MASSACHUSETTS INSTITUTE OF TECHNOLOGY

PROJECT MAC

Artificial Intelligence
Memo No. 196 June 1970

Hypergeometric Functions in MATHLAB

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I. Introduction

This Memo describes some of the more important properties and manipulations of Hypergeometric Functions, ${}_pF_q(.)$, which may be useful in MATHLAB. A convention for representing ${}_pF_q(.)$ is adopted which is readily adaptable to LISP operations.

The most general type of HGF with which we will be concerned is a function of a single independent variable, x, and is parametricized by an "A" list, of length p, and a "B" list, of length q. The latter consist, in general, of atoms; the argument is usually x, but may also be a simple function of x.

The power of the Hypergeometric Formulation arises due to:

(i) Nearly every Tabulated Function can be represented as a HGF, for example the entire families of Bessel functions, and Orthogonal Polynomials, as well as Elliptic, Error, and Incomplete Gamma Functions.

(ii) Linear operations, such as differentiation, integration (definite or indefinte), integral transforms, etc., amount to simple manipulations of the list representations of the HGF.

(iii) Analytic continuation is easily accomplished by operations which allow the variable, x, to map into 1/x, x/(x-1), 1-1/x, 1/(1-x), simultaneously with simple linear operations on the "A" and "B" lists. (This statement should be qualified to the extent that such continuations are <u>always</u> possible for $_2F_1(\cdot)$, are not required for $_pF_q(\cdot)$ where both p and q are less than two, and are not known -- possibly do not exist -- when both p and q are greater than unity.

(iv) The solution of a wide class of homogeneous DE's with nonconstant coefficients, say, of the form:

$$(ax^2 + bx + c)y'' + (dx + e)y' + fy = 0$$

can be represented as a HGF, after an appropriate change of variables.

Other advantages of the approach could be enumerated, as should be clear. In essence, this is also the principal drawback of the technique so far as user-oriented applications are concerned.

Itemizing:

(i) It is time- and core-consuming to recognize and interpret properly into HGF-form an arbitrary function presented by the user. Once this has been accomplished the lowest-level operation are rapid, as is claimed above. However, the result of the latte manipulations must also be represented in HGF notation, and now the process has to be "unwound". In general, this amounts to a fairly long sequence of tests for various linear relationships among members of the "A" and "B" lists, which, if satisfied, identify the HGF as a known tabulated function. If these tests fail, then the HGF must be returned as is. (It should be mentioned that if this happens to be the case, the user now has an easily computable series representation.)

(ii) In the same vein, such an all-or-nothing approach may be foolhardy. For example, if we wished to evaluate:

$$\int xe^{-x}dx ,$$

we could represent it as

$$\int (1 - {}_1F_1(-1;a;ax)) \cdot ({}_pF_p(a,b,c,\ldots;a,b,c,\ldots;x))dx .$$

It works, but who needs it?

(iii) Algebraic and rational functions are difficult and awkard to represent in terms of ${}_pF_q(.)$. Thus, when these are present in a function presented by a user, they must be sorted out and dealt with in a different manner.

(iv) Similarly, elementary transcendental functions (i.e. log(.), and all of those based on e^x) are not to be subject to transformation into HGF representation.

(v) It seems from the above that the user must be subject to some fairly strict, though not overly demanding, ground rules as regards the convention of the ordering of the elements of the function presented. All we ask is that these elements be ordered according to the following hierarchy:

(algebraic, rational)(log, e^x, sin, cos, etc.,)(higher fns.)

For example, if the given function were

$$F(x) = f_1(x) + f_2(x) + \ldots + f_n(x),$$

then each of the $f_k(x)$ would be represented by an ordered triple, e.g.

$$F(x) = x^k \sin x \, I_o(x) + \log(x)$$

((x**k)(sin x)(fhyp ()(1)((x**2)/4)))

+ (()(log x)()),

where $I_o(x)$ is the modified Bessel Function:

$$I_o(x) \stackrel{\Delta}{=} {}_oF_1(\, ;1;x^2/4).$$

This format, or something similar (I am open to suggestions), should assist the interpreter considerably and help to avoid errors.

