrum of the motion also changes, so that these horizontal mixing processes represent energy transformations from one pattern of flow to another.

The details of these processes, in relation to the energy sources and sinks, are an important aspect of this thesis. At this point, a
single property of these advections may be noted. Due to the potential vorticity constraint, the remarks of Chapter I imply that the mixing processes will have only a limited amount of freedom. They will not allow initially broad patterns to evolve into states possessing highly local variability.

Such constraints are not an obvious feature of the geostrophic system when the static stability $\overline{\theta}$ is allowed to vary in time. Lorenz (1960b) has shown that the system (2.1.2 - 2.1.6) satisfies a total energy relation of the form (2.1.13), where $K$ is defined as before. However, the definition of the potential energy $\overline{A}$ in this case is

$$\overline{A} = \frac{2\Delta p}{\overline{\theta}} b^* c_p \frac{\overline{\theta^2}}{\overline{\theta} + \overline{\theta_M}}$$  \hspace{1cm} (2.1.14)$$

Here, from (2.1.5) and (2.1.6), $\overline{\theta_M}$ is a constant given by

$$\overline{\theta_M}^2 = \overline{\theta^2} + \overline{\theta^2}$$  \hspace{1cm} (2.1.15)$$

Comparison of (2.1.14) with (2.1.12), using the fact that $\overline{\theta^2} \leq \overline{\theta_M^2}$, shows that with $\overline{\theta^2}$ given, the system with variable $\overline{\theta}$ contains less potential energy available for conversion to kinetic energy of motion.

Gates (1961) has discussed the implications of this difference for prediction on moderately short time scales. However, it is important to note that the model with constant stability can be derived from a systematic scale analysis (Phillips 1963). There it
is shown that stability variations are only a cumulative result of the quasi-horizontal motions; hence, appreciable variations in \( \tau \) can develop only over a period of time comparable with \( \frac{1}{R_o} \) times the advective time scale, or about two weeks.

It would thus appear that the potential importance of \( \tau \) variations is realized only on relatively long time scales. In fact, Lorenz (1962) has demonstrated the critical importance of this effect on the transitions observed between steady wave regimes in the rotating annulus experiments.

However, the atmospheric flows fluctuate greatly, suggesting that the net effect of slow \( \tau \) variations would be obscured. Also, the results of Chapter V demonstrate that long term mechanisms of change in geostrophic flow develop even when \( \tau \) is constant. In such cases, the inclusion of a variable \( \tau \) would thus represent just one of several small corrections to the quasi-geostrophic (constant stability) theory, and might just as well be neglected.

2.2 Choice of a Spectral Representation

a. The flow domain

Having described the degrees of freedom associated with the vertical structure, let us now consider the horizontal flow domain.

The "beta plane" most accurately represents flow on the spherical earth when the motion is confined to a latitudinal region which is small compared to the earth's radius. Accordingly
we will introduce frictionless walls along lines corresponding to
latitude circles a distance $W$ apart. The longitudinal extent of
the atmosphere will be described by periodicity of the flow at
length intervals of $3W$. Each periodic region will contain one
system of continents and oceans whose meridional coastlines are
equally spaced.

If $W$ were taken to be 5,000 km then the longitudinal period
would be 15,000 km, or one-half of the earth's mid-latitude circum-
ference. Such scaling would thus describe an earth-like system of
two identical pairs of continents and oceans. Results obtained for
flow with this realistic choice of $W$ are analyzed in Chapter 4.
However, it is also true that the North America - Atlantic Ocean
complex has a longitudinal scale closer to $3W = 10,000$ km; a
channel description of this system would thus have a width $W$ equal
to about 3,300 km. Such a system is studied in Chapter 5.

Given these characteristics, the boundary condition must be
considered. First, the longitudinal requirement of periodicity is
simply:

$$\psi(x+3W, y, t) = \psi(x, y, t)$$
$$\zeta(x+3W, y, t) = \zeta(x, y, t)$$
$$\theta(x+3W, y, t) = \theta(x, y, t)$$

(2.2.1)

The influence of the walls on the flow is more subtle. One obvious
effect of them is to allow no flow through the walls:

$$\frac{\partial \psi}{\partial x}(x,0,t) = 0, \quad \frac{\partial \zeta}{\partial x}(x,0,t) = 0$$

(2.2.2)

The simple relation (2.1.2) between $\zeta$ and $\Theta$ means that will be similarly constrained. Hence the effect of continentality must be zero at the walls. (2.2.2) is an important limitation of the model in general, but its implication for the temperature field is the most serious drawback with respect to the purposes of this study.

Finally, following Phillips (1956), it is consistent to assume that the longitudinally averaged east-west velocities are zero at the walls for all time. This may be stated as

$$\frac{\partial \langle \psi \rangle}{\partial y}|_{y=0} = \frac{\partial \langle \psi \rangle}{\partial y}|_{y=w} = 0, \quad \frac{\partial \langle \zeta \rangle}{\partial y}|_{y=0} = \frac{\partial \langle \zeta \rangle}{\partial y}|_{y=w} = 0$$

(2.2.3)

where $\frac{\partial \psi}{\partial y}$ and $\frac{\partial \zeta}{\partial y}$ are the eastward velocities and $\langle (\cdot) \rangle$ represents the longitudinal average.

With these boundary conditions, the equations (2.1.2 - 2.1.6), with suitably simple forms for $Q_\Theta$ and $Q_\zeta$, possess a curious symmetry which is conserved at all times if it is initially satisfied:

$$\psi(x + \frac{3w}{2}, -y, t) = -\psi(x, y, t)$$

$$\zeta(x + \frac{3w}{2}, -y, t) = -\zeta(x, y, t)$$

$$\Theta(x + \frac{3w}{2}, -y, t) = -\Theta(x, y, t)$$

(2.2.4)
This symmetry implies that flows which are symmetric about the central latitude have identical patterns over the continents and oceans, but the signs are reversed. In this case, east coast cyclones would be accompanied by west coast anti-cyclones. Further examples of these symmetries will be noted in Chapter 4, where it was found that they developed spontaneously rather than being imposed by the initial conditions in that special case.

Stackpole (1964) pointed out a similar restriction to the flows in a simpler geostrophic system. In that case, the number of descriptive degrees of freedom was limited, and, more importantly, the advective processes were simplified by allowing the eddies to interact only with the zonal flow. The result was that flows possessing a certain initial asymmetry preserved it for all time. Thus, two distinctly different climates could conceivably evolve from special initial states differing only slightly. This situation is intolerable for the present work. Stackpole showed that the constraint could be relieved by choice of a different flow description. It will later be seen that an alternative procedure would be to allow the direct interaction of the eddies, while retaining the original flow description.

This fact points out the sensitivity of the climate to the mechanism of advective transfer. This influence is of critical importance for this thesis, whose purpose is to gain some understanding of the interrelation of the heating and advection processes.

b. The spectral approach

More specifically, we wish to examine how the zonally asymmetric
heating produces a direct wave response, and how the resulting advecti-
ons redistribute that energy in both space and time. When the
longitudinal heating variations are small compared with the latitudinal
ones, mixing processes by the resulting wave field are negligible, and
the individual scales are excited quasi independently. Such a response
may involve simple dispersive interactions with the symmetric flow.
The natural description of this system would thus be one treating
each scale of motion separately.

In the real atmosphere, the asymmetric heating field may cause
significantly large wave motions which then participate in the sub-
sequent advects. In this case, the patterns associated with each
scale distort each other, thus introducing a complicated interdependence.
In addition, the total flow may contain baroclinic eddies which are not
confined to any geographic locality, a priori. These also enter into
the advective process, by increasing its variability.

It is clear that these physical processes are quite complicated.
The mathematical expressions describing them are also complex, although
the energy integrals and special symmetries previously discussed pro-
vide some information about the solutions. To learn anything more,
an obvious goal here, the complete solution must be found. We content
ourselves with an approximation to it, in which case an acceptable
procedure would be to solve the geostrophic equations using a finite
difference grid to describe spatial variations. The first successful
attempt was made in this manner by Phillips (1956).

Instead of following that procedure, we will instead adopt the
spectral approach first used by Bryan (1959). We express the horizontal flow distribution in terms of a set of specified functions

\[ F_i(x,y) \quad j = 0, 1, 2, \ldots \]

having the property

\[ \mathcal{L}^2 \nabla^2 F_i = -Q_i^2 F_i \quad (2.2.5) \]

where \( \mathcal{L} \) has the units of length and \( Q_i^2 \) is the "wavenumber" or scale measure of \( F_i(x,y) \). By this choice, we see that the individual scales of motion may be kept track of separately. This feature is particularly useful in understanding the results of Chapters 3, 4 and 5, even when the different scales were found to possess certain common modes of behavior.

Let us now specify the form of the functions \( F_i(x,y) \). We take \( L = W/W \) so that \( x' = x/L \) and \( y' = y/L \), the non-dimensional lengths in the eastward and northward directions, take on the values \[ 0 \leq x' \leq 2 \pi \quad \text{and} \quad 0 \leq y' \leq \pi \]

in the periodic region.

Consistent with the boundary conditions given in equations (2.2.1) - (2.2.3) we may follow Lorenz (1963) and define the normalized functions

\[ F_i(x,y) \quad \text{by Table 2.1 and the following definitions:} \]

\[
\begin{align*}
G_{0,0} &= 1 \quad (\text{Horizontal Average}) \\
G_{0,m} &= \sqrt{2} \cos y' \quad m = 1, 2 \quad (\text{Zonal modes}) \\
G_{n,m} &= 2 \sin y' \cos n \left( \frac{\pi}{2} x' \right) \\
G_{n',m} &= 2 \sin y' \sin n \left( \frac{\pi}{2} x' \right) \quad n = 1, 2, 3, (4); m = 1, 2 
\end{align*}
\]

(2.2.6) (2.2.7) (2.2.8)
<table>
<thead>
<tr>
<th>$F_i$</th>
<th>$G_{nm}$</th>
<th>$9 \times a_i^2$</th>
<th>Description of Spatial Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_0$</td>
<td>$G_{00}$</td>
<td>0</td>
<td>Horizontal average</td>
</tr>
<tr>
<td>$F_1$</td>
<td>$G_{01}$</td>
<td>9</td>
<td>Zonal Mode (symmetric)</td>
</tr>
<tr>
<td>$F_2$</td>
<td>$G_{02}$</td>
<td>35</td>
<td>&quot; &quot; (asymmetric)</td>
</tr>
<tr>
<td>$F_3$</td>
<td>$G_{11}$</td>
<td>13</td>
<td>Wave Mode with cosine Phase in $x$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$G_{11}$</td>
<td>13</td>
<td>sine</td>
</tr>
<tr>
<td>$F_5$</td>
<td>$G_{12}$</td>
<td>40</td>
<td>&quot; cosine &quot;</td>
</tr>
<tr>
<td>$F_6$</td>
<td>$G_{12}$</td>
<td>40</td>
<td>&quot; sine &quot;</td>
</tr>
<tr>
<td>$F_7$</td>
<td>$G_{21}$</td>
<td>25</td>
<td>&quot; cosine &quot;</td>
</tr>
<tr>
<td>$F_8$</td>
<td>$G_{21}$</td>
<td>25</td>
<td>&quot; sine &quot;</td>
</tr>
<tr>
<td>$F_9$</td>
<td>$G_{22}$</td>
<td>52</td>
<td>&quot; cosine &quot;</td>
</tr>
<tr>
<td>$F_{10}$</td>
<td>$G_{22}$</td>
<td>52</td>
<td>&quot; sine &quot;</td>
</tr>
<tr>
<td>$F_{11}$</td>
<td>$G_{31}$</td>
<td>45</td>
<td>&quot; cosine &quot;</td>
</tr>
<tr>
<td>$F_{12}$</td>
<td>$G_{31}$</td>
<td>45</td>
<td>&quot; sine &quot;</td>
</tr>
<tr>
<td>$F_{13}$</td>
<td>$G_{32}$</td>
<td>72</td>
<td>&quot; cosine &quot;</td>
</tr>
<tr>
<td>$F_{14}$</td>
<td>$G_{32}$</td>
<td>72</td>
<td>&quot; sine &quot;</td>
</tr>
<tr>
<td>$F_{15}$</td>
<td>$G_{41}$</td>
<td>73</td>
<td>&quot; cosine &quot;</td>
</tr>
<tr>
<td>$F_{16}$</td>
<td>$G_{41}$</td>
<td>73</td>
<td>&quot; sine &quot;</td>
</tr>
<tr>
<td>$F_{17}$</td>
<td>$G_{42}$</td>
<td>100</td>
<td>&quot; cosine &quot;</td>
</tr>
<tr>
<td>$F_{18}$</td>
<td>$G_{42}$</td>
<td>100</td>
<td>&quot; sine &quot;</td>
</tr>
</tbody>
</table>
Note that the $F_c$ form an orthogonal set over the periodic domain, and they each satisfy $\frac{\partial F_c}{\partial y'} = 0$ at the walls $g' = \phi_i n'$.

The total number of degrees of freedom associated with this description is seen to be 19, when all listed values of $n$ and $m$ are considered (see Table 2.1). These result from the horizontal average, two zonal modes, and two phases in $\chi'$ for each of the eight wave modes. Some later results omit $G_{2,0}$ and the possibility $n = 4$, in which case there are then 14 degrees of freedom.

In the above definitions, $N$ is equal to the number of $\chi'$ wavelengths of each mode fitting into the domain. Thus, $n = 1$ represents the longest wave in the longitudinal direction. Substituting (2.2.6 - 2.2.8) into (2.2.5) we see that $q_c^2 = (\frac{2}{3} n)^2 + m^2$. Thus, the ordered pair $(\frac{2}{3} n, m)$ represents the two components of a wave vector describing the shape of the pattern $F_c$, while $q_c^2$ is its squared magnitude. For simplicity we shall identify the modes by $(n, m)_c$.

We next express the flow variables in the $F_c$ set:

\[
\psi'(\chi', y', t) = L_2 f_0 \sum_{i=1}^{N} \psi_i(t) F_i(\chi', y')
\] (2.2.9)

\[
\tau'(\chi', y', t) = L_2 f_0 \sum_{i=1}^{N} \tau_i(t) F_i(\chi', y')
\] (2.2.10)

\[
\Theta'(\chi', y', t) = L_2 f_0 \sum_{i=0}^{N} \theta_i(t) F_i(\chi', y')
\] (2.2.11)

\[
\frac{\omega_2'(\chi', y', t)}{\Delta p} = f_0 \sum_{i=1}^{N} \omega_i(t) F_i(\chi', y')
\] (2.2.12)
The coefficients \( \psi, \tau, \theta, \omega, f_0 \) and the quantity \( b \) are all dimensionless, having used \( L \) and \( f_0^{-1} \) as the units of distance and time, respectively. Using (2.1.2) the coefficients \( \Theta, \zeta \) and \( \tau, \zeta \) are simply related by
\[
\Theta = \tau
\]
The quantity \( L^2 f_0 \) thus represents the basic unit in which temperature is expressed. Note that it also depends upon \( L \).

In reality, reference is made to a finite number \( N \) of terms in (2.2.9-2.2.12), in which case the system is most appropriate for describing flows which vary slowly in \( x' \) and \( y' \). In this case, highly localized fields, such as isolated vortices, cannot be adequately represented.

The truncation to a small finite system also has dynamic restrictions. The most obvious of these is the effect of the restriction \( n \leq 2 \) on the zonal flow. \( \psi, f_0 \) describes an east-west flow of uniform sign in the channel, and is here proportional to the total eastward momentum of the fluid. Hence, it can be changed only by surface friction. However, this flow does contain important variations in \( y' \), so that even the simple advection of a flow pattern \( f(x', y') \) does not amount to a uniform translation in \( x' \).

When \( m = 2 \), \( \psi_2 \) represents a simple asymmetric zonal flow.
corresponding to either high or low latitude westerlies. Only this part of the zonal flow can be changed by a latitudinal redistribution of zonal momentum. Along with $\psi_0$, we thus see that reasonable portrayals of typical mid-latitude zonal wind distributions are possible by the limited representation.

c. Spectral equations

The utility of the description (2.2.6 - 2.2.8) is realized if it holds for all time. Hence, the Jacobians in equations (2.1.3 - 2.1.5) must be expressed in the $F$ set also. We thus define

$$L^2 \mathcal{J}(F_j, F_k) = \sum_{i=0}^{N} C_{i,j} \mathcal{J} F_i \quad (2.2.16)$$

where the coefficients $C_{i,j}$ are defined by:

$$C_{i,j} = \frac{L^2}{F_i} \mathcal{J}(F_j, F_k) \quad (2.2.17)$$

(2.2.16) may be used with (2.2.9 - 2.2.12) and (2.1.2 - 2.1.6) to obtain, after elimination of the $\omega_c$ terms prediction equations for the model:

$$\psi_c = \frac{1}{2} \sum_{j=1}^{N} (a_{j}^2 - a_{h}^2) C_{i,j} \left( \psi_{j} \psi_{h} + \theta_j \theta_h \right) + b_{2h} \frac{1}{a_{i}} \sum_{j=1}^{N} D_{i,j} \frac{1}{b_{j}} \left( \psi_{j} - \theta_j \right) \quad (2.2.18)$$

$$\theta_c = \frac{1}{2} \sum_{j=1}^{N} C_{i,j} \left( (1 + b_{i})^{-1} [\theta_j \psi_{h} + ((B_{i} - B_{h}) - 1) \psi_j \theta_h] \right) + \frac{b_{2h}}{1 + b_{i}} \sum_{j=3}^{N} D_{i,j} \theta_j - \frac{B_{i}}{1 + B_{i}} \left( \theta_c - \psi_{c} \right) - \frac{b_{2h}}{1 + B_{i}} \theta_c + \frac{Q_{i}}{1 + B_{i}} \quad (2.2.19)$$

$$q^*_c = - \sum_{i=1}^{N} \frac{\partial \theta}{\partial \omega_c} + \hat{Q} \quad (2.2.20)$$
Here we have defined the important stability parameter \( B_c \equiv a_0 a_c^2 \)
which represents that given in (1.2.7) for each mode \((n, \hat{m})_c\). Also we define

\[
D_{i,j} = \begin{cases} 
  s_{i,j} + \hat{s}(j, \hat{m}+1) & \text{if } j = \hat{m} \\
  -s_{i,j} & \text{if } j \neq \hat{m}
\end{cases}
\]

where \( s_{i,j} = \begin{cases} 
  1 & \text{if } j = \hat{m} \\
  0 & \text{if } j \neq \hat{m}
\end{cases} \)

By eliminating \( \Theta_c \) instead, the diagnostic relation

\[
\omega_c = \frac{1}{2} \sum_{j=1}^{N_c} C_{i,j} \frac{(\omega B_c)^{-1}}{2} \left\{ -\left[ a_c^2 - (a_j^2 - a_{\hat{m}}^2) \right] \Theta_c \right\}
\]

\[
+ \frac{a_c^2}{\omega B_c} \sum_{j=3}^{N_c} D_{i,j} \Theta_j - \frac{a_c^2}{\omega B_c} \left( \Theta_c - \hat{\Phi} \right) - \frac{a_c^2}{\omega B_c} \frac{\partial \hat{\Theta}_c}{\partial c} - \frac{a_c^2}{\omega B_c} Q_c
\]

is obtained.

It should be noted that (2.2.20) may be omitted from the system

(2.2.18 - 2.2.21) by considering \( \Theta_c \) to be a known constant.

In (2.2.18 - 2.2.21) we have defined \( \kappa \) as the Ekman friction coefficient and \( \kappa' \) as that of internal friction. Both have "units" of inverse non-dimensional time. For the time being, we will leave \( Q_c \) and \( \hat{Q} \) undefined; they represent the non-dimensional potential temperature changes caused by heating.

Since the dot refers to a derivative with respect to dimensionless time, (2.2.18 - 2.2.20) or its counterpart (2.2.18 - 2.2.19) represents a system of ordinary, nonlinear differential equations with time as independent variable. Hence, with initial conditions specified, this system governs the flow for all time.

The nonlinear terms are each multiplied by a coefficient.
whose properties are of great interest. From equation (2.2.17) it is seen to depend upon the three patterns \( F_i \), \( F_j \), and \( F_h \) taken over the entire domain, in such a way that

\[
C_{ijh} = -C_{ijh} \tag{2.2.22}
\]

With the boundary conditions \( \frac{\partial F_i}{\partial x}(Q') = 0 \) and \( \frac{\partial F_i}{\partial y}(Q', y) = 0 \) given earlier, it follows that

\[
C_{ijh} = C_{jih} = C_{hij} \tag{2.2.23}
\]

(2.2.22) and (2.2.23), taken with the system (2.2.18 - 2.2.20) imply that each triple of modes \( F_i, F_j, F_h \) interacts in a manner that conserves the total energy contained in the group. However, a three-way exchange of energy is permitted between the members, which may be thought of as an internal energy transformation between the three separate flow scales. An example of such a system was studied by Lorenz (1960a), which was similar to the choice \( \zeta = \alpha \), \( \eta = \beta \), \( \eta' = \gamma \), \( \eta'' = \delta \) in the present set.

If we take \( j = h \), (2.2.22) shows that \( C_{ijh} \) must be zero; hence, at least three modes are involved in each interaction. This property arises from the Jacobian representation of the advections, which physically is a consequence of the quasi-horizontal nature of the flow. This is in contrast to more general three-dimensional flows, such as those characteristic of non-geostrophic frontal formation (Williams, 1965), where the direct exchange of energy between two modes may take place. For strictly two-dimensional flow, Pedlosky (1962) has commented on the forms of the coefficients \( C_{ijh} \) and considered the implications for energy flow between scales.
Thus in (2.2.18) and (2.2.19) the non-zero $C_{ijh}$ terms represent the nonlinear influence of the flow patterns $F_j$ and $F_h$ on the time changes in the flow pattern $F_i$. The constants $C_{ijh}$ may be termed interaction coefficients for the set of three modes $F_i$, $F_j$, and $F_h$.

If any one of $i$, $j$, or $h$-takes on the value 1 or 2 (see Table 2.1) then the interaction involves the zonal flow as well as the waves. Such coupling will be termed a "zonal interaction." If none of $i$, $j$, or $h$-take on the values 1 or 2 then the zonal flow does not enter, and we have a "wave interaction." Thus, for example, $C_{j02}$ represents a zonal interaction involving the zonal mode $G_{j02}$ and the wave patterns and $G_{j1}$ and $G_{j2}$ (see Table 2.1). Similarly, $C_{307}$ is the wave interaction coefficient coupling the wave mode $G_{321}$ with $G_{11}$ and $G_{12}$.

In (2.2.18 - 2.2.19) these zonal and wave $C_{ijh}$ types could be separated to distinguish between the two advective processes. More importantly, since at each given instant each interaction triad conserves its energy, the sizes of each $C_{ijh}$ could be varied at will without destroying the mathematical consistency of the system. In particular, an individual interaction could be completely neglected by setting its interaction coefficient equal to zero.

Most spectral models have in fact made use of this property to exclude the wave interactions. See, for example, Bryan (1959); Lorenz (1960a, 1962); Kraus and Lorenz (1963); Stackpole (1964). An exception to this was the predictability study by Lorenz (1965), which utilized a system similar to the present one. However, it may be said that the results of this thesis contained the first detailed study of the wave interaction effects for a spectral model.
The flexibility of the model with respect to these interactions is thus the second reason that the spectral approach was taken. It allows a direct comparison of the system behavior with the wave interactions allowed, against the case when they are excluded. Also, the separation of the interaction types allows their individual roles to be examined when they are both present. It should be noted, however, that emphasis will be on the inter-relation of the wave heating and interactions, rather than on the interactions taken as an isolated phenomena.
2.3 Some Characteristics of the Interactions

In this section we shall examine the $C_{ij}h$ in more detail, and outline associated flow phenomena.

Firstly, we note that the orthogonality of the $F_i$ set makes most $C_{ij}h$ equal to zero-in (2.2.17). Those non-zero interactions are indicated in Figure 2.2, where each pair $(n,m)$ represents either $G_{n,m}$ or $G'_{n,m}$ when $h \neq 0$. (See Table 2.1.) The meaning of the diagrams (a)-(j) of Figure 2.2 is the following: each of the small diagrams contains one pair $(n,m)$ whose associated pattern $F_i$ can be changed by interaction with the modes $F_j$ and $F_k$ having "vectors" $(n,r)$; and $(n,m)_k$. The basic mode $(n,m)_i$ is denoted by the symbol $\bigcirc$, while the interacting pair is connected by a line. This diagram thus shows which scale combinations may interact; it does not show which $\kappa'$ phases are involved, however. (This feature confuses the diagrams involving $G_{0j}$, in which case only two scales (but of course three patterns) are involved.

In this case, diagram (a) is interpreted as showing the influence of the two separate phases $G_{n,m}$ on the mode $G_{0j}$. Similarly, diagrams (c)-(j) demonstrate the interaction of $G_{0j}$ and one phase of $(n,m)$ in changing the contribution to the other phase of $(n,m)$.

An example makes these relationships more clear. Referring to diagram (c), we see that flows with patterns $G_{ij}$ or $G'_{ij}$ are affected by interactions with the modes having $(n,m)$ values $(0,1)$; $(0,2)$ and $(1,2)$; $(1,1)$ and $(2,1)$; $(2,1)$ and $(3,2)$; and finally $(2,2)$ and $(3,1)$.
Figure 2.2. Interaction diagrams showing all pairs of waves \((n,m)\) represented by connected dots (•), which may interact to change the "key" mode \((n,m)\) denoted by ◯. Each of parts (a) - (j) refers to a separate "key" mode ◯.
The Roman numerals denote four separate groups of interactions between spatially varying patterns. Group I are zonal interactions involving mode \( (o_y l) \), while group II involve the zonal mode \( (o_y 2) \). Group III representing three interacting wave modes involving values of 1 and 2 only. Group IV interactions involve three wave modes with the three \( N \) values 1, 2, and 3. Note that the wave interactions involving modes with \( \eta=4 \) were specifically excluded, even when those waves were included in the system (2.2.8). This situation occurs in Chapter 4 for a special reason which will be discussed then.

The \( C_{ijh} \) of groups III and IV may be shown to depend upon the different \((\eta_j, \eta)\) combinations quadratically in either of the forms

\[
(n_h m_j + n_j m_h) \quad \text{or} \quad (n_h m_j - n_j m_h)
\]

(2.3.1).

The zonal interaction coefficients are dependent upon \( n_i, n_j, n_h \) in a complicated manner. They are proportional to \( \eta_j \) \((= n_h)\).

The explicit values of the \( C_{ijh} \) for all four interaction groups are listed in Table 2.2. They are shown for all triads of interacting waves pictured in Figure 2.2.

The number of such interactions, when the existence of \( \eta=4 \) in the description is allowed, has the following distribution. For groups I and II there are eight interactions; group III contains four, and group IV contains twelve. When \( \eta=4 \) is ruled out of the system, groups I and II then involve only six interactions each, with III and IV as before.
Table 2.2. Table of $C_{ijk}$ values by groups

Group I - Interaction coefficients of form $C_{1jk}$ for certain $i,j,k$

| $C_{134}$ | $= -0.80028141$ | $C_{156}$ | $= -0.64022513$ |
| $C_{178}$ | $= 2 \times C_{134}$ | $C_{1910}$ | $= 2 \times C_{156}$ |
| $C_{1112}$ | $= 3 \times C_{134}$ | $C_{11314}$ | $= 3 \times C_{156}$ |
| $C_{11516}$ | $= 4 \times C_{134}$ | $C_{11718}$ | $= 4 \times C_{156}$ |

Group II - Interaction coefficients of form $C_{2jk}$ for certain $j,k$

| $C_{236}$ | $= -1.28045026$ | $C_{245}$ | $= +1.28045026$ |
| $C_{2710}$ | $= 2 \times C_{236}$ | $C_{289}$ | $= 2 \times C_{245}$ |
| $C_{21114}$ | $= 3 \times C_{236}$ | $C_{21213}$ | $= 3 \times C_{245}$ |
| $C_{21518}$ | $= 4 \times C_{236}$ | $C_{21617}$ | $= 4 \times C_{245}$ |

Group III - Interaction coefficients of form $C_{3jk}$ or $C_{4jk}$ for certain $j,k$

| $C_{358}$ | $= +1.0000$ | $C_{367}$ | $= -1.0000$ |
| $C_{457}$ | $= -1.0000$ | $C_{468}$ | $= -1.0000$ |

Group IV - Interaction coefficients of form $C_{3jk}$, $C_{4jk}$, $C_{5jk}$ or $C_{6jk}$ for certain $j,k$

| $C_{3714}$ | $= +1/3$ | $C_{3813}$ | $= -1/3$ |
| $C_{4713}$ | $= -1/3$ | $C_{4814}$ | $= -1/3$ |
| $C_{3912}$ | $= +2.0000$ | $C_{31011}$ | $= -2.0000$ |
| $C_{4911}$ | $= -2.0000$ | $C_{41012}$ | $= -2.0000$ |
| $C_{5712}$ | $= -5/3$ | $C_{5811}$ | $= +5/3$ |
| $C_{6711}$ | $= +5/3$ | $C_{6812}$ | $= +5/3$ |
With the latter choice, we see that the system (2.2.7 - 2.2.8) describes fourteen separate patterns of motion $F_i(x')$ representing eight distinct scales $Q^2$, which may interact through twenty-eight separate $C_{i,jh}$ values. The truncated equations are thus too long to write out in full.

Having pointed out the differences between the definitions of the four $C_{i,jh}$ types, let us now examine the general flow characteristics associated with each.

The advections in general produce movement and distortion of initial fields expressed in the $F_i$ set of patterns. We see that those of group I do not in our case involve coupling between wave modes having different $(n, m)$. Each such waveform is thus individually translated along the $\xi'$ axis by this interaction, and may in turn alter the pattern corresponding to $G_{ij}$. Thus, group I interactions are sufficient to describe a steady baroclinic wave regime (Lorenz, 1962).

The zonal interactions of group II allow pre-existing waveforms with given $h$ to be distorted by producing a new $m$ value. They may thus describe a simple "vacillating" wave (Lorenz, 1963). However, no new values of $n$ may appear as a result of this coupling.

The interactions of groups III and IV thus produce maximum distortion of wave patterns, in the sense that new values of both $h$ and $\eta$ may be created. For the group III coupling, a vacillating waveform characterized by a single value of $n$ may thus also produce a new wave with a value $2n$. The group IV interactions admit the simultaneous coupling of waves with $n = 1, 2, \text{and } 3$. 
It would appear that all possible values of \( n \) and \( m \) would occur under the influence of the wave interactions. However, one example where this is not the case is the symmetrical flow given by equation (2.2.4). In that example, only certain combinations of \( n \) and \( m \) exist. Conspicuously absent in our truncated system would be any asymmetries in the zonal flow; group II coupling would then not occur. Also, it can be shown that wave interactions of type III would be absent. Thus, the special symmetry represents an important simplification in the advections which will be used to our advantage in Chapter 4. On the other hand, the climate degeneracy discussed by Stackpole (1964) can be shown to disappear when the wave interactions are included, a happy fact for later results.

In the general case, the system interacts under all four groups I, II, III, and IV. The question may still be asked whether they all play a significant quantitative role in the dynamics. The ever-present latitudinal temperature gradient suggests that the zonal interactions will be important. Indeed, the maintenance of the zonally averaged state through such interactions is defined as the basis for most "general circulation" studies; see, for example, Bryan (1959).

In similar problems, such as turbulent shear flow, it is usually argued theoretically that these zonal interactions are dominant. The basis for this reasoning is that the inter-wave coupling depends upon the detailed nature of the wave state, which is highly variable in time and space. In such a state, it is difficult to maintain coherency between the waves. Hence, it is argued, a net contribution to the flow evolution cannot result from the wave interactions.
Apparently successful descriptions of both turbulent convection (Herring, 1964) and the steady wave regimes in "dishpan" experiments (Lorenz, 1962) have been attained when these fluctuating wave interactions were neglected. However, one has only to look to the theory of the inertial subrange of turbulence for a counter example. In this case, the wave interactions make up the sole energy transfer mechanism for the flow. Only recently has any successful description of the "three way" wave coupling been found (Kraichnan, 1958; 1964).

The reason for the importance of the wave interactions in the latter case lies in the nature of the basic state, which is considered to be a wavy flow concentrated in the "energy containing" scales. On the other hand, the two convective cases mentioned above possess homogeneity of the basic state in at least one direction. These differences suggest that a certain amount of coherency is introduced into the dynamics by spatial variations of the basic state.

Applying this reasoning to the atmospheric case, we might expect that a basic wave state associated with zonally asymmetric heating would play a similar role in organizing the wave coupling. This possibility will be examined semi-quantitatively in the next chapter. For the time being, let us first examine the conditions whereby the wave coupling can be justifiably neglected.

Let us write (2.1.7), for example, in terms of the zonally averaged flow \( \langle \psi_i \rangle \) and a deviation \( \psi'_i \) from it. Subtraction of the \( \psi' \) average of (2.1.7) from itself yields an equation for the eddy field:
(2.3.2) \[
\frac{\partial \hat{q}_i}{\partial t} = -\mathcal{J}(\langle \psi_i', \hat{q}_i' \rangle) - \langle \mathcal{J}(\psi_i', \hat{q}_i') \rangle - \left[ \mathcal{J}(\psi_i', \hat{q}_i') - \langle \mathcal{J}(\psi_i', \hat{q}_i') \rangle \right]
\]

Here \( \langle \hat{q}_i' \rangle \) and \( \hat{q}_i' \) stand for the potential vorticity of the zonal and wave components of flow, respectively. The bracketed term on the right-hand side of (2.3.2) may be written as \( \mathcal{J}(\psi_i', \hat{q}_i') \), and represents the effect of the wave interactions on the wave flow itself.

The first two terms in (2.3.2) arise from the influence of the zonal flow on the waves. We see that the term \( \mathcal{J}(\psi_i', \hat{q}_i') \) can be neglected with respect to these other terms in three cases:

Case (1)--If the \( \hat{q}_i' \) field represents a small asymmetric perturbation on the basic zonal flow, then the wave flow will not directly affect itself by advection. The wave interactions thus enter only after the wave flow is of sufficiently large amplitude.

Case (2)--It may occur that \( \frac{\partial \psi_i'}{\partial y} \) is zero, in which case \( \mathcal{J}(\psi_i', \hat{q}_i') \) is also zero. Such \( \psi_i' \) fields then vary in only one horizontal direction, and represent eddy fields of infinite length in the \( y' \) direction. Due to the boundary condition (2.2.2) at the wall, this is an impossible situation in the channel. However, it may be approximately true when the length scale in the \( x' \) direction is much smaller than in the \( y' \) direction. In this case, \( n \gg m \), and the effect is contained in the dependence of the \( c_{ijh} \) upon \( n \) and \( m \).

Case (3)--If only one wave shape \((n, m)\) exists, then \( \psi_i' \) and \( \hat{q}_i' \) have the same shape and cannot produce new wave patterns by
their interaction. This situation seems relevant to the annulus experiments, but not so much to the atmosphere.

We thus conclude that in flows of bounded $y'$ variation, the waves will not interact appreciably when they represent a small perturbation on the zonally averaged flow. If the wave flow is not so small, it may distort itself if more than one scale of variation is present. This condition is met in the atmosphere, where disturbances in both the planetary and cyclonic scales exist.

In summary, we see that the neglect of the wave interactions introduces a misrepresentation of the advective processes which is of potential importance for the present problem. The omission restricts the degrees of freedom associated with the flow redistribution. By doing so, the relation of the mean and transient flow statistics to the distribution of heat sources is thus altered. The purpose of Chapters 4 and 5 is to examine this question in more detail, taking advantage of the descriptive and dynamic flexibility of the spectral model.

2.4 Heating and Friction Influence

In this section, the simple forms of heating and friction are given and the influence of the vertical motion on their effects is considered.

Let us first discuss the heating influence in the model. Appendix A considers how continentality might be reflected in the heating patterns of a simple atmosphere. A succession of simplifications yields the following form for a mode $F_c$: 
With (2.2.21) and (2.4.1), this produces a contribution to the vertical motion given by

\[ \omega \dot{c} = -\frac{\alpha^2}{H_B c} [\theta^* - \theta^0] \]  

(2.4.2)

The effect of \( \omega \dot{c} \) is to introduce a scale dependence into the total response. With \( b = 0 \), this response is governed by equation (2.2.19):

\[ \dot{c} = \frac{\mathcal{Q}_i}{H_B} \]  

(2.4.3)

The solution of (2.4.3) with initial conditions \( \theta^0 (0) \) is

\[ \theta^0 (t) = \theta^0 \left[ 1 - \frac{\mathcal{Q}_i}{H_B} \right] + \theta^0 (0) \frac{e^{-\frac{\mathcal{Q}_i}{H_B}}}{H_B} \]  

(2.4.4)

We first note that the second term is dependent upon the initial state, and vanishes as \( t \to \infty \). Such a term arises only when \( \mathcal{Q} \to 0 \) (\( \mathcal{R} \to \mathcal{F} \)), and represents the decay of an initial thermal field \( \dot{c}^0 (x, y) \) toward the final equilibrium state \( \dot{c}^0 (x, y) \). The characteristic rate at which this dissipation takes place is given by \( \frac{\mathcal{Q}_i}{H_B} \).

