TIME-DEPENDENT POWER SPECTRA

AND

FIRST PASSAGE PROBABILITIES

by

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ABSTRACT

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The concept of a power spectral density for a nonstationary stochastic process is investigated. Alternate interpretations of this quantity are considered.

The probability distribution of the time to first barrier crossing by a narrow-band stochastic process is expanded to include the concept of an evolutionary power spectrum. Results are given for the response of a lightly damped single degree-of-freedom oscillator subjected to broad-band excitation. The expressions are especially interesting for very small damping levels, including the zero damped case. Approximate closed form solutions are given for first passage probabilities for separable stationary excitation with a Heaviside or exponential deterministic modulating function.

Parameters appropriate to earthquake engineering indicate significant improvement over previous theories through the introduction of the time-dependent power spectrum.

Thesis Supervisor: C. Allin Cornell
Title: Associate Professor of Civil Engineering
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The immeasurable contributions of Professor Erik H. Vanmarcke are also gratefully appreciated.
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Symbols

\( A_D \) probability of no instantaneous barrier exceedance

\( A(t) \) deterministic modulating function of a stochastic process

\( a \) barrier level for first passage considerations

\( f \) cyclic frequency

\( f_{T_0} \) probability density function of time to first barrier crossing

\( G(\omega) \) one-sided power spectrum of a stochastic process

\( G(\omega,t) \) time-dependent power spectrum

\( G_f(\omega) \) one-sided power spectrum of a forcing excitation

\( G_R(\omega) \) one-sided power spectrum of a response process

\( H(\omega) \) transfer function of a linear oscillator

\( h(t) \) oscillator impulse response function

\( L_D(t) \) reliability function: probability that at time \( t \) the process has not yet crossed a given fixed double barrier

\( q \) Vanmarcke's spectral density shape factor

\( R_x(\tau) \) autocorrelation of stationary process \( x(t) \) evaluated with time lag \( \tau \)

\( R_x(t,\tau) \) autocorrelation of general process \( x(t) \) evaluated at time \( t \) with lag \( \tau \)

\( S \) excitation duration

\( T_0 \) time between upcrossings of a fixed barrier level by a process
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<td>$T$</td>
<td>one half time-window for analysis of a stationary process</td>
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<tr>
<td>$t$</td>
<td>general time parameter</td>
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<td>$x(t)$</td>
<td>a stochastic process</td>
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<td>$\alpha$</td>
<td>hazard function in first passage computations</td>
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<td>$\alpha(t)$</td>
<td>time-dependent hazard function in first passage computations</td>
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<td>$\lambda_0, \lambda_1, \lambda_2$</td>
<td>zeroth, first, and second area moments of a power spectrum</td>
</tr>
<tr>
<td>$\lambda(t)$</td>
<td>time-dependent area moments of a power spectrum</td>
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<td>$\mu$</td>
<td>barrier level as a multiple of variance of a zero-mean process</td>
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<tr>
<td>$\nu_0, \nu^+$</td>
<td>representative frequency of a process</td>
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<tr>
<td>$\nu_a, \nu^a$</td>
<td>frequency of upcrossings of the barrier level $a$</td>
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<tr>
<td>$\xi$</td>
<td>per cent of critical damping in an oscillator</td>
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<tr>
<td>$\sigma^2$</td>
<td>variance of a process ($\lambda_0$ for a zero-mean process)</td>
</tr>
<tr>
<td>$\tau$</td>
<td>time lag used in autocorrelation computations</td>
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<tr>
<td>$T$</td>
<td>period of a process, undamped natural period of an oscillator</td>
</tr>
<tr>
<td>$\omega$</td>
<td>circular frequency ($2\pi f$)</td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>natural undamped circular frequency of an oscillator</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>natural damped circular frequency of an oscillator</td>
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<td>$\omega_u$</td>
<td>upper limit of a band-limited spectra</td>
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INTRODUCTION

Engineers are becoming increasingly aware of the importance of an adequate dynamic analysis for the design of many systems. The nondeterministic nature of the motion in a number of applications leads logically to a random vibrations approach. Of major interest is the time to first passage of a fixed threshold. This problem has been studied in some detail for stationary conditions. A frequency domain approach using the power spectral density function (power spectrum) has proved to be very convenient.

This report seeks to extend these concepts to nonstationary motion.

Chapters I and II review certain present first passage theories and outline the difference between stationary and nonstationary vibration. The importance of the power spectrum is discussed, as well as the notion of a time-dependent power spectrum.

Chapter III investigates in detail the concept of this time-dependent power spectrum and relates the intuitive and analytical approaches to such a quantity. A suitable expression is found and compared with the approaches of other investigators.

Chapters IV, V and VI develop the first passage expressions for lightly damped single degree-of-freedom oscillators subjected to deterministically modulated broad-band excitation. Of special
interest is the time-evolving response of an oscillator suddenly exposed to stationary input. The case of a zero damped oscillator is included. Chapter VII presents numerical results for the reliability of an oscillator with this "step" random input. The reliability is defined as the probability that the response has not yet crossed a fixed barrier in a given time interval. The extension to an earthquake-type excitation is also illustrated in the final chapter.
CHAPTER I

FIRST PASSAGE IN RANDOM VIBRATIONS

Section IA Stationary Frequency Analysis

Random vibrations is a relatively young field; yet much of its basis comes from a well established area: signal noise in electrical engineering and communications. Mathematical treatment of the stochastic noise phenomenon dates back at least twenty-five years to Rice [1]. Application of the theory to mechanical oscillators has followed, however, only in the past decade. Much of the work in this application has come from Crandall [2,3,4].

It is assumed in this report that the reader has a working knowledge of the basic concepts of stationary random vibration theory. For clarity, an informal review of general notions is presented in Appendix A. Emphasis is on those quantities that have important generalized forms in nonstationary theory.

A stationary random vibration may be considered to be composed of sines and cosines of various frequencies and amplitudes. If the vibration is periodic, a Fourier series analysis may be made to determine the relative importance of different frequencies. If the vibration is not periodic, a Fourier transformation may be applied [5]. A Fourier transform on a single sample of a random vibration yields the relative frequency content of the actual motion in that sample. If \( X_1(\omega) \) denotes the
Fourier transform of sample i, then

\[ X_i(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t} dt \]  

(IA-1)

Analogous to this, a time-average physical frequency spectrum of sample i may be defined by

\[ F_i(\omega) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X_i(t) e^{-i\omega t} dt \]  

(IA-2)

If the process is ergodic, and sample i is a representative sample, then the time-average physical frequency spectrum of sample i is the time-average physical frequency spectrum of the process in general.

\[ |F_i(\omega)| = |F(\omega)| \]  

(IA-3)

The autocorrelation function of a stochastic process is defined as an ensemble average.

\[ R(\tau) = E[x(t)x(t+\tau)] \]  

ensemble  

(IA-4)

If a Fourier transform is made on the autocorrelation function, a frequency decomposition of the autocorrelation function is obtained.

\[ G(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega \tau} d\tau \]  

(IA-5)

For all real, stationary processes, G(\omega) is an even, non-negative function of \( \omega \). For physical significance it is defined here only for posi-
tive $\omega$. Since $G(\omega)$ is a Fourier transform of $R(\tau)$, the Wiener-Khintchine relation guarantees the inverse transform relation

$$R(\tau) = \int_{-\infty}^{\infty} G(\omega)e^{i\omega\tau}d\omega \quad \text{with} \quad f = \omega/2\pi \quad (IA-6)$$

$G(\omega)$ is interpreted as the mean square spectral density. This can be seen by putting $\tau=0$ in Equations [IA-6] and [IA-4].

$$\int_{-\infty}^{\infty} G(\omega)d\omega = R(0) = E[x^2(t)] \quad (IA-7)$$

For a zero-mean process, $E[x^2(t)]$ equals the variance. The integral of $G(\omega)$ over all frequencies may be interpreted as the zeroth moment of the mean square spectral density.

$$\lambda_0^\Delta = \int_{-\infty}^{\infty} G(\omega)d\omega \quad (IA-8)$$

The temporal autocorrelation function of a sample function, $i$, is defined as

$$\phi_i(\tau) = E_{time}[x(t)x(t+\tau)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_i(t)x_i(t+\tau)d\tau \quad (IA-9)$$

If a Fourier transform is made on $\phi_i(\tau)$, a frequency decomposition of the temporal autocorrelation function is obtained.

$$S_i(\omega) = \int_{-\infty}^{\infty} \phi_i(\tau) e^{-i\omega\tau}d\tau \quad (IA-10)$$

The corresponding inverse relation is

$$\phi_i(\tau) = \int_{-\infty}^{\infty} S_i(\omega) e^{i\omega\tau}d\omega \quad (IA-11)$$
If the process is ergodic, $\phi_i(\tau)$ and $S_i(\omega)$ apply to all samples and to the process in general. Then the temporal averages over a sample of infinite duration are equal to the corresponding ensemble averages.

$$\phi_i(\tau) = R(\tau)$$  \hspace{1cm} (IA-12)

$$S_i(\omega) = G(\omega)$$  \hspace{1cm} (IA-13)

Since $F(\omega)$ is a time-average of a Fourier transform of a sample, and $S(\omega)$ is a Fourier transform of a time-average of a sample, it is not surprising that when the limiting processes are handled carefully $S(\omega)$ and $F(\omega)$ are equal. For a stationary ergodic process, therefore, the interpretation of $G(\omega)$ (the mean square spectral density) as the physical frequency spectrum of the process is given additional physical motivation.
Section IB  First Passage Theory

Of interest in random vibration analysis is the exceedance of a response characteristic above some pre-established level. An approximate expression for the mean crossing rate of a stationary process beyond a fixed barrier level has been derived by Rice[1]. His expression is

$$\nu_a^+ = \frac{1}{dt} \int_{\alpha}^{\infty} dx \int_{a-}^{a+} x p(x, \dot{x}) dx$$ (IB-1)

$\nu_a^+$ is the frequency of crossings of the fixed barrier level $x=a$. The superscript $+$ indicates that only crossings with a positive slope are counted (upcrossings). $p(x, \dot{x})$ is the joint probability density function of the process, $x$, and its first time derivative, $\dot{x}$. If the process is stationary and Gaussian with zero-mean, the joint probability density is

$$p(x, \dot{x}) = \frac{1}{2\pi \sigma_x \sigma_{\dot{x}}} \exp \left[ -\frac{1}{2} \left( \frac{x^2}{\sigma_x^2} + \frac{\dot{x}^2}{\sigma_{\dot{x}}^2} \right) \right]$$ (IB-2)

Then Equation [IB-1] reduces to

$$\nu_a^+ = \frac{1}{2\pi} \frac{\sigma_{\dot{x}}}{\sigma_x} \exp \left\{ -\frac{a^2}{2\sigma_x^2} \right\}$$ (IB-3)

A process that represents the output of a lightly damped single degree-of-freedom oscillator is approximately narrow-band. In a narrow-band process the motion occurs in a narrow range or band of frequencies whose central frequency is large with respect to the bandwidth. For this type of Gaussian process it appears reasonable to
define a representative frequency of the process. This frequency may be obtained by setting $a=0$ in Equation [IB-3].

$$\nu_0^+ = \frac{1}{2\pi} \frac{\sigma_x}{\sigma_x}$$  \hspace{1cm} (IB-4)

The representative period of the process is then

$$T_0 = \frac{1}{\nu_0^+}$$  \hspace{1cm} (IB-5)

Of interest is the time to first passage of a process above a barrier level $x=a$. If it is assumed for the moment that the barrier upcrossings are Poisson arrivals (asymptotically correct for high barrier levels [1]), then the time between upcrossings, $T_0$, has an exponential probability density function with parameter $\nu_a^+$.

$$f[T_0] = \nu_a^+ e^{-\nu_a^+ T_0} \quad T_0 \geq 0$$  \hspace{1cm} (IB-6)

The mean time between upcrossings is then

$$E[T_0] = \frac{1}{\nu_a^+}$$  \hspace{1cm} (IB-7)

which is also the standard deviation of $T_0$.

If one commences observing a stationary process at an arbitrary instant, and if the barrier level is high, it is permissible to neglect the probability that the process will already be above this barrier at the first instant of observation. In this case, the random variable, $T_0$, time between upcrossings, is also the time to first barrier crossing.
The Poisson arrival assumption for barrier crossings implies independence of crossings. In reality, crossings tend to occur in clumps. This is especially true of a narrow-band process, in which the motion may be approximately characterized as an amplitude modulated sinusoid, whose frequency varies in a relatively small range around some central value. A barrier crossing in one cycle implies an increased conditional probability of a crossing in the next cycle. For a fixed mean crossing rate, the clumping of crossings tends to increase the time between clumps. Thus the time, $T_o$, to the first barrier crossing, tends to increase. Early investigations in this area were conducted by Lyon [6] and have more recently undergone additional mathematical development by Vanmarcke [7] and simulation verification by Yanev [8]. Vanmarcke shows that the crossing rate depends on at least one additional parameter, a shape factor of the mean square spectral density of the process. For crossing of a positive barrier level $x=a$ by a stationary Gaussian process, the expected clump size is approximately

$$E[\text{clump size}] = \frac{1}{1 - \exp\left(-aq \sqrt{\frac{2\pi}{\lambda_o}}\right)} \quad (IB-8)$$

The expected rate of clump occurrences is then

$$E[\text{clump rate}] = \frac{v^+_a}{E[\text{clump size}]} = v^+_a \left[1 - \exp\left(-aq \sqrt{\frac{2\pi}{\lambda_o}}\right)\right] \quad (IB-9)$$

where $\lambda_o$ is the root mean square (RMS) of the process, $v^+_a$ is the rate of upcrossings of the barrier level $x=a$, and $q$ is Vanmarcke's spectral
density shape parameter, to be discussed later.

Since the expected rate of zero crossings is $\nu_0^+$, and the expected rate of barrier level $x=a$ crossings is $\nu_a^+$, the expected value of $N$, the number of zero crossings between clumps is

$$E[N] = \frac{\nu_0^+ - \nu_a^+}{E[clump rate]} = \frac{\nu_0^+ - \nu_a^+}{\nu_a^+} E[clump size]$$

$$= \left( \frac{\nu_0^+}{\nu_a^+} - 1 \right) \frac{1}{1 - \exp \left( -a \sqrt{\frac{2\pi}{\lambda_0}} \right)}$$

$$= \frac{a^2/2\lambda_0}{1 - e^{-a \sqrt{2\pi/\lambda_0}}}$$

(IB-10)

For a narrow-band process, the time between clumps is simply

$$T_0 = \frac{2\pi}{\omega_0} N \quad \text{where} \quad \frac{2\pi}{\omega_0} = \frac{1}{\nu_0^+}$$

(IB-11)

As mentioned earlier, $T_0$ is also the time to first barrier crossing for high barrier levels following observation of a stationary process at an arbitrary instant. The probability density function associated with $T_0$ is usually designated $f_{T_0}(t)$.

The notation that will now be used for first passage considerations is consistent with other literature in the field [9,10].

A stationary process, $X(t)$, is considered to be in one of two states: State "0" if it is below the designated barrier and state...
"1" if it is above the barrier - the so-called failure state. Let $L_D(t)$ be the reliability function, i.e. the fraction of sample functions that have not left state "0" when being observed from time zero to time $t$.

$$L_D(t) = P[T_f > t] = 1 - F_{T_f}(t) \quad t > 0 \quad \text{(IB-12)}$$

where the random variable $T_f$ is the time to first barrier crossing and $F_{T_f}(t)$ is the cumulative distribution of $T_f$. The subscript $D$ on $L_D(t)$ signifies that a double sided barrier ($|x| = a$) is being considered. The density function of time to first barrier crossing is

$$f_{T_f}(t) = \frac{d}{dt} F_{T_f}(t) = -\frac{d}{dt} L_D(t) \quad t > 0 \quad \text{(IB-13)}$$

The distribution of time to first barrier crossing includes the consideration of instantaneous barrier exceedance, i.e., the probability that the process is already in state "1" at $t=0$. The function $f_{T_f|T_f>0}(t)$ is defined as the density function of the time to first barrier crossing given that the process was in state "0" at $t=0$.

$f_{T_f|T_f>0}(t)$ is related to $f_{T_f}(t)$ by

$$f_{T_f}(t) = P[T_f > 0] f_{T_f|T_f>0}(t) \quad t > 0 \quad \text{(IB-14)}$$

Using an approach similar to that of Rice and Beer [11] to relate $f_{T_f|T_f>0}(t)$ to $f_{T_0}(t)$ for high barrier levels, yields

$$f_{T_f|T_f>0}(t) = \frac{1}{E[T_0]} \int_t^\infty f_{T_0}(u) du \quad t > 0. \quad \text{(IB-15)}$$

If it is assumed that the holding time in the "0" state is exponen-
tially distributed, then
\[ f_{T_0}(t) = ae^{-at} \quad t > 0 \quad (IB-16) \]
where \( 1/\alpha = E[T_0] \) is the expected holding time in state "0". With this substitution, Equation [IB-15] reduces to
\[ f_{T_f | T_f > 0}(t) = ae^{-at} \quad t > 0 \quad (IB-17) \]
Then, from Equation [IB-14],
\[ f_{T_f}(t) = P[T_f > 0]ae^{-at} \quad t > 0 \quad (IB-18) \]
and from Equation [IB-13]
\[ L_D(t) = P[T_f > 0]e^{-at} \quad t > 0 \quad (IB-19) \]

It is seen that \( \alpha \) is a measure of barrier crossing likelihood. The larger the value of \( \alpha \), the more rapidly the reliability function decreases with time. If the natural logarithm of \( L_D(t) \) is plotted versus time, the graph is a straight line with negative slope \( \alpha \). Because of this, \( \alpha \) is often referred to as the hazard function. For a double sided barrier (a barrier located at \( |x| = a \)) Equation [IB-19] becomes
\[ L_D(t) = A_De^{-\alpha_D t} \quad (IB-20) \]
where \( A_D = P[T_{f,D} > 0] \) for a double sided barrier. For a Gaussian process, the parameters \( \alpha_D \) and \( A_D \) are \([7]\) approximately
\[
\alpha_D = \frac{1}{E[T_{0,D}]} = 2\nu_a^+ \frac{1 - \exp\left\{-n_a/2\nu_a^+\right\}}{1 - \nu_a^+/\nu_0^+}
\]

\[
A_D = P[T_{f,D} > 0] = \frac{E[T_{0,D}]}{E[T_{0,D}] + E[T_{1,D}]}
\]

\[
= 1 - \frac{\nu_a^+}{\nu_0^+}
\]

(1B-21)

where

\(\nu_0^+ = \) representative frequency of the (stationary) process.

\(\nu_a^+ = \) frequency of upcrossings by the process of the fixed double barrier \(|x| = a\) [1].

\(n_a = \) frequency of upcrossings of the envelope of the process of the fixed double barrier \(|x| = a\). This statistic is closely related to the frequency of the clump occurrences.

This definition of \(A_D\) is a slightly conservative one based on the assumption that the process experiences a peak at \(t=0\).

A useful and often appropriate assumption is that the process is Gaussian. For a Gaussian process with a high barrier, Cramer and Leadbetter [12] have related \(\alpha_D\) and \(A_D\) to the properties of the spectral density function of the process. Vanmarcke has extended their work to give better results for narrow-band processes and for low barrier levels, with results dependent on the barrier level \(x=a\), and the first three moments of the spectral den-
density function $G(\omega)$. The zeroth moment was defined in Section IA as

$$\lambda_0 = \int_0^\infty G(\omega) \, df$$  \hspace{1cm} (IB-22) 

$\lambda_0$ is the area under the mean square spectral density function, and

is therefore the mean square value of the process. The square root of $\lambda_0$ is the RMS level.

The first moment is

$$\lambda_1 = \int_0^\infty \omega G(\omega) \, df$$  \hspace{1cm} (IB-23) 

For a narrow-band process, $\lambda_1/\lambda_0$ may be interpreted as a representative frequency of the process.

The second moment is

$$\lambda_2 = \int_0^\infty \omega^2 G(\omega) \, df$$  \hspace{1cm} (IB-24) 

Analogous to sinusoidal motion, $\sqrt{\lambda_2/\lambda_0}$ for a narrow-band process may be interpreted as a process central frequency and is equal to $2\pi \nu^+$. Vanmarcke's spectral density shape factor, $q$, is defined as

$$q = \sqrt{1 - \frac{\lambda_1^2}{\lambda_0 \lambda_2}}$$  \hspace{1cm} (IB-25) 

This parameter is bounded by $q=0$ and $q=1$ and is a measure of the dispersion of the spectral density: The lower the value of $q$ the more concentrated the spectral density about a central frequency. For pure sinusoidal motion at a single frequency, $q=0$. $q$ may also be shown to be equal to the ratio of the RMS of the derivative of the enve-
lope of a process and the RMS of the derivative of the process [13].

For a stationary narrow-band Gaussian process, \( \alpha_D \) and \( A_D \) are

\[
A_D = 1 - \exp \left\{ -\frac{a^2}{2\lambda_0} \right\}
\]

\[
\alpha_D = \frac{2}{2\pi} \sqrt{\frac{\lambda_2}{\lambda_0}} \left[ 1 - \exp \left\{ -aq \sqrt{\frac{\pi}{2\lambda_0}} \right\} \right] 
\]

\[
L_D(t) = \left[ 1 - \exp \left\{ -\frac{a^2}{2\lambda_0} \right\} \right] \exp \left\{ -2t\sqrt{\frac{\lambda_2}{2\pi\lambda_0}} \right\} 
\]

\[
\exp \left\{ \frac{a^2}{2\lambda_0} \right\} - 1
\]

where \( \frac{1}{2\pi} \sqrt{\frac{\lambda_2}{\lambda_0}} = \nu_0^+ \) (IB-26)

For high barrier levels, \( \exp \left\{ -aq \sqrt{\pi/2\lambda_0} \right\} \) and \( \exp \left\{ -a^2 /2\lambda_0 \right\} \) may be neglected with respect to 1. The expression for \( \alpha_D \) then becomes

\[
\alpha_D = \frac{2}{2\pi} \sqrt{\frac{\lambda_2}{\lambda_0}} e^{-\frac{a^2}{2\lambda_0}} \quad (13-27)
\]

which is the hazard function used by Cramer and Leadbetter[12]. Vanmarcke's expression has been shown to give better results for lightly damped oscillators [7,14]. The Cramer and Leadbetter result yields too low an estimate for the reliability.

Vanmarcke extends the first crossing theory to a nonstationary narrow-band Gaussian process by neglecting any time change in the shape of the mean spectral density. Thus, he assumes that the narrow-band representative frequency, \( \nu_0^+ \), and the spectral density shape factor, \( q \),
are independent of time, but the area under the curve, the mean square value of the process, $\lambda_o$, is time-dependent. Following an approach similar to Amin and Ang [15], a time-dependent hazard rate, $\alpha_D(t)$ is introduced. Equation [IB-20] becomes

$$L_D(t) = A_D e^{-\int_0^t \alpha_D(u) du} \quad t \leq s$$  \hspace{1cm} (IB-28)

where $s$ is the duration of the process.

For most applications, a nonstationary process is considered to have started from rest. In this case the probability of instantaneous barrier exceedance is zero, and $A_D=1$. An excellent paper by Caughey and Stumpf [16] discussed RMS response for such an excitation. The reliability in Equation [IB-28], following Vanmarcke, becomes

$$L_D(t) = e^{-\int_0^t \alpha_D(u) du} \quad t \leq s$$  \hspace{1cm} (IB-29)

where $\alpha_D(t)$ is given by

$$\alpha_D(t) = \frac{2}{2\pi} \sqrt{\frac{\lambda_o}{\lambda^2}} \left( 1 - \exp \left\{ -aq \sqrt{\frac{\pi}{2\lambda_o(t)}} \right\} \right) \exp \left\{ \frac{a^2}{\lambda_o(t)} \right\} - 1 \quad t \geq 0$$  \hspace{1cm} (IB-30)

with $\frac{1}{2\pi} \sqrt{\frac{\lambda^2}{\lambda_o}} = \nu_o^+$

and $L_D(t)$ is

$$L_D(t) = \exp \left[ -2\nu_o^+ \int_0^t 1 - \exp \left\{ -aq \sqrt{\frac{\pi}{2\lambda_o(u)}} \right\} \exp \left\{ \frac{a^2}{2\lambda_o(u)} \right\} - 1 \right]$$  \hspace{1cm} (IB-31)
A more detailed development of barrier crossings and first passage problems is given by Vanmarcke \cite{7}. Consideration of double sided barriers and envelope crossings leads to good results for first passage distributions of both stationary and nonstationary processes. This problem has also received attention from Roberts \cite{17}, Rosenblueth and Bustamante \cite{18}, Shinozuka and Yang \cite{19, 20, 21} and Ditlevsen \cite{22}.

This report will be concerned primarily with first passage times of nonstationary processes. In order to do this, the role of spectral density for nonstationary processes will be examined closely in Chapter II. First, however, some first passage results will be examined in the next section in order to determine whether additional analytical development in the field of nonstationary theory is warranted.
Section IC Nonstationarity Effect on First Passage Theory

This section will examine some nonstationary first passage results to see if there are significant phenomena other than those already considered by Amin and Ang [15] of variable RMS and by Vanmarcke [7] of a time-dependent spectral density with a constant shape.

First, some practical examples of nonstationary vibration theory will be reviewed.

Examples of Physical Problems

Many physical problems involve nonstationary random vibrations. Systems in civil engineering, naval architecture, mechanical engineering, aeronautical engineering, and electrical engineering are all liable to be subjected to these excitations. A few references to these fields will be given here.

De Jong [23] has investigated nonlinear ship roll in a "random" sea." His concern is first passage of response beyond a critical barrier.

Several investigators have been concerned with ground interaction effects. Balsara [24] has investigated the effect of blast on buried structures. Some time histories of response characteristics from Balsara's work are shown in Figure [IC-1]. The response characteristics clearly indicate a strong time-dependency. Galbraith
accel. in/sec

36 inch buried arch
pressure = 120 psi

12 inch buried arch
pressure = 209 psi

8 inch buried arch
pressure = 239 psi

(from Balsara, J.P. [24], pages 5-7)

Figure IC-1
and Schreiner [25] and Hudson, Alford, and Housner [26] have considered the response of structures to underground explosion. The rudiments of the deterministic approach for design of structures to resist blast loading may be found in Biggs [27].

The analysis of wind effect on tall buildings has been performed traditionally by modeling the wind as a static lateral pressure or by model testing. The latter method can be very informative for a structure once the design is known, but it is often difficult to extrapolate results. Davenport [28,29,30], Parmelee [31], and others have been considering vibrations of buildings due to wind, and the effect of these vibrations on people. Random vibrations analysis of wind loading normally identifies a mean wind pressure and separates this to leave a stationary zero-mean random process. Actually the process should be modeled as nonstationary because of a time-dependent variance.

A major area for vibrations analysis within civil engineering is in earthquake engineering. Biggs [40] reviews aseismic design from a deterministic viewpoint. Due to the uncertainties associated with earthquakes, however, both in occurrence and characteristics of motion, it is convenient to consider a random vibrations approach. Trifunac [32] has investigated the characteristics of strong motion earthquakes and Rascon and Cornell [33] have analyzed a model for earthquake simulation. Typical large-earthquake accelerograms can be found in Hansen [34] and Wiegel [35]. Time histories of two large
earthquakes are reproduced in Figure [IC-2] from Wiegel. Because of the relatively short duration of an earthquake, nonstationary effects may be important. Even if the ground motion is considered stationary during a substantial portion of its total duration, it may still be necessary to treat the response as nonstationary, especially if the level of structural damping is low. For earthquakes of short duration, as measured in terms of cycles of natural frequency of the structure, response characteristics may reach only a small percentage of their stationary value. The significance of nonstationarity in studying earthquakes has been discussed by Amin and Ang [36]. Shinozuka and Yang [37] and Iyengar and Iyengar [38] have considered the theory behind and the numerical methods necessary for the simulation of nonstationary processes for application in earthquake engineering. Some simulation models have been generated and tested by Hou [39].

Effect of Nonstationarity on First Passage

Concern is centered here on the effect of nonstationarity on first passage times. Section IB has summarized the analytical development of first passage considerations for stationary processes and Vanmarcke's extension to include the first three moments of the spectral density of the response, both for constant and time-dependent RMS.

Some calculated and simulated results will be presented here
El Centro, California
May 18, 1940
N-S component

Taft, California
July 21, 1952
S69E component

(from Wiegel, R.L. [39], pages 405 and 109)

Figure IC-2
to show that nonstationarity can have a major influence on first
passage rates, and that the theories developed heretofore do not
handle the nonstationary problem completely. The motivation for a
more refined analysis of the nonstationary problem is based on these
results.