(v) Concerning Differential Equations (see I (iv) above), the fundamental solutions are indeed linear combinations of ${}_2F_1(.)$'s; there are other solutions, of course, depending on the solutions of the associated indicial equation. However, these can be constructed in a straight-forward manner by means of the transformation and continuation formulas mentioned in I (ii) and I (iii) above. Similar comments hold for the Hypergeometric DE which yields ${}_1F_1(.)$ as its fundamental solution. For other values of p and q, the question is usually rarely encountered, trivial, or unanswerable.

It should be made clear that the purpose of the research effort outlined herein is not intended to supplant or to rewrite substantially the present MATHLAB; rather, the aim is to augment the present capability for symbolic manipulation in order to deal with a wider class of functions which are often encountered in applications, and which are not recognizable by the present system. Indeed, as implied in the comments under "drawbacks", above, such methods should be employed only as a last resort, when other more conventional techniques have fallen short. It is intended to be another tool in the kit, as it were, rather than the be-all and end-all. In its defense, however, it may be noted that the example given in "drawbacks" (v) above* can be dealt with efficiently only be recourse to the HGF notation and operations, if any claim to generality is to be realized.

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*This expression, or its equivalent, occurs with appalling frequency in applications as diverse as MHD, Quantum Mechanics, and Statistical Detection Theory.

The Theory of Hypergeometric Functions is quite extensive, and on the surface, at least, complex. Yet at the same time, it is very compact and very nearly trivial once all of the mathematical superstructure, accoutrements, and other random "elegances" and similar rubbish have been cleared aside. Essentially, an understanding of the entire body of proofs requires no more than a rudimentary familiarity with Taylor's Series and the convergence thereof, and the technique of Analytic Continuation. The rest is Algebra. The most difficult and subtle task is to establish a basic set of "axioms" from which all other properties and operations may be derived. The search is still going on, but it appears that they are at least four in number, and perhaps as many as seven.

*

The principal references used below are Slater (1), Slater (2), Bateman (3), and Bureau of Standards (4).

II. Definition

The Hypergeometric Function is defined by the Series:

$$ {}_pF_q(a_1,a_2,a_3,\ldots,a_p;\ b_1,b_2,b_3,\ldots,b_q;\ x) $$

$$ \triangleq\ {}_pF_q(\underline{A};\underline{B};x) $$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n(\ldots\;)(a_p)_n}{(b_1)_n(b_2)_n(\ldots\;)(b_q)_n} \frac{x^n}{n!}, \qquad (1)$$

where

$(a_1,a_2,\ldots,a_p) = \underline{A}$, is the "A" list, of length p,

$(b_1,b_2,\ldots,b_q) = \underline{B}$, is the "B" list, of length q,

and

$$(c)_k = (c+k-1)!/(c-1)!$$
$$= (c)(c+1)(c+2)\ldots(c+k-1),$$

where we assume that k is an integer, and x is the independent variable. Where c is not an integer, the second definition of $(c)_k$ can always be used; the first definition can be used if the factorial functions are replaced by the Gamma functions of (c+k) and (c), respectively . In various applications, it may be of advantage to adopt one of the three.

III. Elementary Properties

(It should be pointed out beforehand that in nearly all "interesting cases" either p=q or p=q+1.)

(i) Convergence:

(a) if p = q, the series converges uniformly for all real x.

(b) if $p = q + 1$, the radius of convergence of the series is $|x| < 1$; however, analytic continuation can generally be invoked to include the region $|x| > 1$; the singular point lies at $x = 1$; thus either series converges for $|x| = 1$, $\arg(x) \neq 0$.

(c) if $p > q + 1$, the series does not converge in any usual sense*.

(d) if $p < q$, the series converges uniformly for all x.

(ii) Trivial Reductions:

(a) if one or more members of the "A" list are zero, the value of the series is unity.

(b) if one or more members of the "A" list is a negative integer, say $-N$, then the series reduces to a finite polynomial of degree N.

(c) if any member of the "A" list is identical to any member of the "B" list then both of these are deleted from their respective lists, and both p and q are reduced by unity, e.g.