It is seen to contain a weak scale \( (H_B) \) dependence.

In the event that horizontal variations in the heating disappear, the quasi-geostrophic system is unaffected. However, the inclusion of time variable stability necessitates prescribing a form for \( \dot{Q} \) in (2.2.20). Unfortunately \( \dot{Q} \) is a very sensitive function of the state of the atmosphere and underlying surface, in the case of the
real atmosphere. It cannot be accurately described in simple models such as this one. For want of a better choice, we will take

$$\hat{Q} = \left( j + \frac{\hat{K}}{2} \right) (\theta^*_0 - \theta_o) - \frac{\hat{K}}{2} (\theta^*_0 - \theta_o)$$

(2.4.5)

where $\theta^*_0$ and $\theta^*_o$ represent the thermal equilibrium values of $\theta_o$ and $\theta^*_o$, respectively. Here $\theta_o$ is the horizontal average of the potential temperature at the level $P_2$, and is governed by

$$\theta_o = \left( j + \frac{\hat{K}}{2} \right) (\theta^*_o - \theta_o) - \frac{\hat{K}}{2} (\theta^*_o - \theta_o)$$

(2.4.6)

$\hat{K}$ and $j$ are the inverse time scales associated with the heating of the lower layer by the surface and the direct internal heating, respectively.

From theoretical studies of radiative equilibrium, over land and ocean (Manabe and Strickler, 1964; Manabe and M"oller, 1961), and from the fact that sensible heating of the atmosphere is concentrated near the surface, it appears that $\theta^*_o \leq 0$. (2.4.5) and (2.4.6) then contain no mechanism to make the stability positive, so that the long term state of stable stratification must be maintained in this model by the upward heat flux associated with the large-scale baroclinic wave motions. (See equation (2.2.20).)

In summary, we see that the vertical velocity plays a role in the response to both the horizontally and vertically varying heating distributions. In the first case, geostrophic motions are generated by the heating in a manner which favors the larger scales of motion slightly. In the latter case, the strength of the medium-scale wave development must be sufficient to balance the differential heating in the vertical.
Let us now turn to discussion of the frictional mechanism and its influence.

Equations \((2.2.18 - 2.2.19)\) describe the immediate tendency due to the Ekman layer and internal friction when \(\Omega_c \neq 0\). Had \(\Omega_c\) been neglected, the Ekman influence would appear in the vorticity equations, which, replacing \(\zeta\) by \(\Theta\), are

\[
\psi_{cE}^* = -\frac{k}{2} (\psi_c - \Theta_c) \quad \Theta_{cE}^* = +\frac{k}{2} (\psi_c - \Theta_c)
\]

or

\[
(\psi_c + \Theta_c)^*_{E_k} = 0 \quad (\psi_c - \Theta_c)^*_{E_k} = -k (\psi_c - \Theta_c)
\]

The form \((2.4.8)\) shows that, in the absence of vertical motion, the Ekman layer acts as a simple drag on the flow in the lower layer. The constant \(k\) is thus the inverse time constant for this lower layer damping.

The flow at the top of the Ekman layer could have been taken as the geostrophic flow extrapolated to level 4, rather than that at level 3. In this case, \((2.4.7)\) would be replaced by

\[
\psi_{cE}^* = -\frac{k}{2} (\psi_c - 2\Theta_c) \quad \Theta_{cE}^* = +\frac{k}{2} (\psi_c - 2\Theta_c)
\]

Taken by themselves, either \((2.4.7)\) or \((2.4.9)\) would cause an initial flow to decay in the lower layer until the flow at the top of the Ekman layer were zero. The upper flow would remain unchanged, producing the vertical asymmetry expected from a simple surface drag.

With the addition of internal friction,

\[
\psi_{cE}^* = 0 \quad \Theta_{cE}^* = -h^{INT} \Theta_c
\]
such asymmetries would tend to be erased. As the Ekman "spin down" of the lower layer took place, momentum from the upper layer would replenish it. In fluids with constant viscosity, this process is actually very slow and has a negligible effect on the basic Ekman "spin down" (Holton, 1965). In this model, the effect of the internal friction is not necessarily so small.

In reality, both frictional processes induce vertical motion which alter the motion in each layer from that just described. To examine the full case in detail, let us consider the frictional parts of (2.2.18) and (2.2.19) with $b = 0$:

$$
\psi_c = -\frac{\nu}{\alpha}(\psi_c - \theta c) \quad \theta_c = \frac{B_c}{\alpha^2} \int_1^{\frac{h_c}{\alpha}} (\psi_c - \theta c) - \frac{1}{\alpha} \theta_c \right] \quad (2.4.11)
$$

This is a set of linear ordinary differential equations of first order in time. Its two eigenvalues imply the spin-down times for each of two eigenmodes having different vertical structure. The eigenvalues $\lambda_{1,2}$ are:

$$
\lambda_{1,2} = \frac{1}{2\alpha^2} \left[ 1 + \frac{B_c}{\alpha^2} + \frac{B_c}{\alpha^2} \left( 1 + \frac{B_c}{\alpha^2} \right) \right]^{1/2} \quad (2.4.12)
$$

It is seen that either value of $\lambda$ is very sensitive to $B_c$, particularly when $B_c \ll 1$. For given $\sigma_0$, $B_c$ is a measure of the scale of the motion; it will be shown later that the scales of primary interest are those for which $B_c \ll 1$, or for planetary motions, $B_c < 1$.

This scale dependence is most easily seen when we consider the Ekman friction alone. Setting $k' = 0$, the two values of $\lambda$ are

$$
\lambda_1 = -\frac{\nu}{\alpha} \left( 1 + \frac{B_c}{\alpha^2} \right) \quad \text{and} \quad \lambda_2 = 0 \quad (2.4.13)
$$
For $B_\zeta \ll 1$, the second of these has a value near $\lambda_2 \sim -\frac{1}{2}$. The corresponding eigenvector approaches $(\psi_\zeta, \theta_\zeta) = (1, 0)$, implying that the Ekman influence on very long waves is a spin-down of both layers together, and not just the lower layer. The reason for this is that the Ekman outflow sets up a divergence field which in this long wave case is vertically uniform.

In general, it appears that the induced vertical motion can modify the simple tendencies due to friction appreciably. In fact, the spin-down rates for the cases where $\psi_\zeta$ is neglected can be obtained from (2.4.12) by letting $B_\zeta \to \infty$.

In either case, we note that $\lambda_2 = 0$ is also an eigenvalue in the case of pure Ekman friction. Therefore, flow configurations corresponding to the eigenvector for $\lambda_2$ would not spin down at all. The form of this eigenvector is $(\psi_\zeta, \theta_\zeta) = (1, 1)$, corresponding to zero wave amplitude in the lower layer. Similar vertical amplitude distributions are quite common in the real atmosphere, suggesting that an appreciable fraction of the flow is not affected by simple Ekman friction.

Let us now examine the influence of internal friction on the Ekman process. We first note that for very long waves, $B_\zeta \ll 1$, the $k'$ terms in (2.4.12) are very small. Hence, internal friction hardly affects the long waves. Its maximum influence must then be on the short waves. Taking the extreme example $B_\zeta \to \infty$, and assuming the large value $k' = k$, we obtain $\lambda_1 = -1.31 k$, $\lambda_2 = -0.19 k$. Thus, the eventual spinning down of all flow configurations is possible, but those with a minimum amplitude in the lower layer still take a much longer time.
We may summarize these remarks with the following statements:

1. For the moderate scales of interest, \( B \lesssim 1 \), no vertical flow configurations decay as fast as in the case of simple drag. This is true even when an unrealistically large internal friction is present. It does not hold true when the induced vertical motion is neglected.

2. This response dynamics has a scale dependence which acts to minimize the effects of surface and internal friction for very long waves.

3. For such scales, surface friction acts throughout the entire depth of the fluid, at a rate comparable to that for two-dimensional flow.

4. At the same time, an important portion of the flow is virtually unaffected by either surface or internal friction.

Let us now see what use can be made of these facts in choosing parameters for the model.

For an atmosphere with constant eddy viscosity \( \nu_E \), the value of \( h \) can be obtained in a straightforward manner from the theory of the Ekman layer, which states that \( h \) is proportional to \( (\nu_E)^{1/2} \) (Charney and Eliassen, 1949). The internal coefficient of friction \( h' \) can be estimated by assuming the stress between layers to be proportional to \( \nu_E W_z \), where \( W_z \) is the velocity associated with the vertical shear.

For \( \nu_E = 15 \text{ m}^2/\text{sec} \), one thus obtains \( \frac{h}{a} = 0.31 \) and \( h' = 0.22 \) corresponding to inverse decay times of four and six days, respectively. These rates seem reasonable for the effects of small scale turbulence on large scale geostrophic motions. However, the truncation described in section 2.2 for the model ignores all interactions
with eddies having $n > 3$. In addition, the requirement that the motion be periodic in $\varphi'$ over a fraction of the earth's mid-latITUDE circumference does not allow the observed continuous spectrum of motions for $\ldots$

These facts suggest that $h$ and $h'$ be taken larger than the above values. Accordingly, the values $h = \frac{125}{2}$ and $h' = 0.30$ were used in later experiments. While these values seem excessive, it must be remembered that portions of the fluid are hardly affected by the friction, especially in the long wave region where the effects of continentality are most predominant.
3.1 Introductory Remarks

The previous discussion has allowed some general properties of geostrophic motions to be ascertained. For adiabatic, frictionless flow, the governing relationship at level 1 was

\[
\frac{\partial q_i}{\partial t} = -J(\psi_i, q_i)
\]  \hspace{1cm} (3.1.1)

We wish presently to examine in more detail the nature of the dynamics associated with the zonally asymmetric modes of motion.

We first note that equation (2.3.2) of the last chapter showed that the wave flow would evolve adiabatically from an initial state under the dynamics given by (2.3.2):

\[
\frac{\partial q_i^{'}}{\partial t} = -J(\psi_i^{'}, q_i^{'}) - J(\psi_i^{''}, q_i^{''}) - [J(\psi_i^{'}, q_i^{'})']
\]  \hspace{1cm} (3.1.2)

The situations when \([J(\psi_i^{'}, q_i^{'})']could justifiably be neglected were discussed; let us temporarily assume this is so by considering a zonally symmetric flow with small wave perturbations superposed. In this case (3.1.2) is linear in the wave variables \(\psi_i^{'}, q_i^{'}\). (3.1.2) may then be separated into individual equations governing each zonal harmonic. Since the zonal flow is governed by

\[
\frac{\partial}{\partial t} <q_i> = -<J(\psi_i^{'}, q_i^{'})>
\]  \hspace{1cm} (3.1.3)

it depends only upon the separate interactions of each wave mode.
Thus (3.1.3) is then independent of the individual wave phases. Coupled with (3.1.2) this means that the total flow evolves independently of the individual wave phases when the inter-wave coupling is neglected. In such a case, the ensemble of all flows differing only by their individual wave phases evolve similarly in time, despite the variety in flow distribution.

Thus, from the standpoint of the flow field alone, there is then no tendency for the motions to develop particular longitudinal preferences. One objective of this chapter is to show that such behavior is not realistic, in the sense that important space and time fluctuations may arise from the longitudinal structure of the mean flow itself, apart from the direct influence of heating.

One must of course admit that the basic existence of wave fluctuations does not rely upon either the direct or indirect influence of the longitudinal heating variations. Instead, the zonally symmetric flow must be considered a primary source of energy, as first emphasized by Charney (1947). We will devote early discussion in this chapter to the influence of the zonally averaged flow on both the free and forced wave motions.

With the wave interactions omitted (3.1.2) takes the form

$$\frac{\partial \tilde{q}'}{\partial t} + \bar{J}(\bar{q}', \bar{q}') = \frac{\partial \tilde{q}'}{\partial \tilde{\gamma}} \frac{\partial}{\partial \tilde{\gamma}} \bar{q}'$$

(3.1.4)

showing the influence of the zonal flow on the waves*. We note that

* We hereafter drop the primes on the non-dimensional coordinates
\( \frac{\partial}{\partial y} \langle q_1 \rangle \neq 0 \) represents a potentially important dynamical effect, since otherwise \( q'_1 \) is conserved following the zonal flow. In fact \( \frac{\partial}{\partial y} \langle q_1 \rangle \) is of the form

\[
\frac{\partial}{\partial y} \langle q_1 \rangle = \frac{\partial}{\partial y} \left[ \frac{\partial^2}{\partial y^2} \langle q_1 \rangle + \beta y - \frac{f_0}{\theta} \langle \theta \rangle \right]
\]

(3.1.5)

showing that curvature of the zonal flow profile \( \frac{\partial}{\partial y} \langle q_1 \rangle \) produces this dynamic effect, as well as a differential advection. The boundary conditions (2.2.3) suggest that these will always be present in the channel model, even when a less severely truncated flow representation is chosen.

The simple representation given in (2.2.7) limits the \( \frac{\partial}{\partial y} \langle q_1 \rangle \) structure to two possible forms. The \( G_{ci} \) mode represents a latitudinal temperature gradient of uniform sign, and it is also symmetric about the mid-latitude of the channel. In the following analyses, consideration will be restricted to this mode alone.

If we consider single wave shapes \( (\eta, \phi) \), then we may consider the associated variables as elements of a column vector

\[
\mathcal{W}_i = \begin{pmatrix} \psi_i \\ \theta_i \\ \psi_{i+1} \\ \theta_{i+1} \end{pmatrix}
\]

(3.1.6)

where \( i = 3, 5, 7, 9, 11, 13, 15 \) or 17. \( \mathcal{W}_i \) completely describes the amplitude and phase of both the "barotropic" (\( \psi \)) and "baroclinic" (\( \theta \)) vertical modes of the wave. The system (2.2.18, 2.2.19) may then be put in the form

\[
\frac{d}{dt} \mathcal{W}_i = + \mathcal{M}_i \mathcal{W}_i + \mathcal{Q}_i^* \quad (3.1.7)
\]
where \( M_i \) is a 4x4 constant matrix with real elements and in the case of heating \( Q_i^* \) is a constant column vector of the form

\[
Q_i^* = \frac{k}{1 + ik} \left\{ 0, \theta_i^*, 0, \theta_{i+1}^* \right\} \tag{3.1.8}
\]

Those modes \( F_i^{i+1} \) for which \( Q_i^* = 0 \) where \( \theta \) is the null vector, may be termed "free" modes since they are not directly affected by the influence of continentality contained in \( \theta_i^*, \theta_{i+1}^* \). Modes with \( Q_i^* \neq 0 \) will be referred to as "forced" modes.

\( M_i \) depends upon the given basic flow field, physical parameters of the system, and \((n,n)_i\). It has the following form for non-dissipative flow:

\[
\hat{M}_i = \begin{bmatrix}
0 & 0 & -U_i & -T_i \\
0 & 0 & +\hat{T}_i & -\hat{D}_i \\
+U_i & +\hat{T}_i & 0 & 0 \\
-\hat{T}_i & +\hat{U}_i & 0 & 0
\end{bmatrix} \tag{3.1.9}
\]

Here, we have defined

\[
U_i = (1 - \frac{a_i^2}{q_i^2}) \alpha_i \Psi_i - R_i \\
\hat{U}_i = (1 - \frac{B_i}{1 + B_i}) \alpha_i \Psi_i - \hat{R}_i \\
T_i = (1 - \frac{a_i^2}{q_i^2}) \alpha_i \theta_1 \\
\hat{T}_i = \frac{(1 - B_i + B_i)}{1 + B_i} \alpha_i \theta_1 \tag{3.1.10}
\]

\[
\alpha_i = C_i \psi_i + 1, \psi > 0, \text{ where } C_i = \begin{cases} n_i C_{14}, & \psi = 3, 7, 11, 15 \\ n_i C_{165}, & \psi = 5, 9, 13, 17 \end{cases}
\]

\( \psi_i \) and \( \theta_i \) are the barotropic and baroclinic parts of the fixed basic zonal flow.
The matrix elements include non-dimensional frequencies associated with the individual processes. For example

\[ \mathcal{R}_i = \frac{\gamma_i \eta_i}{\Delta_i} \text{ and } \mathcal{R}_i = \frac{\mathcal{B}_i}{1 + \mathcal{B}_i} \]  \hspace{1cm} (3.1.11)

are the frequencies of retrogression of Rossby waves of the forms

\[ \mathcal{W}_i = \begin{pmatrix} \psi_i \\ 0 \\ \theta_i \end{pmatrix} \text{ and } \mathcal{W}_i = \begin{pmatrix} 0 \\ \theta_i \\ 0 \end{pmatrix} \]  respectively.

We have also defined

\[ \mathcal{B}_i = \int_0 a_i^2 \text{ and } \mathcal{B}_1 = \int_0 a_1^2 \]  \hspace{1cm} (3.1.12)

(3.1.7) is an inhomogeneous, linear ordinary differential equation with constant coefficients and the time \( t \) as independent variable. It thus has a solution consisting of a homogeneous part, the "free" solution, which satisfies

\[ \frac{d}{dt} \mathcal{W}_i - \mathcal{M}_i \mathcal{W}_i = 0 \]  \hspace{1cm} (3.1.13)

plus a particular solution \( \mathcal{W}^* \) satisfying

\[ \mathcal{M}_i \mathcal{W}^* = \mathcal{Q}^*_i \]  \hspace{1cm} (3.1.14)

Since \( \mathcal{Q}^*_i \) is a constant, the total solution has the form of a generally transient free wave superimposed upon a forced wave which is fixed in space. When the free mode corresponds to a traveling wave, the total wave pattern is described by a wave varying in amplitude and phase as it travels about a mean state.
A simple but important traveling wave mechanism is that of the "Rossby waves" arising from the beta effect. Their simplest form can be seen by letting $\psi_l$ and $\theta l$ be zero, so that $T_l = \dot{T}_l = 0$, $U_l = -R_l$, $\dot{U}_l = -\dot{R}_l$. In this case the eigenvalues of $\mathbf{H}$ are simply the two frequencies in (3.1.11) which are given by the scale dependent expressions

$$\frac{\sqrt{3} n_c \mathcal{b}}{\alpha_c} \quad \text{and} \quad \frac{\tau_0}{1 + c_i^2 \tau_0} \frac{2}{3} n_c \mathcal{b} \quad (3.1.15)$$

The frequency of retrogression given by $\mathcal{R}_c$ is that of the $\psi$ mode of the wave, while the $\theta$ mode frequency is given in $\mathcal{R}_l$. The first of these corresponds to the non-divergent Rossby wave, while corresponds to the single divergent mode of motion. In a system with more than two layers, or a vertically continuous one, there would be many such divergent modes, of course (Fleagle, 1965).

As will be seen later, $\beta_c < 1$ for the cases of most interest. Thus $\mathcal{R}_c < \mathcal{R}_l$ in these cases, so that the retrogression by the $\theta$ mode is much slower than that of the $\psi$ mode. This can be seen in Figure 3.1, where the dimensional periods, expressed in days, are plotted at the appropriate $(n, m)_c$ values for each wave. These periods were obtained for the scaling $\mathcal{W} = 5,000$ km, so that $n = 1$ corresponds to the second zonal harmonic on the earth. $\mathcal{P}_0$ was taken to be (3 hours)$^{-1}$ and so $\mathcal{L}^2 \mathcal{P}_0 = 250\degree\mathcal{C}$. $\tau_0$ then had the value $0.07$, which is slightly larger than that appropriate for the troposphere. $\beta$ was taken to be $1.7 \times 10^{-11}$ sec$^{-1}$ m$^{-1}$, so $\mathcal{b} = .300$.

Comparison of Figures 3.1(a) and 3.1(b) shows the lengthening of period due to the divergence effect for each wave mode. In fact
Figure 3.1

Fig. 3.1(a) Period of retrogression of non-divergent Rossby wave.
Fig. 3.1(b) Period of retrogression of divergent Rossby wave.

Units: days
Parameters: $W = 5,000$ km $f_0 = (3 \text{ hours})^{-1}$ $b = 0.300 \quad \tau = 0.07$

The longest waves (small $B$) generally retrograde swiftly relative to the shorter wave retrogressions for the $\psi$ mode, but are actually slower in the $\theta$ mode.

One exception to this may be noted in part (a). Comparison of the periods for the waves $(n,m) = (1,1)$ and $(1,2)$ shows that the smaller scale of structure in the $(1,2)$ mode leads to a sharp decrease in the speed of retrogression relative to the wave $(1,1)$. In fact, this $(1,2)$ wave is the least affected by $b$ of any of the waves shown in Figure 3.1, a result of its small value of $n$ and "large" $\eta$ value. Clearly the wave shape as well as its scale is important in determining the dynamical influence of the earth's sphericity.

Let us now examine the influence of the curvature in the zonal
flow upon the waves. We first note that the $U_i$ form in (3.1.10) represents the frequency of the $Y$ mode Rossby wave imbedded in the non-uniform westerly current $\Psi(y) = \Psi F_1(y)$. The effect of the non-uniformity is seen in the coefficient $(1 - \frac{a_i^2}{\alpha_i^2})$, which is equal to 1 when the structure factor $a_i^2$ is set equal to zero.

Thus, we may write $U_i$ as

$$U_i = U_i^* - \frac{a_i^2}{\alpha_i^2} \alpha_i \Psi_i$$

(3.1.16)

where $U_i^*$ is the value of $U_i$ when $a_i^2 = 0$. Thus, for a basic westerly current $\Psi_i > 0$ the extra term $-\frac{a_i^2}{\alpha_i^2} \alpha_i \Psi_i$ represents a retrogression relative to a simple advection. In fact, since $\alpha_i$ is proportional to $\eta_i$, this term resembles that of the $Y$ mode beta effect $R_i$. Further consequences of this will be noted later.

In a similar manner, when $B_i$ is small, $a_i^2$ has a large effect on the term $T_i = \frac{(1 - B_i + B_i)}{(1 + B_i)} \alpha_i \Theta_i$, which represents one of the effects of zonal current baroclinicity upon the wave.

From these remarks it is clear that some quantitative differences are to be expected between this model and a hypothetical atmosphere having a uniform $Y$ distribution of zonal flow. For the most part these differences are not important in the considerations of this chapter, in which case occasional reference will be made to the fictitious possibility $a_i^2 = 0$.

3.2 Free Wave Instability of the Zonal Flow

In this section we briefly review the free solutions of (3.1.7) satisfying

$$\frac{dW}{dt} - \eta_i W = 0$$

(3.2.1)
Our interest is focussed only upon the zonal flow influence on the wave; hence, we consider the adiabatic frictionless form of \( \mathbf{A} \) given in (3.1.9). We temporarily drop the subscripts \( \dot{i} \) on \( \mathbf{A} \) and \( \mathbf{H} \).

With \( \mathbf{A} \) a constant matrix we may assume that (3.2.1) has solutions of the form

\[
\mathbf{u}(t) = \sum_{j=1}^{4} d_j \mathbf{e}_j \exp(\lambda_j t) \tag{3.2.2}
\]

where the \( \lambda_j \) are the four eigenvalues of \( \mathbf{A} \), the \( \mathbf{e}_j \) their corresponding eigenvectors, and the \( d_j \) are obtained from the initial wave state. This expansion is valid when the \( \mathbf{e}_j \) are linearly independent, regardless of the \( \lambda_j \) being distinct. For the cases in this chapter, this was found to be the case.

The \( \lambda_j \) are the roots of the characteristic equation found by setting the determinant \( |\mathbf{A} - \lambda \mathbf{I}| \) equal to zero. Here, \( \mathbf{I} \) is the unit matrix. In general, these roots may be complex. Their imaginary parts correspond to stable oscillations in \( \mathbf{u}(t) \) which may represent, for example, simple traveling waves. The real parts of the \( \lambda_j \) imply exponential growth or decay of the corresponding eigenvector forms.

The special form of \( \mathbf{A} \) in (3.1.9) allows its \( \lambda_j \) to be simply related to the \( 2 \times 2 \) sub-matrix \( \mathbf{A}' \) consisting of the lower left-hand corner elements of \( \mathbf{A} \). In fact, the \( \lambda_j \) of \( \mathbf{A} \) are those of \( \mathbf{A}' \), and hence satisfy

\[
\lambda^2 - \imath \lambda (\mathbf{U} + \mathbf{U}^T) - (\mathbf{U} \mathbf{U}^T + T^T) = 0 \tag{3.2.3}
\]

Here, \( \imath \equiv \sqrt{-1} \).
The two pairs of $\lambda^i$s are then given by

\[
\lambda_{1,2} = \pm \left\{ \hat{\xi} \frac{1}{2} (\hat{U} + \hat{\Theta}) + \left[ T \hat{T} - \left( \frac{U - \hat{U}}{2} \right)^2 \right] \right\}^{1/2}
\]

\[
\lambda_{3,4} = \pm \left\{ \hat{\xi} \frac{1}{2} (\hat{U} + \hat{\Theta}) - \left[ T \hat{T} - \left( \frac{U - \hat{U}}{2} \right)^2 \right] \right\}^{1/2}
\]

(3.2.4)

From (3.2.4) we see that one of the $\lambda^i$s has a positive real part when

\[
\left[ \left( \frac{U - \hat{U}}{2} \right)^2 - T \hat{T} \right] < 0
\]

(3.2.5)

Equation (3.2.5) thus represents the condition for the wave to grow as a free instability of the baroclinic zonal flow. Temporarily setting $b = 0$ and $\lambda_{1} = 0$, this condition becomes $T \hat{T} > 0$ or $B_{\zeta} < 0$. Thus, instability is a possibility only if the waves are of large enough scale: $B_{\zeta} = 1$ then is the "short wave cutoff" which may be deduced from simple physical considerations.

In the channel model with $\lambda_{1} = 0$ but with $b = 0$, the instability criterion (3.2.5) is not so simple for two reasons. First, $(\hat{U} - \hat{\Theta})$ is not zero, contrary to $(\hat{U} - \hat{\Theta})^*$ so that the occurrence of instability depends upon the $\hat{U}_r$ field for given $T$ and $\hat{T}$. Over a long period of time, $\hat{\Theta}$ must be related to $\hat{U}_r$ in a manner making the corresponding flow at the top of the Ekman layer equal to zero. Thus, the instability criterion would also vary with the form of that relation, when possible zonal steady states were considered.

Secondly, even if $\hat{U}_r$ were zero, the cutoff would occur at a shorter wavelength compared to the case without zonal structure:

\[
B_{\zeta} = 1 + B_1 > 1
\]

(3.2.6)

For the purpose of discussion, we may think of $B_{\zeta} = 1$ as the
approximate cutoff point, and limit ourselves to situations where \( B_c < 1 \). For these values of \( B_c \), the beta effect plays an important stabilizing role. The neglect of \( q_1^2 \) in (3.2.5) leads to the condition for marginal instability

\[
T^* \Phi^* = \left( \frac{U^* \lambda^*}{\alpha} \right)^2
\]

or

\[
\left( \alpha_c^2 \theta^1 \right)_{\kappa L} = \left( \frac{R_i}{\alpha} \right) \frac{1}{(1-B_c^2)}
\]

Here, the subscript \( \kappa L \) refers to this "neutral line." In the plane it is defined as a curve in the region \( B_c < 1 \) which separates unstable \( (\alpha_c^2 \theta^1) > (\alpha^2 \theta^1)_{\kappa L} \) from stable \( (\alpha_c^2 \theta^1) < (\alpha^2 \theta^1)_{\kappa L} \) ones. For comparison with later discussion we may also consider the \( \Theta_1, \eta \) plane with different curves for each value of \( \eta \).

Examples of such a curve appear in Figure 3.2 as the dashed lines for \( \eta = 1,2 \). The physical parameters are the same as in Figure 3.1. The other curves in Figure 3.2 have been obtained from the relation (3.2.5) appropriate for the model, rather than from (3.2.7). The parameters are again the same. We have assumed \( \psi = \theta_1 \) as the form of the steady flow. Comparison of the dashed and solid curves shows that the waves for which \( \eta = 1 \) are those most sensitive to the structure of the zonal flow; this is especially true for the longest waves.

Of interest also is the difference between the solid curves for small values of \( \eta \). The stronger influence of \( b \) on the \( \eta = 1 \) waves, anticipated from Figure 3.1, is evident in the much higher values of
Figure 3.2. Baroclinic stability diagram for the model. Heavy solid or dashed lines are "neutral lines" which separate unstable zonal flows (θ̂ large) from neutrally stable ones (θ̂ small) for the case m=1. The thinner solid or dashed lines correspond to the case m=2.

For either m=1 or m=2, the solid lines correspond to the actual model, for which \( \alpha_1^1 = 1 \). The dashed lines correspond to a hypothetical state where \( \alpha_1^1 \geq \sigma \).

The heavy dotted lines represent lines of constant growth rate of unstable disturbances for waves with m=1. They are labelled with their characteristic exponential time scale.

All parameters choices are those shown in Figure 3.1.
\( \theta_l \) required for instability.

For \( \theta_l > (\theta_l)_{mL} \), the roots (3.2.4) are complex, corresponding to traveling waves of a single frequency. Some of these are growing and some are damping as they move. (3.2.5) is used to obtain the growth rates for \( m = 1 \), and are indicated in Figure 3.2 by the dotted lines. The labelling is the characteristic exponential time of growth expressed in days.

With \( \theta_l < (\theta_l)_{mL} \), (3.2.4) gives two roots which are purely imaginary, corresponding to two separate traveling wave modes. These become the separate Rossby wave modes of (3.1.11) when \( \theta_l \rightarrow 0 \), and so may be thought of as Rossby waves in a shearing current, neither of which is non-divergent. At \( \theta_l = (\theta_l)_{mL} \), they coalesce into identical traveling wave modes.

### 3.3 Resonance in Stationary Forced Waves

a. The necessary condition

We now consider the particular solutions \( \mathbf{w}^* \) of (3.1.7) when \( \mathbf{q}^* \) is a constant vector. They satisfy the relation

\[
- \mathbf{\lambda} \mathbf{w}^* = \mathbf{q}^*
\]

(3.3.1)

unless \( \lambda = 0 \) is an eigenvalue of \( \mathbf{\lambda} \), in which case \( (\mathbf{\lambda}^{-1}) \), the inverse of \( \mathbf{\lambda} \) does not exist. In that case, no steady solutions satisfying (3.3.1) are possible. When \( \lambda \) is very small, stationary solutions exist at very large amplitude. Thus \( \lambda = 0 \) is the necessary condition for resonance of the system.

It is not a sufficient condition, however. To see this, we note that if the eigenvectors \( \mathbf{e}_j \), \( j = 1 - 4 \) of \( \mathbf{\lambda} \) are linearly
independent (as we shall assume) then \( \mathbf{M} \) can be expressed as

\[
\mathbf{M} = \mathbf{C}^{-1} \mathbf{D} \mathbf{C}
\]  

(3.3.2)

where \( \mathbf{D} \) is a diagonal matrix of the form

\[
\mathbf{D} = \begin{bmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_4
\end{bmatrix}
\]  

(3.3.3).

\( \mathbf{C} \) has as its \( j \)th column the elements of \( \mathbf{C}_j \):

\[
\mathbf{C} = \begin{bmatrix}
\mathbf{C}_1 \\
\mathbf{C}_2 \\
\mathbf{C}_3 \\
\mathbf{C}_4
\end{bmatrix}
\]  

(3.3.4)

Putting these forms into (3.3.1) and (3.3.2) we obtain

\[
\mathbf{D} (\mathbf{C} \mathbf{W}^*) = \mathbf{C} \mathbf{Q}^*
\]  

(3.3.5)

Using (3.3.3) and (3.3.4) this is equivalent to the four equations

\[
\mathbf{C}_j \cdot \mathbf{W}^* = \frac{1}{\lambda_j} \mathbf{C}_j \cdot \mathbf{Q}^*, \quad j = 1, 2, 3, 4
\]  

(3.3.6)

Here \( \mathbf{C}_j \cdot \mathbf{W}^* \) signifies the vector dot product of \( \mathbf{C}_j \) with \( \mathbf{W}^* \).

In general \( \mathbf{Q}^* \) has a form determined by the specific physical process it represents. For example, when it represents a heating distribution in the wave modes \( \mathbf{F} \) and \( \mathbf{F}^* \), it has the form (3.1.8). \( \mathbf{Q}^* \) may have different forms, however, as when the influence of topography is considered.

From (3.3.6) it is clear that a resonance in \( \mathbf{W}^* \) making the left-hand side large can occur only if the quantity \( \frac{1}{\lambda_j} \mathbf{C}_j \cdot \mathbf{Q}^* \) is large. If \( \mathbf{C}_j \) is not orthogonal to \( \mathbf{Q}^* \), then this occurs if
\[ \lambda_j = \omega \]. This is the necessary condition mentioned above. However, we see that a succession of \( \lambda_j \) yielding smaller values could possibly occur where \( \epsilon_j \cdot \hat{\omega}^k \) became smaller at a faster rate. In this case, resonance would not be said to occur. Such an example will be given later.

While the full solution of (3.3.1) can be studied in detail, let us first concentrate on the resonance condition itself, hoping to gain some insight into the interplay of dynamics and forcing which results in a stationary wave.

The requirement \( \lambda = \omega \) has two immediate implications. First, its real part is zero, so that only neutrally stable waves can resonate in the case of steady forcing. This means that such resonances are observable as mean states for the total wave system, since the free solutions in this case are bounded and periodic. Secondly, the steady resonating wave corresponds to one where this free mode is in fact stationary. Clearly this condition depends strongly upon the advecting current \( \psi_i \) and beta.

To see this relation in detail (3.2.3) implies that \( \lambda = \omega \) when \( U \hat{U} + \hat{T} \hat{\tau} = 0 \) (3.3.7), where we have not restricted ourselves to \( \Omega_i^2 = \omega \). For given values of \( \hat{U} \) and \( \hat{\omega} \) this defines a baroclinic state

\[(T \hat{T})_{RES} = \sim \hat{U} \hat{U} \quad (3.3.8)\]

as the one giving resonance.

In most cases of interest, \( \Omega_i \ll 1 \) and \( T \hat{T} \) is then positive.

We may then define
as the baroclinicity at the neutral line, given \( \mathcal{U} \) and \( \hat{\mathcal{U}} \). Using (3.3.8) it is simple to see that

\[
(\mathbf{T} \hat{\mathbf{T}})_{\mathcal{N}_L} - (\mathbf{T} \hat{\mathbf{T}})_{\mathcal{R}} = \left( \frac{\mathcal{U} - \hat{\mathcal{U}}}{\mathcal{A}} \right)^2
\]

This states that resonance can occur only if \( \mathbf{T} \hat{\mathbf{T}} < (\mathbf{T} \hat{\mathbf{T}})_{\mathcal{N}_L} \), it verifies our earlier remark about the incompatibility of instability and steady resonance. It is interesting to note that these maximum forced wave amplitudes are associated with relatively small values of \( \Theta \), rather than large ones. Thus, the presence of a zonal source of potential energy is not necessarily reflected in correspondingly large forced wave amplitudes.

b. Dependence of the necessary condition on the zonal flow

A convenient description of the above relations is presented in Figure 3.3, which is similar to Figure 3.2. By neglecting \( \mathcal{A}^2 \) from now on, we plot \( |\Theta| \) vs. \( \mathcal{B} \), which, for given \( \mathcal{L} \), is proportional to \( \mathcal{A}^2 \). For \( \mathcal{B} \), a stable region (I) and unstable one (II) are indicated. Resonance may occur in I or on the line \( \mathcal{N_L} \); for a given scale \( \mathcal{B} \) (3.3.10) indicates that the resonant \( |\Theta| \) value relative to \( |\Theta|_{\mathcal{N}_L} \) depends upon \( \mathcal{U} \) and \( \hat{\mathcal{U}} \), or \( \Psi \) and \( b \) for a given wave mode \((n, m)\). Note that we have restored the model subscripts \( \hat{\mathcal{U}} \).

Since \( \mathcal{B} \hat{\mathcal{U}} < 1 \), \( \mathcal{U} \hat{\mathcal{U}} \) is negative, and so \( \mathcal{U} \) and \( \hat{\mathcal{U}} \) are of opposite sign. Therefore, by the definitions (3.1.10), \( \mathcal{U} \) must be negative and \( \hat{\mathcal{U}} \) positive. In this case \( \Psi \) lies between the two
values \( \hat{\Psi}_i = \frac{R_i}{\alpha_i} \) and \( \hat{T}_i = \frac{R_i}{\alpha_i} \) \( (3.3.11) \)

These values correspond to the two values of \( \Psi_i \) which would yield stationary Rossby waves of the \( \Psi \) and \( \Theta \) modes, respectively, in the absence of \( \Theta_i \). Thus, resonance is in this case impossible for easterly flow.

The particular choice \( U_i = \hat{U}_i \) along with \( (3.3.10) \) gives

\[
(\hat{T}_i \hat{\Psi}_i)_{\text{NL}} = (\hat{T}_i \hat{\Psi}_i)_{\text{NL}}
\]

In this case, resonance occurs for \( \Psi_i \) values on the neutral line \( \text{NL} \), when \( \Psi_i \) takes on the single value

\[
\left( \hat{\Psi}_i + \frac{\hat{\Psi}_i}{2} \right)
\]

\( (3.3.12) \)

This critical value is largest for the long waves, as can be verified from \( (3.3.11) \).

