ELECTROSTATIC INSTABILITIES IN
THE OUTER MAGNETOSPHERE

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Electrostatic instabilities in magnetospheric plasmas are considered. The plasma models used cover the range of possible magnetospheric plasmas. The plasma parameters are adjusted so that the predicted spectrum of the instabilities matches the observed spectrum in the magnetosphere (Kennel, et al., 1970). The observed waves are mainly near 3/2 of the electron gyro-frequency with some observations at higher half harmonic and harmonic frequencies as well. These waves are observed mainly in the trapping region between L=4 and 10. The instability criteria and the observed wave spectrum of these waves are found to require rather special plasma parameters. One must have a cold as well as a warm species of electrons, with a density ratio of greater than $10^{-2}$. The temperature ratio of the two species is approximately 0.1. The warm species has an anomalous distribution function which has to be non-monotonic in $v_e$. These parameters are consistent only with those for the trapping region.

Particle diffusion in velocity space due to these electrostatic waves is also studied. The preliminary results show that the diffusion is strong for lower energy electrons ($\leq 1$ kev) during periods of moderate wave amplitude ($\leq 10$ mv/m). Electrons with energy up to a few tens of kev are strongly diffused only when the wave amplitude is large ($\geq 100$ mv/m). It is predicted that substantial acceleration as well as pitch-angle diffusion of the electrons can be caused by these waves.

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Chapter 1

INTRODUCTION

Electrostatic waves are usually more important than the electromagnetic waves in laboratory plasmas. However, because of electronic technology difficulties, electrostatic waves were not successfully observed in the magnetosphere until recently (Kennel et al, 1970; Scarf et al, 1971). The observed wave frequency, $\omega$, is found to be larger than the local electron gyrosfrequency, $\Omega$. The waves occurring most frequently have $1.25 \leq \omega/\Omega \leq 1.75$, while high frequency waves, $\omega/\Omega \sim n > 1$, have also been observed occasionally. Cyclotron resonant modes, $1 < \omega/\Omega < 1.25$, have been observed only infrequently, and lower frequency waves have not been reported in any case. The wave length and wave vector orientation are not known. The wave amplitude ranges from a few tenths of a mv/m to 10 mv/m, and large amplitudes as high as 100 mv/m have also been observed during magnetic substorms. The waves seem to be confined in the region of $\pm 10^\circ$ geomagnetic latitude, and between the magnetic shells having equatorial distances of four to ten earth radii. They have been observed on more than sixty per cent of the satellite crossings through this region. The waves seem to occur more frequently on the morning side of the magnetosphere than on the afternoon side.

Extremely strong half harmonic waves ($\geq 100$ mv/m) at $\omega/\Omega = 1.5$ have been observed during the expansion phase.
of a substorm (Scarf et al, 1971), when the satellite was on
the magnetic shell of the auroral oval. Simultaneous observation
of a high flux of energetic electrons ($>49$ kev) and corre-
lated changes in the pitch-angle distribution are also re-
ported. Following the peak amplitude of the half harmonic
wave, a whistler chorus is observed with considerable intensi-
ty. Scarf and co-workers (1971) suggest that the observed elec-
trostatic waves are strong enough to control the local dis-
tribution function, and that the following electromagnetic
whistler chorus is a consequence of the distribution function
being changed by the electrostatic waves.

The purpose of this thesis is to make an extensive
theoretical investigation of these electrostatic waves and
to explore the importance of these waves in the magnetosphere.
The relevant observations of the magnetospheric plasma will
be reviewed in the second chapter for later reference.
Various instabilities and waves in the magnetospheric plasma
will also be discussed in the second chapter. The third
chapter discusses the linear analysis of the possible flute-
like and non-flute instabilities in different frequency
ranges. Some of the wave characteristics and plasma para-
eters are calculated theoretically, and some experimental
observations are checked against the theoretical results.
Hence, not only definite identification of the waves can be
made, but also some important characteristics of the plasma
can be determined by using the wave characteristics obtained.
The fourth chapter discusses the quasi-linear theory of the
wave–particle interaction and the associated diffusion tensor. These calculations show that the electrostatic waves are definitely important in accounting for the magnetospheric electron acceleration and pitch-angle diffusion. A conclusion and proposed future work are the subject of the final chapter.
Chapter 2
PLASMAS AND WAVES IN THE
MAGNETOSPHERE

2.1 Introduction

The spatial and time variation of plasmas in the magnetosphere is very complicated. The kinetic energy per particle ranges from about 1 ev to greater than 1 Mev. Satellite data in the past have been mainly conducted in the higher energy range (≥40 Kev for electrons and ≥1 Mev for protons). Only in recent years has accurate detection of the lower energy particles been made possible by the use of electrostatic analyzers and channeltron particle detectors (Frank, 1967). However, data on cold particles (~1 ev) and at higher latitudes (≥40°) are still either inadequate or unavailable. Information on the velocity distribution function of particles, which is needed for kinetic analysis of instabilities, is also scarce.

Several distinct regions of intense particle flux can be identified by the order of their distances from the earth (see Fig. (2-1)). The innermost region, L<4, where L is the McIlwain parameter (Hess, 1963), is the densely populated plasmasphere, which is enclosed by a region (4≤L≤10) having much smaller density, called the trapping region. Beyond this, an unstable zone, i.e. the cusp, extends out to the magnetopause on the day side, and out to L=15 on the night side. The natural extension of the cusp into the
Fig. (2-1) Schematic drawing of the magnetosphere enclosed by the streaming solar wind from the sun. The geomagnetic south pole which sits near the geographic north pole is pointing upward. The geomagnetic axis and the sun are in the plane of the diagram. The plasmasphere(I), trapping region (II), cusp(III), plasma sheet(IV), and the magnetotail(V) are indicated on the diagram.
magnetotail is the plasma sheet. It is approximately 4 to 6Re thick (1Re ≈ 6400 km is the radius of the earth), and stretches across the magnetotail with the embedding neutral sheet near its plane of symmetry.

For detailed information on the magnetosphere, we refer the reader to some of the standard reference books (McCormac, 1966; McCormac, 1968; Hess, 1968; Carovillano, 1963). In this work we will review in the next section only those properties which are closely related to the subject of this thesis, e.g. particle density, density variation, temperature variation. Based on that information, a reliable plasma model can be established, and our theoretical conclusions in the following chapters can be checked and extended. Waves and instabilities other than the type of interest to us will be reviewed in the third section so that the importance of the electrostatic instabilities in the magnetospheric plasma can be assessed relative to other kinds of possible waves and instabilities.
2.2 Magnetospheric Plasmas

The solar wind and the ionosphere are the two main sources of particles in the magnetosphere. Because of the vast temperature difference between these two particle sources, the particles in the magnetosphere can be roughly divided into two categories, namely one warm and one cold species.

The particle population within the plasmasphere is predominantly that of a cold species (~1 ev). The density of these cold particles falls off in the radial direction something like the inverse third or fourth power of the distance from the earth's center (Carpenter, 1966). A sudden drop of almost two orders of magnitude has usually been detected at the plasmapause by both whistler dispersion analysis (Carpenter, 1966) and satellite observation (Taylor et al, 1965). The equatorial density just inside the plasmapause is about 400/cm$^3$. Information on the cold particle density along the field line is less extensive. However, the work by the above authors indicates that the density discontinuity is a field aligned feature. It is estimated that the density along a field line falls off like the inverse third or fourth power of the distance along the field line from the top of the ionosphere.

The warm particles of solar wind origin (>> 1 ev), whose density in the plasmasphere is much less than that of the cold particles, carry a kinetic energy much larger than that of their cold counterpart. The total kinetic energy
density in the outer region of the plasmasphere \((3 \leq L \leq 4)\) is roughly an order of magnitude smaller than the local magnetic energy density during a magnetic storm (Frank, 1967), and is at least another order of magnitude smaller during quiet periods.

Going outward across the plasmapause, the cold particle density drops sharply, while the warm particle density variation is less apparent. Comparisons between the satellite data and the whistler results show that the cold particle density outside the plasmapause \((\sim 1 \text{ cm}^{-3})\) is no longer dominant, but is only a small fraction of the warm particle density. Typically, the warm particle density is about \(1 \text{ cm}^{-3}\), but it can be as high as about \(10 \text{ cm}^{-3}\) during magnetic substorms. Fig.(2-2) shows a schematic drawing of the radial variation of the total density measured by the whistler technique at the equatorial plane.

The particle flux, which is a measurement primarily of the warm particles, is quite different in the trapping region from that in the plasmasphere. Numerous minor fluctuations are more evident in the trapping region (Forbush et al, 1962), while only major disturbances can produce significant effects inside the plasmapause. According to the convection theory, (Axford, 1969; Nishida, 1968; Freeman, 1968), the convected plasma flow from the magnetotail can penetrate deep into the trapping region. Therefore, any disturbance of the electric field, or particle flux, in the magnetotail is expected to
Fig. (2-2)
Average measurement of equatorial electron density by the whistler technique in the region near the earth and by satellite measurement in the region far away in the plasma sheet (Ref. Vasyliunas, 1970; Angerami and Carpenter, 1966). The radial distance is plotted in terms of earth radius.
cause some corresponding fluctuation of the particle flux in the trapping region. The fresh warm particles (1~3 kev) from the plasma sheet are more or less continuously supplied to the trapping region, while only a sporadic supply of such particles is transported inward across the plasmapause during the less frequent magnetic storms.

The plasma sheet is a region of high particle flux. The energy spectrum of the particle flux is nearly monoenergetic (1~3 kev) and similar to that of the auroral precipitation over the auroral oval. The warm particle density in the plasma sheet is ~1 cm⁻³, and the average kinetic energy density is the order of the average magnetic energy density.

Cold particles of ionospheric origin have never been successfully measured in the plasma sheet. The difficulty appears to lie in the large fluxes of ~1 ev electrons produced near the satellite by photoelectric and secondary emission. Measurement of the total plasma density by the technique of whistler dispersion is successful within the plasmasphere and trapping region. However, such measurements do not extend far enough out to provide data on densities within the plasma sheet. Nevertheless, a comparison between the electron density measurements beyond the plasmapause by the whistler technique and the density of the warm component within the plasma sheet by satellite indicates (Vasyliunas, 1970) that the ionospheric electrons either do not exist within the plasma sheet or constitute a component at most comparable to
the density of the warm electron population.

Except for sporadic electron spikes, or so-called electron islands, particle fluxes were until recently undetectable in the regions of the magnetotail outside the plasma sheet. Upper limits on the electron density can be deduced from the absence of a measurable flux in the OGO-3 Faraday cup. The density limit can be determined if the particle mean energy is given (Vasyliunas, 1970). If the mean energy is chosen as \( \sim 50 \) ev, then the density is \( \lesssim 5 \times 10^{-2} \text{ cm}^{-3} \), which agrees well with the data (mean energy \( \lesssim 100 \) ev, and density \( \lesssim 4 \times 10^{-2} \text{ cm}^{-3} \)) detected by Bame (1968). Hence, only a very low density of a quite cold species exists in the magnetotail outside the plasma sheet. If we assume that the cold electron density in the plasma sheet is the same order as that in the tail, the ratio of the cold particle density to the warm particle density in the plasma sheet is presumably comparable to or less than a few per cent.
2.3 Waves and Instabilities in the Magnetosphere

The inward convection of plasma is prohibited from penetrating into the "forbidden region" (Alfvén, 1963). The cross-tail electric field (~50 kv) is not strong enough to supply the necessary additional kinetic energy for those particles of >10 kev from the magnetotail to enter the region of L≤10. However, particles with energy even as high as a few Mev are present as trapped radiation inside L=10. Indeed, energetic particles of 10 to about 100 kev account for most of the kinetic energy of the ring currents, which are at L≈4 to 6. Hence, local acceleration and inward diffusion have to be responsible for these energetic particles. Without such mechanisms to replenish the trapped high energy flux, the radiation belts would disappear through the loss cone in a period of a few weeks in the plasmasphere and a few days in the trapping region (Van Allen, 1969).

We will not be particularly concerned with the radial diffusion, which has been shown both theoretically (Fälthamar, 1968) and experimentally (Vernov, et al, 1969) to be more effective for the equatorial mirroring particles than for those mirroring to high latitude. Rather, various kinds of waves and instability and the associated wave-particle interaction and velocity space diffusion will be discussed. The potential importance of the electrostatic waves will then be clear.

Wave-particle interaction can change the first two
adiabatic invariants of the particles, and hence cause pitch angle diffusion and/or acceleration. The waves can be either externally imposed or internally excited. The external source can be strokes of lightning, solar wind-magnetopause interaction, or interplanetary noise, with the resultant waves transmitted into and absorbed in the magnetosphere. Internal sources can be waves generated by unstable plasma states, such as sharp density gradients (Chamberlain, 1963; Hasegawa, 1969), internal currents (Swift, 1965), unfavorable magnetic curvature, or velocity distribution anisotropies (Chang and Pearlstein, 1965; Kennel and Petscheck, 1966; Cornwall, 1966; Eviatar, 1966). All these possible wave sources are probably present simultaneously or individually. Detailed studies have not yet clarified which or if any of these sources is dominant, and exactly how the wave-particle interaction changes the particle distribution in the magnetosphere.

The most commonly observed waves in the magnetosphere are the whistlers, (Stix, 1962), which belong to a family of electromagnetic waves of low frequency, $10^2-10^5$ Hz (Burtis and Helliwell, 1969; Helliwell, 1969). Kennel and Petscheck (1966) have calculated the pitch-angle diffusion coefficient caused by the unstable whistler mode. They have solved a simplified Fokker-Planck equation governing a steady state plasma ($\frac{\partial}{\partial t} = 0$) with the assumption of an equatorial particle source, and obtained an estimate of the upper limit on the trapped equatorial particle flux. Energetic electron
fluxes $\geq 40$ kev (Frank, 1965) and proton fluxes $\geq 120$ kev (Davis and Williamson, 1963) obey the calculated upper limit with only occasional exceptions.

Roberts and Schulz (1968) have shown that electromagnetic waves, e.g. the compressional Alfvén wave, at harmonics of the particle bounce frequency can efficiently change the second invariant of those particles mirroring near the equator. Roberts proposes (1969) that the combination of bounce resonance, effective at large pitch angles, and whistler diffusion, effective at smaller pitch angles, can account for the observed pitch-angle diffusion of radiation zone electrons.

However, whistler diffusion is effective only for high energy electrons ($\geq 10$ Kev) (Kennel, et al, 1970), while the main component of the plasma sheet electrons has much lower energy ($\leq 3$ Kev). Also, the statistical spatial distribution of whistlers (Russell et al, 1968; Dunckel and Helliwell, 1969) resembles that of the $\sim 40$ Kev electron precipitation more than it does the soft auroral electron precipitation (Jelley and Brice, 1967) of a few kev. Therefore, while cyclotron resonant whistlers and bounce resonant VLF (very low frequency waves) may be sufficient to explain the pitch-angle diffusion of $\sim 40$ Kev electrons in the magnetosphere, other mechanisms are needed to explain the strong precipitation of the soft auroral electrons and the energization of these soft electrons to much higher energy to supply the source of particles for the energetic precipitation associated with whistler emission.
3.1 Introduction

In kinetic theory, the linear dispersion relation is derived by solving the Vlasov equation and the complete set of Maxwell's equations self-consistently. The problem usually turns out to reduce to solving an integro-differential equation which is made up of a time integral and a triple integral in velocity space. The time integration is performed for a particle along its spiral orbit in the background magnetic and electric fields. In linear theory, the orbits are calculated ignoring the perturbations due to the wave field. The time integration gives the cumulative distribution perturbation, which is caused by destruction of the particles constants of motion by the wave perturbation. The density perturbation is obtained by integrating this distribution perturbation over velocity space. Similarly, the perturbations of the first and second moments of the velocity give the perturbations in the current and energy respectively. The dispersion relation is then obtained by requiring the perturbed fields, density, and current to satisfy Maxwell's equations. When the waves are electrostatic modes, Poisson's equation is used to replace Maxwell's equations.

The electrostatic approximation greatly simplifies the algebra. It is possible to decouple the electrostatic
perturbation from the electromagnetic component only if $(\omega_{pe}/c)^2 << 1$, (Bernstein, 1958; Callen and Guest, 1971), which corresponds to the cases of low density and short wavelength. This is approximately satisfied for the half harmonic waves in the outer magnetosphere if the electron density is $\leq 10^3 \text{cm}^{-3}$.

The dispersion relation obtained (Stix, 1962; Hall, Heckrotte and Kammash, 1965; Dory et al., 1965; Pearlstein et al., 1966) gives the relation between the frequency, $\omega$, and the wave-vector, $k$, for all allowed modes in the plasma. Some modes can grow in amplitude by drawing energy from the plasma and hence are called unstable modes. These unstable modes can arise from initial perturbations due to thermal noise and particle discreteness effects. Usually, the most interesting modes in plasmas are unstable ones.

It will be made clear in the following section that the model relevant to the electrostatic instabilities in the trapping region is an infinite, homogeneous and magnetized plasma. The general linear dispersion relation for such a model is the Harris dispersion relation (Harris, 1959)

$$I = \frac{1}{k^2} \sum_{\text{species}} \left( \frac{\omega^2}{k^2} \int d^3V \sum_{n=-\infty}^{\infty} \frac{J_n(k_o \frac{\partial V}{\partial x})}{k_n V_n + \eta \Omega_n \omega} \times \left[ \frac{\partial^2}{\partial V_n \partial x} - \frac{n \Omega_n}{\eta} \frac{\partial^2}{\partial V_n \partial x} \right] \right)$$

(3.1.1)

where $j$ signifies the $j^{\text{th}}$-species. The subscripts $\perp$ and $\parallel$ represent respectively the perpendicular and parallel components of a vector with respect to the background magnetic
field. Here, \( \omega_p = \sqrt{\frac{4\pi n_e e^2}{m}} \), \( \omega_p \), \( \Omega \), and \( v \) are the plasma frequency, wave frequency, wave-vector, gyrofrequency, and particle velocity respectively. Particle number density and particle mass are represented by \( N \) and \( m \).

The previously described time integration has been performed with the assumption that the imaginary part of \( \omega \) is positive so that the perturbation is exponentially small at \( t = -\infty \). Therefore, the dispersion relation is valid for temporally growing modes which are solutions in the upper-half complex \( \omega \)-plane. Analytic continuation is applied to extend the functions into the lower \( \omega \)-plane. The singularity of (3.1.1) at \( \frac{\hbar \Omega_p}{\Omega_i} + n \Omega_i - \omega = 0 \) is the manifestation of wave-particle resonance, and the infinite number of harmonics are the result of the finite wavelength to gyro-radius ratio. For zero magnetic field, \( \Omega_i = 0 \), (3.1.1) can be reduced to a simple one-dimensional integration along the direction of wave propagation, and all except the zeroth harmonic term, \( n = 0 \), vanish.