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*There are ways of handling this situation, however, by recourse to the theory of MacRobert's E-function or Meijer's G-function; we will not consider these at present.

$$_2F_2(a,b;b,c;x) = {_1F_1}(a;c;x) ,$$

$$_3F_2(1,2,3;2,3;x) = {_1F_0}(1; ;x) .$$

(d) <u>every</u> $_1F_0(\cdot)$ is a simple algebraic function:

$$_1F_0(a; ;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n$$

$$= (1-x)^{-a}. \tag{2}$$

(e) <u>every</u> $_0F_1(\cdot)$ is a Bessel Function:

$$\left.\begin{aligned} {_0F_1}(;a;x) &= \frac{\Gamma(a)}{x^{\frac{a-1}{2}}} I_{a-1}(2x^{\frac{1}{2}}), \\ {_0F_1}(;a;-x) &= \frac{\Gamma(a)}{x^{\frac{a-1}{2}}} J_{a-1}(2x^{\frac{1}{2}}). \end{aligned}\right\} \tag{3}$$

(f) if one or more members of the "B" list is zero or a negative integer, and cannot be cancelled by an an identical member from the "A" list, as in (c) above, then the series is undefined.

It is worthwhile to digress briefly in order to amplify some of the consequences of this restriction (f). It must be emphasized that when at least one of the b_k is zero or a negative integer, we cannot represent the function in <u>power-series</u> form meaningfully -- no more, no less. This may or may not have a bearing on the question as to whether or not the function exists in some other sense. The purpose of this aside is to illustrate some of the subtleties of the decision processes involved in the HGF formulation, and to point out that it has some limitations of an unexpected kind.

Consider the function F(s;n) defined by the integral:

$$F(s;n) = \frac{\Gamma(2n)}{\Gamma(n)\Gamma(n)} \int_0^\infty \frac{t^{n-1}}{(1+t)^{2n}} e^{-st} dt ,$$

(which happens to be the Characteristic Function of Fisher's F-Distribution (4)). There is no doubt that the integral converges for $n \geqslant 1$ and $Re(s) > 0$, but does it possess a Taylor's Series? Well, sort of.

Let us attempt to formally expand F(s;n) as

$$F(s;n) = \sum_k \left(\frac{\partial}{\partial s}\right)^k F(s;n) \Bigg|_{s=0} \frac{s^k}{k!} .$$

It is not difficult to show that this turns out to be

$$F(s;n) = \sum_k \frac{(n)(n+1)\ldots(n+k-1)}{(n-1)(n-2)\ldots(n-k)} \frac{(-1)^k}{k!} s^k$$

$$= \sum_k \frac{(n)(n+1)\ldots(n+k-1)}{(-n+1)(-n+1+1)\ldots(-n+k)} \frac{s^k}{k!}$$

$$= \sum_k \frac{(n)_k}{(1-n)_k} \frac{s^k}{k!}$$

$$\equiv {}_1F_1(n;1-n;s),$$ by "definition", at least in the purely formal sense. However, as is easily seen from the

first line above, the term in the series for $k = n$ blows up, as do all subsequent terms. The whole point of this exercise is to show that a HGF with one or more zero or negative integer members in its "B" list may or may not represent a meaningful function. To repeat, this simply means that no valid Taylor's (or Laurent) series exists. However, integral representations often exist in such cases, as the present example demonstrates. A subsequent section will list some of the more important of these.

At the risk of appearing to lapse into pedantry, one further comment must be made -- and this is probably the most important of all. Most of us with a reasonable working knowledge of higher mathematics would probably take it for granted that a function such as $F(s;n)$, which is clearly well-behaved for $n \geqslant 1$ and $\mathrm{Re}(s) > 0$, should possess a Taylor's (or other type of) series representation. Not so. The sleeper in this argument is the condition "$\mathrm{Re}(s) > 0$". $F(s;n)$ possesses an <u>essential singularity everywhere</u> in the left-half complex plane; furthermore, it is not analytic, and cannot be continued into the left-half plane. In brief, there exist well-behaved functions which can be represented only in integral form, and the HGF's encompass a wide class of these, at least formally.

IV. Derivatives

Essentially the only simple general formula which can be given concerning derivatives of HGF's is

$$\left(\frac{d}{dx}\right)^n \left\{ {}_pF_q(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;x) \right\}$$

$$= \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n} \cdot {}_pF_q(a_1+n,a_2+n,\ldots,a_p+n;$$

$$b_1+n,\ b_2+n,\ldots,\ b_q+n;x). \qquad (4)$$

The formulas given below for ${}_1F_1(\cdot)$ and ${}_2F_1(\cdot)$ can be generalized for any particular p,q pair with varying degrees of algebraic effort. Their development involves principally ad hoc methods for manipulation of series, and is difficult to systematize.