For a wide-band stationary excitation suddenly applied to an
oscillator at rest, the response of the oscillator is a special type
of evolutionary response. This particular evolutionary response to
a step input starts at zero and approaches the stationary steady-
state level for long times. Caughey and Stumpf [22] consider the
RMS value for such an evolutionary process. At $t=0$ the oscillator
is exposed to a stationary excitation for all future time. The im-
pulse response function of the oscillator is denoted $h(t)$, the excit-
ing acceleration, $X(t)$, and the response displacement of the oscil-
lator, $Y(t)$. Then

$$Y(t) = \int_{0}^{t} h(t-\tau) X(\tau) d\tau \quad \text{(IC-1)}$$

If the exciting acceleration is a zero-mean stochastic pro-
cess ($E[X(t)] = 0$), then the output is also a zero-mean stochastic
process.

$$E[Y(t)] = 0 \quad \text{(IC-2)}$$

The variance of the output is
\[ \text{Var}[Y(t)] = E[Y^2(t)] - E^2[Y(t)] \]
\[ = E[Y^2(t)] \text{ for a zero-mean process} \]
\[ = \lambda_o(t) \quad (\text{IC-3}) \]

Using the impulse response function, this becomes
\[ \lambda_o(t) = \text{Var}[Y(t)] = E[\int_0^t h(t-\tau)X(\tau)d\tau \int_0^t h(t-\tau')X(\tau')d\tau'] \]
\[ = \int_0^t \int_0^t h(t-\tau)h(t-\tau')E[X(\tau)X(\tau')]d\tau d\tau' \quad (\text{IC-4}) \]

Since \(X(t)\) is a stationary process, \(E[X(\tau)X(\tau')]\) is \(R_F(\tau-\tau')\). The second Wiener-Khintchine relation was used by Caughey and Stumpf to replace \(R_F(\tau-\tau')\)
\[ R_F(\tau-\tau') = \int_0^\infty G_F(\omega)e^{i\omega(\tau-\tau')}d\omega \quad (\text{IC-5}) \]

For a real process, \(e^{i\omega(\tau-\tau')}\) may be replaced by its real part, \(\cos[\omega(\tau-\tau')]\). Equation [IC-5] is now substituted into Equation [IC-4]. The order of integration may be reversed since the integrals are convergent. The resulting expression for the mean square response (variance) is
\[ \lambda_o(t) = \text{Var}[Y(t)] = \int_0^\infty G_F(\omega) \int_0^t \int_0^t h(t-\tau)h(t-\tau') \cos[\omega(\tau-\tau')]d\tau d\tau' d\omega \quad (\text{IC-6}) \]

\(h(t)\) for a linear one degree-of-freedom oscillator is given as
\[ h(t) = \begin{cases} 
0 & t \leq 0 \\
-\omega_0^2t \sin(\omega_1t) & t > 0
\end{cases} \quad (\text{IC-7}) \]

If the excitation is approximately white noise, \(G_F(\omega)\)
is almost constant with respect to \( \omega \), and it may be replaced in Equation [IC-6] by \( G_F(\omega_o) \). Equation [IC-6] then becomes

\[
\lambda_o(t) = \text{Var}[Y(t)] = \frac{G_F(\omega_o)}{8\xi\omega_o^3} \left[ 1 - e^{\frac{-2\omega_o\xi t}{\omega_1^2}} \right] \left\{ \omega_1^2 + \omega_o\omega_1\xi \sin(2\omega_1 t) + 2\omega_o^2\xi^2 \sin^2(\omega_1 t) \right\} 
\]  

(IC-8)

This result is plotted in Figure [IC-3] from Caughey and Stumpf.

In order to compare the difference in first passage probabilities for an evolutionary response to step input and a stationary process, results have been computed using Cramer's hazard function, \( \alpha_D = 2\nu_0^+ e^{-a^2/2\lambda_o} \), as given in Equation [IB-27]. For the stationary case, \( \alpha_D \) is independent of time, and the reliability function is

\[
L_D = e^{-\alpha_D t} 
\]  

(IC-9)

For the transient response to step input, \( \lambda_o \) is given by Equation [IC-8] and the reliability function is

\[
L_D(t) = e^{-\int_0^t \alpha_D(u)du} \quad t \leq s 
\]  

(IC-10)

Barrier levels considered are \( r = 1.2, 2.0, \) and \( 3.0 \) times the RMS at the end of the excitation (not the steady-state RMS). An excitation duration of \( 30\omega_o \) is considered, where \( \omega_o \) is the undamped natural circular frequency of the oscillator. Damping levels are one-tenth of one per cent and ten per cent. The results are plotted in Figures [IC-4] to [IC-9]. These figures give the reliability probabilities as a function of time.
Mean Square Response

\[ \frac{4\omega_0^3 \lambda_0(\omega_0 t)}{G_f(\omega_0)} \]

(\text{from Caughey and Stumpf[16], Figure 1})

Figure IC-3
Reliability: Single DOF Oscillator - Broad-Band Excitation

$\xi = 0.001$

$R = 1.2$

$S = 30$
Reliability: Single DOF Oscillator - Broad-Band Excitation

$\xi = .001$

$R = 2$

$S = 30$

Figure IC-5

L$_D$(t)

stationary Poisson

nonstationary Poisson
Reliability: Single DOF Oscillator - Broad-Band Excitation

\[ \xi = 0.001 \]
\[ R = 3 \]
\[ S = 30 \]

nonstationary Poisson

stationary Poisson

\( L_0(t) \)
Reliability: Single DOF Oscillator - Broad-Band Excitation

\[ \xi = 0.1 \]
\[ R = 1.2 \]
\[ S = 30 \]
Reliability: Single DOF Oscillator - Broad-Band Excitation

\[ L_D(t) \]

\[ \xi = 0.1 \]
\[ R = 2 \]
\[ S = 30 \]

nonstationary Poisson
stationary Poisson
Reliability: Single DOF Oscillator - Broad-Band Excitation

\[ L_D(t) \]

\[ \xi = 0.1 \]
\[ R = 3 \]
\[ S = 30 \]

Simulation (Chandiramani[55])

Nonstationary Vanmarcke

Nonstationary Poisson

Poisson
It may make an important difference if the barrier level is in terms of the RMS at the end of the excitation or the steady-state RMS. If the duration is short enough that the response has not had time to near the steady-state level, then a barrier of a given multiple of the steady state RMS will appear to be much higher with respect to the actual motion. Thus an oscillator is safer for a given barrier level if this level is a multiple of the steady-state RMS rather than the same multiple of the end-of-excitation RMS.

The more rapid the decline in the reliability function, the less safe an oscillator is. On the other hand, for a given set of parameters the lower the curve, the safer it is to use it as an estimate of the reliability (i.e. it is more conservative). For the evolutionary response to step input, the probability of instantaneous barrier exceedance is zero. For the $\alpha_D$ equal to a constant case, there is a small nonzero probability of instantaneous barrier exceedance. This probability has been neglected in the graphs in order to compare more clearly the difference in time histories of the two solutions.

Using the transient solution yields a smaller barrier exceedance probability because it takes into account that the RMS response starts from zero, thereby giving little chance of barrier exceedance during the initial seconds. Barnoski and Maurer [41] have verified Caughey and Stumpf's results indicating that for white noise input,
suddenly applied to an oscillator at rest, the response RMS builds up to its steady-state value. They also found, however, that for a non-white excitation the response RMS may overshoot its stationary value before finally settling down at steady state. For this latter case, the use of stationary analysis could be unconservative. As damping increases, the RMS at the end of the excitation, for a given duration, more nearly approaches the steady-state RMS. Therefore, the difference between a barrier defined as a multiple of steady-state RMS or end-of-excitation RMS lessens. Holman and Hart [42] are currently working on the RMS response to nonstationary excitation.

Figures [IC-4] to [IC-9] show that the consideration of nonstationarity makes a significant difference in the reliability values, especially for small damping levels. This is to be expected since the lower damping implies a longer time until stationarity is reached. Figure [IC-9] makes it clear that while present nonstationary theories are much closer to simulated results than traditional steady-state analysis, there is still a significant discrepancy. It is this difference that warrants the further analysis of the nonstationary problem. It is believed a remaining significant effect is the time variation of the shape of the power spectral density function. This leads immediately into the question of the definition of a power spectrum for a nonstationary process. This will be the subject of the next chapter.
THE ROLE OF A TIME-DEPENDENT POWER SPECTRUM

The time-average physical frequency spectrum was introduced in Section IA for a stationary process. It describes the mean square motion present at each frequency for a sample. For an ergodic, stationary process, an equivalent spectrum may be determined by performing a Fourier transform on the autocorrelation function of the process. It will be the purpose of this chapter to investigate the extension of these concepts to a nonstationary process. This area has been the subject of recent investigations in civil engineering by Liu [43], Shinozuka [44], and Shinozuka and Brant[45].
Section IIA  The Evolutionary Response Spectrum for Step Input

Consider again an oscillator that is at rest for $t<0$, and exposed to a stationary excitation for all $t>0$. For $t<0$ the appropriate power spectrum of the response is zero. The oscillator response will pass through a transient stage and finally (asymptotically) reach a stationary condition. For $t\to\infty$ the power spectrum is the familiar physical description of the frequency decomposition of the (asymptotically) stationary response. This latter spectrum may be found by taking the Fourier transform of the asymptotic autocorrelation function of the motion. Both the magnitude (RMS) and the relative frequency content of the response evolve from their values at time zero to those of the stationary level. At any given time, one may envision a power spectrum that physically represents the frequency decomposition at that time. This notion leads to the concept of an evolutionary spectrum. The fundamentals of this type of evolutionary response to random excitation were outlined in the last section. The resulting expression for $\lambda_0(t)$ was given in Equations [IC-6] and [IC-8]. This former equation is

$$\lambda_0(t) = \int_\infty^t G_F(\omega) \int_\infty^t h(t-\tau)h(t-\tau')\cos[\omega(\tau-\tau')]d\tau d\tau' df \quad \text{(IIA-1)}$$

Taking a lead from this result, if one defines a function, $G(\omega,t)$, of frequency and time as

$$G(\omega,t) \triangleq G_F(\omega) \int_\infty^t h(t-\tau)h(t-\tau')\cos[\omega(\tau-\tau')]d\tau d\tau' \quad \text{(IIA-2)}$$

then $G(\omega,t)$ has the convenient properties that it is a function only
of time up to the present, and when integrated over all frequencies it gives the mean square response.

\[ \lambda_0(t) = E[Y^2(t)] = \int_0^\infty G(\omega, t) df \]  

satisfying the second Wiener-Khintchine relation with \( \tau = 0 \). It is thus appropriate to call \( G(\omega, t) \) a time-dependent output power spectral density.\(^*\) An examination of \( G(\omega, t) \) shows that this time-dependent spectrum is initially zero for all frequencies, and for long time it approaches the steady-state transfer function of a one degree-of-freedom oscillator.

This evolutionary spectrum for step input has been computed for several different times for an initially quiescent one degree-of-freedom oscillator exposed to stationary white noise input at time \( 0 \). The oscillator has either one per cent or ten per cent of critical damping. The results are plotted in Figures [IIA-1] and [IIA-2]. It is seen that the spectrum increases monotonically at \( \omega = \omega_0 \) whereas it tends to remain fairly constant or even to decrease slightly at frequency ratios not near unity. The spectrum experiences sinusoidal-like fluctuations, both with respect to frequency for a given time & with respect to time at a given frequency. These fluctuations will be shown to be of the form \( \sin(\omega_1 t) \), \( \sin^2(\omega_1 t) \) and \( \sin(\omega t) \). For long times, these fluctuations decay and the transfer function for steady-state is, as expected, smooth. The growth

\(^*\) This definition is not the only one that can be considered, as will be discussed in Chapter III.
Evolutionary Spectrum - Step Excitation
Normalized input spectrum: $G_f(\omega_0)/2\pi = 1$

Figure IIA-1
Evolutionary Spectrum - Step Excitation

Normalized input spectrum: \( G_f(\omega_0)/2\pi = 1 \)
of the spectrum deviates greatly from that which would be associated with proportional or constant-shape growth. Therefore, not only $\lambda_0$ but also Vanmarcke's spectral density shape factor, $q$, is a function of time. This may have significant implications with respect to first passage probability estimation since the time-dependent barrier exceedance rate depends on $q$ as well as on $\lambda_0$. The influence of time changes in $q$ has not been considered before.

* Figure [IIA-3] shows the evolution of $G(\omega, t)$ implied by constant shape ($q$) growth.
Constant Shape Spectrum Growth

\[ G(\omega) = 10\% \]

Normalized input spectrum: \( G_{\xi}(\omega_0)/2\pi \)

Figure IIA-3
Section II B  The Evolutionary Response Spectrum for Deterministic Modulated Stationary Input

There is no reason to limit the concept of an evolutionary spectrum to the type associated with response to a suddenly applied stationary excitation. Intuitively, it would seem that a frequency decomposition of a random process could be made at any instant, yielding an instantaneous spectral analysis. One type of suitable process is the response of an oscillator to arbitrary random input. The specific procedure for performing the spectral analysis, either physically or theoretically, introduces a number of potential difficulties and conflicting definitions. These problems will be discussed in Chapter III.

Utilizing the same approach as the one that resulted in $G(\omega,t)$ for the evolutionary response to step input, it is straightforward to find an integral expression for the evolutionary output spectrum of the response of an oscillator to an input described by a separable time-dependent spectrum of the form $G(\omega)f(t)$. $G(\omega)$ is a stationary normalized (i.e. $\lambda_0$ of $G(\omega) = 1$) power spectrum and $f(t)$ is an arbitrary deterministic function of time. This form forces the relative frequency content of the input to remain constant, but allows the total amount of power to change. For this kind of process, Vanmarcke's spectral density shape factor, $q$, of the (input) spectrum is constant, but the mean square, $\lambda_0(t)$, is not. In the particular case where $f(t)$ is exponential in form, the expression for $G(\omega,t)$ may be evaluated after some simple algebra. The closed form expressions will be discussed in Chapter V.
Section IIC  Equivalent Time-Dependent Damping

In Section IIA it was seen that the power spectrum of the response of an oscillator to a suddenly applied stationary white noise random excitation built up from zero to the stationary level. For a given damping at a given time, the power spectrum looks similar to that which would be associated with the stationary spectrum of an oscillator with higher damping. The investigation of this observation leads to some interesting insights.

Consider first a quiescent oscillator subjected to a very narrow-band input. This excitation is approximated by $F_0 \sin(\omega_0 t)$ where $\omega_0$ is the central frequency of the excitation. The circular frequency $\omega_0$ is assumed to be the same as the natural frequency of the oscillator. The power spectrum of the stationary input, $G_F(\omega)$, is approximately a delta function at $\omega = \omega_0$.

$$G_F(\omega) = \begin{cases} 0 & \text{otherwise} \\ G_0 & \frac{\Delta \omega}{2} \leq \omega \leq \omega_0 + \frac{\Delta \omega}{2} \end{cases} \quad (IIC-1)$$

The area under the input spectrum equals the mean square value of the input. The power spectrum of the output is $G_R(\omega, t) = |H(\omega, t)|^2 G_F(\omega)$. The time-dependent mean square response is the area under the output power spectrum.

$$\text{Var}[R] = \int_0^\infty G_R(\omega, t) df = \int_0^\infty G_F(\omega)|H(\omega, t)|^2 df \quad (IIC-2)$$
\[ G_F(\omega) = 0 \] except for \( \omega_0 + \Delta \omega / 2 \), where \( \Delta \omega \) is the input band width. Since \( \Delta \omega \) is small, it is permissible to pull the transfer function out of the integral, using the value of \( H(\omega, t) \) at \( \omega = \omega_0 \).

\[
\text{Var}[R] = |H(\omega_0, t)|^2 \int_0^\infty G_F(\omega) d\omega
\]

The area under \( G_F(\omega) \) is \( G_0 \Delta \omega \).

\[
\text{Var}[R] = |H(\omega_0, t)|^2 G_0 \Delta \omega
\]

|H(\omega, t)|^2 is simply Equation [IIA-2] for unit white noise input evaluated for one degree-of-freedom oscillator parameters. The result is

\[
|H(\omega, t)|^2 = \frac{1}{|Z(\omega)|^2} \left[ 1 + e^{-2\omega_0 \xi t} \left\{ 1 + 2\xi \sin(\omega_1 t) \cos(\omega_1 t) - \right. \\
- e^{\omega_0 \xi t} \left\{ 2 \cos(\omega_1 t) + \frac{2\omega_0 \xi}{\omega_1} \sin(\omega_1 t) \right\} \cos(\omega t) - \\
- e^{\omega_0 \xi t} \frac{2\omega}{\omega_1} \sin(\omega_1 t) \sin(\omega t) + \frac{(\omega_0 \xi)^2 - \omega_1^2 + \omega^2}{\omega_1^2} \sin^2(\omega_1 t) \right] \]

where \(|Z(\omega)|^2 = (\omega_0^2 - \omega^2) + (2\omega_0 \xi)^2\) (IIC-5)

Letting \( \omega_1 = \omega_0 \), neglecting small terms for lightly damped systems, and evaluating at \( \omega = \omega_0 \), leads to
\[ |H(\omega_0, t)|^2 = \frac{1}{4\omega_0^4 \xi^2} \left[ 1 + e^{-2\omega_0 \xi t} \left\{ 1-e^{-\omega_0 \xi t} \left[ 1 - 2\omega_0 \xi t \cos^2(\omega_0 t) - e^{\omega_0 \xi t} \sin^2(\omega_0 t) \right] \right\} \right] \]

\[ = \frac{1}{4\omega_0^4 \xi^2} \left[ 1 + e^{-2\omega_0 \xi t} \left\{ 1 - 2e^{-\omega_0 \xi t} \right\} \right] \]

\[ = \frac{1}{4\omega_0^4 \xi^2} \left[ 1 - e^{-\omega_0 \xi t} \right]^2 \quad \text{(IIC-6)} \]

With this substitution, the mean square response is given as

\[ \text{Var}[R] \approx \frac{G_o \Delta \omega}{4\omega_0^4 \xi^2} \left[ 1 - e^{-\omega_0 \xi t} \right]^2 \quad \text{(IIC-7)} \]

The equivalent time-dependent damping is defined as that time-dependent damping that would be necessary with steady-state response to yield the same mean square response as Equation [IIC-7]. Let \( \xi_{ne}(t) \) represent the effective time-dependent damping level for a narrow-band input. The above definition yields

\[ \text{Var}[R] = \frac{G_o \Delta \omega}{4\omega_0^4 \xi^2} \left[ 1 - e^{-\omega_0 \xi t} \right]^2 \approx \frac{G_o \Delta \omega}{4\omega_0^4 \xi_{ne}(t)^2} \]

\[ \frac{1}{\xi_{ne}(t)} = \frac{1}{\xi} \left[ 1 - e^{-\omega_0 \xi t} \right] \]

\[ \xi_{ne}(t) = \frac{1}{\xi} \left[ \frac{1}{1 - e^{-\omega_0 \xi t}} \right] \quad \text{(IIC-8)} \]

This relation is seen to reduce to two known boundary conditions for deterministic vibration problems. First, as \( t \to \infty \), \( \xi_{ne}(t) \to \xi \). That
Time-Dependent Damping: Narrow-Band Excitation

\[ \xi(t) \]

\[ \omega_0 t \]
is, the effective damping approaches the actual damping for long time. Second, as $\xi \to 0$, $e^{-\omega_0 \xi t} \to 1 - \omega_0 \xi t$, and

$$
\xi_{ne}(t) = \xi \frac{1}{1 - (1 - \omega_0 \xi t)} = \xi \frac{1}{\omega_0 \xi t} = \frac{1}{\omega_0 t} \quad (IIC-9)
$$

Recall that for $\xi = 0$ it is no longer appropriate to talk about a steady-state response [46]. From Equation [IIC-8], the RMS response is inversely proportional to $\xi_{ne}(t)$, and from Equation [IIC-9], the RMS response of an undamped oscillator is proportional to $\omega_0 t$.

Figure [IIC-1] shows $\xi_{ne}(t)$ as a function of time for two different damping levels.

The same oscillator is secondly considered to be subjected to a broad-band input. The input power spectrum is again denoted $G_F(\omega)$. The mean square response, from Caughey and Stumpf, is

$$
\text{Var}[R] = \int_{-\infty}^{\infty} G_F(\omega)|H(\omega, t)|^2 \, df \quad (IIC-10)
$$

The reasonable assumption is made that the power spectrum of the input is level as compared to the transfer function of a lightly damped one degree-of-freedom oscillator around its natural frequency. Then $G_F(\omega)$ may be evaluated at $\omega = \omega_0$.

$$
\text{Var}[R] = G_F(\omega_0) \int_{-\infty}^{\infty} |H(\omega_0, t)|^2 \, df \quad (IIC-11)
$$

The result, given by Caughey and Stumpf, is

$$
\text{Var}[R] = \frac{G_F(\omega_0)\pi}{4\xi \omega^3} \left[ 1 - e^{-2\omega_0 \xi t} \left( 1 + \frac{2\omega_0^2 \xi^2}{\omega_i^2} \sin^2(\omega_1 t) + \frac{\omega_0 \xi}{\omega_i} \sin(2\omega_1 t) \right) \right] \quad (IIC-12)
$$
which for small damping yields

\[ \text{Var}[R] \approx \frac{G_F(\omega_0)^\pi}{4\omega_0^3 \xi} \left[ 1 - e^{-2\omega_0 \xi t} \right] \]  

(IIC-13)

As with the narrow-band input, this response variance is set equal to the steady-state response for an effective time-dependent damping, designated \( \xi_{be}(t) \) for effective broad-band damping.

\[ \text{Var}[R] = \frac{G_F(\omega_0)^\pi}{4\omega_0^3 \xi} \left[ 1 - e^{-2\omega_0 \xi t} \right] \]

\[ \frac{1}{\xi_{be}(t)} = \frac{1}{\xi} \left[ 1 - e^{-2\omega_0 \xi t} \right] \]

\[ \xi_{be}(t) = \xi \left( \frac{1}{1 - e^{-2\omega_0 \xi t}} \right) \]  

(IIC-14)

As \( \xi \to 0, \xi_{be}(t) \to 1/(2\omega_0 t) \).

This result is different from the one found for narrow-band input. The effective broad-band damping approaches the true damping at a faster rate than does the effective narrow-band damping. Figure [IIC-2] shows \( \xi_{be}(t) \) as a function of time for two different damping levels.

For the broad-band input, it is seen from Equation [IIC-14] that the RMS response is inversely proportional to the square root of damping. Thus, for a forced undamped system subjected to broad-band excitation, the RMS response is proportional to the square root
Time-Dependent Damping: Broad-Band Excitation

\[ \xi_{be}(t) \]

\[ \omega_0 t \]

\[ \xi = 0.05 \]

\[ \xi = 0.01 \]
of $\omega_0 t$. The above results are summarized in Table [IIC-1].

For a lightly damped oscillator subjected to a narrow-band random input, the response may actually experience both types of growth during its evolution. As discussed in Section IIA, during the initial stages of growth, the transfer function of the response is flat. The narrow-band input will then effectively cover the peak region of the transfer function. The RMS response will grow in proportion to the growth of the transfer function peak, which is narrow band growth: the RMS response will initially grow proportional to time. As the transfer function continues to evolve, however, it becomes very peaked, and the power spectrum of the input will effectively cover all of the significant region of the transfer function. The RMS response will then grow in proportion to the growth of the area under the transfer function, which is broad-band growth: now the RMS response will grow proportional to the square root of time. These considerations are illustrated qualitatively in Figure [IIC-3].
<table>
<thead>
<tr>
<th>Excitation</th>
<th>Effective Damping</th>
<th>Effective Damping of an Undamped Oscillator</th>
<th>Relation of Steady State RMS Response to Damping</th>
<th>Growth of RMS Response of an Undamped Oscillator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Narrow-Band &quot;step&quot;</td>
<td>( \frac{\xi}{1-e^{-\omega_0 \xi t}} )</td>
<td>( \frac{1}{\omega_0 t} )</td>
<td>( \frac{1}{\xi} )</td>
<td></td>
</tr>
<tr>
<td>Broad-Band &quot;step&quot;</td>
<td>( \frac{\xi}{1-e^{-2\omega_0 \xi t}} )</td>
<td>( \frac{1}{2\omega_0 t} )</td>
<td>( \frac{1}{\sqrt{\xi}} )</td>
<td>( \sqrt{\omega_0 t} )</td>
</tr>
</tbody>
</table>

Effective Time-Dependent Damping
Figure IIC-3

RMS Response

proportional to time

transition region

proportional to √time

Figure IIC-3
ANALYTICAL DEVELOPMENT: TIME-DEPENDENT SPECTRUM

It will be the purpose of this chapter to consider the alternate mathematical definitions of a power spectrum for a nonstationary process. Section IIIA presents the various approaches, Section IIIB applies these to a physical example, and Section IIIC evaluates the relative merits of the different definitions.
Section IIIA  Alternate Definitions

For the stationary ergodic case, there is no ambiguity in the definition of \( G(\omega) \), the mean square spectral density. It can be interpreted as a frequency decomposition of an infinitely long sample function, and it can be derived as the Fourier transform of the autocorrelation function of the process. The transformation involves an integration over all values of time lag from \(-\infty\) to \(+\infty\). The spectral density is a non-negative function. It provides a frequency decomposition of the mean square value, \( \sigma \), of a process. A clear development of this is given by Davenport and Root [47].

For a nonstationary process, the statistical parameters change with time. In the stationary case the autocorrelation function turns out to be a function only of a time lag, \( \tau \), but in the nonstationary case it is a function of \( t_1 \) and \( t_2 \), or of \( t \) and \( \tau \). For a nonstationary process, \( R(t_1,t_2) \) may be defined in general as

\[
R(t_1,t_2) = E[x(t_1)x(t_2)] \tag{IIIA-1}
\]

Analogous to the stationary case, the lag \( \tau \) is defined as \( t_1 - t_2 \).

\[
\tau = t_1 - t_2 \tag{IIIA-2}
\]

\( t_1 \) and \( t_2 \) may now be defined in terms of a reference time \( t \).

\[
t_1 = t + a\tau \quad 0 < a < 1
\]
\[
t_2 = t -(1-a)\tau \quad 0 < a < 1 \tag{IIIA-3}
\]

Then \( R(t_1,t_2) \) may be written as

\[
R(t_1,t_2) = R[t + a\tau, t - (1-a)\tau] \quad 0 < a < 1 \tag{IIIA-4}
\]
Mark [48] has suggested using $a = 1/2$. In this case, the nonstationary autocorrelation function is an even time function of $\tau$. To simplify notation, $R[t+a\tau, t-(1-a)\tau]$ will be designated $R(t, \tau)$. Where important, it will be noted whether $R(t, \tau)$ is a forward ($a=1$), backward ($a=0$), or central ($a = 1/2$) autocorrelation function.

For the first approach to a nonstationary power spectrum, the spectrum may be defined by the following Wiener-Khintchine relation:

$$G_1(\omega, t) = \int_{-\infty}^{\infty} R(t, \tau)e^{-i\omega\tau} \, d\tau$$

$$R(t, \tau) = \int_{0}^{\infty} G_1(\omega, t)e^{i\omega\tau} \, df$$

where $f = \omega/2\pi$

Since the lag, $\tau$, takes on all values between $\pm \infty$, $G_1(\omega, t)$ is dependent on the value of the autocorrelation at all lags and therefore statistical characteristics of the process at all times. It is thus somewhat contradictory to label $G_1(\omega, t)$ as an instantaneous power spectrum. Kouskoulas, Hart, and Hurty [49] have been working in this area to define a meaningful evolutionary spectrum.

A second approach to a time-dependent spectral density definition is the proposed extension of the approach used by Caughey and Stumpf and Barnoski and Maurer [41] utilizing mean square response. This definition was given in Equation [IIA-2] for step input and the approach outlined in Section IC.
Recall by that definition, \( G(\omega,t) \) for step input is

\[
G_2(\omega,t) = G_F(\omega) \int_0^t \int_0^t h(t-u)h(t-v)\cos[\omega(u-v)]dudv \quad \text{(IIIA-6)}
\]

Consider the application of this second approach for the more general input of the form \( F(t) = A(t)X(t) \) where \( A(t) \) is a deterministic envelope function and \( X(t) \) is a stationary process. Then

\[
R_F(u,v) = E[F(u)F(v)] = A(u)A(v)R_X(u-v) \quad \text{(IIIA-7)}
\]

where

\[
R_X(u-v) = E[X(u)X(v)] \quad \text{(IIIA-8)}
\]

\( R_X(u-v) \) may be replaced by its Fourier transform

\[
R_X(u-v) = \int_0^\infty G_X(\omega)e^{i\omega t} df \quad \text{(IIIA-9)}
\]

This then produces the general result for the second definition of a time-dependent spectrum

\[
G_2(\omega,t) = G_X(\omega) \int_0^t \int_0^t h(t-u)h(t-v)A(u)A(v)\cos[\omega(u-v)]dudv \quad \text{(IIIA-10)}
\]

A third approach to define an evolutionary spectrum is based on a partial Fourier transform to the output covariance. Employing Equation [IC-1], the covariance of the output is given as

\[
R(t_1,t_2) = \text{cov}(t_1,t_2) = E[Y(t_1)Y(t_2)]
\]

\[
= \int_0^{t_1} \int_0^{t_2} h(t_1-u)h(t_2-v)R_F(u,v)dudv \quad \text{(IIIA-11)}
\]

\( R_F(u,v) \) may be replaced by Equation [IIIA-7] to yield
For this third approach it is convenient to follow the example set by Mark [48] and introduce the concept of a time-window function. It is desired to have a frequency spectrum that gives the relative importance of various frequencies in a motion at a particular time. The time-window concept limits consideration to statistical characteristics of the process in a limited time range at any particular instant. As this range decreases (smaller time-window length) the spectrum more truly becomes an "instantaneous" picture. The time-dependent spectrum is now defined as the truncated Fourier transformation of the autocorrelation function. The Fourier integral is over only a limited time region rather than over all time lags. The equation for \( G(\omega) \) then becomes

\[
G_3(\omega, t) = \int_{-T}^{T} R_y(t, \tau) e^{-i\omega \tau} d\tau
\]

(IIIA-13)

where \( 2T \) is the window length. The decreasing time-window also has a countering effect. The nature of a Fourier decomposition limits the validity of a spectrum to frequencies greater than \( 2\pi/2T \). This is clear by viewing the Fourier integral as a limiting process of the Fourier series. A series analysis is applied to a function that is assumed to be periodic over a period of time, \( 2T \). The lowest frequency in the Fourier series, excluding the constant term, has a frequency of \( 2\pi/2T \). Normally with a Fourier integral the time lag,
2T, \to \infty, and all frequencies are considered, with the function not necessarily being periodic in finite time.