Suppose the distribution function \( f_j(\mathbf{v}) \) is separable in terms of the perpendicular velocity \( \mathbf{v}_\perp \) and the parallel velocity \( \mathbf{v}_\parallel \) for each species. Hence

\[
f_j(\mathbf{v}) = f_{j\parallel}(\mathbf{v}_\parallel) \times f_{j\perp}(\mathbf{v}_\perp).
\]

The Harris dispersion relation can then be written as

\[
\epsilon = 1 - \omega^2_p H(\omega, k) = 0,
\]
where $\mathcal{E}$ is the dielectric constant, $\omega_p$, is the plasma frequency of a reference species, and

$$H(\omega, \mathbf{R}) = \sum_{\mathbf{R}} \frac{\mathbf{\omega}_p}{\mathbf{R}} \sum_{n=0}^{\infty} \left\{ C_n(\lambda_i) \gamma(\delta_{n,i}) \right\}$$

is the dispersion function. $C_n(\lambda)$ and $D_n(\lambda)$ are respectively the usually defined $C$- and $D$-functions of the $n^{\text{th}}$ order (Guest and Dory, 1965):

$$C_n(\lambda) = \int_0^{\infty} 2\pi \mathbf{U}_i \, d\mathbf{U}_i \, J_n^1 \left( \sqrt{2} \lambda \frac{\mathbf{U}_i}{\alpha} \right) f(\mathbf{U}_i)$$

and $Y(z)$ is defined as

$$Y(\delta_{n,i}) = \delta_{n,i} \int_{-\infty}^{\infty} \frac{\mathbf{f}_i(u) du}{u - \delta_{n,i}}, \quad \delta_{n,i} = \frac{\mathbf{w} - \mathbf{\eta} \mathbf{\Omega}}{\mathbf{\kappa}_n \delta_{n,i}}, \quad \delta_{n,i} = \frac{\mathbf{w} - \mathbf{\eta} \mathbf{\Omega}}{\mathbf{\kappa}_n \delta_{n,i}}$$

where $\delta_{n,i}$ and $\delta_{n,i}$ are respectively the perpendicular and parallel thermal speeds, and $a_i = \sqrt{\frac{\mathbf{U}_i}{\alpha}}$ is the gyro-radius. When $f_i$ is non-monotonic, $\delta_{n,i};$ will be chosen as the speed at which $f_i(\mathbf{U}_i)$ is peaked. When $f_i(\mathbf{U}_i)$ is taken as Maxwellian, $Y(z)$ becomes the tabulated plasma dispersion function (Fried and Conte, 1961). Detailed characteristics of the parametric functions $Y$ and $C_n$, $D_n$ are described in Appendices A and B respectively.

An analysis of the dispersion relation is usually done
by first taking both $k$ and $\omega$ real. The dispersion relation is real with singularities at each harmonic of the gyro-frequency for flute-like modes (i.e. $k_{||}=0$). For modes with finite wavelengths along the magnetic field ($k_\perp \neq 0$), an imaginary part is introduced by the singular integral $\gamma(z)$, and the dispersion function, $H(\omega, k)$, remains finite at the harmonics. The imaginary part of the dispersion function, $\text{Im}H$, is proportional to the real part of the plasma conductivity, $\text{Re}\sigma$. The real part of the dispersion function, $\text{Re}H$, is proportional to the imaginary part of the plasma conductivity, $\text{Im}\sigma$. When $\text{Re}\sigma$ is positive, which corresponds to negative $\text{Im}H$, the plasma is positively dissipative to the wave. Conversely, negative $\text{Re}\sigma$ and positive $\text{Im}H$ correspond to a negatively dissipative plasma.

The most accurate and dependable method to study the possible unstable modes for a given $k$ is to use the Penrose criteria (Penrose, 1960). By defining

$$ F(u) \equiv \sum_j \omega_j^2 \int d^3V \int_j (\nu) \delta (u - \nu \cdot k / \lambda) $$

for an unmagnetized plasma, the necessary and sufficient criteria are

$$ F'(u) = 0 \quad \text{with} \quad \left\{ \begin{array}{l} F'(u_0 + \delta) > 0 \\ F'(u_0 - \delta) < 0 \end{array} \right. $$

and

$$ \frac{1}{\lambda} \int_{-\lambda}^{\lambda} \frac{F'(u)}{u-u_0} du > 0 $$
for a plasma to be marginally stable at \( \omega/k = U_0 \), stabilized at \( \omega/k = u_0 - \delta \), and unstable at \( \omega/k = u_0 + \delta \), where \( \delta \) simply represents a small positive deviation. For a uniformly magnetized plasma, an algebraically complicated but physically similar expression for \( F(u) \) has been worked out by McCune (McCune, 1965; McCune and Callen, 1970). The criteria (3.1.9), (3.1.10) remain unchanged for this case as long as the proper \( F(u) \) is used.

It can be shown that criterion (3.1.10) is equivalent to the requirement that \( \Re H > 0 \) for a pair of real \( \omega \) and \( k \), while (3.1.9) is equivalent to \( \Im H(\omega, k) = 0, \Im H(\omega + \delta, k) > 0, \) and \( \Im H(\omega - \delta, k) < 0 \). Therefore, on a Nyquist diagram, which is a plot of the complex \( H \) obtained by changing \( \omega \) from \(-\infty \) to \(+\infty \), marginal stability is at the point where the complex \( H \) contour crosses the positive \( \Re H \)-axis from the lower half plane. This criterion is used in numerical calculations of the marginal stability boundaries (Callen, 1968).

When the growth rate is small and slowly varying, a rough estimate of \( \Im \omega \) can be obtained by series expansion of the dispersion relation near a real frequency \( \omega_r \):

\[
\gamma(\omega, k) = -\left[ \Im H \frac{\partial \Re H}{\partial \omega} \right]_{\omega_r}.
\]

(3.1.11)

For a positive-energy wave, which has negative \( \partial \Re H/\partial \omega \), positive \( \Im H \) corresponds to instability and negative \( \Im H \) corresponds to damping. In contrast, for a negative-energy
wave, which has positive $\omega \, \frac{\partial R_{z} H}{\partial \omega}$, positive $\text{Im} H$ corresponds to damping and negative $\text{Im} H$ corresponds to instability (Guest, 1970). It is to be emphasized that (3.1.11) is valid only for slowly growing modes, and cases in which $H$ is a very slowly varying function of $\omega$. For the present case in the region of lower half-harmonic frequencies ($\omega \sim (n+1/2) \omega_\alpha$, $n=1,2,3$), the modes are positive-energy and relatively slowly growing. Hence, (3.1.11) is a good approximation.

The problem of convective versus absolute instability is of special interest to plasmas of finite geometry (Bers and Briggs, 1963; McCune and Callen, 1970). When a convectively unstable wave has such a large group velocity that it can propagate out of the system before its amplitude has grown to a significant value, the mode can be considered unimportant as far as the instability effects are concerned. However, the geometry of the magnetosphere can be safely considered as infinitely large, and all the convective as well as absolute instabilities will have an equal chance to develop measurable amplitudes.
3.2 Models of the Distribution Function

It is indicated by the satellite observations (Kennel et al., 1970) that the half harmonic waves are locally produced. Theoretically, the wavelengths, and hence the characteristic damping length of these fast oscillations cannot be very large compared with the electron gyro-radius. Therefore, local emissions rather than long distance propagation from outside have to be responsible for the waves. The conclusion of local emissions implies that the warm electron distribution function has to have some free energy available to drive the instabilities.

When plasma stability is studied by using the Vlasov equation, the selection of the model of the distribution function is of great importance. Distribution functions and/or the geometry of the plasma have to be either anomalous in $v$-space or inhomogeneous in $r$-space to make instabilities develop by extracting free energy from the plasma. Different types of instabilities can be identified by their corresponding distribution function anomalies or plasma inhomogeneities. For example, there are those which develop from a loss-cone distribution, a double-hump distribution, two-stream distributions, temperature anisotropy, temperature gradients, or density gradients. While different types of anomalies support unstable modes of different characteristics in frequency, wavelength, band width, and growth rate, combinations of anomalies tend to blur the distinctions.
All kinds of anomalous distributions persist almost everywhere in the magnetosphere. However, spatial inhomogeneity will be neglected, and only the velocity anomalies will be considered in our analysis. We justify this simplification as follows.

For a spatial inhomogeneity to be of any significance, the dominant particles interacting with the waves have to "feel" the inhomogeneity in the characteristic time period. In other words, the inhomogeneity across the characteristic length measured by the traveling particle in one characteristic time period has to be significant. For the waves with frequency comparable with the electron cyclotron frequency $\Omega_e$, the characteristic time is of order $\Omega_e^{-1}$. For an electron of 1 kev kinetic energy, the characteristic length in a magnetic field of 100G is

$$\sim 2 \times 10^4 \text{ km-sec}^{-1/2} \times 10^4 \text{ sec}^{-1} \sim 1 \text{ km}.$$ 

The characteristic scale lengths of variation for the magnetic field, particle flux, and density in the magnetosphere are typically more than $10^3$ km. Except near the plasmapause where drift waves caused by a density gradient are possible (Coroniti and Kennel, 1970), or in the neutral sheet where the sharp reversal of magnetic field may induce a sheet pinch instability (Coppi et al., 1966), the spatial inhomogeneity is negligible, at least for the higher frequency modes.

The geo-electric field of the convection motion can be removed by Lorentz transformation once the magnetic
field is considered homogeneous. Moreover, the $E \times B$ drift velocity is only the order of tens of km/sec, which is negligibly small even compared with the thermal speed ($\gtrsim 600$ km/sec) of the cold electrons of ionospheric origin. Hence, the instabilities are solely caused by velocity anisotropy, and the magnetized plasma can be considered as infinite and homogeneous with no background electric field.

When the wave frequency is the order of the electron cyclotron frequency, the ions are not responsive to the fast oscillation of waves and can be considered as a motionless background. Conversely, for the wave frequency in the order of ion cyclotron frequency, the electrons behave like a massless fluid responding to the waves with little inertia.

Satellite observations reveal that the electron distribution function can be very different for different energy ranges, at different places, times, and for different intensities of geo-magnetic activity. Hence, even for a fixed place in the magnetosphere, the distribution function for given magnetic activity can be very complicated. The usually assumed Maxwellian, Dory-Guest-Harris (see Appendix B), and Lorentzian distribution functions, which facilitate the complicated dispersion relation analysis, cannot be expected to realistically model the real distribution function even by decomposing the entire real distribution into a few separate species.

Equatorial observations by the geostationary satellite
ATS-5 at 6.6 $R_e$ cover a wide range of energy (50 ev~50 kev) of the electron and proton distributions parallel and perpendicular to the background magnetic field (DeForest and McIlwain, 1971). In Fig.(3-la) and Fig.(3-lb), we show velocity distributions for electrons and protons respectively, which have been obtained by transforming the differential energy flux data from satellite ATS-5. Only the perpendicular electron distribution constantly shows a non-monotonic feature. It has a region of positive slope not necessarily caused by the pitch-angle loss cone of the open ends of the magnetic field, but more likely related to the injection mechanism. The non-monotonic $\mathcal{V}$-distribution function has a sharp positive slope, a peak at about 100 ev, and a long tail extending beyond 40 kev. The corresponding differential energy flux is approximately constant between 100 ev and 40 kev, and it drops sharply at about 80 ev and slowly beyond 40 kev. The ratio of the flux at 100 ev and 50 ev is usually 10 to 50. Unfortunately, because of the photoelectric effect which builds up an electric potential over the surface of the satellite, data at low energy in Fig.(3-1) may be not dependable (McIlwain, 1971). Particle energy spectrum data below $\sim 10^2$ ev, as far as we know, is yet to be collected by any other satellite.

The fine details of the distribution function in the magnetosphere are probably not really essential for the instability analysis. This is because the physics behind
Fig (3-1a) Differential energy flux, $G$, from ATS-5 (DeForest and McIlwain, 1971), and the corresponding distribution function, $F$, of electrons. Subscripts $\perp$ and $\parallel$ represent the components measured perpendicular and parallel to the background magnetic field, respectively. $F$ is proportional to $\epsilon^2$ divided by the square of the corresponding energy component. The sharp slope in $F_\perp$ does not appear in $F_\parallel$ or in the distribution functions for protons.
Fig (3-1b) Differential energy flux, $G$, and the corresponding distribution function, $F$, of protons. No positive slope region can be detected, for this case, typical for protons.
the development of instabilities is usually not the fine
details, but rather some prominent features of the distri-
bution function. Those prominent features must possess a
large proportion of the free energy accessible to the waves.
The plasma should approach a more isotropic state if the
accessible free energy is being extracted by the waves.
Moreover, it will become clear later in the analysis that
the wave characteristics, which can be checked against ex-
perimental observation, are determined by the gross features
of the distribution and the experimental data will show that
the choice of a non-monotonic distribution function is the
correct one. Hence, we will proceed with our analysis by
assuming a non-monotonic $V_i$-distribution for the warm electrons,
and consider other types of unstable distribution functions
in due course.

Let us now turn to the cold species of ionospheric
origin. It is understood from the previous chapter that the
cold species dominates the density in the plasmasphere and
seems almost to be non-existent in the magnetotail and the
plasma sheet. In the trapping region where field lines are
closed and the warm particles convected from the plasma
sheet are fresh and abundant, the cold species is probably
a small fraction of the total density. Their energy is the
order of 1 ev, and can be higher when heated by the energetic
species. Their distribution function has not been experi-
mentally determined.
In conclusion, we assume an infinite, homogenous plasma in a uniform magnetic field. The electron distribution function can be regarded as being composed of two species: one cold species and one warm species. The cold species has a temperature of a few ev and its distribution function will be shown to be unimportant. Its density is taken to be between 0.1 and a few particles per cm$^3$. The warm species is taken as non-monotonic in $V/L$ and peaked at 100 ev. It has a long energetic tail in both the perpendicular and parallel directions. The density range lies between 0.1 and 10 per cm$^3$. It will be clear that it is the ratio of the temperatures of the two species rather than the magnitudes of the individual temperatures that counts for the instability. Thus, if the peak of the perpendicular distribution of electron appears at any energy higher than 100 ev, the energy range considered as cold should be proportionally increased; the corresponding "cold" electrons refer then not only to those of ionospheric origin but may also refer to those having lower energy which are of solar wind origin.

We will show in the linear dispersion analysis that the high-energy tail is of little importance for the occurrence of the instabilities. Nor does the distribution function of the cold species matter. The only important features are the existence of the cold species and a warm species, which has free energy accessible to the waves in the form of velocity-space anisotropies in the distribution function.
3.3 A Singular Case: The Flute-Like Mode

A realistic model of the plasma in the trapping region has been discussed in detail in Section 3.2. The model is an infinite homogeneous plasma in a uniform magnetic field. The electron distribution function is composed of two species: one cold and one warm species. Their temperature ratio is the order of $10^{-3} \sim 10^{-1}$. A closer estimate of their temperature ratio can be derived from the linear stability analysis and the wave observation. The warm species has a density between 0.1 to 10 cm$^{-3}$, while the cold species density is usually a fraction of it. However, cold particle densities less than 0.1 cm$^{-3}$ are believed to occur only infrequently in the trapping region. The warm species distribution is taken as non-monotonic in $v_\perp$ (i.e. "loss-cone like"). It has a long energetic tail in the perpendicular and parallel directions to the background magnetic field. We further assume that the distribution functions are separable as in (3.1.2) in terms of the perpendicular velocity, $v_\perp$, and the parallel velocity, $v_\parallel$, for each species. This assumption, although unrealistic, is necessary to facilitate the algebraic work. Finally, ions are neglected throughout the analysis.

The model under investigation is electrostatic, and hence expressions (3.1.2) to (3.1.7) will be used. The related notations have been described in the associated paragraphs. The subscript $j$ will be replaced by $c$ for the cold electrons and by $w$ for the warm electrons. The temperature anisotropy,
\( \alpha_{\perp} / \alpha_{\parallel} \), is assumed to be the order of unity. The quantities \( \alpha_{\perp} \) and \( \alpha_{\parallel} \) are the thermal speeds at which \( f_{\perp}(\omega) \) and \( f_{\parallel}(\omega) \) are respectively at their maxima. The warm electrons are chosen as the reference species, so that \( \omega_{p}^{\parallel} \) in (3.1.4) is replaced by \( \omega_{p}^{\perp} \). Detailed characteristics of the parametric functions \( Y \), and \( C_n, D_n \) are described in Appendix A and B.

The experimentally observed wave frequency is half harmonic (i.e. \( \omega_{n+1/2, n = 1, 2, 3, \ldots} \)). The most interesting range of frequency is the second half harmonic mode, i.e. \( n = 1 \), but higher harmonics will also be discussed.

The flute-like mode \( (k_\parallel = 0) \), which is the simplest case mathematically, will be discussed in this section and proved irrelevant to the unstable half harmonic mode, while the more complicated non-flute mode \( (k_\parallel \neq 0) \) will be discussed in Sections 3.4 and 3.5. The flute-like mode is separated from the rest of the regular cases because its instability criterion is different and the Penrose criteria in (3.1.10) and (3.1.11) are not applicable. Setting \( k_\parallel = 0 \), the general dispersion relation can be written as

\[
H(\omega, k_\perp) = \sum_{j = \text{species}} \frac{\alpha_{\perp}^{2}}{\omega_{j}} \sum_{n = -\infty}^{\infty} \frac{n_\perp \omega_{n} D_n(\lambda_j)}{\omega - n_\perp \omega_{n}} \tag{3.3.1}
\]

by keeping the first term of the series expansion in (A.2) for any type of distribution. Note that \( H(\omega, k_\perp) \) is real for real \( (\omega, k_\perp) \). When \( D_n(\lambda_j) \) is positive \( H_j \) approaches positive infinity near \( n_\perp \omega_{n} \) from the higher frequency side, and negative infinity from the other side. When \( D_n(\lambda_j) \) is
negative, the signs of the divergences are reversed. An unstable mode, for which $\text{Im}\,H = 0$ and $\text{Im}\,\omega > 0$, is possible only if there exists a minimum of $H(\text{Re}\,\omega, k_\perp)$ (See Fig.(3-2a)). The relevant instability criterion for an unstable mode lying between $n\lambda_2$ and $(n+1)\lambda_2$ is approximately that we have $D_n > 0$ and $D_{n+1} < 0$. It can be shown by a Taylor expansion near the frequency at the minimum $H$ that lower $H$ can be matched on the right side of $(3.3.1)$ by adding a small positive growth rate to the real frequency. Hence, marginal stability is reached at a density corresponding to the minimum $H$, and a further increase of the density destabilizes the mode.

The flute-like mode with no cold species has been studied extensively (Young et al, 1971 -- see Appendix C) for both narrow and broad widths of distribution functions and various positive slope regions. The numerical results show (see Fig.(C-1a) and (C-1b)) that for distribution functions having narrow positive slope regions instability is possible at $\omega \approx 1.1 - 1.2\lambda_2$. If the distribution function has a broad width, which seems to be the normal case in the magnetosphere, the threshold density is found to be relatively high ($\omega^{/\lambda_2^2} \geq 50$) even for very sharp positive slopes. Hence, besides the requirement of a sharp positive slope, instability is possible in the trapping region only if the warm density is higher than $\sim 5$ particles per c.c.. This is certainly not the normal case in the magnetosphere, and thus the flute-like mode with no cold electrons can be ruled out as a potential candidate for the very common half harmonic
Fig. (3-2a) A typical sketch of the dispersion function for the flute-like mode with an unstable mode at $\omega_m$. The corresponding threshold density is proportional to $H_w$. Fig. (3-2b) A typical sketch of the contribution, $H_c$, by the cold species to the dispersion function. The magnitude of $H_c$ is large compared with that of the warm species shown in Fig. (3-2a), if the density of the cold species is larger than one per cent of that of the warm electrons.
The flute-like mode with a cold species is obtained by adding the cold species contribution in Fig. (3-1b) to the warm species dispersion function in Fig. (3-1a). The dominant feature of the cold contribution above the first harmonic and away from the vicinity of each of the harmonics is the first harmonic contribution

\[ \left( \frac{\omega_{p_c}}{\omega_{p_w}} \right)^2 \frac{1}{\lambda_c} \frac{D_j(\lambda_c)}{\alpha_{j-1} \lambda_c} \]

which moves the minimum \( H, H_m \), between the first and the second harmonics upward and the corresponding frequency, \( \omega_m \), proportionally toward the right. \( H_m \) and \( \omega_m \) increase with a decreasing \( \alpha_c^2/\alpha_w^2 \). By increasing the density ratio \( N_w/N_c \) and the magnitude of negative \( D_2(\lambda_w) \), \( \omega_m \) can be decreased.