(i) For $p = q = 1$, we have

(a) $$\left(\frac{d}{dx}\right)^n \left\{ x^{a+n-1}\ {}_1F_1\ (a;b;x) \right\}$$

$$= (a)_n x^{a-1}\ {}_1F_1\ (a+n;b;x), \qquad (5)$$

(b) $$\left(\frac{d}{dx}\right)^n \left\{ e^{-x}\ {}_1F_1\ (a;b;x) \right\}$$

$$= \frac{(-1)^n(b-a)_n}{(b)_n} e^{-x}\ {}_1F_1\ (a;b+n;x)\ , \qquad (6)$$

(c) $$\left(\frac{d}{dx}\right)^n \left\{ e^{-x} x^{b-a+n-1} \; {}_1F_1(a;b;x) \right\}$$
$$= (b-a)_n \; e^{-x} x^{b-a-1} \; {}_1F_1(a-n;b;x) \quad , \qquad (7)$$

(d) $$\left(\frac{d}{dx}\right)^n \left\{ x^{b-1} \; {}_1F_1(a;b;x) \right\}$$
$$= (-1)^n (1-b)_n x^{b-1-n} \; {}_1F_1(a;b-n;x) \quad . \qquad (8)$$

These four relations demonstrate how either a or b can be increased or decreased by an arbitrary integer. If we wish to change both a and b, the requisite operators are concatenated (see below, under <u>Continguous Functions</u>).

(ii) for $p = 2$, $q = 1$, the analogous four relations are

(a) $$\left(\frac{d}{dx}\right)^n \left\{ x^{a+n-1} \; {}_2F_1(a,b;c;x) \right\}$$
$$= (a)_n x^{a-1} \; {}_2F_1(a+n,b;c;x) \quad , \qquad (9)$$

(b) $$\left(\frac{d}{dx}\right)^n \left\{ (1-x)^{a+b-c} \; {}_2F_1(a,b;c;x) \right\}$$
$$= \frac{(c-a)_n (c-b)_n}{(c)_n} \; (1-x)^{a+b-c-n} \; {}_2F_1(a,b;c+n;x) \quad , \qquad (10)$$

(c) $$\left(\frac{d}{dx}\right)^n \left\{ x^{c-a+n-1} (1-x)^{a+b-c} \; {}_2F_1(a,b;c;x) \right\}$$
$$= (c-a)_n x^{c-a-1} (1-x)^{a+b-c-n} \; {}_2F_1(a-n,b;c;x) \quad , \qquad (11)$$

(d) $$\left(\frac{d}{dx}\right)^n \left\{x^{c-1}\,{}_2F_1(a,b;c;x)\right\}$$

$$= (c-n)_n x^{c-n-1}\,{}_2F_1(a,b;c-n;x) \quad . \qquad (12)$$

The last comment of the previous subsection applies here.

V. Integrals

Most indefinite integrals which can be evaluated exactly in terms of HGF's are simply inversions of the results of the previous section. For example, the integral

$$\int_0^t x^{\nu} e^{-sx}\,{}_pF_q(\underline{A};\underline{B};x)\,dx$$

cannot in general be represented as a ${}_pF_q(\cdot)$.

(i) Two General Integrals

A vast number of definite integrals involving HGF's are tabulated (see Bateman (3), Tables of Integral Transforms, Vols. I, II). Two typical, and fairly general, examples are

$$\int_0^{\infty} t^{r-1} e^{-st}\,{}_pF_q(\underline{A};\underline{B};\lambda t)\,dt$$

$$= \frac{\Gamma(r)}{s^r}\,{}_{p+1}F_q(a_1,a_2,\ldots,a_p,r;\; b_1,b_2,\ldots,b_q;\lambda/s), \quad (13)$$

(LAPLACE TRANSFORM)

$$\int_0^1 (1-x)^{\nu-1} x^{s-1} {}_pF_q(\underline{A};\underline{B};\lambda x)\,dx$$

$$= \frac{\Gamma(\nu)\Gamma(s)}{\Gamma(\nu+s)} \; {}_{p+1}F_{q+1}(a_1,a_2,\ldots,a_p,s;\; b_1,b_2,s+\nu;\lambda) . \quad (14)$$

(MELLIN TRANSFORM)

(ii) Example:

We present an example below which demonstrates the power of the Hypergeometric formulation. The integral under consideration can be evaluated by more conventional techniques, but only by means of "intuitive" reasoning.