It will be instructive at this point to compare the time-window Fourier transform for a stationary process for several window lengths, including the infinite length window. The autocorrelation function of the stationary response of an oscillator to zero-mean white noise input is [3]

$$R_y(\tau) = \frac{G_F}{4\xi\omega_0^3} e^{-\xi\omega_0|\tau|} \left[ \cos(\omega_1|\tau|) + \frac{\xi\omega_0}{\omega_1} \sin(\omega_1|\tau|) \right]$$

with \( \tau = \text{time lag} \)

- \( G_F = (\text{constant}) \text{ input power spectrum} \)
- \( \xi = \text{the oscillator fraction of critical damping} \)
- \( \omega_0 = \text{oscillator undamped natural circular frequency} \)
- \( \omega_1 = \text{oscillator damped natural circular frequency} \)

(IIIA-14)

The window output spectrum may be found from the partial Fourier transform

$$G(\omega) = \int_T^{T} R_y(\tau) e^{-i\omega \tau} d\tau$$  \hspace{1cm} (IIIA-15)

After some tedious algebra, the following result is obtained:

$$G(\omega) = \frac{G_F(\omega_0)}{\left(\omega_0^2 - \omega^2\right) + 4\xi^2\omega_0^2 \omega^2} \left[ 1 + e^{-\xi\omega_0 T} \left\{ \omega^4 \omega_0^4 - 2\xi^2\omega_0^2 \omega_0^2 \omega^2 \right\} \frac{\sin(\omega_1 T) \cos(\omega T)}{2\xi\omega_0^3 \omega_1} + \right.$$

$$\left. \left\{ 4\xi^2\omega_0^2 - \omega_0^2 + \omega^2 \right\} \frac{\omega \sin(\omega T) \cos(\omega_1 T)}{2\xi\omega_0^3} + \left\{ 4\omega_0^2 \xi^2 - 3\omega_0^2 + \omega^2 \right\} \frac{\omega \sin(\omega_1 T) \sin(\omega T)}{2\omega_0^2 \omega_1} \right]$$

$$\cos(\omega T) \cos(\omega_1 T) \right\} \right]$$

(IIIA-16)
This equation is plotted in Figures [III-1] to [III-4] for several different values of $T$ for both one per cent and ten per cent of critical damping. As $T \to \infty$, the spectrum becomes the usual stationary transfer function for a one degree-of-freedom oscillator (see Appendix A, Equation [A-8]). From the figures it is seen that the window spectrum experiences significant fluctuations and even takes on negative values. This latter aspect is especially disconcerting when attempting to interpret $G_3(\omega)$ as a power spectral density.

Table [III-1] presents results from numerical integration to obtain the moments of $G(\omega)$ defined by Equation [III-16]. The table indicates that except for relatively short window lengths ($\omega_\text{o} T < 2/\xi$) the first three moments and Vanmarcke's spectral density shape factor are quite close to their respective values for a complete Fourier transform ($T = \infty$). By Vanmarcke's theory, it is only through the first three moments that the power spectrum influences first passage considerations.

A fourth spectrum definition is also based on a truncated Fourier transform involving the autocorrelation function. The autocorrelation function of the output was given in Equation [III-11] as

$$R(t_1, t_2) = \int_0^t \int_0^t h(t_1-u)h(t_2-v) R_F(u,v) \, du \, dv \quad (\text{III-17})$$

Using Equation [III-7] and Equation [III-9], $R_F(u,v)$ may be replaced by

$$R_F(u,v) = A(u) A(v) \int_0^\infty G_X(\omega) e^{i\omega(u-v)} \, df \quad (\text{III-18})$$
Figure IIIA-1
$G(\omega)/G_f(\omega_0)$

$\xi = 0.01$

Figure IIIA-2
Figure IIIA-3
$G(\omega)/G_f(\omega_0)$

$T = \pm \omega_0 t$

$T = \pm 25 \omega_0 t$

$T = \pm 10 \omega_0 t$

$\xi = .1$

Figure IIIA-4
<table>
<thead>
<tr>
<th>damping</th>
<th>moment</th>
<th>time</th>
<th>window</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$5\omega_0 t$</td>
<td>$10\omega_0 t$</td>
</tr>
<tr>
<td>1%</td>
<td>$\lambda_0$</td>
<td>78.4</td>
<td>77.6</td>
</tr>
<tr>
<td>1%</td>
<td>$\lambda_1$</td>
<td>77.9</td>
<td>74.0</td>
</tr>
<tr>
<td>1%</td>
<td>$\lambda_2$</td>
<td>66.3</td>
<td>54.9</td>
</tr>
<tr>
<td>1%</td>
<td>$q$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10%</td>
<td>$\lambda_0$</td>
<td>7.9</td>
<td>7.8</td>
</tr>
<tr>
<td>10%</td>
<td>$\lambda_1$</td>
<td>7.4</td>
<td>7.2</td>
</tr>
<tr>
<td>10%</td>
<td>$\lambda_2$</td>
<td>7.0</td>
<td>6.6</td>
</tr>
<tr>
<td>10%</td>
<td>$q$</td>
<td>.07</td>
<td>.08</td>
</tr>
</tbody>
</table>
Then Equation [IIIA-17] becomes
\[ R(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} h(t_1-u)h(t_2-v)A(u)A(v) \int_0^{\infty} G_x(\omega) e^{i\omega(u-v)} \, df \, du \, dv \] (IIIA-19)

The following substitutions are made:
\[ t_1 = t + a \tau \quad 0 < a < 1 \]
\[ t_2 = t - (1-a) \tau \quad 0 < a < 1 \] (IIIA-20)

A change of variables is now introduced.
\[ s = u - a \tau \]
\[ r = v + (1-a) \tau \] (IIIA-21)

With this substitution, Equation [IIIA-19] becomes
\[ R(t_1, t_2) = \int_{-a \tau}^{t} \int_{(1-a) \tau}^{t} A(s+a \tau)A(r-(1-a) \tau)h(t-s)h(t-r) \int_0^{\infty} G_F(\omega) e^{i\omega(s-r+\tau)} \, df \, ds \, dr \] (IIIA-22)

Changing the order of integration (permissible since the integrals are convergent) yields
\[ R(t_1, t_2) = \int_0^{\infty} e^{i\omega(s-r+\tau)} G_F(\omega) \int_{-a \tau}^{t} \int_{(1-a) \tau}^{t} A(s+a \tau)A(r-(1-a) \tau)h(t-s)h(t-r) \, ds \, dr \, df \] (IIIA-23)

The exponential term may be factored to yield
\[ R(t_1, t_2) = \int_0^{\infty} e^{i\omega t} G_F(\omega) \int_{-a \tau}^{t} \int_{(1-a) \tau}^{t} A(s+a \tau)A(r-(1-a) \tau)h(t-s)h(t-r)e^{i\omega(s-r)} \, ds \, dr \, df \] (IIIA-24)
The third approach to an evolutionary spectrum defined a time-window spectrum from Equation [IIIA-13].

\[ G_3(\omega, t) = \int_0^T R_y(t, \tau) e^{-i\omega \tau} d\tau \]  

(IIIA-25)

This is the first Wiener-Khintchine relation for a time-window analysis of the problem. The fourth approach to an evolutionary spectrum is based on satisfying the second Wiener-Khintchine relation:

\[ R(t, \tau) = \int_0^{\infty} G_4(\omega, t) e^{i\omega \tau} d\omega \]  

(IIIA-26)

For a frequency window analysis consistent with the time-window, only frequencies greater than 1/2T should be used, where 2T is the window length in the time domain. The lower limit of zero will be left on the frequency integral with the understanding that \( G_4(\omega, t) \) is only valid for \( \omega > 2\pi/2T \). Equation [IIIA-24] is seen to satisfy the second Wiener-Khintchine relation (Equation [IIIA-5]) if \( G_4(\omega, t) \) is defined by

\[ G_4(\omega, t) = G_F(\omega) \int_{-aT}^t \int_{-(1-a)T}^t A(s+a\tau)A(r-(1-a)\tau)h(t-s)h(t-r)e^{i\omega(s-r)}dsdr \]  

(IIIA-27)

Note that this \( G_4(\omega, t) \) is, in general, different from that obtained from satisfying Equation [IIIA-25] unless the window length is infinite. As long as \( t \) is large and the window length is fairly small, time lags, \( \tau \), are small with respect to \( t \), and the lower limits on the integrals may be set equal to zero without significant error.
Then
\[ G_4(\omega, t) = G_F(\omega) \int_0^t \int_0^t A(s + aT)A(r - (1-a)T)h(t-s)h(t-r)e^{i\omega(s-r)}dsdr \]  \hspace{1cm} (IIIA-28)

Similarly, \( A(s+aT) \) and \( A(r-(1-a)T) \) may be approximated by \( A(s) \) and \( A(r) \) respectively if the statistical properties of the input are slowly varying or for regions more than \( T \) away from abrupt changes in \( A(t) \). Then
\[ G_4(\omega, t) = GF(w) \int_0^t \int_0^t A(s)A(r)h(t-s)h(t-r)e^{i\omega(s-r)}dsdr \]  \hspace{1cm} (IIIA-29)

For the Caughey and Stumpf case of step excitation, \( A(s) = A(r) = 1 \) for \( s \) and \( r > 0 \). Then
\[ G_4(\omega, t) = GF(\omega) \int_0^t \int_0^t h(t-s)h(t-r)e^{i\omega(s-r)}dsdr \]  \hspace{1cm} (IIIA-30)

Since the spectrum of a real process is real and even, \( e^{i\omega(s-r)} \) may be replaced by \( \cos[\omega(s-r)] \). Then
\[ G_4(\omega, t) = G_F(\omega) \int_0^t \int_0^t h(t-s)h(t-r)\cos[\omega(s-r)]dsdr \]  \hspace{1cm} (IIIA-31)

Comparison of Equation [IIIA-31] with Equation [IIIA-10] shows that the two definitions of a time-dependent spectrum are equivalent. Thus the spectrum based on Caughey and Stumpf is approximately the same, for a slowly varying input and time-window small with respect to \( t \), as the window spectrum determined by using Mark's concept of a time-window function and the second Wiener-Khintchine relation.

A fifth definition of a time-dependent spectrum follows from
Priestley [50] and Hammond [51]. For a stationary process, the change from the time domain to the frequency domain is made by a Fourier transform.

\[ x(t) = \int_{-\infty}^{\infty} e^{i\omega t} dX(\omega) \quad \text{(IIIA-32)} \]

where the \( dX \) are orthogonal random increments. For a nonstationary process, the Fourier transformation will not produce a result with orthogonal increments, and therefore cannot be interpreted as the energy decomposition over frequency. Priestley defines an "oscillatory" function, \( A(t,\omega) \), such that the orthogonal transformation may be made

\[ x(t) = \int_{-\infty}^{\infty} A(t,\omega) e^{i\omega t} dZ(\omega) \quad \text{(IIIA-33)} \]

where the \( dZ(\omega) \) are orthogonal increments. If \( A(t,\omega) \) varies slowly as a function of \( t \), then \( A(t,\omega) e^{i\omega t} \) retains the significance of amplitude modulated sine and cosine waves of frequency \( \omega \). The special type of evolutionary case considered by Caughey and Stumpf consists of stationary input (for all \( t > 0 \)) and nonstationary output. If \( x \) is defined as the input, and \( y \) the output, the equation of motion for a one degree-of-freedom oscillator is

\[ x(t) = y + 2\xi_0 \dot{y} + \omega_0^2 y \quad \text{(IIIA-34)} \]

where each dot above a variable represents one differentiation with respect to time. The transformation relations for \( x \) and \( y \) are

\[ x(t) = \int_{-\infty}^{\infty} A(t,\omega) e^{i\omega t} dX(\omega) \quad \text{(IIIA-35)} \]
Substitution of these transformations into the equation of motion yields
\[ \int_{-\infty}^{\infty} A(t,\omega) e^{i\omega t} d\omega = \int_{-\infty}^{\infty} \left\{ \hat{B}(t,\omega) + 2\hat{B}(t,\omega)[i\omega+\xi] \right\} e^{i\omega t} d\omega \]

The term in the wavy brackets is defined as \( D(t,\omega) \). Then
\[ \int_{-\infty}^{\infty} A(t,\omega) e^{i\omega t} d\omega = \int_{-\infty}^{\infty} D(t,\omega) e^{i\omega t} d\omega. \]

Equating the power in the two expressions at each frequency yields
\[ |A(t,\omega)|^2 |dX(\omega)|^2 = |D(t,\omega)|^2 |dY(\omega)|^2 \]

For the general case of nonstationary response to nonstationary excitation, Priestley assumes \( D(t,\omega) \) and \( A(t,\omega) \) vary with time in the same manner. This seems to restrict the solution to slowly varying input, but according to Priestley is taken only for mathematical convenience and does not have to be assumed. The power spectrum of the input is
\[ G_x(t,\omega) = |A(t,\omega)|^2 G_x(\omega) \]

and of the output,
\[ G_y(t,\omega) = |B(t,\omega)|^2 G_y(\omega) \]
respectively of the associated stationary processes \( A(t,\omega) = B(t,\omega) = 1 \). For the time-dependent response to step input, Hammond determines a solution from the requirement that \( G(t,\omega) \to 1 \) for long time. The right side of Equation [IIIA-37] contains a differential expression for \( B(t,\omega) \). This may be solved for \( B(t,\omega) \) by setting up a differential equation for \( B(t,\omega) \) that yields \( B(t,\omega) \to 1 \) as \( t \to \infty \). This equation is

\[
\ddot{B} + 2B(i\omega + \xi \omega_o) + B(\omega_o^2 - \omega^2 + 2i\xi \omega_o) = (\omega_o^2 - \omega^2 + 2i\xi \omega_o). \tag{IIIA-42}
\]

The solution of this equation yields

\[
|B(t,\omega)|^2 = \left\{ 1 + e^{-2\omega_o \xi t} \left[ 1 + \frac{2\xi \omega_o}{\omega_1} \sin(\omega_1 t) \cos(\omega_1 t) + \left( \frac{\xi^2 \omega_o^2 + \omega^2 - \omega_o^2}{\omega_1^2} \right) \sin^2(\omega_1 t) \right] - 2e^{-\omega_o \xi t} \frac{\xi \omega_o}{\omega_1} \cos(\omega t) \sin(\omega t)
- \frac{-\xi \omega_o t}{\omega_1} \cos(\omega t) \cos(\omega_1 t) - \frac{2\omega}{\omega_1} e^{-\omega \xi t} \sin(\omega t) \sin(\omega_1 t) \right\}. \tag{IIIA-43}
\]

The output power spectrum is

\[
G_S(t,\omega) = |B(t,\omega)|^2 G_Y(\omega) \tag{IIIA-44}
\]

The stationary spectra are related by

\[
G_Y(\omega) = |H(\omega)|^2 G_X(\omega) \tag{IIIA-45}
\]

Then
\[ G_s(t, \omega) = |B(t, \omega)|^2 |H(\omega)|^2 \quad \text{III A-46} \]

where \( H(\omega) \) is the stationary transfer function (Equation [A-8]).

The time-dependent power spectrum is then

\[
G_s(t, \omega) = \frac{G_X(\omega)}{(\omega_o^2 - \omega^2)^2 + 4\xi^2 \omega_o^2} \left[ 1 + e^{-2\xi \omega_o t} \left[ 1 + \frac{2\xi \omega_o}{\omega_1} \sin(\omega_1 t) \cos(\omega_1 t) + \left( \frac{\xi^2 \omega^2 + \omega_o^2 - \omega_1^2}{\omega_1^2} \right) \sin^2(\omega_1 t) - 2e^{-\xi \omega_o t} \frac{\xi \omega_o}{\omega_1} \cos(\omega t) \sin(\omega_1 t) - 2e^{-\xi \omega_o t} \cos(\omega t) \cos(\omega_1 t) - \frac{2\omega}{\omega_1} e^{-\xi \omega_o t} \sin(\omega t) \sin(\omega_1 t) \right] \right] \quad \text{III A-47}
\]

This definition of a time-dependent spectrum is exactly the same as \( G_2(\omega, t) \) (Equation [III A-6]), the spectrum derived from an extension of Caughey and Stumpf.
Several definitions of an evolutionary power spectrum, including some from the previous section, will be illustrated in this section for a sample case. The example chosen is a lightly damped one degree-of-freedom oscillator that has been experiencing stationary response to a zero-mean white noise input for all $t > -\infty$. At $t = t_0$, the forcing function is removed and the oscillator is allowed to experience free vibration for all $t > t_0$. This behavior is shown in Figure [IIIB-1]. Attention is focused on the behavior of the evolutionary power spectrum in the neighborhood around $t_0$.

Three different definitions of the evolutionary power spectrum are examined. First, one based on a purely intuitive idea of the spectrum as a representation of the instantaneous frequency content of the motion. Second, one based on Caughey and Stumpf's work. And third, one based on a time-window function to define a spectrum as a partial Fourier transform of the actual autocorrelation function of the motion.

**Intuitive Spectrum**

For the "intuitive spectrum" the motion is treated as if it existed with the same statistical parameters for all past and future time. For the example being considered in this section, two regions must be defined. For the first region, a stationary analysis of the random motion is performed, and the resulting spectrum is taken...
Oscillator Response

Figure IIIB-1
to apply for all \( t < t_0 \). The stationary autocorrelation function of the response to zero-mean white noise input is given by Crandall and Mark [4] as

\[
R_{yy}(\tau) = \frac{G_F(\omega_0)}{4\omega_0^2\xi} \ e^{-\xi\omega_0|\tau|} (\cos(\omega_1|\tau|) + \frac{\xi\omega_0}{\omega_1} \sin(\omega_1|\tau|)) \quad (IIB-1)
\]

where

- \( G_F \) = power spectrum of the (stationary) forcing function
- \( \xi \) = per cent of critical damping for the oscillator
- \( \omega_0 \) = natural undamped circular frequency of the oscillator
- \( \omega_1 \) = natural damped circular frequency of the oscillator
- \( \tau \) = time lag

The response power spectrum, obtained by a Fourier transformation, is also given there as

\[
G(\omega) = \frac{G_F(\omega_0)}{(\omega_0^2 - \omega^2)^2 + 4\xi^2\omega_0^2} \quad t \leq t_0 \quad (IIB-2)
\]

For the second region, \( t > t_0 \), the "intuitive spectrum" is a Dirac delta at \( \omega = \omega_1 \) multiplied by a decaying exponential. The equation is

\[
G(\omega,t) = \frac{G_F(\omega_0)\pi}{4\omega_0^2\xi} \ e^{-2\omega_0\xi(t-t_0)} \quad t \geq t_0 \quad (IIB-3)
\]

This equation represents pure sinusoidal motion, decaying with time. The factor of 2 in the exponential causes the RMS response (the square root of the area under the power spectrum) to decay as

\[
\exp \left[ -\omega_0\xi(t-t_0) \right], \text{ which is consistent with deterministic theory.}
\]
This intuitive result cannot be found from any sort of stationary analysis since the damping prevents the free vibration from being stationary. At \( t = t_0 \), the total power in the two spectra are equal.

**Caughey and Stumpf Spectrum**

For the Caughey and Stumpf type spectrum, Equation [IIIA-10], the region \( t < t_0 \) gives the same result as the intuitive spectrum, since their result is asymptotically stationary. For \( t > t_0 \), the spectrum is based on a further extension of the Caughey and Stumpf approach for time-limited step input. The theoretical development of this is given in Appendix D. The following result is a special case of the expression derived there:

\[
G(\omega, t) = \frac{G_F(\omega_0) e^{-2\omega_0 \xi (t-t_0)}}{4\omega_1^2 [\omega_0^2 - \omega^2]^2 + (2\xi \omega_0 \omega)^2} \left[ 2\omega_1^2 + 2\omega^2 + 2\xi^2 \omega_0^2 + \right.
\]

\[
\left. \left\{ \sin^2(\omega_1 t) - \cos^2(\omega_1 t) \right\} + \left\{ 4\xi \omega_0 \omega_1 \sin(2\omega_1 t_0) - 2(\omega_1^2 - \omega^2) \cos(2\omega_1 t_0) + 2\xi^2 \omega_0^2 \cos(2\omega_1 t_0) \right\} + \right.
\]

\[
\left. \left\{ \sin(2\omega_1 t) \right\} \right\} \left\{ 4\xi \omega_0 \omega_1 \cos(2\omega_1 t_0) + 2[(\omega_1^2 - \omega^2) - \right.
\]

\[
\left. \xi^2 \omega_0^2 \right] \sin(2\omega_1 t_0) \right\} \right] \quad (\text{IIIB-4})
Time-Window Spectrum

For the time-window spectrum, four regions must be handled separately. A window is defined to span from \( t - T \) to \( t + T \) for each value of time \( t \). The window length is then \( 2T \). For the first region, \( t \) and \( t + T \) are both less than \( t_0 \). This is shown in Figure [III-2(a)]. The autocorrelation function, \( R \), of the response in this region is then given by Equation [IIIB-1]. The spectrum is given by a truncated Fourier transform over the time-window.

\[
G_1(\omega) = \int_{-T}^{T} R_{yy}(\tau) e^{-i\omega \tau} \, d\tau \quad (\text{IIIB-5})
\]

This result is given in Equation [IIIA-16].

For the second region, \( t < t_0 \), but \( t + T > t_0 \). This is shown in Figure [III-2(b)]. In this region, two different expressions must be used for the autocorrelation function, depending on whether or not \( t + \tau \) is greater than or less than \( t_0 \). For \( t + \tau < t_0 \), the autocorrelation function is the same as that for the first region. At time \( = t_0 \), the response becomes free damped sinusoidal motion with initial conditions \( y_0 = y(t_0) \) and \( \dot{y}_0 = \dot{y}(t_0) \), where these values are a result of the forced response up to time \( = t_0 \). Therefore, the motion for time \( > t_0 \) is given as

\[
y(t) = e^{-\omega_0 t(t+t_0)} \left\{ y(t_0) \cos[\omega_1(t+t_0-t_0)] + \frac{\dot{y}(t_0)}{\omega_1} \sin[\omega_1(t+t_0-t_0)] \right\} \quad (\text{IIIB-6})
\]
Figure IIIB-2
The autocorrelation for times $t$ and $t + \tau$ when $t < t_0$ and $t + \tau > t_0$ is

$$R_{yy}(t, t+\tau) = E[y(t)y(t+\tau)]$$

$$= E\left[ \int e^{-\omega_0 \xi(t+\tau-t_0)} \left\{ y(t)y(t_0) \cos[\omega_1(t+\tau-t_0)] + \frac{y(t)y(t_0)}{\omega_1} \sin[\omega_1(t+\tau-t)] \right\} \right]$$

$$= \left\{ R_{yy}(t_0-t) \cos[\omega_1(t+\tau-t_0)] + R_{yy}(t_0-t) \frac{1}{\omega_1} \sin[\omega_1(t+\tau-t_0)] \right\} e^{-\omega_0 \xi(t+\tau-t_0)}$$

and

$$R_{yy}(t_0-t) = \frac{d}{dt} R_{yy}(t_0-t)$$

Then

$$R_{yy}(t, t+\tau) = e^{-\omega_0 \xi(t+\tau-t_0)} \left[ R_{yy}(t_0-t) \cos[\omega_1(t+\tau-t_0)] + \frac{d}{dt} R_{yy}(t_0-t) \frac{1}{\omega_1} \sin[\omega_1(t+\tau-t_0)] \right]$$

The window spectrum for this second region is again the result of taking the truncated Fourier transform of the autocorrelation function. The transform integral must now be divided into two integrals, each using the autocorrelation function appropriate to that zone. The first integral contains both positive and negative values of $\tau$ while the second integral contains only positive values.

$$G_2(\omega, t) = \int_{-t}^{t_0-t} R_{yy}(\tau) e^{-i\omega \tau} d\tau + \int_{t_0-t}^{T} R_{yy}(t, t+\tau) e^{-i\omega \tau} d\tau$$

(IIIB-10)
Because of the unsymmetric limits on the integrals $G_2(\omega,t)$ is no longer real. The resulting expression for $G_2(\omega,t)$ in the second region is

$$
G_2(\omega,t) = \frac{G_F(\omega_0)}{4\omega_0^3} \left\{ \text{sec}(\omega_1, \xi\omega_1, \omega_0, t_0 - t, 0) + \text{sec}(\omega_1, -\xi\omega_0, \omega_0, t_0 - t, 0) - \text{sec}(\omega_1, \omega_0, \omega_1, t_0 - t, 0) - \text{sec}(\omega_1, -\omega_0, \omega_1, t_0 - t, 0) \right\}
$$

$$
+ \frac{\xi\omega_0}{\omega_1} \left[ \text{sec}(\omega_1, \xi\omega_0, \omega_0, t_0 - t, 0) - \text{sec}(\omega_1, -\xi\omega_0, \omega_0, t_0 - t, 0) \right] + \left[ \cos[\omega_1(t_0 - t)] + \frac{\xi\omega_0}{\omega_1} \sin[\omega_1(t_0 - t)] \right] \left\{ \cos[\omega_1(t - t_0)] \right\}
$$

$$
+ \text{sec}(\omega_1, \omega_0, t_0 - t, 0) - \text{sec}(\omega_1, -\omega_0, \omega_1, t_0 - t, 0) \right\}
$$

$$
+ \left[ \cos[\omega_1(t_0 - t)] + \frac{\xi\omega_0}{\omega_1} \sin[\omega_1(t_0 - t)] \right] \left\{ \cos[\omega_1(t - t_0)] \right\}
$$

$$
+ \text{sec}(\omega_1, \omega_0, t_0 - t, 0) - \text{sec}(\omega_1, -\omega_0, \omega_1, t_0 - t, 0) \right\}
$$

$$
+ \left[ \cos[\omega_1(t_0 - t)] + \frac{\xi\omega_0}{\omega_1} \sin[\omega_1(t_0 - t)] \right] \left\{ \cos[\omega_1(t - t_0)] \right\}
$$

$$
+ \text{sec}(\omega_1, \omega_0, t_0 - t, 0) - \text{sec}(\omega_1, -\omega_0, \omega_1, t_0 - t, 0) \right\}
$$

$$
+ \left[ \cos[\omega_1(t_0 - t)] + \frac{\xi\omega_0}{\omega_1} \sin[\omega_1(t_0 - t)] \right] \left\{ \cos[\omega_1(t - t_0)] \right\}
$$

$$
+ \text{sec}(\omega_1, \omega_0, t_0 - t, 0) - \text{sec}(\omega_1, -\omega_0, \omega_1, t_0 - t, 0) \right\}
$$

$$
+ \left[ \cos[\omega_1(t_0 - t)] + \frac{\xi\omega_0}{\omega_1} \sin[\omega_1(t_0 - t)] \right] \left\{ \cos[\omega_1(t - t_0)] \right\}
$$

$$
+ \text{sec}(\omega_1, \omega_0, t_0 - t, 0) - \text{sec}(\omega_1, -\omega_0, \omega_1, t_0 - t, 0) \right\}
$$

$$
+ \left[ \cos[\omega_1(t_0 - t)] + \frac{\xi\omega_0}{\omega_1} \sin[\omega_1(t_0 - t)] \right] \left\{ \cos[\omega_1(t - t_0)] \right\}
$$

$$
+ \text{sec}(\omega_1, \omega_0, t_0 - t, 0) - \text{sec}(\omega_1, -\omega_0, \omega_1, t_0 - t, 0) \right\}
$$

$$
+ \left[ \cos[\omega_1(t_0 - t)] + \frac{\xi\omega_0}{\omega_1} \sin[\omega_1(t_0 - t)] \right] \left\{ \cos[\omega_1(t - t_0)] \right\}
$$

$$
+ \text{sec}(\omega_1, \omega_0, t_0 - t, 0) - \text{sec}(\omega_1, -\omega_0, \omega_1, t_0 - t, 0) \right\}
$$

$$
+ \left[ \cos[\omega_1(t_0 - t)] + \frac{\xi\omega_0}{\omega_1} \sin[\omega_1(t_0 - t)] \right] \left\{ \cos[\omega_1(t - t_0)] \right\}
$$

$$
+ \text{sec}(\omega_1, \omega_0, t_0 - t, 0) - \text{sec}(\omega_1, -\omega_0, \omega_1, t_0 - t, 0) \right\}
$$

$$
+ \left[ \cos[\omega_1(t_0 - t)] + \frac{\xi\omega_0}{\omega_1} \sin[\omega_1(t_0 - t)] \right] \left\{ \cos[\omega_1(t - t_0)] \right\}
$$

$$
+ \text{sec}(\omega_1, \omega_0, t_0 - t, 0) - \text{sec}(\omega_1, -\omega_0, \omega_1, t_0 - t, 0) \right\}
$$

$$
+ \left[ \cos[\omega_1(t_0 - t)] + \frac{\xi\omega_0}{\omega_1} \sin[\omega_1(t_0 - t)] \right] \left\{ \cos[\omega_1(t - t_0)] \right\}
$$

$$
+ \text{sec}(\omega_1, \omega_0, t_0 - t, 0) - \text{sec}(\omega_1, -\omega_0, \omega_1, t_0 - t, 0) \right\}
$$

$$
+ \left[ \cos[\omega_1(t_0 - t)] + \frac{\xi\omega_0}{\omega_1} \sin[\omega_1(t_0 - t)] \right] \left\{ \cos[\omega_1(t - t_0)] \right\}
$$

$$
+ \text{sec}(\omega_1, \omega_0, t_0 - t, 0) - \text{sec}(\omega_1, -\omega_0, \omega_1, t_0 - t, 0) \right\}
$$

$$
+ \left[ \cos[\omega_1(t_0 - t)] + \frac{\xi\omega_0}{\omega_1} \sin[\omega_1(t_0 - t)] \right] \left\{ \cos[\omega_1(t - t_0)] \right\}.
$$