We have studied the marginal stability analysis numerically for a wide range of parameters. Various cases have been studied by changing \( \alpha_c^2/\alpha_w^2 \) from \( \sim 10^{-1} \) to \( \sim 10^{-3} \), and \( N_c/N_w \sim 2 \) to \( \sim 10^{-3} \), for the modified Dory, Guest and Harris distribution function in (B-7) for different values of the functional parameters \( v_0 \) and \( s \). The results are summarized as follows.

The threshold density can be greatly decreased by adding a small amount of \( N_c \). For the cases of sharp positive slope \( (e.g. v_0=0.8, s=10 \) and \( j=1 \) in (B-7)) and \( N_c/N_w \gtrsim 0.2 \), the wave frequency at minimum \( H \) is slightly less than \( 2 N_2 \), and decreases only slowly with decreasing \( N_c/N_w \). Unstable modes below the half harmonic are possible only when \( N_c/N_w \) is small \( (\ll 10^{-1}) \) and \( N_w \gtrsim 2 \text{ cm}^{-3} \). When \( N_c/N_w \) becomes very small,
the threshold density becomes higher than \( \sim 5 \text{ cm}^{-3} \) and the corresponding frequency approaches \( 1.1 - 1.2 \omega_{\alpha} \). For the cases of broader positive slope regions (i.e. \( s=0 \) in (B-7)), unstable modes below \( 1.5 \omega_{\alpha} \) are not feasible unless \( N_C/N_w < 10^{-2} \) and \( N_w > 3 \text{ cm}^{-3} \). Comparing the answer for the cases of \( s \neq 0 \) with those of \( s=0 \) for other parameters being equal, the threshold density and the corresponding \( \omega \) for the latter case are usually higher. This is because the negative magnitude of \( D_2(\omega_{\alpha}) \) for the distributions having broad positive slope regions is smaller than those for sharper positive slope. The second harmonic contribution, which is proportional to the magnitude of \( D_2 \), is thus less significant for the former cases than for the latter. It can be seen in Fig. (3-2a) that by decreasing \( D_2(\omega_{\alpha}) \) to zero, the positive branch of the first and the third harmonics will move toward the right and left respectively, and become connected near the second harmonic. The above topological description remains true for cases with and without cold species.

Hence, unstable flute-like modes above the second half harmonic are possible under normal magnetospheric conditions if the warm electron distribution has a sharp enough positive slope (e.g. \( f_{\text{DGH}}; j=1 \)). Unstable modes below the second half harmonic cannot be flute-like unless the positive slope of the warm species is very sharp and \( N_C/N_w \) is small (\( << 10^{-1} \)). Satellite observation of the second half harmonic mode, \( \omega/\omega_{\alpha} \sim 3/2 \), seems to indicate that the wave
frequency lies above the half harmonic as often as below it. Such lack of preferential frequency range above the half harmonic indicates that the observed second half harmonic waves are predominantly not flute-like.
3.4 Imaginary Part of the Dispersion Function

Following the study of the flute-like modes in section (3.3), the non-flute modes will be studied extensively for the same plasma model in the rest of this chapter. Though we will not explicitly mention it at each stage of the discussion, the qualitative and quantitative results obtained in the remainder of this chapter compare quite favorably with numerical computations of the various modes obtained from previously developed computer codes (Callen, 1968). This section will be devoted to the study of the imaginary part of the dispersion function, which has to be positive to ensure that the wave-particle resonances can be destabilizing. The real part of the dispersion relation, which determines the particle densities at which the modes occur, will be discussed in the next section.

Consider the near resonant frequency ranges, i.e. \(|\omega_n - \omega| \ll \Omega_5\), where \(\text{Im} \chi(\omega, k)\) is dominated by the contributions from the \(n^{th}\) harmonic. Hence, (3.1.4) is reduced to

\[
H = \sum_{j=c, w} \frac{\omega_{p_j}^2}{k^2 a_{nj}^2} \left[ C_n \left( \nu_{nj} \right) I_m \chi(\nu_{nj}) - \frac{2n}{\nu_{nj}} \left( \frac{\nu_{nj}}{\nu_{nj}} \right)^2 D_n \nu_{nj} I_m \chi(\nu_{nj}) \right].
\]

(3.4.1)

Let us take \(f_{\|j}(\nu_n)\) to be a Maxwellian distribution for illustration. The following approximations for small growth rates, from (A.4) and (A.5), are valid for both warm and cold species.
Since the positive slope of the warm electrons is the source of free energy, \( k_\perp \) has to be chosen to have the interaction between the waves and the warm electrons cause instabilities. This is possible only if \( k_\perp q_w > 1 \) in general. Mathematically, instabilities with \( n < \omega / \nu_w < n + 1, n > 0 \), are possible for a non-monotonic \( \nu \) -distribution if \( D_n \) is negative, which first occurs when the peak of \( f_{w\perp} \) or \( f_{w\perp}' \) coincides with the first maximum of \( J_n \left( \frac{\nu_w}{\nu_{\perp w}}, \frac{\nu_w}{\nu_{\perp w}} \right) \) given by (B.3), i.e.,

\[
\bar{k}_\perp \alpha_{w \perp} \geq 0.8 + 1.1 \eta \tag{3.4.4}
\]

With the above criterion in mind, let us consider \( C_n \) and \( D_n \) for the cold electrons. \( C_n \) is always positive, and \( D_n = C_n \) for a Maxwellian \( f_{lc} \), which will be chosen for the cold electrons. It can be seen in Fig. (B-3a) that if a weak non-monotonic \( \nu \) -distribution was chosen for \( f_{lc} \), \( D_n (\lambda_c) \) would still be positive and \( \Pi \) would not be greatly different from that of a Maxwellian \( f_{lc} \) if

\[
\bar{k}_\perp \alpha_{lc} < 0.8 + 1.1 \eta \tag{3.4.5}
\]

Hence, taking \( \alpha_{w \perp} = \alpha_{lc} = \alpha_c \), the imaginary contribution of the cold species to (3.4.1) can be written as
\[ \text{Im}H_c \sim \pi \frac{1}{2} \left( \frac{\omega_p}{\omega_{pu}} \right)^2 \frac{C_0(\lambda_c)}{k^2 \alpha_c^2} \left[ -2 \frac{\omega - n\Omega_2}{R_{\parallel} \alpha_c} - \frac{2n\Omega_0}{R_{\parallel} \alpha_c} \right] e^{-\left( \frac{\omega - n\Omega_2}{R_{\parallel} \alpha_c} \right)^2} \]

or,

\[ \text{Im}H_c \sim -\pi \frac{1}{2} \left( \frac{\omega_p}{\omega_{pu}} \right)^2 \frac{C_0(\lambda_c)}{k^2 \alpha_c^2} \frac{2\omega}{R_{\parallel} \alpha_c} e^{-\left( \frac{\omega - n\Omega_2}{R_{\parallel} \alpha_c} \right)^2} \]  

by using (3.4.2) and (3.4.3), where the symbol "Im" reads "the imaginary part of". Because of the factor \( \alpha_c^2 \alpha_c \) in the denominator of (3.4.6), the maximum magnitude of \( \text{Im}H_c \) is much larger than its warm counterpart for at least the first few harmonics. Because the cold electrons only tend to damp the waves, \( \text{Im}H_c \) is negative as expected. Fig.(3-3a) shows a sketch of \( \text{Im}H_c \) for \( N_c/N_w = 0.2, a_c/a_w = 0.3, k_\parallel a_w = 0.5, \) and \( k_\perp a_w = 3.0 \) on a relative scale ten times smaller than that of \( \text{Im}H_w \) (Fig.(3-3b)) from the warm electrons. The damping due to the cold species, which is centered near \( \omega/\Omega_2 = n \), has a width of about \( k_\parallel a_c \), and becomes exponentially small at

\[ \left| \frac{\omega - n\Omega_2}{R_{\parallel} \alpha_c} \right| \geq 1.5 - 2, \text{ or,} \]

\[ R_{\parallel} \alpha_c \leq 0.25 \sim 0.3. \]  

(3.4.7)

This approximate criterion, which sets an upper limit on \( \lambda_c \) for the unstable waves, prevents the neighboring harmonic cold contributions from overlapping and hence damping all possible instabilities. The ratio of the peak magnitudes of \( \text{Im}H_c \) at the \((n+1)\)th and \(n\)th harmonics is about \( C_{n+1}/C_n \). If \( \lambda_c < 1 \), as is the case for small or moderate \( k_\perp \), series expansion of \( C_n \) and \( C_{n+1} \) gives \( C_{n+1}/C_n \sim \lambda_c/(2(n+1) \ll 1 \), and hence the peak magnitude of \( \text{Im}H_c \) at successive harmonics
Fig. (3-3a) \( \Im H_c \) plot on a scale ten times smaller than that for \( \Im H_w \) in Fig. (3-3b). We have assumed \( N_c/N_w = 0.2 \), \( a_c/a_w = 0.3 \), \( \kappa_{\parallel a_w} = 0.5 \) and \( \kappa_{\perp a_w} = 3.0 \). The damping by the cold electrons has a frequency width of \( \sim \kappa_{\parallel a_c} \) and is centered near each harmonic.

Fig. (3-3b) \( \Im H_w \) plot for the same parameters as in Fig. (3-3a). A typical non-monotonic \( v_L \)-distribution has been assumed for the warm species, and the corresponding parametric functions are \( D_2 = -D_1 = 6.4 \times 10^{-2} \), and \( D_2 = 0 \). The distribution function is approximately \( f \), with \( j = 1 \), as defined in (B.2).
falls off rapidly as shown in Fig. (3-3a). The wave damping by the cold species, which is dominant at lower harmonics, is not important beyond \( \omega / \omega _c \sim 3 \), and completely negligible at higher harmonics. However, if we have \( \lambda _c > 1 \), such as the case for large \( k_L \), the peak magnitude of \( \text{Im} H_c \) at consecutive harmonics only decreases slowly, i.e. \( c_{n+1} / c_n \ll 1 \).

The imaginary contribution from the warm species, \( \text{Im} H_w \), is more complicated because \( \lambda _w > 1 \) and \( D_n \) can be negative. If \( \alpha _{\parallel w} = \alpha _{\perp w} \) the second term in the right hand bracket of (3.4.1) will generally dominate the first one. This is true especially when \( k_{\parallel} a_w \ll 1 \). Fig. (3-3b) shows a sketch of \( \text{Im} H_w \) for a non-monotonic \( f_{\perp w} \) when (3.4.1) is valid and \( k_{\parallel} a_w \ll 1 \) so that the first term in the bracket of the right side of (3.4.1) is negligible. The broken curves are for the individual harmonic contributions from (3.4.1) and the solid one is the total \( \text{Im} H_w \). Assuming \( D_2 = D_1 = 6.4 \times 10^{-2} \) and \( D_2 = 0 \), we show in the diagram that when \( k_{\parallel} a_w \ll 1 \) the contribution by \( C_n Z_t^* \), antisymmetric with respect to the \( n \)th harmonic, will appear only if \( D_n = 0 \) (where \( n \) is taken as 2). The harmonic contribution for a positive \( D_n \) (\( n=3 \)) is a negative hump centered near \( \omega / \omega _c = n \), and it is a positive hump for a negative \( D_n \) (\( n=1 \)). Hence, in the regions of small cold cyclotron damping, which must be away from the harmonics, instability is made possible by a negative \( D_n (\lambda _w) \). At higher harmonics, positive \( \text{Im} H \) will appear only if \( k_{\perp} \) is large enough to satisfy (3.4.4).
Because of the exponential decrease of $\tilde{f}_{\parallel w}$ for large arguments, $\text{Im}\tilde{H}_w$ is small if \( \frac{\omega - n\Omega_{\omega}}{\tilde{a}_{\parallel w}} \gg 1 \). Hence, the growth rate will be insignificant near the half harmonics unless \( \tilde{a}_{\parallel w} \gg 0.2 \). \hspace{1cm} (3.4.8)

This is a lower limit on $k_{\parallel}$ for unstable waves of significant growth rate. However, if a high-energy tail exists in $\tilde{f}_{\parallel w}$, as is seen by satellite observation in the trapping region, the lower limit for $k_{\parallel}$ will be proportionally smaller.

Suppose, for example, the peak of $\tilde{f}_{\parallel w}$ is located at $\sim 100$ ev while $\tilde{f}_{\parallel w}$ has a measurable tail out to about 10 kev. Then, the lower limit on $k_{\parallel}$ is given by \( \frac{\omega - n\Omega_{\omega}}{\tilde{a}_{\parallel}(10\text{kev})} \ll 1 \), or, $k_{\parallel} \tilde{a}_{\omega}(100 \text{ ev}) \gg 0.04$. Hence, for non-exponential $\tilde{f}_{\parallel w}$, (3.4.8) becomes

\[ \tilde{a}_{\parallel} \gg \left| \frac{\omega}{\Omega_{\omega}} - n \right|, \]  

or,

\[ \tilde{a}_{\parallel} \gg \frac{\tilde{a}_{\parallel w}}{a_{\omega}} \left| \frac{\omega}{\Omega_{\omega}} - n \right|, \] \hspace{1cm} (3.4.9)

where $n\Omega_{\omega}$ is the harmonic closest to the wave frequency and $a_{\omega}$ is the gyro-radius of the fastest particle in the tail for which $\tilde{f}_{\parallel w}(\nu_{\parallel} = \omega_{\omega})$ still has a measurable value, say, a few per cent of $\tilde{f}_{\parallel w}(\nu_{\parallel} = \omega_{\omega})$.

If $k_{\parallel} \tilde{a}_{\parallel w} > 1$, the overlapping of the warm harmonic contributions to $\text{Im}\tilde{H}$ is important and the term $C_n Z'$ becomes important. The approximate expression for $\text{Im}\tilde{H}_w$ in (3.4.1) is thus invalid for large values of $k_{\parallel}$. However, because of the overlapping of the cold cyclotron damping at large
values of \( k_{||} \), this case is stabilized and of little interest to us.

On the other hand, \( C_n Z' \) will dominate if \( f_{\omega \nu} \) is highly temperature anisotropic, i.e. \( (\alpha_{\omega \nu}/\alpha_{\parallel \nu}) > 1 \). Therefore, the first term on the right side of (3.4.1) is usually called the "temperature-anisotropy term", while the second term is the so called "loss-cone term".

If \( f_{\omega \nu} \) is not non-monotonic as we have assumed all along but temperature anisotropic, \( \text{Im} \text{H}_w \) will look like the sketch in Fig. (3-4a) where we assume that \( (\alpha_{\omega \nu}/\alpha_{\parallel \nu})^2 > 10 \), \( k_{||} a_{||\nu} = 0.5 \), \( C_1 = 6.4 \times 10^{-2} \), \( C_2/C_1 = C_3/C_2 = C_4/C_3 = 0.9 \) and \( D_1 = D_2 = 2D_3 = 2D_4 = C_1 \). The temperature-anisotropy term, \( C_n Z' \), dominates the frequency range lower than the fifth harmonic. Because of the antisymmetric character of \( \text{Im} Z' \) (see Fig. (A-2)), \( \text{Im} \text{H}_w \) is now negative above each harmonic and positive below it. \( \text{Im} \text{H}_w \) changes from positive to negative near each harmonic and from negative to positive near each half harmonic. The minimum frequency for \( \text{Im} H_w > 0 \) between two neighboring harmonics, \( n\omega_\nu \) and \( (n+1)\omega_\nu \), is determined by \( (\omega - n\omega_\nu) C_n (\omega_\nu) + (\omega - (n+1)\omega_\nu) C_{n+1} (\omega_\nu) \approx 0 \). It lies below the half harmonic if \( C_n < C_{n+1} \), and above the half harmonic if \( C_n > C_{n+1} \). If \( f_{\omega \nu} \) is a monotonically decreasing function we will have \( C_n > C_{n+1} \) for all \( n \), while \( C_n < C_{n+1} \) is possible only if \( f_{\omega \nu} \) has a positive slope region. In any case, the fastest growing mode will occur above each half harmonic, i.e.
Fig. (3-4a) \( \text{Im} H \) plot for a warm species having "strong" temperature-anisotropy. We have assumed \((\alpha_{\perp}/\alpha_{\parallel})^2 > 10, \) 
\( k_{\parallel}a_{\perp} = 0.5, \) 
\( C_1 = D_1 = D_2 = 2D_3 = \epsilon D_4 = 6.4 \times 10^{-2}, \) and 
\( C_2/C_1 = C_3/C_2 = 0.9. \) 
The positive \( \text{Im} H \) regions are confined solely to regions above each half harmonic. The dashed curves represent each individual harmonic contribution.

Fig. (3-4b) \( \text{Im} H \) for a warm species of "weak" temperature-anisotropy. We have assumed the same parameters as those in Fig. (3-4a) except \((\alpha_{\perp}/\alpha_{\parallel})^2 = 4. \)
\[ n + 0.5 \leq \omega / \omega_2 \leq n + 1, \quad (3.4.10) \]

if temperature anisotropy dominates. Slowly growing modes below the half harmonics are possible only if a positive slope region coexists with the temperature anisotropy. Positive \( \text{Im}H_w \) regions will be centered near the half harmonics only if the temperature anisotropy is "weak" compared with the non-monotonic feature.

A rough estimate (Soper and Harris, 1965) states that the temperature anisotropy dominates the first \( n \) harmonics for \( k_2 a_{2w} \) not very small compared to unity, and for

\[ \alpha_{-2w}^2 / \alpha_{+2w}^2 > 2n. \quad (3.4.11) \]

At the same time, higher harmonics are dominated by the "loss-cone terms". All of those terms will be negative if \( f_{-w} \) is a monotonically decreasing function. The gradual disappearance of the temperature-anisotropy character can be seen in Fig. (3-4b) beyond \( \omega / \omega_2 \sim 2 \).

By adding \( \text{Im}H_c \) and \( \text{Im}H_w \), we get the total \( \text{Im}H \), as shown in Fig. (3-5a) and (3-5b), which corresponds to the cases of positive slope in Fig. (3-3b) and the temperature anisotropy in Fig. (3-4a), respectively. It can be seen in these diagrams that in the near harmonic region, \( |\omega / \omega_2 - n| \leq k_2 a_c \) for \( n = 1,2 \), \( \text{Im}H \) is dominated by the damping due to the cold species which becomes negligibly small at higher harmonics. The positive-slope region of \( f_{-w} \) results in a negative \( D_1 \) and
Fig. (3-5a) ImH plot obtained by combining ImH_c in Fig. (3-3a) and ImH_d in Fig. (3-3b). The potentially unstable region lies for such cases of a non-monotonic warm species both above and below the half harmonic frequency.