Elsewhere in region I there exist two resonant \( \Psi_i \) values.

For example, when \( (\hat{T}_i \hat{\Psi}_i)_{\text{NL}} \approx 0 \), two values of \( \Psi_i \) occur, given by \( \hat{\Psi}_i \) and \( \hat{\Psi}_i \). In this case, the apparent resonant modes are simply the two stationary Rossby wave modes. Again the largest values of \( \Psi_i \) at resonance occur for the longest waves. It is thus apparent that the very long waves \( (B_i \ll 1) \) resonate for a wide range of \( \Psi_i \) and do so with at least one relatively large value of \( \Psi_i \). The shorter baroclinic waves \( (B_i \ll 1) \) resonate for small individual values of \( \Theta_i \) and \( \Psi_i \).

For a given scale (value of \( B_i \)), the range of \( \Psi_i \) values between \( \hat{\Psi}_i \) and \( \hat{\Psi}_i \) are possible resonances, depending upon the vertical shear \( \Theta_i \). We may thus refer to that region of \( \Psi_i \) values as a "conditionally resonant" one.
Figure 3.3. A resonance diagram for the model. Dotted areas indicate possible resonant combinations of $|\theta_1|$, $B_\nu$

$\lambda$ and $b$ are considered as given by the values of Figure 3.1.
Region I - The necessary resonance condition is satisfied if $\Psi_i$ takes on either of two values, both satisfying $\hat{\Psi}_i < \Psi_i < \tilde{\Psi}_i$.
Region II - Resonance is impossible.
Region III - The necessary resonance condition is satisfied by either of two $\Psi_i$ values satisfying $\Psi_i < \hat{\Psi}_i$ and $\Psi_i > \tilde{\Psi}_i$, respectively.
Let us now expand the resonant condition (3.3.7) using the definitions (3.1.10) and again neglecting $\alpha_i^2$. We obtain

$$\left(\alpha_i \psi_i - \bar{R}_i \right)\left(\alpha_i \psi_i - \bar{R}_i \right) = \frac{B_i - 1}{B_i + 1} \left(\alpha_i \theta_i\right)^2$$

(3.3.13)

For given values of $b, \xi$, this represents a complicated relation between $\psi_i, \theta_i$ and $B_i$, being quadratic in each of these.

This relation is valid for all values of $B_i$; having discussed the case $B_i < 1$, let us turn to consideration of the shortest waves, for which $B_i > 1$. In this case $\theta_i > 0$, so $U_i$ and $\bar{U}_i$ must be of the same sign for resonance to occur, using (3.3.7). Therefore $\psi_i$ must lie outside of the conditionally resonant region for long waves defined above. Thus, resonance may now occur for negative values of $\psi_i$, as well as for both small and large positive values.

This is indicated in region III of Figure 3.3 where very large $\theta_i$ values resonate with large absolute values of $\psi_i$. As $\theta_i > 0$ the resonant $\psi_i$ values approach $\bar{U}_i$ and $\bar{U}_i$, as in region II.

The above remarks concerning the important dependence of resonance upon $\psi_i$ are more easily seen in Figure 3.4. This figure contains six regions, separated by the lines $\chi, \gamma, B_i = \omega$ and $B_i = 1$, $\Theta_0$ is again considered as fixed. Only the stippled regions I, II and III of possible resonance are identified.

Due to the form of (3.3.13), given values of $\psi_i$ and $B_i$ yield a single value of $|\theta_i|$ giving resonance. Thus, this diagram more easily shows how the resonance condition is altered by a change in $|\theta_i|$. 
The line labelled $\chi$ satisfies the stationary wave condition
$$\psi_i = \bar{\psi}_i,$$
while $\gamma$ satisfies
$$\psi_i = \tilde{\psi}_i.$$  \hfill (3.3.14)

Each of these values makes the left-hand side of (3.3.13) zero, so that along these lines $|\theta_i|_{RE}=0$. It follows that $|\theta_i|_{RE}=0$ in regions I, II and III.

For a given $B_c < 1$ \hfill (3.3.12) immediately shows that the maximum resonant value of $|\theta_i|$ occurs for $\psi_i$ values midway between the lines $\chi$ and $\gamma$. This corresponds to the dashed neutral line of Figure 3.3. It is labelled as line $\gamma N$ here; along it, the resonant $|\theta_i|$ values vary with $B_c$, and in fact $|\theta_i|_{RE} \to \infty$ as $B_c \to 0$ on it.

For $B_c > 1$, regions II and III define possible resonant combinations of $\psi_i$ and $B_c$. The resonant $|\theta_i|$ values at each point are zero along $\chi$ and $\gamma$, and increase away from them into the interiors both of II and III. Therefore, particularly as $B_c \to \infty$, most values of $\psi_i$ are capable of resonating with a sufficiently large $|\theta_i|$. This occurs physically because as $B_c \to \infty$, the motion in each layer becomes more uncoupled from its neighbor, and both are unaffected by beta. In this case, the resulting phase speeds are those of disturbances in each individual layer. Hence, resonance depends only upon $\psi_i \pm \theta_i$, and not upon $\psi_i$ or $\theta_i$ individually.
c. Scale dependence of the resonance condition

Up to this point, we have examined the influence of \( \Psi_i \) and \( \Theta_i \) upon the occurrence of resonance for various values of \( B_i \). In the model, we have seen that \( \Psi_i \) and \( \Theta_i \) are not independent over long periods of time. In the real atmosphere, the zonal wind undergoes smaller seasonal variations near the surface than higher up, suggesting a similar qualitative relation. This suggests that the influence of \( B_i \) on resonance be examined for sequences of specific \( \Psi_i, \Theta_i \) states.

Referring to Figure 3.4 the points \( P_x \) and \( P_y \) correspond to a particular zonal state \( \Psi_i, \omega_i, k \Theta_i = 0 \). From the standpoint of the necessary resonance condition (3.3.13), it is satisfied at the two separate \( B_i \) values corresponding to the points above.

This possibility of two resonating scales can be examined for the general case by rewriting (3.3.13) in terms of \( \alpha_i^2 \) (\( = \frac{B_i}{\omega_i} \)):

\[
\left( \alpha_i^2 \right)^{\frac{1}{4}} \left( \frac{1}{\alpha_i^2} - \frac{1}{\alpha_i^2} \right)^{\frac{1}{2}} \left( \frac{1}{\alpha_i^2} - \frac{1}{\alpha_i^2} \right)^{\frac{1}{2}} \left[ \alpha_i^2 \Psi_i \Theta_i - \frac{2}{3} b \right] = 0
\]

(3.3.15)

This expression has two real or two complex roots. By its definition, only values of \( \alpha_i^2 \) which are real and positive have any meaning. Thus, if the roots are complex, no resonant values exist. If they are real, then either zero, one or two resonant scales exist, depending upon the signs of the roots.

This possibility of two resonant scales is not in agreement with
Figure 3.4. A resonance diagram for the model. Dotted areas I, II and III represent possible combinations of resonance. Along lines X and Y, the necessary resonance condition is satisfied for $|\theta_1| = 0$. Elsewhere in I, II and III resonance may occur for $|B_i| \neq 0$. In region I, $|\theta_1|$ must be moderate or small, with an exception near $B_i = 0$ and when the resonant value is large near line LN. $\theta_1$ and $\theta$ take the values given in Figure 3.1.
the conclusions of previous investigators. For example, Smagorinsky's (1952) work on a continuous model or Saltzman's (1965) similar version interpreted the single resonance condition as corresponding to a single resonating wavelength. Gilchrist (1952) analyzed a two level model similar to this one and also noted only one resonant value of $\lambda$. Contrary to those conclusions, the present analysis suggests the possibility of two resonances in the two layer model, and possibly more in the case of many layers, or a continuous model.

d. Simple stationary waves forced by heating

Having discussed the necessary resonance condition, let us now examine the solution of (3.3.1), which has the form

$$\mathbf{w} = -\mathbf{H}^{-1} \mathbf{Q}$$

(3.3.16)

In general, $\mathbf{Q}$ represents all inhomogeneous effects influencing this particular wave mode set having $(\eta, \eta')$. Thus, apart from the heating terms arising from continentality, it may also include the effects of $\mathbf{w}_h$ due to the flow of a zonal current over surface longitudinally varying topography. In the former case (3.1.8) shows that $\mathbf{Q}$ may contain non-zero elements in the second and fourth members only; in the latter case, all elements of $\mathbf{Q}$ could be non-zero. While our primary interest is centered on the influence of heating asymmetries, we will find reference to the mountain forcing to be useful.

In the case of heating alone $\mathbf{Q}$ has the form given by (3.1.8) and represents a fixed heat source containing a weak scale dependence. The variable, or dissipative, heating may be included
by appending to the form (3.1.9) of \( \tilde{m} \) the matrix:

\[
\mathbf{H} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & -\frac{b}{\ell + \beta} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{b}{\ell + \beta}
\end{bmatrix}
\]  

(3.3.17)

As a first example, we will neglect the inclusion of \( \mathbf{H} \). For simplicity we will also take \( \theta_c^0 = 0 \) in (3.1.8). Assuming \( \tilde{m} \) to be non-singular, we find that (3.3.16) gives the particular solution

\[
\Psi_i = \frac{b}{\ell + \beta} \theta_i^* \left\{ \frac{T_i}{U_i \theta_c^* + T_i} \right\} \\
\Psi_{i+1} = 0
\]

(3.3.18)

As a special case of (3.3.18) we first note that if \( \theta_1 \) is taken to be identically zero, then \( \Psi_i \) is zero while

\[
\theta_i = \frac{b}{\ell + \beta} \theta_i^* \left\{ \frac{1}{\theta_i^*} \right\}
\]  

(3.3.19)

Thus, a vertically averaged wave response develops only as a result of its coupling to the thermal mode \( \theta_i \) through \( \theta_1 \). In addition, we see that (3.3.19) is independent of \( U_i \) the frequency of the non-divergent Rossby wave. Thus, resonance occurs only with \( \dot{U}_i = \omega \) (3.3.20), corresponding to the divergent Rossby mode being stationary. This is in contrast to the result given by the necessary resonance
condition alone, which in this case would imply that $U_i = 0$ would also yield resonance. The difference is accounted for by the mathematical reasons given in Section 3.1.

This conclusion is somewhat contrary to the qualitative argument given by Rossby (1939), which carried the implication that a resonant response to heating would occur when $U_i = 0$ for the simple case $\Theta_i = 0$. Similar arguments have been given since. The present results show that internal heating resonance in a barotropic zonal current occurs only when $\dot{U}_i = \Theta_i = 0$; i.e., when the divergent Rossby wave is stationary, rather than the non-divergent one. For large scales the speed of the current giving resonance is much less than it would be in the case of the non-divergent Rossby wave resonance.

However, when the basic current is baroclinic, the situation changes somewhat. For this case, qualitative arguments of the above sort are noticeably lacking in the literature. In the solution (3.3.18), with $\dot{\Theta}_i \neq 0$, one finds that resonance may occur when $\dot{U}_i$ is not too different from $\dot{\Theta}_i$. Resonance is then permitted because the mode of the wave is coupled to the thermal mode, in which case the non-divergent beta response dominates the wave movement. Thus, the baroclinicity is seen to play an important part in creating a second resonance mode through its implied vertical wind shear, rather than through its available potential energy.

e. Stationary mountain waves

Some further insight into these two resonant points may be gained by considering the response to flow over a mountain system.
In this case we choose $Q^*$ to be of the form

$$Q^* = \{ \omega, 0, \omega \hat{R}_c - \hat{R}_c \} \quad (3.3.21)$$

where $|\lambda_c| \ll |\hat{R}_c|$. We will not be concerned with the detailed forms of $\lambda_c$ and $\hat{R}_c$. Neglecting the contribution of $H$ to $M$ and including $\theta_i \neq 0$ the general solution is

$$\Psi_c = -\left[ \frac{T_c \hat{R}_c + \hat{U}_c \hat{R}_c}{U_c \hat{U}_c + T_c \hat{T}_c} \right], \quad \Psi_c^{+1} = 0$$

$$\Theta_c = -\left[ \frac{T_c \hat{R}_c - U_c \hat{R}_c}{U_c \hat{U}_c + T_c \hat{T}_c} \right], \quad \Theta_c^{+1} = 0 \quad (3.3.22)$$

Naturally the same necessary condition for resonance is found as in the heating case: $U_c \hat{U}_c + T_c \hat{T}_c = 0$. However, here it is also a sufficient condition, for, by taking $\theta_i = 0$ (3.3.22) yields

$$\Psi_c = -\frac{\hat{R}_c}{U_c} \quad \text{and} \quad \Theta_c = +\frac{\hat{R}_c}{\hat{U}_c} \quad (3.3.23)$$

Thus, the flow of a barotropic current over a mountain system may excite large amplitudes when either Rossby wave mode is stationary:

$$\Psi_i = \hat{\Psi}_c \quad \text{or} \quad \Psi_i = \hat{\Psi}_c$$

Mathematically the difference between this and the heating case is due to their separate $Q^*$ forms. For the heating case, (3.1.8) had a form which was orthogonal to that of the non-divergent Rossby wave mode. Physically this lack of matching between this Rossby wave and heating field occurs because the wave heating field creates
differential motion between the layers, while the non-divergent Rossby wave involves their joint behavior. Only when the zonal vertical shear is present to "tip back" an initially baroclinic wave mode may a quasi non-divergent resonant mechanism operate.

f. Influence of dissipation

Aside from these features of resonance, additional points of interest in the heating solution (3.3.18) may be noted. Firstly, the ratio of the amplitudes of $\psi_i$ to $\theta_i$ is

$$\left|\frac{T_i}{U_i}\right|$$

(3.3.24)

This shows how strongly the existence of a barotropic wave response is tied to the baroclinicity $\theta_i$. This ratio, depending upon $\psi_i$, $\theta_i$, $b$ and $c_i$, partially describes the vertical structure of the wave.

The description is completed by knowledge of the phase difference between the $\psi$ and $\theta$ fields. In the simple case shown in (3.3.18) these fields are in phase with each other for $U_i > 0$ and ninety degrees out of phase with the heating field. As a resonant state is passed, the denominator $U_i \hat{\theta}_i + T_i \hat{T}_i$ changes sign, implying a discontinuous phase shift of 180 degrees for both modes of the wave.

The above $\psi - \theta$ phase relations are characteristic of "equivalent barotropic" wave states in which the wave transports no sensible heat to the north or south. Hence the zonal flow represents neither a source nor a sink of available potential energy, in agreement with our earlier comments. The out of phase relation between $\theta$ and the heating implies that the heating does not generate available potential energy of the wave, and is consistent with the fact that no dissipative
mechanism is contained in $M$ as given by (3.1.9).

While these solutions are interesting from the purely dynamical point of view, certain of their features are not in agreement with the real atmosphere. In particular, both the wave phase and its vertical structure seem more constrained than is observed. We may therefore ask how the solutions might be changed by the addition of dissipative processes into $M$.

To start with, we will simply add $H$, given by (3.3.17) to $M$ given by (3.1.9). The solution is then

$$\psi_c = \frac{b}{1 + B_c} \left\{ \psi_{c+1} \right\} \mathcal{T}_c \left( \psi_c \mathcal{U}_c + \mathcal{T}_c \mathcal{T}_c \right) / \left[ (\psi_c \mathcal{U}_c + \mathcal{T}_c \mathcal{T}_c)^2 + \psi_c^2 \left( \frac{b}{1 + B_c} \right)^2 \right]$$

$$\psi_{c+1} = \frac{b}{1 + B_c} \left\{ \psi_c \mathcal{U}_c + \mathcal{T}_c \mathcal{T}_c \right\} / \left[ (\psi_c \mathcal{U}_c + \mathcal{T}_c \mathcal{T}_c)^2 + \psi_c^2 \left( \frac{b}{1 + B_c} \right)^2 \right]$$

$$\theta_c = \frac{b}{1 + B_c} \left\{ \theta_c \mathcal{U}_c + \mathcal{T}_c \mathcal{T}_c \right\} / \left[ (\theta_c \mathcal{U}_c + \mathcal{T}_c \mathcal{T}_c)^2 + \theta_c^2 \left( \frac{b}{1 + B_c} \right)^2 \right]$$

$$\theta_{c+1} = \frac{b}{1 + B_c} \left\{ \theta_c \mathcal{U}_c + \mathcal{T}_c \mathcal{T}_c \right\} / \left[ (\theta_c \mathcal{U}_c + \mathcal{T}_c \mathcal{T}_c)^2 + \theta_c^2 \left( \frac{b}{1 + B_c} \right)^2 \right]$$

(3.3.25)

The squared amplitudes of $\psi$ and $\theta$ are

$$|\psi_c^2 + \psi_{c+1}^2| = \frac{\left\{ \frac{b}{1 + B_c} \right\}^2 \theta_c^2 \mathcal{T}_c^2}{(\psi_c \mathcal{U}_c + \mathcal{T}_c \mathcal{T}_c)^2 + \psi_c^2 \left( \frac{b}{1 + B_c} \right)^2}$$

$$|\theta_c^2 + \theta_{c+1}^2| = \frac{\left\{ \frac{b}{1 + B_c} \right\}^2 \theta_c^2 \psi_c^2}{(\theta_c \mathcal{U}_c + \mathcal{T}_c \mathcal{T}_c)^2 + \theta_c^2 \left( \frac{b}{1 + B_c} \right)^2}$$

(3.3.26)
We first note that resonance of the stationary wave is now prohibited. In fact, \( \left| \Theta_i^z + \Theta_{\ell,t}^z \right| \) never exceeds the thermal equilibrium value \( \left| \Theta_{\ell,t}^z \right| \). It is thus clear that the wave amplitude is primarily determined by the total heating process, with the dynamics playing a crucial modifying role. It is not surprising then that the \( \psi \) and \( \theta \) modes are again in phase; the dissipative heating is unable to induce energy exchange with the zonal flow.

On the other hand, the phase of the wave is now variable, contrary to the case without this dissipation. In fact, the \( \Theta \) field is no longer 90° out of phase with that of the \( \Theta^x \) field. Of course, it is out of phase with the total heating field \( k(\Theta^x - \Theta) \). The latter is now a variable quantity, both in magnitude and phase.

The introduction of dissipation also influences the solution (3.3.25) when the original resonance condition \( (\psi \hat{\Theta}_i + \Theta_i \hat{T}_i) = 0 \) is satisfied. In this case \( \psi_i, \theta_i \to 0 \) and the solution is then out of phase with the original non-dissipative one. At that point the solution represents the mid-point of a continuous phase shift as resonance is passed. The detailed states at such points are of course extremely sensitive to the dissipation.

Finally, we note that the inclusion of internal friction does not add any new elements to \( \hat{\mu} \); it merely changes the values of each of the terms in (3.3.17). Thus, most of the above points are valid observations when the heating is a fixed quantity and the dissipation is of a simple internal friction nature.

These results are still unrealistic in one sense; the vertical
tilt of the wave is still zero. It appears that the mechanism of Ekman friction is responsible for the vertical tilt of the stationary forced waves. This is in agreement with the more complete analyses of Smagorinsky (1952) and Saltzman (1964) and the comments of Wiin-Nielson (1961). It is physically reasonable, since we have seen that Ekman friction acts as a drag which is stronger in the lower layer for all except the longest waves, thus introducing vertical asymmetry. Dynamically, this influence on the vertical tilt has great potential importance, for it allows the wave to transport heat to the north or south, and thus exchange energy with the zonal flow.

Due to the complexity introduced by the Ekman elements into the solutions will not be shown here. The results of the following section contain their influence. Before proceeding to them, let us briefly summarize the key points of this section.

We have seen that a general fixed heating field may excite large wave amplitudes in those free modes which are both stationary and stable on the given zonal flow. In this model, there are generally two such waves.

The "long" waves, of greater scale than that corresponding to the short wave cutoff, may resonate under the proper conditions. These are met when $\theta_i$ is not too large and when $\lambda_i \omega$ is in the correct range of values. Both conditions are critically dependent upon $b$. In fact, with $\theta_i \sim \omega$, the resonating wave corresponds to the stationary, divergent Rossby Wave.

The "short" waves, which possess no baroclinic source of energy
in the zonal flow, may nevertheless resonate freely for a wide range of $\psi$ and $\theta$. $b$ is not an important influence on their development.

The primary "dynamic" influence on resonance is thus a simple wave translation by the zonal current $\psi$, with $b$ and $\theta$ influencing the wave speed relative to that current. The effect of energy sources and sinks, apart from the fixed heating field, is of secondary importance to the development of resonance.

On the other hand, the vertical structure of the forced wave away from resonance is greatly influenced by dissipative processes. In particular, only Ekman friction seems capable of producing vertical tilt in the waves. Hence, the baroclinic energy of the zonal flow becomes available to the waves only in the presence of this surface friction.
3.4 Examples of Stationary Wave Solutions

The results of this section portray more realistic solutions than those given in section 3.3, since they contain the influence of all dissipative mechanisms acting simultaneously. The calculations were performed on a high speed digital computer, so that a wide range of all the parameters could be investigated.

For the most part, however, the dissipative elements were not varied. Their values then corresponded to those of the main experiment discussed in Chapter V. Some of these were rather large. \( h \) had the value of \( h = 0.25 \). \( K \) was taken to be \( 0.66 \) while \( \kappa = 0.3 \) was not so large. These values accentuate the effects of dissipation on the solutions, which thus compliment the simpler solutions of section 3.3.

Only a few of the many solutions will be shown here; as expected the curves can be somewhat complicated due to the interplay of the several physical effects. The solutions differed according to the separate choices of \( \xi_0 \), \( b \), \( W \), \( h \), and \( m \). \( \Psi_1 \) and \( \Theta_1 \) were varied together from zero to large values following the relation \( \Psi_1 = 2 \Theta_1 \) which is appropriate for steady zonal flow when the Ekman effect is obtained from the flow extrapolated to \( P \).

As a first example, we choose \( W = 33.3 \) km, so that \( n = 1 \) is equivalent to earth wave three. We take \( \epsilon_0 = 0.14 \) and \( b = 2 \). The solution for \( (\eta, m) = (4, 2) \) is shown in Figure 3.5(a), where the \( \Psi_1 \) values label the curves for the \( \Psi \) (continuous line) and \( \Theta \) (dashed line) wave modes. The arrow indicates the state of thermal equilibrium \( \Theta_1 \), toward which the system is being driven by the heating.
**Figure 3.5**

Steady wave solutions for waves (1,2) and (1,1).

Lines define sequences of steady wave states computed for different prescribed zonal flows $\Psi_i$ and $\Theta_i$. Note that a portion of the solutions in part (b) are missing. Wave states are normalized with respect to their forcing magnitude $\left| \Theta_{i+1}^* \right|$. Solid lines are wave $\Psi$ modes. Dashed lines are wave $\Theta$ modes. ▽ denotes the thermal equilibrium field $\Theta^*(x,y) = \Theta_i(x,y) + \Theta_{i+1}^* F(x,y)$ for the wave $(n,m)$ i.e. $\Psi_i (2\Theta_i)$ values are indicated along each solution sequence.

Parameters: $W = 3333$ km, $\alpha = .14$  $\beta = .20$

$k = .125$  $k' = .030$  $\rho = .066$

Ekman drag calculated from the interior flow extrapolated to level $f^4$.

Part (a) - $(n,m) = (1,2)$  $(\dot{\epsilon} = 5)$

Part (b) - $(n,m) = (1,1)$  $(\dot{\epsilon} = 3)$
We note that the two resonant combinations of $\psi, \theta$ which might be expected without dissipation are not discernible here. Instead, an entire range of resonant $\psi$ values occurs within the conditionally resonant region defined before. In this range, the phase of the wave changes rapidly from positions upstream from the fixed heating for small $\psi$ to downstream positions for larger $\psi$. At the same time, the vertical tilt of the wave changes sign; at low $\psi$, the wave tilts eastward with height, while it tilts westward for large values of $\psi$. Such characteristics were pointed out by Smagorinsky (1952), and their relevance to seasonal changes of wave patterns was discussed by Gilchrist (1953). In accord with this behavior, the thermal part of the wave is seen to exhibit a more nearly constant phase than the barotropic part.

Figure 3.5(b) shows the solution for wave $\left(1,1\right)$ for the same value of $\omega$. This wave mode is of significantly larger scale than the previous one. Thus, the greater width of the conditionally resonant region of $\psi$ is reflected by the occurrence of two regions of relatively large amplitudes, with the dominant one at large $\psi$ and the other at small $\psi$. These may be associated with the resonant modes of the last section, allowing for some alteration by the influence of dissipation. Also, dissipation apparently limits the phase shift and structure change to just the larger $\psi$ resonance.

For this same wave mode, we may temporarily abandon the constraint between $\psi$ and $\theta$ by setting $\theta = 0$ and varying $\psi$ only. The solid and dashed lines of Figure 3.6(a) show the result for $\psi$ and $\theta$, respectively. As expected from section 3.3, the "upper" resonance
Steady wave solutions for waves \((1,1)\) and \((3,2)\).

Lines define sequences of steady wave states computed for different prescribed zonal flows \(\Psi_i\) and \(\Theta_i\). Wave states are normalized with respect to their forcing magnitude \(\Theta_{\zeta+i}^*\). Wave states are normalized with respect to their forcing magnitude \(\Theta_{\zeta+i}^*\). Values are indicated along each solutions sequence. \(\nabla\) denotes \(\Theta_{\zeta+i}^*\) for the wave \((\eta, \kappa)\).

Part (a) - solutions for wave \((1,1)\) for two values of \(b\).

Other parameters are those given in Figure 3.5.

Solid line: \(\Psi\) mode solution in the case \(b = .2\).
Dashed line: \(\Theta\) mode solution in the case \(b = .2\).
Dotted line: \(\Psi\) mode solution in the case \(b = 0\).
Dot-dashed line: \(\Theta\) mode solution in the case \(b = 0\).

Part (b) - solutions for wave \((3,2)\) for the parameter choices given in Figure 3.5. \(b = .2\).

Solid line: \(\Psi\) mode solution
Dashed line: \(\Theta\) mode solution
disappears, with maximum amplitudes appearing only at low $\psi_f$ values. Referring to the dotted counterparts of these lines, we see the result of further setting $b = \omega$. In this case, the wave phase is never upstream from $\theta^*_c + l$. Comparing these lines with those for which $b \neq 0$, and focusing upon the case $\psi_f = 0$, we thus see the effect of $b$ on a simple "monsoonal" model of the forced wave response. It is seen that for such long waves, the beta effect exerts a large effect on both the amplitude and phase response of the wave, especially in the barotropic mode.

Next, in Figure 3.6(b), we consider a shorter wave for which $\beta_c = 1.3$. This wave lies very near the short wave cutoff scale when we take $\alpha^2 = 1$. Here, $\gamma_m$ are given by (3.2), and $W$ is again 3,300 km. We again constrain $\psi_f = 2\theta_f$. As expected, the maximum wave amplitudes now occur over a narrow region at small values of $\psi_f$. Further, the relative response of this wave is always smaller than that of the much longer waves considered above. The scale dependence of the heating response contained in (3.1.8) seems sufficient to account for this difference.

Additional solutions, not shown here, were found for different values of $W, h$ and $\gamma_0$. With $W = 5,000$ km and $\gamma_m$ taken as $(1,1)$ the resonance separation shown in Figure 3.5(b) became an extreme. In fact, for realistic values of $\psi_f$ and $\theta_f$, only the "lower" resonance occurred.

With $h$ decreased by a factor of 5, but with $W, \gamma_0, n$ and $m$ taken as in Figure 3.5(b), the solution was changed only quantitatively at points away from resonance. However, in the resonant range
the solution loop was closed in the reverse manner to that shown in Figure 3.5(b).

We may consider this as an extreme example of the sensitivity of, resonant solutions to the physical processes. In addition, a similar phase sensitivity was noted in certain solutions when the stability $\Omega$ was varied, showing that the details of the basic state hold a similar importance.

In the real atmosphere, fluctuations of many time scales cause a wide range of dynamic and dissipative changes in $\mathcal{M}$ to occur, even during a given time of the year. In accord with Rossby's (1939) comments, the present results suggest that the linear mechanisms contained in $\mathcal{M}$ could account for a significant portion of the observed variety in the forced wave motions. In later chapters, we shall examine some nonlinear mechanisms by which the atmosphere might generate space-time irregularity when the elements of $\mathcal{M}$ are not so variable.

3.5 Barotropic Waves on a Basic Wavy Flow

The results of the last section suggest that the long waves have a maximum response to zonally asymmetric heating. In addition, these stationary wave motions do not generate large transient components by baroclinic growth on the zonal flow. These characteristics suggest that a combination flow made up of the zonal plus long wave ones might be considered as fixed. In this section we assume this to be the case, and examine the influence of the long wave upon smaller scale motion systems.

Since our purpose here is that of demonstration, we choose the simplest dynamical set exhibiting a wave influence by considering all
to be zero in the expansion \((2,2,10,2,11)\). We thus consider a barotropic fluid; neglecting dissipation, such a flow is governed by the relation

\[ \psi_{ij} = \sum_{j,k=1}^{N} \tilde{a}_{i}^{j,k} \psi_{j} \psi_{k} + \frac{2}{\Omega} \sum_{j=3}^{N} \tilde{b}_{i,j} \psi_{j} + \mathcal{L}_{i} \]  \tag{3.5.1}

where the \(\mathcal{L}_{i}\) arises from the topographically-produced \(\omega \psi\). Let us choose a basic state described by \(\tilde{\psi}(x,y) = \psi_{1}(x,y) + \psi_{3}(x,y)\)

where \(\psi_{1}\) and \(\psi_{3}\) are fixed constants. We then ask what wave motions having \((n,m) \neq (1,1)\) may occur. To answer this, we will assume that the qualitative behavior of the system may be seen by considering (3.5.1) for specific modes \(j,h\) for which \(C_{ijh} \neq 0\). One of the simplest such systems is obtained with

\[ (n,m) = (2,1) \quad \text{and} \quad (3,2) \]  \tag{3.5.2}

Their direct wave interaction with wave \((1,1)\) is pictured in Figure 2.2(h).

The linear equations for this set are of the form

\[ \frac{d\psi}{dt} = \mathcal{N} \psi + \mathcal{L} \]  \tag{3.5.3}

where we define

\[ \psi = (\psi_1, \psi_3, \psi_8, \psi_{14}) \]  \tag{3.5.4}

For the moment, \(\mathcal{L}\) is left arbitrary, while \(\mathcal{N}\) has the form

\[ \mathcal{N} = \begin{bmatrix}
0 & 0 & -\psi_{17} & +\psi_{17} \\
0 & 0 & +\psi_{13} & -\psi_{13} \\
+\psi_{17} & -\psi_{17} & 0 & 0 \\
-\psi_{13} & +\psi_{13} & 0 & 0
\end{bmatrix} \]  \tag{3.5.5}
Here, $\nu_1$ and $\nu_{13}$ are the frequencies of non-divergent Rossby waves imbedded in the zonal current $\nu_1$, and are defined in (3.1.10).

Assuming $\nu_3 > 0$, $\nu_\gamma$ and $\nu_{13}$ are now defined as

$$\nu_\gamma \equiv \frac{(\nu_3^2 - \nu_1^2)}{2\sigma_3} \gamma_3 > 0, \quad \nu_{13} \equiv \frac{(\nu_3^2 - \nu_1^2)}{\sigma_3} \gamma_3 > 0 \quad (3.5.6)$$

they represent the coupling between the wave modes $(n, m) = (2,1)$ and $(n, m) = (3,2)$ through the fixed wave flow $\nu_3$. $\nu_\gamma$ and $\nu_{13}$ are both positive, indicating that the wave system $\nu_3 F_3(y, \gamma)$ is barotropically stable to infinitesimal perturbations of the form

$$\psi(y, \gamma) = \psi_\gamma F_\gamma(y, \gamma) + \psi_3 F_3(y, \gamma) + \psi_{13} F_{13}(y, \gamma) + \psi_{14} F_{14}(y, \gamma) \quad (3.5.7)$$

The influence of the fixed wave $\nu_3 F_3$ upon the transient behavior of the shorter waves is seen by momentarily neglecting $\gamma_2$ in (3.5.3) and considering the eigenvalues of $\gamma_2$. We note first that $\gamma_2$ has the form assigned to $\gamma_1$ in equation (3.1.9). Therefore we may replace the ordered terms $(\nu_\gamma \nu_{13}, -\nu_\gamma + \nu_{13})$ in equation (3.1.9) by $(\nu_\gamma \nu_{13}, -\nu_\gamma + \nu_{13})$

(3.5.8)

and then consult equations (3.2.3) and (3.2.4) to obtain the two pairs of eigenvalues:

$$\lambda_{1,2} = \pm i \left\{ \frac{1}{2} (\nu_\gamma + \nu_{13}) \pm \sqrt{(\nu_\gamma \nu_{13} + \frac{(\nu_\gamma - \nu_{13})^2}{2})} \right\} \quad (3.5.9)$$

$$\lambda_{3,4} = \pm i \left\{ \frac{1}{2} (\nu_\gamma + \nu_{13}) - \sqrt{(\nu_\gamma \nu_{13} + \frac{(\nu_\gamma - \nu_{13})^2}{2})} \right\}$$

Since $\nu_\gamma \nu_{13}$ is positive, all eigenvalues are imaginary, and can be shown to correspond to travelling wave systems. In fact,
when $\psi_3 = 0$, the eigenvalues reduce to $\lambda = \pm U_1 \pm U_3$, with each separate frequency then associated with simple travelling waves of a single scale. However, when $\psi = \psi_0$, the eigenvalues are simply $\lambda = \pm i (\psi_1 \psi_3)^\frac{1}{2}$, $\lambda = \pm i (\psi_1 \psi_3)^\frac{1}{2}$. In this case, the solutions represent travelling waves whose individual amplitudes undergo two periods of maxima and minima during each cycle. The two different scales then share the same period of oscillation, however.

This same characteristic holds when $\psi = 0, b \neq 0$, so that the associated eigenvectors may be said to represent a "wave packet", for which a unique scale is difficult to define. The reason for this is that any local measure of the scale would depend upon the phases of the two waves, which vary in time. These time variations are of a special sort, as can be seen from the following: the average longitudinal phase velocity over one period is proportional to $\lambda/n_c$, and is thus smaller for the shorter wave, since $\lambda$ is a constant. Thus, the two wave trains move relative to each other, periodically changing the shape of the wave pattern as a whole.

For the case considered here, the coupled waves are the modes $(\eta, m) = (2, 1)$ and $(3, 2)$. Since both the $\eta$ and $m$ values are different, the system undergoes shape changes in both $\eta$ and $\gamma$ as the system progresses. Such distortions may be thought of as arising from the gross "steering" effect by the fixed wave, coupled with a distortion caused by its non-uniform structure.

This distortion and steering is of a purely transient nature, with no time-mean effect evident. An interesting question is how might direct steady state relations between the waves develop. To
answer this, let us consider particular solutions of (3.5.3) when \( \frac{d}{dt} \psi = 0 \) and \( \mathcal{R} \neq 0 \). \( \mathcal{R} \) here represents topographical forcing of any phase in either or both of the shorter scales.

For example, \( \mathcal{R} \) could have the simple form

\[
\mathcal{R} = (0, 0, \mathcal{R}_8, \mathcal{R}_{14})
\]  

(3.5.10)

This appears in the same mathematical form as the forcing for the solution (3.3.22). However, in the present case, this choice of \( \mathcal{R} \) determines two particular phases of the forcing, one for each scale. These phases are not arbitrarily chosen, since they are now measured relative to the basic wave flow \( \psi^*_3 F_3(\psi, y) \). Thus, the above choice of \( \mathcal{R} \) represents a special one, in the sense that the position and shape of the forcing field are those of a particular situation.

The steady solution of (3.5.3) follows from (3.5.5), the correspondence (3.5.8), (3.1.9) and (3.3.21-3.3.22):

\[
\psi_7 = -\left[ \frac{+\psi_7 \mathcal{R}_{14} + \psi_13 \mathcal{R}_8}{\psi_7 \psi_13 + \psi_17 \psi_{13}} \right], \quad \psi_8 = 0
\]

(3.5.11)

\[
\psi_{13} = -\left[ \frac{+\psi_7 \mathcal{R}_8 + \psi_17 \mathcal{R}_{14}}{\psi_7 \psi_{13} + \psi_17 \psi_{13}} \right], \quad \psi_{14} = 0
\]

Suppose now that the form of \( \mathcal{R} \) were simplified in such a way that the forcing occurred in only one scale. For example, we may take \( \mathcal{R}_{14} = 0 \). In this case a stationary response is nevertheless produced in both scales, the response in the unforced scale
We thus see that a stationary forced wave system imbedded in a given wavy flow will itself contain energy in more than one scale, even when only one scale is forced.

This coupling is a feature which is found in later, more realistic results. Its importance stems from the fact that, without wave interactions, the forced flow is excited only in those scales which are contained in the forcing field. Each such response may differ from the impressed pattern of that scale in amplitude and phase, as we saw in the last section. However, with the inclusion of wave coupling, new scales may also be introduced into the response.