\text{(IIIIB-11)}
where

\[ CEC (a, b, c, d, e) = \int_{d}^{e} e^{bx} \cos(ax)\cos(cx) \, dv \]

\[ = \frac{e^{bx}}{2[b^2 + (a-c)^2]} \left[ (a-c)\sin[(a-c)x] + b \cos[(a-c)x] \right] \]

\[ + \frac{e^{bx}}{2[b^2 + (a+c)^2]} \left[ (a+c)\sin[(a+c)x] + b \cos[(a+c)x] \right] \]

(IIIB-12)

\[ SEC (a, b, c, d, e) = \int_{d}^{e} e^{bx} \sin(ax)\cos(cx) \, dx \]

\[ = CES (c, b, a, d, e) \]

\[ = \frac{e^{bx}}{2[b^2 + (a-c)^2]} \left[ b \sin[(a-c)x] - (a-c)\cos[(a-c)x] \right] \]

\[ + \frac{e^{bx}}{2[b^2 + (a+c)^2]} \left[ b \sin[(a+c)x] - (a+c)\cos[(a+c)x] \right] \]

(IIIB-13)

\[ SES (a, b, c, d, e) = \int_{d}^{e} e^{bx} \sin(ax)\sin(cx) \, dx \]

\[ = \frac{e^{bx}}{2[b^2 + (a-c)^2]} \left[ (a-c)\sin[(a-c)x] + b \cos[(a-c)x] \right] \]

\[ - \frac{e^{bx}}{2[b^2 + (a+c)^2]} \left[ (a+c)\sin[(a+c)x] + b \cos[(a+c)x] \right] \]

(IIIB-14)
For the third region, $t > t_0$, but $t - T < t$. This is shown in Figure [IIIB-2(c)]. Again, two expressions for the autocorrelation must be used, depending on the value of $\tau$. The integral expression for $G_3(\omega,t)$ is again Equation [IIIB-10]. In this region, the second integral contains only negative values. The expression for $G_3(\omega,t)$ is

$$G_3(\omega,t) = \frac{G_F(\omega_0)}{4\xi_0^2} \left[ \cos^2[\omega_1(t-t_0)] \right] \left\{ \text{SEC}(\omega_1,\epsilon_0,\omega,-T,t_0-t) - \right.$$  

$$\left. \text{iCES}(\omega_1,\epsilon_0,\omega,-T,t_0-t) \right\} + \cos[\omega_1(t_0-t)] \sin[\omega_1(t_0-t)]$$

$$+ \cos[\omega_1(t-t_0)] \sin[\omega_1(t_0-t)] \left\{ \frac{\xi_0}{\omega_1} \left\{ \text{SEC}(\omega_1,\epsilon_0,\omega,-T,t_0-t) - \right. \right.$$  

$$\left. \text{iCES}(\omega_1,\epsilon_0,\omega,-T,t_0-t) \right\} - \cos^2[\omega_1(t-t_0)] \frac{\xi_0}{\omega_1} \right.$$  

$$\left. \left\{ \text{SEC}(\omega_1,\epsilon_0,\omega,-T,t_0-t) - \text{iSES}(\omega_1,\epsilon_0,\omega,-T,t_0-t) \right\} + \right.$$  

$$\left. \frac{1}{\omega_0} \sin[\omega_1(t-t_0)] \cos[\omega_1(t_0-t)] \left\{ \text{SEC}(\omega_1,\epsilon_0,\omega,-T,t_0-t) - \right. \right.$$  

$$\left. \text{iSES}(\omega_1,\epsilon_0,\omega,-T,t_0-t) \right\} - \frac{1}{\omega_0} \sin^2[\omega_1(t-t_0)]$$

$$- \left\{ \text{SEC}(\omega_1,\epsilon_0,\omega,-T,t_0-t) - \text{iSES}(\omega_1,\epsilon_0,\omega,-T,t_0-t) \right\} -$$  

$$\frac{\xi_0^2}{\omega_1} \sin^2[\omega_1(t-t_0)] \left\{ \text{SEC}(\omega_1,\epsilon_0,\omega,-T,t_0-t) - \right.$$  

$$\text{iCES}(\omega_1,\epsilon_0,\omega,-T,t_0-t) \right\} - \frac{\xi_0^2}{\omega_1} \sin[\omega_1(t-t_0)] \cos[\omega_1(t_0-t)]$$

$$\left\{ \text{SEC}(\omega_1,\epsilon_0,\omega,-T,t_0-t) - \text{iSES}(\omega_1,\epsilon_0,\omega,-T,t_0-t) \right\} -$$
\[
\frac{\omega_1}{\omega_o} \sin^2[\omega_1(t-t_0)] \left\{ \text{CEC}(\omega_1, \xi \omega_0, \omega, -T, t_0 - t) - \text{iCES}(\omega_1, \xi \omega_0, \omega, -T, t_0 - t) \right\} - \\
\frac{\omega_1}{\omega_o} \sin[\omega_1(t-t_0)] \cos[\omega_1(t_0 - t)] \left\{ \text{SEC}(\omega_1, \xi \omega_0, \omega, -T, t_0 - t) - \right. \\
\left. \text{iSES}(\omega_1, \xi \omega_0, \omega, -T, t_0 - t) \right\} - \xi \sin[\omega_1(t-t_0)] \cos[\omega_1(t_0 - t)] \\
\left\{ \text{CEC}(\omega_1, \xi \omega_0, \omega, -T, t_0 - t) - \text{iCES}(\omega_1, \xi \omega_0, \omega, -T, t_0 - t) \right\} \xi \sin^2[\omega_1(t-t_0)] \\
\left\{ \text{SEC}(\omega_1, \xi \omega_0, \omega, -T, t_0 - t) - \text{iSES}(\omega_1, \xi \omega_0, \omega, -T, t_0 - t) \right\} + e^{-2\omega_0 \xi t} \\
\left\{ \text{CEC}(\omega_1, -\omega_0 \xi, \omega, t_0 - t, T) + \text{iCES}(\omega_1, -\omega_0 \xi, \omega, t_0 - t, T) \right\} \\
\]  
(IIIB-15)

For the fourth region, \( t > t_0 \) and \( t - T > t_0 \), as shown in Figure [IIIB-2(d)]. In this case, the response of decaying sinusoidal nature. \( y(t) \) is then given as 

\[
y(t) = e^{-2\omega_0 \xi t} \left( y_0 \cos(\omega_1 t) + \frac{y_0}{\omega_1} \sin(\omega_1 t) \right) \\
\]  
(IIIB-16)

The autocorrelation function is 

\[
R_{yy}(t, t+\tau) = E[y(t)y(t+\tau)] \\
\]  
(IIIB-17)

and its truncated Fourier transform is 

\[
G_4(\omega, t) = e^{-2\omega_0 \xi t} \int_T^T R_{yy}(t, t+\tau) e^{-i\omega \tau} \, d\tau \\
= \frac{G_F(\omega_0)}{4\omega_0^3 \xi} e^{-2\omega_0 \xi t} \left[ \text{CEC}(\omega_1, -\omega_0 \xi, \omega, -T, T) + \right. \\
\left. \text{iCES}(\omega_1, -\omega_0 \xi, \omega, -T, T) \right] \\
\]  
(IIIB-18)
Note that \( G_n(\omega, t) \) is complex, because the damping destroys the symmetry of \( R(t, t+\tau) \) with respect to \( \tau \).

In order to compare the three definitions of a power spectrum, a specific set of parameters is evaluated. A window length of \( 2T = 20/\omega_0 \) is chosen, where \( \omega_0 \) represents the oscillator's undamped natural frequency of \( \omega_0 \) radians per second. The difference between damped and undamped natural frequency is ignored (\( \omega_0 = \omega_1 \)). The damping level chosen is \( \xi = 0.1 \) (ten per cent). The results are plotted in Figure [IIIB-3].

The interpretation of the spectrum as a power decomposition over frequency leads one to the conclusion that in the case of a complex spectrum, the magnitude of the total quantity (not just the real part) should be most consistent with physical intuition. Both the real and imaginary parts have been calculated for spectra that are complex. The imaginary parts proved to be small compared to the real parts. In Figure [IIIB-3] only the real parts are plotted.

The curves show significant differences among the three definitions. The intuitive approach differs from the others because of the instantaneous change to a delta function at \( t = t_0 \). The time-window spectrum differs from the intuitive spectrum in two areas. Firstly, the peak of the window spectrum is not as high. This was seen in Figures [IIIA-1] to [IIIA-4]. An increased damping level or longer time window would improve the agreement. Secondly, the tire-
Alternate Response Spectra for Sample Nonstationary Example

\[ \xi = 0.1 \quad T = \pm 10/\omega_0 \]

Diagram showing response spectra for different time windows and comparison between Caughey and Stumpf and Intuitive methods.
window spectrum seems somewhat erratic for times within the second and third regions, although it is all right at the ends of these regions. It is within these regions that the long algebraic expressions of Equation [IIIB-11] and Equation [IIIB-15] do not simplify. This latter effect may be due to an undiscovered error in these equations.
Section IIIC  Relative Advantages of Various Power Spectrum Definitions

Since there is no unique definition of the power spectrum for a nonstationary process, one must be aware that the use of any one approach may be in conflict with other equally reasonable analyses. It is the desire here to keep the results computationally simple and to maintain a clear physical interpretation.

The traditional approach of a double Fourier transform (see Crandall [3]) is mathematically exact. However, the algebra becomes complicated, and physical significance of the power spectrum is not clear since the spectrum is a function of two frequency variables. The double transform spectrum is found from the time-dependent autocorrelation function as

$$G(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(t_1, t_2) e^{-i\omega_1(t_1 + t_2)} dt_1 dt_2 \quad \text{(IIIC-1)}$$

The approach of Priestley and Hammond attempts to maintain the physical significance associated with a time-dependent spectrum. In the case of a process with slowly varying statistical parameters, the transform they use retains the concept of a decomposition of a sample function into sines and cosines. The loss of this concept in the general case is not necessarily a weakness of their approach. It may not be reasonable to think in terms of sines and cosines for a process with rapidly varying statistical characteristics. The mathematics invoked by Priestley and Hammond,
however, becomes very lengthy in all but the simplest cases, and the concept of a transformation with an "oscillatory" function is unfamiliar to most engineers. Clearly there is much potential in their investigations, and it is hoped that future investigators continue to develop applications of their theory.

The use of a window function to perform spectral decomposition in a limited region surrounding the time of interest (see Mark [48]) leads, after smoothing, to a spectrum with immediate physical significance. The procedure can never be considered "exact." A large time-window yields a more complete spectral analysis, while a shorter time-window allows less dependence on motion remote from the instant of interest, and the computed spectrum then changes less sluggishly to changes in the statistical parameters. Mark discusses the physical interpretation and measurement implications of the use of window functions.

The window approach seems to be especially well suited to oscillators subjected to excitation with fairly smooth continuous modulating functions.

Mark [48] calculates spectra based on the window approach (his "physical spectrum") and also by Fourier transform of the autocorrelation function (his "instantaneous spectrum"). For the latter he uses a central autocorrelation function. That is, the nonstationary autocorrelation function for a zero-mean process is
When a Fourier transformation is applied to this function, the resulting power spectrum is not positive definite, and it exhibits what appears to be spurious behavior. Following Cramer and Leadbetter [18], it is straightforward to show that the transform of an autocorrelation function should be non-negative. Let \( R(t_j, t_k) \) be the autocorrelation function of a zero-mean random process. Then

\[
R(t_j, t_k) = E[x(t_j)x(t_k)]
\]  

(IIIC-3)

where \( \overline{x(t_k)} \) is the complex conjugate of \( x(t_k) \), and is simply equal to \( x(t_k) \) for a real process. A transformation, such as a Fourier transformation, may be expressed as a summation over orthogonal increments, designated \( z_j \) and \( z_k \).

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} R(t_j, t_k) Z_j Z_k = E \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} E[x(t_j)x(t_k)] Z_j Z_k \right]
\]

(IIIC-4)

Because the increments are orthogonal, this becomes

\[
E \left[ \sum_{j=1}^{n} \left| x(t_j) Z_j \right|^2 \right]
\]

(IIIC-5)

which is non-negative. Cramer and Leadbetter note,"This [non-negative behavior] is ... a characteristic property of the class of all covariance functions." [18]. The above kind of transformation yields a double frequency spectrum. It is desirable, however, to
have the same characteristic for a time-frequency spectrum in order to retain physical interpretation as a mean square spectral density.

The problem in deriving a spectrum from a central autocorrelation function can be seen in a different way by recalling that

\[ R(t_1, t_2) = \mathbb{E}[x(t_1)x(t_2)] \] (IIIC-6)

A central autocorrelation, with \( t = \frac{1}{2}(t_1 + t_2) \) and \( \tau = t_1 - t_2 \) is then

\[ R(t, \tau) = \mathbb{E}[x(t - \frac{\tau}{2})x(t + \frac{\tau}{2})] \] (IIIC-7)

Clearly, the two times \( t - \frac{\tau}{2} \) and \( t + \frac{\tau}{2} \) are not independent in a transformation with respect to the variable \( \tau \). A Fourier transform with respect to \( \tau \) of a central autocorrelation function does not then result in orthogonal increments. The physical significance of the transformation as a power spectrum is then questionable.

Bendat and Piersol also note that \( R(t, \tau) \) "is not a conventional correlation function." [52]. They caution that the Fourier transform of \( R(t, \tau) \) gives an instantaneous spectrum, "which should not be confused with the physically meaningful time-varying power spectrum." [52]

Despite these theoretical problems, Mark [48] shows that the instantaneous spectrum achieved by transforming the central autocorrelation is very nearly equivalent to the spectrum \( G_3(\omega, t) \) achieved by using the time-window function (Equation [IIIA-13]) when the latter is appropriately smoothed.
The physical spectrum \( G_w(\omega,t) \) found by using a time-window function so that the Fourier transform of \( G(\omega,t) \) yields the auto-correlation function seems physically reasonable. It was shown earlier that for slowly varying input and a small time-window, this spectrum is approximately the same as the spectrum obtained by the extension of the Caughey and Stumpf principle (Equation [IIIA-10]). Furthermore, the calculations involved in determining the latter involve only simple algebra, although somewhat tedious. The resulting expressions, when specialized to small damping levels, are fairly compact.

For these reasons, it has been decided to use the spectrum based on Caughey and Stumpf for all first passage computations in this work. The behavior of this spectrum when an input ceases should be noted. After an excitation is removed, an oscillator experiences free vibration. The Caughey and Stumpf approach produces a continuous spectrum that decays as \( \exp[-2\omega \xi t] \). As was seen in the previous section, the intuitive spectrum is a delta function at \( \omega = \omega_1 \) decaying at the same rate as the Caughey and Stumpf spectrum.

It is clear that if the process is a decaying sinusoid, it will either fail in the first half cycle or not at all, since nothing is random about the process once the amplitude and phase at \( t_0 \) are known. Thus, if an oscillator is excited by a random force that suddenly ceases, the oscillator, given that it has not failed
up to the time of input cessation, will either first cross a given barrier in the next half cycle or not at all. Whether or not this crossing occurs depends on the magnitude of the oscillator displacement and velocity when the input stops. If these are independent Gaussian variables, the energy in the process is Rayleigh distributed. Barrier exceedance in the next half cycle is a function only of this energy. Neglecting damping, the probability of exceedance is then Rayleigh, but conditioned on the fact that the energy has not been sufficient to cause a crossing of the barrier during the history of the motion.

The probability of no barrier exceedance associated with the intuitive spectrum will thus be constant after the next half cycle of motion, while that associated with the Caughey and Stumpf-based spectrum will level off exponentially. For lighter damped systems, the levelling off will be slower. These results are shown qualitatively in Figure [IIIC-1]. For random vibrations analysis in earthquake engineering this discrepancy is less serious because the excitation does not stop abruptly, but rather decays smoothly.
Figure IIIC-1
This chapter will consider the time-dependent moments of the evolutionary power spectrum defined in Equation \([IVA-3]\) and motivated by the work of Caughey and Stumpf. This is the power spectrum of the response to a suddenly applied, quasi-stationary random input; this will be called a step input here.

The stationary moments were defined in Equations \([IB-11]\), \([IB-12]\), and \([IB-13]\) in terms of the stationary power spectrum. Their importance in first passage computations was also developed in that section.

The analysis by Caughey and Stumpf resulted in an expression for the RMS response of a linear oscillator subjected to an approximately white noise step input. Their expression was given in Equation \([IC-6]\) as

\[
\Var{y(t)} = \int_{0}^{\infty} G_F(\omega) \int_{0}^{t} \int_{0}^{t} h(t-\tau)h(t'-\tau')\cos(\omega(\tau-\tau'))d\tau d\tau' \quad (IVA-1)
\]

For a zero-mean process, the variance is equal to the mean square response, which is the area under the power spectral density curve. The variance is, therefore, the "zero\(^{th}\) moment" of the power spectrum, \(\lambda_0\). The derivation of \(G(\omega,t)\) was summarized in Section IIIA.
G(\omega,t) was defined there as

\[ G(\omega,t) = G_F(\omega) \int_0^t \int_0^t h(t-\tau)h(t-\tau')\cos(\omega(\tau-\tau'))d\tau d\tau' \]  

(IVA-2)

For a one degree-of-freedom oscillator, this expression becomes

\[ G(\omega,t) = \frac{G_F(\omega)}{\left(\omega_0^2-\omega^2\right)^2 + (2\omega_0 \xi)^2} \left[ 1 + e^{-2\omega_0 \xi t} \left\{ 1 + \frac{2\omega \xi}{\omega_1} \sin(\omega_1 t)\cos(\omega_1 t) \right. \right. \]

\[ - e^{-\omega_0 \xi t} \left( 2\cos(\omega_1 t) + \frac{2\omega_0 \xi}{\omega_1} \sin(\omega_1 t) \right) \cos(\omega t) - e^{-\omega_0 \xi t} \frac{2\omega}{\omega_1} \]

\[ \frac{(\omega_0 \xi)^2 - \omega_1^2 + \omega^2}{\omega_1^2} \sin(\omega_1 t)\sin(\omega t) + \frac{\xi^2}{\omega_1^2} \sin^2(\omega_1 t) \} \]  

(IVA-3)

where \( \omega_0 \) = oscillator natural undamped frequency

\( \omega_1 \) = oscillator natural damped frequency

\( \xi \) = oscillator fraction of critical damping

The time-dependent zeroth moment of the power spectrum, designated \( \lambda_0(t) \), is obtained by performing the following integration:

\[ \lambda_0(t) = \int_0^\infty G(\omega,t) \, df \]  

(IVA-4)

The integration has been carried out by Caughey and Stumpf for white noise input using contour integration. This method is permissible because the integrand meets two requirements. The first is that the integrand is even. This is necessary since contour integration yields the result for limits of \(-\infty\) and \(+\infty\). For an even integrand, the integral from 0 to \(\infty\) is one-half the integral from \(-\infty\) to \(+\infty\),
but for an odd integrand, the integral from $-\infty$ to $+\infty$ is zero. The second requirement is that the integration is convergent over the limits $-\infty$ and $+\infty$.

The result is

$$\lambda_0(t) = \frac{G_F(\omega_0)}{8\xi \omega_0^3} \left[ 1 - e^{-2\omega_0 \xi t} \left\{ 1 + \frac{2\xi^2 \omega_0^2}{\omega_1^2} \sin^2(\omega_1 t) + \frac{\omega_0^2 \xi}{\omega_1^2} \sin(2\omega_1 t) \right\} \right]$$

(IVA-5)

This result, which is exact for white noise input, may also be used when the power spectrum of the input, $G_F(\omega)$, is slowly varying in the region $\omega = \omega_0$ as compared to the transfer function. Note that for true white noise input, $G_F(\omega) = G_f$, independent of $\omega$.

The second time-dependent moment, $\lambda_2(t)$, is defined as

$$\lambda_2(t) = \int_0^\infty \omega^2 G(\omega, t) \, df$$

(IVA-6)

In expanded form, this becomes

$$\lambda_2(t) = \int_0^\infty \frac{G_F(\omega)}{\left( \omega_0^2 - \omega^2 \right) + (2\omega_0 \xi)^2} \left[ \omega^2 + e^{-2\omega_0 \xi t} \left\{ \omega^2 + \frac{2\omega_0 \omega^2 \xi}{\omega_1^2} \right\} \right] df$$

(IVA-7)

An interesting problem now arises. Previously, this second moment
has been calculated for steady-state conditions only. This moment can be found from the above expression by setting $t = \infty$, and then carrying out the indicated integration. In this case, no convergence problems arise. However, if it is attempted to carry out the integration first for arbitrary $t$, the integration does not converge. The cause is certain terms in $G(\omega,t)$ that do not decay with $\omega$ faster than $|1/\omega|$. These terms decay exponentially with time, $t$. This lack of convergence is not observed in simulation studies because the input spectral density, $G_f(\omega)$, goes to zero for large $\omega$. True white noise input (constant for $0 < \omega < \infty$) would require infinite power, something that cannot be produced in the laboratory or in real situations.

In order to avoid this lack of convergence, two approaches can be taken. The first is to carry out the integration with an upper limit of $\omega = \omega_u (f = f_u = 2\pi \omega_u)$. Then, in the resulting expression, one can look at the effect of various values of $\omega_u$. The integration for an upper limit, other than $\infty$, however, is not straightforward. The second approach, and the one adopted here, is to separate out the terms that do not converge and perform two separate integrations.

\[
\lambda_2(t) = \int_0^\infty (\text{convergent terms})df + \int_0^{f_u} (\text{divergent terms})df
\]  
(IVA-8)

The result can be evaluated for various values of $\omega_u$. The value for any finite $\omega_u$ by this second approach is not exactly the same.
as that of the first approach, because the convergent terms have been integrated up to \( \omega_u = \infty \) here.

The integrand for \( \lambda_2(t) \), Equation [IVA-7], indicates that the contribution to \( \lambda_2(t) \) from frequencies much larger than \( \omega_0 \) is not significant, except for the last term. This last term is

\[
G_F(\omega_0) \int_0^\infty \frac{1}{(\omega_0^2 - \omega^2)^2 + (2\omega_0\xi)^2} \frac{\omega^4}{\omega_1^2} \sin^2(\omega_1 t) \, df \quad [IVA-9]
\]

The simplest approach is to split this term into two terms. This is accomplished as follows:

\[
\frac{G_f(\omega_0)\omega^4\sin^2(\omega_1 t)e^{-2\omega_0 \xi t}}{\omega_1^2[(\omega_0^2 - \omega^2)^2 + (2\omega_0\xi)^2]} = \frac{\omega^4}{\omega_0^4 - 2\omega_0^2\omega^2 + \omega^4 + 4\omega_0^2\omega_0^2\xi^2} \quad \frac{G_f(\omega_0)\sin^2(\omega_1 t)e^{-2\omega_0 \xi t}}{\omega_1^2}
\]

\[
= \frac{\omega^4 - (2\omega_0^2 - 4\omega_0^2\xi^2)\omega^2 + \omega^4}{\omega_0^4 - (2\omega_0^2 - 4\omega_0^2\xi^2)\omega^2 + \omega^4} \quad \frac{G_f(\omega_0)\sin^2(\omega_1 t)e^{-2\omega_0 \xi t}}{\omega_1^2}
\]

\[
= \frac{[\omega^4 - (2\omega_0^2 - 4\omega_0^2\xi^2)\omega^2 + \omega^4]}{\omega_0^4 - (2\omega_0^2 - 4\omega_0^2\xi^2)\omega^2 + \omega^4} + \frac{(2\omega_0^2 - 4\omega_0^2\xi^2)\omega^2 - \omega^4}{\omega_0^4 - (2\omega_0^2 - 4\omega_0^2\xi^2)\omega^2 + \omega^4} \frac{G_f(\omega_0)\sin^2(\omega_1 t)e^{-2\omega_0 \xi t}}{\omega_1^2}
\]
Thus the two terms are

\[
G_f(\omega_0)\omega_0^4\sin^2(\omega_1 t)e^{-2\omega_0 \xi t} \cdot \frac{\omega_1^2}{\omega_1^2[(\omega_0^2-\omega^2)^2+(2\omega_0 \xi)^2]} = G_f(\omega_0)\sin^2(\omega_1 t)e^{-2\omega_0 \xi t} \cdot \frac{\omega_1^2}{\omega_1^2}
\]

\[
+ \frac{(2\omega_0^2-4\omega_0 \xi^2)^2}{\omega_0^4-(2\omega_0^2-4\omega_0 \xi^2)^2+\omega_0^4} \cdot G_f(\omega_0)\sin^2(\omega_1 t)e^{-2\omega_0 \xi t} \cdot \frac{\omega_1^2}{\omega_1^2}
\]

(IVA-11)

The second term can be included in the convergent integral. \( \lambda_2(t) \) is then the sum of a convergent integral and a term which diverges for pure white noise input. All terms in the convergent integral are even, so the method of contour integration may be used. The result of this integration for the convergent part of \( \lambda_2(t) \) is

\[
\lambda_2(t) = \frac{G_f(\omega_0)}{1-e^{-2\omega_0 \xi t}} \left[ 1 - \frac{2\omega_0 \xi^2}{\omega_1^2} \sin^2(\omega_1 t) + \frac{\omega_0 \xi}{\omega_1} \sin(2\omega_1 t) \right]
\]

convergent

(IVA-12)

where, as for \( \lambda_0(t) \), \( G_f(\omega_0) = G_f \) if the input is true white noise.

The rather surprising result is that except for the sign of the \( \xi^2 \) term, \( \lambda_2(t) \) is simply \( \omega_0^2 \lambda_0(t) \). It is interesting to note that in the case of deterministic sinusoidal input, the stationary output consists simply of motion at \( \omega=\omega_0 \), and then \( \lambda_2(t)=\omega_0^2 \lambda_0(t) \).
The divergent last term is independent of \( \omega \), so its integral is simply

\[
\lambda_2(t) = \int_0^\infty \frac{G_f(\omega)\sin^2(\omega_1 t)e^{\omega_0 \xi t}}{\omega_1^2} \, df
\]

\[
= \frac{G_f(\omega)\sin^2(\omega_1 t)e^{\omega_0 \xi t}}{\omega_1^2} \frac{\omega_u}{2\pi}
\]  

(IVA-13)

This term grows linearly with \( \omega_u \), the upper cut-off frequency. For band-limited input that does not have a flat power spectrum, this becomes

\[
\lambda_2(t) = \frac{\sin^2(\omega_1 t)}{\omega_1^2} e^{-2\omega_0 \xi t} \int_0^\infty G_F(\omega) \, df
\]

\[
= \frac{\sin^2(\omega_1 t)}{\omega_1^2} e^{-2\omega_0 \xi t} \lambda_{of}
\]

(IVA-14)

where \( \lambda_{of} = \) zero th moment of power spectrum of forcing process

\[= \) area under the input power spectrum.