Fig. (3-5b) ImH plot obtained by combining ImH_c in Fig. (3-3a) and ImH_d in Fig. (3-4a). The potentially unstable region for such cases of large temperature-anisotropy is always above the half harmonics.
hence a positive $\text{Im}H$ region between $1.3 \lesssim \omega/\Omega_k \lesssim 1.7$. This is a region of potential instability if the real part of the dispersion relation can also be satisfied there.

The positive $\text{Im}H$ region for the case of temperature-anisotropy is shown to lie above the half harmonic (Fig. (3-5b)), and this will be the case for any temperature-anisotropic $f_{1,w}$ with no positive slope region. The reason has been explained in the paragraph discussing Fig. (3-4).

Satellite observation of the second half harmonic mode, $\omega/\Omega_k \sim 3/2$, seems to indicate that the wave frequency lies above the half harmonic as often as below it. Such lack of a preferential frequency range above the half harmonic indicates that temperature anisotropy cannot be the main source of free energy for these waves, and thus a positive slope should be the dominant feature.

Let us now turn our attention to the high frequency region ( $\omega/\Omega_k \gg 1$). Considering a negative $D_n(\lambda_w)$, there are two cases with different characteristics of $\text{Im}H$, i.e. the "resonant" and the "non-resonant" cases. The criterion which differentiates the above two cases is given by the ratio, $R$, between the peak magnitudes of the relevant $n^\text{th}$ harmonic contributions of the cold and warm species, i.e.

$$R = \frac{N_c}{N} \left( \frac{\lambda_w}{\lambda_c} \right)^2 \left| \frac{D_n(\lambda_c)}{D_n(\lambda_w)} \right|$$

(3.4.12)

for $\omega/\Omega_k \sim n$. A case will be called "resonant" if $R < 1$ and $D_n(\lambda_w) < 0$, and it is called "non-resonant" if $R > 1$ and $D_n(\lambda_w) < 0$. Cases of positive $D_n(\lambda_w)$ are obviously of no
interest for a non-monotonic distribution functions, because no instability is possible. \( R \) is less than unity for the cases where the temperature ratio between the cold species and the warm species is very small or the magnitude of \( |D_n(\lambda_w)| \) can be made large. Assuming a Dory, Guest and Harris distribution function for the warm electrons (see Fig. (3-6)), we have found that for \( j=1 \) and \( \frac{\partial^2 \omega_c}{\partial \omega_r^2} \geq 10^{-2} \) the corresponding high frequency region is "non-resonant". The high frequency region is "resonant" for \( j=2, \omega_r \approx 10^{-2} \). The ImH's for the "resonant" and "non-resonant" cases are sketched in Fig. (3-6a) and (3-6b) respectively. For the "non-resonant" case, the Penrose criterion (3.1.9) can be satisfied at frequencies marked by \( A_1, A_2, \) and \( A_3 \) in Fig. (3-6b). Hence, as far as the first Penrose criterion is concerned, instabilities are possible in the vicinity above \( A_1, A_2, \) and \( A_3 \). However, it can been seen in Fig. (3-6a) that (3.1.9) cannot be satisfied for the "resonant" case, and no instability is possible.

The criterion on \( k_z \) for instability given by (3.4.4) remains valid in the high frequency region. The lower limit (3.4.9) on \( k_n \) for instability is valid for the "non-resonant" case. The reason is analogous to that for criterion (3.4.9). Namely, \( \text{Im} H_w \) will be too small for significant instability if \( \left| \frac{\omega / n_c - n}{\omega_w} \right| \geq 2 \). The upper limit (3.4.7) derived from the requirement to prevent the neighboring harmonic contributions from overlapping for the "non-resonant" case is reduced by a factor which depends on the relevant ratio.
Fig. (3-6) Sketch of ImH for comparison between the "resonant" case (3-6a) and "non-resonant" case (3-6b). The cold electron contribution at each high order harmonic, $n \gg 1$, is a negative "drop" of the ImH curve. The first Nyquist criterion (3.1.9) is satisfied at $A_1$, $A_2$, and $A_3$. 
(3.4.12) of the cold and warm electron contributions. As for the "resonant" case, because of the pure loss-cone characteristic both criteria (3.4.7) and (3.4.9) are invalid.

The important results of this section can be summarized as follows. The upper and lower limits on $k_\parallel$ for unstable lower frequency modes ($\omega \sim (n+1/2)\Omega_\perp$, $n=1, 2, 3$) have been found in (3.4.7) and (3.4.9) respectively. The lower limit on $k_\parallel$ has been shown to depend on the shape and extent of the high energy portion of $f_{\parallel\omega}$. The lower limit of $k_\perp$ for a non-monotonic $f_{\perp\omega}$ is given by (3.4.4). A marginal stability analysis has been performed numerically for different wave-numbers and plasma parameters, and the results support the above analytic criteria. The lack of a preferential frequency of the observed wave between $1<\omega/\Omega_\perp<2$ indicates that a positive slope rather than temperature anisotropy is the main source of free energy for the wave, but, the possibility of a "weak" temperature anisotropy in coexistence with a positive-slope is not ruled out. As for the high frequency waves ($\omega \gg \Omega_\perp$), we have found that for $j=1, d\xi/d\omega \gg 10^{-2}$, $\text{Im} H$ is "non-resonant" and the first Nyquist criterion (3.1.9) can be satisfied above each of the harmonics.
3.5 Real Part of the Dispersion Function

In the previous section, ImH has been studied in detail at both low and high harmonics of the electron gyro-frequency. Instability criteria with regard to the wavenumber have been obtained in (3.4.4), (3.4.7) and (3.4.9). Because of the existence of a long high-energy tail, the lower limit on the unstable \( k_{\parallel} \), given by (3.4.9) is much lower than that which would have been expected for an exponentially decreasing \( f_{\parallel\omega}(v_{\parallel}) \). The corresponding cold electron resonant damping, which overwhelmingly dominates ImH within a frequency half width of about \( k_{\parallel} \alpha_{\parallel c} \), is then only nonzero very close to the harmonics. Outside the narrow resonant damping region, the imaginary part of the dispersion function, ImH\(_w\), due to the non-monotonic nature of the warm electron distribution, can keep the total imaginary part of the dispersion function, ImH, positive. In other words, as far as ImH is concerned, except for the narrow resonant region, instability is possible for the entire frequency spectrum. Therefore, we have to find some other way to further restrict the unstable region to meet the experimental observations that only waves between \( 1.25 \omega_{\parallel c} < \omega_{\parallel c} < 1.75 \) have been frequently observed. It will be seen below that the answer lies in a consideration of the real part of the dispersion relation for the plasma densities of interest.

As in the last section, the most interesting frequency range is near the second half harmonic, \( \omega_{\parallel c} \sim 1.5 \), but higher
frequency regions will also be discussed later in the section.

Let us first consider the cold contribution, \( R_{eH_c} \), and concentrate on the lower frequency range. Using the asymptotic expressions (A.2) and (A.3) for \( Z \) and \( Z' \) in (3.1.4.), we can write,

\[
R_{eH_c} \approx \left( \frac{\omega_{pe}}{\omega_{pe}} \right)^2 \left( \frac{1}{k} \frac{a_{Tc}}{a_{Tc}} \right)^2 \sum_{n=-\infty}^{\infty} \left[ C_n(\lambda_c) R_n a_{Tc}^2 + D_n(\lambda_c) \left( \frac{\omega_{pe}}{\omega_{pe}} \right)^2 \frac{2n}{\omega_{pe} - n} \right]
\]

\[
= \frac{N_c}{N_w} \left( \frac{2}{k} \frac{a_{Tc}}{a_{Tc}} \right)^2 \left[ \sum_{n=1}^{\infty} \left( C_n(\lambda_c) R_n a_{Tc}^2 + \sum_{n=1}^{\infty} \left[ D_n(\lambda_c) \frac{2}{\omega_{pe}^2 a_{Tc}^2} \frac{\omega_{pe}^2 a_{Tc}^2}{\omega_{pe}^2 a_{Tc}^2 - n^2} \right] \right)
\]

\[
|\omega_{pe} - n| \gg k a_{Tc}
\]

for a Maxwellian \( f_{e\perp c} \), where \( N_c/N_w \) is the density ratio of the cold and warm electrons. By neglecting terms that are of order \( k a_{Tc}^2 \), we obtain from the above expression,

\[
R_{eH_c} \approx \frac{N_c}{N_w} \left( \frac{2}{k} \frac{a_{Tc}}{a_{Tc}} \right)^2 \sum_{n=1}^{\infty} \frac{2n}{\omega_{pe}^2 a_{Tc}^2 - n^2} D_n(\lambda_c),
\]

\[
|\omega_{pe} - n| \gg k a_{Tc}, (3.5.1)
\]

For a Maxwellian \( f_{e\perp c} \), we have \( D_n(\lambda_c) \sim \frac{1}{n!} (\frac{\lambda_c}{2})^n \), for \( \lambda_c < 1 \), and the ratio of the \( n \)th with respect to the \( (n+1) \)th harmonic contributions in (3.5.1) for non-resonant \( \omega \) is

\[
\frac{D_{n+1}}{D_n} \sim \frac{n+1}{n^2} \frac{\lambda_c}{2} \ll 1, \quad (3.5.2)
\]
If $\lambda_c < 1$, the contributions from the first harmonic dominate and, except for the second one, all the higher order harmonics are negligible. By taking a Maxwellian $f_{\perp c}$ for convenience of illustration, we have

$$R_e H_c \approx \frac{N_c}{N_{wr}} \left( \frac{R_e}{R} \right)^2 \left( \frac{1}{\omega \beta_{\perp}^2 - 1} + \frac{\lambda_c}{\omega \beta_{\perp}^2 - 4} \right),$$

$$| \omega / n_e - \eta | > k_{||} a_c \quad (3.5.3)$$

from (3.4.1), where the first term in the bracket represents the upper hybrid mode and the second term gives the most significant cold thermal correction.

Expression (3.5.3) is valid for all frequency ranges as long as $\lambda_c < 1$. It is also approximately valid for different shapes of $f_{\perp c}$ if the inequality in (3.5.2) is satisfied. Note that the minimum $k_{\perp}$ required for positive $\text{Im} H_w$ is proportional to $\omega$ and becomes large in the high frequency range (see (3.4.4)). Hence, the corresponding $\lambda_c$ for unstable high frequency waves can be larger than unity and thus make (3.5.3) invalid. However, in considering the lower frequency region, the case of large $k_{\perp}$, which invalidates (3.5.3), is not of interest to us because the growth rate (3.1.11) for a given real frequency decreases roughly as $k_{\perp}^{-3}$ for $k_{\perp}$ larger than the minimum value required for positive $\text{Im} H_w$. This rapid decrease arises from the combination of the $k_{\perp}^{-2}$ factor on the right side of (3.1.4), and a $k_{\perp}^{-1}$ factor from $C_n(\lambda_w)$ or $D_n(\lambda_w)$ for $\lambda_w > n^2$.

The real contribution by the warm electrons, $\text{Re} H_w$, is
not dominated by any particular harmonic. The magnitude of Re$H_w$ in the low frequency region is much smaller than that of Re$H_c$. Only when $\alpha_c/\alpha_w$ or $N_c/N_w$ is small can the second harmonic contribution of the warm electrons be significant compared with that of the cold electrons. The first harmonic contribution of the warm electrons is always negligible. The general form of Re$H$ can be obtained by adding the second harmonic contribution of the warm electrons to (3.5.3),

$$\text{Re}H = \frac{N_c}{N_w} \left( \frac{k}{\delta} \right)^2 \left[ \frac{1}{\omega^2 \nu^2 - 1} + \frac{\lambda_c}{\omega^2 \nu^2 - 4} \right] + \frac{D_2(\lambda_w)}{\delta^2 a_w} \frac{16}{\omega^2 \nu^2 - 4}. $$

$$|\omega/\nu - n| > \frac{k}{\delta} a_w (3.5.4)$$

and

$$\text{Re}H = \frac{N_c}{N_w} \left( \frac{k}{\delta} \right)^2 \left[ \frac{1}{\omega^2 \nu^2 - 1} + \frac{\lambda_c}{\omega^2 \nu^2 - 4} \right] + \frac{D_2(\lambda_w)}{\delta^2 a_w} \frac{8}{\nu^2 a_w} \frac{\omega^2 \nu^2 - 2}{\nu^2 a_w} \frac{1}{k a_w}, $$

$$\frac{k}{\delta} a_w > |\omega/\nu - n| > \frac{k}{\delta} a_c (3.5.5)$$

for $k a_w < 1$ and $a_{\parallel w} = a_{\perp w} = a_w$. Note that the warm electron contribution has different forms in the two regions separated by $|\omega/\nu - 2| = k a_w$. Similarly, in the resonant regions where $|\omega/\nu - n| < k a_{\parallel c}$, a series expansion replaces the large argument asymptotic expansion for the corresponding $n$th harmonic contribution to Re$H_c$. However, because of the large cold damping, these regions are of less interest to us.

By comparing the second and the third terms in (3.5.4),
we can see that the ratio of the second harmonic contributions of the warm electrons to the cold electrons is approximately

\[
\frac{\delta D_2(\lambda_w) N_w}{\lambda_w N_c (a_w/a_c)^2} \quad (3.5.6)
\]

When \( D_2(\lambda_w) \) is positive, the warm and cold second harmonic contributions work in the same direction in modifying the dominant cold upper hybrid feature. When \( D_2(\lambda_w) \) is negative, the two second harmonic contributions tend to cancel each other. Taking \( N_w/N_c = 5, a_c/a_w = 0.3 \) for example, the second harmonic contributions of the warm electrons is less than the cold electrons if \( 8D_2(\lambda_w)/\lambda_w^2 \lesssim 2 \times 10^{-2} \). Apparently, this may not be satisfied by the lower range of \( k_\perp \) for which \( D_1(\lambda_w) < 0 \) but \( D_2(\lambda_w) > 0 \). In other words, when \( D_2(\lambda_w) > 0 \) and hence the warm and cold second harmonic contributions work in the same direction in modifying the dominant cold hybrid feature, the warm contribution to \( \text{Re} H \) can easily be made larger than that of the cold for a wide range of \( N_w/N_c \) and \( a_w/a_c \). However, when \( D_2(\lambda_w) \) has a significant negative value (typically of order \( -5 \times 10^{-2} \)), in order to alter the cold second harmonic contribution, the corresponding \( \lambda_w \) is generally large enough for a broad non-monotonic \( f_{\perp,w} \) to keep the warm contribution only a small fraction of the cold contribution. Comparing Figs.(B-3a), (B-4b), (B-6a), and (B-6b), it can be seen that unless \( f_{\perp,w} \) has a very sharp positive-slope region, such as those in Fig.(B-6b), the above statement that \( (3.5.6) \) is less than unity remains valid. The
ReH obtained by multiplying (3.1.4) by $\omega_{p,w}/\omega_{pe}$ is shown (as the solid curve) in Fig.(3-7) in terms of the cold electron density in the lower frequency region. Here, we have taken $N_c/N_w=0.2, (a_{\perp c}/a_{\perp w})^2=0.09, k_{\parallel}a_{\parallel w}=0.5, k_{\perp}a_{\perp w}=3,$ and $a_{\perp w}=a_{\parallel w}$. In obtaining this diagram, we have assumed that $D_2(\lambda_w)=0$ in (3.4.5) to avoid complications due to the warm second harmonic contribution. The value of $k_{\perp}$ is so chosen that $D_1(\lambda_w)$ is negative and hence the corresponding $\text{Im}H_w$ can be positive for $\omega$ in the range $1<\omega/\omega_2<2$.

Because of the fast decrease ($\sim k_{\perp}^{-3}$) of the growth rate at large $k_{\perp}$, we expect the upper limit on $k_{\perp}$ for the unstable second half harmonic mode to be not much larger than the value we have chosen. The value $k_{\parallel}a_{\parallel w}=0.5$ is chosen to lie between the upper and lower limits for $k_{\parallel}$ defined by (3.4.7) and (3.4.9). $N_c/N_w=0.2$ is simply a rough guess at the normal physical condition in the trapping region. We will show that $(a_{\perp c}/a_{\perp w})^2=0.09$ is a dependable result deduced from a comparison of the linear analysis and the satellite observations.

The vertical axis is scaled in terms of $(\omega_{pc}/\omega_e)^{-2}$, which is proportional to the inverse of the cold electron density $N_c$, in order to show that the real frequency along the horizontal axis is solely determined by the density of the cold electrons. Note that for $B\approx 100r$, we have $(\omega_{pc}/\omega_e)^2 \sim 10 \cdot N_c$, where $N_c$ is to be given as the number of particles per cubic centimeter. Assuming $N_c \geq 0.1 \text{ cm}^{-3}$,
Fig. (3-7) Diagram for ReH measured in terms of $(\omega_{pe}/n_e)^2$ in the lower frequency region. The cold electron first harmonic contribution is represented by the dashed curve. Typical magnitudes have been assumed for the parameters: $N_c/N_\nu = 0.2$, $\alpha_c^2/\alpha_\nu^2 = 0.09$, $k_{||}a_\nu = 0.5$, and $k_{\perp}a_\nu = 3$. 
as is presumably the normal condition in the trapping region, we obtain $\omega/\Omega_e < 1.35$ from the undistorted portion of the cold upper hybrid branch in Fig. (3-7). This agrees well with the satellite observation that wave frequencies have seldom been observed in the range $1 < \omega/\Omega_e < 1.25$. Because of the cold electrons, the real part of the dispersion relation does not allow the resonant wave, $1 < \omega/\Omega_e < 1.25$, to occur in the trapping region.