For the scales which are forced, the solutions (3.5.11) show that the presence of wave interactions alters the response expected in its absence. This is more clearly seen when the condition for resonance

$$U_7 U_{13} - V_7 V_{13} = 0$$

is examined. For $$V_{13} = 0$$, or simply by neglecting the wave coupling by setting $$C_{3 y_{13}} = C_{3 y_{13}} = 0$$, the system resonates when $$U_7 = 0$$ or $$U_{13} = 0$$. These situations correspond to the stationary non-divergent Rossby waves of each wave. When $$V_{13} \neq 0$$, resonance does not occur at these points, but at neighboring states satisfying $$U_7 U_{13} = V_7 V_{13} = 0$$. In summary, we conclude that the presence of longitudinal variations in a fixed flow of large scale may alter the mean response of
the shorter waves to the forcing, especially by the introduction of "new" scales of response. At the same time, the transient components of these shorter waves are advected by the fixed flow in a manner producing variable movement and distortion of initial patterns.

3.6 Baroclinic Instability of a Wave Flow

The preceding section described the influence of a fixed wave pattern upon a wave system supposed initially given. The fixed flow was barotropically stable and hence did not represent an energy source for the case considered. We now consider the possibility of the thermal field of a fixed long wave acting as a source of available potential energy for the shorter waves.

Let us consider a barotropic zonal current \( Y, \) \( F, (y), \) with \( \Theta = \omega. \) From section 2.2 it is clear that this zonal flow contains no source of energy for any of the waves. Instead, let us assume that a fixed wave pattern \( \Theta, \) \( F, (k, y), \) exists in the thermal field alone; we therefore are considering a simpler stationary wave state than those given in sections 3.3 or 3.4. We then ask whether this wave field can support baroclinically unstable waves of other scales.

The general mathematical analysis of such a system is extremely complicated, reflecting that of the physical process. We will elude full mathematical rigor by again referring the problem to a simpler system of the homogeneous form

\[
\frac{d}{dt} \nu = \mathcal{P} \nu
\]

(3.6.1)

Here \( \mathcal{P} \) contains the zonal current \( Y, \) and the thermal wave mode
We restrict \( \mathcal{V} \) to be a vector containing variables for two scales, and we again choose these to be waves \((n, m) = (2, 1)\) and \((3, 2)\), so that

\[
\mathcal{V} = (\psi_7, \theta_7, \psi_8, \theta_8, \psi_{13}, \theta_{13}, \psi_{14}, \theta_{14})
\]  

(3.6.2)

For simplicity, we ignore the effects of heating or dissipation.

\( \mathcal{P} \) is then given by the 8 x 8 matrix

\[
\begin{pmatrix}
0 & 0 & -U_7 & 0 & 0 & 0 & 0 & +T_7 \\
0 & 0 & 0 & -\hat{U}_7 & 0 & 0 & -\hat{T}_7 & 0 \\
+U_7 & 0 & 0 & 0 & 0 & -T_7 & 0 & 0 \\
0 & +\hat{U}_7 & 0 & 0 & +\hat{T}_7 & 0 & 0 & 0 \\
0 & 0 & 0 & +T_{13} & 0 & 0 & -U_{13} & 0 \\
0 & 0 & -\hat{T}_{13} & 0 & 0 & 0 & 0 & -\hat{U}_{13} \\
0 & -T_{13} & 0 & 0 & +U_{13} & 0 & 0 & 0 \\
+\hat{T}_{13} & 0 & 0 & 0 & 0 & +\hat{U}_{13} & 0 & 0 \\
\end{pmatrix}
\]

(3.6.3)

The elements \( T_i \) and \( \hat{T}_i \) are not of the form (3.1.10), but are here defined by:

\[
T_7 = \frac{a_1^2 - a_2^2}{a_1 a_3} C_{3717} \theta_3 > 0, \quad \hat{T}_7 = \frac{(1 - B_1 + B_2)}{(1 + B_2)} C_{3714} \theta_3
\]  

(3.6.4)

\[
T_{13} = \frac{a_1^2 - a_{13}^2}{a_{13} a_3} C_{3717} \theta_3 > 0, \quad \hat{T}_{13} = \frac{(1 - B_1 + B_3)}{(1 + B_{13})} C_{3714} \theta_3
\]  

(3.6.4)
Here we have taken $\theta_3 > 0$. Note that these elements are similar in form to those defined in (3.1.10), when the basic flow amplitude and structure is generalized from the zonal to the wave modes. Thus, $\hat{T}_7$ and $\hat{T}_{13}$ are positive for sufficiently small values of $\mathcal{F}_0$.

The quantities $U_1$, $\hat{U}_1$, $U_{13}$, and $\hat{U}_{13}$ are defined as in (3.1.10). Hence, they again may be identified with the Rossby wave phase speeds of $\mathcal{U}$ and $\hat{\mathcal{U}}$ modes of both wave scales.

We now wish to find the conditions under which (3.6.1) may admit exponentially growing solutions. To do this, we examine the eigenvalues $\lambda$ of $\mathcal{P}$, which are found to satisfy the relations

\[ \lambda^2 = \left[ \frac{\hat{T}_7 \hat{T}_{13}}{2} \right] + \frac{1}{2} (U_{13} \hat{U}_{13}) \left[ 1 - \frac{q T_{13} T_7}{(U_{13} \hat{U}_{13})^2} \right]^{1/2} \]

(3.6.5)

\[ \lambda^2 = \left[ \frac{\hat{T}_7 \hat{T}_{13}}{2} \right] + \frac{1}{2} (U_{13} \hat{U}_{13}) \left[ 1 - \frac{q T_{13} T_7}{(U_{13} \hat{U}_{13})^2} \right]^{1/2} \]

These may be factored to yield the eight values

\[ \lambda = \pm \left\{ \pm \frac{1}{2} (U_{13} \hat{U}_{13}) + \left[ \hat{T}_7 \hat{T}_{13} \left( \frac{U_{13} \hat{U}_{13}}{2} \right) \right]^{1/2} \right\} \]

(3.6.6)

\[ \lambda = \pm \left\{ \pm \frac{1}{2} (U_{13} \hat{U}_{13}) - \left[ \hat{T}_7 \hat{T}_{13} \left( \frac{U_{13} \hat{U}_{13}}{2} \right) \right]^{1/2} \right\} \]

(3.6.7)

Equations (3.6.6) and (3.6.7) each contain two pairs of $\lambda$'s. Each such pair represents the frequency of a wave system having the form of the corresponding eigenvectors of $\mathcal{P}$. Generally speaking, these eigenvectors involve components of both wave modes.
By comparing the forms (3.6.6) and (3.6.7) with (3.2.4), we see that interpretation of (3.6.6) and (3.6.7) is similar to that given for equations (3.2.4). In the present case, the terms now contain mixtures of quantities depending upon the two wave scales. For example, the beta influence included in the terms $U_7$ and $U_{13}$ is seen in equations (3.6.6) to give an influence involving the non-divergent Rossby wave speed of wave $(\eta, \eta) = (2,1)$ and the divergent one of $(\eta, \eta) = (3,1)$.

More importantly, even when $b$ and $\alpha_i\zeta$ are neglected, we note that the translation by $\gamma_i$ of the wave system past the fixed wave field lessens the possibility of instability, because $(U_7 - U_{13}) \neq 0$. In this case $\gamma_i$ decreases the coherency in the wave coupling, and hence damps the growth mechanism. In a sense, the zonal advection may be thought of as forcing the wave packet to sample both favorable and unfavorable regions of growth relative to the fixed wave $\Theta_3 \Theta_3(\eta, \eta)$.

While effects such as these are of great importance, let us now examine the necessary condition for instability by neglecting both beta and $\gamma_i$. Using (3.6.4), we see that a real positive $\lambda$ may then exist if either of two conditions is satisfied:

$$1 - B_7 + B_3 > 0 \iff B_7 < 1 + B_3 \quad (3.6.8)$$

and

$$B_{13} < 1 + B_3 \quad (3.6.9)$$

In a sense, it appears that we are now dealing with two modes of baroclinic energy release, rather than one. Each one corresponds
in a general sense to the "short wave cutoff" of equation (3.2.6).
Since \( \mathcal{B}_3 > \mathcal{B}_1 \), the critical value of \( \mathcal{B}_n \) or \( \mathcal{B}_3 \) is increased over the values found for the zonal flow in (3.2.6). This suggests that a wider range of baroclinically unstable waves might be excited when the basic baroclinic flow is distributed unevenly in space.

The actual occurrence of instability, given sufficiently small values of \( \mathcal{B}_7 \) or \( \mathcal{B}_{13} \), depends upon the influence of \( b \) and \( \psi \), upon the radicals in expressions (3.6.6) and (3.6.7). In them, \( \vartheta_3 \) must be large enough to satisfy either of the inequalities

\[
\frac{\mathcal{T}_7 \mathcal{T}_{13}}{\mathcal{T}_7 \mathcal{T}_{13}} > \left( \frac{\mathcal{U}_7 - \mathcal{U}_{13}}{2} \right)^2
\]

(3.6.10)

or

\[
\frac{\mathcal{T}_{13} \mathcal{T}_7}{\mathcal{T}_{13} \mathcal{T}_7} > \left( \frac{\mathcal{U}_{13} - \mathcal{U}_7}{2} \right)^2
\]

(3.6.11)

The mathematical similarity of this form to the condition (3.2.5) is obvious, in a general sense.

We may end these remarks by a further observation. Consider the possibility that the fixed wave has a scale intermediate to those contained in the perturbation vector \( \mathcal{U} \). In this case, Fjortoft (1953) has shown that purely two dimensional flow could be unstable, neglecting any influences of beta or other flow components. Such a state of two-dimensionality is approximately true when \( \mathcal{T}_0 \) is very large in this model. In this case, one might expect an outflow of kinetic energy from the fixed wave. However, for very small values of \( \mathcal{T}_0 \), one also expects the fixed wave to give up its available potential energy.
It is thus clear that the eddies of intermediate scale are dominated by a complex combination of barotropic and baroclinic effects. On the other hand, the earlier results of this section suggest that the longer waves may play a more constant role as baroclinic energy sources for the shorter waves.

3.7 Final Remarks

The results of this chapter have been obtained in a manner lacking mathematical generality. However, some qualitative insight has been gained about the nature of the steady wave response to forcing, and to its subsequent influence on both forced and free motions in other wave scales.

The results suggest that, for a realistic range of zonal states, the stable long waves would develop as stationary motions with greatest amplitude. This would seem particularly true when it is noted that the large scale heating field is dominant in those wavelengths.

The shorter baroclinically unstable waves would be mainly transient since their steady parts were far from resonance for realistic zonal flows. Hence, due to heating asymmetries alone, they would exhibit little preference for one longitudinal phase over another.

These conclusions followed from a simple linear analysis which allowed interactions of the waves with the zonal flow only. Sections 3.5 and 3.6 discussed the possible phenomena which might be introduced by the wave interactions. It was found that the large scale thermal wave pattern seemed potentially capable of generating growing waves in preferred localities. During their lifetime, such motions
would be further guided and distorted by this long wave field, thereby introducing further longitudinal variations in the transient flow field.

The steady motions of intermediate scale would be altered somewhat by their interaction with the long wave flows. In particular, unforced waves could develop through their coupling to a forced wave. In such cases, the shapes and phases of the steady patterns would be related to both the forcing field and to the long wave field.

In reality, the transient and steady modes of response cannot be separated as in these simple results. Instead, they evolve together, each influencing the other. To learn anything more, the true nonlinear coupling must be allowed to develop free of any mathematical constraints by studying a close approximation to the system.

Such solutions were obtained by step-wise numerical integration over a long period of time. They are presented and discussed in Chapters 4 and 5. Appendix B discusses the choice of a computational method capable of insuring the correctness of qualitative features of the flow. It also summarizes the characteristics of several methods as applied to both linear and non-linear systems.

The method finally decided upon was a three step procedure performed at each three hour time step. The solution was thus obtained by twenty-four extrapolations per day of real time. As a by-product of the numerical solutions, certain functionals of them were also computed. These included energy transformations for each wave mode of the system, as well as assorted energies. Data cards from the
original solutions were used in later more detailed analyses. These included mapping and spectral routines.

The computer programs were set up for flexibility in studying the influences of individual physical and dynamical processes. In particular, the instantaneous advective rates of change were separated into those involving zonal interactions and those involving wave interactions. In this manner, the wave interactions could be completely neglected, or their effects singled out at each time step.

A large choice of realistic parameter combinations was possible. The preceding analysis has demonstrated the types of dynamical behavior most likely to occur. Thus, the numerical solutions concentrated upon the non-linear corrections to the preceding analysis for a very restricted range of parameters. Special attention will be paid to the role of inter-wave coupling and its influence on longitudinal climatic variations in later discussion.
CHAPTER 4. SOLUTIONS WITH PERIODIC ELEMENTS

4.1 Introduction

The results to be presented in this chapter represent an attempt to isolate the principal qualitative behavior of the solutions to the governing set of equations (2.4.6 and 2.2.18 - 2.2.20). These solutions may be thought of as sequences of points in a 38-dimensional phase space \( \mathcal{R} \) whose coordinates are the variables \( \psi_i, \theta_i, \xi_i \) and \( \theta_i \). Here \( i = 1, 2, 3, \ldots, 18 \). Each state of the system may be thought of as a 38-dimensional vector extending from the origin of \( \mathcal{R} \) to that state point. This vector changes in time in a unique manner described by the governing equations. The total solution may thus be thought of as the continuous trajectory of a particle moving through \( \mathcal{R} \) with a single trajectory through each point. The system is then said to be deterministic.

In Chapter 3 we examined the properties of certain linear analogues to the system. Some of the solutions corresponded to stationary waves of given \( (n_m) \), and were thus represented by a single point in \( \mathcal{R} \). For each such wave the projection of the state vector onto the planes \( (\psi_i, \psi_{i+1}) \) and \( (\theta_i, \theta_{i+1}) \) described the wave state completely. Such representations also are permissible when the waves are time varying, in which case they trace out curves in these polar diagrams. For certain simple curves, the corresponding space-time fluctuations described by the wave system are also simple. To see this, we note that a wave system described by
\[ \psi(\chi, \eta, t) = \psi_i(\ell) \bar{f}_i(\chi/\eta) + \psi_{i+1}(\ell) \bar{f}_{i+1}(\chi/\eta) \]  

(4.1.1)

has the form

\[ \psi(\chi, \eta, t) = \sin m_i \psi \left[ \psi_i(\ell) \cos \frac{2}{3} (\eta, \chi) + \psi_{i+1}(\ell) \sin \frac{2}{3} (\eta, \chi) \right] \]  

(4.1.2)

with \( i = 3, 5, \ldots, 17 \). At each latitude \( \eta \), this describes a sinusoidal disturbance with time-varying amplitude and phase:

\[ \psi(\chi, \eta, t) = \sin m_i \psi \left| \psi_i(\ell) + \psi_{i+1}(\ell) \right| \cos \left( \frac{2}{3} n_i \chi - \xi_{(n, m)}(\ell) \right) \]  

(4.1.3)

Here \( \xi_{(n, m)}(\ell) \) is defined by

\[ \xi_{(n, m)}(\ell) = \tan^{-1} \left( \frac{\psi_{i+1}(\ell)}{\psi_i(\ell)} \right), \quad \left| \psi_i^2 + \psi_{i+1}^2 \right| \]

is the length of the state vector in the \( (\psi_i, \psi_{i+1}) \) plane, while

\( \theta_{(n, m)}(\ell) \) is the angle it forms with the \( \psi_i \) axis. \( \xi_{(n, m)}(\ell) \) is the phase angle of the wave measured in units relative to its own wavelength.

In the special case \( \psi_i(\ell) = \alpha \cos \omega t, \quad \psi_{i+1}(\ell) = \alpha' \sin \omega t \)

where \( \alpha \) and \( \alpha' \) are constants, the trajectory is periodic; with \( \alpha = \alpha' \), it has constant amplitude \( \alpha \) and steady phase progression

\[ \frac{d}{dt} \xi_{(n, m)}(\ell) = \omega = \text{constant}. \]

In real space, expression (4.1.3) then takes the form

\[ \psi(\chi, \eta, t) = \alpha \sin m_i \psi \cos \frac{2}{3} n_i (\chi - C_i t) \]

where \( C_i = \omega / \sqrt{3} n_i \) is the eastward phase speed of this simply traveling wave. With \( \alpha \neq \alpha' \), neither the amplitude nor the phase speed is constant, since the \( \gamma \) trajectory then describes an ellipse.

If instead \( \psi_i(\ell) = \alpha \cos \omega t \) and \( \psi_{i+1}(\ell) = \alpha' \cos \omega t \) the amplitude oscillates with frequency \( \omega \) in time, with the phase
\[ c_n^2 + \frac{1}{\omega_n} \frac{\lambda_2}{\omega'_n} = \text{constant.} \] In this case, expression (4.1.3) is seen to represent a standing oscillation of the wave pattern at a particular longitude.

Most wave systems do not behave so simply as either of these cases. Indeed, they need not be periodic. For example, the initial growth of wave perturbations as an instability, described in sections 3.2 or 3.6 would be represented by trajectories spiralling outward in the polar diagram. In reality, such growths are eventually halted by either dissipation or the non-linearities of the governing laws. Let us now discuss the phase space properties of such a system; in particular, we consider systems exhibiting various degrees of regularity, or lack thereof.

A completely irregular system is of course a non-periodic one. For such systems a precise analogue is impossible. Lorenz (1963a) has shown that the trajectories of quasi-analogue states ultimately diverge from one another in this case. That is, the evolution of a non-periodic system is unstable to slight changes in its initial state. The important implications of this for the problem of predictability of atmospheric motion were also discussed by Lorenz.

This mathematical instability has its physical counterpart - it may be regarded as a continuous process of "cyclogenesis" somewhat similar to that of sections 3.2 or 3.6. For such a physical system, the sensitivity of the behavior to initial conditions obscures the effects arising from externally imposed energy sources. It is quite possible that this sensitivity is felt for weeks or even months, and
could account for the difficulty one finds when trying to account for fluctuations in mean monthly flow patterns. Part of Chapter 5 is devoted to consideration of this question.

In contrast to the non-periodic flow, some systems exhibit "perfect regularity" in the sense that they are periodic. Observable periodicities are in some sense stable, for the trajectory of a state initially adjacent to the periodic motion remains a neighbor to the closed trajectory for all time. However, it need not remain close to its initial neighbor on the closed trajectory. Instead, we merely require that the periodic trajectory be approached by the deviating state as \( t \to \infty \) In such a case, the system's stability is of a special type known as "asymptotic orbital stability."

Systems fulfilling these criteria are said to possess a stable limit cycle. These cycles occur in non-linear, non-conservative systems such as our present one. The necessity of the latter property is clear enough: all initial states with their individual energy levels must eventually approach the closed trajectory, and hence take on a single specific value of the energy at each instant. Once this trajectory is reached, the periodicity demands that the energy sources and sinks must be such that they cancel out when averaged over one period of the cycle. At a given moment they are not in equilibrium, and hence are important in determining the specific form of the limit cycle.

The limit cycle description is completed by knowledge of the energy conserving dynamics, some insight into which may be gained
by the following reasoning. Since the limit cycle is orbitally stable, a slight deviation from it may nevertheless grow. It may do so in a very special manner, however, for its trajectory must always lie very nearly along that of the periodic solution. Since the latter is bounded and closed, so must be the deviation, which this possesses a maximum.

Put in this way, it seems reasonable to think of the limit cycle as an instability process of the energy-conserving dynamics which is limited by the time-varying energy sources and sinks. In our system, the conservative processes are those of advection. Apart from the beta effect, they appear in the non-zero coefficients $C_{i,j}$. The generation and dissipation mechanisms are represented by $k^p$, $k^r$, and $k'$. The initial conditions, after a sufficiently long time, do not influence the (increasingly) periodic solution.

We thus see that the single periodic limit cycle yields information about the governing laws alone, and is not contaminated by the influence of initial states. Studying the limit cycle is thus an excellent way to reach our goal of isolating the inter-relations of heating, advective redistribution and dissipation. In addition, all information is contained in a single period of the system, dismissing the need for replacing precise analysis by statistical analysis.

The remaining portions of this chapter are devoted to the study of solutions whose essential properties correspond to those of the limit cycle.
4.2 Choice of Parameters

In this section we briefly consider the parameters and implied flow regimes for the experiments of this chapter. We also list some of the observed characteristics of each experiment. See Table 4.1 for this information.

Experiments I-IV had a great deal in common, and their results are analyzed in sections 4.3 - 4.5. The choice $\mathcal{W} = 5,000$ km was made for these experiments, corresponding to longitudinal wave numbers of 2, 4, 6 and 8 on the earth. Experiment V was a special exception, where the choice $\mathcal{W} = 10,000$ km was made, corresponding to zonal harmonics 1, 2 and 3. The fourth harmonic was omitted there for reasons given in section 4.6, when that experiment is considered separately.

When $b \neq 0$, its value was computed from $\beta = \frac{\partial f}{\partial y} \bigg|_{y=\phi_0} = 1.7 \times 10^{-11} \text{ sec}^{-1} \text{ m}^{-1}$ except in experiment V, where the inappropriateness of $W$ led to the choice $\beta = 1.28 \times 10^{-11} \text{ sec}^{-1} \text{ m}^{-1}$.

The frictional parameters $k$ and $k'$ were discussed in section 2.4. The "non-extrapolated" form of Ekman friction given in equation (2.4.8) was used in all experiments here.

The wave interactions, when allowed, were restricted to those shown in Figure 2.2. Waves (4,1) and (4,2) could not interact directly with any other waves.

In all five experiments, $\mathcal{R}$ was allowed to vary in time, following the system (2.2.20, 2.2.21, 2.4.5 and 2.4.6). $\mathcal{F}$ was taken to be zero. $k$ and $j$ were the inverse time scales associated with
Table 4.1. Input parameters and miscellaneous information for experiments of Chapter 4.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W$ (km)</td>
<td>5,000</td>
<td>5,000</td>
<td>5,000</td>
<td>5,000</td>
<td>10,000</td>
</tr>
<tr>
<td>$f_0^e$ (hrs)</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
</tr>
<tr>
<td>$T_d$ (°C)</td>
<td>250</td>
<td>250</td>
<td>250</td>
<td>250</td>
<td>1000</td>
</tr>
<tr>
<td>$b$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.30</td>
<td>0.00</td>
<td>0.45</td>
</tr>
<tr>
<td>Wave interact.</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$h$</td>
<td>.120</td>
<td>.120</td>
<td>.120</td>
<td>.120</td>
<td>.120</td>
</tr>
<tr>
<td>$h'$</td>
<td>.120</td>
<td>.120</td>
<td>.120</td>
<td>.120</td>
<td>.040</td>
</tr>
<tr>
<td>$\sigma^x$</td>
<td>+.120</td>
<td>+.120</td>
<td>+.120</td>
<td>+.120</td>
<td>+.030</td>
</tr>
<tr>
<td>$\theta^x$</td>
<td>-.060</td>
<td>-.060</td>
<td>-.060</td>
<td>-.060</td>
<td>-.011</td>
</tr>
<tr>
<td>$\theta^e$</td>
<td>-.020</td>
<td>-.020</td>
<td>-.020</td>
<td>0.000</td>
<td>-.008</td>
</tr>
<tr>
<td>$\theta^i$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>-.007</td>
</tr>
<tr>
<td>$\theta^k$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>-.016</td>
</tr>
<tr>
<td>$\theta^o$</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>+.250</td>
</tr>
<tr>
<td>$\sigma^o$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Average Obs. (Range V=1 in.)</td>
<td>6.5</td>
<td>9.7</td>
<td>14.3</td>
<td>9.7</td>
<td>12.4</td>
</tr>
<tr>
<td>Period (days)</td>
<td>.126</td>
<td>.125</td>
<td>.125</td>
<td>.126</td>
<td>.0358</td>
</tr>
<tr>
<td>$(A+k)$</td>
<td>.097</td>
<td>.102</td>
<td>.090</td>
<td>.099</td>
<td>.0364</td>
</tr>
<tr>
<td>$\theta^M$</td>
<td>.060</td>
<td>.071</td>
<td>.058</td>
<td>.058</td>
<td>0.000</td>
</tr>
<tr>
<td>$\theta^p$</td>
<td>.066</td>
<td>.054</td>
<td>.067</td>
<td>.067</td>
<td>0.000</td>
</tr>
<tr>
<td>$\theta^o$</td>
<td>1.013</td>
<td>1.011</td>
<td>1.011</td>
<td>1.014</td>
<td>.0252</td>
</tr>
<tr>
<td>$\theta^o$</td>
<td>.039</td>
<td>.032</td>
<td>.032</td>
<td>.041</td>
<td>.0024</td>
</tr>
<tr>
<td>$\theta^o$</td>
<td>.053</td>
<td>.052</td>
<td>.054</td>
<td>.058</td>
<td>.0305</td>
</tr>
<tr>
<td>$(\pm 0.007)$</td>
<td>$(\pm 0.007)$</td>
<td>$(\pm 0.004)$</td>
<td>$(\pm 0.004)$</td>
<td>$(\pm 0.004)$</td>
<td>$(\pm 0.004)$</td>
</tr>
</tbody>
</table>
the direct heating influence on stability variations. In accord
with , the inverse time scale for spatially variable heating,
they were taken to be excessively large. , , and had
dimensional values of , , and days respectively.

The field of thermal equilibrium was the same for experiments I-IV. Its zonal average was assumed to represent a constant latitudinal temperature gradient. The choice thus represented it reasonably well, corresponding to a temperature difference between the walls of °C. This value does not seem particularly extreme for mid-latitude winter conditions.

Along a given latitude was assumed to take on a single low value over land and a high one over the ocean. The land area had the longitudinal extent , and covered the latitudinal breadth of the channel. The oceanic area covered the remainder. Between the land and ocean, an abrupt change of °C was assumed to occur in . This large value was adopted to compensate for the constraint of the walls and to accentuate the longitudinal heating effects.

The spectral representation of thus took the form

\[ \Theta^x(y,\varphi) = \Theta_1^x F_1(y) + \Theta_4^x F_4(y,\varphi) + \Theta_{12}^x F_{12}(y,\varphi) \]  \hspace{1cm} (4.2.1)

where . The longitudinal differences were thus described by two scales. The first of these involved \[ \Theta_4^x F_4(y,\varphi) \]
the planetary scale heating field whose longitudinal period coincided with that of the continent-ocean system. The second scale \( \Theta_{12}^* F_{12}(x, y) \) allowed a crude representation of the sharp jump in \( \Theta^*(x, y) \) at the two coastlines.

Due to the choice of large \( \lambda \) and the pronounced nature of \( \Theta^*(x, y) \) the heating rates resulting in these experiments were excessive. In fact, they probably were strong enough to make the assumed geostrophic balance of even the long waves of borderline validity. However, it may be noted that, without \( b \), the quasi-geostrophic equations (2.2.18 and 2.2.19) with \( \Theta^0 \) held constant have the property that the form of the solution is unchanged by a multiplication of all amplitudes and external parameters by the same factor. Only the time scale is changed. Thus, the-dimensional heating rates are changed by the square of this factor. In particular, if the influence of \( \Theta^0 \) time variations was considered to be of secondary importance in these experiments, a factor of three decrease of \( \Theta_{12}^* \) and \( \Theta_{12}^* \) and the parameters \( \lambda \), \( h \) and \( \lambda' \) would yield the same form of limit cycle as was observed, except that the period would be increased by a factor of three. In this case, the heating rates would be decreased by a factor of nine.

Even disregarding such a rescaling, we should keep in mind that the goals of this chapter are not to find a set of suitable dynamics for this case of extreme heating. Instead, this large heating, along with the dissipation, is the mechanism ensuring the stable limit cycle. Hence, only with these large non-adiabatic effects are we able to
study the secrets of the geostrophic dynamics so conveniently. In addition, a secondary effect of the large heating is to accentuate the geographical differences in the resulting flow fields.

The parameter choices shown in Table 4.1 for experiments I-IV do not exhaust all possibilities giving different flow regimes for equations (2.2.18 - 2.2.21). In fact, many of the resultant regimes here had several points in common, some of which were simply studied in Chapter 3. These included:

1) The zonal flow $\theta_1$, with reasonable $\beta$ values, always represented a baroclinic energy source for some waves in the system.

2) The forced wave $(1,1)$ was not by itself "unstable" when $b = .3$. With $b = 0$, it did receive available potential energy from $\theta_1$.

3) The shorter forced wave $(3,1)$ was a baroclinically active traveling wave. It often possessed a significant mean component, but never a dominant one.

4) The wave having $(4,2)$ represented the most unstable baroclinic wave on the zonal flow. (See Figure 3.2)

These points together imply that no other waves $(n, m)$ would be expected to develop under the influence of zonal advections alone. However, section 3.5 would seem to suggest that a mean response in the mode $(2,2)$ would be forced by the interaction of the forced waves $(1,1)$ and $(3,1)$. Section 3.6 suggests that the thermal mode of wave $(1,1)$ might support instability in any of the other modes.
with which it interacts. However, the guaranteed existence of the forced wave (3,1) suggests that the free wave (2,2) would then be favored. In this case section 3.5 further suggests that the advection of the transient component of wave (2,2) by wave (1,1) would alter the transient characteristics of wave (3,1).

With these "predictions" in mind, we now investigate experiments I, II-IV, and finally V. We will concentrate interest on the following points:

1) The roles of the zonal and wave interactions and their qualitative influence upon the climate of the model atmosphere. (See experiments I, III and IV.)

2) The differences in the climate arising when the wave interactions are excluded. (See experiment II.)

3) The nature of the solution when the beta effect is present. (See experiment III.)

4) The sensitivity of the advective processes to the shape of the $\theta^x(\nu, y)$ field. (See experiment IV.)

5) The quasi-steady distortions arising from interactions between forced waves of planetary scale. (See experiment V in section 4.6.)
4.3 Experiment I

This experiment, with the parameter choices indicated in Table 4.1, most clearly portrayed the joint influence of heating asymmetries and interwave coupling on the model behavior. $\theta$ was taken to be zero. The impressed heating $\theta^x_k(\nu, \gamma)$ contained the "edge effect" $\theta^x_{12} f_{12}(\nu, \gamma)$. All interactions shown in Figure 2.2 were allowed.

The solutions were found numerically using the three-step computational procedure discussed earlier, starting from a uniform distribution of small amplitudes in the 36 variables associated with spatial variations. $T_0(0)$ was .08 while $T_0(0) = 1.00$. The precise size of the computational error was uncertain for reasons to be discussed later. However, the periodicity of the resulting solution gave some assurance as to its reliability.

After several weeks of unsteady adjustment, the solution settled toward the final limit cycle. The waves which developed and persisted were those predicted in the last section: waves $(1,1)$, $(2,2)$, $(3,1)$ and $(4,2)$. All other wave modes eventually decayed to zero amplitude. Only the zonal mode $(0,1)$ remained in the final solution. As may be verified from the forms $F_\nu(\nu, \gamma)$ in (2.2.6 - 2.2.8), all states of the limit cycle possessed the special symmetry given in (2.2.4). The resulting wave patterns were thus simply described.

More importantly, the number of interactions was reduced considerably. All group II (zonal) interactions and group III wave interactions were absent. Only the simple group I zonal interactions and a single one of group IV interactions remained. Thus, only the
portions \((a), (c), (f), (j)\) of Figure 2.2 were relevant to this solution.

The wave mode \((4, 2)\) represented a large amplitude baroclinic wave which was slightly unsteady in time, but nevertheless periodic. This wave, being unable to interact directly with the other waves, was thus considered to here represent the general background of simple baroclinic waves to the sub-system involving the directly coupled waves. Its presence reduced the truncation error which would have arisen when only \(\eta = 1, 2, 3\) were allowed in this particular case \(W = 5,000\) km.

The detailed behavior of this wave was not the object of this analysis. However, it is interesting to note that it participated in the limit cycle in this first experiment. This evidently was an example of the phenomenon termed "parametric excitation" (Minorsky, 1962), whereby wave oscillations may be excited by indirect interaction with other waves through the unsteady zonal flow. This periodic behavior was not a feature of other experiments, and so this wave \((4, 2)\) was generally considered to be an extraneous member to the subsystem of modes having \(\eta = 0, 1, 2\) or 3.

Therefore, in the strict sense, most of these results were not described by limit cycles. Only the subsystem of zonal flow and directly coupled waves behaved in an essentially periodic manner. We will nevertheless refer to such solutions as being periodic in this special qualitative sense.
Table 4.1 shows the time mean values $\overline{\mathcal{E}_0}$ and $\overline{\Theta_4}$ as well as their range of variation over one cycle of experiment I. Here, the overbar denotes a time average. (Recall that $\overline{\Psi} = \overline{\Theta_4}$ here.) Also shown are the mean energy levels. The available potential energy was defined as in (2.1.14); the corresponding statistics for that defined by (2.1.12) at each instant were found to be $11.1 \pm 0.29$. The differences of these mean energies indicate the possible importance of the $\mathcal{J}_0(+)\overline{}$ variations, while the larger fluctuations in the second form suggest that this was the case. However, the role of the fluctuations in $\mathcal{J}_0(+)\overline{}$ was believed to be a quantitative one only; the important qualitative features of the experiment were thought to depend upon $\mathcal{J}_0$ only.

With either energy definition, the large fluctuations implied unsteadiness in the waves. Figure 4.1 shows their polar trajectories during one cycle. Part (a) shows that the planetary scale forced wave $(1,1)$ oscillated somewhat about a large mean state. The vertical wave tilt was to the west. The $\Theta$ mode was strongly locked to a position near that of pure thermal equilibrium.

Figure 4.1 (c) shows the forced wave of baroclinic scale $(3,1)$. We see that it was much more transient than its planetary counterpart, having experienced periodic growth and decay at preferred phases with eastward movement at an irregular rate. At nearly all times its tilt was toward the west with height; however, around day 5 the wave was briefly barotropic. This was evidently due to the heating influence at that time. (Note the amplitude and phase of thermal equilibrium...
Figure 4.1

Projection of phase space trajectories for experiments I and II onto polar diagrams for individual waves \((\eta, \theta)^{\circ}\).

Symbol \(\nabla\) denotes the wave forcing \(\theta^{\ast}\).

Solid lines: Behavior of \(\psi_i, \psi_{i+1}\) for experiment I.

Dashed lines: Behavior of \(\theta_i, \theta_{i+1}\) for experiment I.

Dotted lines: Behavior of \(\psi_i, \psi_{i+1}\) for supplement to exp. I.

Dot-dash lines: Behavior of \(\theta_i, \theta_{i+1}\) for supplement to exp. I.

(a) Behavior of the wave \((1,1)\). Solid and dashed lines: exp. I.

Symbol \(\Box\): Steady state \(\psi_3, \psi_4\) for exp. II.

Symbol \(\times\): Steady state \(\Theta_3, \Theta_4\) for exp. II.

(b) Behavior of wave \((2,2)\) for exp. I.

Behavior of wave \((2,2)\) for supplement to exp. I.

(c) Behavior of wave \((3,1)\) for exp. I.

Behavior of wave \((3,1)\) for supplement to exp. I.

(d) Behavior of wave \((2,1)\) for exp. II.

Time in days is indicated on time-varying trajectories.
(a) Wave $(1, 1)$

(b) Wave $(2, 2)$
in the figure.) An appreciable mean state for this wave obviously existed.

Figure 4.1 (b) shows the free wave (2,2) behavior. We immediately note the great similarity in it compared to that of the forced wave (3,1). In particular, the trajectories again represent unsteady waves with preferred phases in both the $\psi$ and $\theta$ vertical modes. The wave usually exhibited a westward tilt with height. A significant mean state is again indicated, despite the lack of forcing in that wave mode.

The similarity in behavior of the waves (2,2) and (3,1) was no mere coincidence. It was a consequence of their dynamical interaction through the quasi-stationary planetary wave. This coupling, along with the analysis given in sections 3.5 and 3.6 suggests that these waves could be considered part of a single wave packet consisting of the two separate scales of motion. The internal coupling of the packet members then produced a mean response in one mode when the other was forced. In addition, their growths near day 1, accompanied by a decrease in the $\theta$ amplitude of wave (1,2), suggested that the development was an inter-wave baroclinic instability of the shorter wave packet on the planetary wave.

A further characteristic of this wave packet is interesting. Each member had the same period of time to traverse one wavelength. Therefore the average eastward translation speed of the longer wave in $\psi$, the (2,2) mode, was actually faster than that of the shorter $\psi$ wave (3,1). Accordingly, the superposition of such traveling waves could be expected to yield a time varying interference field, and hence
an unsteady wave packet shape in real space. This particular geostrophic non-linearity thus did not lock the individual waves into fixed positions relative to each other, contrary to the tendency for some non-geostrophic cascades (Williams, 1965). Instead, the "locking" of the two waves was in phase space only, giving each the same average frequency.

These phase speed differences reflected a special "non-linear dispersion" of the wave packet, which introduced a strong time variability into its spatial form. The non-linearity of it was reflected in the dependence of its shape upon the planetary wave. In this manner, the planetary scale of forcing appeared sufficient to alter the spatial characteristics of the shorter baroclinic waves.