The total expression for \( \lambda_2(t) \) for broad-band input is then

\[
\lambda_2(t) = \frac{G_f(\omega_0)}{8\xi \omega_0} \left[ 1 - e^{-2\omega_0 \xi t} \left\{ 1 - \frac{2\omega_0^2 \xi^2}{\omega_1^2} \sin^2(\omega_1 t) + \frac{\xi \omega_0}{\omega_1} \sin(2\omega_1 t) \right\} \right] + \frac{\sin^2(\omega_1 t)}{\omega_1^2} e^{-2\omega_0 \xi t} \lambda_{of}
\]

(IVA-15)

For small damping (\( \xi < 10\% \) of critical) this moment becomes

\[
\lambda_2(t) \approx \omega_0^2 \lambda_0 + \frac{\sin^2(\omega_1 t)}{\omega_1^2} e^{-2\omega_0 \xi t} \lambda_{of}
\]

(IVA-16)
The first moment, $\lambda_1(t)$, is defined as

$$\lambda_1(t) = \int_0^\infty \omega G(\omega, t) \, df$$

$$= \int_0^\infty \frac{\omega G_f(\omega)}{(\omega^2 - \omega_0^2)^2 + (2\omega_0 \xi)^2} \left[ 1 + e^{-2\omega_0 \xi t} \left\{ 1 + \frac{2\omega_0 \xi}{\omega_1} \sin(\omega_1 t) \cos(\omega_1 t) \right\} \right]$$

$$- e^{-\omega_0 \xi t} \left\{ 2\cos(\omega_1 t) + \frac{2\omega_0 \xi}{\omega_1} \sin(\omega_1 t) \right\} \cos(\omega t) - e^{-\omega_0 \xi t} \frac{2\omega_0}{\omega_1}$$

$$\sin(\omega_1 t) \sin(\omega t) + \frac{(\omega_0 \xi)^2 - \omega_1^2}{\omega_1^2} \sin^2(\omega_1 t) \right\} \right\}$$

As with $\lambda_2(t)$, there is a convergence problem with the last term in $\lambda_1(t)$. This last term is

$$\int_0^\infty G_f(\omega) \frac{1}{(\omega^2 - \omega_0^2)^2 + (2\omega_0 \xi)^2} \frac{\omega^3}{\omega_1^2} \sin^2(\omega_1 t) \, df \quad \text{(IVA-18)}$$

This may be expanded to

$$\int_0^{\infty} \frac{\sin^2(\omega_1 t)}{\omega_1^2} e^{-\omega_0 \xi t} \frac{G_f(\omega)}{\omega} \, df +$$

$$\int_0^{\infty} \frac{(2\omega_0^2 - 4\omega_0^2 \xi^2)\omega^2 - \omega_0^4}{\omega^2 - (2\omega_0^2 - 4\omega_0^2 \xi^2)\omega^2 + \omega_0^4} G_f(\omega) \sin^2(\omega_1 t) \frac{\omega^2}{\omega_1^2} e^{-2\omega_0 \xi t} \, df \quad \text{IVA-19)}$$

In addition to the problem with divergence as $f_u \to \infty$, both integrals in Equation [IVA-19] diverge for the lower limit of $\omega = 0$. 
The two divergent parts cancel, however, and the sum is convergent with respect to this lower limit. For convenience, the lower limit of \( \omega = \omega_0 \) will be used in Equation [IVA-19]. In this case, the only divergence in \( \lambda_1(t) \) is with respect to the upper limit of \( \omega \). The divergent part, for (band-limited) white noise input, may be written as

\[
\lambda_1(t) = \frac{G_f(\omega_0) \sin^2(\omega_1 t)}{\omega_1^2} \cdot e^{-2\omega_0 \xi t} \int_{f_0}^{f_u} \frac{df}{f}
\]

This term grows logarithmically with the upper cutoff frequency.

Unfortunately, integration of the convergent parts of \( \lambda_1(t) \) is not straightforward. Contour integration may not be used because the integrand is odd. No exact, closed-form expression could be found for the general \( \lambda_1(t) \) for the case of evolutionary response to step input. Several special cases can be solved, however, and these, along with numerical integration of the general case, lead to a satisfactory approximate result.

Consider first the special case of zero damping. The expression for \( \lambda_1(t) \) becomes

\[
\lambda_1(t) = \int_{0}^{f_u} w \cdot \frac{G_f(\omega_0)}{(\omega_0^2 - \omega^2)^2} \left[ 2 - 2\cos(\omega_0 t)\cos(\omega t) - \frac{2\omega}{\omega_0} \sin(\omega_0 t)\sin(\omega t) + \frac{\omega^2 - \omega_0^2}{\omega_0^2} \sin^2(\omega_0 t) \right] df
\]

(IVA-21)
The solution for white noise, after some tedious algebra, is

\[
\lambda_1(t) = \frac{G_f(\omega_0)}{8\omega_0^2} \left[ 2\omega_0 t - \sin(2\omega_0 t) - \frac{4\cos(\omega_0 t)}{\pi} + \frac{4\cos(\omega_0 t)}{\pi} \right. \\
+ \left. \frac{4\cos(\omega_0 t)}{\pi} \right] \sin(\omega_0 t) + \frac{4\sin(\omega_0 t)}{\pi} \frac{\sin^2(\omega_0 t)}{2\pi\omega_0} \int_0^{\omega_0 t} \frac{\omega G_f(\omega) d\omega}{(\omega - \omega_0^2)} 
\]

where \( Si(\omega_0 t) \) is the sine-integral and \( Ci(\omega_0 t) \) is the cosine integral,

\[
Si(\omega_0 t) = \int_0^{\omega_0 t} \frac{\sin(u)}{u} \, du - \frac{\pi}{2} \\
Ci(\omega_0 t) = -\int_{\omega_0 t}^{\infty} \frac{\cos(u)}{u} \, du
\]

and both are widely tabulated [53]. The outline of the derivation of Equation [IVA-22] is given in Appendix B. The sine-integral and cosine-integral are plotted in Figure [IVA-1] for reference.

The last term in \( \lambda_1(t) \) for zero damping for true white noise must be handled carefully since the integration passes a singularity in the function. The correct expression is

\[
\frac{\sin^2(\omega_0 t)}{2\pi\omega_0^2} \int_0^{\omega_0 t} \frac{\omega G_f(\omega_0) d\omega}{\omega^2 - \omega_0^2} = \frac{\sin^2(\omega_0 t)}{4\pi\omega_0^2} \ln \left( \frac{\omega^2}{\omega_0^2} - 1 \right)
\]

For response after the first cycle (\( \omega_0 t > 2\pi \)), little error is intro-
duced by setting \( \sin(\omega_0 t) = \cos(\omega_0 t) = 0 \).

\[
\lambda_1(t) = \frac{G_f(\omega_0)}{8\omega_0^2} \left[ 2\omega_0 t - \frac{4}{\pi} \right] + \frac{\sin^2(\omega_0 t)}{4\pi\omega_0^2} G_f(\omega_0) \ln \left( \frac{\omega_0^2}{\omega_0^2} - 1 \right)
\]

(IVA-24)

Consider now critical damping. The expression for \( \lambda_0(t) \) becomes

\[
\lambda_1(t) = \int_{0}^{\infty} \frac{\omega G_f(\omega)}{(\omega_0^2 + \omega^2)} \left[ 1 + e^{-2\omega_0 t} + 2\omega_0 t e^{-2\omega_0 t} + 
-2(1+\omega_0 t)\cos(\omega t)e^{-\omega_0 t} - 2\omega t \sin(\omega t)e^{-\omega_0 t}
+ (\omega_0^2 + \omega^2) t^2 e^{-2\omega_0 t} \right] df
\]

(IVA-25)

The solution for white noise, after more tedious algebra (see Appendix B) is

\[
\lambda_1(t) = \frac{G_f(\omega_0)}{4\pi\omega_0^2} \left[ \omega_0^2 + e^{-2\omega_0 t} \left( 1 + 2\omega_0 t + \omega_0^2 \right) \right] - \omega_0 t
+ 2\omega_0 t e^{-2\omega_0 t} E_i(\omega_0 t) + 2\omega_0 t^2 e^{-2\omega_0 t} \int_{0}^{\omega_0} \frac{e^{\omega^2} \text{d}\omega}{(\omega^2 + \omega_0^2)}
\]

(IVA-26)

where \( E_i(\omega_0 t) \) is the exponential-integral,

\[
E_i(\omega_0 t) = \int_{-\infty}^{\omega_0 t} \frac{e^u}{u} \text{d}u
\]

and is widely tabulated \[53\]. \( E_i(\omega_0 t) \) is shown in Figure [IVA-2].

Consider a third special case. The steady state solution for \( \lambda_1 \) has been used by Crandall, Chandiramani, and Cook \[54\], and by Van-
Figure IVA-2
markke [7] in stationary frequency analysis. It can be derived from Equations [IV-A3] and [IVA-4] by setting $t = \infty$. Then

$$\lambda_1 = \int_0^\infty \frac{G_f(\omega_0) \omega \, df}{(\omega_0^2 - \omega^2)^2 + (2\omega_0 \xi)^2} \quad \text{(IVA-27)}$$

The solution is

$$\lambda_1 = \frac{G_f(\omega_0)}{8\pi \omega_0^2 \xi \omega_1} \left( \frac{\pi}{2} - \tan^{-1} \left( \frac{2\omega_0 \xi^2 - \omega_0}{2\xi \omega_1} \right) \right) \quad \text{(IVA-28)}$$

$\xi \leq 1$

For small damping, this becomes

$$\lambda_1 = \frac{G_f(\omega_0)}{8\pi \omega_0^2 \xi} \left( \frac{\pi}{2} - \left[ - \frac{\pi}{2} + 2\xi \right] \right) = \frac{G_f(\omega_0)}{8\pi \omega_0^2 \xi} (\pi - 2\xi) \quad \text{(IVA-29)}$$

Based on these special cases and insight gained from looking at the effect of various terms in the general expression, the following approximate expression for $\lambda_1(t)$ for approximate white noise input is assumed:

$$\lambda_1(t) = \frac{G_f(\omega_0)}{8\xi \omega_0^2} \left[ \frac{\omega_0}{2\omega_1} - \frac{\omega_0}{\omega_1} \tan^{-1} \left( \frac{2\omega_0 \xi^2 - \omega_0}{2\xi \omega_1} \right) \right] \left( 1 - e^{-\omega_0 \xi t} \right) +$$

$$- \frac{G_f(\omega_0)}{2\pi \omega_0^2} \left[ 1 + \omega_0 \xi t \right] e^{-\omega_0 \xi t} +$$
For small damping levels ($\xi < 10\%$) and times such that $\omega_o t > \pi$, the above expression simplifies to

$$
\lambda_1(t) = \frac{G_f(\omega_o)}{8\pi\xi\omega_o} \left[ 1 - \frac{2\xi}{\pi} \right] \left[ 1 - e^{-2\omega_o\xi t} \right] - \frac{G_f(\omega_o)}{2\pi\omega_o^2} e^{-\omega_o\xi t} + \frac{G_f(\omega_o)\sin^2(\omega_1 t)}{2\pi\omega_1^2} \frac{e^{-2\omega_o\xi t}}{\ln \left( \frac{\omega_o}{\omega_0} \right)}
$$

In order to verify this assumed small damping expression, it was evaluated for three different damping levels and compared with results from numerical integration of the integrand indicated in Equation [IVA-17]. To facilitate the comparison the divergent term was not included since the divergent term is exact in Equation [IVA-31]. These results are given in Figures [IVA-3] to [IVA-6]. The agreement between the numerical integration and the approximate integral improves as damping decreases and time increases. Equation [IVA-31] is exact for $\xi = 0$. Equation [IVA-31] neglects the oscillatory terms, which are significant for small $t$ and damping greater than a few per cent. From Figures [IVA-3] to [IVA-6] it is
\[ \frac{\lambda_1(\omega_0 t)}{G_f(\omega_0)}/2\pi \]

Figure IVA-5

- **Numerical Integration**
- **Equation [IVA-31], \( \omega_u = \omega_0 \)**
\[
\frac{\lambda_1(\omega_0 t)}{G_f(\omega_0) / 2\pi}
\]

Figure IVA-6

- Numerical integration
- Equation [IVA-31], \( \omega_y = \omega_0 \)
concluded that the convergent part of the integral for \( \lambda_1(t) \) may be approximated by Equation [IVA-31] for damping not greater than 10% of critical and times such that \( \omega_0 t > \pi \).

Sample values of \( \lambda_0(t) \), \( \lambda_1(t) \), and \( \lambda_2(t) \) are plotted in Figures [IVA-7] to [IVA-10]. It is observed that for all damping levels, the moments initially grow linearly with time. For long time they eventually approach an asymptotic level for all damping levels greater than zero. The nature of the growth differs slightly for each moment. This difference in growth is reflected in the growth of Vanmarcke's spectral density shape factor, as will be seen in the next section.

The first and second moments contain oscillatory terms whose magnitude is related to the upper cutoff frequency for band-limited input. The oscillatory terms diverge for infinite band-width. Therefore an engineer cannot specify true white noise input, but must give the upper band limit. Chapter VII will show that first passage probabilities are not too sensitive to the cutoff frequency.
Figure IMA-7

\[ \frac{\lambda(\omega_0 t)}{G_f(\omega_0)}/2\pi \]
\[
\frac{\lambda_1(\omega_0 t)}{G_f(\omega_0)}/2\pi
\]

Figure IVA-8
\frac{\lambda_2(\omega_0 t)}{\omega_0^2} / G_f(\omega_0) / 2\pi
Section IVB  Time-Dependent Spectral Shape Factor

Since the first three time-dependent moments of the power spectrum are now known, the time-dependent Vanmarcke spectral density shape factor, \( q(t) \), may be determined by analogy to the stationary definition (Equation [IB-25]).

\[
q(t) = 1 - \sqrt{\frac{\lambda_1^2(t)}{\lambda_0(t)\lambda_2(t)}} \quad (IVB-1)
\]

with \( \lambda_0(t) \) given in Equation [IVA-5], \( \lambda_1(t) \) given in Equation [IVA-30], and \( \lambda_2(t) \) given in Equation [IVA-15].

For the case where damping is small and where the "white noise" input is band limited to frequencies not more than four or five times the natural frequency of the system (i.e. \( \omega_0 < 5\omega_0 \)), higher order terms in \( \xi \) may be neglected, and the divergent last terms in \( \lambda_1(t) \) and \( \lambda_2(t) \) may be neglected. In this case, the moments are simply

\[
\lambda_0(t) = \frac{G_f(\omega_0)}{8\xi_0\omega_0^2} \left[ 1 - e^{-2\omega_0\xi t} \right]
\]
\[
\lambda_1(t) = \frac{G_f(\omega_0)}{8\xi_0\omega_0^2} \left[ 1 - e^{-2\omega_0\xi t} \right] - \frac{G_f(\omega_0)}{4\pi\omega_0^2} \left[ 1 + e^{-2\omega_0\xi t} \right]
\]
\[
\lambda_2(t) = \frac{G_f(\omega_0)}{8\xi_0\omega_0^2} \left[ 1 - e^{-2\omega_0\xi t} \right] \quad (IVB-2)
\]

Then
The shape factor is independent of $G(\omega)$.  

For steady state ($t = \infty$) this reduces to

$$q = \frac{2}{\sqrt{\pi}} \sqrt{1 - \left(1 - \frac{\xi}{\pi}\right)^2}$$

The last radical is approximately one since small damping is being
considered, so the steady state, small damping $q$ may be approximated simply as

$$q = \frac{2}{\sqrt{\pi}} \sqrt{\xi} \quad \text{(IVB-5)}$$

which agrees with Vanmarcke's Equation [I-5.18] with $k = \sqrt{\frac{2\pi}{\xi}}$.

Equation [IVB-3] is plotted in Figure [IVB-1] for $\xi = 0.001, 0.01$, and 0.1. It is observed that $q(t)$ starts near $q(t) = 1$ for $\omega_0 t = 1$. Then $q$ decreases monotonically. The value of $q(t)$ is independent of $\xi$ until $q(t)$ has almost reached its stationary value. In words, this means the shape of the evolving spectrum to step input is independent of the damping level until the spectrum has almost reached stationarity. It therefore does not seem appropriate to use the stationary value of $q$ and an effective duration of a process unless the duration, $s$, is long enough such that $\omega_0 s \gg 1/\xi$. 
Figure IVB-1
CHAPTER V

RESPONSE TO SEPARABLE INPUT

In order to analyze first passage probabilities for a more general class of forcing functions, in this chapter the first three time-dependent moments of the output power spectral density will be determined. This will be done by finding the time-dependent variance in a manner parallel to that used by Caughey and Stumpf for the white noise, quasi-stationary (or step) input. This result will be used to determine the time-dependent output spectra, \( G(\omega, t) \). Finally the moments \( \lambda_0(t), \lambda_1(t), \) and \( \lambda_2(t) \) of this function will be determined by integration.

In general, the autocorrelation of the input depends not only on the lag, \( u-v \), but on both \( u \) and \( v \) separately. Then

\[
R_X(u,v) = \mathbb{E}[X(u)X(v)]
\]

Equation [IC-4] for the response variance becomes

\[
\text{Var}(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t-u) h(t-v) R_X(u,v) \, du \, dv
\]

A major assumption will now be introduced. In order to simplify the analyses, the input will be assumed to be of the separable stationary type. That is:

\[
X(t) = A(t)F(t)
\]

where \( F(t) \) is a stationary zero-mean random process and \( A(t) \) is a
deterministic modulating function. This is a common model for earthquake motions [35]. The autocorrelation then becomes

\[ R_X(u,v) = E[A(u)F(u)A(v)F(v)] \]

\[ R_X(u,v) = A(u)A(v)E[F(u)F(v)] \]

\[ = A(u)A(v) R_f(u-v) \] \hspace{1cm} (V-4)

For \( u = v \), \( \text{Var}[X] = A^2(t)\text{Var}[f] \), so \( A(t) \) is equal to the nonstationary intensity of the input.

The expression for the response variance (Equation [IC-4]) then becomes

\[ \text{Var}(Y) = \int_0^\infty \int_0^\infty h(t-u)h(t-v)A(u)A(v)R_f(u-v)du \, dv \] \hspace{1cm} (V-5)

\( R_f(u-v) \) may be replaced by its Fourier transform,

\[ R_f(u-v) = \int_0^\infty G_f(\omega) \, e^{i\omega(u-v)} \, df \] \hspace{1cm} (V-6)

Then

\[ \text{Var}(Y) = \int_0^\infty \int_0^\infty h(t-u)h(t-v)A(u)A(v) \int_0^\infty G_f(\omega) e^{i\omega(u-v)} df \, du \, dv \] \hspace{1cm} (V-7)

Interchanging the order of integration yields

\[ \text{Var}(Y) = \int_0^\infty G_f(\omega) \int_0^\infty h(t-u)h(t-v)A(u)A(v) e^{i\omega(u-v)} du \, dv \, df \] \hspace{1cm} (V-8)

For a one degree-of-freedom oscillator, this becomes
\[ \text{Var}(Y) = \int_{0}^{\infty} G_f(\omega) \int_{0}^{\infty} A(u)A(v) \frac{\sin(\omega_1(t-u))}{\omega_1} e^{-\omega_0 \xi(t-u)} \]

\[ \times \frac{\sin(\omega_1(t-v))}{\omega_1} e^{-\omega_0 \xi(t-v)} e^{i\omega(u-v)} du \, dv \, df \]

\[ = \int_{0}^{\infty} G_f(\omega) \frac{1}{\omega_1^2} e^{-2\omega_0 \xi t} \int_{0}^{\infty} \int_{0}^{\infty} \sin(\omega_1(t-u))\sin(\omega_1(t-v)) \]

\[ e^{i\omega(u-v)} A(u) e^{\omega_0 \xi u} A(v) e^{\omega_0 \xi v} \]

\[ du \, dv \, df \]  

(V-9)

with \( df = \frac{d\omega}{2\pi} \)

An alternate derivation of this result is given in Appendix C through the use of superposition of step functions.

For the Caughey and Stumpf type case (i.e., evolutionary response to step input), \( A(t) = 1, \ t > 0 \). For white noise input the result was given in Equation [IC-8].

As discussed in Chapter III, the frequency integrand in Equation [V-9] is the definition of the time-dependent response power spectrum being used in this report:

\[ G(\omega,t) = \frac{G_f(\omega)e^{-2\omega_0 \xi t}}{\omega_1^2} \int_{0}^{\infty} \int_{0}^{\infty} \sin(\omega_1(t-u))\sin(\omega_1(t-v)) \]

\[ e^{i\omega(u-v)} e^{\omega_0 \xi u} e^{\omega_0 \xi v} A(u)A(v) du \, dv \]  

(V-10)

For the transient response to step input, \( G(\omega,t) \) was given in Equation [IVA-3]. The following portion of that result is de-
fined to be \( I(\omega_0 \xi) \):

\[
I(\omega_0 \xi) = \int_0^t \int_0^t \sin(\omega_1(t-u))\sin(\omega_1(t-v))e^{i\omega(u-v)}e^{\omega_0 \xi u}e^{\omega_0 \xi v} du \, dv \\
= \frac{1}{(\omega_0^2-\omega^2)^2+(2\omega \omega_0 \xi)} \left[ \omega_1^2 e^{\omega_0 \xi t} + \omega_1^2 \left\{ 1+ \frac{2\omega_0 \xi}{\omega_1} \sin(\omega_1 t)\cos(\omega_1 t) + \\
e^{\omega_0 \xi t} \left[ 2\cos(\omega_1 t)+\frac{2\omega_0 \xi}{\omega_1} \sin(\omega_1 t) \right] \cos(\omega t) - e^{\omega_0 \xi t} \frac{2\omega}{\omega_1} \sin(\omega_1 t) \\
\sin(\omega t) + \frac{(\omega_0 \xi)^2-\omega_1^2+\omega^2}{\omega_1^2} \sin^2(\omega_1 t) \right\} \right] \\
\text{ (V-11)}
\]

In terms of this function, \( G(\omega, t) \) for the step input can be written

\[
G(\omega, t) = \frac{G_f(\omega)e^{-2\omega_0 \xi t}}{\omega_1^2} \quad I(\omega_0 \xi)
\]

If, on the other hand, \( A(t) \) is of exponential form, this result can be used by replacing the \( \omega_0 \xi \) terms in \( I(\omega_0 \xi) \) by an effective damping. Let \( A(t) \) be of the form \( C_1 + C_2 e^{C_4 t} + C_3 e^{-C_5 t} \). By choosing the constants properly this form can be used to approximate many increasing-decreasing non-stationary input intensities versus time. An example will be evaluated in Chapter VII. Then by inspection of Equation [V-10]

\[
G(\omega, t) \approx \frac{G_f(\omega)e^{-2\omega_0 \xi t}}{\omega_1^2} \left[ C_1 I(\omega_0 \xi) + C_2 I(\omega_0 \xi+C_4) + C_3 I(\omega_0 \xi-C_5) \right] \\
\text{ (V-12)}
\]

This result is approximate and should be used only for \(|\omega_0 \xi+C_4|\) and \(|\omega_0 \xi-C_5| \) less than 0.5.
The time-dependent variance for this case is

$$\lambda_0(t) = \int_0^\infty G(\omega, t)df = \int_0^\infty G_f(\omega)e^{-2\omega_0\xi t} \left[ c_1 I(\omega_0 \xi) + 
\right.$$  
$$+ c_2 I(\omega_0 \xi + c_4) + c_3 I(\omega_0 \xi - c_5) \right] df \quad (V-13)$$

For the transient response to a step input the moments, $\lambda_i$, were defined in Chapter IV. It is convenient to use these results in defining moments of $I(\omega_0 \xi)$.

$$J_i(\omega_0 \xi) = \int_0^\infty \omega^i I(\omega_0 \xi)df \quad (V-14)$$

In particular, using Equation [IIA-3], one may write

$$J_0(\omega_0 \xi) = \int_0^\infty I(\omega_0 \xi)df = \frac{1}{8\omega_0^3 \xi} \left[ \frac{\omega_0^2}{\omega_1^2 \xi} \right] e^{2\omega_0 \xi t} +$$  
$$- \left\{ \omega_1^2 + \omega_0 \omega_1 \xi \sin(2\omega_1 t) + 2\omega_0^2 \xi^2 \sin^2(\omega_1 t) \right\} \right] \quad (V-15)$$

For small damping levels ($\xi < 10\%$) the following approximation for $J_0(\omega_0 \xi)$ may be used:

$$J_0(\omega_0 \xi) \approx \frac{\omega_0^2}{8\omega_0^3 \xi} \left[ e^{2\omega_0 \xi t} - 1 \right] \quad (V-16)$$

For $\xi < 10\%$ and white noise (or broad-band) input with $A(t) = c_1 + c_4 t - c_5 t$,

$$c_2 e^{-c_2 t} + c_3 e^{-c_3 t},$$
\[ \lambda_0(t) = \frac{G_f(\omega_0)}{\omega^2} e^{-2\omega_0 \xi t} \left[ C_1 J_0(\omega_0 \xi) + C_2 J_0(\omega_0 \xi + C_4) + C_3 J_0(\omega_0 \xi - C_5) \right] \]

(V-17)

The first moment, \( \lambda_1 \), may be found similarly. For the transient response to step input, \( \lambda_1 \), was given in Equation [IVA-30]. From that equation, one finds

\[ J_1(\omega_0 \xi) = \int_0^\infty \omega I(\omega_0 \xi) \, df \]

\[ = \frac{\omega^2 e^{\omega_0 \xi t}}{8 \omega_0^2} \left[ \frac{\omega_0}{2\omega_1} - \frac{\omega_0}{\omega_1 \pi} \tan^{-1} \left( \frac{2\omega_0 \xi^2 - \omega_0}{2\xi \omega_1} \right) \right] \left[ 1 - e^{-2\omega_0 \xi t} \right] - \]

\[ \frac{\omega^2 e^{\omega_0 \xi t}}{2\pi \omega_0^2} \left[ 1 + \omega_0 \xi t \right] e^{-\omega_0 \xi t} + \frac{\xi \omega_0^2 e^{\omega_0 \xi t}}{4\omega_0^2} \left[ \frac{\omega_0}{2\omega_1} - \right. \]

\[ \frac{\omega_0}{\omega_1 \pi} \tan^{-1} \left( \frac{2\omega_0 \xi^2 - \omega_0}{2\xi \omega_1} \right) \left[ 1 - e^{-2\omega_0 \xi t} \right] - \frac{\omega_0}{\omega_1 \pi} \tan^{-1} \left( \frac{2\omega_0 \xi^2 - \omega_0}{2\xi \omega_1} \right) \left[ 1 - e^{-2\omega_0 \xi t} \right] - \]

\[ \text{Ei}(\omega_0 \xi t) + \frac{\sin^2(\omega_1 t)}{2\pi} \ln \left( \frac{\omega_0}{\omega_0} \right) \]  

(V-18)

For small damping levels, \( J_1(\omega_0 \xi) \) is given approximately as

\[ J_1(\omega_0 \xi) \approx \frac{\omega^2 e^{\omega_0 \xi t}}{8 \omega_0^2} \left[ 1 - \frac{2\xi}{\pi} \right] \left[ 1 - e^{-2\omega_0 \xi t} \right] - \frac{\omega^2}{2\pi \omega_0^2} \omega_0 \xi t + \]

\[ \frac{\sin^2(\omega_1 t)}{2\pi} \ln \left( \frac{\omega_0}{\omega_0} \right) \]  

(V-19)

For \( \xi < 10\% \) and white noise (or broad-band) input with \( A(t) = C_1 + C_4 t - C_5 t \),

\[ C_2 e^{-\omega_0 \xi t} + C_3 e^{-\omega_0 \xi t} , \]
\[
\lambda_1(t) = \frac{G_f(\omega_0)}{\omega_1^2} e^{-2\omega_0\xi t} \left[ C_1 J_1(\omega_0\xi) + C_2 J_1(\omega_0\xi+C_4) + C_3 J_1(\omega_0\xi-C_5) \right]
\]  
(V-20)

Similarly, \( \lambda_2 \) may be found in terms of the evolutionary results to step input, Equation [IVA-6]. From that,

\[
J_2(\omega_0\xi) = \int_0^\infty \omega^2 I(\omega_0\xi) \, df
\]

\[
= \frac{1}{8\omega_0^5} \left[ \omega_0^2 e^{2\omega_0\xi t} - \omega_1^2 \left[ 1 - \frac{2\omega_0^2\xi^2}{\omega_1^2} \sin^2(\omega_1 t) \right. \right. \\
+ \frac{\xi\omega_0}{\omega_1} \sin(2\omega_1 t) \left. \right] + \omega_0 \sin^2(\omega_1 t) 
\]

For small damping levels,

\[
J_2(\omega_0\xi) \approx \omega_0^2 J_0(\omega_0\xi) + \sin^2(\omega_1 t) \omega_u 
\]  
(V-21)

Again for \( \xi < 10\% \) and white noise (or broad-band) input with \( A(t) = C_4 t + C_5 t \),

\[
\lambda_2(t) = \frac{G_f(\omega_0)}{\omega_1^2} e^{-2\omega_0\xi t} \left[ C_1 J_2(\omega_0\xi) + C_2 J_2(\omega_0\xi+C_4) + C_3 J_2(\omega_0\xi-C_5) \right]
\]  
(V-23)

For other than broad-band input one must integrate the product of \( G(\omega,t) \) in Equation [V-12] and \( \omega^i \) to obtain \( \lambda_1(t) \).

Using the same approach, the time-dependent moments for the case of a time limited step input are given in Appendix D. This type
of excitation is zero for \( t < t_0 \) and \( t > t_1 \), and "stationary" for \( t_0 \leq t \leq t_1 \). The results obtained there for the response moments following the cessation of the motion reduce to very compact expressions for small damping. As discussed in Chapter III, however, the time-dependent spectrum adopted in this report loses physical significance for the case of suddenly removed input, so those results must be used with caution.