Recall that $\text{Im} H_w$ can be positive below $\omega = \Omega_e$ (see Figs. (3-5a), (3-5b)). Without the cold electrons, unstable modes at $\omega/\Omega_e \sim 0.7$ would be expected to develop. The fact that such electrostatic modes have not been observed in the trapping region indicates that the cold electrons, which contribute a negative upper hybrid branch below $\omega/\Omega_e = 1$, must be the dominant factor in determining $\text{Re} H$ there. A comparison between $\text{Re} H_c$ and $\text{Re} H$ shows that this mode will be suppressed from being unstable by the cold species only if

$$N_e/N_w > 10^{-2}. \quad (3.5.7)$$

Moving toward frequencies higher than $1.5\Omega_e$ in Fig. (3-7) the thermal correction of the cold electrons gradually overtakes the smoothly decreasing character of the upper hybrid branch, shown as the dashed curve, and shows a structure antisymmetric with respect to $\omega/\Omega_e = 2$. The resonant portion which intersects the upper hybrid branch at $\omega/\Omega_e = 2$ is not described by the (3.5.3). Instead of the asymptotic expansion
which is valid for $|\omega/\Omega_\infty - n| > \beta_{ii}^2 \alpha_{ii} \epsilon$, a series expansion has to be used for $Z$ and $Z'$ near each harmonic. Similar structure due to the change of the appropriate expansion also exists near the first harmonic, but this is not shown in the diagram. The two second harmonic extrema are shown in Fig. (3-7) to be located at $\omega/\Omega_\infty \approx 2 \pm k_{ii} a_{ii} \epsilon$. The height of the corresponding deviation from the upper hybrid branch is about $\left| \frac{N_e}{N_\infty} \frac{\lambda_{ce}}{4 \beta_{ii} a_{ii} \epsilon} \right|$ (if the warm electrons are used as the reference species). Similarly, one positive and one negative extremum are located, but not shown in the diagram, respectively, above and below the first harmonic (anti-symmetrically) at $\omega/\Omega_\infty \approx 1 \pm k_{ii} a_{ii} \epsilon$, with a peak magnitude of $\approx \left| \frac{1}{2 \beta_{ii} a_{ii} \epsilon} \right|$ Note that the cold electrons have been used as the reference species in Fig. (3-7) to emphasize the relevance of the cold species in determining the wave frequency in the potentially unstable region. Therefore, in Fig. (3-7), the height of the humps near the first and second harmonics are $\approx \left| \frac{1}{2 \beta_{ii} a_{ii} \epsilon} \right|$ and $\approx \left| \frac{\lambda_{ce}}{4 \beta_{ii} a_{ii} \epsilon} \right|$, respectively.

The limiting frequency, $\omega_c$, at infinite density for $1 < \omega/\Omega_\infty < 2$ can be obtained by setting (3.5.4) equal to zero:

$$\omega_c/\Omega_\infty = \left[ (4 + b)/(1 + b) \right]^{1/2}, \quad (3.5.8)$$

where

$$b \equiv \lambda_c + \frac{N_\infty}{N_e} \frac{D_2(\lambda \omega)}{\lambda \omega} \quad (3.5.9)$$

is smaller than unity but remains positive for all unstable $k_\perp$. The first term on the right side of (3.5.9) is the second harmonic cold contribution while the second term
arises from the warm electrons. The maximum $\omega_c$ is obtained at the positive minimum value of $b$ under the conditions that $k_\perp$ is large enough to keep $D_1(\lambda_w)$ negative and $\omega_c^2/\alpha_{1w}^2$ is large enough to keep $b$ positive. From the paragraph following (3.5.6) we know that because a significant negative $D_2(\lambda_w)$ can generally be obtained only at large $\lambda_w$, the $k_\perp a_{1w}$ chosen for a positive minimum $b$ must lie between $1 < k_\perp a_{1w} < 4$, for which $D_1(\lambda_w) < 0$ and $D_2(\lambda_w) = 0^+$. The cold electron density in the trapping region is expected to be high (e.g., one particle per c.c.) at least occasionally. Also, satellite observation shows that the second half harmonic mode is always below $\omega_c/\Omega_c \approx 1.75$. Thus, the temperature ratio of the cold species to the warm species, $\alpha_{1c}^2/\alpha_{1w}^2$, can be estimated by using (3.5.8) and (3.5.9), and by setting $\omega_c/\Omega_c = 1.75$, $k_\perp a_{1w} = 3$, and $b = \lambda_c$: 

$$\alpha_{1c}^2/\alpha_{1w}^2 \approx 0.1.$$  

(3.5.10)

Assuming $f_{DGH, j = 1}$, for $f_{1w}$ (see (B.2)) and $N_w/N_c = 5$, $b$ is plotted in Fig. (3-8a) as a function of $k_\perp a_{1w}$ with $\omega_c^2/\alpha_{1w}^2 = 0.05$, 0.1 and 0.2 as parameter. Given the maximum frequency of $\omega_c/\Omega_c = 1.75$, the corresponding minimum $b (\approx 0.45)$ is used to identify the temperature ratio of the two species. It can be seen in the diagram that $\alpha_{1c}^2/\alpha_{1w}^2 = 0.1$ is an accurate estimate.

Suppose the positive-slope region of the warm electrons is practically filled up as defined in (B.6) with $p = 0.2$. Then, a similar diagram for $b$ can be obtained. Although the
Fig. (3-8a) A plot of (3.5.9) for $\alpha_{L} / \alpha_{W} = 0.2, 0.1,$ and 0.05. We have chosen $f_{L_{g} w}, j=1$, for the warm electrons. The limit of $\omega = 1.75 - \omega_{c}$ corresponds to $b \approx 0.45$. The diagram shows that 0.1 is a good estimate of the temperature ratio between the two species.
Fig. (3-8b) A plot for $b$ similar to Fig. (3-8a). Here, a partially filled up non-monotonic $f_\perp$ defined in (B.6) for $p=0.2$ has been chosen. The diagram shows that 0.1 is again a good estimate of the temperature ratio for a distribution quite different from that in Fig. (3-8a).
distribution function is very different from the case for Fig. (3-8a), the temperature ratio is again found to be about 0.1. Note that in the definitions (B.6), the characteristic particle speed, \( \alpha_{th} \), is related to the speed, \( \alpha_w \), of maximum \( f \) by \( \alpha_{th}^j = (1-\gamma) \alpha_w^j \), for \( j=1 \).

The approximate growth rate given by (3.1.16) is valid here because \( \text{Im}H \) is smoothly varying. A typical example of the growth rate corresponding to the \( \text{Im}H \) diagram in Fig. (3-3a) is given by Fig. (3-9). For the unstable mode the value of \( k \lambda \) lies between 0.2 and 1.5 as expected, where the lower limit will be smaller if \( f_w \) is chosen to have a long tail. The frequency range is narrow (\( \sim 0.1 n_a \)), and the growth rate is about \( 10^{-2} n_a \), which corresponds to a growth time \( \sim 5 \times 10^{-3} \) sec in a \( \sim 10^2 \gamma \) magnetic field. The wave frequency for such a typical growth rate diagram is mainly determined by the cold electrons as shown in (3.5.4).

Such a large growth rate cannot remain long in the plasma. The wave amplitude will rapidly become saturated by nonlinear processes, which are beyond the scope of this thesis. On the other hand, the growth rate will also be nearly zero if the plasma is only marginally unstable. It seems that the observed wave amplitude in the trapping region is often nearly constant which implies that either some non-linear saturation processes must be at work or the plasma is only marginally unstable.

When \( N_c \) is large enough to satisfy (3.5.7), the upper hybrid mode, \( \omega / \Omega_a = 1 + \omega / \Omega_a \), obtained by neglecting the
Fig. (3-9) Typical temporal growth rates for the second half harmonic mode caused by a non-monotonic $f_{\perp \omega}$ coexisting with a cold species. The density is fixed, while $k_{\perp} a_{\omega}$ is the implicit parameter. We have chosen $N_c/N_w=0.2$, $k_{\perp} a_{\omega}=3$, $T_e/T_w=0.09$, $D_1=-D_2=-64 \times 10^{-2}$ and $D_3=0$, as in Fig (3-3b).
thermal correction in (3.5.3), is unstable at very low density. This is because at any low density, the corresponding upper hybrid frequency can have a \( k_u \) at which the Penrose criteria are satisfied. The relation between \( \omega \) and \( k_u \) can be obtained by requiring that \( \text{Im} H \) vanishes. Thus, we obtain

\[
\frac{\omega/\Omega - 1}{k_u a_{lw}} \sim \frac{a_{sc}}{a_{lw}} \left[ \frac{\Omega_c}{\Omega_w} \frac{N_c}{N_w} \frac{\lambda_w}{D_1(\lambda_w)} \right]^{1/2} \tag{3.5.11}
\]

Our numerical calculation agrees with the above expression. Therefore, the threshold density between \( \Omega_2 \) and \( 2\Omega_2 \) is made very low by the existence of the cold species.

When \( N_c/N_w \) is much less than \( 10^{-2} \), the characteristics of the unstable modes are different from our half-harmonic modes. Unstable modes may appear below the first harmonic and just below each of the higher harmonics (Guest and Sigmar, 1970). In addition, because there is no significant cold contribution to \( \text{Re} H \), the threshold density for modes between \( \Omega_2 \) and \( 2\Omega_2 \) is greatly increased. For the Dory, Guest, and Harris distribution with \( j=1 \), the threshold density is the order of \( \omega^2/\Omega^2_2 \sim 10 \). If the positive slope is less steep, the first appearance of negative \( D_1(\lambda_w) \) requires higher \( \lambda_w \) and hence the threshold density will be even higher.

Therefore, considering the normal magnetosphere conditions, it appears that if there is no cold species available to lower the threshold density, non-monotonic \( f_{lw} \)'s having mild positive-slope regions will not cause instability.

Following the foregoing research in the low frequency
region, the high frequency region \( (\omega / n_x >> 1) \) will be briefly discussed in the rest of this section. Because the high frequency waves are observed infrequently and the wave characteristics are not well recorded, it is difficult to draw such definite conclusions as those for the low frequency case.

The dispersion function in the high frequency range is determined by the "loss-cone" term. For small \( k_u \), a typical unstable mode is shown in Fig.(3-10) between A and B, where at point A the Penrose criteria are satisfied. This type of unstable mode is similar to those at low frequency in that \( \text{Im}H = 0 \) is possible because of a balance between a negative \( \text{Im}H_c \) and a positive \( \text{Im}H_w \). Numerical results show that for \( f_{DG}, j=1, \) and \( \omega / n_x > 4, 5, 6, \ldots \), the wave spectrum for small \( k_u \) is narrowly peaked above the various harmonics and the threshold density is moderate \( (N_w \gtrsim 1 \text{ cm}^{-3}) \). For larger \( k_u \), the wave spectrum shifts proportionally toward the half harmonics and the threshold density increases rapidly. Very high densities \( (N_w \gtrsim 10 \text{ cm}^{-3}) \) are required to obtain the high frequency half harmonics.

The existence of a sharp positive slope can give \( D_{n-1} > 0 \) and \( D_n < 0 \), and make possible another type of unstable mode of broader spectrum and lower threshold density. It is sketched in Fig.(3-10) as the region between point A' and B'. If the positive slope were not sharp and hence \( D_{n-1} \) were negative, the curve for \( \text{Im}H \) below point A' would be
Fig. (3-10) Sketch of ReH and ImH plotted respectively in the upper and lower diagrams. The dispersion function near $\omega/\omega_{ce}=n$ is a "non-resonant" case. Hence, marginal stability is obtained at point A, and an unstable mode is possible between A and B. The warm electron distribution function has a very sharp positive slope. Hence, $D_{n-1} > 0$ and $D_n < 0$, and an unstable half harmonic mode exists between point A' and B'.
inverted and instability would no longer be possible between \((n-1)\omega_k\) and \(n\omega_k\). However, the existence of such a sharp positive slope as required for \(D_n > 0\) and \(D_{n+1} < 0\), at \(n = 4, 5, 6,...\) seems difficult to explain for the magnetospheric plasma.

Besides the above two possibilities, sharp small irregularities in the distribution function such as a delta function spike upon the background distribution may also be able to induce the high frequency mode. However, whether the free energy of such small irregularities is large enough to drive such instabilities remains to be investigated.

The important results of this section can be summarized as follows:

The wave characteristics agree well with the known data. Due to the lack of observation of waves between \(1.75 < \omega_k < 2\), the temperature ratio between the cold and warm species is found to be about 0.1. A lower bound on the density ratio, \(N_c/N_w \geq 10^{-2}\), is required to stabilize the first half harmonic mode, which has not been observed in the trapping region. The infrequent observation of the near resonant waves, i.e. \(1 < \omega_k \omega_k < 1.25\), indicates that \(N_c \geq 0.1\ \text{cm}^{-3}\) should be the normal cold electron density in the trapping region. Finally, high density, sharp positive slopes, and sharp irregularities in the distribution function have been discussed as possible causes of the high frequency modes. Without a sharp positive slope, high-frequency half harmonic modes would require a very high density \((N_w \geq 10\ \text{cm}^{-3})\) to be unstable.
3.6 Conclusion and Further Geophysical Implications

An infinite, homogeneous, plasma model has been establish in Section 3.2, where all the spatial inhomogeneities were shown to be negligible. The electron distribution function is made up of two species, a cold species and a warm species.

Based on this model, the half harmonic modes for different frequency ranges and wave vectors have been studied in the preceding three sections. The flute-like mode has been shown in Section 3.3 to require a sharp positive slope for the warm species $v_A$-distribution, plus the presence of a cold species to be unstable at reasonably low density. For the wave to be unstable below $\omega/\Omega_A = 1.5$, the density ratio between the cold and warm species is required to be less than $10^{-2}$, a value smaller than that expected for normal magnetospheric conditions. Because the flute-like waves are not likely to occur below $1.5\Omega_A$ while the observed wave frequencies seem to be equally distributed above and below $1.5\Omega_A$, the majority of the observed waves cannot be flute-like.

The modes having finite wave lengths along the magnetic field ($k_\parallel \neq 0$) have been studied in Section 3.4 and 3.5. These modes are found to be unstable between consecutive harmonics of the cyclotron frequency. We have found that the threshold density in the lower frequency range ($\omega \sim (n+1/2)\Omega_A$, $n=1,2,3$) is low and these modes are unstable in the trapping region. It has been shown that for the half harmonic modes to be
unstable for frequencies in the range $1.25 \leq \omega n_2 \leq 1.75$
without having any apparent preference within this frequency
range, the warm electrons must have a non-monotonic rather
than temperature anisotropic distribution.

Upper and lower limits on $k_\parallel$ for the unstable lower
frequency modes have been found ((3.4.7), (3.4.9)), and a
lower limit of $k_\perp$ has also been obtained (3.4.4). For large
$k_\perp$, the growth rate decreases rapidly like $\sim k^{-3}$. The
temperature ratio and the lower bound on the density ratio
between the cold and the warm species have been found to be
about 0.1 and $10^{-2}$ respectively. We also concluded that
because the near resonant regime $(1 < \omega n_2 < 1.25)$
corresponds to the case where the cold electron density is
approximately less than 0.1 cm$^{-3}$, the rare appearance of
such waves indicates that the cold electron density in the
trapping region is normally higher than 0.1 cm$^{-3}$.

Based on $f_{DGH}$, $j=1$, the threshold density for the high
frequency waves ($\omega n_2 \gg 1$) has been found the order of 1 cm$^{-3}$
or higher, if the temperature ratio between the two species is
larger than or comparable to $10^{-2}$. The corresponding fre-
quency range has been found only slightly above each harmonic,
and it requires high density ($\geq 10$ cm$^{-3}$) to obtain the unstable
half harmonic modes. The wavelengths of these high frequency
waves are short, and the criterion (3.4.4) for $k_\parallel$ has to be
satisfied. On the other hand, if $\alpha^2/\alpha^2$ is much less than
$10^{-2}$ or the parameter $j$ for $f_{DGH}$ is chosen to be 2, both
analytic and numerical analyses show that no high frequency waves (e.g. \( \omega / n_e \geq 5, 6, \ldots \)) can occur. Besides high density, the only way we have found to obtain the half harmonic high frequency instabilities is to have a sharp positive slope in \( f_{ew} \). Because the high frequency waves are observed only occasionally and the wave frequencies have not been accurately recorded, definite conclusions about the cause of these instabilities cannot be given.

Recall that we have concluded in Chapter 2 that the trapping region has a constant supply of fresh warm and cold electrons from the plasma sheet and the ionosphere respectively, while the plasmasphere is well shielded from the incoming warm electrons except during occasional magnetic storms. The plasma sheet has plenty of warm electrons but seems to have only a very few cold electrons if any. Since the necessary and sufficient condition for these waves to develop is a sufficient amount of cold electrons (see (3.5.7)) and the existence of a non-monotonic warm electron \( v_1 \)-distribution, these waves are not expected to be frequently observed in either the plasmasphere or the plasma sheet. Without a constant supply of fresh warm electrons from the plasma sheet, any non-monotonic feature of the warm electrons in the plasmasphere probably dissipates quickly and becomes monotonic during the long period between successive magnetic storms. The cold electron density in the plasma sheet under normal conditions is too low, and even if the warm species there is
non-monotonic the required threshold density without a cold species is probably too high for the half harmonic modes to be unstable. The trapping region ($4 < L < 10$) is the only region where a cold species and an anisotropic warm species constantly coexist, and thus it is also understandably the region where the electrostatic waves are frequently observed.

Because of the asymmetric pattern of the convection, incoming warm electrons from the plasma sheet are predominantly convected in a counterclockwise direction around the plasmasphere through the morning side. This is consistent with the fact that the half harmonic waves seem to be observed mostly in the morning side of the trapping region. Recall that the half harmonic waves are confined within $\pm 10^\circ$ near the magnetic equatorial plane. The most probable explanation of this effect is that the non-monotonic $v_\perp$-distribution exists only at low latitudes. However, the question of why the non-monotonic feature should exist only at low latitudes remains unanswered.

Recall that the energy spectra of the particle flux in the plasma sheet and of the auroral precipitation over the auroral oval are similar. Also, the auroral oval is suggested (Kennel, 1970) to be approximately the region where the incoming plasma is strongly diffused in pitch-angle. Thus, it seems that the electrostatic waves we have studied may offer the required mechanism for the acceleration and pitch-angle diffusion for the auroral electrons. Because the
intensity of the waves is closely related to the amount of free energy associated with the non-monotonic feature of the warm species, the wave intensity is expected to be the strongest when the incoming non-monotonic warm electrons first encounter the cold species which helps to "trigger" the instability. The strongest wave amplitude observation \((\gtrsim 100 \ \text{m}_f)\) occurred during the expansion phase of a substorm, and the location seems to be on the magnetic shell corresponding to the auroral oval (Scarf et al, 1971). The wave intensity is expected to be smaller as the plasma is convected further inward, because there is less free energy available in the plasma for the unstable wave. A quantitative discussion of the velocity space diffusion caused by the electrostatic waves will be presented in the next chapter.

Only the electrostatic waves involving the electron cyclotron motion have been studied in this thesis. Their counterpart involving the ion cyclotron motion has not been studied because there are not yet any wave experiments in the low frequency range to detect the ion cyclotron waves. However, our best guess is that the threshold density for the ion "loss-cone" instability is probably about a thousand times lower than that of the electron loss-cone instability. Therefore, if a non-monotonic \(v_f\)-distribution exists in the plasma sheet for both ions and electrons, the density there is so high for the ion loss-cone instability that ion cyclotron waves will quickly diminish the positive slope in the ion
\(v_\perp\)-distribution even without the presence of a cold species (see Fig. (3-1b)). Hence, the wave characteristics and the spatial distribution of the ion electrostatic waves must be quite different from that of the electrostatic waves caused by the electrons.
4.1 Introduction

In the last chapter we have thoroughly investigated the half harmonic modes for various ranges of plasma parameters and wave frequency using a linear analysis. All the important plasma and wave characteristics which have been found in the analysis are consistent with experimental observations. The next important stage of theoretical study is the quasi-linear diffusion of the particles in velocity space caused by the unstable waves whose growth rates are linearly determined by the instantaneous distribution function. The velocity-space diffusion of the particles provides the distribution function with a mechanism for time evolution through which energy is exchanged between waves and particles. Within quasi-linear theory, a quasi-steady equilibrium is expected to exist asymptotically among the weakly unstable particle distribution, the saturated waves, and a self-consistent diffusion in velocity space. The diffusion coefficient in velocity space is parameterized by the detailed characteristics of the plasma and the wave spectrum. The energy exchange between the wave and the particles should then be balanced macroscopically and also microscopically so that the quasi-steady state can be maintained. However, this problem is very difficult not only because the linear dispersion relation is complicated and the diffusion is two-
dimensional in velocity space, but also because the general theory of the nonlinear saturation of the wave amplitude is still inadequate. Faced with the tremendous complexity of this problem, we have to be satisfied with some less ambitious goals in this chapter. An introduction is given in the first section to discuss the general characteristics and validity of the quasi-linear theory and the diffusion equation. A diffusion coefficient is obtained and discussed in the second section by assuming a wave spectrum model which is roughly consistent with the linear dispersion relation. Also, affirmative supporting data from the satellite and the physical implications of the diffusion coefficient obtained are discussed in the second section.