To demonstrate this, a supplementary experiment was performed in which the initial state was taken to be that of experiment I at day 1, except for the wave (1,1). This planetary mode simply had its phase changed, while it retained its initial amplitude. Solutions were found for three separate cases of phase change $\Delta \xi_{1,1} = +90^\circ + 180^\circ$.

Figure 4.2 shows the initial portions of the resulting $\psi$ mode solutions for the traveling waves. The closed curve represents the limit cycle $\Delta \xi_{1,1} \equiv 0^\circ$. The important influence of the planetary wave phase upon the initial growth and movement of the traveling waves is clearly seen.

While such phase influence could have conceivably arisen indirectly through the zonal flow, this was not the case here. With the same deviating initial states as above, a similar solution was
Initial behavior of the \( \gamma \) mode of waves (2,2) and (3,1) for the supplement to experiment I. Dotted lines are solutions for the 3 changes in initial phase of wave (1,1):
\[
\Delta \varepsilon_{\gamma}(1) = \pm 90^\circ, +180^\circ.
\]
Solid lines: Limit cycle solution \( \Delta \varepsilon_{\gamma}(1) = 0^\circ \).

(a) wave (2,2)
(b) wave (3,1)
found for the case with the wave interactions excluded. In contrast to the results in Figure 4.2, it was found that the traveling waves were initially unaffected by the planetary wave phase shift. This implies that the planetary wave phase was directly influencing the behavior of the other scales through their wave interaction, and seems to verify our earlier remarks given in section 3.1.

Having examined the time variations of the individual waves, let us now examine their energetics to learn more of the processes controlling them. We first introduce the energy forms appropriate for the spectral model.

In section 2.1 it was shown that the continuous equations (2.1.2 - 2.1.6) could be written as equations (2.1.9) and (2.1.10) for the time changes in and . Here, and are the available potential energy and kinetic energy defined by (2.1.14) and (2.1.11), respectively. These quantities can be expressed in terms of the spectral variables by introducing the expansions (2.2.9 - 2.2.13) and using the harmonic and orthonormal properties of the functions . Hereafter referring to the non-dimensional forms of the energies, they become (4.3.1), where (4.3.2).

Defining and (4.3.3),(4.1.1) is . For experiment I, we have seen that many of the were zero. The non-zero thus represents stores of potential energy in the waves . Clearly advective distortions of the temperature field would represent changes in the spectral distribution
of the $\mathbf{\theta}_i$, and hence an exchange of $\mathbf{\theta}_i$ between the separate
wave modes.

In a similar manner, we have

$$k = \frac{\bar{a}_i}{\bar{c}_z}, \quad k_i = \bar{c}_z + \bar{k}_w$$

(4.3.4)

where

$$k_i = \frac{a_i^2}{2} (\psi_i^2 + \Theta_i)$$

(4.3.5).

The $k_i$ represent kinetic energy in either of the vertical modes $\psi$ or $\Theta$ at the horizontal scale $a_i^2$.

Prediction equations for $\mathbf{\theta}_i$ and $k_i$ can be obtained for the
system with variable $\mathbf{\theta}_o$ (Gates, 1961). They are similar to those
shown in (2.1.9) and (2.1.10), except that the transformation arising
from the horizontal redistribution of potential energy contains an
extra term from that given in (2.1.10). This term is proportional
to the $\mathbf{\theta}_o$ changes accompanying the $\theta$ advections.

Putting the spectral equations (2.2.18 - 2.2.20) into the forms
(2.1.3 - 2.1.6) by use of the $\omega_i$ as given in (2.2.21) the energy
budgets take the form

$$\frac{\partial}{\partial t} A_{\eta,m} = <q_{\eta,m} A_{\eta,m}> + <a_z A_{\eta,m}> + <w A_{\eta,m}> - <A_{\eta,m} k_{\eta,m}>$$

(4.3.6)

$$\frac{\partial}{\partial t} k_{\eta,m} = <k_z k_{\eta,m}> + <k_w k_{\eta,m}> + <A_{\eta,m} k_{\eta,m}> - <k_{\eta,m} D_{\eta,m}>$$

(4.3.7)

Here we logically define

$$A_{\eta,m} = A_i f_{\eta,m}, \quad A_{\eta,m} = A_i + A_{i+1} f_{\eta,m}, \quad i = 1, \ldots, 17$$

(4.3.8)
so that \( \Delta_2 = \Delta_0 + \Delta_0 \), and \( \Delta_2 = \Delta_0 + \Delta_0 \). Similar definitions hold for the \( k_{n,m} \). The \( \langle \rangle \) terms represent the energy "transformations" of each physical process. For example, \( \langle Q_{n,m} \cdot A_{n,m} \rangle \) is the signed generation of \( A_{n,m} \) by the heating field \( Q_{n,m} \) of that scale. \( \langle A_{n,m} \cdot k_{n,m} \rangle \) represents the internal conversion of \( A_{n,m} \) to \( k_{n,m} \) for a given wave \( (n,m) \), and is proportional to \( \omega \cdot \delta t + \omega \cdot \delta t + \omega \cdot \delta t \). (See equations (2.1.9) and (2.1.10).)

The term \( \langle A_2 \cdot A_{n,m} \rangle \) stands for a transformation of zonal flow potential energy \( (A_0 + A_0) \) to \( A_{n,m} \) of a particular \((n,m)\), while \( \langle A_{W} \cdot A_{n,m} \rangle \) represents the production of \( A_{n,m} \) by the interaction of all other waves with the mode \((n,m)\). Naturally \( \langle A_2 \cdot A_{0,m} \rangle = 0 \) and \( \langle A_{W} \cdot A_{0,m} \rangle = 0 \) in this notation.

Similar remarks hold for the terms in (4.3.7); the term \( \langle k_{n,m} \cdot d_{n,m} \rangle \) there represents the dissipation of \( k_{n,m} \) by frictional processes.

The principal interest of this thesis is in the quantities \( \langle A_{W} \cdot A_{n,m} \rangle \) and \( \langle k_{W} \cdot k_{n,m} \rangle \), and secondarily in the "general circulation" exchanges \( \langle A_2 \cdot A_{n,m} \rangle \) and \( \langle k_2 \cdot k_{n,m} \rangle \). Both potential energy transformations contain a principal part \( \langle \rangle \) proportional to the advective change in the temperature variance \( \Theta_2 \), as in equation (2.1.9). These changes were seen to be large in experiment I. (See Figure 4.1.) A secondary portion of the transfer involves the \( A_{n,m} \) changes caused by the upward heat flux of all other modes, and does not represent the primary process of horizontal energy exchange. Therefore, it will be neglected here, so that
reference will be made only to the primary exchanges $\langle \hat{A}_m', \hat{A}_n \rangle$ and $\langle \hat{A}_m, \hat{A}_n \rangle$. Similarly the transformation $\langle \hat{Q}_{m,n}, \hat{A}_n \rangle$ will be considered alone, corresponding to neglect of the influence of heating changes in $S_0$ of $\nabla M$ on $\hat{A}_n$. 

Due to these simplifications, a complete energy budget is impossible here. However, this is only a quantitative shortcoming of the present analysis. It does not hamper our learning more about the horizontal redistribution processes, which is our goal here.

The transformations associated with these processes may be illustrated for a simple case. We consider the kinetic energy processes which take place in a barotropic flow by setting $\theta_i \cdot 0$, $\dot{\mathcal{C}} = 1-18$. If, for example, $\dot{\mathcal{C}}$ takes on one of the values $\dot{\mathcal{C}} = 3, 4, \ldots, 13$, then multiplication of (2.2.18) by $\Psi_i$, and repeating the whole process for $\Psi_{i+1}$ yields the form (4.3.7), where $\langle \hat{A}_{m,n}, \hat{K}_{m,n} \rangle = 0$ and the 6 terms cancel. Here the dissipation has the form

$$\langle k_{m,n} \cdot \chi_{m,n} \rangle = \frac{b}{2} \Psi_i^2$$

The $C_{i,j,k}$ could be separated into the zonal and wave interactions, in which case we have

$$\langle k_{x} \cdot k_{n,m} \rangle = \frac{1}{2} \sum_{j,k=1}^{2} (a_j^2 - c_k^2) C_{i,j,k} \Psi_i \Psi_j \Psi_k \tag{4.3.9}$$

and

$$\langle k_{y} \cdot k_{n,m} \rangle = \frac{1}{2} \sum_{j,k=3}^{14} (a_j^2 - c_k^2) C_{i,j,k} \Psi_i \Psi_j \Psi_k \tag{4.3.10}$$

The triple products $(\Psi_i, \Psi_j, \Psi_k)$ depend upon the amplitudes and phases of each of the waves $(\eta_{j,m}), (\eta_{j,n})$, and $(\eta_{m,n})$. In (4.3.9) only two wave variables appear, so that the phase influence
on this product is not so strong as in (4.3.10), where three separate wave phases appear. The requirement for a persistent direction of energy transfer between the waves in (4.3.10) thus depends critically upon sustained coherence of these wave phases.

We next ask how this might arise. In the case of no continentality, none of the waves possess preferred phases a priori and the triple products must develop freely. In this case, they have been termed the "fluctuating interactions," and then often dropped. However, with continentality present, the phases of some of the waves are statistically fixed. This reduces the degrees of freedom in the triple product to two, one or none.

In addition, the interactions may themselves induce phase relations between the waves, further decreasing the degrees of freedom. While the latter process may be operative even in a fluctuating flow (Kraichnan, 1958) it is especially so when one of the waves is fixed, as in this problem.

An example of a strongly forced interaction is the development of the mean state in the free wave (2,2) in experiment I. An interaction of the latter type will be pointed out in experiment IV. For either process, it seems possible at this point that an important influence of continentality is to induce preferred directions of energy flow between scales. This is an important conclusion of this thesis and it will be demonstrated more conclusively in later sections.

Having now defined these energy concepts, let us examine the observed energetics of experiment I. Some mean energy transformations
are shown in Figure 4.3. Note the "units" of energy transfer.

Part (a) illustrates the total heating field as an external source of $A_{\eta_1 \eta_2}$. It is interesting to note that, in the mean, the relatively fixed flow consisting of the zonal flow and planetary wave received energy from the heating field. On the other hand, the fluctuating waves of cyclone scale, $(2,2)$, $(3,1)$ and $(4,2)$ were all dissipated by the heating. In particular, the excitation of wave $(3,1)$ by the edge heating did not represent a net process of genesis.

Part (b) shows that the conversions $\left< A_{0_1}, A_{\eta_1 \eta_2} \right>$ were positive for all waves, in accord with their vertical tilts noted earlier. Of special interest is wave $(2,2)$, whose gain of potential energy from the zonal flow was not sufficient to balance the heating dissipation. The question arises as to what its energy source was.

The question is answered in part (c), which shows the simultaneous flow of potential energy between the three interacting waves. Certainly one can say that wave $(2,2)$ was being maintained by the planetary wave $(1,1)$, but one can not necessarily say that it was a direct transfer between the two waves. At any rate, we see that a fluctuating wave $(2,2)$ of "small" scale was maintained by the more steady planetary wave $(1,1)$. This mean energy transfer thus represented a net exchange between differing scales of flow in both space and time.

Part (d) of this figure shows that the net kinetic energy exchange between the waves was small. (Of course, that of the zonal flow and the waves was zero, since the mode $(0,2)$ was absent.) The small net transfer which did occur was mainly between the two traveling
Figure 4.3. Some mean energy transformations of experiment I for individual modes \( (\eta, m) \). The transformations are normalized with respect to the total mean energies \( \bar{A} \) and \( \bar{K} \), respectively. Units are day\(^{-1}\). Arrows show direction of energy transfer.

\[
\begin{align*}
(a) &\quad \int_c \frac{Q_{nm} \cdot A_{\eta m}}{\bar{A}} \\
(b) &\quad \int_c \frac{A_{c1} \cdot A_{\eta m}}{\bar{A}} \\
(c) &\quad \int_c \frac{A_{w'} \cdot A_{\eta m}}{\bar{A}} \\
(d) &\quad \int_c \frac{K_{w'} \cdot K_{\eta m}}{\bar{K}}
\end{align*}
\]
wave members.

This exchange appears more prominent when one considers the fluctuating parts of these transformations shown in part (b) of Figure 4.4. We first refer to part (a), which shows the time variation of the inter-wave \( \frac{1}{\mu} \varepsilon_{\eta} \) exchanges. We see that if the wave (3,1) was receiving potential energy and passing it on to wave (2,2), then it was doing so by storing almost none of it at each instant.

Such behavior seems improbable, especially in view of the constant signs of \( \langle A_{w} \cdot A_{1,1} \rangle \) and \( \langle A_{w} \cdot A_{2,2} \rangle \). We may thus tentatively conclude that the wave (3,1) played a catalytic role in the exchange \( \langle A_{1,1} \cdot A_{2,2} \rangle \).

The fluctuations in magnitude of \( \langle A_{w} \cdot A_{1,1} \rangle \) and \( \langle A_{w} \cdot A_{2,1} \rangle \) arose primarily from the changing phases and amplitudes of the waves (2,2) and (3,1), which presumably were induced by the "edge" heating of (3,1). In fact, it was found that the short wave phase angles measured relative to the planetary field varied in such a manner as to optimize the planetary wave instability at about days 1 to 2, when the maximum energy transfer did in fact occur.

Turning to part (b) of Figure 4.4, we see that the kinetic energy exchange between the wave packet members was oscillatory. The planetary wave was quite inactive. Thus, the two traveling waves traded kinetic energy back and forth as they moved past the planetary mode, which was almost fixed in space. This corresponded to a periodic distortion of the wave packet which was thus ultimately related to its geographical position.
Figure 4.4. Some periodic energy transformations between waves for experiment I involving
(a) available potential energy
(b) kinetic energy
A final feature of Figure 4.4 is important. We note that only the wave (2,2) was an active participant in both the potential and kinetic interwave exchanges. Since it interacted only with forced zonal or wave modes, it thus represented a sort of middleman between the fields of advection and heating. We therefore study its energetics a bit further, concentrating especially on its growth processes.

Figure 4.5 exhibits the time variations of several transformations for wave (2,2). Part (a) shows its potential energy exchange rates as well as the dissipation by heating. The latter is simply proportional to the squared amplitude of the thermal mode, which was a minimum at day 0. Part (b) shows two parts of the conversion \( \left< A_{z,2}, K_{z,2} \right> \), along with the exchange \( \left< K_{w,2}, K_{z,2} \right> \) and frictional dissipation \( \left< K_{z,2}, D_{z,2} \right> \). This dissipation rate is related mainly to the amplitude of the wave in the lower layer.

Referring first to part (a) we see that the thermal part of the wave began to grow first in response to the interaction \( \left< A_{w,2}, A_{z,2} \right> \). This represented the growth of wave (2,2) at the expense of the planetary wave's store of available potential energy. This exchange increased to a maximum at day 2, by which time the wave was also draining potential energy from the zonal baroclinicity \( \Theta_f \). After this the wave interaction decreased rapidly; the zonal flow alone could not maintain the large thermal amplitude which had developed, and so the wave decayed to small amplitude at day 5.

Considering the kinetic energy cycle in Figure 4.5 (b), we see that the conversion \( \left< A_{z,2}, K_{z,2} \right> \), \( \text{WAVE} \) initiated its growth. This particular part of the potential to kinetic energy conversion
Figure 4.5. Some periodic energy transformations for wave (2,2) involving (a) available potential energy (b) kinetic energy.
arose from the vertical velocity \( \omega_z \) induced by the wave interaction \( \langle \hat{A}_w, \hat{A}_{2,2} \rangle \). At first this conversion just balanced the loss of energy \( \langle k_w, k_{2,2} \rangle < 0 \) to wave (3,1), but this loss becomes a gain at day 1, after which the conversion \( \langle \hat{A}_{2,1}, k_{1,1} \rangle \) joined in to generate more kinetic energy.

At day 3 the production of \( k_{2,2} \) by the direct and indirect influence of the wave interactions began to decline, as did itself. After day 4 the wave lost kinetic energy to wave (3,1) again and the conversion \( \langle \hat{A}_{2,1}, k_{1,1} \rangle \) decreased, allowing swift decay of \( k_{2,2} \) toward its final minimum.

In summary, the relations shown in Figure 4.5 indicate that the wave interactions were instrumental both in initiating the growth of the free wave, and in allowing its subsequent decay. The zonal flow represented an effective energy source only after the wave had reached its "finite amplitude" stage. This suggests that the wave could not have been maintained if the wave interactions were suddenly "turned off".

To test this hypothesis, the system was allowed to evolve from its initial state at day 1 under the influence of only the zonal interaction type. The resulting trajectories for the traveling waves are shown as deviations from the periodic solution in Figure 4.1 (b) and (c), and are denoted by the dotted and dot-dash lines there. It is seen that, since the immediate deviations of the zonal flow were observed to be small, the net effect of the zonal interactions and dissipation was to favor the forced wave (3,0) over the free mode (2,2). In fact, the latter is seen to have quickly decayed, while
the former grew at a greater rate than before. Apparently the presence of the wave interactions in experiment I inhibited the optimum vertical structure for the interaction of the wave \((3,1)\) with the zonal flow. More importantly, we see that the zonal flow influence on the wave \((2,2)\) was alone insufficient to halt the dissipation of the wave, as was indicated before.

Thus, we conclude that both the existence and behavior of the free mode \((2,2)\) depended critically upon the wave interactions. Their neglect would have amounted to ignoring an important element in the evolution of the total field of motion.

In the next section we turn to consideration of the spatial description of the solution of experiment I. Our goal there is to translate the important conclusions of this section into ones relating the distribution of the model's climatic factors to that of the continents and oceans.

4.4 The Climate of Experiment I

In this section we present maps of some of the climatic elements corresponding to the solution of experiment I. The immediate purpose is to relate these distributions to the simple system of continents and oceans. No detailed comparison with the real atmosphere is intended, for it cannot be justified. A second purpose is to show the climate for this particular experiment so that it may be compared with those of experiments II-IV in the next section.

In this description the influence of the "extraneous" wave \((4,2)\) is excluded, for we focus attention upon the subsystem of waves
Simple climatic maps for experiment I. (Cold)

(a) Map of mean flow $\overline{\psi} (x, y)$ given by thin lines.
Isolines at non-dimensional intervals of .15.
L stands for low pressure, H for high pressure.
Continuous heavy lines show tracks of moving cyclone systems.
The integers along them refer to their position at that day of the limit cycle.
Similarly, dashed heavy lines show tracks of anticyclonic systems.

(b) Map of mean daily standard deviation $\left[ \overline{\psi'^2} (x, y) \right]^{1/2}$.
Isolines labelled in non-dimensional units.

(c) Map of mean temperature $\overline{\theta} (x, y)$ given by thin lines.
Isolines labelled in °C departure from horizontal mean.
L stands for low temperature, H for high temperature.
Continuous heavy lines: tracks of moving cold masses.
Dashed heavy lines: tracks of moving warm masses.
Positions at days 0, 2, 4 labelled along tracks.

(d) Map of mean daily standard deviation $\left[ \overline{\theta'^2} (x, y) \right]^{1/2}$.
Isolines labelled in °C.
discussed in the last section. Since the mean state of the wave \((4,2)\) was zero, the only effect of this wave on the climate would have been in the distribution of fluctuating statistics, which presumably would have no net longitudinal variations.

Figure 4.6 shows the maps of \(\bar{\psi}\) and \(\bar{\Theta}\) and their standard deviations \(\sigma_\psi^2\) and \(\sigma_\Theta^2\). Here \(\psi' = \psi - \bar{\psi}\), \(\Theta' = \Theta - \bar{\Theta}\). The dotted areas represent the continental regions and the clear areas the oceanic ones. Superposed on the mean maps (a) and (c) are the tracks of moving systems in the \(\psi\) and \(\Theta\) fields, respectively. The numbers written along these tracks refer to the corresponding day of the limit cycle.

The maps have one feature in common; they all possess the special symmetry given by equation (2.2.4). Despite this, these large-amplitude waves were not simply related to the continent-ocean system.

Part (a) shows \(\bar{\psi}(\chi,\gamma)\), which is equivalent to either the vertically averaged mean flow or the mean flow at 600 mb. A deep mean flow vortex is seen to have occurred at high latitudes over the east coast of the continent. The maximum mean circulation speeds about it was 30 m/sec. A secondary portion of the vortex was found over the center of the continent. More importantly, a third center of low pressure was found at relatively low latitudes in the center of the ocean. This trough represented a local feature of medium scale which was not present in the impressed heating distribution \(\Theta^x(\chi,\gamma)\). This was a result of the non-linear advective responses to the continentality. The feature has probably been accentuated
somewhat by the truncated spectral representation; however, the
extistence of a mean component in the free wave (2,2) guaranteed its
basic validity.

Due to the symmetry of the flow, similar comments could be made
about the anti-cyclonic systems, interchanging references to con-
tinents and oceans above. In the discussion to follow, reference
will be centered mainly around the cyclonic systems, but it is to be
understood that analogous remarks hold for the anti-cyclones.

The \( \psi \) map is complemented by the tracks also shown in part (a).
We see that two cyclonic systems generated in the ocean to the east
of the vortex merged in mid-ocean and moved onto the continent. At
this point the system changed shape rapidly with the center appearing
at a lower latitude. It then moved eastward and northward, merging
with the semi-permanent east coast vortex.

Comparing these tracks to the mean streamlines, we see that there
was a broad correspondence between the two. This tendency for steering
of the transient centers by the meandering mean flow was consistent
with the relative persistence of the main vortex.

There were of course important deviations from the mean flow as
shown in Figure 4.6(b). The outstanding feature of this diagram is
the lack of homogeneity of the fluctuations, not only between the con-
tinents and oceans, but between the latitudes as well. Some areas, such
as the north-eastern part of the continent, are seen to have been
sheltered. This was not related to any obvious feature of the forcing
field. Instead, its location was near that of the main vortex.
The band of maximum variability was related to the tracks, but it was more closely connected to the meandering "jet stream" of the mean flow. It almost appears as though the statistics of the traveling systems, rather than their centers, were carried along by $\mathcal{U}$. This band shows some tendency for the fluctuations to have been greatest at certain longitudes, suggesting that local generation processes were operative, in agreement with the tracks of Figure 4.6(a) and the results discussed in section 4.3.

Parts (c) and (d) of Figure 4.6 show the mean and transient statistics of the temperature field. The mean field (c) shows that the cold and warm masses were confined to their source regions over the continents and oceans, respectively. In fact, the maximum land-ocean thermal contrast was 50°C, a significant fraction of the 80°C difference imposed by $\mathcal{D}^*(\mathcal{V}, \mathcal{Y})$. Neither feature is surprising, owing to the large heating rates in the experiment. These mean rates were as large as 30°C/day, several times larger than those observed in the real atmosphere. In fact, comparable rates are found only for short periods of time in shallow layers of the atmosphere at oceanic localities near the east coast of Asia in wintertime (Manabe, 1958).

The tracks of cool and warm pools of air are shown in part (c). Contrary to $\mathcal{D}$, they exhibited more of the influence of the dynamics on the temperature field. First of all we see that a source region for a traveling mass of cold air lay over the northern ocean. This was associated with the traveling system shown in part (a), which evidently grew at the expense of the thin band of baroclinicity in that region. That system traveled eastward to the continent, where it was absorbed.
into the more permanent source of cold air at a lower latitude.

This larger mass was associated with the (1,1) wave; we have seen earlier it was being continuously depleted by the traveling wave packet. That transfer reached a maximum at days 1-2. Reference to the cold air tract of part (c) shows that this energy release was associated with an outbreak of cold air which crossed the eastern continental coastline at that time, later to be dissipated by the oceanic heating. Similar bursts of warm air onto the high-latitude continents from the eastern ocean also occurred, of course.

These observations point out the role that the longitudinal variations in heating played in dissipating energy. The heating, being dependent upon the temperature distribution, was indirectly set up by the temperature advections. In particular, we have seen that the wave interactions allowed the increasing longitudinal temperature differences to be discharged laterally into a dissipative heating field.

Referring now to part (d) we see that the thermal variability was a maximum near each coast. However, the latitudes of these centers were different for the two coasts. Thus, the $\Theta$ fluctuations occurred near the continuous bands of larger temperature gradients which meandered through the channel. However, such baroclinic zones were not everywhere associated with maximum fluctuations, as for example in the western ocean. This suggests that the initiation and decay of these fluctuations was related to both the wave interactions and heating field. This conclusion is in agreement with the "outburst" example above and the comments of section 4.3.
4.5 Experiments II, III and IV

In this section we consider the results of three separate experiments, each grossly similar to experiment I but differing from it in a single important manner. Referring to Table 4.1, these differences were the exclusion of wave interactions, the inclusion of the beta effect, and the smoothing of the $\Theta(\psi, \gamma)$ field, respectively.

a. Experiment II

Section 4.3 gave much insight into the role that the wave interactions played in determining the flow characteristics of experiment I. Experiment II was performed as a repeat of it, except that the wave interactions were omitted from the start. The initial conditions were as in experiment I. The flow states which evolved formed a periodic set in the modes having $\ell=0, 1, 2, 3$, as before. The wave $(4, 1)$ was somewhat unsteady, having a time of travel of seven days for one wavelength.

The periodicity in the remaining modes was less spectacular than in experiment I. Firstly, the free mode $(2, 1)$ was totally absent, as expected from the supplementary experiment discussed in section 4.3. The energetic importance of this wave was demonstrated there, and its absence here therefore reflects an important change in the flow characteristics of the other scales.

For example, the planetary wave $(1, 1)$ was very nearly in a steady state despite the moderate oscillations of the zonal flow. This state is shown in Figure 4.1(a) along with the same wave's behavior for experiment I. Comparing the two solutions, we see that the mean amplitude was larger in experiment II. In fact, the thermal
field was then very nearly in a state of thermal equilibrium. The \( \Psi \) phase is seen to have been closer to the "locked in" \( \Theta \) mode, resulting in a smaller vertical wave tilt. In short, it is evident that the wave interactions in experiment I produced a feedback onto the quasi-steady planetary wave. This was characterized by moderate fluctuations about the mean state and moderate changes in the planetary phase of \( \overline{\Psi} \). Neither feature was predicted by the linear analyses of Chapter 3, and thus each represents a truly non-linear effect.

The forced wave \((3,1)\) for experiment II is shown in Figure 4.1(d). Contrary to its behavior in Figure 4.1(c), its amplitude was greater and nearly steady without the wave interactions. Noticeably lacking was the pronounced growth at certain phases found in experiment I. However, the phase speed of the wave did fluctuate here, being influenced by the heating field. This unsteadiness was accompanied by an oscillation in the zonal flow variables \( \overline{\Psi}(t) \) and \( \overline{\theta}(t) \) having the same period.

Interestingly enough, \( \overline{\Psi} \) and \( \overline{\theta} \) were about the same as in experiment I, and yet the mean phase speed of the wave \((3,1)\) was smaller than before by about 50 per cent. This implies that the wave interactions played a very significant role in determining the wave phase speeds in experiment I. (An example of this effect was given in section 3.5.) In fact, we see that the "longer" travelling wave moved eastward about twice as fast with the wave interactions as the "shorter" wave \((3,1)\) did without them.
Further comparison of parts (c) and (d) shows that the mean states of wave \( (3,1) \) in experiments I and II differed, being more dominant in the former case. Thus, we conclude that this forced wave in experiment I was influenced jointly by its own heating field and the planetary wave, as was predicted qualitatively in section 3.5. To the extent that the planetary flow reflected the planetary scale heating, we see that the response of the cyclone wave \( (3,1) \) depended upon the total heating distribution described by both scales. That is, it was sensitive to the shape of the continent-ocean system.

From Figure 4.1, it is clear that the energetic relations between the waves and zonal flow were similar in these two experiments. One exception was that the planetary heating field represented a sink of available potential energy in experiment II, of about the same size as the corresponding source indicated in Figure 4.3(a) for experiment I. This was accomplished by an increase in the amplitude of wave \( (1,1) \), which could occur only in the absence of the dissipative wave interactions of experiment I. Broadly speaking, this wave seemed to adjust itself to a state where it dissipated potential energy through either its interaction with other waves (experiment I), or via its dissipative heating (experiment II).

A second exception to Figure 4.1 was that the wave \( (3,1) \), having possessed a steadier structure and greater amplitude in the absence of wave interactions, was then able to play a more outstanding role as a baroclinic sink for the zonal flow.

We now turn to the climatic maps for experiment II, shown in Figure 4.7. Part (a) shows the mean \( \overline{\Psi} \) field. The waves were again
Figure 4.7. Climatic maps for experiment II.
See Figure 4.6 for explanation.
of large amplitude, but with their form changed from that in Figure 4.6(a). This time, the main vortex was over the center of the continent, with its secondary center over the coast. The mid-ocean trough of experiment I was completely missing this time; in fact, an anticyclonic regime prevailed there. In short, the mean flow was more simply related to the pattern of continentality when the dynamics were "linearized" to exclude the wave interactions.

The individual patterns at a given instant were that of a baroclinic wave almost uninfluenced by the "edge" heating, travelling through a stationary planetary wave pattern which was dominated by the large scale heating. Neither wave was visibly affected by the other, so that the system was nearly linear in that sense. This was in stark contrast to experiment I, where a continuous distortion of the travelling wave about a mean state took place.

These differences are reflected in the tracks shown in part (a). We first note that they were continuous, with no areas of generation for new centers. Secondly, they underwent relatively minor changes in latitude which were simply associated with the superposition of the two waves on the zonal flow. No steering of the transient centers by the mean flow was evident. The mean coastal vortex, a quasi-permanent feature of experiment I, was not so steady in this case. Here, it was a secondary feature arising from the slower phase speeds of the travelling wave in that region.

The fluctuation fields shown in part (b) are seen to have possessed imperceptible longitudinal preferences, contrary to those
shown in Figure 4.6(b). Instead, the transient motions were almost unaffected by the longitudinal variations of either the heating or mean flow. There was no tendency for unusually active or quiet areas, aside from the sheltering effect near the walls.

The field shown in Figure 4.7(c) differs little from that in Figure 4.6(c), due to the dominant heating process. The latitudinal temperature difference was only slightly smaller, reflecting the increased efficiency of the single baroclinic wave. The maximum land-ocean difference was now larger, being nearly that of the thermal equilibrium field.

The tracks of moving thermal features are seen to have been continuous and simple, following a path which tended to minimize the heating dissipation. Noticeably lacking was any cold air "outburst" onto the warm ocean, which relied upon the wave coupling absent in this experiment.

The standard deviation of temperature in part (d) illustrates the comparative inability of the continentality to generate longitudinal variations in the temperature fluctuations in the absence of wave interactions. The difference between this pattern and that of Figure 4.6(d) is striking, especially since the two solutions possessed such similar $\overline{\theta (v,g)}$ distributions.

The reason for this is that the heating field for experiment I included source regions for the time-mean temperature, which was then redistributed by the wave and zonal interactions. In experiment II, the heating dissipated thermal contrasts everywhere, which, with fewer advective degrees of freedom, resulted in a decrease in differences of fluctuations between localities.
b. Experiment III

Experiments I and II simply demonstrated the heating-advection relation in the absence of the beta effect. We now consider the results of experiment III obtained by setting $b = B$ and allowing the waves to interact with each other.

Starting from the same initial conditions as before, the system eventually evolved to a limit cycle in the interacting waves $(1,1)(2,1), a \sim (3,1)$. The period was $14.3$ days. Wave $(4,2)$ was slightly unsteady and again played only a background role.

The behavior of wave $(1,1)$ is shown by the closed curves in Figure 4.8(a). The behavior of $\psi$ and $\theta$ was similar to that in experiments I and II. With these mean values $\psi_i$ and $\theta_i$, the results of sections 3.3 and 3.4 indicate that the steady state solution shown by the two points $\bigcirc$ and $\bigotimes$ would be expected. This wave solution would tilt eastward with height. Comparison of that steady solution with the observed response shows that the vertical wave tilt was instead toward the west when the wave interactions were included. This was evidently a result of a direct feedback of the shorter baroclinic waves upon the longer wave. The effect was largest on the $\psi$ mode, which lay further to the east. The amplitude in this case was again smaller due to the inter-wave feedback. The directions of both changes were those noted in experiments I and II. They assumed more importance here because the sign of the vertical tilt had been changed, implying that the planetary wave was now a sink of zonal baroclinicity, rather than a source.
Figure 4.8

Polar trajectories of individual waves $\eta, m$ for experiment III.

Time in days is indicated by integers.
Symbol $\nabla$ denotes the wave forcing
Solid lines: Solutions for $\psi$ mode
Dashed lines: Solutions for $\theta$ mode

(a) Wave (1,1) behavior for experiment III. Also shown is a separate steady state solution with wave interactions excluded.
- $\odot$ denotes steady $\psi$ mode
- $\Box$ denotes steady $\theta$ mode
(b) Wave (2,2) behavior
(c) Wave (3,1) behavior
(b) Wave \((2,2)\)

(c) Wave \((3,1)\)
Referring to Figure 4.8(b), (c), we see that the forced wave \((3,1)\) and the free mode \((2,2)\) developed less marked phase preferences in experiment III than in experiment I. In fact, their individual behavior was not too different from that of wave \((3,1)\) in experiment II. However, a joint behavior of the waves here existed like that in experiment I, with the \((2,2)\) wave progressing eastward at a faster rate than wave \((3,1)\). Curiously, the individual phase speeds fluctuated together during the cycle in a manner inversely related to the speed of the zonal flow \(\psi_1\). This was contrary to the behavior of the single travelling wave \((3,1)\) in experiment II, whose phase speed varied directly with \(\psi_1\). It is thus apparent that the coupling of the wave packet \((2,2)\) and \((3,1)\) to the planetary mode \((1,1)\) introduced an irregular translation of the packet as a whole, in addition to changing its shape.

As expected from Figure 4.8, the energy exchanges for this experiment were quite steady in time, and their directions resembled those of the mean energy flows for experiment I. The rates were somewhat larger. The conspicuous lack of strong wave fluctuations evidently reflected the stabilizing influence of the beta effect upon the planetary wave instability which was seen in section 3.6.

We now turn to consideration of the mean maps for experiment III. Figure 4.9(a) shows \(\overline{\psi}(\lambda,y)\). We first note that the "fine structure" of the main vortex has disappeared, with the mean wave flow consisting primarily of the planetary wave. This resulted from the transient nature of the wave packet at each longitude, which
Figure 4.9. Climatic maps for experiment III. See Figure 4.6 parts (a) and (c) for explanation.
instead influenced the tracks shown in the figure.

These tracks show that cyclonic systems were emitted at the continental east coast which travelled at high latitudes across the ocean onto the continent. At this point an abrupt change of shape placed a new center at a lower latitude. This was followed immediately by a similar event at the next downstream trough, which then fed into the high latitude system. Inspection of the daily maps, not shown, shows that the latter adjustment restored the system to a "normal" state possessing two travelling cyclonic centers at high latitudes, rather than just one.

Referring to part (b), we see that the small amplitude of $\theta_4$ was reflected in a correspondingly small land-ocean temperature difference. Despite this, the tracks did exhibit longitudinal temperature differences. For example, the lower latitudes of the land area were regions of occasional cold air outbreaks from the north. This did not occur over the ocean.

The maps of flow variability will not be shown here, for they were similar to those in Figure 4.6. This is not surprising, considering the fact that the travelling wave packet occurred in both experiments I and III, and accounted for the major part of the fluctuating wave field.

c. Experiment IV

Let us now turn to a final variation of experiment I. Up to this point we have considered a continent-ocean system with an abrupt change in $\theta^* (\nu, t)$ at the coastlines. It was seen in equation (4.2.1)
that it was adequately described by the two wave scales \((1,1)\) and \((3,1)\). Let us now consider an artificially smooth transition region at the coasts by setting \(\theta_{12}^{*} = 0\). The heating input may now only occur on the single planetary scale \((1,1)\).

Except for this change in \(\theta_{12}^{*}\), the other parameters for this experiment IV were the same as in experiment I. Starting from the same initial conditions, the solution attained a limit cycle behavior with period \(9.5\) days in the limited wave set \((1,1), (2,2), (3,1)\) and the zonal flow \((0,1)\). The extraneous wave \((4,2)\) had a period of \(10.0\) days.

Contrary to experiment I, the zonal flow was nearly constant in time, as was the wave \((1,1)\), as shown in Figure 4.10(a). This behavior is consistent with that anticipated in Chapter 3. Comparing Figure 4.10(a) with Figure 4.1(a), we see that the feedback of the baroclinic waves onto the planetary mode was altered. Firstly, the wave \((1,1)\) no longer fluctuated about its mean position; secondly, its mean \(\Psi\) phase was changed by about 45 wave degrees.

The reason for this is seen in the same figure, where the two free wave solutions are shown. First of all, the waves are seen to have progressed at constant speed toward the east, with no preference for one phase over another. Such behavior could have arisen even in the absence of the planetary wave. However, other features of this wave packet were dependent upon its interaction with the planetary mode. For example, the existence of the free wave \((2,2)\) was again a result of its direct coupling to the waves \((3,1)\) and \((1,1)\).

In this case, the wave \((3,1)\) was unforced, so that the phase difference of about \(90^\circ\) which existed between its \(\Psi\) and \(\Theta\) modes
Polar trajectories of waves (2,2), (i=9) and wave (3,1), (i=11) for experiment IV. Time indicated in days.