The approximate small damping results for the convergent parts of the spectral moments for the evolutionary response to a broad-band step input, a broad-band time-limited step input, a separable broad-band exponential input, and a general, separable nonstationary input are summarized in Table [V-1].
<table>
<thead>
<tr>
<th>Excitation</th>
<th>( \lambda_0(t) ) (small damping)</th>
<th>( \lambda_1(t) ) (small damping convergent part)</th>
<th>( \lambda_2(t) ) (small damping convergent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step Excitation</td>
<td>( G_f(\omega_0) \left( 1 - e^{-\omega_0 t} \right) )</td>
<td>( \omega_0 \left( 1 - \frac{2\xi}{\pi} \right) \lambda_0(t) - \frac{G_f(\omega_0)}{2\omega_0^2} e^{-\omega_0 t} )</td>
<td>( \omega_0^2 \lambda_0(t) )</td>
</tr>
<tr>
<td>Time-Limited Step Excitation</td>
<td>( t &lt; t_1 ) same as step excitation</td>
<td>( t &lt; t_1 ) same as step excitation</td>
<td>( t &gt; t_1 ) same as step excitation</td>
</tr>
<tr>
<td></td>
<td>( t &gt; t_1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( G_f(\omega_0) \left( e^{-\omega_0 t_1} - e^{-\omega_0 t_2} \right) e^{-\omega_0 t} )</td>
<td>( \omega_0 \left( 1 - \frac{2\xi}{\pi} \right) \lambda_0(t) - \frac{G_f(\omega_0)}{2\omega_0^2} e^{-\omega_0(t_2 + t_1 - 2t)} )</td>
<td>( \omega_0^2 \lambda_0(t) )</td>
</tr>
<tr>
<td>Separable Broad-Band Exponential</td>
<td>Equation [V-17]</td>
<td>Equation [V-20]</td>
<td>Equation [V-23]</td>
</tr>
<tr>
<td>Separable Broad-Band in General</td>
<td>( \int_0^\infty G_f(\omega) \int_0^\infty h(t-u) h(t-v) A(u) A(v) \cos[\omega(u-v)] , du , dv ) with ( \omega = 2\pi f )</td>
<td>( \int_0^\infty G_f(\omega) \int_0^\infty h(t-u) h(t-v) A(u) A(v) \cos[\omega(u-v)] , du , dv ) with ( \omega = 2\pi f )</td>
<td>( \omega_0^2 \lambda_0(t) )</td>
</tr>
</tbody>
</table>
CHAPTER VI

FIRST PASSAGE FORMULAS

Section VIA  Hazard Rate and Reliability

With the analytical results obtained in Chapters IV and V for time-dependent spectral density moments, the expressions for first passage probabilities from Chapter I may be evaluated for the non-stationary response of one degree-of-freedom oscillators excited by Gaussian input. The reliability function for nonstationary motion was given in Equation [IB-28] as

\[ L_D(t) = A_D e^{-\int_0^t \alpha_D(u) du} \]

where \( \alpha_D(u) \) may be interpreted as the time-dependent hazard function, and \( A_D \) is the probability of no instantaneous barrier exceedance (i.e., the value of \( L_D(t) \) at \( t = 0 \)).

\( L_D(t) \) is the probability that in the time interval 0 to \( t \), the random process will not exceed a fixed barrier. For an evolutionary response to step input (response starts from rest), \( A_D = 1 \). For lightly damped processes, Vanmarcke's expression for \( \alpha_D(t) \) (Equation [IB-26]) is

\[ \alpha_D(t) = 2\nu_0 \frac{1 - \exp \left\{ -aq(t) \sqrt{\frac{\pi}{2\nu_0(t)}} \right\}}{\exp \left\{ \frac{a^2}{2\nu_0(t)} \right\} - 1} \]  

\( q(t) \) was given in Equation [IVB-1] as
\[ q(t) = \sqrt{1 - \frac{\lambda_1^2(t)}{\lambda_0(t)\lambda_2(t)}} \]  
(VIA-3)

where \( \nu_0(t) \) is \( \frac{1}{2\pi} \sqrt{\lambda_2(t)/\lambda_0(t)} \) and \( a \) is the barrier level.

Knowing \( \lambda_0(t) \), \( \lambda_1(t) \) and \( \lambda_2(t) \), the time-dependent hazard function may be evaluated and integrated to give the reliability of a single degree-of-freedom oscillator. In general the integration must apparently be done numerically.

A closed form solution for this integration for the transient response to step input has been found for medium to high barrier levels applied to lightly damped oscillators in cases where the spectral density of the input does not have significant values for frequencies greater than, say, about ten times the natural frequency of the oscillator being excited. Under these conditions, the convergent terms of the small damping expressions for the spectral moments may be used. These are

\[ \lambda_0 = \frac{G_f(\omega_0)}{8\omega_0^3\xi} \left[ 1 - e^{-2\omega_0\xi t} \right] \]

\[ \lambda_1 = \frac{G_f(\omega_0)}{8\omega_0^4\xi} \left[ 1 - e^{-2\omega_0\xi t} \right] - \frac{G_f(\omega_0)}{4\pi\omega_0^2} \left[ 1 + e^{-2\omega_0\xi t} \right] \]

\[ \lambda_2 = \frac{G_f(\omega_0)}{8\omega_0^4\xi} \left[ 1 - e^{-2\omega_0\xi t} \right] \]  
(VIA-4)

Substitution of these moments yields approximate results for \( \nu_0(t) \), \( q(t) \), and finally \( \omega_0(t) \).
The resulting approximate expression for $\int_0^t \alpha_D(u)\,du$ is

$$
\int_0^t \alpha_D(u)\,du = \frac{\nu_0}{\omega_0} \left[ E_1(z_1) - E_1(z_2) + e^{-\alpha_1} E_1(z_3) - e^{-\alpha_2} E_1(z_4) \right]
$$

(VIA-5)

where $E_1(z)$ is one of the widely tabulated exponential integrals, (and is different from the exponential integral $E_i(x)$ discussed in Chapter IV.)

$$
E_1(z) = \int_\frac{z}{u}^\infty \frac{e^{-u}}{u} \, du
$$

The derivation of Equation [VI-5] is given in Appendix E. A graph of $E_1(z)$ is shown in Figure [VIA-1].

The parameters in Equation [VIA-5] are

$$
\alpha_1 = \frac{\mu^2}{2} \left[ 1 - e^{-2\omega_0 t} \right] - 2\omega_0 \xi t
$$

$$
\alpha_2 = \alpha_1 + \mu \sqrt{2\xi} \sqrt{1 - e^{-2\omega_0 t}}
$$

$$
z_1 = \frac{\alpha_2}{-2\omega_0 \xi t} \left[ 1 - e^{-2\omega_0 \xi t} \right]
$$

$$
z_2 = \frac{\alpha_1}{-2\omega_0 \xi t} \left[ 1 - e^{-2\omega_0 \xi t} \right]
$$

$$
z_3 = \frac{\alpha_1}{2\omega_0 \xi t} \frac{e^{-2\omega_0 \xi t} - 1}{e^{-2\omega_0 \xi t} - 1}
$$

$$
z_4 = \frac{\alpha_2}{2\omega_0 \xi t} \frac{e^{-2\omega_0 \xi t} - 1}{e^{-2\omega_0 \xi t} - 1}
$$

(VIA-6)
in which \( \mu \) is the barrier level, \( a \), normalized with respect to the end-of-excitation RMS (\( \mu = a/\sqrt{\lambda_0} \)). It may be desired to specify a barrier with respect to the stationary RMS associated with \( G_f(\omega) \) (\( \mu' = a/\sqrt{\lambda_0} \)). In this latter case \( \mu' \) should be multiplied by \( \sqrt{\lambda_0/\lambda_0} \) to get the value of \( \mu \) to be used in Equation [VIA-6]. The distinction between \( \mu \) and \( \mu' \) may be particularly important in cases involving lightly damped oscillators responding to forcing functions of short duration. In these cases the value of \( \lambda_0(t) \) at the end of the excitation may be much smaller than the stationary level.

For the transient response to step input for a lightly damped single degree-of-freedom oscillator subjected to Gaussian banded white noise input, \( \alpha_D(t) \) may be evaluated by adding to the expression for the moments (Equation [VIA-4]) the divergent terms. Then

\[
\lambda_0 = \frac{G_f(\omega_0)}{8\omega_0^2 \xi} \left[ 1 - e^{-2\omega_0 \xi t} \right]
\]

\[
\lambda_1 = \frac{G_f(\omega_0)}{8\omega_0^2 \xi} \left[ 1 - e^{-2\omega_0 \xi t} \right] - \frac{G_f(\omega_0)}{4\pi \omega_0^2} \left[ 1 + e^{-2\omega_0 \xi t} \right]
\]

\[
+ \frac{\sin^2(\omega_1 t)}{\omega_1^2} e^{-2\omega_0 \xi t} \frac{G_f(\omega)}{2\pi} \ln(\omega_u)
\]

\[
\lambda_2 = \frac{G_f(\omega_0)}{8\omega_0^2 \xi} \left[ 1 - e^{-2\omega_0 \xi t} \right] + \frac{\sin^2(\omega_1 t)}{\omega_1^2} e^{-2\omega_0 \xi t} \frac{G_f(\omega)}{2\pi} \omega_u
\]

(VIA-7)
where $\omega_u$ is the upper cutoff frequency above which the input "white" noise does not have significant value. $q$ for this case should be evaluated from Equation [VIA-3] directly.

Numerical integration of the resulting expression for $\alpha_d(t)$ yields results applicable to the evolutionary response to step input of a lightly damped one-degree-of-freedom oscillator for virtually all barrier levels and for all band widths of the input. Numerical results will be shown in Chapter VII.
Chandiramani [54,55] has performed numerical simulation of the first passage problem for oscillators subjected to white Gaussian excitation. One of the notable features of his results was the occurrence of time-decaying fluctuations on the first passage rates. A graph from Chandiramani's thesis is shown in Figure [VIB-1]. He comments, "There is a marked cyclic fluctuation in the first passage probability... The frequency of fluctuation is twice the natural frequency... of the oscillator."

From the development herein it seems clear that these fluctuations arise in the hazard function, $\alpha$, through Vanmarcke's spectral density shape factor, $\gamma$. The use of time-dependent spectral density moments to evaluate $\gamma$ has introduced divergent terms for white noise input. Observation of the divergent terms in $\lambda_1$ (Equation [IVA-20]) and $\lambda_2$ (Equation [IVA-14]) shows that they are of the form $\sin^2(\omega t)e^{-2\omega^2 t}$. Therefore they are time decaying and they have a period of oscillation one-half that of the natural damped period of the oscillator. These properties are consistent with Chandiramani's observations.

The magnitude of the fluctuations should be directly related to the upper cutoff frequency on the "white" noise input. See equations [VIA-2], [VIA-3], and [VIA-7]. Unfortunately, this problem has not been recognized previously: it was generally thought that any upper frequency limit substantially greater than the natural fre-
Figure VIB-1

Scale for $L_D(t)$

$\omega_u = 8 \quad \xi = .08 \quad \mu = 3$

Scale for $\alpha_D(t)$

$2.0 \times 10^{-3} \quad 1.5 \times 10^{-3} \quad 1.0 \times 10^{-3} \quad 0.5 \times 10^{-3}$
frequency of the oscillator being excited would produce similar simulation results. This conclusion is the case for the RMS response, but there is an \( \omega_u \) influence of a transient nature (for damped oscillators) in the first passage-related functions \( \alpha_D(t) \) and \( L_D(t) \).

In his work, Chandiramani has assumed an upper cutoff frequency of eight times that of the oscillator undamped natural frequency.
CHAPTER VII

RESULTS AND CONCLUSIONS

Section VIIA Computed First Passage Results

This section will examine first passage probabilities for lightly damped single degree-of-freedom oscillators subjected to Gaussian white noise step input. The reliability is the probability that the oscillator has not yet crossed a fixed barrier in the time interval 0 to s. This is given by

\[ L_D(t) = A_D e^{-\int_0^t \alpha_D(t) \, dt} \]  

(VIIA-1)

where \( A_D \) is the probability of no instantaneous barrier exceedance and \( \alpha(t) \) is the time-dependent hazard function. Input duration and oscillator damping and period typical for real civil engineering structures subjected to seismic activity will be used. Results will be computed by different approaches with varying degrees of sophistication, and their effect on the first passage probabilities will be examined.

The simplest approach is to assume stationary theory (Equation [IB-20] and Cramer's theory of independent Poisson distributed threshold crossings (Equation [IB-27])). It is implied in the Poisson assumption that barrier crossings are point processes and \( A_D \) should be set equal to one. For this case, \( \alpha_D \) is not a function of time. \( L_D(t) \) is given as
151

$$L_D(t) = e^{-\alpha t} \quad \text{with} \quad \alpha = \frac{2}{2\pi} \sqrt{\frac{\lambda_2}{\lambda_0}} e^{-\frac{a^2}{2\lambda_0}} \quad (\text{VIIA-2})$$

A refinement of the above procedure is to introduce the concept of a time-dependent RMS value. Since \(\alpha_D(t)\) is now a function of time, Equation [IB-29] must be used, with \(\alpha_D(t)\) given by Equation [IB-27] and \(\lambda_0(t)\) given in Equation [IVA-5]. \(L_D(t)\) is now given as

$$L_D(t) = e^{\int_0^t \alpha_D(u) du}$$

with

$$\alpha_D(u) = \frac{2}{2\pi} \sqrt{\frac{\lambda_2(u)}{\lambda_0(u)}} e^{-\frac{a^2}{2\lambda_0(u)}} \quad (\text{VIIA-3})$$

$$\sqrt{\frac{\lambda_2(u)}{\lambda_0(u)}} = \omega_0(u)$$

and was shown in Chapter IV to be essentially independent of time for damping less than 10% of critical and input frequency band-limited to about ten times the stationary natural frequency, \(\omega_0\), of the oscillator. Previous theories have not taken into account the possibility of a time-dependent \(\omega_0\), but that is not a serious drawback.

The two approaches above are based on the Poisson assumption for high barrier levels. Vanmarcke [7] uses a Markov assumption to achieve improved results for lower barriers or when crossings occur in clumps. The simplest approach here is to adopt the stationary theory, Equation [IB-26]. Since the actual response starts at rest, there is no probability of instantaneous barrier exceedance, and \(A_D\) is taken equal to one. \(L_D(t)\) in this case becomes
This theory above may be extended as Vanmarcke suggested to include the concept of an evolutionary RMS (but stationary shape factor, \(q\), and natural frequency, \(\omega_0\)), in which case Equation [IB-29] should be used, and \(L_D(t)\) is given as

\[
L_D(t) = e^{-\alpha_D t} \quad \text{with} \quad \alpha_D = \frac{2}{2\pi} \sqrt{\frac{\lambda_2}{\lambda_0}} \left[ \frac{1 - \exp \left\{ -aq \sqrt{\frac{\pi}{2\lambda_0}} \right\}}{\exp \left\{ \frac{a^2}{2\lambda_0} \right\} - 1} \right].
\]

(VIIA-4)

The above approaches may be used for very low damping levels and/or short duration if the RMS levels and the barrier levels are referred to the end of excitation. For the stationary approaches, this means using the end-of-excitation RMS based on evolutionary analysis as the equivalent stationary RMS. For the approaches using Vanmarcke's spectral density shape factor, \(q\), however, the failure rate will be too small when the stationary shape factor is used.

The most complete case, in the present context, is where \(q\) and \(\omega_0\) are considered to be time-dependent. \(q(t)\) is given in Equation [IVB-1] with the small damping approximation for the moments given in Equation [VIA-7] and \(\omega_0(t) = \sqrt[\lambda_2(t)/\lambda_0(t)}\). If the "white noise" (or broad-band) input does not in fact have significant
values for frequencies greater than about ten times the natural frequency of the oscillator, the expressions for the moments given in Equation [VIA-4] may be used. \( L_D(t) \) takes the following form

\[
L_D(t) = e^{-\int_0^t \alpha_D(u) du}
\]

with \( \alpha(u) = \frac{2}{\pi} \sqrt{\frac{\lambda_2(u)}{\lambda_0(u)}} \left[ \frac{1 - \exp \left\{ -aq(u) \sqrt{\frac{\pi}{2\lambda_0(u)}} \right\}}{\exp \left\{ \frac{a^2}{2\lambda_0(u)} \right\} - 1} \right]
\]

and \( q(u) = \sqrt{1 - \frac{\lambda_1^2(u)}{\lambda_0(u)\lambda_2(u)}} \) \hspace{1cm} (VIIA-6)

Equation [VIIA-6] may be used for the zero-damped case.

Equation [VIIA-6] still does not give the exact solution, however. Vanmarcke has shown that his stationary expression, Equation [IB-26], while more accurate than Cramer's, is still conservative [56]. In Chapter I it has been argued that the clumping of barrier crossings increases the average time between clumps and hence the mean first passage time. This kind of clumping is accounted for in Vanmarcke's approach but not in Cramer's. It has been observed that, for low barrier levels, the clumps themselves tend to clump together, thus further increasing the average time to first passage. This kind of clumping is not accounted for in either Cramer's or Vanmarcke's theory.

The approaches described above have been used to compute the reliability of lightly damped oscillators subjected to suddenly
applied "stationary" white noise (band-limited) excitation. The results are presented in Figures [VIIA-1] to [VIIA-26]. The barrier level, $\mu$, in all cases is a number to be applied to the RMS response at the end of the excitation duration. For the stationary analyses the (constant) RMS is not the one obtained from steady-state theory. Instead, it is the level that actually occurs at the end-of-excitation as determined from an evolutionary analysis. Barrier levels are then referred to this (assumed constant) RMS.

The following notation is used in the figures.

CA  Cramer and Leadbetter stationary analysis

\[
Reliability = e^{-\alpha t} \quad \alpha = \frac{2}{2\pi} \sqrt{\frac{\lambda_2}{\lambda_0}} e^{-\frac{a^2}{2\lambda_0}}
\]

CB  Cramer and Leadbetter with Caughey and Stumpf evolutionary RMS

\[
Reliability = e^{-\int_0^t \alpha(u) du} \quad \alpha(u) = \frac{2}{2\pi} \sqrt{\frac{\lambda_2}{\lambda_0}} e^{-\frac{a^2}{2\lambda_0(u)}}
\]

CC  Vanmarcke stationary analysis

\[
Reliability = e^{-\alpha t} \quad \alpha = \frac{2}{2\pi} \sqrt{\frac{\lambda_2}{\lambda_0}} \left[ 1 - \exp \left\{ -\frac{a\sqrt{\pi}}{\lambda_0(u)} \right\} \right] \frac{\exp \left\{ \frac{a^2}{2\lambda_0(u)} \right\}}{\exp \left\{ \frac{a^2}{2\lambda_0(u)} \right\} - 1}
\]

CD  Vanmarcke with Caughey and Stumpf evolutionary RMS

\[
Reliability = e^{-\int_0^t \alpha(u) du} \quad \alpha(u) = \frac{2}{2\pi} \sqrt{\frac{\lambda_2(u)}{\lambda_0(u)}} \left[ 1 - \exp \left\{ -\frac{a\sqrt{\pi}}{\lambda_0(u)} \right\} \right] \frac{\exp \left\{ \frac{a^2}{2\lambda_0(u)} \right\}}{\exp \left\{ \frac{a^2}{2\lambda_0(u)} \right\} - 1}
\]
\[ L_D(t) \]

\[ \xi = 0.01 \]
\[ \mu = 1.2 \]
\[ \omega_0 S = 15 \]
\[ \frac{RMS(\omega_0 t=15)}{RMS(\omega_0 t=\infty)} = 0.030 \]
\[
\xi = 0.001 \\
\mu = 1.2 \\
\omega_0 S = 30 \\
\frac{\text{RMS}(\omega_0 t=30)}{\text{RMS}(\omega_0 t=\infty)} = 0.058
\]
\( \xi = 0.001 \)

\( \mu = 2 \)

\( \omega_0 s = 15 \)

\[ \frac{\text{RMS}(\omega_0 t=15)}{\text{RMS}(\omega_0 t=\infty)} = 0.030 \]
\[ L_D(t) \]

\[ \xi = 0.001 \]
\[ \mu = 2 \]
\[ \omega_0 S = 30 \]
\[ \frac{RMS(\omega_0 t=30)}{RMS(\omega_0 t=\infty)} = 0.058 \]
\[ L_D(t) \]

\[ \xi = 0.001 \]
\[ \mu = 2 \]
\[ \omega_0 s = 30 \]
\[ \frac{\text{RMS}(\omega_0 t=30)}{\text{RMS}(\omega_0 t=\infty)} = 0.058 \]

CE1, CE2(\omega_u = 4\omega_0)
CE2(\omega_u = 10\omega_0)
CE2(\omega_u = 20\omega_0)
CE2(\omega_u = 50\omega_0)
CE2(\omega_u = 100\omega_0)
\[ L_D(t) = 0.001 \]
\[ \text{RMS}(\omega_0 t = 15) = 0.030 \]

\[ \xi = 0.001 \]
\[ \mu = 3 \]
\[ \omega_0 S = 15 \]
$L_D(t)$

$\xi = 0.01$

$\mu = 3$

$\omega_0 S = 30$

$\frac{RMS(\omega_0 t=30)}{RMS(\omega_0 t=\infty)} = 0.058$
\[ L_D(t) = 0.01 - 0.6^n = 1.005 \]

\[ \text{RMS}(\omega_0 t=15) = 0.259 \frac{\text{RMS}(\omega_0 t=\infty)}{162} \]

\[ \xi = 0.01 \]

\[ \mu = 1.2 \]

\[ \omega_0 S = 15 \]
\[ \begin{align*}
\xi &= 0.01 \\
\mu &= 1.2 \\
\omega_0 S &= 30 \\
\frac{RMS(\omega_0 t=30)}{RMS(\omega_0 t=\infty)} &= 0.451
\end{align*} \]
\[ \xi = 0.01 \]
\[ \mu = 2 \]
\[ \omega_0 S = 30 \]
\[ \frac{\text{RMS}(\omega_0 t=30)}{\text{RMS}(\omega_0 t=\infty)} = 0.451 \]
\[ L_D(t) \]

\[
\begin{align*}
\xi &= .01 \\
\mu &= 2 \\
\omega_0 S &= 30 \\
\text{RMS}(\omega_0 t=30) &= .451 \\
\text{RMS}(\omega_0 t=\infty) &=
\end{align*}
\]

Figure VIIA-11

- \( CE_1 \)
- \( CE_2(\omega_u=4\omega_0) \)
- \( CE_2(\omega_u=10\omega_0) \)
- \( CE_2(\omega_u=20\omega_0) \)
- \( CE_2(\omega_u=50\omega_0) \)
- \( CE_2(\omega_u=100\omega_0) \)
\[ L_D(t) = \text{RMS}(\omega_0 t = 30) / \text{RMS}(\omega_0 t = \infty) = 0.451 \]

\[ \varepsilon = 0.01 \]

\[ \mu = 3 \]

\[ \omega_0 S = 30 \]
\[ L_0(t) \]

\[ \xi = .05 \]
\[ \mu = 1.2 \]
\[ \omega_0 S = 15 \]
\[ \frac{\text{RMS}(\omega_0 t=15)}{\text{RMS}(\omega_0 t=\infty)} = .777 \]
\[ L_D(t) \]

\[ \xi = 0.05 \]

\[ \mu = 1.2 \]

\[ \omega_0 S = 30 \]

\[ \frac{\text{RMS}(\omega_0 t=30)}{\text{RMS}(\omega_0 t=\infty)} = 0.950 \]
\[ \zeta = 0.05 \]
\[ \mu = 2 \]
\[ \omega_0 S = 30 \]
\[ \frac{\text{RMS}(\omega_0 t=30)}{\text{RMS}(\omega_0 t=\infty)} = 0.950 \]
\[ L_D(t) \]

\[ \xi = .05 \]
\[ \mu = 2 \]
\[ \omega_0 S = 30 \]
\[ \text{RMS}(\omega_0 t=30) = .950 \]
\[ \text{RMS}(\omega_0 t=\infty) = .950 \]

Figure VIIA-16

CE1
CE2(\omega_u=4\omega_0)
CE2(\omega_u=10\omega_0)
CE2(\omega_u=20\omega_0)
CE2(\omega_u=50\omega_0)
CE2(\omega_u=100\omega_0)

\omega_0 t
\[ L_D(t) \]

\[
\xi = 0.05
\]

\[
\mu = 3
\]

\[
\omega_0 S = 30
\]

\[
\frac{\text{RMS}(\omega_0 t=30)}{\text{RMS}(\omega_0 t=\infty)} = 0.950
\]
Figure VIIA-18

\[ \xi = 0.1 \]
\[ \mu = 1.2 \]
\[ \omega_0 S = 15 \]

\[ \frac{RMS(\omega_0 t=15)}{RMS(\omega_0 t=\infty)} = \text{value} \]
\[ L_D(t) \]

Figure VII.A-19

\[ \xi = .1 \]
\[ \mu = 1.2 \]
\[ \omega_0 S = 30 \]
\[ \frac{\text{RMS}(\omega_0 t=30)}{\text{RMS}(\omega_0 t=\infty)} = .9975 \]
\( L_D(t) \)

\( \xi = 0.1 \)

\( \mu = 2 \)

\( \omega_0 S = 30 \)

\[
\frac{\text{RMS}(\omega_0 t=30)}{\text{RMS}(\omega_0 t=\infty)} = 0.9795
\]
$L_D(t)$

\[ \xi = 0.1 \]
\[ \mu = 2 \]
\[ \omega_0 S = 30 \]

\[
\begin{align*}
\text{RMS}(\omega_0 t=30) &= 0.9975 \\
\text{RMS}(\omega_0 t=\infty) &= 0.9975
\end{align*}
\]

Figure V1A-21

- $CE1, \ 2(\omega_u=4\omega_0)$
- $CE2(\omega_u=10\omega_0)$
- $CE2(\omega_u=20\omega_0)$
- $CE2(\omega_u=50\omega_0)$
- $CE2(\omega_u=100\omega_0)$

$\omega_0 t$
\[ L_D(t) \]

Figure VIIA-22

\[ \xi = 0.1 \]
\[ \mu = 3 \]
\[ \omega_0 S = 30 \]
\[ \frac{RMS(\omega_0 t=30)}{RMS(\omega_0 t=\infty)} = 0.9975 \]
\[ \xi = 0.001 \]
\[ \mu = 2 \]
\[ \omega_0 S = 30 \]
\[ \frac{\text{RMS}(\omega_0 t=30)}{\text{RMS}(\omega_0 t=\infty)} = 0.058 \]
\( \xi = 0.01 \)
\( \mu = 2 \)
\( \omega_0 S = 30 \)
\[ \frac{\text{RMS}(\omega_0 t=30)}{\text{RMS}(\omega_0 t=\infty)} = 0.451 \]
$L_{\theta}(t)$

Figure VIA-26

$\xi = 0.1$

$\mu = 2$

$\omega_0 S = 30$

$\frac{RMS(\omega_0 t=30)}{RMS(\omega_0 t=\infty)} = 0.9975$
Vanmarcke with the evolutionary RMS and with the time-dependent shape factor, \( q \), proposed in this report.

CE1 convergent parts of spectral moments only

CE2 complete spectral moments

\[
\text{Reliability} = e^{\int_0^t \alpha(u) \, du} = 2 \sqrt{\frac{\lambda_2(u)}{\lambda_0(u)}} \frac{\exp\left\{-aq(u)\sqrt{\frac{\pi}{2\lambda_0(u)}}\right\}}{\exp\left\{\frac{a^2}{2\lambda_0(u)}\right\} - 1}
\]

\[ q(u) = \sqrt{1 - \frac{\lambda_2^2(u)}{\lambda_0(u)\lambda_2(u)}} \]

For small damping, \( \sqrt{\frac{\lambda_2(u)}{\lambda_0(u)}} = \sqrt{\frac{\lambda_2}{\lambda_0}} = \frac{\nu_0}{2\pi} \)

The following influences in the different cases should be noted. Case CB versus CA and case CD versus CC is the influence of nonstationary versus stationary RMS of the response. Case CC versus CA and case CD versus CB is the influence of stationary clumping (Vanmarcke approach) versus Poisson occurring crossings (Cramer approach). Case CE versus CD is the influence of a time-dependent versus stationary spectral density shape factor.

In all cases, the most conservative estimate for reliability is given by case CA, the Cramer and Leadbetter approach based on Rice's stationary analysis. The reliability in this case is of exponential form. A less conservative estimate is obtained when it is taken into
account that the response RMS starts at zero and asymptotically approaches its stationary value (case CB). Relative to steady state conditions, there is little probability of failure during the period that the RMS is much less than its stationary level.

The stationary analysis with Vanmarcke's hazard rate (case CC) is also less (over) conservative than the stationary analysis utilizing Rice's hazard rate (case CA). For higher values of damping (large $q$) and higher barrier levels (large $\mu$) Vanmarcke's rate approaches Rice's. This was shown in Section IB and is verified here by the graphs.

If the Caughey and Stumpf evolutionary RMS is introduced into Vanmarcke's hazard rate, case CC becomes case CD, which is less conservative.

The final modification extends case CD to case CE by using time-dependent spectral moments to evaluate $q$.

Case CE gives a lower estimate of reliability than case CD. This is because of the nature in which the response power spectrum evolves when an oscillator is subjected to white noise input. The spectrum becomes more peaked as it evolves, reducing the parameter $q$ from a high level to its stationary level. Because large values of $q$ increase the hazard function (see Equation [IB-26]), the $q$ effect tends to increase the hazard rate during the transient period.

The figures show that the "complete solution" (case CE) often
differs markedly from the other approximations. In particular, the difference from Vanmarcke's solution with constant $q$ and evolutionary RMS is very significant for very light damping and low barrier levels. Under these conditions a major fraction of the total risk of barrier exceedance is incurred during the transient part of the response, when $q$ is significantly different from its stationary value. The isolation of this effect of time-dependent spectral density has been a major objective of this report.

Figure [VIIA-27] compares one of Chandiramani's simulation results [55] with results obtained here. The agreement with the complete theory is quite good. The oscillations have been smoothed in all cases. More results could not be compared because most of Chandiramani's results were for different starting conditions or barrier definitions.

Two types of complete solutions are shown in Figures [VIIA-1] to [VIIA-22]. Case CE1, used generally, does not include the divergent terms in the first and second moments ($\lambda_1$ and $\lambda_2$). These results are valid for "white noise" type input, but band-limited to frequencies less than, say, ten times the natural frequency of the oscillator ($\omega_u < 10\omega_0$). Case CE2 includes the divergent terms in the moments. CE2 results are given, therefore, for various upper cutoff frequencies ($\omega_u$) of the input.