The initial growth rates of unstable waves in a quiescent plasma are adequately described by the linear dispersion relation. During a time scale of typically a few wave growth periods, the amplitude of individual unstable modes grow exponentially like $A \cdot e^{\tau t}$, where $A$ is the initial amplitude caused by thermal noise or the discreteness of particles, and $\tau$ is the linearly determined wave growth rate. In contrast to theories for finite amplitude waves, where the perturbed particle orbit is taken into consideration in the orbital integration, the wave amplitude does not appear in any equation of the linear theory. Also, the particle distribution function is considered to change much more slowly than the wave growth rate so that the distribution function can be regarded as
nearly constant. However, for cases where the plasma is considered as an isolated system, it is known by the conservation of energy that when a wave is growing the total particle kinetic energy must be decreasing at the same rate as the wave energy is increasing. The kinetic energy loss must be modifying the particle distribution function in such a way that the plasma kinetic energy content is decreasing. The modification of the distribution function is equivalent to a particle diffusion in velocity space, where the number of particles is conserved. The cumulative effect of the diffusion can change the distribution function significantly so that the wave growth rate determined by the instantaneous distribution function may be reduced significantly even to zero. The general theory dealing with particle diffusion in velocity space and the self-consistent reduction of the wave growth rate with constant total energy is called the quasi-linear theory. (Vedenov et al, 1961; Drummond and Pines, 1962; Dupree, 1966; Galeev, 1966).

The wave amplitude in the trapping region is generally less than about 30 mv/m, which is equivalent to an energy density of about $4 \times 10^{-14}$ erg/cm$^3$. The ratio of the maximum wave energy to kinetic energy (for a density of 1 per c.c., and a temperature of 100 ev) is less than $10^{-3}$. Therefore, the high order wave amplitude effects such as wave-wave coupling and non-linear orbit effects can be neglected and the quasi-linear approximation is likely to be the one
applicable to the problem.

Recall that the wave amplitude detected in the trapping region is nearly constant over a long period of time. Namely, it is typically constant for a few minutes, which is equivalent to about $10^6$ electron cyclotron periods. Evidently, the corresponding effective growth rate must be kept nearly zero. The most probable way of doing it has been described above. Namely, regardless of the initial distribution, marginal stability is always reached by particle diffusion in velocity space, and the growth rate is so small that a saturation level of wave amplitude is maintained.

Interactions between waves and particles can be expressed in terms of the real and imaginary parts of the dispersion function in the linear theory. Here, the real part represents a reversible interaction with no net energy exchange between wave and particles over a long period of time, and the interaction involves the non-resonant particles. The imaginary part represents an irreversible interaction with energy exchange and involves only the resonant particles satisfying

$$\vec{F}_n \cdot \vec{u}_n - \omega + n \chi = 0 \quad (4.1.1)$$

when the growth rate is small. However, when the growth rate is not small, significant irreversible interaction may also exist between the wave and the non-resonant particles whose velocity is far from that satisfying the resonant
Analogous to the above description, two types of velocity space diffusion can be identified. When the growth rate $\tau$ is small so that the Dirac relation (Stix, 1962)

$$\lim_{\tau \to 0^+} \frac{1}{\mathcal{V}_\parallel - \frac{\omega - n\Omega_\parallel}{\mathcal{K}_\parallel}} = \frac{P}{\mathcal{V}_\parallel - \frac{\omega - n\Omega_\parallel}{\mathcal{K}_\parallel}} + \pi \delta (\mathcal{V}_\parallel - \frac{\omega - n\Omega_\parallel}{\mathcal{K}_\parallel})$$

(4.1.2)

is applicable, only the resonant particle diffusion need be considered. This case is called resonant diffusion. On the other hand, when the growth rate is large so that (4.1.2) is not applicable and the particle diffusion involves both resonant and non-resonant particles, this case is called non-resonant diffusion. The correction to the resonant diffusion with a finite growth rate can be estimated by comparing the delta function in (4.1.2) with the imaginary part of the exact expression

$$\frac{1}{\mathcal{V}_\parallel - \frac{\omega - n\Omega_\parallel}{\mathcal{K}_\parallel}} = \frac{\mathcal{V}_\parallel - \frac{\omega - n\Omega_\parallel}{\mathcal{K}_\parallel}}{\left(\mathcal{V}_\parallel - \frac{\omega - n\Omega_\parallel}{\mathcal{K}_\parallel}\right)^2 + \frac{\tau^2}{\mathcal{K}_\parallel^2}} + \frac{i \tau}{\mathcal{K}_\parallel^2}.$$

(4.1.3)

It can be seen that only when $\tau \to 0^+$ are (4.1.2) and (4.1.3) identical. However, (4.1.2) is a good approximation to (4.1.3) if $\tau \ll |\omega - n\Omega_\parallel|$. Obviously, the growth rate of the half harmonic mode is small in this sense and thus the resonant diffusion approximation is valid.
4.2 Velocity Space Diffusion

In quasi-linear theory, there are two time scales for the variation of the distribution function and field perturbations. A short time scale corresponds to a rapid variation due to waves, and a long time scale corresponds to the slow variation due to the cumulative change of the distribution function. If the phases of the waves are random and the amplitudes are small, the reaction of the particles is approximately a Brownian motion in velocity space, i.e. small random steps of velocity variation with a short autocorrelation time. No corresponding spatial diffusion is discernible for the case of an infinite homogeneous plasma. The analysis is mathematically similar to that of the van der Pol method.

Taking the phase average of the Vlasov equation over a period long compared with the short time scale and the wave-particle autocorrelation time, the resulting equation representing the slow variation of the electron distribution function, f, for the case of electrostatic waves in a magnetized plasma is (Kennel and Engelman, 1966)

\[
\frac{\partial f}{\partial t} = \frac{2}{\partial \mathbf{v}} \left( \mathbf{D} \cdot \frac{\partial f}{\partial \mathbf{v}} \right),
\]

(4.2.1)

where

\[
\mathbf{D} = 8 \pi i \sum_n \frac{e^2}{m} \int d^3 \mathbf{k} \frac{J_n^2 \mathcal{E}^2 \left( \frac{n \Omega_{e} \mathbf{v}_{\perp} + \mathbf{R}_n \mathbf{e}_n}{R^2 (\omega_r - \mathbf{k} \cdot \mathbf{v}_{||} + n \Omega_{e})} \right)^2}{R^2 (\omega_r - \mathbf{k} \cdot \mathbf{v}_{||} + n \Omega_{e})},
\]

(4.2.2)

and
Eq. (4.2.1) is the Fokker-Planck diffusion equation which determines the time evolution of the electron distribution in three dimensional velocity space. (The axial symmetry with respect to the magnetic field in the z direction actually reduces the equation to two dimensions, $v_\parallel$ and $v_\perp$). $\mathcal{E}$ is the diffusion tensor for the electrons and will be discussed in detail in this section. $\mathcal{E}_k$ is the electrostatic wave energy density associated with the mode $k$, and related to the total wave energy density by Parseval's theorem,

$$\int |E_\mathbf{k}|^2 d^3 \mathbf{r} = (2\pi)^3 \int |E|^2 d^3 r.$$  \hspace{1cm} (4.2.4)

The wave energy growth is governed by the equation

$$\frac{\partial \mathcal{E}_k}{\partial t} = 2 \tau \mathcal{E}_k,$$  \hspace{1cm} (4.2.5)

where the linear growth rate $\tau$ for the $k^{th}$-mode is determined by the instantaneous dispersion relation. It is not surprising that the slow time variation of $f$ has been found to obey the Fokker-Planck diffusion equation. This is the result of the fast oscillation and small amplitude of the randomly phased waves which tends to work against any derivative of the particle distribution in velocity space just as the small but frequent random molecular collisions tend to gradually smooth out the density gradient of a soluble dye in still water.

However, unlike the isotropic molecular collisions, the discreteness of the infinite number of harmonic contributions
to the diffusion coefficient (4.2.2) introduces discrete sets of diffusion paths for each point in velocity space. The total energy and momentum of the waves and particles are conserved only along the diffusion paths. The differential equation for the diffusion paths, derived from the conservation of energy and momentum during wave-particle interaction is

\[(v_{ii} - \omega / k_{ii}) dv_{ii} + v_{li} dv_{li} = 0, \quad \text{or,} \]

\[\frac{\omega}{n_{l}} \cdot v_{l} dv_{l} - v_{ii} dv_{ii} = 0. \quad (4.2.6)\]

In integrating (4.2.6), the resonance condition (4.1.1) has to be used in combination with the solutions, \(\omega(k)\), of the instantaneous dispersion relation in order to express \(\omega\) in terms of \(v_{ii}\) by eliminating the dependence on \(k_{ii}\).

Comparing (4.1.1), (4.2.2) and (4.2.6), one can see a one-to-one correspondence for each \(n\) between the resonance condition, the diffusion tensor, and the paths of diffusion. In other words, each particle in velocity space can have a denumerable infinity of diffusion paths to move along. The \(n^{th}\) component of the diffusion tensor is associated with the \(n^{th}\) diffusion path, while the corresponding resonance condition is maintained implicitly in the \(k_{ii}\)-dependence of the \(n^{th}\) component of the diffusion tensor. As a matter of fact, the above description can be illustrated best by rewriting the tensor form of the Fokker-Planck equation (4.2.1) as

\[
\frac{\partial f}{\partial t} = \sum_{n} \frac{\partial}{\partial v_{n}} \left( D_{n} \frac{\partial}{\partial v_{n}} f \right). \quad (4.2.7)
\]

Here, the diffusion coefficient is given by

\[D_{n} = 8 \pi i \frac{e^{2}}{m} \int d^{3}k \int_{-\infty}^{\infty} \frac{E_{n} \left[ \left( \frac{n_{l}}{v_{l}} \right)^{2} + \left( \frac{\omega - n_{l}}{v_{l}} \right)^{2} \right] \text{}}{k} (\omega - k_{ii} v_{ii} + n_{l} n_{l}) , \quad (4.2.8)\]
where $\Delta v_n$ is the differential velocity increment along the $n^{th}$ order diffusion path determined by (4.2.6). To obtain (4.2.7) and (4.2.8), the following substitution

$$\frac{n \cdot \alpha}{v_L} e_\perp + k_{\parallel} \cdot e_{\parallel} = \left[ \left( \frac{n \cdot \alpha}{v_L} \right)^2 + \left( \frac{\omega - n \cdot \alpha}{v_L} \right)^2 \right]^{\frac{1}{2}} e_n \quad (4.2.9)$$

has been used, where $e_n$ is the unit vector at $(v_{\parallel}, v_\perp)$ along the $n^{th}$ order diffusion path.

When a source or sink of particles exists in real space, an additional term has to be added to (4.2.1) to account for the extra particle fluxes. For example, when the particle loss at the ends of a magnetic tube is taken into account, particles residing in the velocity loss cone will be lost in less than a half bounce period. This loss rate is not considered in the velocity diffusion process, but it certainly changes the diffusion speed, especially inside the velocity loss cone. Also, if fast radial diffusion exists in the magnetosphere, a source has to be assumed in (4.2.1) to account for the additional particle fluxes. In addition to the above complication of the need for particle sources and sinks, the real part of wave frequency, $\omega_r$, is a function of $k$. Hence, a reasonably self-consistent solution of $f$ is very difficult to obtain. However, given a rough shape of $f$, even an approximate $\bar{\Phi}$, which is strongly velocity dependent, can yield some valuable information on the instantaneous response of $f$ to the waves.

In order to study $\bar{\Phi}$, knowledge of the wave spectrum is indispensable. We assume here a model of the wave spectrum which is consistent with the type of waves studied in Chapter 3, namely

$$|E_k|^2 = A \left( \frac{R_i}{R_{10}} \right)^2 \left( \frac{R_i}{R_{10}} \right)^{2j} e^{-\left( \frac{R_i}{R_{10}} \right)^2} \Phi_k(k_{\parallel}), \quad (4.2.10)$$
where $A$ is a normalization constant determined by (4.2.4). The factor $k^2$ on the right side of (4.2.10) is used to cancel the $k^2$ in the denominator of (4.2.2) in order to simplify the algebra. The $k_\perp$ dependence of $|E_k|^2$ is chosen such that the wave energy is bell shaped around a nonzero $k_\perp\omega$. The main $k_\parallel$ dependence of $|E_k|^2$ is not specified in (4.2.10). The reason is simply because for resonant diffusion the integration of $\mathcal{D}$ in (4.2.2) can easily be done with a quite arbitrary choice for the $k_\parallel$ dependence of $|E_k|^2$. In contrast, no better choice than that given in (4.2.10) for the $k_\perp$ dependence can be made without resulting in complicated algebra. However, short of completely solving (4.2.1) to (4.2.5) as an initial value problem we have no way of knowing the wave spectrum precisely, and any bell shaped spectrum is generally an equally acceptable assumption.

Using the Dirac relation (4.1.2) for resonant diffusion in the integral for $\mathcal{D}$, a real part evaluated at a singularity of the delta function of (4.1.2) can be obtained by integrating over $k_\parallel$. The $k_\parallel$ integration for the principal value part vanishes because of the antisymmetry condition, $\omega_r(k)=-\omega_r(-k)$. As for the $k_\perp$ integration, it gives a $C$-function (see Appendix B) for the chosen $|E_k|^2$ in (4.2.10), if the $k_\perp$ dependence of $\omega$ is negligible. The wave frequency is strictly independent of $k$ only for the flute-like mode ($k_\parallel=0$) and when the electron density is low. The wave is then represented by the first term in (3.4.4), i.e. the upper
hybrid branch, with \( k_\perp = k \). The thermal corrections represented by the second and third terms on the right side of (3.4.4) are relatively small if the frequency is near or below the second half harmonic. This is clearly shown in Fig. (3-7) by the smallness of the deviation of ReH from the dashed curve for the cold upper hybrid branch. Moreover, because \( k_\perp \gg k_\parallel \) for the unstable mode, the factor \( (k_\perp / k)^2 \) in (3.4.4) can be taken approximately as unity. Therefore, an approximate formula \( \omega_\tau = (\omega_{2k}^2 + \Omega_{2}^2)^{1/2} \) will be used in (4.2.2) for the evaluation of the diffusion tensor. Substituting (4.2.10) into (4.2.2) and performing the integration by using the above simplification, we obtain,

\[
D_n = D_0 \sum_n C_n \left( \frac{k_{20}^2 \Omega_{2}^2}{\Lambda_2^2} \right) \phi_n \left( \frac{k_\parallel}{k_{10} \nu_\parallel} = \frac{\omega - n \Omega_{2}^2}{\nu_\parallel} \right) \left( \frac{n \Omega_{2}^2}{k_{10} \nu_\parallel} \epsilon_\perp + \frac{\omega - n \Omega_{2}^2 \epsilon_\parallel}{k_{10} \nu_\parallel} \right) \left( \frac{n \Omega_{2}^2}{k_{10} \nu_\parallel} \epsilon_\perp + \frac{\omega - n \Omega_{2}^2 \epsilon_\parallel}{k_{10} \nu_\parallel} \right) \left( \frac{n \Omega_{2}^2}{k_{10} \nu_\parallel} \epsilon_\perp + \frac{\omega - n \Omega_{2}^2 \epsilon_\parallel}{k_{10} \nu_\parallel} \right),
\]

or, by the definition in (4.2.7),

\[
D_n = D_0 \sum_n C_n \left( \frac{k_{20}^2 \Omega_{2}^2}{\Lambda_2^2} \right) \phi_n \left( \frac{k_\parallel}{k_{10} \nu_\parallel} = \frac{\omega - n \Omega_{2}^2}{\nu_\parallel} \right) \left( \frac{n \Omega_{2}^2}{k_{10} \nu_\parallel} \epsilon_\perp + \frac{\omega - n \Omega_{2}^2 \epsilon_\parallel}{k_{10} \nu_\parallel} \right) \left[ \left( \frac{n \Omega_{2}^2}{k_{10} \nu_\parallel} \epsilon_\perp + \frac{\omega - n \Omega_{2}^2 \epsilon_\parallel}{k_{10} \nu_\parallel} \right)^2 \right].
\]

Here,

\[
D_0 = j\frac{4\pi^{3/2}}{\Lambda_2 \sqrt{\Omega_{2}^2}} \frac{A k_{10}^2 k_{0}^2}{8 \pi \sqrt{2} \nu_\parallel^3} \epsilon^2 \left( \frac{m^2}{\Lambda_2^2} \right),
\]

and \( k_{10} \) is a characteristic \( k_\parallel \) at which \( \phi_n(k_\parallel) \) is maximized.

\( C_n \left( j_\perp \right) \) has been defined in (3.1.5) where the function \( f_\perp \) is replaced by \( f_{DGH} \) in (B.2) for \( j = j_\perp \). For the simplified
wave spectrum (4.2.10), the integration of (4.2.6) for the diffusion paths can be performed easily to obtain

\[ \frac{\omega_{M\omega}}{n} \nu_{\perp}^2 - \nu_{n}^2 = \text{constant}, \quad n \neq 0 \]  

(4.2.14)

and

\[ \nu_{\perp}^2 = \text{constant}, \quad n = 0. \]  

(4.2.15)

For low frequency waves, \( \omega_{M\omega} < 1 \), all the \( n \neq 0 \) diffusion paths are ellipses. However, for wave frequencies higher than the first harmonic, i.e. \( (M+1) > \omega_{M\omega} > M > 0 \), the positive lower order diffusion paths, i.e. \( M > n > 0 \), are hyperbolas, and the other harmonic contributions are ellipses. The diffusion paths (4.2.15) corresponding to the Cerenkov resonance (\( n = 0 \)) are independent of \( \omega \), and involve changes of \( \nu_{n} \) only. The first order, \( n = 1 \), diffusion paths for the second half harmonic wave are shown in Fig. (4-la) as a set of hyperbolas, while the diffusion paths of all the other orders except \( n = 0, 1 \), are shown in Fig. (4-lb) as ellipses. For large \( n \), the ratio of the elliptic axes approaches unity, which corresponds to circular paths of pure pitch-angle diffusion. Comparing Fig. (4-la) and (4-lb), it is clear that the diffusion associated with the elliptic paths is mainly of the pitch-angle type with little energy change for each individual particle. In contrast, the hyperbolic diffusion paths are strongly associated with energy-exchange. It is especially apparent in Fig. (4-la) that the hyperbolic diffusion paths for those particles residing near the asymptotes
Fig. (4-1) Two types of paths of diffusion for $\omega \approx 1.5 \Omega_e$.
For $n=1$, the corresponding paths of diffusion are hyperbolic.
For $n \neq 0, 1$, the paths are elliptic.
result in either acceleration or deceleration without significant change in the pitch-angle.