- Solid line: $\psi$ mode of wave (2,2)
- Dashed line: $\Theta$ mode of wave (2,2)
- Dotted line: $\psi$ mode of wave (3,1)
- Dot-dash line: $\Theta$ mode of wave (3,1)

Also shown is the (steady) state of wave (1,1) for experiment IV.

- $\bigcirc$ denotes $\psi$ mode
- $\otimes$ denotes $\Theta$ mode
- $\blacktriangledown$ denotes $\theta^f$ for wave (1,1).
was the optimum difference for its interaction with $\Theta_1$. However, the free mode (2,2) had a phase difference of $45^0$, which did not correspond to the most efficient interaction of it with the $\Theta_1$ field. This evidently resulted from the coupling of this "secondary" free wave to the "primary" wave (3,1) and the planetary wave.

From these observations, it is clear that the fine structure in the forcing field of experiment I led to several non-linear effects connected with the wave interactions:

1) The system was more unsteady, as seen in
2) the preferred phases of growth and decay of the individual traveling waves, and hence
3) a change in mean state of all wave modes, both free and forced.

Without the "edge forcing," the system was more steady, with the traveling components showing no longitudinal preferences. However, the longitudinal structure of the planetary wave was still able to

1) excite the unforced "secondary" mode to finite amplitude, and
2) set a fixed phase relation between the traveling waves (2,2) and (3,1).

It can be verified that the latter implied that a unique wave packet shape at each longitude of the channel for each instant of time existed.

From these properties, it is not surprising that the energy transformations were nearly steady in time. They were similar to the mean exchanges of experiment I, except that the secondary wave (2,2) here received a larger portion of potential energy from both the planetary
and zonal fields. Presumably this resulted from the fixed phase relation of the free waves, which was not an interaction with optimum phase differences, due presumably to the influence of the zonal current translation, as discussed in section 3.6.

The mean flow patterns and tracks for experiment IV will not be shown here. The above remarks indicate their similarity to those in Figure 4.5, with the following exceptions:

1) The "fine structure" in \( \psi \) and \( \sigma \) was absent when the "edge" forcing was omitted.

2) The \( \psi \) phase of the planetary wave was further to the east, over the coast.

3) The tracks resembled those of experiment III more than they did experiment II.

In summary, we may state that very broad forcing fields of a single scale excited a mean flow response which was also broad, and simply related to the continent-ocean distribution. However, the fluctuating flows were less simply related to either the surface geography or the mean flow. Their distribution was controlled by the interaction of the transient flows with the stationary planetary wave.

When a more realistic \( \mathcal{F}(\psi, \gamma) \) field with sharper features was added, both the mean and fluctuating circulations were strongly influenced by the wave coupling. In this case, the fine structure also appeared in the mean state, but its relation to its input pattern was strongly obscured by the resulting wave interactions.
4.6 Experiment V: Planetary Scale Interactions

In this final experiment of Chapter 4 we temporarily free our parameter choices from the constraint $\mathcal{L}/\mathcal{C} << 1$. See section 1.1. We choose the channel to have $W = 10,000$ km, comparable with the distance from equator to pole on the earth. The longitudinal period of the system then corresponds to the mid-latitude circumference of the earth.

The great latitudinal width means that the "beta-plane" approximation is no longer a good one, for all waves having $n = 1, 2, 3$ are then of planetary scale. As Burger (1958) has shown, the geostrophic balance involves the fully variable $\mathcal{F}(\phi)$ where $\phi$ = latitude, contrary to the type 1 quasi-geostrophic motions considered here. Nevertheless, it will be seen shortly that these equations can give forced solutions which are nearly steady, thus exhibiting one of the primary characteristics of the true planetary waves.

It thus seems reasonable to consider the equations (2.2.18 – 2.2.20) as capable of crudely describing the behavior of the planetary scales of motion. In this case, the unit temperature $\mathcal{C} = 1,000^\circ\text{C}$ and the stability value $\mathcal{Q}_0$ appropriate for the atmosphere is then about .015. The corresponding $\mathcal{E}_i$ values for waves having $n = 1, 2$ or 3 then range from about .02 to .12. That is, the dynamic stability effect on these planetary waves is negligibly small. The implications of this for the heating and friction responses have been discussed in section 2.4.

The beta effect is represented by $\mathcal{B} = .45$, a large non-dimensional
value. The effect of this increase over previous experiments is to raise the neutral curve of Figure 3.2; coupled with the smaller values of $\Theta^*$, this means that the planetary waves are not unstable on the zonal flow $\Theta$. The resulting lack of a wave energy sink for $\Theta^*$ is thus an artificiality of this experiment which forces $\Theta^*$ to be near its equilibrium value $\Theta^*_0$. Also, these waves cannot provide a large upward heat flux, so that $C^0 \sim C^*$ is an artificiality of this experiment which forces $\Theta^*$ to be near its equilibrium value $\Theta^*_0$. This means that the planetary dynamics are essentially those of the "advective model," which is known to over-emphasize the instability of baroclinic waves. However, the smallness of $\Theta^*$ for our planetary waves shows that this error is negligible in the present case.

The heating coefficients are chosen to be moderately large. The thermal equilibrium distribution is described by

$$\Theta(\theta, \phi) = \Theta^* F_1 + [\Theta^*_2 F_2 + \Theta^*_3 F_3] + [\Theta^*_4 F_4 + \Theta^*_5 F_5] + \Theta^*_6 F_6.$$  \hspace{1cm} (4.6.1)

Its spatial pattern is shown in Figure 4.12(a). It is seen that the maximum land-ocean temperature differences are then comparable with the pole-to-equator difference of 85°C. Although the continental and oceanic areas extend from equator to pole, they do not all have the same width. Instead, the distribution is seen to mimic that of the earth's Northern Hemisphere in summer. (See the geographical identifications shown in Figure 4.12.)

This distribution contains the waves (1,1), (2,1) and (3,1). Thus none of the zonal harmonics $\eta = 1, 2$ or 3 tilt in the $\theta, \phi$ plane.
in this equilibrium state. In addition, their amplitudes are a maximum at the center of the channel. With this pattern of continentality, we now ask how these various planetary scales can interact, and if so, what will be their influence on the total flow.

Part of the answer is shown in Figure 4.11 for the waves having $\eta = 1, 2$ or 3 in experiment V. Waves (4,1) and (4,2) were omitted from the experiment. Part (a) shows the polar trajectories of the barotropic modes of the waves which developed from initially small amplitudes in all modes. It is seen that, in addition to the forced waves having $\eta = 1$, the free modes $\eta = 2$ also developed. Our previous results suggested that these free waves would have decayed in the absence of wave interactions. Thus, persistence of these modes must be considered a result of the internal distortion of the wave field by itself. Their presence in the final slow description thus had two consequences: 1) the longitudinal variations were no longer a maximum at the mid-latitude, and 2) the phases of the zonal harmonics varied with latitude, contrary to those in the $\theta(x, y)$ field.

The polar orbits of all wave components represented retrogressions about a dominant mean state, and had a period of 13 days. The "locking" of the free waves to fixed positions was a further consequence of the wave interactions, and it was in qualitative agreement with the analysis of section 3.5. However, those results predicted only a pure equilibrium, indicating that the slight unsteadiness which actually was present arose from the wave coupling and beta effect.
Figure 4.11. Polar diagrams for the six waves of experiment V. 
(n,m) values labelled for each wave solution. 
(a) \( \psi \) mode, with time in days indicated. \( \odot \) denotes \( \psi \) mode steady state. 
(b) Steady state \( \theta \) modes denoted by \( \otimes \). \( \theta^* \) indicated by \( \triangleleft \).
Referring now to part (b) of Figure 4.11, we see that the thermal field of each forced mode had nearly the phase of the thermal equilibrium state. The corresponding amplitudes were considerably larger than those of part (a), contrary to the case in the atmosphere. Hence, it is not surprising that the influence of advections upon these $\theta$ waves was small. Despite this, the wave interactions did generate smaller mean components in the free modes, as shown.

Comparison of parts (a) and (b) shows that the vertical wave tilts were slightly toward the east. The resulting southward heat flux thus maintained $\frac{\partial}{\partial t}$ at a value slightly in excess of $\frac{\partial}{\partial t}$. Of more interest was the fact that the mode $\psi_2$ also developed to a steady state value of $\psi_2 = -0.0115$. Along with the zonal mode $\psi = +0.0305$, this described a latitudinal variation of the mean zonal wind which had a maximum to the north of the central latitude.

Since this asymmetry was not forced $(\theta^*_{2,1} - 0)$, it was in this case a result of the internal dynamics. In fact, since the modes with $m = 2$ could develop only through the wave interactions, it is clear that these same interactions were indirectly responsible for maintaining $\psi_2$.

Such a feedback represented a completely new result from any of the analyses in Chapter 3, and arose from the fully non-linear character of the system. It represented a certain duality of the advective response to heating, in which the forced development of the wave motions to finite amplitude led to a simultaneous distortion of themselves and the zonal flow.

The spatial representation of these features is shown in the
mean maps of Figure 4.12. Parts (a) and (b) show the thermal field for the cases of no advection (pure thermal equilibrium) and full advections allowed, respectively. The influence of the advections was not particularly noticeable here, except that some tendency for the tilting of trough and ridge lines in the $(\psi, y)$ plane did occur.

The advective distortion was much more evident in the $\psi$ field shown in part (b). At high latitudes, where the maximum zonal current lay, the wave amplitudes were small. Thus, the influence of continentality at these latitudes was paradoxically to maintain the zonally symmetric flow, rather than to disrupt it. At lower latitudes, the influence of continentality was present, but the local scales of response were shifted from those forced by $\theta^*$; the oceanic troughs were sharp, and the continental ridges were broad.

These points illustrate the complex relation that even steady flows could adopt relative to their energy sources. Thus, the importance of the shape of the continents and oceans upon the flow is seen to have been important on the very large scales, as well as on the cyclonic ones.
Figure 4.12. Climatic maps for experiment V.
Dotting denotes (summertime) continental areas.
Geographical correspondence to earth is indicated.
(a) $\Theta^*(\gamma, \beta)$, isolines in $^\circ$C.
$L$ and $H$ indicate low and high temperatures.
(b) $\bar{\Theta}(\gamma, \beta)$, isolines in $^\circ$C.
(c) $\bar{\Psi}(\gamma, \beta)$, with isolines labelled in non-dimensional units. $L$ and $H$ indicate low and high pressure, respectively.
CHAPTER 5. NON-PERIODIC FLOW

5.1 Introduction

In Chapter 4 we saw that the climate of the model atmosphere for essentially periodic flows depended upon the role of the waves as elements participating fully in the advection processes. We now wish to examine this role for a more realistic flow regime which possesses a distinct lack of periodicity. This irregularity increases the difficulty of analysis, so that we must resort to the use of new mathematical tools to gain meaningful results.

The bulk of this chapter is devoted to analysis of a single experiment whose behavior resembled that of the real atmosphere more closely than any of the other three experiments. The particular parameter choices for each experiment, along with pertinent statistics of the resulting flows, are shown in Table 5.1. Comparing with Table 4.1 we see that the following changes were made to increase the realism of the flow regimes:

1) $\ell$ was decreased to a more moderate dimensional value of about $(2 \text{ days})^{-1}$. Thus, the contribution of the dissipative heating toward stabilizing the irregularities was lessened.

2) The "beta effect" was included, with $\beta$ assuming the realistic value $1.7 \times 10^{-11} \text{ m}^{-1} \text{sec}^{-1}$.

3) The "extraneous" wave modes $(4,1)$ and $(4,2)$ were completely excluded. Thus, only six separate wave shapes were used to describe the flow, but the full interaction between these
Table 5.1. Input parameters and some observed quantities of experiments VI – IX.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Exp. No.</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
<th>IX</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3.333</td>
<td>3.333</td>
<td>3.333</td>
<td>3.333</td>
</tr>
<tr>
<td>( W ) (kcm)</td>
<td></td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
</tr>
<tr>
<td>( L_0 ) (hrs)</td>
<td></td>
<td>111.1</td>
<td>111.1</td>
<td>111.1</td>
<td>111.1</td>
</tr>
<tr>
<td>( a ) (%)</td>
<td></td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
</tr>
<tr>
<td>( b )</td>
<td></td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>( T_0 )</td>
<td></td>
<td>0.140</td>
<td>0.140</td>
<td>0.140</td>
<td>0.140</td>
</tr>
<tr>
<td>( k )</td>
<td></td>
<td>0.655</td>
<td>0.655</td>
<td>0.655</td>
<td>0.655</td>
</tr>
<tr>
<td>( k' )</td>
<td></td>
<td>0.125</td>
<td>0.125</td>
<td>0.125</td>
<td>0.125</td>
</tr>
<tr>
<td>( h' )</td>
<td></td>
<td>0.030</td>
<td>0.030</td>
<td>0.030</td>
<td>0.030</td>
</tr>
<tr>
<td>( \theta_{1*} )</td>
<td></td>
<td>+.1500</td>
<td>+.1500</td>
<td>+.1500</td>
<td>+.1500</td>
</tr>
<tr>
<td>( \theta_{2*} )</td>
<td></td>
<td>-.1220</td>
<td>-.1220</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>( \theta_{3*} )</td>
<td></td>
<td>+.0800</td>
<td>+.0800</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>( \theta_{4*} )</td>
<td></td>
<td>-.0400</td>
<td>-.0400</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>( \theta_{5*} )</td>
<td></td>
<td>+.0280</td>
<td>+.0280</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>Comp. Error Rate (% Tote En/day)</td>
<td>- .16</td>
<td>-.13</td>
<td>-.08</td>
<td>-.05</td>
<td></td>
</tr>
<tr>
<td>Avg. Value and Range Var. (A+K)</td>
<td>.115 (+.030)</td>
<td>.132 (+.005)</td>
<td>.1096 (+.003)</td>
<td>.0750 (+.001)</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>.043 (+.015)</td>
<td>.046 (+.003)</td>
<td>.0530 (+.005)</td>
<td>.0308 (+.0005)</td>
<td></td>
</tr>
<tr>
<td>k</td>
<td>.072</td>
<td>.086</td>
<td>.057</td>
<td>.0442</td>
<td></td>
</tr>
<tr>
<td>( \Theta_1 )</td>
<td>+.079 (+.045)</td>
<td>+.070 (+.001)</td>
<td>+.117 (+.001)</td>
<td>+.072 (+.001)</td>
<td></td>
</tr>
<tr>
<td>( \Theta_2 )</td>
<td>+.0005 (+.1100)</td>
<td>-.0345 (+.0075)</td>
<td>.0000 (+.0480)</td>
<td>eventually decays</td>
<td></td>
</tr>
</tbody>
</table>
members was then possible.

4) The results obtained with this choice (3) were then meaningful only if \( W \) was taken to be 3,300 km. In this case, the wave system contained two values of \( \eta \) whose wave modes lay in the baroclinic range. Only the wave \((1,1)\) could then be said to be of planetary scale.

For convenience, most experiments also had the following properties:

1) \( \bar{\eta}_0 = .14 \) was taken to be a given constant, corresponding to a stratification slightly larger than that observed in the "standard atmosphere." With this simplification the energy transformations assumed a simpler form, and so a complete energy budget was possible. The reliability of the computed solutions could then be checked.

2) A more realistic vertical wind profile was made possible by taking the top of the Ekman layer to lie at \( \eta = 1000 \) mb. Thus, the equations (2.4.9 ) became the appropriate ones for the Ekman influence. With \( \eta \) and \( \eta' \) taken as before, this increased the "spin-down" rate by a small fraction of fifty percent when the vertical circulation was taken into account. Also, since \( \eta' > \eta \) it can be shown that the truncation error due to the vertical extrapolation could at no time give an anomalous increase in kinetic energy due to frictional processes.

3) A more realistic distribution of \( \theta^*(\chi,\gamma) \) was taken, in which the latitudinal \( \theta^* \) variations were of different forms over
the land and ocean. This was accomplished by representing \( \Theta \) in wave modes having \( M = 1, 2 \).

Sections 5.2 – 5.4 discuss the results of experiment VI, in which the influence of both continentality and wave interactions was retained. Section 5.5 compares the energetic and climatic properties of this experiment with those of experiment VII, in which the wave interactions were artificially suppressed. Section 5.6 discusses the behavior of the model in the absence of continentality, and comparisons are made with experiments VI and VII.
5.2 Experiment VI

The parameters of experiment VI are shown in Table 5.1. Of special note is the representation

$$\sum_{k}^{x} - x_{4} + x_{4} + x_{6} + x_{2} x_{4} + x_{6}$$

where $x_{4} = \frac{1}{2} x_{4}$, $x_{12} = \frac{1}{3} x_{4}$, and $x_{14} = -\frac{1}{3} x_{12}$.

This describes a continent which is almost everywhere colder than the ocean at the same latitude, particularly at high latitudes. A sharp transition region at the meridional coastlines takes place again. This field thus contains forcing in four different wave modes, and does not possess the symmetry noted in Chapter 4. However, waves $(2,1)$ and $(2,2)$ are again "free" modes.

The solution to the governing equations was found by numerical integration as in Chapter 4, starting from small initial amplitudes in all modes. The length of the integration extended to 700 days of real time, during which no state was observed to repeat itself. As seen in Table 5.1, the total energy, defined using the constant stability definition of $A_{1}$, oscillated between moderately wide extremes. This irregularity was considered to be a true property of the continuous solution, for the energy change unaccounted for by heating and dissipation amounted to only a fraction of a percent per day. (See Table 5.1.)

As in earlier experiments, the zonal flow $\theta_{1}$ oscillated even more strongly about its mean value above the "neutral line" of Figure 3.2. Contrary to the experiments of Chapter 4, the $\psi_{2}$ mode was present at large amplitude on a transient basis, with an apparent period near three days. At no time did $\psi_{2}$ dominate $\psi_{1}$, and so
it represented a high frequency oscillation of the latitudinal position of the maximum zonal flow.

The $\psi$ modes of all waves fluctuated rather strongly in both amplitude and phase. Over a substantial length of time all wave modes attained at least a temporarily large amplitude. Even the planetary wave ($l=1$) was mainly transient about its mean state, in agreement with the planetary wave behavior at 500 mb in the real atmosphere (Saltzman and Fleisher, 1962). The thermal modes of the $n=1$ waves were more nearly fixed, so that the vertical wave tilts were determined mainly by the $\psi$ phase. The free wave ($l=2$) possessed a moderately large mean component, but it underwent large fluctuations similar to those of the other baroclinic waves ($l=3$), $\omega(3,2)$.

Due to the irregularity, the detailed wave behavior must be considered from the statistical point of view. A measure of the amplitude oscillations is afforded by the power spectra of the $\psi$ mode kinetic energies of each mode ($\eta, \omega$), some of which are shown in Figures 5.1 and 5.2. They were computed from 500 days of data, using the technique described in Appendix C. For some of the flow variables the daily sampling frequency was not often enough, since periods somewhat shorter than two days were in evidence. However, it is believed that the true spectral densities did fall off rapidly beyond the "folding frequency" of $(2 \text{day})^{-1}$, in which case the aliasing error was confined to the higher frequencies.

Figure 5.1(a) shows the spectra for the two zonal flow variables $\psi_l$ and $\psi_2$. We see that $\psi_l$ had an extremely strong preference for low frequencies, reflecting the long-period "index cycle" which
Figure 5.1. Relative power spectral estimates for some non-dimensional $\psi$ mode kinetic energies for experiment VI. Units are same for parts (a) and (b). One standard deviation scale is shown for each estimate, which are each based upon 500 daily values. (a) spectra for zonal modes $(0,1)$ and $(0,2)$. (b) spectrum for total kinetic energy $K_\psi$ of $\psi$ flow.
occurred in \( \hat{\Theta}_1 \), to be discussed later. On the other hand, \( \psi_2 \) contained a relatively large high frequency content, indicating less organized fluctuations, which arose mainly from momentum exchange with the waves. The spectrum of the total kinetic energy at \( \psi_2 \) is shown in part (b).

Figure 5.2 shows the \( \psi \) spectrum of wave \((1,1)\), which had many features in common with those of the other waves. In particular, a tendency for the longer periods to overshadow the shorter ones is evident. This tendency was stronger for wave \((3,2)\), which was of small enough scale to make it nearly baroclinically stable. Its low frequency fluctuations were due to its observed intermittent periods of growth and decay.

Figure 5.2 also shows the smoothed spectrum of the kinetic energy of the geostrophic meridional flow at 500 mb in the real atmosphere, taken from Shapiro and Ward (1960). Since this kinetic energy occurs for wave flows only, the general similarity of its time spectrum to the wave amplitude spectra of the model suggests the realism of this experimental solution.

Further information about the wave behavior is given by their phase changes in time. A convenient description of the direction and speed of \( \psi \) mode phase propagation is given by the cross-power spectrum of \( \psi_{\psi_i(t)} \) and \( \psi_{\psi_j}(t) \) (Deland, 1964). Here, we consider wave modes only: \( i = 3,5,\ldots,13 \). The coherency magnitude indicates the existence of a linear relation between \( \psi_i \) and \( \psi_j \) for a given frequency band; i.e., it indicates to what extent the band-filtered polar trajectories form coherent patterns. For coherent
Figure 5.2. Relative power spectral estimates for the non-dimensional $\Psi$ mode kinetic energies of waves (1,1) and (3,2) of experiment VI. Units are the same for waves, and coincide with those of Figure 5.1. Scale indicates one standard deviation of the estimates, which were obtained from 500 daily values. Also shown is the smoothed spectrum of the total meridional kinetic energy at 500 mb. in the earth's atmosphere. It is expressed in arbitrary units but shown on the same scale as the other spectra for easy comparison of spectral shapes.
Figure 5.3. Amplitude and phase behavior of waves (1,1) and (3,1) in experiment VI obtained from cross spectra of $\psi_3(t)$ and $\psi_{41}(t)$, and $\psi_{41}(t)$ and $\psi_{41}(t)$, respectively. Estimates obtained from daily states in the interval days 125-625. Parts (a) and (b) show coherency magnitudes for each wave; dashed lines lie at $\pm 1$ standard deviation from the estimated value. Parts (c) and (d) show coherency phases; $-90^\circ$ signifies eastward traveling wave, $+90^\circ$ a westward moving one. Arrow denotes the $\psi_1'$ mode Rossby wave frequency for wave (1,1) calculated from the mean zonal current $\overline{\psi}$. 
frequencies, the coherency phase gives additional information. In our case, it corresponds to the phase difference between the temporal oscillations of $\psi_i$ and $\psi_{i+1}$. A value of $-90^\circ$ corresponds to a counter-clockwise movement of the trajectory, and hence indicates an eastward-moving wave.

Part (c) of Figure 5.3 shows the coherency magnitude for a typical wave of the system-wave (3,1). For frequencies lower than about (12 days)$^{-1}$, the wave trajectory behavior was evidently ill-defined, since the coherency values are low. However, for intermediate periods of three to seven days, the behavior is more apparent. Reference to part (d) shows that the wave (3,1) moved toward the east with these periods.

Referring to part (a), we see that the planetary wave (1,1) fluctuated about its mean state in a less coherent manner than the traveling wave (3,1). However, its behavior for certain frequencies was definite. For example, at short periods near three days, it resembled a wave traveling toward the east. At longer periods, especially near two to three weeks, the wave moved in a coherent fashion toward the west. Interestingly enough, this was also the approximate period of the non-divergent Rossby wave (1,1) imbedded in the mean current $\overline{\psi}$. Finally, at the two lowest frequencies, there is indication that the transient planetary wave drifted eastward with a period of a few months. This behavior is verified later in this chapter.

In short, the planetary wave is seen to have undergone rather
complicated fluctuations characterized by

1) incoherent behavior at some frequencies,

2) coherent phase movement in both longitudinal directions at one of three frequency bands, and

3) long period phase movement. In no case was its behavior characterized by a simple standing oscillation (coherency phase of zero degrees).

Let us now turn to consideration of the energy exchanges. Figure 5.4 shows a two month sample of the time fluctuations of the transformations \( \langle \hat{A}_2 \cdot \hat{A}_n \rangle \) and \( \langle \hat{A}_w \cdot \hat{A}_n \rangle \) for each wave \((n,m)\). We note first that almost all waves had \( \langle \hat{A}_2 \cdot \hat{A}_n \rangle > 0 \). An exception was wave \((1,1)\) whose fluctuating vertical tilt tended to prefer one sign over another when periods of a few weeks were considered.

The inner-wave exchanges \( \langle \hat{A}_w \cdot \hat{A}_n \rangle \) had a more variable sign, except for wave \((1,1)\), which lost \( \hat{A}_{1,1} \) to the other waves most of the time. Occasionally this loss became small or was even turned into a gain, so that this planetary wave was effectively under the influence of an intermittent eddy conductivity due to the shorter waves. In fact, the sign of \( \langle \hat{A}_w \cdot \hat{A}_{l,1} \rangle \) was nearly always positive, suggesting that the planetary wave \((1,1)\) provided potential energy to the free wave \((2,1)\) on a more or less continuous basis, as in Chapter 4.
Figure 5.4. Available potential energy exchanges of experiment VI for indicated waves \((n,m)\). Units: day\(^{-1}\)

Dashed line: \(\int_0^T \langle A_w \cdot A_{n,m} \rangle / A\)

Solid line: \(\int_0^T \langle A_z \cdot A_{n,m} \rangle / A\)
Other energy exchanges, not shown here, had the following properties:

1) $\langle \hat{A}_{n,m} \cdot \hat{K}_{n,m} \rangle$ was almost always positive; this was true for the conversions forced by both the zonal and wave interactions.

2) The transformations $\langle \hat{K}_{2} \cdot \hat{K}_{n,m} \rangle$ tended to fluctuate less than $\langle \hat{K}_{w} \cdot \hat{K}_{n,m} \rangle$ which underwent rather violent changes in magnitude and sign, especially for the shorter waves. This was primarily connected with a direct transfer between waves $(2,2)$ and $(3,1)$.

3) The coupling between waves $(1,1)$, $(1,2)$ and $(2,1)$ was such that wave $(2,1)$ stored residual amounts of energy, although the primary exchange was between the waves $(1,1)$ and $(1,2)$.

The time mean transformations will be given later in section 5.9. For the above comments, it is clear that the transient energy exchanges were an important indicator of the behavior of the system. We next consider some of their longer-period characteristics.
5.3 Low Frequency Fluctuations

To study the longer-period flow characteristics in more detail, let us consider the "monthly" time means taken over 25 day intervals, starting with day 125 and ending with day 625. The slow variations of the two zonal modes $\psi_1$ and $\psi_2$ will be discussed later; generally speaking, only the mode (0,1) exhibited significantly organized oscillations from month-to-month.

Figure 5.5 shows the monthly variation of the $\psi$ mode wave states, numbered according to the monthly interval number. The less variable $\theta$ mode positions at each period are simply represented by dots. We first note the relatively large differences in monthly wave positions; these were especially marked for the planetary wave (1,1), which underwent large amplitude and phase changes. Interestingly, the general sense of these slow changes about the mean state was an eastward drift, in agreement with the cross-spectrum analysis discussed earlier.

The other waves, including the two free modes (2,1) and (2,2), are seen to have favored one position even on these monthly means. In fact, these free modes possessed differences which were no larger than those of their forced wave counterparts (3,1) and (3,2).

With these changing wave patterns, it is not surprising that the monthly mean energy transformations exhibited even more striking variations. Figure 5.6 shows the inter-wave and zonal exchanges of available potential energy and kinetic energy for the waves as grouped.
Figure 5.5. "Monthly" mean wave positions for consecutive twenty-five day intervals starting at day 125 of experiment VI. Integers denote $\psi$ mode positions for that "month". Monthly $\psi$ modes are denoted by (•) in parts (a) and (b), and by (λ) in parts (c) - (f).
Figure 5.6. Some mean "monthly" energy transformations grouped according to zonal harmonic \( \eta \) for experiment VI.
according to their \( n \) values. We see that the ranges of monthly variations were considerable. In fact, the power spectrum of was computed and found to contain its maximum energy at periods longer than about two weeks.

The slow variations in the monthly standard deviations of energy transfer, not shown here, were similarly variable, implying a certain long-period intermittency of the daily energy exchanges. In some cases these high-frequency exchanges seemed related to the slower variations, but a general relationship was difficult to pin down.

The slow variations themselves appeared to be inter-related only in that the wave exchanges in parts (b) and (d) showed negative correlations between the planetary waves, a source of energy, and the shorter waves. However, we also note a tendency for the cyclone waves \( n = 2,3 \) to have exchanged kinetic energy between themselves. This behavior was similar to that found in Chapter 4 on a shorter time scale.

Given these fluctuating waves and their interactions, let us now turn to their influence on the model climate. We first consider the maps of monthly mean statistics, shown in Figure 5.7. This figure excludes the \( \psi \) field, whose monthly means did not vary significantly. Part (a) shows the mean \( \psi \) field for five consecutive months. While they all had certain broad-scale features in common, their differences were quite significant.

These differences are more obvious in the "high frequency" patterns in parts (b) and (c). Comparison of these monthly standard deviations with the more slowly varying mean flow yielded no obvious relations,
Figure 5.7. "Monthly" mean statistics for five consecutive 25-day periods of experiment VI. 

(a) monthly mean of $\Psi(y, t)$ expressed in 100's of m. 
(b) monthly standard deviation of $\Psi(y, t)$ expressed in 100's of m. 
(c) monthly standard deviation of $\Theta(y, t)$ expressed in °C.
with one exception. The maps for the time interval days 225-250 show that the strongest fluctuations were confined to relatively high latitudes in contrast to the other cases shown. During this time the mean flow contained only a small contribution from the planetary wave. This suggests that the fluctuating states were most dependent upon the planetary wave state than any other single feature.

The considerations of Chapters 3 and 4 in fact strongly suggested that such a relation might exist. In addition, they also indicated that the longitudinal structure of this quasi-permanent planetary wave might induce geographical differences in the growth, movement, and decay of the transient systems. Thus, these fluctuating flows could be considered to be a sort of inhomogeneous turbulence, in which case the time spectra might be expected to possess geographical variations. In fact, the distributions just shown in Figure 5.7 give some indication of this.

For the five points indicated in Figure 5.8 (e), the low resolution power spectra shown in parts (a) - (d) were computed. A common feature of most of the spectra is the high power level at low frequencies. This is particularly true for the $\psi$ fluctuations at station 1, whose spectrum seems to possess a general "noise" level at all frequencies plus an important low frequency component at periods longer than one week. Comparison of this $\psi$ spectrum with that at station 2 shows that the latter contains more power at periods of one week, suggesting that cyclone-like eddies were prevalent at station 2.
Figure 5.8. Relative power spectra of $\Psi$ fluctuations (parts (a) and (b)) and $\Theta$ fluctuations (parts (c) and (d)). Taken from experiment VI at selected geographical positions as shown in part (e).
Part (c) shows the $\Theta$ spectra for stations 2, 4 and 5. Since each of these exhibits high power levels at high frequencies, we focus our attention upon the lower ones which were uncontaminated by the implied aliasing. Station 5 is a somewhat singular case since its low frequency spectral content is actually smaller than that at most other frequencies. However, both it and the spectrum at station 4 possess high power levels in a band whose periods run from one to two weeks. The $\Theta$ spectrum at station 2 does not exhibit this clear-cut frequency preference.

Samples of other spectra are shown in parts (b) and (d). While the comparative features contrast less, we note that both the $\psi$ and $\Theta$ spectra at station 3 are noticeably lacking in high frequencies.

In summary, it is clear that the spectral shapes varied significantly between different geographical areas, and apparently reflected the geographical preferences for each separate physical process which contributed to the fluctuating state of the atmosphere. From our previous findings, a significant portion of these processes must have arisen from the internal non-linearities of the space-variable flow itself.

5.4 Planetary Fluctuations and Energy Exchange

In this section we consider further the slow variations of the flow patterns and energy transformations in an attempt to define more clearly their relation to the direct and indirect influence of continentality.

Let us start by examining the relations between the planetary wave (1,1) and the cyclone waves, considered first on a daily basis. Figure
5.9 (a) shows the distribution of daily \( \psi \) states of the wave \((1,1)\) for days 125-625. While there were rather large variations in the position of this wave over that length of time, the definite tendency for a mean preferred state is seen. The phase of this state is seen to have been near that of the quasi-permanent \( \theta \) state, shown by a cross. Thus, to a large extent, the fluctuations of \( \psi \) phase about its mean state were reflected in similar changes in the sign of the vertical wave tilt. This relation, not shown, was such that the \( \psi \) states in quadrant IV of the polar plane were associated with a westward-tilting wave, and those in quadrant III with an eastward tilting wave.

In Figure 5.4 a tendency for an inverse relation between \( \langle A_{2, \psi} A_{1,1} \rangle \) and \( \langle A_{w, \psi} A_{1,1} \rangle \) was indicated. This is seen more clearly in part (b) of Figure 5.9, which shows the position of the \( \psi \) mode of wave \((1,1)\) on days when \( \langle A_{w, \psi} A_{1,1} \rangle \) deviated strongly from its mean negative value. We see that the effect of the shorter waves upon the fixed \( \phi \) mode corresponded to that of a non-linear eddy conductivity with rather peculiar properties. These were that the magnitude of the eddy effects, and even their sign, depended upon the position of the barotropic mode of the planetary wave flow. In a sense these interactions were being "turned on and off" by the phase changes of the planetary wave.

In Chapter 4 we noted a feedback of the short waves onto this planetary mode, and presumably this could account for these phase changes. However, the manner in which these changes might have induced a feedback onto the planetary wave thermal field is not so
Figure 5.9 (a) Distribution of daily $\psi$ mode positions of wave $(1,1)$ for days 125-625 of experiment VI. Units are the number of occurrences during that period which lay within the indicated square region about each wave state $P$. The mean state of the $\psi$ mode of wave $(1,1)$ is denoted by (X).

(b) $\psi$ mode positions of wave $(1,1)$ on days when $\langle \frac{\mathbf{A}_w \cdot \mathbf{A}_{1,1}}{\mathbf{A}} \rangle$ deviated extremely from its mean value. Taken from days 250-700 of experiment VI.

(*) $\int_{p}^{c} \langle \frac{\mathbf{A}_w \cdot \mathbf{A}_{1,1}}{\mathbf{A}} \rangle \geq 0 \text{ day}^{-1}$ (x) $\int_{p}^{c} \langle \frac{\mathbf{A}_w \cdot \mathbf{A}_{1,1}}{\mathbf{A}} \rangle \leq -0.25 \text{ day}^{-1}$
simple; the logical mechanism would seem to be that the barotropic planetary mode somehow would change the relation of the shorter waves to its own thermal field.

To examine this possibility, we refer to Figure 5.10 which shows the \( \Phi \) states of the shorter waves (2,1), (2,2) and (3,1) on days when \( \langle A_{\omega} \cdot A_{11} \rangle \) departed strongly from its mean value, as in Figure 5.9 (b). While the relationships are not so clear as before, one point stands out. The wave phases, but not their amplitudes were segregated according to the direction of the deviation of \( \langle A_{\omega} \cdot A_{11} \rangle \). That is, the growth of the short waves at the expense of the planetary wave depended strongly upon their own phases, in accord with the results of Chapter 4. This behavior was especially clear for the two free waves (2,1) and (2,2), shown in parts (a) and (b). Taken with the results of Figure 5.9 (b), this shows that the energy exchanges depended upon the over-all configuration of all scales of motion relative to the geography. Again, we see a correspondence of the irregular behavior to that found in the periodic solutions of Chapter 4.

Let us now turn to the system behavior as described by the variations in the monthly means of some variables, pictured in Figure 5.11. Parts (a) and (b) show that the transformations affecting \( A_{11} \), namely \( \langle A_{\omega} \cdot A_{11} \rangle \) and \( \langle A_{2} \cdot A_{11} \rangle \) tended to compensate each other, so that their sum fluctuated less than either of them taken individually. Comparison of part (b) to part (a) shows that the \( \Phi \) phase of wave (1,1) was related to \( \langle A_{2} \cdot A_{11} \rangle \) on even this monthly basis. Parts (d) and (e) suggest that the monthly level of zonal baroclinicity \( \Phi \).
Figure 5.10. $\psi$ mode positions of some cyclone waves on days when $<A_w'\cdot A_{y_1}'>$ deviated extremely from its mean value. Taken from days 250-700 of exp. VI.

(a): $f_o <\frac{A_w'\cdot A_{y_1}}{A}> > 0$ day$^{-1}$ (X); $f_o <\frac{A_w'\cdot A_{l_1}}{A}> \leq -0.25$ day$^{-1}$
Figure 5.11. Parts (a)-(f) show the variation of selected "monthly" mean variables at one "month" intervals for experiment VI.
Part (g) shows schematically the energy transformations (proportional to arrow length) and associated zonal flow and planetary wave states during two stages of the idealized "long-period" cycle of experiment VI.
was determined primarily by the transformation \( \langle \tilde{A} \cdot \tilde{A} \rangle \)
for all waves \( (\eta, m) \). However, comparison of (d) with (f), the transformation involving just the shorter waves \( (\eta, m) = (1, 1) \) indicates that their northward advective heat flux was nearly constant on a monthly basis, since \( \langle \tilde{A} \cdot \tilde{A} \rangle \) is proportional to this heat flux times \( \Theta \). Thus the oscillations in \( \Theta \) appear to have been driven by the interaction \( \langle \tilde{A} \cdot \tilde{A} \rangle \).