The oscillations in these curves have been smoothed. It is
\[ \xi = 0.08 \]
\[ \mu = 3 \]
\[ \omega_0 S = 30 \]
\[ \frac{RMS(\omega_0 t=30)}{RMS(\omega_0 t=\infty)} = 0.992 \]
seen that for upper cutoff frequencies no more than ten or twenty times the natural frequency of the oscillator ($\omega_u = 20 \omega_0$) the divergent terms do not alter the reliability appreciably for short to medium durations. Even for upper frequencies 100 times the oscillator natural frequency ($\omega_u = 100 \omega_0$) the divergent term effect is not exceptionally large for moderate times. The fluctuations on the moments decay exponentially with time as $\exp[-\omega_0 \xi t]$. In the region where the fluctuations are large (small values of $\omega_0 \xi t$) the moments themselves are growing approximately linearly with $\omega_0 t$. Therefore, the effect of the fluctuations as a percentage of the total moment decreases rapidly. Since the hazard function is small in this region (small $\lambda_0(t)$), the effect of the fluctuations on the reliability is diminished.

For most civil engineering structures, the natural period is in the range 0.1 seconds to 10 seconds. The natural circular frequency ($\omega_0$) is then between 0.6 and 60 radians per second ($\omega_0 = 2\pi/\tau_p$). Past earthquake records indicate significant motion up to approximately 60 radians per second ($\omega = 60$). The presence of higher frequencies is hard to measure with most recording instruments presently in use. Thus, $\omega_u$ varies between one and one-hundred $\omega_0$ ($\omega_0 < \omega_u < 100 \omega_0$).

The duration of the strong motion portion of historical earthquake records is of the order of ten to thirty seconds ($s = 30$). Thus, the nondimensional time parameter $\omega_0 s$ varies between
6(\omega_0 \tau = 0.6 \times 10) and 1800 (\omega_0 \tau = 60 \times 30). It is longer for very short period structures.

(Typical values of the above parameters are \(T_0 = 4\) seconds \((\omega_0 = 2\pi/4 = 1.57)\) and \(s = 20\) seconds. Then

\[\omega_u = 60 = 40 \omega_0 \quad \text{and} \quad \omega_0 \tau = 30\]

One final example is presented utilizing the results of Chapter V for motions with modulating functions which may be expressed as sums of exponential functions. The oscillator is considered to have parameters appropriate to a steel frame high-rise building. Its damping is chosen to be one per cent of critical, and its natural period, five seconds \((\omega_0 = 2\pi/5 = 1.255)\). The input is typical of that for a strong motion earthquake on rock. It is a band-limited white noise with an upper cutoff frequency of 60 radians per second \((\omega_u = 60/\omega_0 = 47.5)\). The deterministic modulating function of the input is given in Figure [VIIA-28].

Results for five approaches are given in Figure [VIIA-29]. The barrier level is 0.5 times the end-of-excitation evolutionary RMS response that would exist if the input RMS was constant and equal to one. First, the actual motion is replaced by an equivalent stationary one, and the stationary results used (case \(CC_{\text{equiv}}\)). The (assumed stationary) input variance is such that the total duration is the same and the total power is preserved (i.e. the input RMS is equal to the average value of the modulating function of
Excitation Modulating Function (RMS)

\[
3e^{-2\omega_0 t} (1 - e^{-3\omega_0 t}) \quad \text{where } \omega_0 = 0.4\pi
\]
Figure [VIIA-28], which is 0.478. The effective barrier is then $0.5/0.478 = 1.045$. The reliability for this case plots as a straight line on semi-log paper. This approach is of doubtful value because the results reflect only the average input RMS and not the peak value.

The second approach is similar to the one above, except that now a unit input RMS value is assumed (case $CC_{\text{unit}}$). The reliability in this case is much lower than case $CC_{\text{equiv}}$ because of the larger RMS of the input. Case $CC_{\text{unit}}$ is a more conservative approach.

The third approach is also based on a stationary analysis. In this approach (case $CC_{\text{actual}}$) the stationary RMS response to a unit input is multiplied by the modulating function of the actual input. This approach becomes "exact" as the input modulating function becomes more slowly varying.

The fourth approach (case CD) involves a nonstationary analysis which uses the time-dependent response RMS, but the stationary value of the spectral density shape factor, $q$. This approach may be considered to be unconservative because of the use of the stationary value of $q$.

The fifth approach is another nonstationary analysis, case CE, with a time-dependent response RMS and spectral density shape factor, $q(t)$. The reliability for this approach is well below that of case CD. Case CE for the excitation of Figure [VIIA-28] may be vastly
conservative. The power spectrum used throughout this report broadens as the input excitation decreases. As was pointed out in Chapter III, this broadening effect is not consistent with intuitive reasoning. Appendices C and D discuss in detail the behavior of the response for a decreasing and truncated input, respectively.

The other reliability curve, case $CE_{\text{approx}}$, shown in Figure [VIIA-29] is an approximation. For the growth period of the input modulating function, case $CE_{\text{approx}}$ is equal to case $CE$. For times beyond this, the response spectral density shape factor, $q$, continues to decrease. Eventually it becomes less than the stationary forced value of $q$ because most of the motion is free, damped sinusoidal (see Appendices C and D). It is felt that further analysis on this approach is warranted.
Section VII B  Conclusions

The frequency domain approach to the study of oscillator response to random excitation has proved to be a very convenient one. The former theoretical drawback of limiting this approach to stationary analysis has been partially removed. The concept of a time-dependent response spectrum has made this possible. The analytical development of a time-dependent spectrum in Chapter III has raised conflicting definitions and mathematical complexity.

The physical notion of the power spectral density as a frequency decomposition of a sample function of a process has led to a definition of an intuitive spectrum. The concept of a spectrum as a Fourier transform of the (nonstationary) autocorrelation function has led to another spectrum definition. Additionally, a spectrum has been defined such that its Fourier transform yields the nonstationary autocorrelation function. The concept of a partial Fourier transform for that latter spectrum has been introduced along with the concept of a time-window [48]. A generalized transform [50] approach has led to another form of a time-dependent spectrum. Finally, the evolutionary spectrum adopted in this report, $G(\omega, t)$, was developed from an extension of the concept of an evolutionary RMS response [16].

Relatively simple expressions for the evolutionary spectrum and the spectral moments have been obtained for the response of lightly damped oscillators subjected to wide-band input whose
deterministic modulating function is either Heaviside or exponential, but whose relative frequency content is independent of time.

It has been seen that time-dependent shape changes of $G(\omega,t)$ are quite significant for Heaviside random excitation for times much less than those required to reach a steady state. These shape changes lead to an interesting concept of time-dependent damping for an oscillator subjected to random excitation.

First passage probabilities are extremely useful as a design criterion for oscillators subjected to random excitation. Combining the work of many earlier investigators with the concept of a time-dependent power spectrum for the response has led to meaningful results for first passage of nonstationary response for problems of practical engineering interest. In first passage computations, the time-dependent shape of the response spectrum influences the first passage rate through Vanmarcke's spectral density shape factor, $q$. For Heaviside band-limited random excitation, $q$ decreases from a high value to its stationary level. The increased shape factor increases the first passage rate for a given RMS because larger $q$ implies less clumping of crossings. In particular, the very low damping (0.1% and 1%) results obtained by this approach are of interest. Many approaches cannot handle zero damping, but this one can.

Analytical results have been prepared for oscillators subjected to Heaviside wide-band excitation. The oscillators have 0.001, 0.01, 0.05, and 0.10 fraction of critical damping. Barrier levels
are 1.2, 2.0, and 3.0 times the RMS response level at the end of the excitation. Excitation durations are \( \omega_0 t = 15 \) and 30 (in terms of radians of the undamped oscillator). The above parameters are typical to aseismic design of civil engineering structures. For many of the cases considered, especially those with low damping and low threshold, results obtained yield reliabilities significantly different from those based on previous theories. They also confirm that stationary analysis tends to be vastly over-conservative for many sets of parameters.

For cases in which the response is near stationarity for a major part of the duration, reasonable results are obtained using a stationary analysis if a shortened duration is used. This can be seen by noting that the reliability curves for the stationary and nonstationary analyses become parallel for times long enough that the response is approximately stationary. This effect is a consequence of the exponential form of the reliability curve for stationary conditions. The effective duration is dependent on the damping and barrier levels. There is also a certain dependence on duration, related to the fact that barrier levels are given with respect to the end-of-excitation response RMS. The difference between actual duration and effective duration increases with barrier level and duration, and decreases with damping. Further numerical examples will be needed before definite quantitative values can be put on these effects.
The examples earlier in this chapter and the expressions for time-dependent spectral moments and first passage rates in Chapters IV and VI indicate that nonstationary effects for Heaviside wide-band input can be neglected if the excitation duration is greater than about $15/\omega_0 \xi$ where $\omega_0$ is the oscillator undamped natural frequency and $\xi$ is the fraction of critical damping. For shorter durations and damping levels on the order of a few per cent or less, the time-dependent variations in the shape of the response power spectral density function are significant as well as the time-dependent RMS of the response. It seems reasonable to assume that for damping greater than about five per cent of critical the shape of the output power spectrum does not depend on time, although the response RMS does.
REFERENCES


BIOGRAPHY

The author is a native of New Jersey. He was born on January 15, 1945, and attended Moorestown Friends' School from 1949 to 1963.

He received the Bachelor of Science degree in Civil Engineering from M.I.T. in 1967, and the Master of Science degree in 1968, while holding a National Science Foundation Graduate Fellowship. His research was coordinated with the National Bureau of Standards on statistical consideration of building live loads.

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APPENDIX A

REVIEW OF STATIONARY RANDOM VIBRATION THEORY

A random vibration is a stochastic process whose first and second order properties are described by a quantity called the autocorrelation function. The autocorrelation function of a stochastic process, $x(t)$, evaluated at times $t_1$ and $t_2$ is defined as

$$R_X(t_1,t_2) = E[x(t_1)x(t_2)] - E[x(t_1)]E[x(t_2)]$$  \hfill (A-1)

where the expectation is an ensemble average: i.e., an average taken over all samples of the process. If the process has a mean equal to zero, then

$$R_X(t_1,t_2) = E[x(t_1)x(t_2)]$$  \hfill (A-2)

By symmetry

$$R_X(t_1,t_2) = R_X(t_2,t_1)$$  \hfill (A-3)

In general, assuming the process has a zero-mean value, the autocorrelation function exhibits a general decreasing trend as $|t_1 - t_2|$ increases. As $|t_1 - t_2| \to \infty$, the autocorrelation function converges to zero.

An important concept in random vibrations is stationarity. A stochastic process is stationary if its underlying probability distributions are independent of a shift in time. For a stationary process, the autocorrelation is a function only of the time difference between $t_1$ and $t_2$. The autocorrelation function may satisfy this
condition for a process that is not strictly stationary. Such a process is termed "weakly" stationary. The difference between \( t_1 \) and \( t_2 \) is commonly denoted \( \tau \). The invariant nature of a stationary process ensures that the autocorrelation function is an even function of \( \tau \).

\[
R(t,t+\tau) = R(\tau) = R(t,t-\tau) = R(-\tau)
\] (A-4)

This property of evenness can be retained for nonstationary processes only by a redefinition of parameters that somewhat alters the physical interpretation of autocorrelation. This will be seen in Chapter III.

A stationary process is usually assumed to satisfy the condition of ergodicity. Theoretically, an infinite number of samples of a stochastic process are required to completely describe the process. If it is ergodic, however, one sample of infinite duration is sufficient. Ensemble averages can then be replaced by time averages over a single infinitely long sample.

The study of stationary oscillator response to random excitation may be made in either the time or frequency domain. The latter approach has two distinct advantages: the associated mathematics is simpler, and the results are in a convenient form for parameter variation studies.

In the frequency domain, an oscillator may be characterized by its (complex) transfer function, \( H(\omega) \). \( H(\omega) \) is the response of
an oscillator to a sinusoidal excitation of unit amplitude and frequency $\omega$. For a linear oscillator, the principle of superposition may be applied over the full frequency range of the excitation. Let $x(t)$ be a sample time history of a stationary excitation, and $X(\omega)$ its Fourier transform. Care must be exercised here since $X(\omega)$ exists only if $|x(t)| \to 0$ at least as fast as $1/|t|$ as $|t| \to \infty$. $x(t)$ cannot, therefore, be strictly stationary. This is not a problem in real applications since the Fourier transform exists for all realizable processes (those with less than infinite power). Potential problems can be avoided by careful handling of limits.

Let $y(t)$ be the time history of the stationary response of an oscillator characterized by a frequency transfer function $H(\omega)$, and $Y(\omega)$ the Fourier transform of $y(t)$. The following well-known results then hold:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t} \, dt$$

$$Y(\omega) = H(\omega)X(\omega)$$

$$y(t) = \int_{-\infty}^{\infty} Y(\omega)e^{i\omega t} \, df \text{ with } f = \omega/2\pi$$

The extremely useful result here is that, in the frequency domain, the output is the product of the input and the transfer function. If a time history is desired, it may be obtained by applying a Fourier transform to the frequency response.
The oscillator impulse response function, here designated \( h(t) \), gives the time history of response of an oscillator subjected to a unit impulse excitation. Linear superposition produces the general result

\[
y(t) = \int_{-\infty}^{t} x(\tau) h(t-\tau) \, d\tau
\] (A-6)

In the time domain the input is convolved with the impulse response function to obtain the output, while in the frequency domain, the input is multiplied by the transfer function to obtain the output. It is not surprising that \( h(t) \) and \( H(\omega) \) form a Fourier transform pair.

\[
H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} \, dt
\]
\[
h(t) = \int_{0}^{\infty} H(\omega) e^{i\omega t} \, d\omega
\] (A-7)

The derivation of \( h(t) \) and \( H(\omega) \) for both single degree-of-freedom oscillators and multiple degree-of-freedom oscillators is straightforward and well known. The expressions for a one degree-of-freedom oscillator are

\[
h(t) = \begin{cases} 0 & t < 0 \\ -e^{-\omega_0 \xi t} \frac{\sin(\omega_1 t)}{\omega_1} & t \geq 0 \end{cases}
\]

\[
H(\omega) = \frac{-1}{\omega_0^2 - \omega^2 + 2i\xi\omega_0 \omega}
\] (A-8)

where the oscillator parameters are shown in Figure [A-1], and the governing differential equation of motion is
\[
\sqrt{\frac{k}{m}} = \omega_0, \text{ undamped natural oscillator frequency}
\]

\[
\frac{c}{2\sqrt{km}} = \xi, \text{ fraction of critical oscillator damping}
\]

\[
\omega_0\sqrt{1-\xi^2} = \omega_1, \text{ damped natural oscillator frequency}
\]
\[ \ddot{m}y + c\dot{y} + ky = -m\ddot{x} \quad (A-9) \]

Each dot over a quantity indicates a differentiation with respect to time.

The input and output power spectral densities of a stationary process are related by

\[ G_y(\omega) = H(-\omega)H(\omega)G_x(\omega) \quad (A-10) \]

where \( G(\omega) \) is the one-sided power spectral density of a stationary process. An examination of \( H(-\omega) \) shows that it is \( H^*(\omega) \), the complex conjugate of \( H(\omega) \). Using the relation

\[ H^*(\omega)H(\omega) = |H(\omega)|^2 \quad (A-11) \]

allows the slightly simpler form for the spectral densities

\[ G_y(\omega) = |H(\omega)|^2G_x(\omega) \quad (A-12) \]

The mean square response may be immediately derived from Equation [IA-7].

\[
\mathbb{E}[y^2(t)] = \int_0^\infty G_y(\omega) \, df = \int_0^\infty |H(\omega)|^2G_x(\omega) \, df \quad (A-13)
\]
Appendix B

Derivation of $\lambda_1(t)$ for Zero and Critical Damping

Following is the derivation of $\lambda_1(t)$ for the zero damped case as referred to in Chapter IV.

$$
\lambda_1(t) = G_f(\omega_0) \int_0^\infty u \frac{\omega}{(\omega^2 - \omega_0^2)^2} \left[ 2 - 2 \cos(\omega_0 t) \cos(\omega t) - \frac{2\omega}{\omega_0} \sin(\omega_0 t) \sin(\omega t) + \frac{\omega^2 - \omega_0^2}{\omega_0^2} \sin^2(\omega_0 t) \right] d\omega
$$

$$
= \frac{G_f(\omega_0)}{8\pi \omega_0} \left[ \int_0^\infty \frac{2\omega d\omega}{(\omega^2 - \omega_0^2)^2} - \int_0^\infty \frac{2\omega \cos(\omega_0 t) \cos(\omega t) d\omega}{(\omega^2 - \omega_0^2)^2} - \int_0^\infty \frac{2\omega^2 \sin(\omega_0 t) \sin(\omega t) d\omega}{\omega_0 (\omega^2 - \omega_0^2)^2} \right] + \frac{G_f(\omega)}{2\pi} \int_0^\infty \frac{\omega (\omega^2 - \omega_0^2) \sin^2(\omega_0 t) d\omega}{\omega_0 (\omega^2 - \omega_0^2)^2}
$$

$$
= \frac{G_f(\omega_0)}{8\pi \omega_0} \left[ \int_0^\infty \frac{2d\omega}{(\omega - \omega_0)^2} - \int_0^\infty \frac{2d\omega}{(\omega + \omega_0)^2} - \int_0^\infty \frac{2\cos(\omega_0 t) \cos(\omega t) d\omega}{(\omega - \omega_0)^2} + \int_0^\infty \frac{2\cos(\omega_0 t) \cos(\omega t) d\omega}{(\omega + \omega_0)^2} - \int_0^\infty \frac{2\omega \sin(\omega_0 t) \sin(\omega t) d\omega}{\omega_0 (\omega - \omega_0)^2} + \int_0^\infty \frac{2\omega \sin(\omega_0 t) \sin(\omega t) d\omega}{\omega_0 (\omega + \omega_0)^2} + \frac{G_f(\omega_0)}{2\pi} \int_0^\infty \frac{\omega \sin^2(\omega_0 t) d\omega}{\omega_0 (\omega^2 - \omega_0^2)} \right]
$$

(B-1)
A change of variables is now made.

\[
\lambda_1(t) = \frac{G_f(\omega_0)}{8\pi \omega_0} \left[ \int_{-\omega_0}^{\omega_0} \frac{2du}{u^2} - \int_{-\omega_0}^{\omega_0} \frac{2dv}{v^2} - \int_{-\omega_0}^{\omega_0} \frac{2\cos(\omega_0 t) \cos((u+\omega_0) t) du}{u^2} \right. \\
+ \int_{\omega_0}^{\infty} \frac{2\cos(\omega_0 t) \cos((v-\omega_0) t) dv}{v^2} - \int_{-\omega_0}^{\omega_0} \frac{2(u+\omega_0) \sin(\omega_0 t) \sin(u+\omega_0) du}{u^2 \omega_0} \\
+ \left. \int_{\omega_0}^{\infty} \frac{2(v-\omega_0) \sin(\omega_0 t) \sin((v-\omega_0) t) dv}{v^2 \omega_0} \right] + \frac{G_f(\omega_0)}{2\pi} \\
+ \int_{0}^{\infty} \frac{\omega \sin^2(\omega_0 t) d\omega}{\omega^2 (\omega+\omega_0)(\omega-\omega_0)} \\
(B-2)
\]

Evaluating the integrals that are straightforward and expanding and simplifying the other terms leads to

\[
\lambda_1(t) = \frac{G_f(\omega_0)}{8\pi \omega_0} \left[ -\frac{4}{\omega_0} + \frac{4\cos(\omega_0 t)}{\omega_0} + 2t \int_{-\omega_0}^{\omega_0} \frac{\sin(yt) dy}{y} \right. \\
- \frac{2\sin(\omega_0 t) \cos(\omega_0 t)}{\omega_0} \int_{-\omega_0}^{\omega_0} \frac{\sin(yt) dy}{y} - \frac{2\sin^2(\omega_0 t)}{\omega_0} \\
\left. - \int_{-\omega_0}^{\omega_0} \frac{\cos(yt) dy}{y} - \frac{2\sin^2(\omega_0 t)}{\omega_0} \int_{-\omega_0}^{\omega_0} \frac{\cos(yt) dy}{y} \right] + \\
+ \frac{G_f(\omega_0)}{2\pi} \frac{\sin^2(\omega_0 t)}{\omega_0} \int_{0}^{\omega_0} \frac{\omega d\omega}{\omega^2 - \omega_0^2} \\
(B-3)
\]
The following functions (both widely tabulated) must now be used:

\[ S_i(\omega_0 t) = \int_{0}^{\omega_0 t} \frac{\sin(u)du}{u} - \frac{\pi}{2} \quad (B-4) \]

\[ C_i(\omega_0 t) = -\int_{\omega_0 t}^{\infty} \frac{\cos(y)dy}{y} \quad (B-5) \]

Then

\[
\lambda_1(t)_{\text{zero damping}} = \frac{G_f(\omega_0)}{8\pi\omega_0} \left[ -\frac{4}{\omega_0} + \frac{4\cos(\omega_0 t)}{\omega_0} + 2\pi t + 4t S_i(\omega_0 t) + \right.
\]

\[
-\frac{2\pi\sin(\omega_0 t)\cos(\omega_0 t)}{\omega_0} - \frac{4\sin(\omega_0 t)\cos(\omega_0 t)}{\omega_0} S_i(\omega_0 t) +
\]

\[
+ \frac{2\sin^2(\omega_0 t)}{\omega_0} C_i(\omega_0 t) + \frac{2\sin^2(\omega_0 t)}{\omega_0} C_i(\omega_0 t) +
\]

\[
+ \frac{\sin^2(\omega_0 t)}{\omega_0^2} \int_{0}^{\omega_0} \frac{u\omega d\omega}{\omega^2-\omega_0^2} \right] \quad (B-6)\]

and simplifying,

\[
\lambda_1(t)_{\text{zero damping}} = \frac{G_f(\omega_0)}{8\omega_0^2} \left[ 2\omega_0 t \sin(2\omega_0 t) - \frac{4}{\pi} \right. + \frac{4\cos(\omega_0 t)}{\pi} +
\]

\[
+ \left( \frac{4\omega_0 t}{\pi} - \frac{4\sin(\omega_0 t)\cos(\omega_0 t)}{\pi} \right) S_i(\omega_0 t) +
\]

\[
+ \frac{4\sin^2(\omega_0 t)}{\pi} C_i(\omega_0 t) + \frac{\sin^2(\omega_0 t)}{2\pi\omega_0^2} \int_{0}^{\omega_0} \frac{\omega G_f(\omega) d\omega}{\omega^2-\omega_0^2} \right] \quad (B-7)\]
Following is the derivation of $\lambda_1(t)$ for the critically damped case.

$$
\lambda_1(t)_{\text{critical damping}} = \frac{G_\ell(\omega_0)}{2\pi} \int_0^\infty \frac{\omega^2}{(\omega^2 + \omega_0^2)} \left[ 1 + e^{-2\omega_0 t} \left( 1 + 2\omega_0 t + \omega_0 t (2 + 2\omega_0 t) \cos(\omega t) - \omega_0 t (2\omega_0 t) \sin(\omega t) + (\omega_0^2 + \omega_0^2) t^2 \right) \right] d\omega
$$

$$
= \frac{G_\ell(\omega_0)}{2\pi} \left[ 1 + e^{-2\omega_0 t} \left( 1 + 2\omega_0 t + \omega_0 t (2 + 2\omega_0 t) \cos(\omega t) - \omega_0 t (2\omega_0 t) \sin(\omega t) + (\omega_0^2 + \omega_0^2) t^2 \right) \right] \int_0^\infty \frac{\omega d\omega}{(\omega^2 + \omega_0^2)^2} +
$$

$$
= \frac{G_\ell(\omega_0)}{2\pi} \left[ 1 + e^{-2\omega_0 t} \left( 1 + 2\omega_0 t + \omega_0 t (2 + 2\omega_0 t) \cos(\omega t) - \omega_0 t (2\omega_0 t) \sin(\omega t) + (\omega_0^2 + \omega_0^2) t^2 \right) \right] \int_0^\infty \frac{\omega^2 \sin(\omega t) d\omega}{(\omega^2 + \omega_0^2)^2} +
$$

$$
= \frac{G_\ell(\omega_0)}{2\pi} \left[ 1 + e^{-2\omega_0 t} \left( 1 + 2\omega_0 t + \omega_0 t (2 + 2\omega_0 t) \cos(\omega t) - \omega_0 t (2\omega_0 t) \sin(\omega t) + (\omega_0^2 + \omega_0^2) t^2 \right) \right] \int_0^\infty \frac{\omega^3 d\omega}{(\omega^2 + \omega_0^2)^2} +
$$

$$
= G_\ell(\omega_0) \left[ 1 + e^{-\omega_0 t} \left( 1 + 2\omega_0 t + \omega_0 t (2 + 2\omega_0 t) \cos(\omega t) - \omega_0 t (2\omega_0 t) \sin(\omega t) + (\omega_0^2 + \omega_0^2) t^2 \right) \right] \int_0^\infty \frac{d\omega}{(\omega^2 + \omega_0^2)^2} - \int_0^\infty \frac{d\omega}{(\omega - i\omega_0)^2} - \int_0^\infty \frac{d\omega}{(\omega + i\omega_0)^2} +
$$

$$
= G_\ell(\omega_0) \left[ 1 + e^{-\omega_0 t} \left( 1 + 2\omega_0 t + \omega_0 t (2 + 2\omega_0 t) \cos(\omega t) - \omega_0 t (2\omega_0 t) \sin(\omega t) + (\omega_0^2 + \omega_0^2) t^2 \right) \right] \int_0^\infty \frac{\cos(\omega t) d\omega}{(\omega - i\omega_0)^2} - \int_0^\infty \frac{\cos(\omega t) d\omega}{(\omega + i\omega_0)^2} +
$$

$$
= G_\ell(\omega_0) \frac{(1 + \omega_0 t)}{4i\pi \omega_0} e^{-\omega_0 t} \int_0^\infty \frac{\cos(\omega t) d\omega}{(\omega - i\omega_0)^2} - \int_0^\infty \frac{\cos(\omega t) d\omega}{(\omega + i\omega_0)^2} +
$$

$$
= G_\ell(\omega_0) \frac{\omega_0 t}{4i\pi \omega_0} e^{-\omega_0 t} \int_0^\infty \frac{\omega \sin(\omega t) d\omega}{(\omega - i\omega_0)^2} - \int_0^\infty \frac{\omega \sin(\omega t) d\omega}{(\omega + i\omega_0)^2} +
$$

$$
= G_\ell(\omega_0) \frac{\omega_0 t^2}{2\pi \omega_0^2} e^{-\omega_0 t} \int_0^\infty \frac{u \omega^3 d\omega}{(\omega^2 + \omega_0^2)^2} \quad (B-8)
$$

A change of variables is now made.
\[
\lambda_1(t) \frac{\text{critical damping}}{G_f(\omega_0)} = \frac{1 + e^{-\omega_0 t}}{8i\pi\omega_0} \left[ \int_{-i\omega_0}^{i\omega_0} \frac{du}{u^2} - \int_{i\omega_0}^{\infty} \frac{dv}{v^2} \right] + 
\]

\[
\frac{-2\omega_0 t}{8i\pi\omega_0} \left[ \int_{-i\omega_0}^{\infty} \frac{\cos((u+i\omega_0)t)du}{u^2} - \int_{i\omega_0}^{\infty} \frac{\cos((v-i\omega_0)t)dv}{v^2} \right] + 
\]

\[
\frac{2\omega_0 t e^{-\omega_0 t}}{8i\pi\omega_0^2} \left[ \int_{-i\omega_0}^{\infty} \frac{(u+i\omega_0)\sin((u+i\omega_0)t)du}{u^2} \right] + 
\]

\[
\int_{i\omega_0}^{\infty} \frac{(v-i\omega_0)\sin((v-i\omega_0)t)dv}{v^2} \right] + \frac{\omega_0^2 t^2 e^{-2\omega_0 t}}{2\pi\omega_0^2} \int_{i\omega_0}^{\infty} \frac{\cos((v-i\omega_0)t)dv}{(\omega_0^2 + v^2)^2} 
\]

\[
\frac{-2\omega_0 t}{1 + e^{-\omega_0 t}} \left[ 1 + 2\omega_0 t + \omega_0^2 t^2 \right] = \frac{1 + e^{-\omega_0 t}}{8i\pi\omega_0} \left[ \int_{-i\omega_0}^{i\omega_0} \frac{du}{u} \right] + 
\]

\[
\frac{2(1+\omega_0 t)e^{-\omega_0 t}}{8i\pi\omega_0} \left[ \cos(i\omega_0 t) - \frac{\cos(ut)}{u} \right]_{i\omega_0}^{\infty} - t - tS_1(i\omega_0 t) + 
\]

\[
\frac{t\pi}{2} \left[ \sin(ut) \right]_{i\omega_0}^{\infty} + tC_1(i\omega_0 t) - t\pi + \frac{\cos(ut)}{v} \right]_{i\omega_0}^{\infty} - tS_1(i\omega_0 t) + 
\]

\[
\frac{t\pi}{2} \left[ \sin(vt) \right]_{i\omega_0}^{\infty} + tC_1(i\omega_0 t) - \frac{2\omega_0 t e^{-\omega_0 t}}{8i\pi\omega_0} \left[ \cos(i\omega_0 t) \right]_{i\omega_0}^{\infty} 
\]

\[
\left\{ S_1(i\omega_0 t) + \frac{\pi}{2} \right\} + \sin(i\omega_0 t) \left\{ -C_1(i\omega_0 t) + \pi \right\} + i\omega_0 \cos(i\omega_0 t) \]
\[
\begin{align*}
\{ -\frac{\sin(ut)}{u} \bigg|_{-i\omega_0}^\infty - tC_i(i\omega_0t) + i\pi t \} + \sin(i\omega_0t) \{ -\frac{\cos(ut)}{u} \bigg|_{-i\omega_0}^\infty - \\
-tS_i(i\omega_0t) - \frac{\pi t}{2} \} + \cos(i\omega_0t) \left\{ S_i(i\omega_0t) - \frac{\pi t}{2} \right\} - \sin(i\omega_0t)C_i(i\omega_0t) + \\
+ \cos(i\omega_0t) \left\{ -\frac{\sin(vt)}{v} \bigg|_{i\omega_0}^\infty - tC_i(i\omega_0t) \right\} - \sin(i\omega_0t) \left\{ -\frac{\cos(vt)}{v} \bigg|_{i\omega_0}^\infty \right\} + \\
+ tS_i(i\omega_0t) - \frac{t\pi}{2} \right\} \} + \frac{\omega_0^2 t^2 e^{-\omega_0 t}}{2\pi \omega_0^2} \int_0^\omega \frac{\omega^3 d\omega}{(\omega^2 + \omega_0^2)^2} \\
\end{align*}
\]