To consider the characteristic diffusion coefficient $D_o$, let us take $\phi_k(k||) = \left(\frac{\pi}{k_{||0}}\right)^\beta e^{-\left(\frac{k_{||}}{k_{||0}}\right)^\beta}$ for convenience of illustration. Also, we use (4.2.10) for $|E_k|^2$ in (4.2.4), which becomes

$$
\frac{A}{k_{||0}} \frac{k_{||0}}{\sqrt{\pi}} \int_{i\Omega} \left(\frac{\pi}{k_{||0}}\right)^\beta e^{-\left(\frac{k_{||}}{k_{||0}}\right)^\beta} \sim W
$$

(4.2.16)

where

$$W = \frac{1}{\sqrt{\pi}} \int |E|^2 d^3\gamma
$$

(4.2.17)

is the wave energy density in real space. If another functional form of $\phi_k$ were chosen, the magnitude of the constant relating $A$ with $W$ in (4.2.16) would not be much different.

When the wave dispersion relation is unknown, an overall rough estimate of the diffusion intensity is given by $D_o$ (Kennel et al., 1970) which, by using (4.2.16), is

$$D_o \sim \frac{1}{\sqrt{\pi}} \frac{|E|^2}{m} \sim C^2 \Omega e \left(\frac{\text{Wave Energy Density}}{\text{Magnetic Energy Density}}\right)
$$

(4.2.18)

For a wave amplitude of about 30 mv/m, the corresponding $D_o$ is about $2.5 \times 10^{18} \text{cm}^2/\text{sec}^3$, which is equivalent to a root-mean-square velocity spreading of about $10^9 \text{cm/sec per second}$. However, because of the strong velocity dependence of the diffusion tensor, (4.2.18) is at best an estimate of the maximum diffusion coefficient in the velocity space, and the actual value can be much smaller.

In considering the general velocity dependence of $D_n$,
the $v_{\parallel}$ dependence is mainly determined by $\phi_{\parallel}$. Recall that the upper and lower limits of $k_{\parallel}$ for the unstable half harmonic modes have been found in (3.4.7) and (3.4.9) respectively. The lower and upper limits of $v_{\parallel}$ for the resonant electrons can thus be found by using (3.4.7) and (3.4.9) respectively in the resonance condition for given $n$ without specifying the functional form for $\phi_{\parallel}(k)$. The resonant velocity $v_{\parallel R}$ corresponding to the $n^{th}$ order diffusion path is then found to be confined in a strip,

$$0.3 \frac{\alpha_{\parallel R}}{\alpha_{\parallel C}} \geq \left| \frac{v_{\parallel R}}{\omega_{\parallel} - \Omega_{\perp}} - n \right| \geq 0.5 \frac{\alpha_{\parallel R}}{\alpha_{\parallel T}},$$

parallel to the $v_{\perp}$-axis. For example, the upper limit of $v_{\parallel R}$ for the first and second order diffusion paths is about $\alpha_{\parallel T}$, if $\omega_{\parallel}/\Omega_{\perp} = 1.5$. For high mode number diffusion paths, the width of the strip of $v_{\parallel R}$ and its upper and lower limits increase linearly with $n$.

Because of the first (zeroth) order diffusion and the overlapping of resonant diffusion regions, the pitch-angle diffused resonant particles are also accelerated (decelerated). As soon as more particles are accelerated to the region of slightly higher $v_{\parallel}$, the upper limit of $v_{\parallel R}$ for diffusion is extended further and hence this process becomes more effective in diffusing the particles in the "new tip" of the tail, and so on. This process is presumable consistent with the existence of a high energy tail in the magnetospheric plasma.

For a fixed $v_{\parallel}$, $\mathcal{D}_n$ first increases and reaches a maximum as $v_{\perp}$ increases from zero. Passing the maximum, $\mathcal{D}_n$ decreases.
like $v_L^{-2}$ when the first term in the bracket of the right side of (4.2.12) dominates. Then, $D_n$ decreases only slowly after crossing $v_L/v_{\parallel} \sim 1$. A complete study of the diffusion tensor has not been done in this thesis, but some numerical estimates have been made by calculating the most important terms. The results indicate that the rough estimate given by $D_0$ can be off by two orders of magnitude in some regions of velocity space.

For large $v_{\parallel}/\alpha_{\parallel}\omega$ ($\sim 2N, N \gg 1$), and small $v_L/v_{\parallel}$, the diffusion paths for different $n$ are almost parallel to each other and hence an effective diffusion coefficient $D$ can be obtained by summing up the series over $n$. Thus, for exponential $\phi_{\parallel}$,

$$D_\parallel \sim D_n/N. \quad (4.2.20)$$

The corresponding pitch-angle diffusion is

$$D_{\perp} \approx D_n \cdot v_{\parallel}^{-2} = \left(\frac{E}{10 \text{ keV}}\right)^2 \frac{0.1}{|\mathcal{E}|^{1/2}}. \quad (4.2.21)$$

Suppose the wave is spatially confined near the equator so that mirroring particles are diffused by the wave in 10% of their bounce time. Let us loosely define strong pitch-angle diffusion as the case where $D_\perp \approx 10^3 \text{ rad}^2/\text{sec}^3$ (Kennel et al., 1970). Particles with parallel kinetic energy $\mathcal{E}_{\parallel} \leq 1 \text{ keV}$ near the loss cone will experience strong pitch-angle diffusion when the wave amplitude $E$ is of order 10 mV/m. For particles of higher $\mathcal{E}_{\parallel}$, the diffusion coefficient decreases as $(\mathcal{E}_{\parallel}/\text{keV})^{-3/2}$, i.e. very rapidly. During substorms, waves
can develop large amplitudes (e.g., $E \geq 100$ mV/m) and particles with parallel energy up to 30 kev or greater can be strongly diffused. Recall the simultaneous observation of an extremely strong half harmonic wave, the correlated changes of the local pitch-angle distribution, and the high flux of energetic electrons during a substorm (Scarf, et al., 1971). It seems that the change in the pitch-angle distribution can be easily explained by the large pitch-angle diffusion coefficient. Also, the particle acceleration for the observation of the high flux of energetic electrons may be due to the $n=0,1$ components of diffusion and the overlapping of the resonant regions. Although the time evolution of the entire distribution function is not known at this stage, the observation of whistlers after appearance of electrostatic waves (Scarf, et al., 1971) is not inconsistent with our understanding of the diffusion tensor. If electrons are accelerated more along the $v_\perp$ direction than the $v_\parallel$ direction, a temperature anisotropy responsible for the development of whistlers may result from these electrostatic waves.

In summary, the general features of the diffusion tensor have been discussed and an expression (4.2.11) for the diffusion tensor useful for further theoretical study of the quasi-linear diffusion has been obtained. Physical meaning has been attached to the lower limit on $k_\parallel$ and the evolution of the distribution. A rough estimate indicates that $D_0$ is not a good estimate of the magnitude of the pitch-angle
diffusion coefficient, while a better estimate has been given in (4.2.21). The lower energy warm electrons ($E_{\parallel} < 1 \text{ kev}$) are expected to be diffused strongly all the time, while strong diffusion for energetic electrons ($> 10 \text{ kev}$) is possible only during substorms when the wave amplitude is large ($E \gtrsim 100 \text{ mv/m}$).
5.1 Summary

In order to study the electrostatic instabilities in the magnetosphere, we have based our theoretical analysis on both wave observations (Kennel et al, 1970; Scarf, et al, 1971) and the magnetic field, electric field, and plasma in the magnetosphere. The wave observations have been summarized briefly in Chapter 1. Useful information on the electron density, temperature, and particles of different origin in different regions of the magnetosphere as well as other kinds of waves and instabilities in the magnetosphere have been studied extensively in Chapter 2. Based on this information, we have concluded that velocity space anomalies in the electron distribution function rather than spatial inhomogeneity must be the source of the free energy driving the observed waves. In Section 3.2, we have assumed an infinite, homogeneous, and magnetized plasma with two species, one cold and one warm, of electrons for the plasma model in the trapping region. The temperature ratio and the density ratio between the cold and the warm species have been respectively assumed to be about a few per cent and greater than a few per cent. The distribution function of the warm species has been assumed to be unstable, and the cold species has been assumed to have no free energy available for the instability. The unstable
velocity distribution of the warm species was expected to either have a temperature anisotropy or be non-monotonic in $v_\perp$ (i.e. loss-cone like).

We first studied the flute-like ($k_n=0$) mode in Section 3.3 and found that for the assumed plasma parameters this mode can be unstable above the second half harmonic (i.e. $\omega > 1.5 \Omega_\perp$) if the positive slope of the warm species distribution is sufficiently sharp. However, because the second half harmonic frequency modes observed in the trapping region are about equally probable above and below $1.5 \Omega_\perp$, we have concluded that the waves are not predominantly flute-like modes.

The analysis of the non-flute modes in Sections 3.4 and 3.5 has shown that these modes can be unstable near the lower half harmonics of the electron gyro-frequency ($\omega / \Omega_\perp \approx n + \frac{3}{2}, n=1,2,\ldots$). Among these modes, the second half harmonic mode ($\omega / \Omega_\perp \approx 3/2$) is the one which has the largest relevant range of density for instability, and hence occurs most frequently in the magnetosphere. In those two sections, we have obtained some important parameters for the waves and the plasma. In order to obtain agreement with the nature of the waves observed in the trapping region, $f_w$ has been shown to be of the non-monotonic type rather than of the temperature anisotropic type. We have found the temperature ratio between the two species to be approximately $0.1$, and the lower bound for the density ratio, $N_c / N_w$, to be $10^{-2}$. The cold electron density is normally above $0.1 \text{ cm}^{-3}$. The upper and lower limits on
for the unstable lower frequency waves have been found in
(3.4.7), (3.4.9) respectively. The lower limit on \( k \) is given
by (3.4.4). The wave growth rate decreases like \( k^{-3} \) for
large \( k \). Hence, the essential shape of the wave spectrum
has been obtained, and except for the precise sharpness of
the positive slope region in the warm \( \nu_\perp \)-distribution, the
other important plasma parameters and the type of distribution
function have also been made clear.

For the high frequency range \( (\omega_\perp \gg 1) \), unstable waves
with frequencies slightly above each harmonic can occur at
reasonable densities \((\gtrsim 1 \text{ cm}^{-3})\). The only way we have found,
besides going to high density, to obtain the high frequency
instabilities at or above the half harmonics is to have a
sharp positive slope in the non-monotonic \( \nu_\perp \)-distribution.

Some geophysical implications have been discussed in
Section 3.6. The instability criterion, which requires a cold
species and an anomalous warm species, predicts, in good agree-
ment with experimental observation, that these electrostatic
waves should occur most frequently on the morning side of the
trapping region. We have conjectured that the extremely strong
waves on the auroral magnetic shell during substorms are the
result of an extra amount of incoming warm electrons during
substorms. The location of the largest wave amplitude should
be the place where the incoming warm electrons first coexist
with the cold species, which triggers the instability.

We have studied the quasi-linear particle diffusion in
velocity space in Chapter 4. An expression for the diffusion
tensor which may be useful for further theoretical study
of the quasi-linear diffusion has been obtained. Using the
obtained diffusion tensor, we have shown that in the trapping
region the frequently observed wave amplitude ($E \lesssim 10 \text{mv/m}$) is
large enough to cause strong pitch-angle diffusion for the
lower energy electrons ($\lesssim 1 \text{ kev}$). However, for high energy
electrons ($>10 \text{ kev}$), the diffusion is weak unless the wave
amplitude is as large as that observed on the auroral magnetic
shell during substorms (ie. $E \gtrsim 100 \text{ mv/m}$). Based on the
characteristics of the diffusion tensor, the pitch-angle
diffusion has been found to be accompanied by a substantial
particle acceleration, which has indeed been observed in
satellite experiments.
5.2 Proposed Future Work

We have thoroughly studied the lower frequency modes in this thesis, and obtained quite a bit of information about the plasma parameters and wave characteristics. The high frequency waves, whose wavelengths have been found to be much shorter than those of the lower frequency waves, are closely related to the smaller scale anomalies of the distribution function. If future satellite observations can cover higher frequency ranges with better accuracy, the theoretical study can be enhanced by an understanding of the smaller scale character of the distribution function, such as the sharpness of the positive slope and small irregularities in the distribution function. Also, some of the theoretical assumptions and predictions, such as the two species model, the non-monotonic feature, the predicted temperature ratio and density ratio, the predicted wavelength and wave orientation, and the conclusion about the spatial distribution of the electrostatic waves, are all waiting to be checked further by more satellite observations.

Another interesting subject for future work is a self-consistent study of the quasi-linear diffusion by these electrostatic waves and the time evolution of the distribution function. The wave spectrum can be obtained linearly from the instantaneous distribution function, and the diffusion tensor is then determined by the wave spectrum as we have done in Chapter 4. The temporal variation of the distribution
changes the wave spectrum and hence the diffusion tensor. Thus, a quasi-steady state for the distribution function can be obtained along with a saturation level for the wave amplitude. The latter can be compared with the detected wave amplitude, and the former has to be checked by future satellite observations of the detailed structure of the distribution function. The anisotropy of the temporally evolving distribution function will be useful in understanding the question of whether the electrostatic waves are controlling the local distribution function and can destabilize the whistler mode. At the same time, a comparison between the pitch-angle diffusion and particle acceleration is important in understanding the cause of the high energy electrons which must be locally accelerated somewhere inside the magnetosphere.

Finally, electrostatic ion cyclotron waves are yet to be studied. Satellite experiments can be done by extending the wave detection to low frequency, and theoretical work parallel to the procedure in this thesis should also be carried out.
Appendix A

PARAMETRIC FUNCTION: Y

The general dispersion function (3.1.4) for an infinite, homogeneous, magnetized plasma is largely determined by the parametric functions $\mathcal{C}_n (\lambda)$, $\mathcal{D}_n (\lambda)$, and $Y(z_n)$ defined in (3.1.5), (3.1.6) and (3.1.7) respectively. It is important to understand the character of these parametric functions which represent the collective as well as resonant response of the plasma to the various modes. Therefore, the characteristics of these parametric functions for different kinds of distribution functions and wavelengths will be discussed in this and the next appendix.

If a Maxwellian form is chosen for $f_\parallel (\nu_\parallel)$, i.e.

$$f_\parallel (\nu_\parallel) = \frac{1}{\pi^{\frac{1}{2}} \sigma} \exp\left[-(\nu_\parallel / \nu_\parallel)^2\right],$$

(A.1)

$Y(z)$ simply turns out to be the tabulated plasma dispersion function, $Z(z)$ (Fried and Conte, 1961). Its asymptotic and series expansions are

$$Z(z) \sim \frac{-i}{\delta} \left(1 + \frac{1}{2 \delta^2} + \frac{3}{4 \delta^4} + \cdots \right) + i \pi^{\frac{1}{2}} \sigma e^{-\frac{z^2}{\delta^2}}$$

(A.2)

$$Z'(z) \sim \frac{1}{\delta^2} \left(1 + \frac{3}{2 \delta^2} + \frac{15}{4 \delta^4} + \cdots \right) - i 2 \pi^{\frac{1}{2}} \sigma \frac{z}{\delta} e^{-\frac{z^2}{\delta^2}}$$

(A.3)

for $|z| >> 1$, and
The three asymptotic expansions for $Z(z)$ at $\text{Im}z = 0$ are consistent with the fact that the actual function $Z(z)$ is continuous across the real axis, since the discontinuities in the asymptotic expansions occur only in the exponentially small imaginary part. This is an example of the Stokes phenomenon. It is of interest to note that the imaginary parts of $Y(z)$ and $Y'(z)$ for $\text{Im}z = 0$ are proportional to $f_{\|}(v_\|/\alpha_{\|} = z)$ and $f_{\|}'(v_\|/\alpha_{\|} = z)$ respectively. It is shown in Fig. (A-1) that for $f_{\|}(v_\|)$ with a long tail the corresponding $\text{Im}Y(z)$ extends proportionally further along the $z$-axes compared to the $\text{Im}Y(z)$ corresponding to a Maxwellian $f_{\|}(v_\|)$. As for $\text{Re}Y(z)$ and $\text{Re}Y'(z)$, the existence of a long tail in $f_{\|}(v_\|)$ does not change their shape much. For a Maxwellian $f_{\|}$, the corresponding $\text{Re}Z$ and $\text{Re}Z'$ are shown in Fig. (A-2).

Physically, the imaginary part of $Y(z)$ can be regarded as representative of the irreversible resonant response of the plasma to the wave. The real part of $Y(z)$ is associated with the reversible nonresonant response of the plasma to the wave. For the case
Fig. (A-1) Plots for Im $Y$ and Im $Y'$. The usual plasma dispersion function $Z$ and $Z'$ for a Maxwellian distribution function are plotted in the solid curves, while the dashed curves are for the distribution function with a high-energy tail.
Fig. (A-2) Plots for $\text{Re}Y$ and $\text{Re}Y'$. The high-energy tail in the distribution function does not make any recognizable difference as in the plots Fig. (A-1) for $\text{Im} \imath$ and $\text{Im} \imath'$. 
of zero magnetic field, the particles contributing to the resonant response are those satisfying the resonant condition \( \mathbf{v} \cdot \mathbf{k} - \omega = 0 \). For a magnetized plasma, the resonant condition is \( v_n k_n - \omega + n \alpha = 0 \), \( n = -\infty \) to \( +\infty \). Hence, only a small part of the total particle population can interact with the wave resonantly. The rest of the particles which are perturbed non-resonantly by the wave are the majority of the particle population, and hence, their response is not dominated by any particular group of particles but rather represented as a collective phenomenon for the total population.

Mathematically, when the wave growth rate is small the resonant response of the plasma is represented by the residue part of the \( v_n - \) integration evaluated at the resonant singularity, \( k_n v_n - \omega + n \alpha = 0 \). The residue part is represented by the last term in the expressions (A.2) and (A.4) for \( i \). The non-resonant response is represented by the principal part of the \( v_n - \) integration indicated by the series expressions in (A.2) and (A.4) for \( i \). If the growth rate is large, significant irreversible interaction between the wave and the non-resonant particles begins to show up as imaginary contributions from the series expressions in (A.2) and (A.4).
Appendix B
PARAMETRIC FUNCTIONS: \( C_n \) AND \( D_n \)

The shapes of \( C_n (\lambda) \) and \( D_n (\lambda) \) defined in (3.1.5) and (3.1.6), respectively, depend on the shape of \( f_\lambda (V_\lambda) \). At large values of \( \lambda \), \( C_n (\lambda) \) and \( D_n (\lambda) \) are sensitive to the smaller scale variations, and especially to those at small \( v_\lambda \), of \( f_\lambda (V_\lambda) \) and \( f_\lambda ' (V_\lambda) \). In contrast, \( C_n (\lambda) \) and \( D_n (\lambda) \) at small \( \lambda \) are sensitive only to the gross features and average characteristics of \( f_\lambda (v_\lambda) \) and \( f_\lambda ' (v_\lambda) \).