Given these points, let us now try to construct a coherent picture of the slow variations, whose energetic properties are pictured schematically in part (g). This diagram represents the fact that the oscillating transformations affecting \( A_{\eta} \) and \( A_{\eta} \) tend to be compensated in such a way that the net advective energy fluxes into each of them was more constant. The individual exchanges did fluctuate slowly in a manner related to those of the zonal flow and planetary wave, as indicated.

Their characteristics were such that when \( \Theta \) was large, the important interactions were those involving only the zonal flow, with the planetary wave tilting eastward with height. Interestingly enough, this wave state was similar to that of a steady wave solution (as given in Chapter 3), which had the prescribed property that the wave could interact with the zonal flow alone. However, a further curiosity is that the relation between the wave tilt and \( \Theta \) was opposite to that expected from the linear steady solutions of section 3.4.

On the other hand, when \( \Theta \) was relatively small, it continued to provide potential energy to the cyclone waves. In this case, the planetary wave \( (1, 1) \) did the same via its direct interactions with the shorter waves.
The most reasonable explanation of these two stages must be capable of describing the relation between $\langle \hat{A}_w, \hat{A}_{1,1} \rangle$ and $\langle \hat{A}_{2,1}, \hat{A}_{1,1} \rangle$. A possible one could involve oscillations in the energy levels of the cyclone waves. However, such organized variations were not noted in this experiment. The remaining possibility would then seem to involve the planetary wave as a keystone.

We have noted that the phase fluctuations of its $\psi$ mode changed its vertical tilt and ultimately the $\hat{\Theta}_l$ level. Along with the constancy of the cyclone-scale northward heat flux, this would account for the variations in $\langle A_{0,1} \rangle$, $\langle A_{0,1,1} \rangle$, and $\langle A_{0,1}, A_{1,1} \rangle$ of part (g). The variations in $\langle \hat{A}_w, \hat{A}_{1,1} \rangle$ are less easily accounted for, but a plausible argument would be that the $\psi$ phase relation of wave $(1,1)$ determined not only its interactions with $\hat{\Theta}_l$, but also that with the cyclone waves. In Chapter 4 it was seen that the interactions $\langle \hat{A}_w, \hat{A}_{1,1} \rangle$ involving the planetary thermal field were sensitive to even a simple mean current. To the extent that a similar influence could arise from the steering influence by the $\psi$ mode of wave $(1,1)$, the relation between its phase and $\langle \hat{A}_w, \hat{A}_{1,1} \rangle$ would then be explained.

These idealized relations thus allow the following description of the long-period cycle. A "normal" state defined over one "period" would be characterized by a wavy flow of planetary scale consisting of the zonal mode $(0,1)$ and the forced planetary wave $(1,1)$. Its $\Theta$ field would be nearly fixed in space, with the space-variable heating field continually maintaining it against losses to the transient cyclones.
However, the $n$ phase of this flow would swing slowly back and forth, thereby altering the effect of the cyclone eddies on the planetary store of potential energy. With $\Theta_i$ large, this energy would first be converted to $\tilde{A}_{ij}$ and then to the potential energy of the shorter waves. On the other hand, with $\Theta_i$ smaller $\tilde{A}_{ij}$ would act as a direct energy source for the cyclones.

This description is not adequate for fully understanding the long-period behavior, since we have been able to account only for the internal dynamic consistency of each stage. However, this has pointed out that the cycle depended critically upon the two central topics of this thesis: the existence of long waves forced by the continentality and the importance of the resulting inter-wave interactions.

At the present time, the mechanism which caused the transition between the two stages is unknown; we know only that it was an intrinsically non-linear process which operated at frequencies much lower than the natural linear frequencies of the system.
5.5 Comparative Solutions for Three Representations of Advection

In this section we compare the results of experiment VI with those of two simpler solutions. The first of these is that of experiment VII, which differed from experiment VI only by exclusion of the wave interactions (groups III and IV). Interactions of groups I and II remained, however. The second variation amounted to further excluding the group II zonal interactions and solving the resulting steady state equations obtained by setting all time derivatives equal to zero. This solution was found by linearizing the equations about the mean zonal state $\bar{u}$ and $\bar{v}$ of experiment VI.

The initial conditions for experiment VII were the same as those of experiment VI. The resulting solutions differed strongly in one important way, however. Without the wave interactions, it settled into a quasi-periodic behavior which resembled the solutions of Chapter 4 much more than the irregular behavior of experiment VI. This can be seen in the estimates of energy variability shown in Table 5.1. Thus, the increase in the degrees of freedom associated with the advections resulted in a similar increase in the irregularity. This result is not surprising, but it is interesting, for the long period fluctuations of experiment VI did not develop when the wave interactions were excluded. The longest periods seemed to be about 12 days, so that monthly variations were negligible.

The zonal flow $\bar{v}$ was remarkably steady, in stark contrast to its behavior in experiment VI. $\bar{w}$ exhibited only moderate fluctuations about its mean value. Thus, the total zonal flow was nearly
steady, with the maximum zonal velocities occurring to the north of the central latitude, contrary to experiment VI.

The waves in experiment VII contained a larger portion of the total energy, as Table 5.1 would suggest. For example, the planetary wave \((l_1, 1)\) was nearly stationary, and had a larger amplitude. Its vertical tilt was toward the east. The wave \((l_2, 2)\) was more affected by the \(\mathcal{U}_2\) fluctuations, and it oscillated about its mean state with the same period of about three days. Its amplitude was also somewhat larger than in experiment VI.

The forced baroclinic modes were hardly affected by their \(\Theta^x\) fields. Both were travelling waves with periods of about three days. Their amplitudes were also larger in this case, with the wave \((3_1, 1)\) dominating wave \((3_1, 3)\).

The free mode \((3_1, 1)\) had almost the same large amplitude as wave \((3_1, 3)\). It drifted eastward with a period of about 12 days, showing no tendency for any preferred phase. The wave \((2_1, 2)\) exhibited the same behavior, except that its amplitude was considerably smaller.

Taken together, these observations suggest that the larger value of \(\Theta_1\) in experiment VI was a consequence of the wave interactions acting as an increased friction on the most unstable waves, effectively raising the neutral line shown in Figure 3.2. In support of this, we note that a dominant wave mode was difficult to find in that experiment; the wave interactions appeared to spread out the energy in such a manner that some of the relatively stable waves were forced to
interact with the zonal flow. In any case, it is clear that even in the presence of continentality the wave interactions were able to alter the zonal flow \( \Theta_1 \) indirectly. In addition, the appearance of \( \Omega_2 \neq 0 \) in experiment VII indicates that a similar influence of the wave coupling on the zonal kinetic energy exchanges took place.

A part of the differences between the three solutions is indicated by their mean wave states, shown in Figure 5.12. The simplest solutions were the linear steady states denoted by the "L" solution. The unsteady "Z" solution corresponded to experiment VII, while the fully non-linear solution "W" was that of experiment VI.

The \( \Theta \) modes seemed rather insensitive to the detailed advective mechanisms. The main differences arose from the wave interactions. For example, the thermal amplitudes of the long waves \( (1,1) \) and \( (1,2) \) were both reduced by the wave interactions. The free wave \( (3,1) \) had a mean thermal state induced by the wave coupling.

The relations were not so simple for the \( \Psi \) modes. Except for the wave \( (3,2) \), these mean states were sensitive to the advections. For example, the difference of the \( L \) and \( Z \) solutions was in general due to the influence of \( \Omega_2 \). These differences were not large for most waves, except for wave \( (1,2) \), whose \( \Psi \) phase and amplitude were altered greatly by the zonal flow asymmetry in the \( Z \) solution. This resulted in an equivalent barotropic mean wave state which thus changed its mean state interaction with \( \Theta_1 \).

The most significant differences between the \( Z \) and \( W \) solutions did not arise from the influence of \( \Omega_2 \), but from the influence of
Mean states for experiment VI which allowed wave interactions (W): Experiment VII which allowed only zonal interactions (Z): the steady linear solution corresponding to experiment VI (L).

[,] denotes the $\psi$ mode; $\circ$ denotes the $\theta$ mode. $\triangledown$ denotes $\theta_0$ for waves (1,1) and (1,2); not indicated are $\theta_0 = 0$, $\theta_0^x = -0.400$ for wave (3,1) and $\theta_0 = 0$, $\theta_0^x = +0.280$ for wave (3,2).

**Figure 5.12**
the wave interactions. The difference was rather striking for waves $(l,1)$ and $(l,2)$, whose mean phases lay further to the east in the $W$ case, and resulted in a mean state transfer of potential energy by the $\eta_{2l}$ harmonic out of $\overline{\Theta_l}$. Again, we see the appearance of short-wave feedback onto the planetary waves and indirectly onto the zonal flow.

As in the previous examples, the free waves $(2,1)$ and $(2,2)$ possessed important mean $\varphi$ components which of course owed their existence to the wave interactions. In this case, the mean $\varphi$ amplitude of wave $(2,1)$ was nearly as large as that of the maximally forced modes $(l,1)$ and $(l,2)$. The mean $\varphi$ state of the forced wave $(3,1)$ was also affected by the wave coupling, but in that case the mean amplitudes were small.

Further contrasts between these three solutions were evident in the mean energy transformations. Figure 5.13 shows some of these for experiment VI. We consider first the mean exchanges with the zonal flow, shown in parts (a) and (b).

We see that $\Theta_l$ acted as a mean source of potential energy for all waves except the planetary mode $(l,1)$, whose transient states transferred energy into $\Theta_l$. Taken with mode $(1,2)$, this implied that the conversion between the harmonic $\eta_{2l}$ (earth wave three) and the zonal flow was not a dominant one, contrary to the case observed in the earth's atmosphere. See, for example, Murikami and Tomatsu (1965) and Wiin-Nielsen et al. (1963).

Deviations from observed behavior were also evident in the
Figure 5.13. Some mean energy transformations for experiment VI for individual modes \((n,m)\). The transformations are normalized with respect to the mean energies \(\overline{A}\) and \(\overline{K}\), respectively, taken from experiment VI. Units are day\(^{-1}\). Arrows indicate direction of energy transfer.

\[
\begin{align*}
(a) \int_{c} \frac{\langle A_{2} \cdot A_{n,m} \rangle}{\overline{A}} & \quad (b) \int_{c} \frac{\langle k_{c,1} \cdot k_{n,m} \rangle}{\overline{K}} & \quad (c) \int_{c} \frac{\langle A_{w} \cdot A_{n,m} \rangle}{\overline{A}} & \quad (d) \int_{c} \frac{\langle k_{w} \cdot k_{n,m} \rangle}{\overline{K}}
\end{align*}
\]
kinetic energy exchanges. Along with its interaction with wave \((3, \lambda)\), the mode \((0, \lambda)\) also lost energy to waves \((1, 1)\) and \((1, \lambda)\). This is in disagreement with the findings of Saltzman and Teweles (1964) for the earth's atmosphere. Based upon nine years of data, they found that all waves supplied kinetic energy to the zonal flow. This discrepancy in the present model probably arises from the truncated \((n \leq \lambda)\) representation of the flow field, which does not allow a true "jet" to form. Inclusion of the zonal mode \((0, 3)\) would probably alleviate this problem. At any rate, it is interesting to note that this apparently anomalous conversion is in agreement with the observations of Murikami and Tomatsu, which dealt with data for a single year.

Parts (c) and (d) show the direct energy exchange between the waves in this experiment. The exchanges of part (c) show that the planetary wave \((1, 1)\) was an important source of potential energy for all waves except \((2, 1)\). This wave, despite the fact that it was unforced, also contributed its potential energy to the other waves. A separate energy study, not shown here, indicated that this transfer was in fact one arising from the mean state interactions alone. It represented a net loss by the mean wave \((2, 1)\) and a gain by the mean wave \((1, \lambda)\). Other features of part (c) are in agreement both with the results of Chapter 4 and the winter observations of Murikami and Tomatsu, and again suggest the importance of \(\Pi_{1, 1}\) as a wave energy source.

Curiously, this wave also was a source of kinetic energy for the other waves. Part (d) in fact shows that the energy transfers
between the zonal harmonics \( n = 1, 2, \text{and } 3 \) were similar to those potential energy exchanges of part (c). On the surface, such directions of kinetic energy flow seem contrary to the simple barotropic argument that the medium scale baroclinic waves should provide kinetic energy for both the larger and smaller scales. Murikami and Tomatsu and Saltzman and Fleisher (1960) both found apparent observational agreement with this hypothesis. Looking at the results of part (d) more closely, it was found that the energy transformations arising from the transient flows did possess this property. On the other hand, the large mean state transformations between waves \((1, 1)\), \((1, 2)\) and \((2, 1)\) was responsible for a net flow of energy into the medium scale, which for these three waves was the mode \((1, 2)\). It is possible that this fact could explain the behavior of the second harmonic in the earth's atmosphere, which, according to Saltzman and Teweles, exports kinetic energy to the shorter waves.

We may end this discussion of Figure 5.13 by noting one further point. We see that the wave interactions were the dominant mechanism of energy transfer for the long waves \((1, 1)\) and \((2, 2)\), while wave \((1, 2)\) was affected equally by both zonal and wave interactions. Wave \((2, 1)\) received potential energy from the zonal flow but also received kinetic energy from its mean state "cascade" interaction with waves \((1, 1)\) and \((1, 2)\). The baroclinic waves \((3, 1)\) and \((3, 2)\) were influenced mostly by the zonal flow alone.

Let us now turn to the mean transformations for experiment VII, shown in Figure 5.14, parts (a) and (b). We see that the zonal flow lost potential energy and gained kinetic energy, as in experiment VI
Figure 5.14. Some mean energy transformations for experiment VII (parts (a) and (b)) and the steady linear solutions (part (c)). All transformations are normalized with respect to \( \overline{\epsilon} \) or \( \overline{K} \) of experiment VI to facilitate comparison with Figure 5.13. Units are day\(^{-1}\). Arrows indicate direction of transfer.

(a) \( \int_0^1 \frac{\langle \hat{A}_z \cdot \hat{A}_m \rangle}{\overline{\epsilon}} \) for exp. VII

(b) \( \int_0^1 \frac{\langle K_{02} \cdot \hat{K}_n \rangle}{\overline{K}} \) for exp. VII

(c) \( \int_0^1 \frac{\langle \hat{A}_{01} \cdot \hat{A}_m \rangle}{\overline{\epsilon}} \) for steady solution.
and the real atmosphere. Contrary to experiment VI, the wave \((l, l)\) here lost potential energy to the zonal at a large rate. Hence, its behavior in the absence of wave interactions was even further from that observed in the real atmosphere.

Except for the wave \((3, 1)\), the transformations \(\langle k_{0,2} \cdot k_{h, m} \rangle\) of part (b) here reversed their signs from those shown in Figure 5.13(b) for experiment VI. Clearly the wave interactions exerted a profound effect on even the zonal mode \(\psi_2\) in that experiment. In the present one, \(\psi_2\) was maintained by the longest wave harmonic \(\eta_{1,1}\), since its scale was intermediate to those of waves \((l, l)\) and \((l, 2)\). A separate investigation again showed that this transfer arose primarily from the mean state "tilt" of the harmonic \(\eta_{1,1}\).

We also note that the harmonic \(\eta_{2,2}\) received kinetic energy from the zonal flow. While this is consistent with the governing equations, it is contrary to the situation observed in the atmosphere.

In short, we note that the exclusion of the wave interactions actually forced the model to depart even more strongly from behavior exhibiting atmospheric-like energy relations between the waves and the zonal flow. In addition, it also amounted to ignoring the entire set of interwave energy exchanges which are observed to be important in the behavior of the real atmosphere.

With these facts in mind, it is not surprising that the linearized steady state equations, with wave interactions ignored, yielded
a solution whose zonal energy exchanges were a further departure from reality. Part (c) of Figure 5.14 shows the potential energy exchanges of each forced wave with $\Phi_i$. It is seen that the forced planetary wave $(l,1)$ dominated this "mean state" exchange, feeding its energy into the zonal flow at a greater rate than its companion wave $(l,2)$ could extract it. (Since the system was linear, that energy flow should not be interpreted as a gain of energy by the wave $(l,2)$ at the expense of $(l,1)$.) Finally, we note that no kinetic energy exchanges took place, since $\Psi_2$ was taken to be zero.

Having considered the behavior of the waves taken individually, let us now see how their joint behavior resulted in particular climates for each of the three solutions. Figure 5.15 shows $\Theta^{x}(x,y)$ for these cases. We see that in the absence of geostrophic motion, the thermal field would have corresponded to a mass of very cold air over the northern part of the continent, with somewhat warmer air elsewhere. In particular, within the bounds of the truncated representation, the ocean would have been uniformly warm, except at high latitudes. Between the cold and warm areas a well-defined band of large temperature gradient would have existed, and it would have been especially strong at the high-latitude coasts.

With this field driving the system, the geostrophic flow evolved to produce the mean patterns shown in Figures (5.16 - 5.18) for the three solutions.

Part (a) of Figure 5.16 shows $\Phi(x,y)$ for experiment VI, which included the influence of wave interactions. This pattern is less simple than that given in Chapter 4, and its relation to
Figure 5.15. $\theta^*(\xi, \eta)$ for experiments VI, VII and the steady linear solution. Units are in °C. Continental areas are denoted by dotting.
the $\Theta^\star(y_1,y)$ field is quite obscure. For example, at mid-latitudes we note the existence of two mean troughs, while at low latitudes a single ridge stands out.

The trough over the continent reflected to some extent the store of cold air there. However, the ridge and trough over the ocean were not obviously related to oceanic features in the $\Theta^\star$ field. It is interesting to note that this cyclonic area over the ocean has its counterparts in the real atmosphere in January. See, for example, Lahey et al (1958a).

Since the two mean troughs were tilted in opposite directions, their tendency to influence the $\Psi_2$ zonal mode was small. However, analysis of the mean state did indicate a transfer of kinetic energy from waves $(l_1,l)$ and $(l_1,l)$ to wave $(2_1,1)$. Since wave $(2_1,1)$ accounted for the oceanic trough in $\Psi(y_1,y)$, it appears that it must be considered to have been the downstream reflection of a self-distortion of the continental trough system. In any case, we see that even a single forced harmonic $h$ in this case $h=1$, can "cascade" to produce a mean response in the harmonic $2h$.

These dynamical features combined to form a generally confluent mean flow over the continent, with a maximum flow velocity near the eastern continental coast, followed by a diffluent pattern over the ocean. These three characteristics are each typical of the observed mean flow in January, and suggest that the wave dynamics forming these patterns were qualitatively correct.

The $\Theta(y_1,y)$ field for experiment VI is shown in part (b). While the cold air was somewhat contained over the continent, we see
Figure 5.16. Mean flow maps for experiment VI. (Wave interactions included.)
that it did extend out over the ocean at higher latitudes, with a maximum band of large temperature gradient lying away from the northern boundary of the channel. As in the $\overline{\psi}$ field, the $\overline{\Theta}$ pattern was somewhat diffluent over the ocean, as is observed in the earth's atmosphere during winter (Peixoto, 1960).

The maximum land-ocean thermal contrast was about $20^\circ C$, approximately $5\%$ larger than that observed. The mean latitudinal temperature difference of $21^\circ C$ over the channel was more nearly in agreement with observed values, however. The magnitudes of the corresponding heating rates were larger than those in the earth's atmosphere by about a factor of two, being typically about $5^\circ C/day$. Only in local areas making up about $1\%$ of the total area did the rates ever exceed $10^\circ C/day$.

Having noted the similarities of the mean flow of experiment VI to the real atmosphere, let us now consider the disparities between the observed characteristics and the results of the experiment which excluded the wave interactions. Figure 5.17 shows the mean flow for experiment VII. Comparing part (a) to Figure 5.16 (a), we see a crude resemblance between the two $\overline{\psi}$ fields. However, large differences appear over the ocean, where in the eastern part the flow directions are almost at right angles to each other. Connected with this is the fact that the cyclonic and anti-cyclonic regions are confined to the continents and oceans, respectively, being associated with the single wave train of planetary scale, contrary to Figure 5.16 (a). We again see that the mean flow patterns were importantly
\( \overline{\psi}(x, y) \) expressed as geopotential, in m.

\( \overline{\Theta}(x, y) \) expressed in °C.

Figure 5.17. Mean flow maps for experiment VII.  
(Wave interactions excluded.)
influenced by the direct inter-wave coupling.

Comparing Figure 5.17(b) to Figure 5.16(b), we see that the differences were not so important for $\theta(\chi, \gamma)$. The main differences arose over the northwestern ocean, and were such that the bulk of the oceanic atmosphere was at a more uniform temperature when the wave interactions were excluded. In this case, no diffuseness in the thermal pattern was noticeable. In short, the $\bar{\theta}$ pattern of Figure 5.17(b) resembled a simpler distortion of the imposed $\bar{\theta}^*(\chi, \gamma)$ field than did its counterpart in experiment VI.

Figure 5.18(a) shows the $\bar{\theta}$ field obtained for the steady linear solution. As expected, it resembles $\bar{\theta}$ for experiment VII more closely than it does that of experiment VI. We see that the mean latitude of the jet, which meandered on the planetary scale only, was higher in the non-linear solution of experiment VII, due to the mean wave tilt there.

Part (b) shows $\bar{\theta}(\chi, \gamma)$ for this steady solution. It is seen that the temperature gradient was again confined to lie nearer the $\bar{\theta}^*(\chi, \gamma)$ field.

Having noted the relation of the individual $\bar{\theta}$ and $\bar{\Theta}$ patterns to $\bar{\theta}^*$, let us now consider a measure of their joint relation. The mean flow at the top of the Ekman layer, given by $(\bar{\theta} - 2\bar{\Theta})$, is shown in Figure 5.19 for all three experiments. We see that all patterns had common features, such as the oceanic low pressure area and the continental high pressure. None of the maps is a particularly good representation of the observed surface flow in January (Lahey et al, 1958b). In particular, the two solutions for which the wave
Figure 5.18. Steady forced flow computed from $\bar{\Psi}$, $\bar{\Theta}$, of experiment VI. (Interaction with zonal flow linearized; wave interactions excluded.)
Figure 5.19. Mean streamfunction at the top of the Ekman layer (1000 mb) for the three solutions of this section, expressed as geopotential, in m.
Streamfunction obtained from \( \overline{\Psi}(r, \theta) - 2 \overline{\overline{\Psi}}(r, \theta) \).
Figure 5.20. Standard deviations for experiment VI, computed from daily values. (Wave interactions allowed.)
Figure 5.21. Standard deviations for experiment VII. (Wave interactions excluded.) Estimates computed from daily values.
interactions were neglected possessed anti-cyclonic centers over the eastern continent, contrary to the observed state, which resembles that of part (a) most closely. In short, the mutual relations of \( \bar{\varphi}, \bar{\theta}, \alpha, \lambda, \theta^* \) seemed most realistic for experiment VI.

Finally, let us turn to the standard deviation maps for experiments VI and VII, shown in Figures 5.20 and 5.21, respectively. The general level of the \( \varphi \) variability for both experiments was about that observed in the real atmosphere. However, only in experiment VI did appreciable longitudinal differences arise. In that case the distribution resembled that observed in the earth's atmosphere, but the contrasts between longitudes were not large enough.

The \( \theta \) fluctuations, whose statistics are shown in Figures 5.20(b) and 5.21(b) were of the same magnitude as those in the earth's atmosphere, for both experiments. However, the longitudinal differences again appeared only in the case with wave interactions. Their pattern seems in reasonable agreement with that given by Peixoto, since both possessed a minimum near the west coast and a maximum in the vicinity of the eastern continental coast.

5.6 Solutions Without Continentality

Our previous efforts have been directed toward understanding the atmospheric behavior when the influence of continentality was present. A final question of some importance is that of its characteristics when the longitudinal differences are not impressed by the heating. In particular, we may wonder how the wave developments change, particularly on the planetary scale. Also, how will the
energy transfers between scales be affected? To answer these questions, two experiments were performed in which \( \Theta^k(k, y) \) had the simple symmetric form \( \Theta^k(k, y) = \Theta^k_i F_i(y) \). All other aspects of these experiments corresponded to either of experiments VI or VII. See Table 5.1.

The first experiment VIII was one in which the interactions between waves were allowed. It was run for 250 days, starting from the same initial conditions as before. After about ten weeks, the total energy reached a nearly constant level. This was accompanied by a rather regular behavior of the flow modes of largest amplitude. The solution was not periodic, but it certainly did not exhibit the marked aperiodicities noted in the case with continentality (experiment VI).

Two features of great importance were: 1) the steadiness and large amplitude of the zonal flow mode \( \Theta_i \), and 2) the lack of significant energy residing in the longest waves \((1, 1)\) and \((1, 2)\). These points will be discussed later in this section. Also of interest was the fact that the waves \((2, 1)\), \((2, 2)\) and \((3, 2)\) dominated the wave field. The wave \((2, 2)\) was of largest amplitude, and its steady progression resembled that of a "dishpan" wave.

Aside from the dominant wave interactions with the zonal flow, some small but somewhat persistent exchanges of energy took place between the waves. For example, those involving available potential energy were for the most part directed opposite to those observed in experiment VI; the planetary wave received a small amount of energy from wave \((3, 1)\) via the dominant wave \((2, 2)\). The somewhat
larger kinetic energy exchanges were such that residual amounts flowed to wave \((l, l)\), with the dominant transfer being from wave \((3, l)\) to wave \((2, l)\).

These inter-wave energy flows suggested that the nature of the wave state might be sensitive to the wave interactions, even in the absence of continentality. To examine this possibility, experiment IX was performed, in which the wave interactions were excluded. Starting from the same initial conditions as experiment VIII, it was found that the two solutions were nearly the same until about day 14, by which time the wave amplitudes were appreciable. At this point, wave \((2, l)\) did not grow so fast as in experiment VIII. Its interaction with the pre-established mode \((2, l)\) thus led to a change in the oscillation of \(\Psi_2\), which then altered wave \((3, l)\) and ultimately the basic baroclinic energy source \(\theta_1\). About ten days after the first development of differences, the wave \((2, l)\) then grew to a dominant amplitude.

Thus we see that the presence of the wave interactions in experiment VIII was sufficient to indirectly shift the spectral distribution of wave energy from the wave \((2, l)\) for the case without interactions to wave \((2, l)\) when they were included. This corresponded to a significant shift in the \(y\) scale of the dominant disturbance. A further direct influence was also the tendency for broadening of the spectrum about this most unstable wave when the inter-wave coupling was present. Finally, as can be seen by comparing \(\theta_1\) and \(\bar{A}\) in Table 5.1, we see that the wave energy levels were lower relative to the zonal flow in experiment VIII. This is in agreement.
with the role the wave interactions played as an "equivalent viscosity" on the most unstable wave when continentality was present, as noted previously.

Let us finally turn to a direct comparison of experiments VIII and IX with experiments VI and VII by considering Table 5.1. Study of the kinetic energy levels for experiments VI and VIII, along with $\vartheta_l$ and $\mathcal{A}$ shows that the level of eddy energy relative to the zonal flow was larger in experiment VI. This was verified by inspection of the wave amplitudes as well. Generally speaking, this relative energy level was larger by about a factor of three. A similar comparison for experiments VII and IX showed that the increase was then by a factor nearer two. In either case, we see that the influence of continentality was such as to cause enhanced development of the wave flow, which is an intuitively reasonable conclusion.

A portion of this increased eddy energy could be accounted for by the longest waves, which developed appreciably only in the case with continentality. An important conclusion of this section is thus that the planetary waves of the atmosphere are not an important feature of quasi-geostrophic flows a priori, but rely primarily upon external energy sources for their maintenance. However, when they do exist in a developed state, they influence the atmospheric energy transfer in a very important manner.

Another consequence of continentality was that the zonal flow was then smaller. In fact, comparing experiments VI and VIII, we see that the effect was strikingly large, amounting to a fifty percent difference. It is interesting to note that this factor is the
same as that for the observed difference between the zonal flow in
the Southern Hemisphere and Northern Hemisphere in both seasons
(Obasi, 1963). This leads to the conjecture that the zonal flow
differences between hemispheres are mainly of a dynamic origin
which is induced by differences in continentality, rather than of
a frictional origin.

Taken together, these results show that if the ultimate dissi-
pactive sink of energy is simply related to the amplitude distribution,
the influence of continentality is such as to increase the dissipation
taking place in the wave flow. However, at the same time, the effect
of the wave interactions is in the opposite direction, indirectly
favoring the zonally symmetric flow.
6.1 Summary of Relationships

The primary results of this thesis have been presented and discussed in Chapters 3-5. Let us now examine the total picture.

A convenient starting point is that of the resonance discussion of section 3.3. There it was noted that only those waves whose free modes were both stable and stationary could exhibit a resonant response to steady forcing. Apart from being of interest for its own sake, this conclusion had important implications for the entire set of remaining results, for it stated that the unstable waves do not resonate, but that stable ones do. Thus, linear analysis suggested that the forced geostrophic flow superimposed upon a baroclinic westerly current contains two wave energy sources, namely the heating field and the available potential energy of the zonal flow. More importantly, the dispersive nature of the medium separates the scales at which these individual processes are most efficient. The primary conclusion of this thesis is that the resulting direct interaction of these scales of motion greatly influences the climate.

Before studying these inter-wave interactions, the role of advective interactions between a single forced wave and the zonal flow was examined. It was found that consideration of the necessary resonance condition yielded much insight into the inter-relation between the heating and resulting advections. We saw that the development of large amplitudes for the non-dissipative case was a direct result of the heating. The baroclinicity of the zonal flow represented a possible source of energy for the wave only in the presence of skin friction. Hence, its primary influence was a kinematical one
affecting the wave speed, and not so much a dynamical one. For the long waves the beta effect was of great importance; in fact, heating resonance in a barotropic zonal current was then found to occur when the free wave coincided with the divergent Rossby wave mode of this two-level model, rather than the non-divergent one. However, in the presence of zonal baroclinicity resonance could develop in modes near the non-divergent Rossby wave mode, also. In accord with these results, it was found that some zonal states would allow the simultaneous existence of two scales of resonating forced waves in the two-level model.

These results suggested that the forced planetary waves would tend to develop large amplitudes in the steady state. In addition, the linear stability of these waves would not allow their migration, for the free travelling modes would be dissipated. On the other hand, the shorter baroclinic waves would develop as travelling waves, and the linear forced wave analysis suggested that their mean components would be relatively small.

Taken together, we see that the linear theory was capable of describing the generation of wave motions whose space and time scales were segregated. On the one hand, the continentality would favor the long waves with infinite period, while on the other the zonal flow baroclinicity would generate shorter cyclone waves with periods of several days. These clear-cut traits further allowed linear studies of the interactions between these scales. The expected spectral broadening in the space and time domains due to the (assumed)
fixed planetary wave was demonstrated for barotropic flow. It was suggested that the joint behavior of two scales of motion would be an important feature in describing both the time mean and transient states of the flow. Similarly, a simple stability analysis suggested that the fixed planetary thermal field might prove to be an important source of energy for the shorter scales.

With these predictions in mind, numerical experiments were performed to examine the unsimplified behavior of the non-linear system. Results were obtained for periodic motions, which yielded many of the qualitative conclusions in simplest form, and for non-periodic motions, which resembled those of the atmosphere more closely.

Experiment IV yielded a characteristically simply behaving solution, which arose under the influence of impressed heating on the planetary scale alone. It was found that the forced planetary wave \((l,1)\) was stationary as expected. The primary baroclinic wave \((3,1)\) interacted with it to produce a free wave \((2,1)\) of intermediate scale which in fact was of dominant amplitude. This wave travelled at constant speed and amplitude toward the east, as did its companion wave \((3,1)\). However, the wave speeds were synchronized so that each wave moved one wavelength during the same period of time. This had the effect of changing the shape of the wave "packet" comprised of these two modes periodically in such a way that their interaction with the fixed wave \((l,1)\) was steady. In addition, their individual interactions with the zonal flow were
steady. Thus, this solution had the characteristics of an atmosphere whose "general circulation" consisted of a travelling wave system of two waves which moved in such a manner as to maintain the sum of the zonal and planetary wave components as a steady wavy flow. This system then represented a generalization of the general circulation of an atmosphere without continentality, for which a single wave is sufficient to balance the mean flow.

Thus the influence of continentality was seen to have indirectly introduced new degrees of freedom into the space and time variations, as represented by a "new" baroclinic wave. This mode had a constant wave phase relative to its travelling companion, which led to two important effects. Firstly, the energy exchanges between all waves were constant in time. Secondly, the travelling wave packet described a system whose transient statistics depended strongly upon the planetary flow pattern, and hence indirectly upon the pattern of continentality. Thus, these two points implied a correspondence between spatial variations in the climate and the constant energy exchange between different scales of motion. In retrospect it is not difficult to understand the basis for this relation, for the steering effect of a wave system by another wave pattern necessarily amounts to a distortion of the former, which is reflected in a redistribution of energy between scales. This result is one of the most important for this thesis.

Intuition further suggests that if the wave packet were also influenced by its own imposed heating field, the energy transfer
between it and the planetary wave would become unsteady. In fact, since this short wave pattern of continentality would have a distribution of its own, this unsteadiness would give preferred regions of growth and decay via the wave interaction. This was observed to be the case for both the periodic and non-periodic flows; interestingly, the effect was most clearly evident for the free wave, which of course possessed no phase preferences of its own a priori.

Thus, growth and decay of the individual waves at the expense of the planetary wave represented a generalized process of cyclogenesis at preferred locations which was missing in the absence of small scale continentality effects. However, the energy source in these cases ultimately was the planetary scale wavy flow (planetary wave plus zonal flow), so that the detailed behavior of the short waves was not simply related to their heating field, as was emphasized by the case of the free wave growth. Nevertheless, even in the case of irregular flow, a statistical relation between the short wave phases and their interaction with the planetary wave was found.

In short, we see that the combination of planetary scale forced flow and smaller-scale heating was able to produce bounded instabilities which resulted in a changed character of both the nature and distribution of the shorter-wave fluctuations. The growth, movement, and decay of the transient flow systems thus resembled a sort of "inhomogeneous turbulence" of the geostrophic flow whose mean and transient properties were a complicated function of the continentality pattern and even its spectrum.
From this description, it is not surprising that these large-amplitude baroclinic waves fed back onto the planetary flow, which then exhibited some variability. Interestingly enough, it was also found that a steady feedback mechanism also existed in either the fluctuating or stationary cases, which thus altered the mean planetary flow distribution. However, the effects depended importantly upon the heating distribution of the smaller scale. Thus, taken with the results quoted above for the short waves, we see that heating on both the planetary and cyclonic scales significantly influenced those flows through the resulting non-linear interactions. This was attended by complicated geographical variations in the intensity and frequency distribution of the unsteady flow statistics.

The interactions of the planetary wave and two baroclinic modes were not the only ones producing the major climatic features. For example, in the irregular flow, the waves \((\ell_1, 1)\) and \((\ell_2, 1)\) together represented a planetary harmonic \(n=1\) which tilted in the \(\nu, \gamma\) plane, resulting in a mean state cascade of energy into the free wave \((2, 1)\). This was reflected in the existence of a cyclonic area in the mean flow which was not present in the continentality pattern.

Similarly, in Chapter 4 it was shown that quasi-steady forced planetary waves could interact among themselves to distort the structure of each wave. The resulting mean tilts were seen in that case to maintain the maximum zonal flow at a high latitude. This offered a clear example of how the feedback of the waves onto the
zonal flow could, in the presence of continentality, be influenced by their interactions among themselves.

A similar influence on the vertical wave structures was found in other experiments. In general, the wave interactions appeared to decrease their gain of zonal available potential energy. These interactions also produced spectral broadening, and even a spectral shift, compared to the case when they were excluded. Taken with the above, we see that the wave coupling influenced not only the distribution of energy among the waves, but that the partitioning between the wave and zonal flows was also influenced. Thus, the ultimate spectral distribution of energy dissipation was influenced by the wave interactions. The importance of this was demonstrated in Chapter 2, where it was shown that the dynamical adjustments associated with frictional flow dissipation were strongly scale dependent.

Finally, we note that the feedback relations between the zonal flow, the planetary wave, and the baroclinic waves led to the development of long period fluctuations of the wavy planetary scale flow. This was characterized by slow changes in the monthly mean flow patterns which were accompanied by similar fluctuations in the energy transfer. The description of the two extreme states of the "oscillation" were seen to depend critically upon the interactions of the forced planetary wave with the transient waves and the zonal flow.
6.2 Further Remarks

In addition to the results discussed in section 6.1, a number of investigations were made for both the periodic and aperiodic regimes in which artificial assumptions were purposefully introduced.

The first of these amounted to systematic exclusion of the wave interactions. It was found that the planetary waves were then nearly in a steady state, even when the rest of the flow was rather unsteady. Significantly, no long period variations took place. The planetary wave amplitudes were larger without the interactions, and the phase of their barotropic mode differed substantially in that case. In addition, when the beta effect was included, the vertical tilt was never toward the west, contrary to the case with the wave interactions included. Thus the planetary wave role in the "general circulation interactions" was altered.