Evaluating and simplifying leads to

\[
\frac{2\pi \lambda_1(t)}{G_f(\omega_0)} \bigg|_{\text{critical damping}} = \frac{1}{2} \left[ 1 + e^{-2\omega_0 t} \left\{ 1 + 2\omega_0 t + \omega_0^2 t^2 \right\} \right] - (1 + \omega_0 t) e^{-\omega_0 t} + \\
+ i \frac{\omega_0 t}{2} \sin(i\omega_0 t)C_i(i\omega_0t)(2 + 2\omega_0 t)e^{-\omega_0 t} - \frac{1}{2} \sin(i\omega_0 t) \\
+ C_i(i\omega_0t)2te^{-\omega_0 t} + \frac{\omega_0 t}{2\omega_0} \cos(i\omega_0 t) C_i(i\omega_0t) 2te^{-\omega_0 t} + \\
- \frac{i\omega_0 t}{2\omega_0} \cos(i\omega_0 t)S_i(i\omega_0t) \left\{ 2e^{-\omega_0 t} + 2\omega_0 t e^{-\omega_0 t} \right\} + \\
- \frac{i}{2} \cos(i\omega_0 t) S_i(i\omega_0t)2te^{-\omega_0 t} + \frac{\omega_0 t}{2\omega_0} \sin(i\omega_0 t) \\
S_i(i\omega_0t)2te^{-\omega_0 t} + \frac{\pi_0^0 t}{4\omega_0} \sin(i\omega_0 t) \left\{ 2e^{-\omega_0 t} + 2\omega_0 t e^{-\omega_0 t} \right\} + \\
- \frac{\pi}{4} \sin(i\omega_0 t)2te^{-\omega_0 t} - i \frac{\pi_0^0 t}{4\omega_0} \cos(i\omega_0 t)2te^{-\omega_0 t} + 
\]
\[
\lambda_1(t) = \frac{G_f(\omega_0)}{4\pi} \left[ -2\omega_0 t e^{-\omega_0 t} \left( 1 + 2\omega_0 t + \omega_0^2 t^2 \right) \right] +
\]
\[
- \frac{G_f(\omega_0)}{2\pi} (1 + \omega_0 t) e^{-\omega_0 t} + \frac{G_f(\omega_0)}{2\pi} t^2 e^{-\omega_0 t} E_i(\omega_0 t) +
\]
\[
+ \frac{G_f(\omega_0)}{2\pi} t^2 e^{-\omega_0 t} \int_0^\infty \frac{\omega u \omega^3 d\omega}{(\omega^2 + \omega_0^2)^2}
\]
\[
= \frac{G_f(\omega_0)}{4\pi \omega_0} \left[ \omega_0^2 + \omega_0^2 e^{-2\omega_0 t} \left( 1 + 2\omega_0 t + \omega_0^2 t^2 \right) \right] -
\]
\[
2\omega_0 (1 - \omega_0 t) e^{-\omega_0 t} + 2\omega_0^2 e^{-2\omega_0 t} E_i(\omega_0 t) +
\]
\[
+ 2\omega_0^2 e^{-2\omega_0 t} \int_0^\infty \frac{\omega u \omega^3 d\omega}{(\omega^2 + \omega_0^2)^2}
\]

(E-12)
STEP FUNCTION DERIVATION OF G(\omega,t) FOR SEPARABLE INPUT

In analogy with the deterministic approach of decomposing a forcing function into positive and negative unit heaviside functions, a stochastic process may be decomposed into stationary processes with unit deterministic heaviside modulating functions. The simplest case is shown in Figure [C-1].

F is a zero mean stochastic process. It can be considered to be the linear superposition of two forces, F_1 and F_2. The input enters power spectrum computations through the autocorrelation, which is a measure of the variance of the input. The variance is a probabilistic measure of energy, and is thus non-negative. Therefore, there must be some other measure to ensure that F_2 completely removes F_1 for t > t_0. This measure is the cross-correlation between F_1 and F_2. Forces F_1 and F_2 must be perfectly negatively correlated. Then F_1(t) = -F_2(t), t > t_0.

\[
E[F_1(t)F_1(t+\tau)] = E[-F_1(t)F_1(t+\tau)] = -E[F_1(t)F_1(t+\tau)] \quad (C-1)
\]

If

\[
R_1(\tau) \triangleq E[F_1(t)F_1(t+\tau)] \quad (C-2)
\]

and

\[
R_{12}(\tau) \triangleq E[F_1(t)F_2(t+\tau)] \quad (C-3)
\]

then

\[
R_{12}(\tau) = -R_1(\tau) \quad (C-4)
\]

The response to two inputs, for a linear system, is the sum of the response to each input taken separately.
Excitation Superposition
\[ Y(t) = \int_0^t h(t-u)F_1(u)\,du + \int_0^t h(t-v)F_2(v)\,dv \]  
(C-5)

Since \( u \) and \( v \) are dummy variables, Equation [C-5] may be written as one integral.

\[ Y(t) = \int_0^t h(t-u) [F_1(u) + F_2(u)] \,du \]  
(C-6)

The variance of the response is then

\[ \text{Var}(Y) = \mathbb{E}\left\{ \int_0^t h(t-u)[F_1(u)+F_2(u)]\,du \int_0^t h(t-v)[F_1(v)+F_2(v)]\,dv \right\} \]

\[ = \int_0^t \int_0^t h(t-u)h(t-v)\mathbb{E}[F_1(u)F_1(v)+F_1(u)F_2(v)+F_2(u)F_1(v)\]
\[ +F_2(u)F_2(v)]\,du\,dv \]

\[ = \int_0^t \int_0^t h(t-u)h(t-v)\{R_1(u,v)+R_{12}(u,v)+R_{21}(u,v)+R_2(u,v)\}\,du\,dv \]  
(C-7)

The autocorrelation and cross-correlation for the case shown in Figure [C-1] for the various regions of \( u \) and \( v \) are as follows:

\[ R_1(u,v) = R_1(\tau) \quad 0 < u < t \quad 0 < v < t \]
\[ R_2(u,v) = R_1(\tau) \quad t_0 < u < t \quad t_0 < v < t \]
\[ R_{12}(u,v) = -R_1(\tau) \quad 0 < u < t \quad t_0 < v < t \]
\[ R_{21}(u,v) = R_1(\tau) \quad t_0 < u < t \quad 0 < v < t \]  
(C-8)

All autocorrelations and cross-correlations are zero in the other regions. The variance is then
\[ \text{Var}(Y) = \int_0^t \int_0^t h(t-u)h(t-v) \left[ R_1(u-v) \right] \, du \, dv \]

\[ + \int_0^t \int_0^t h(t-u)h(t-v) \left[ R_1(u-v) + R_{12}(u-v) \right] \, du \, dv \]

\[ + \int_0^t \int_0^t h(t-u)h(t-v) \left[ R_1(u-v) + R_2(u-v) + R_{12}(u-v) + R_{21}(u-v) \right] \, du \, dv \quad \text{(C-9)} \]

where only nonzero values of correlation are shown. Substituting the values from Equation [C-8] one gets

\[ \text{Var}(Y) = \int_0^t \int_0^t h(t-u)h(t-v) R_1(\tau) \, du \, dv \]

\[ + \int_0^t \int_0^t h(t-u)h(t-v) \left[ R_1(\tau) - R(\tau) \right] \, du \, dv \]

\[ + \int_0^t \int_0^t h(t-u)h(t-v) \left[ R_1(\tau) - R_1(\tau) \right] \, du \, dv \]

\[ + \int_0^t \int_0^t h(t-u)h(t-v) \left[ R_1(\tau) + R_1(\tau) - R_1(\tau) - R_1(\tau) \right] \, du \, dv \]

\[ = \int_0^t \int_0^t h(t-u)h(t-v) R_1(\tau) \, du \, dv \quad \text{(C-10)} \]

which is
\[
\text{Var}(Y) = \int_0^t \int_0^t h(t-u)h(t-v) A(u)A(v) R_1(\tau) \, dudv
\]

since \( A(t) = \begin{cases} 1 & 0 < t < t_0 \\ 0 & \text{otherwise} \end{cases} \) (C-11)

Next is considered the more complicated case shown in Figure [C-2]. In order to remove forces properly, \( F_3 \) must be perfectly negatively correlated with \( F_1 \) or \( F_2 \), and \( F_4 \) must be perfectly negatively correlated with the other. The solution is dependent on whether \( F_3 \) removes \( F_1 \) or \( F_2 \). The simplest case is to assume that \( F_3 \) and \( F_4 \) are perfectly correlated with each other, and \( F_1 \) and \( F_2 \) are perfectly negatively correlated with both. This assumption is the same as requiring \( F \) to be separable stationary. Using the same procedure as in the previous example, the response variance is

\[
\text{Var}(Y) = \int_0^{t_0} \int_0^{t_0} h(t-u)h(t-v) R_1(\tau) \, dudv
\]

\[
+ \int_0^{t_0} \int_0^{t_1} h(t-u)h(t-v) 2R_1(\tau) \, dudv
\]

\[
+ \int_0^{t_0} \int_0^{t_2} h(t-u)h(t-v) R_1(\tau) \, dudv
\]

\[
+ \int_0^{t_1} \int_0^{t_0} h(t-u)h(t-v) 2R_1(\tau) \, dudv
\]
Figure C-2
\[ \begin{align*}
&+ \int_{t_0}^{t_1} \int_{t_0}^{t_1} h(t-u)h(t-v) \ 4R_1(\tau)dudv \\
&+ \int_{t_0}^{t_2} \int_{t_0}^{t_1} h(t-u)h(t-v) \ 2R_1(\tau)dudv \\
&+ \int_{t_0}^{t_2} \int_{t_0}^{t_1} h(t-u)h(t-v) \ R_1(\tau)dudv \\
&+ \int_{t_1}^{t_2} \int_{t_0}^{t_1} h(t-u)h(t-v) \ R_1(\tau)dudv \\
&+ \int_{t_1}^{t_2} \int_{t_1}^{t_1} h(t-u)h(t-v) \ R_1(\tau)dudv \\
\end{align*} \]  
\text{(C-12)}

which is

\[ \text{Var}(Y) = \int_{0}^{t} \int_{0}^{t} h(t-u)h(t-v) A(u) A(v) R_1(\tau)dudv \]

Since \( A(t) \)

\[ \begin{cases} 
1 & 0 < t < t_0 \\
2 & t_0 < t < t_1 \\
1 & t_1 < t < t_2 \\
0 & t_2 < t < \infty 
\end{cases} \]  
\text{(C-13)}

Chapter V uses the result of Equation [C-13] for arbitrarily varying input. Physical significance has been shown through the use of step functions for a non-decreasing input. For decreasing input, the result does not retain complete physical significance. As was discussed in Chapter III, if the input is suddenly removed,
the response vibrates sinusoidally at its natural damped frequency, and the response power spectrum is then a decaying dirac delta at \( \omega = \omega_1 \). Using the above facts, one can utilize a step function approach and take the limit of smaller and smaller steps. The following results are obtained for the convergent parts of the small damping moments for the general input shown in Figure [C-3]. \( G_f(\omega) \) is a stationary unit area power spectrum and \( A(t) \) is a deterministic non-increasing modulating function.

\[
\lambda_0 = \frac{G_f(\omega_0)}{8\omega_0^3\xi} \left[ A(t) \left\{ 1 - e^{-2\omega_0\xi t} \right\} + \right.
\]

\[
- \int_0^t \frac{dA(t)}{dt} \left. \left\{ e^{-2\omega_0\xi(t-u)} \left\{ 1 - e^{-2\omega_0\xi u} \right\} \right. \right] du \left. \right|_{t=u}
\]

\[
\lambda_1 = \omega_0\lambda_0 - \frac{A(t) G_f(\omega_0)}{2\pi\omega_0^2} e^{-\omega_0\xi t}
\]

\[
\lambda_2 = \omega_0^2\lambda_0 \quad (C-14)
\]

For \( A(t) \) equal to a constant, those reduce to the step function results.
General Non-Increasing Modulating Function

Figure C-3
RESPONSE TO TIME-LIMITED STEP INPUT

A case of interest, especially in earthquake engineering applications, is a time-limited step modulating function, as shown in Figure [D-1]. For \( t < t_1 \) and \( t_0 = 0 \), the results will be the same as the transient case for step input (Chapter IV), because the Caughey and Stumpf approach produces a time-dependent spectrum dependent only on past time. \( G(\omega,t) \) for \( t < t_1 \) was given in Equation [IIA-2]. For \( t > t_1 \), \( G(\omega,t) \) becomes

\[
G(\omega,t) = G_f(\omega_0) \int_{t_0}^{t_1} \int_{t_0}^{t_1} h(t-u)h(t-v)\cos(\omega(u-v))du dv \tag{D-1}
\]

The result of the double integration is

\[
G(\omega,t) = \frac{G_f(\omega)}{4\omega_1^2 \left[ \omega_0^2 - \omega^2 \right]^2 + (2\omega_0\omega_1)^2} \left[ 2(e_{t_0} + e_{t_1}) - 4e_{t_0}t_1 \cos(\omega_1(t_0-t_1)) \right]
\]

\[
+ \left[ \omega_1^2 + \omega^2 + \xi^2 \omega_0^2 \right] \left[ -8\omega_1 \omega e_{t_0}t_1 \sin(\omega_1(t_0-t)) \right]
\]

\[
+ \sin(\omega(t_0-t_1)) + \left\{ \sin^2(\omega_1 t) - \cos^2(\omega_1 t) \right\} \left[ 4\xi \omega_0 \omega_1 \right]
\]

\[
+ \left[ e_{t_0} \sin(2\omega_1 t_0) + e_{t_1} \sin(2\omega_1 t_1) - 2e_{t_0}t_1 \sin(\omega_1(t_0+t_1)) \right]
\]

\[
+ \cos(\omega(t_0-t_1)) + \left[ 2(\omega_1^2 - \omega^2) \right] \left[ -e_{t_0} \cos(2\omega_1 t_0) - e_{t_1} \cos(2\omega_1 t_1) \right]
\]

\[
+ 2e_{t_0}t_1 \cos(\omega_1(t_0+t_1)) \cos(\omega(t_0-t_1)) \right]
\]
Time-Limited Step Excitation

Figure D-1
\[ + \left[ 2\xi^2\omega_0^2 \right] \begin{align*} & e_{t_0} \cos(2\omega_1 t_0) + e_{t_1} \cos(2\omega_1 t_1) - 2e_{t_0} e_{t_1} \cos(\omega_1(t_0 + t_1)) \\
& \cos(\omega(t_0 - t_1)) \end{align*} \]
\[ + \left\{ \sin(2\omega_1 t) \right\} \begin{align*} & \left[ 4\xi\omega_0 \omega_1 \right] e_{t_0} \cos(2\omega_1 t_0) + \\
& e_{t_1} \cos(2\omega_1 t_1) - 2e_{t_0} e_{t_1} \cos(\omega_1(t_0 + t_1)) \cos(\omega(t_0 - t_1)) \end{align*} \]
\[ + 2 \left[ \omega_1^2 - \omega_2^2 - \xi^2\omega_0^2 \right] \begin{align*} & e_{t_0} \sin(2\omega_1 t_0) + e_{t_1} \sin(2\omega_1 t_1) - 2e_{t_0} e_{t_1} \\
& \sin(\omega_1(t_0 + t_1)) \cos(\omega(t_0 - t_1)) \end{align*} \] \hfill (D-2)

where
\[ e_{t_0} = e^{2\omega_0 \xi t_0} \]
\[ e_{t_1} = e^{2\omega_0 \xi t_1} \]
\[ e_{t_0} e_{t_1} = e^{(t_0 + t_1)\omega_0 \xi} \] \hfill (D-3)

Equation [D-2] may be expressed more compactly as
\[ G(\omega, t) = \frac{1/(4\omega_1^2)}{(\omega_0^2 - \omega^2)^2 + (2\xi\omega_0 \omega)^2} \begin{align*} & \left[ \beta + \mu \cos(\omega(t_1 - t_0)) \right] \\
& - \epsilon \sin(\omega(t_1 - t_0)) + \lambda \omega^2 + \tau^2 \omega \cos(\omega(t_1 - t_0)) \end{align*} \] \hfill (D-4)

where
\[ \beta = (2e_{t_0} + 2e_{t_1})(\omega_1^2 + \omega_2^2) + [\sin^2(\omega_1 t) - \cos^2(\omega_1 t)] \]
\[ \left[ 4\xi\omega_0 \omega_1 e_{t_0} \sin(2\omega_1 t_0) + 4\xi\omega_0 \omega_1 e_{t_1} \sin(2\omega_1 t_1) - \\
2\omega_1^2 e_{t_0} \cos(2\omega_1 t_0) - 2\omega_1^2 e_{t_1} \cos(2\omega_1 t_1) + 2\xi^2\omega_0^2 e_{t_0} \cos(2\omega_1 t_0) + \\
\right. \]
\[+ 2\xi^2 \omega^2 e_{t_1} \cos 2\omega t + 2 \sin \omega t \cos \omega t [4\xi \omega^2 \omega e_t \cos 2\omega t_0 +
\]
\[+ 4\xi \omega^2 \omega_0 e_{t_1} \cos 2\omega t_1 + 2\omega^2 e_{t_0} \sin 2\omega t_0 - 2\xi^2 \omega^2 e_{t_0} \sin 2\omega t_0 + 2\omega^2 e_{t_1} \sin 2\omega t_1 - 2\xi^2 \omega^2 e_{t_1} \sin 2\omega t_1\]

\[\mu = \gamma + \delta \sin^2 \omega t - \delta \cos^2 \omega t + \rho \sin \omega t\]
\[\epsilon = 8\omega e_{t_0 t_1} \sin(\omega(t_1-t_0))\]
\[\lambda = 2e_{t_0} + 2e_{t_1} + \eta \sin^2 \omega t - \eta \cos^2 \omega t - \phi \sin^2 \omega t\]
\[\tau = -4e_{t_0 t_1} \cos(\omega(t_1-t_0)) - 4e_{t_0 t_1} \sin^2 \omega t \cos(\omega(t_0+t)) + 4e_{t_0 t_1} \cos^2 \omega t \cos(\omega(t_0+t)) + 4e_{t_0 t_1} \sin^2 \omega t \sin(\omega(t_0+t))\]
\[\gamma = -4\omega^2 e_{t_0 t_1} \cos(\omega(t_1-t_0)) - 4\xi^2 \omega^2 e_{t_0 t_1} \cos(\omega(t_1-t_0))\]
\[\delta = -8\xi \omega \omega_0 e_{t_0 t_1} \sin(\omega(t_1+t_0)) + 4\omega^2 e_{t_0 t_1} \cos(\omega(t_1+t_0))\]
\[-4\xi^2 \omega^2 e_{t_0 t_1} \cos(\omega(t_0+t))\]
\[\eta = 2e_{t_0} \cos 2\omega t_0 + 2e_{t_1} \cos 2\omega t_1\]
\[\rho = -8\xi \omega \omega_0 e_{t_0 t_1} \cos(\omega(t_0+t_1)) - 4\xi^2 \omega^2 e_{t_0 t_1} \sin(\omega(t_0+t_1)) +
\]
\[+ 4\xi^2 \omega^2 e_{t_0 t_1} \sin(\omega(t_0+t_1))\]
\[\phi = 2e_{t_0} \sin(2\omega t_0) + 2e_{t_1} \sin(2\omega t_1)\]

(D-5)

The variance for \( t > t_1 \) is
$$\lambda_0 = \frac{G_f(\omega_0)}{32\omega_0^2\omega_1^3} \left[ e^{-\omega_0\xi(t_1-t_0)} \sin(\omega_1(t_1-t_0)) \left\{ \mu - \frac{\omega_0}{\xi} e^{-\omega_0^2\tau} \right\} + e^{-\omega_0\xi(t_1-t_0)} \cos(\omega_1(t_1-t_0)) \left\{ \frac{\omega_0}{\xi} \frac{\omega_1\omega_0}{\xi} \right\} + \frac{\omega_1}{\omega_0^2} + \frac{\omega_1\omega_0}{\xi} \lambda \right] \quad (D-6)$$

For small damping levels, Equation [D-6] takes approximately the following form:

$$\lambda_0 \approx \frac{G_f(\omega_0)}{8\xi\omega_0^3} \left[ e^{2\omega_0\xi(t_1-t)} - e^{2\omega_0\xi(t_0-t)} \right] \quad (D-7)$$

The convergent part of the first moment is assumed by analogy to the evolutionary response to step input. For small to moderate damping levels and $t > t_1$,

$$\lambda_1 \approx \frac{G_f(\omega_0)}{8\xi\omega_0^2} \left( 1 - \frac{2\xi}{\pi} \right) \left[ e^{2\omega_0\xi(t_1-t)} \left\{ 1 + \frac{2\omega_0\xi}{\omega_1} \sin(\omega_1 t_1) \cos(\omega_1 t_1) \right\} - e^{2\omega_0\xi(t_0-t_1)} - \frac{G_f(\omega_0)}{2\pi\omega_0} \left\{ 1 + \frac{\omega_0\xi}{\omega_1} \sin(\omega_1(2t-t_0-t_1)) \right\} \right] e^{\omega_0\xi(t_0+t_1-2t)} \quad (D-8)$$

For small damping levels Equation [D-8] becomes

$$\lambda_1 \approx \frac{G_f(\omega)}{8\xi\omega_0^2} \left( 1 - \frac{2\xi}{\pi} \right) \left[ e^{2\omega_0\xi(t_1-t_0)} - e^{2\omega_0\xi(t_0-t)} \right] + \frac{G_f(\omega_0)}{2\pi\omega_0^2} e^{\omega_0\xi(t_0+t_1-2t)} \quad (D-9)$$
The convergent part of the second moment for \( t > t_1 \) is

\[
\lambda_2 = \frac{G_f(\omega_o)}{32w_1^2} \left[ e^{-\omega_0 \xi (t_1-t_0)} \sin(\omega_1(t_1-t_0)) \right] \left[ \frac{e^{-\rho \rho \xi (t_1-t_0)}}{\xi} + 3\omega_o^2 \tau + 4\omega_o^2 \xi^2 \tau \right] + e^{-\omega_0 \xi (t_1-t_0)} \text{cps}(\omega_1(t_1-t_0)) \left[ \frac{\omega_1}{\omega_0} \xi \mu \frac{\omega_1 \omega_0}{\xi} \beta + \frac{\omega_1 \omega_0}{\xi} \lambda - 4\omega_1 \omega_0 \xi \lambda \right]
\]

For small damping levels Equation [D-10] reduces to

\[
\lambda_2 \approx \frac{G_f(\omega_o)}{8 \xi \omega_o} \left[ \frac{2\omega_o \xi (t_1-t)}{e^{2\omega_o \xi (t_1-t)}} - \frac{2\omega_o \xi (t_0-t)}{e^{2\omega_o \xi (t_0-t)}} \right]
\]

\[\approx \omega_o^2 \lambda_0 \]

With Equations [D-6], [D-8], and [D-10] for the time-limited step function, it is tempting to solve a nonstationary problem by breaking it up into separate time regions, and solving each region separately. The fallacy is that unless the input in each time interval is independent from other intervals (orthogonal input intervals) the response in the different regions is not independent. Therefore, the response cannot in general be solved in each region without taking into account the correlation from one interval to the next due to the correlation of the excitation.
APPENDIX E

DERIVATION OF THE RELIABILITY FUNCTION

The small damping convergent expressions for the moments given in Equation [VIA-4] are used.

$q(t)$ was given in Equation [IVB-1] and $\alpha_0(t)$ in Equation [IB-30]. For the reliability function, $q(t)/\sqrt{\lambda_0(t)}$ is needed.

\[
q(t) = \frac{q(t)}{\sigma(t)} = \sqrt{\frac{4\xi}{\pi}} \left[ 1 + e^{-2\omega_0 \xi t} \right] \left[ 1 - \frac{\xi}{\pi} \left( 1 + e^{-2\omega_0 \xi t} \right) \right] \left[ 1 - e^{-2\omega_0 \xi t} \right] \frac{G_f(\omega_0)}{8\omega_0^3 \xi^2} \left[ 1 - e^{-2\omega_0 \xi t} \right]
\]

(E-1)

Equation [E-1] simplifies to

\[
q(t) = \sqrt{\frac{4\xi^2}{\pi} \left[ \pi - \pi e^{-2\omega_0 \xi t} - \xi e^{-2\omega_0 \xi t} - \xi e^{-4\omega_0 \xi t} \right] \frac{G_f(\omega_0)\pi^3}{4\omega_0^3} \left[ 1 - e^{-2\omega_0 \xi t} \right]^3}
\]

(E-2)

If $\xi$ is neglected with respect to $\pi$, Equation [E-2] becomes

\[
q(t) = \sqrt{\frac{16\omega_0^3 \xi^3}{\pi^3 G_f(\omega_0)} \left[ \pi - \pi e^{-2\omega_0 \xi t} \right] \left[ 1 - e^{-2\omega_0 \xi t} \right]^3}
\]

(E-3)

which further simplifies to
\( g(t) = \frac{2\xi \omega_0}{1 - e^{-2\omega_0 \xi t}} \sqrt{\frac{8\omega_0}{\pi G_f(\omega_0)}} \left[ 1 + e^{-2\omega_0 \xi t} \right] \) \( \text{(E-4)} \)

\( \alpha_D(t) \) is then

\[
\alpha_D(t) = 2\nu_0 \frac{1}{\exp \left\{ \frac{a^2}{2\lambda_0(t)} \right\} - 1}
\]

\[
= 2\nu_0 \frac{1}{\exp \left\{ \frac{4\xi a^2 \omega_0^3}{G_f(\omega_0)} \left[ 1 - e^{-2\omega_0 \xi t} \right] \right\} - 1} \]

\( \text{(E-5)} \)

The reliability is

\[
L_D(t) = e^{-\int_0^t \alpha_D(u) du}
\]

\( \text{(E-6)} \)

The approximate expression for \( \alpha_D(t) \) given in Equation [E-5] is technically only valid for \( t > \pi/\omega_0 \xi \), since the expression for \( \lambda_1(t) \) is very approximate for times less than this. An examination of the error introduced in using Equation [E-5] in Equation [E-6] for all \( 0 < u < t \) has indicated that the error in the reliability is negligible for all times greater than \( T_\omega/2\pi \) where \( T_\omega \) is the natural period of the oscillator.

In order to obtain an analytical solution for the integral of Equation [E-5] some approximating assumptions had to be made. One of
these approximations is

\[ \sqrt{1 + e^{-2\omega_o \xi t}} \approx 1 \]  
(E-7)

in the exponent in the numerator. Another is

\[ \exp \left\{ \frac{\mu^2}{2} \left[ \frac{1 - e^{-2\omega_o \xi t}}{1 - e^{-\xi t}} \right] \right\} \gg 1 \]  
(E-8)

where \( s \) is the duration of the excitation and \( \mu \) is defined in Chapter IV as a nondimensional barrier level. Equation [E-8] is good for \( \mu > 2 \). \( \alpha_D(t) \) then becomes

\[ \alpha_D(t) = 2v_0 \left\{ 1 - \exp \left\{ \frac{-4a\xi\omega_0}{1 - e^{-2\omega_o \xi t}} \sqrt{\frac{\omega_0}{G_f(\omega_0)}} \right\} \right. \]
\[ \left. \exp \left\{ \frac{4\xi a^2 \omega_0^3}{G_f(\omega_0)} \left[ -2\omega_o \xi t \right] \right\} \right\} \]  
(E-9)

Equation [E-9] expression may now be integrated by using the exponential integral, \( E_1(z) \), defined in Chapter VI. The result is

\[ \int_0^t \alpha_D(u) du = \frac{v_0}{\omega_o^{\xi}} \left[ E_1(z_1) - E_1(z_2) - e^{-\gamma_1} E_1(z_2) - e^{-\gamma_2} E_1(z_3) \right] \]
\[ \Delta = I_\alpha(t) \]  
(E-10)

where \( \gamma_1, \gamma_2, z_1, z_2, z_3, \) and \( z_4 \) are defined in Equation [VIA-6]. The reliability function is now

\[ L_D(t) = A_D e^{-2v_0 I_\alpha(t)} \]  
(E-11)