From the definition of \( C_n (\lambda) \) in (3.1.5), it is clear that \( C_n (\lambda) \) is positive for all \( \lambda \) and oscillates somewhat like \( J_n^2 \) does. When \( f_\lambda (v_\lambda) \) is a monotonically decreasing function, \( D_n (\lambda) \) is also positive for all \( \lambda \). In particular, for a Maxwellian \( f_\lambda (v_\lambda) \),

\[
C_n(\lambda) = D_n(\lambda) = I_n(\lambda) e^{-\lambda} \geq 0 , \quad (B.1)
\]

where \( I_n (\lambda) \) is the modified Bessel function.

Consider the case where \( f_\lambda \) has a positive slope region. A typical example is the Dory, Guest and Harris distribution function,

\[
\int_{DGH} = \frac{1}{\pi^\frac{d}{2} \chi _{th}^2} \left( \frac{V_\lambda}{\chi _{th}} \right)^2 \left( \frac{V_\lambda}{\chi _{th}} \right)^2 e^{-\frac{(V_\lambda)^2}{2\chi _{th}^2}} , \quad (B.2)
\]
where \( j \) is a positive integer and \( \lambda_\text{th} \) is a characteristic particle speed. \( f_{\text{DGH}} \) vanishes at the origin and has a maximum at \( v_1/\lambda_\text{th} = \frac{j^{\frac{1}{2}}}{\ell} \). To study the non-monotonic \( \nu - \) distribution in general let us first note the location of the maximum, \( p_{1,\nu} \), and the first zero, \( q_{1,\nu} \), of \( J_n^2 \), namely

\[
p_{1,\nu} \approx 0.8 + 1.1 \, n, \quad n > 0
\]  

and

\[
q_{1,\nu} \approx 2.25 + 1.45 \, n, \quad n \geq 0.
\]  

For a typical non-monotonic \( \nu - \) distribution \( (f_1(0)=0) \), \( D_n(\lambda) \) can be estimated by matching the curves of \( J_n^2 (\sqrt{\frac{\pi}{2}} \frac{\nu_1}{\lambda}) \) and \( f'_1 (v_1) \) as in Fig. (B-1a). There \( n \) is taken as 1. The diagram is sketched on the scale of \( \sqrt{\frac{\pi}{2}} \frac{\nu_1}{\lambda} \) so that by increasing \( \lambda \) from \( \lambda_1 \) to \( \lambda_2 \) the dashed curve for \( f'_1 (v_1) \) flattens gradually from curve (1) to curve (3). Clearly, \( D_n (\lambda_1) \) is positive and \( D_n (\lambda_2) \) has reached its negative extremum by the matching of the peak of \( f'_1 (v_1) \) with the first maximum of \( J_n^2 \). For all larger \( \lambda \) \( D_n \) is negative but approaches zero gradually. Therefore, by increasing \( \lambda \) from zero, the corresponding \( D_n \) increases from zero to a positive value, reaches a maximum, then decreases to a negative value and reaches a negative extremum when the peak of \( f'_1 (v_1) \) coincides with the first peak of \( J_n^2 \). For the \( f_1 \) having a broad positive slope region (e.g. \( j = 1 \) for \( f_{\text{DGH}} \)), a further increase of \( \lambda \) brings the corresponding \( D_n \) asymptotically toward zero from below. A change of sign of \( D_n \) from negative to positive is possible.
**Fig. (B-1a)**  Plot of a non-monotonic $f'$ against a fixed $J_1^2$ curve. The $f'$ is shown as curve (1), (2), and (3) as $\lambda$ increases from $\lambda_1$ to $\lambda_2$ to $\lambda_3$. This diagram shows how the sign of $D_1$ changes from positive to negative for increasing $\lambda$.

**Fig. (B-1b)**  Plot of a non-monotonic $f'$ with sharp positive slope against fixed $J_1^2$ and $J_2^2$. The diagram shows how to obtain $D_1 > 0$ and $D_2 < 0$ for a sharp positive-slope $f_1$. 

---

$\lambda$ 

$\sqrt{2/\lambda}$ 

$\nu_1/\alpha_2$ 

5.0 

10.0
only if the positive slope region of $f_\perp$ is narrow enough. The distribution $f_\perp$ itself does not have to be narrow (see App. C). Positive $D_n(\lambda)$ can thus be obtained by matching the peak of $f_\perp'(\nu_\perp)$ with the higher order zeros of $J_n(\overline{\frac{\mu \nu_\perp}{\alpha_\perp}})$ as shown in Fig. (B-1b). At the same time, because $J_n$ and $J_{n+1}$ are roughly sinusoidal and 90° out of phase, a negative $D_{n+1}$ can be obtained for the same $\lambda$ (see Appendix C). Hence, we can have

$$D_n(\lambda) > 0, \quad D_{n+1}(\lambda) < 0, \quad n > 0. \quad (B.5)$$

Such pairing of $D_n > 0$ and $D_{n+1} < 0$ has special importance for the Dory, Guest, and Harris flute-like mode. Also, they are important for higher half harmonic modes (i.e. $\omega \sim (n+\frac{1}{2}) \lambda$).

From the above demonstration we conclude as follows.

For a monotonically decreasing $f_\perp(\nu_\perp)$, the corresponding $C_n(\lambda)$ and $D_n(\lambda)$ are positive. Fig. (B-2) shows examples of $C_n(\lambda)$ for the case of a Maxwellian $f_\perp(\nu_\perp)$. When $f_\perp'(\nu_\perp)$ has a small positive peak (e.g. $f_1(0) > 0$), $D_n(\lambda)$ can be slightly negative. The magnitude of the negative $D_n(\lambda)$ increases gradually as the positive slope sharpens and $f_\perp(0)$ approaches zero. The correspondence between an increasing positive slope and the increasing magnitude of the negative $D_n(\lambda)$ is illustrated in Fig. (B-3a), where we assume that

$$f_\perp(\nu_\perp) = \frac{1}{1+\rho} \left[ \rho \times f'(\nu_\perp) + f(\nu_\perp) \right], \quad (B.6)$$
Fig. (B-2) Plots of $C_n(\lambda)$, $n=1, 3, 5, 7$ for a Maxwellian $f_\perp$. 
Fig. (B-3a) D-functions for a partially filled up non-monotonic $f_4$ defined in (B.6).
\( f_1(V_1) \) is taken as Maxwellian (i.e. \( j=0 \) for \( f_{DGH} \)) and \( f_2(V_1) \) is taken as the first order (\( j = 1 \)) \( f_{DGH} \). The parameter \( p \) in (B.6) is used for modifying the height of the positive peak of \( f_1'(V_1) \). For a fixed \( \hat{\rho}_L \), \( D_1 \) is shown in Fig. (B-3a) to become less negative as the height of the positive peak decreases (i.e. \( p \) increases). At the same time, the corresponding variation for \( C_1 \) as shown in Fig. (B-3b) is less significant. Because of the close association between the negative \( D_n \) and a positive slope region, which is usually equivalent to a so-called "loss-cone like" distribution, the terms related with \( D_n \) in the dispersion relation (3.1.4) are usually called the "loss cone terms".

Curves of \( C_n, D_n, n = 1, 3, 5, 7 \) for the standard \( f_{DGH}, j = 1 \), are shown in Fig. (B-4a) and (B-4b) for comparison.

If the positive slope region of \( f_1(V_1) \) is sharp and narrow enough, (B.5) can be satisfied for some values of \( \lambda \).

To illustrate this let us take the distribution as

\[
\hat{f}_1(V_1) = 1 \cdot q(V_1) \cdot f_{DGH}(V_1), \tag{B.7}
\]

where

\[
q(V_1) = 1 + \tanh \left[ S(\tfrac{1}{2} \theta_{\text{th}} - V_0) \right], \tag{B.8}
\]

\( V_0 < (\tfrac{1}{2} \theta_{\text{th}})^{-1} \).
Fig. (B-3b) Plots of C-functions for the same p as in Fig. (B-3a)
Fig. (B-4)  C-functions and D-functions for Dory, Guest, Harris distribution function, $f_{DGH}$, $j=1$. 
and $I$ is a normalization constant. $f_{DGH}(V_1)$ is defined in (B.2), and $g(V_1)$ is a parametric function which can modify the location and width of the positive slope region of $f_{DGH}$ by taking different values of $v_0$ and $s$, while leaving the high velocity side of $f_{DGH}$ not much affected. This class of distribution functions also allows us to model more closely the actual laboratory plasma distributions (Coensgen et al, 1969). A comparison between the usual $f_{DGH}$ with $j = 1, 2, 3,$ and 6, and the modified $f_{DGH}$ defined in (B.7) with $j = 1$, $v_0 = 0.8$, and $s = 5, 8, 15$ is shown in Fig. (B-5). Apparently, this class of modified $f_{DGH}$ distributions has broad thermal widths but narrow positive slope regions.

Fig. (B-6a) and Fig. (B-6b) show the curves of $D_1(\lambda)$ and $D_2(\lambda)$ corresponding to the modified loss-cone type distribution defined in (B.7) with $v_0 = 0.9$, $s = 10$, and $j = 1$, and 2 respectively. The corresponding curves of $C_1(\lambda)$ and $C_2(\lambda)$ are shown in Fig. (B-7a) and Fig. (B-7b). Comparing Fig. (B-6a), (B-6b) with Fig. (B-3a), which is derived from a class of broad loss-cone distributions, we can see that the narrow positive slope region is responsible for the possibility of having positive $D_n(\lambda)$ and negative $D_{n+1}(\lambda)$. 
Fig. (B-5) A comparison between $f_{DGH}$ and the modified $f_{DGH}$ defined in (B.7). The latter has a nearly positive slope at a broad width.
Fig. (B-6a) $D_1$ and $D_2$ for modified $f_{D4M}$ defined in (3.7). Here we define $j=1$, $v_0=0.9$, and $a=10$.

Fig. (B-6b) $D_1$ and $D_2$ for modified $f_{D4M}$, $j=2$, $v_0=0.9$, and $a=10$. 
Fig. (B-7a). Plots of C-functions for the same modified $f_2$ as in Fig. (B-6a).
FIG. (B-7b)  Plots of C-functions for the same modified $f_m$ as in Fig. (B-6b).
Appendix C

FLUTE-LIKE INSTABILITIES IN MIRROR CONFINED PLASMAS WITH BROAD $v_e$-DISTRIBUTIONS

The paper reproduced on the following pages was produced in the course of this thesis research and is reproduced here to simplify the discussion of the flute-like modes in Section 3.3. It is to be published in the Physics of Fluids.
ABSTRACT

It is demonstrated that it is the width of the positive derivative region of the $v_\alpha$ distribution function which determines if the flute-like microinstabilities of a magnetized plasma occur at high densities.
Flute-like \((k_z = 0)\) instabilities in a homogeneous, magnetized, one-species plasma, first discussed by Dory, Guest and Harris, are known to appear when the perpendicular velocity distribution function, \(f(v_z)\), is sufficiently sharply peaked. Generally speaking, broadening of the distribution is usually thought to provide stabilization, and, conversely, the narrowness of the distribution function is frequently regarded as responsible for the instabilities. The main theme of this note is the observation that these instabilities can develop in plasmas having broad as well as narrow \(v_z\) distributions whenever the width of the positive derivative (or, as we will refer to it, "positive-slope") region of the distribution function is sufficiently narrow. When the positive-slope region is narrow enough for instability to occur, there remains a (slight) dependence of the threshold density on the width of the distribution function.

The approximate criteria for occurrence of flute-like instabilities of the \(n\)th mode are

\[
D_n > 0 \quad \text{and} \quad D_{n+1} < 0
\]

where \(D_n\) is defined as

\[
D_n = -\frac{\alpha^2}{2} \int_0^{\infty} 2\pi v_z \, dv_z \, \frac{\partial f}{\partial v_z} \, J_0^2 \left( \frac{k_z v_z}{\Omega} \right).
\]

When (1) and (2) are fulfilled, the right side of the dispersion relation

\[
\frac{\omega_p^2}{\omega^2} = \frac{\hbar}{k^2 a^2 \Omega} \sum_{n=1}^{\infty} \frac{n^2 \Omega^2}{\omega^2 - n^2 \Omega^2} \, D_n
\]

has a positive minimum (as a function of frequency \(\omega\) for given \(k_z\)) and hence instability obtains. Standard notation has been used in the above expressions.
Consider a distribution function $f_1$ such that a positive peak of $\frac{3f_1}{3v_1}$ occurs. If this peak is narrow and has a location (in $v_1$) such that it fits one of the valleys of $J_n^2(k_1v_1/\Omega)$, then the magnitude of the negative contribution to the integral of $J_n^2$ in (3) can be reduced to a small value, and we may obtain a positive $D_n$. At the same time, because $J_n$ and $J_{n+1}$ are roughly sinusoidal and about 90° out of phase, a negative $D_{n+1}$ can easily be obtained for the same $k_1$. This argument suggests the potential importance of the positive-slope region.

In order to study the relevance of the positive-slope region to the flute-like instabilities, let us use a parametric function $g$ to modify the usual Dory, Guest and Harris loss-cone distribution $f_o$:

$$f_1 = Ngf_o^{(j)}$$

$$f_o^{(j)} = \frac{1}{\pi a_1^2} \left( \frac{v_1}{a_1} \right)^2 \exp\left(-\frac{v_1^2}{a_1^2}\right)$$

$$g = 1 + \tanh \left[ s \left( \frac{v_1}{a_1} - v_0 \right) \right].$$

$N$ is a normalization constant. Note that it is mainly the location and width of the positive-slope region which are modified by the parameters $v_0$ and $s$, while the high velocity side of the distribution is not significantly affected. This class of distribution functions also allows us to much more closely model actual experimental distributions than is possible with $f_o^{(j)}$ alone.

In Fig. 1a, the threshold densities, minimized with respect to $k_1$, of the first harmonic mode for various values of $v_0$, $s$, and $j$, are plotted against the relative width, $\delta v/v_f$, of the positive-slope region, which in turn is a function of $v_0$, $s$, and $j$. Here, $\delta v_f$ is defined as

$$\int_{v_f - 0.56\delta v_f}^{v_f + 0.56\delta v_f} f_1'(v_1) dv_1 = 0.8 \int_{0}^{v_f} f_1'(v_1) dv_1$$

(8)
i.e.,

\[ f_{\perp}(v_f', + 0.58v_f, ) - f_{\perp}(v_f' - 0.58v_f,) \equiv 0.8f_{\perp}(v_f') \]  

(9)

where \( v_f' \) and \( v_f '' \) are the velocities at the maxima of \( f_{\perp}(v_f') \) and \( f_{\perp}(v_f'') \), respectively. Thus, \( \delta v_f ' \) is defined as the width of the region encompassing 80% (by area) of the positive-slope region. The seven curves in Fig. la represent a wide variety of distribution functions and, except for the lowest one which is obtained by using the usual distributions (i.e. \( s = 0 \)), all have large thermal spreads (i.e., \( j < 5 \), the critical value for instability of these first harmonic modes according to the thermal width criterion). Thus, we see that instabilities exist at finite density if the relative width of the positive slope region satisfies

\[ \frac{\delta v_f'}{v_f'} \leq 0.39 \pm 0.03. \]  

(10)

These small (\( \pm 0.03 \)) variations in the critical \( \delta v/v_f' \) are caused by the effects of thermal spread, the slope of \( f_{\perp}(v_f') \) on the high velocity side, and the details of \( f\pm(v_f) \). Note that when (10) is satisfied, the threshold density for instability does depend slightly on the thermal spread \( (j) \) of the distribution function. The wavenumbers at the threshold densities are found to be in the range

\[ \frac{k_{\perp}v_f'}{\alpha} = 3.7 \pm 0.25 \]  

(11)

for all the distributions. In other words, for the first harmonic mode to be unstable the corresponding \( k_{\perp} \) is so determined at threshold density that the maximum of \( f_{\pm}(v_f') \) occurs near the first zero of \( j^2(k_{\perp}v_f'/\alpha) \). This result, valid for all except very narrow distributions \( (j \geq 20) \), not only provides an easy way of predicting the wavelength at threshold density but also indicates a close association between Doré, Guest and Harris instabilities and the positive-slope region.
To further verify the above statement we have also modified the distributions \( f_1(v_1) \) by cutting out their low energy tails. Specifically, the low velocity side of the positive slope region is linearly extrapolated to zero with a line tangent to \( f_1(v_1) \) at the point of maximum \( f'_1 \). The results for these "cut-off" distribution show (Fig. 1b) that the above criteria concerning the relative width and threshold wavelength are still valid.

The zero-frequency mode has also been studied. The corresponding threshold density curves are shown in Fig. 2, where the broken curves are for the cut-off distributions. The distributions we have used lead to finite threshold densities for this mode if

\[
\frac{\delta v}{v}\bigg|_{f_1'<0} < 0.55 \pm 0.03. \tag{12}
\]

Except for very narrow distributions \((j > 15)\), the wavenumbers at threshold are found to be in the range

\[
\frac{k_1 v_{f_1'}}{\Omega} = 2.35 \pm 0.15, \tag{13}
\]

which is again close to the first zero \((\sim 2.4)\) of the relevant Bessel function.

We have not studied the higher harmonic modes or the bands of higher wavenumbers for the lower modes. However, similar results can be expected; moreover, the lowest bands of the first and zero-frequency modes are usually the most important because their occurrence usually requires the lowest threshold density and they are compatible with the least narrow positive-slope region.

It is known that the (electron) distribution function of the plasma in the magnetosphere is generally very broad but may have large positive derivatives. When the effects of only a very small amount of cold "impurity" are added to the dispersion relation, the above type of instabilities, associated with a narrow positive-slope region for the "hot" electrons, become one of the candidates.
for explanation of the electrostatic emissions observed at $\omega = (n + 1/2)\Omega_e$, by Kennel et al.\textsuperscript{6} on the morning side of the outer radiation belts. In addition, the above criteria concerning the threshold densities, wavelengths, and the widths of the positive-slope regions offer an easier means of identification of the flute-like mode observed in laboratory plasmas once the $f_L(v_L)$ distribution is known.\textsuperscript{4}

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5. The critical value is given as 6 in Ref. 1. We calculate the D's from a numerical integration scheme and find instability occurring with j = 5 for $\omega_p^2/\Omega^2 \geq 250$, $\omega = 1.06 \Omega$, and $k_y v_f/\Omega$ in the narrow range of 4.72 $\pm$ 0.06.
Figure Captions

Fig. 1. Minimum threshold density for instability versus thermal width of the positive-slope region for the $\omega = 1.2 \Omega$ instabilities. $J, v_0$ are parameters defined in Eqs. (6) and (7). (a) Curves for distributions given in Eqs. (5) - (7). (b) Curves for "cut-off" distributions.

Fig. 2. Minimum threshold density for instability versus thermal width of the positive-slope region for the $\omega = 0$ instabilities. Solid lines are for distributions given in Eqs. (5) - (7), dotted lines indicate results for cut-off distributions.
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BIOGRAPHICAL NOTE

The author received his undergraduate education at National Taiwan University, Taiwan, The Republic of China, and received a B.S. in Physics in 1963. He attended graduate schools in the Geophysical Institute at University of Alaska during the period 1964 to 1966 (M.S.) and at M.I.T. (Ph.D.) during 1967 to 1971. The author has a semester of teaching experience in Taiwan during 1967 at National Taiwan University and National Central University.