Without direct wave coupling the shorter wave spectrum was not so broad, with only those forced modes and the primarily unstable ones existing. The forced modes were quite insensitive to their heating field and showed no tendency for sudden growth. The unforced short waves were unaffected by the slight variations in the forced waves.

Thus, contrary to the case with wave interactions, both the space and time fluctuations were considerably simpler. In fact, the transient flow fields were seen to exhibit almost no longitudinal differences, in contrast with the real atmosphere and the results obtained with the wave interactions included. The lack of long period fluctuations, either in the planetary waves or the cyclones,
suggested that a necessary mechanism for spontaneous monthly climate was the interaction of the planetary and cyclone scale waves. This result is in some agreement with the qualitative remarks of Namias (1954), although in the present results the planetary oscillations were a result of the non-linear feedback from the short waves alone.

Consideration of the time mean flows showed that the solutions which excluded the wave interactions resembled those mean wave states obtained from a simple linear solution of the steady-state equations. Neither of these was a particularly good representation of the observed atmospheric flow patterns, contrary to the case when the wave interactions were included. Thus, the dominant interactions producing longitudinal climate variations were again seen to be those between the waves.

Finally, the question was asked whether these interactions could produce significant mean energy transfer between scales even in the absence of continentality. It was found that the wave interactions contributed negligibly to the net energy flow in this case. This was partially due to the conspicuous lack of energy in the planetary waves.

It thus appears that the external agents, such as the heating field, considered here, are responsible for maintaining the very long waves observed in the earth's atmosphere, and that the net energy exchange between scales depends importantly upon these waves.

In addition, it was also found that the zonal flow contained relatively more energy when the effects of continentality were absent,
which led to the speculation that the large observed differences of zonal circulation between the Northern Hemisphere and the Southern Hemisphere are caused by differences in the wave flows resulting from continentality.
6.3 Final Remarks and Suggestions

In summary, we have seen that the effects of continentality on the atmospheric climate are importantly influenced by the complicated advection processes. In effect, it was found that consideration of the inter-wave interactions was a necessary step in describing the development of the longitudinal and latitudinal variations in climate.

This influence was felt even for the case of essentially steady flow, where the exclusion of the wave interactions amounted to neglecting part of the advecting flow for a given wave. To the extent that this neglected part was driven by the heating field, this meant that part of the flow "memory" of the heating distribution would be ignored. Clearly the proper influence of the full heating field would be attained only by consideration of the full advecting field.

For the transient behavior, the relations were less obvious, for the motion was governed by complicated wave instabilities. In this case they implied a redistribution of energy from the space and time scale of the heating field to other scales. An important example was seen in the case of the planetary wave, which represented a source of available potential energy to the cyclone waves.

We saw that straight forward considerations led to the expectation of significant interactions between these scales. In addition, simple linear analyses suggested some of the characteristics of these interactions, and their dependence upon the zonal flow or beta effect. However, some important results were obtained only by resorting to numerical integration of the governing equations. The most important
of these involved the fully non-linear processes of feedback from the waves to the planetary wave and zonal flows.

Let us finally consider how these results might be improved. Firstly, they were obtained for a very simple heating law and for unrealistically large heating rates. It would be interesting to see how some of the detailed behavior would be changed by introducing a more earth-like representation. However, the system behavior in such cases would be much more difficult to analyze. A supplementary suggestion would be to examine the climate when the long waves are forced by topographical variations, rather than heating. We have not tried to settle the question concerning their relative importance; instead the point of view has mainly been to examine the climate-producing processes when some mechanism of "continentality" was operative.

To the author, some extensions of this work seem more important than these suggested improvements, however. For example, the possibility that widely varying initial states might persist should be examined, for the influence of anomalous flow patterns of one season on those of the following one are not understood.

A second example concerns the sensitivity of the climate to changes in the heating distribution. Some of the results of Chapter 4 indicated that the smaller scale heating field could feed back onto the transient and mean flows in all scales. This question should be considered in more detail, possibly utilizing flow representations in grid-point space, rather than the spectral approach used here.

The resonance results of Chapter 3 suggest that the resonance
conditions for steady forced flows should be re-examined.

Finally, and most importantly, a concerted effort to understand
the mechanics of inter-wave interactions must be made. Possibly
further linear analyses such as those of Chapter 3 would give initial
insight into the problem. An intermediate step might involve studying
the properties of more general linear equations with variable coef-
ficients. However, it seems clear that to understand not only the
directions of inter-wave energy exchange, but also their fluctuations,
the problem must be treated essentially as one involving geostrophic
turbulence. In this respect, further study of the adiabatic con-
straints should be made with the hope of eventually constructing
an idealized interaction model, possibly following the lines of
Kraichnan's work on three-dimensional turbulence.
APPENDIX A. DISCUSSION OF SIMPLIFIED HEATING LAWS

In this section, we discuss some heating laws which have relevance to the earth's atmosphere, and examine simplifications of them suitable for inclusion into the two level spectral model. Much of the discussion is similar in spirit to that given by Mintz (1958).

For this study, the main requirement of the heating function is that it differentiate between continental and oceanic areas. These areas will be assumed to differ in one important respect: the heat capacity of the surface underlying the atmosphere. Over land, it will be assumed negligibly small; over the ocean it will be considered infinite.

Such an assumption simplifies the boundary influence on the atmosphere, but says nothing of the internal heating mechanisms, which are quite intricate and variable. Rather than to study the flow under an ensemble of heating laws, the purpose of this thesis is to examine the manner in which a single heating law introduces variety into the resulting flow. To achieve this end, we now hunt for a simple means of introducing heating asymmetry which has important features in common with the atmosphere.

In dimensional form, the first law of thermodynamics for an ideal gas may be written as

\[ \frac{d}{dt} (\Theta) = \left[ \frac{1}{c_p} \left( \frac{\rho_0}{\rho} \right)^n \right] Q \]  

(A.1)
where $\Theta$ is the potential temperature, $Q$ the heating rate per unit mass, and $P$ and $P_r$ the pressure and reference surface pressure, respectively. $R$ is the universal gas constant for air, and $C_p$ is the specific heat at constant pressure of the atmosphere.

From (2.1.5), the spatially varying temperature field appears only in the variable $\Theta$, which may be regarded as the vertically averaged potential temperature. Thus the quantity $Q_{\Theta}$ represents a vertically averaged heating function for the two-level model, and hence it is given by

$$Q_{\Theta} = Q_{\text{I}} + F_{B} - F_{T} \quad (A.2)$$

Here $Q_{\text{I}}$ is the average solar heating in situ. $F_{B}$ and $F_{T}$ are the upward heat fluxes at the bottom and top boundaries of the atmosphere.

For our purpose, $Q_{\text{I}}$ will be restricted to a known function of latitude and season: $Q_{\text{I}} = Q_{\text{I}}^{*}(\gamma, t)$. It follows that the longitudinal variations in $Q_{\Theta}$ may arise only through the fluxes $F_{B}$ and $F_{T}$ into the atmosphere.

We now assume that $F_{T}$ consists of long wave emission primarily from atmospheric water vapor and secondarily from the earth's surface, clouds, and other gaseous components. When the atmosphere is relatively dry, the outgoing long wave radiation originates at a lower vertical level than when it is wet. The observed lapse of temperature in the earth's atmosphere then implies that the radiative source is at a warmer temperature, other things being the same.
However, other things are not the same. A general feature of the atmosphere in mid latitudes is the tendency for dry air to be relatively cool, when a given atmospheric level is considered. We thus see that the two effects tend toward compensation. A perfect matching would imply that the source region for outgoing radiation were an isothermal surface, in which case the outward water vapor flux would be nearly constant in space and time. We will assume this to be the case, assuming the actual variability to be small compared to other effects to be noted shortly. In this case, it also seems reasonable to neglect systematic relations between clouds and temperature, as well as variations in mid-latitude surface window radiation. The first of these may play a significant quantitative role in the real atmosphere (Suomi and Shen, 1963) but its influence in this model would be obscured by inaccuracies associated with the crude vertical structure. We thus will take the outward radiative flux at the top of the atmosphere as a known constant.

With this simplification, longitudinal variations in $Q_0$ may occur only if $F_B$ contains such variations; i.e., if the thermal interaction of the atmosphere with the surface is longitudinally variable. Let us now see how variations in heat capacity could bring this about.

Over land areas, the condition of zero heat capacity demands that the total heat storage at any instant be zero. Thus, a balance of energy sources and sinks exists, given by

$$F_B = S_L - W_S$$

(A.3)

$S_L$ and $W_S$ are the absorbed solar radiation flux and outgoing
window radiation. Consistent with the remarks above, $S_L$ is taken to be a known function $S_L^* (y, t)$ . If the variations in $\nabla S$ are smaller than those in $S_L$, then $F_B$ is seen to approximate a known function of latitude and season also.

$F_B$ represents the total interaction of the land surface and the atmosphere, and we see that, under the assumptions above, it may be specified without identifying the mechanism involved. In the real atmosphere, these mechanisms are primarily radiative and sensible heat transfer. A simple-minded approximation to these or other processes might produce a linear relation of the form

$$F_B = -a \theta + b T_g + c$$

(A.4)

where $a$, $b$, and $c$ are constants and $T_g$ is the temperature of the ground. In this case, longitudinal and time variations in $\theta$ would have to be accompanied by similar ones in $T_g$ to maintain $F_B$ at its constant value for that latitude and season. That is, a feedback relation between the air and ground would exist, causing them to be relatively warm and cold together. Such behavior is not contrary to experience, and suggests that a linear relation of the form (A.4) is approximately valid over land areas.

Even if this linear law did not hold, the longitudinally fixed nature of $F_B$ and $Q_\perp$ would nevertheless mean that the over-land heating field was independent of the temperature field. In this case, temperature anomalies would not necessarily be dissipated by such heating.

Over the ocean, such behavior does not occur, to our order of
approximation. To see this, we note that the infinite heat capacity of the ocean allows unlimited storage of heat without a corresponding temperature change. Clearly this property rules out any possible air-sea coupling of the feedback type envisioned by Namias (1963).

Thus, its surface temperature \( T_o \) may be taken to be a known quantity for each latitude and season \( T_o^*(y,t) \). This condition then replaces equation (A.3) of the land case.

If we again assume \( Q^* \) is a known constant, then the heating \( Q_\theta \) over the ocean is obtained only after first specifying a law relating \( F_B \) to \( \theta \) and \( T_o^* \). We again assume a simple linear form as in (A.4). Charney (1959) has shown that such a relation is a reasonable approximation for radiative processes in a grey atmosphere, where we replace \( T_g \) by \( T_o^* \) in (A.4). The vertical flux of sensible and latent heat seems to satisfy such a law also, in which case \( \mathcal{C} \) may be chosen positive to parameterize the dependence of the process on static stability, thereby yielding an upward flux of energy even when the temperature dependent terms cancel.

Thus, over the ocean, (A.2) takes the form

\[
Q_\theta(y,t) = Q^*(y,t) + \mathcal{C} - \theta(y,t) + bT_o^*(y,t) - F_T^* \tag{A.5}
\]

where \( F_T^* \) is a constant. We may further write (A.5) in the form

\[
Q_\theta = \mathcal{K} \left[ \theta^*(y,t) - \theta(y,t) \right] \tag{A.6}
\]

by defining

\[
\mathcal{K} = \mathcal{C}, \quad \theta^*(y,t) = \frac{1}{\mathcal{C}} \left[ Q^*(y,t) + bT_o^*(y,t) + \mathcal{C} - F_T^* \right] \tag{A.7}
\]
Since \( \psi_0 = 0 \) when \( \Theta = \Theta^* \), we see that \( \Theta^* (\psi_0, t) \) represents a thermal equilibrium value of \( \Theta \) over the ocean; it is seen to depend upon the ocean temperature and solar radiation as shown in (A.7).

To describe the atmospheric heating over a region containing both continental and oceanic areas, a generalized form of (A.6) could be adopted:

\[
Q_\Theta (\kappa, \eta, t) = \int_0^{\Theta^* (\kappa, \eta, t) - \Theta^* (\kappa, \eta, t)}
\]

(A.8)

Here we see that the heating function \( Q_\Theta \) contains both fixed and variable dependences upon longitude. The fixed variations are found in the fields \( \Theta^* (\kappa, \eta, t) \) and \( \Theta^* (\kappa, \eta, t) \). Those of \( \Theta^* \) arise from the land-sea differences in the prescribed parts of the surface fluxes, which are determined mainly by the solar surface radiation and ocean surface temperature, respectively. This can be considered the primary mechanism producing continentality in this model.

The fixed variations in \( \Theta^* (\kappa) \) are simple: we define \( \Theta^* (\kappa) \) as being one over the ocean and zero over land. This influence of continentality must be considered a secondary one, for its influence in (A.8) depends upon the pre-existing temperature field \( \Theta \).

Thus, as a final simplification, we may fix attention on the representation (A.8) when \( \Theta^* (\kappa) \) is replaced by one everywhere. By doing so, the total influence of continentality is restricted to a prescribed effect, independent of the flow itself. By reference to
equation (A.6), this form of equation (A.8) is seen to be similar to replacing the land areas by equivalent oceanic ones of different surface temperature from that of the true ocean.

Such a "two ocean" representation of the impressed heating has been adopted many times before. The stationary wave solutions found by Doos (1962) and the spectral model used by Kraus and Lorenz (1963) are some recent examples. It is hoped that the above discussion leading up to this choice makes its relevance to heating in the real atmosphere more clear.
APPENDIX B

COMPARATIVE PROPERTIES OF SOME COMPUTATIONAL SCHEMES

In this section we discuss briefly the problem of computational stability and compare the properties of certain "deterministic" methods for both linear and non-linear advective oscillations. The final choice of the three-step "double forward-centered" method is made.

The governing equations for our system are abstractly represented in the form

\[ \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \]  

(B.1)

where \( \mathbf{x} \) and \( \mathbf{F} \) are vectors. The solutions described by (B.1) are represented by continuous trajectories threading their way through the phase space \( \mathbb{R}^n \). The goal of numerical integration is to follow this continuous trajectory as closely as possible with a "hopping" trajectory between the discrete states of the approximate solution.

For a realistic non-periodic system, the true solutions at neighboring points eventually diverge. The numerical approximation must certainly do the same. We thus content ourselves with a solution which nearly conserves certain functionals of the true solution. For example, the solution convergence to a stable limit cycle in the case of periodic motion should be reproducible by the computational scheme. In a more irregular case, the accurate representation of slow fluctuations over a long period of time should not be obscured by computational effects.
In the present case, a further requirement is that the detailed wave response should be qualitatively correct. In the case where the wave interactions are excluded, all waves would be interacting with the same zonal flow, and no problem would arise. However, in the more realistic case with direct inter-wave coupling, each wave would be influenced by a different over-all flow. Presumably the computational errors would then differ between waves. Such a situation should clearly be avoided, since the relations of the waves to each other and to the geography form an important part of the thesis analysis.

Let us now consider methods which might preserve some of these features for us. Since our interest is focussed upon the control of the errors over great lengths of time, it is thus only slightly connected with the local generation of error, which may be found from a Taylor's Series analysis. Instead, the accumulation of the error is determined by the integrated truncation effects, and hence depends upon the character of the solution itself.

An obvious way to decrease this "instability" is to decrease the local truncation. However, if past data is used to provide the additional information needed for higher differences, the order of the difference equation exceeds that of the differential equation. In this case, extra "parasitic" solutions arise, which may obscure the single solution of interest. Linear analysis would imply that methods could be devised whereby these extra solutions would damp out, but this could not be guaranteed when non-linear oscillations might occur. An example of this problem is given in the work of Baer (1964).
One may instead obtain this higher order information from suitable estimates of the future data. This can be accomplished by using the initial state alone to generate approximate states for $t > t_0$ in which case the method is a deterministic one. The price paid is high, for the generation of each "pseudo-state" requires a separate solution of the equation (B.1). From the standpoint of the amount of work demanded, this is equivalent to using a time step approximately equal to $\Delta t / N$, where $N$ is the number of pseudo states and $\Delta t$ is the true time step. Happily, the error arising in the description of linear oscillating systems rapidly decreases as increases at a disproportionate rate when $\Delta t$ is sufficiently small.

To see that this is so, let us consider a simple analogue to the full system of equations. Reference to equations (2.2.18) and (2.2.19) shows that the adiabatic, frictionless flow represented by

$$\Psi(x, y) = \Psi_1(t) F_1(y), \quad \Theta(x, y, t) = \Theta_1(t) F_2(y) + \Theta_2(t) F_3(y)$$

(B.2)

is governed by the simple triad of equations:

$$\begin{align*}
\dot{\Psi} & = C_{134} \left(1 + B \right)^{-1} \dot{\Theta} \Psi_4 \\
\dot{\Theta} & = - C_{134} \left(1 + B \right) \left(1 + B^2 + B^4 \right) \Theta_1 \Psi_4 \\
\Theta_3 & = C_{134} \left(1 + B \right)^{-1} \left(1 + B^2 - B^4 \right) \Theta_1 \Psi_4
\end{align*}$$

(B.3)

This representation is that of a baroclinic zonal flow $\dot{\Theta}_1 F_1(y)$ with a simple vertically tilting wave superposed.
If for the moment $\mathcal{J}_0$ is taken artificially large, so that 
\[(1 + \beta_3, -\theta_4) < 0\]
then the flow $\Theta_1$ is stable to perturbations represented by $\Theta_3$ and $\Psi_4$. If we then take $\Theta_1$ to be fixed, the two perturbation equations take the linear form

\[\eta^* = i \omega \eta\]  \hspace{1cm} (B.4)

where

\[\eta = \left\{ \left[ c_4 (\frac{b_3 - 8_4}{b_3}) \theta_3 + i \left[ c_4 \left( \frac{b_4 - b_1 - 1}{1 + b_3} \right) \theta_1 \right] \Psi_4 \right) \right\} \]  \hspace{1cm} (B.5)

and

\[\omega^2 = c_1^2 \beta_4 \left( \frac{b_3 - 8_4}{b_4} \right) \left( \frac{b_4 - b_1 - 1}{1 + b_3} \right) \]  \hspace{1cm} (B.6)

is the frequency of the stable oscillation. Here $\omega = \sqrt{1 - \beta^2}$. For initial states $\Theta_3(0)$ and $\Psi_4(0)$, the solution is

\[\eta(t) = \eta(t_0) \exp \left( i \omega n \Delta t \right)\]  \hspace{1cm} (B.7)

Here $\Delta t$ is the length of the time step and $n = \frac{t - t_0}{\Delta t}$ is the number of the time step measured from $t_0$. This solution has the property that, in the $\eta$ plane, the trajectory is a circular one with frequency $\omega$. That is, we have at $t_0 = 0$

\[|\eta(t)| = (\eta^* \eta) \frac{1}{\nu_2} |\eta(0)| \frac{1}{\nu_2} |\eta(0)| \]  \hspace{1cm} (B.8)

and so

\[\eta(t) = \eta(t_0) \exp \left[ i \omega t \right]\]  \hspace{1cm} (B.9)
where $\alpha(t)$ is the phase angle given by

$$\alpha(t) = \alpha(t_0) + (\omega \Delta t) \eta \tag{B.10}$$

Here $\eta^*$ denotes the complex conjugate.

Let us now consider how the finite difference solutions differ from this exact one. For example, a single forward difference of (B.4) would give

$$\eta_{t+1} = \eta_t + i(\omega \Delta t) \eta_t \tag{B.11}$$

over several time steps one would have

$$\eta_{t+n} = G^n \eta_t$$

where

$$G = \left[ 1 + i(\omega \Delta t) \right] \tag{B.12}$$

Here the subscript $t$ refers to the initial time step and $t+n$ to a later one. We see that the approximate solution (B.12) would agree with the exact one (B.7) if $G = |G| e^{i\alpha G}$ were equal to

$$G_e = |G_e| e^{i\alpha G_e} = G \eta_p [i(\omega \Delta t)] \tag{B.13}$$

Note that $|G_e|^2 = 1$ and $\alpha G_e = (\omega \Delta t)$. For this forward difference procedure, $G \neq G_e$. In fact, we would have

$$|G|^2 = 1 + (\omega \Delta t)^2 > 1 \tag{B.14}$$

and

$$\alpha G_e = \tan^{-1}(\omega \Delta t) \neq \alpha G_e (\omega \Delta t)$$

Since $(\omega \Delta t)$ is generally small, we see that this procedure gives a small erroneous amplification at each time step, which is, however,
accumulative. Also, to within order $(\omega \Delta t)^2$, the percentage phase error at each step is

$$\frac{\Delta \theta}{\theta} \sim (\omega \Delta t)^2 \quad (B.15)$$

Similar analyses hold for all deterministic methods, of which the above is the simplest. The solutions are all of the form (B.12), and hence differ only by their respective forms of $G$. From the above expressions, we see that a substantial reduction in the magnitude error could be achieved by increasing the exponents on the $(\omega \Delta t)$ terms; this is possible for more complicated methods, in which case more work would be required at each time step. However, the smallness of $(\omega \Delta t)$ suggests that such a procedure would be well worth the time investment, for the large decrease in the errors would allow a much longer time integration.

Let us now consider the less simple behavior of the full non-linear system (B.3). First, the formation of the combination

$$\frac{1}{\rho_e} (1 + B_1) \Theta^i \theta_i^* + \frac{1}{\rho_0} (1 + B_3) \Theta^3 \theta_3^* + \omega_4^* \psi_4^*$$

yields the conservation of total energy:

$$\frac{d}{dt} E = 0, \quad E = \frac{1}{2} \left( q_i^2 + \frac{1}{\theta_i} \right) + \frac{3}{2} \left( q_3^2 + \frac{1}{\theta_3} \right) + \frac{1}{2} \omega_4^2$$

(B.16)

In a similar manner, the total squared potential vorticity is conserved:

$$\frac{d}{dt} V = 0, \quad V = \left[ \Theta_1 (q_i^2 + \frac{1}{\theta_i}) \right]^2 + \left[ \Theta_3 (q_3^2 + \frac{1}{\theta_3}) \right]^2 + \left[ \omega_4^2 \psi_4^2 \right]^2$$

(B.17)
It follows that the ratio $S = \frac{V}{E}$ is also conserved; it has the "units" of $A_i^2$ or $\frac{1}{d_0}$, and thus is a constant measure of the three-dimensional scale of the system.

If we now consider the more realistic case $\phi_0 = \gamma_0$, $\bar{\theta}_1$ is unstable to the wave perturbations and the linear analysis eventually fails to be meaningful. However, Lorenz (1960a) has shown that the above constraints may be used to obtain the complete solution to (B.3) in terms of elliptic functions of time. If we choose as initial conditions $\theta_1(0) = 0.000\,\text{and}\,\theta_3(0) = 0.034\,\text{and}\,\psi(0) = 0$, the solution is

$$
\begin{align*}
\theta_1(t) &= 0.000\,S\sin[kt + k] \\
\theta_3(t) &= 0.020\,\cos[kt + k] \\
\psi(t) &= 0.0582\,\sin[kt + k]
\end{align*}
$$

(B.18)

Here $k = 2.504\,\text{and}\,\Lambda = 0.049\,\text{.}$

This solution represents the slow growth of the disturbance variables $\theta_3$ and $\psi$ at the expense of $\theta_1$, which decreases to zero and becomes negative. The vertical tilt of the wave also changes sign, forcing the system back to its original state after a length of time $\int_0^{4K} = 30.9\,\text{days}$. The non-linear nature of this exchange is obvious; however, the time variations resemble trigonometric ones, with fundamental periods of 30.9 days for $\theta_1$, and $\psi$ and one-half that for $\theta_3$. However, the non-linearities here produce higher frequencies of all integral multiples of these fundamental ones (Davis, 1962).

In summary, the full system (B.3) describes a simple energy
exchange similar to that occurring in the atmosphere. Hence the solution (B.18) can be compared with the approximate one; in particular, this yields an estimate of the phase error for the non-linear case. The over-all energy error can be obtained from the variations of $E(t)$, while the correctness of the spectral energy distribution among $\Theta_1$, $\Theta_2$, and $\psi$ can be ascertained from the constancy of $S(t)$.

Let us now consider the methods which were actually studied in this manner. All methods had the common property that the total changes in $\tilde{X}$ over one time step were expressible in terms of the initial state $\tilde{X}_t$ and the set of approximate future states derived from it. The names below are the author's terminology only.

1) Method SFC - the 'single forward, centered' scheme.
\[
\tilde{X}_{t+1} = X_t + \frac{\Delta t}{2} F(X_t), \quad \tilde{X}_{t+2} = X_t + \Delta t \frac{F(X_t + \Delta t F(X_t))}{2}
\]

2) Method DF - the 'double forward approximation' scheme used by Lorenz (1963b).
\[
\tilde{X}_{t+1} = X_t + \Delta t F(X_t), \quad \tilde{X}_{t+2} = \tilde{X}_{t+1} + \Delta t F(\tilde{X}_{t+1})
\]
\[
\tilde{X}_{t+1} = \frac{1}{2}(\tilde{X}_{t+1} + X_t) = X_t + \frac{\Delta t}{2} \left( F(X_t) + F(\tilde{X}_{t+1}) \right)
\]

3) Method GDF - This was similar to DF except for the following:
   
   if a typical non-linear term in $F(X)$ was of the form $B \xi$, then DF had the form
   \[
   \tilde{\xi}_{t+1} = \xi_t + \frac{\Delta t}{2} \left( B \xi_t + \tilde{B}_{t+1} \tilde{\xi}_{t+1} \right)
   \]
   Method GDF differed slightly from this, being of the nature of a "geometric mean" approximation:
   \[
   \tilde{\xi}_{t+1} = \xi_t + \frac{\Delta t}{2} \left( \frac{\tilde{B}_{t+1} \xi_t + B \xi_t \tilde{\xi}_{t+1}}{2} \right)
   \]
4) Method IMP - This method was the implicit scheme given by:
\[ X_{z+1} = X_z + \frac{\Delta t}{2} \left[ F(X_z) + F(X_{z+1}) \right] \]
Here \( N \) is the number of iterations required for the difference \( X_{z+1} - X_z \) to decrease to a value smaller than the arbitrary convergence factor. For \( N \approx 1 \), we see that the method IMP then coincided with method DF.

5) Method DFC - The "double-forward, centered" method which was used in the computations of this thesis. In it, method DF was used to obtain \( \tilde{X}_{z+1/2} \), and this was then followed by a third extrapolation:
\[ \tilde{X}_{z+1} = X_z + \Delta t F(\tilde{X}_{z+1/2}) \]

6) Method RK4 - This was a fourth-order method attributed to Kutta; the method was obtained from the minimization of the truncation error, which in this case was \( \| \sigma (\omega \Delta t)^5 \| \), the smallest of any of the methods shown here. The computation was governed by the series of steps:
\[
\begin{align*}
X_{t+1} &= X_t + \frac{\Delta t}{6} \left[ F(X_t) + 2F(\tilde{X}_{t+1/2}) + 2F(\tilde{X}_{t+1}) + F(\tilde{X}_{t+1/2}) \right] \\
\tilde{X}_{t+1/2} &= X_t + \frac{\Delta t}{2} F(\tilde{X}_t) \\
\tilde{X}_{t+1/2} &= X_t + \frac{\Delta t}{2} F(\tilde{X}_{t+1/2}) \\
\tilde{X}_{t+1} &= X_t + (1) \Delta t F(\tilde{X}_{t+1/2})
\end{align*}
\]

Some computational properties of these six methods are shown in Table B.1. We see that the differences between the various methods were smaller in the non-linear case than would have been expected.
er

Tab k B.1


from linear considerations. Also, the reduction of the time step by a factor of $\frac{1}{3}$ for the method DF is seen to have yielded a solution whose stability characteristics improved only slightly, contrary to the factor .067 which would have been expected for the linear case.

We see that the methods involving a greater number of extrapolations per time step still possessed stability advantages over the simpler ones, even when the increase in computational time was taken into account. One exception to this was the method RK4, which was not significantly better than method DFC, and it required 33 per cent more computational effort.

Since SFC and DF exhibited equally poor stability characteristics in the non-linear case, consideration was thus limited to methods DFC and IMP.

Several other methods were however examined, in which each member of $X$ was extrapolated in time using the most recent estimates of the other components. These methods had the disadvantage that the solution depended upon the order in which the elements of $X$ were solved. In addition, the first or last elements tended to be influenced by information from different times, when the order of solution was reversed at each time step. However, in this case the computational properties were remarkably good. Nevertheless, the method did not seem to be an attractive one from the above properties, and so it was excluded from further consideration. Further tests for a similar method for the case where the order of solution
of the elements of \( X \) was randomized yielded poor computational characteristics.

Other methods involving unequal time steps, both random and in fixed sequences, were investigated. As might be expected from the non-linear dependence of the errors on \( (\omega \Delta t) \), the resulting solutions for both the linear and non-linear systems were worsened by this procedure.

The final decision to use the method DFC instead of IMP was based upon the fact that the maximum errors during the cycle were comparable for the two methods, even though the average errors for method IMP were smaller. Since IMP usually required more than three extrapolations per time step, it was neglected in favor of the method DFC.

Method DFC was finally tested on the full dynamical equations as given in Chapter 2 for frictionless, adiabatic conditions. The computational change in the total energy amounted to less than 3 per cent per 100 days, suggesting that even this conservative system could have been studied accurately out to several years of real time.
APPENDIX C. SPECTRAL TECHNIQUES

In this section we discuss the methods used for obtaining the spectral and cross-spectral estimates. These techniques differed somewhat from more commonly used ones.

Let us first consider the steps one goes through in obtaining the power spectrum \( \hat{F}(\omega) \) of a single function of time \( \hat{a}(t) \).

**Step 1:** The data \( \hat{a}(t) \) is sampled at discrete intervals \( \Delta t \) of time. This captures all spectral information for frequencies less than \( \frac{1}{2} \omega \), the "folding frequency." Frequencies higher than \( \frac{1}{2} \omega \) are "folded" or "aliased" into the lower frequency region. Thus, if very high frequencies are present, the spectral estimate one obtains may be meaningless. However, if the higher frequencies lie just outside \( \frac{1}{2} \Delta t \), only the spectral estimates near \( \omega = \frac{1}{2} \Delta \omega \) will be affected.

**Step 2:** Even in the absence of aliasing, the finite data length reminds us that the computed spectrum will be only a statistical estimate of the true spectrum of \( \hat{a}(t) \). This automatically induces a complicated smoothing of the spectrum, so that a minimum frequency resolution exists.

**Step 3:** This involves rejection of phase information at each frequency, in favor of finding the amplitude information alone. One common method has been to compute the autocovariance of the time series (Blackman and Tukey, 1958). This is the equivalent of squaring
the absolute value of the Fourier transform $F_a(\omega)$, which is defined as

$$F_a(\omega) = \int_{-\infty}^{\infty} a(t) e^{-i\omega t} dt$$

Step 4: The "window shaping" step is the most crucial one of any spectral analysis. Up to this point the spectral estimate corresponds to that of a simple "periodogram" analysis, which is not a statistically stable one, for the fractional variance of the estimate is equal to one. This arises because the spectral resolution increases in proportion to the data length. Therefore, to reduce the number of frequency points needed to describe the spectrum and hence reduce the variance of the spectral estimate, further smoothing is needed.

In the method of Blackman and Tukey this is accomplished by considering only part of the autocorrelation, which is then multiplied by a smooth function ("lag window") which decreases as the lag increases. This multiplication has the effect of smoothing the spectral estimate in an effective manner. However, it has the disadvantage that the autocorrelation truncation introduces "side lobes" into the spectral smoothing operator. In practice, such contamination has been alleviated by the use of "prewhitening" and special filtering techniques.

Such procedures could be avoided by allowing computation of the entire autocorrelation, but this erases the computational time advantage which the autocorrelation approach possesses over other methods. However, in the event that time was not important, the smoothed spectrum without these side lobes could be calculated, in which case the smoothing function would be of the "boxcar" type. Such a spectral
estimate is known as a Daniell power spectrum.

Step 5: This last step arises when the autocorrelation method is used. It consists of the cosine transformation of the weighted autocorrelation to obtain the amplitude spectrum.

Let us now turn to the case of two time series \( \hat{a}(t) \) and \( \hat{b}(t) \) whose joint behavior may be of interest. Additional information is then provided by the cross-spectrum \( \Phi_{ab}(\omega) \). A convenient form for this information is that of the "coherency," which is a complex number at each frequency defined

\[
\tilde{\gamma}(\omega) = \frac{\Phi_{ab}(\omega)}{|\Phi_{a}(\omega)||\Phi_{b}(\omega)|}^{1/2}
\]

The maximum value of its magnitude is one, and this occurs when the relation between \( \hat{a}(t) \) and \( \hat{b}(t) \) at that frequency can be described by a linear relation. When such a relationship does not exist, the coherency magnitude is zero.

Non-zero coherencies arise when a consistent phase relation between \( \hat{a}(t) \) and \( \hat{b}(t) \) exists for a frequency band. (Naturally such a relation always holds for a single frequency.) In this case the real and imaginary parts of the cross-power \( \Phi_{ab}(\omega) \) produce the real and imaginary parts of the coherency, for which a phase angle can be defined. This phase angle measures the angular lag between \( \hat{a}(t) \) and \( \hat{b}(t) \) at each particular frequency.

From the above definition it is clear that the calculation of the coherency is performed in a similar manner to that of the power spectrum. The main difference arises in step 3, where the lagged cross-correlation must be computed when one uses the Blackman and Tukey method. Since lags of both signs are needed, the time required to produce \( \Phi_{ab}(\omega) \) is nearly that spent in obtaining \( \Phi_{a}(\omega) \) and \( \Phi_{b}(\omega) \) together.
On the other hand, if one uses the Fourier transform approach, \( \mathcal{F}_a(\omega) \) and \( \mathcal{F}_b(\omega) \) are already known, and their complex product then contributes to \( \mathcal{F}_{ab}(\omega) \). Thus only a single multiplication at each frequency is needed, in contrast to the further laborious analysis demanded by the autocorrelation method. For this reason the conventional autocorrelation method was rejected here. Instead, the Fourier transform approach currently in use by Madden (1963) was utilized for both the power and cross-power spectral estimates.

These were obtained by the following procedure: 1) The data were broken up into two sections to reduce the labor involved in taking the Fourier transforms. Such sectioning is known as the Bartlett procedure. 2) The time mean of each section was removed from \( \mathcal{A} \) and \( \mathcal{B} \). 3) The Fourier transforms of each section were carried out for both \( \mathcal{A} \) and \( \mathcal{B} \). 4) The products \( \mathcal{F}_a(\omega) \mathcal{F}_a^*(\omega) \), \( \mathcal{F}_b(\omega) \mathcal{F}_b^*(\omega) \), and \( \mathcal{F}_a(\omega) \mathcal{F}_b^*(\omega) \) were computed. 5) These products were then averaged in frequency and between the two sections to produce the smoothed estimates \( \mathcal{I}_{ab}(\omega) \), \( \mathcal{I}_{bb}(\omega) \), and \( \mathcal{I}_{aa}(\omega) \), respectively. 6) The approximate mathematical equivalence of the autocorrelation and Fourier transform methods allowed the fractional variance of the estimates to be taken as \( \frac{M}{N} \), where \( M \) was the number of frequency points (10 or 40 here) and \( N \) was the number of data points (500 here). These estimates seemed reasonable since most spectra were relatively smooth. 7) A similar argument allowed the fractional variance of the coherency magnitude to be given approximately by

\[
\left( \frac{M}{N} \right)^{1/2} \left( 1 - |\text{coh}|^2 \right)^{1/2}.
\]
BIBLIOGRAPHY


Eliassen, A., Slow thermally or frictionally controlled meridional circulations in a circular vortex, Astro. Norvegica 5(20), 1952.

Fjortoft, R., On the changes in the spectral distribution of kinetic energy for two-dimensional, non-divergent flow, Tellus 5, 225-230, 1953.


Lorenz, E.N., Maximum simplification of the dynamic equations, Tellus 12, 243-254, 1960a.


Phillips, N.A., A simple three-dimensional model for the study of large-scale extratropical flow patterns, J. Meteor. 8, 381-


BIOGRAPHY

The author was born in Washington, D. C., on July 4, 1939. His childhood was spent in suburban Arlington, Virginia, where he attended public schools. He graduated from Washington-Lee High School in 1957.

From 1957 to 1961 he attended Miami University in Oxford, Ohio, where he majored in physics. He obtained his Bachelor of Arts degree in 1961, graduating Magna Cum Laude with Honors in physics.

During the seven summers 1956-1962 he worked in six separate sections of the Office of Meteorological Research of the U.S. Weather Bureau. A by-product of the work in 1961 was the article: "The measurement of vertical gradients near the ground", by A. B. Bernstein and J. A. Young, which appeared in the Journal of Applied Meteorology, vol. 1, p 438, 1962.

From September 1961 until February 1966 the author was in continuous residence as a graduate student in the Department of Meteorology at M.I.T. Financial support during the first three years was given by a Ford Foundation fellowship.

The author was married to the former Julie Satkamp in June 1962. They presently have one child, David, aged 2½ years.