Consider

$$I = \int_0^\infty x^k \operatorname{erf}(x)\, e^{-ax^2} dx,$$

where erf(x) is the Error Function:

$$\operatorname{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt .$$

It happens to be true that (see <u>Tabulation</u>, below)

$$\operatorname{erf}(x) = \frac{2x}{\sqrt{\pi}} e^{-x^2} {}_1F_1(1;3/2;x^2) .$$

Substituting,

$$I = \frac{2}{\sqrt{\pi}} \int_0^\infty x^{k+1} e^{-(a+1)x^2} {}_1F_1(1;3/2;x^2)\,dx .$$

Let $x^2 \to t$:

$$I = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{k/2} \; e^{-(a+1)t} \; {}_1F_1(1;3/2;t)\,dt \; .$$

But this is identical in form to (13). Plugging in, we obtain

$$I = \frac{1}{\sqrt{\pi}} \frac{\Gamma(1 + k/2)}{(a+1)^{1 + k/2}} \; {}_2F_1(1,1 + k/2;\; 3/2;\; (a+1)^{-1}) \; .$$

Many readers will be singularly unimpressed by this result, inasmuch as one problem (the evaluation of the original integral) has been replaced by one equally obscure. However, there are several replies to this objection, namely:

(a) The integral has been reduced to a form which is easily computable numerically, should the need arise.

(b) For the particular case, k = 1, one of the members of the "A" list is equal to one of the members of the "B" list (= 3/2), so that the ${}_2F_1(.)$ reduces to a ${}_1F_0(.)$ and the result is trivial (see Elementary Properties, above).

(c) The function ${}_2F_1(.)$ is subject to a number of variable transformations (Analytic Continuation, below) and simple manipulation of the parameters A and B (Contiguous Functions, below). These may be invoked for both numerical and theoretical requirements.

One of these happens to transform the second expression on p. 16 into

$$I = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(1 + k/2)}{a\cdot(a+1)^{k/2}} \; {}_2F_1(1, \tfrac{1-k}{2}; 3/2; -\tfrac{1}{a}) \; .$$

It follows that if k is a <u>positive odd integer</u>, then a member of the "A" list is a <u>negative integer</u>, and that the ${}_2F_1(.)$ is a polynomial of finite degree (see <u>Elementary Properties</u>, above).

(d) It turns out that even if k is a <u>positive even integer</u> there are still some tricks left in the bag, to be described subsequently.

(e) If $k \neq 0,1,2...,n...$, we can be fairly well assured that the representation given in paragraph (c) above or the second on p. 16 is the best available.

For the record, it is not difficult to show that the original integral (middle p. 15), using more <u>ad hoc</u> methods,* can be reduced to:

$$I = \frac{\Gamma(1 + k/2)}{\sqrt{\pi}\, a^{k+1}} \int_0^{\tan^{-1}(1/a)} \cos^{k+1}(\theta)\, d\theta.$$

*by writing erf(x) in its integral representation, switching to polar coordinates and performing the r-integral.

For arbitrary values of a and k, no compact closed form is known for this integral.

(iii) It is usually desirable that the argument of the HGF be the independent variable, x, itself, rather than some f(x). If not, then, in general we perform the substitution y = f(x) and simplify throughout. If this restriction proves to be impractical then it can probably be eased somewhat to allow arguments of the form $\pm ax$ or $\pm ax^2$.

(iv) The cases in which the integrand involves the product of two or more HGF's lies beyond the scope of the present descriptions. When only two ${}_pF_q(.)$'s are involved the integral, if it converges in any reasonable sense, can be expressed in terms of the aforementioned E- and G-Functions. For three of more HGF's in the same integrand closed form results are available only in special cases.

(In particular, the last integral given in Bateman (4), Integral Transforms, Vol. II (and if a "God" integral exists, this has to be the closest known approximation) is

$$\int_0^\infty G^{mn}_{pq}\left(\alpha x \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) \cdot G^{kl}_{rs}\left(\beta x \,\middle|\, \begin{matrix} c_1,\ldots c_r \\ d_1,\ldots d_s \end{matrix}\right) dx$$

$$= \alpha^{-1}\, G^{k+n,\,l+m}_{q+r,\,p+s}\left(\beta/\alpha \,\middle|\, \begin{matrix} -b_1,-b_2,\ldots,-b_m,c_1,\ldots c_r,-b_{m+1},\ldots,-b_q \\ -a_1,-a_2,\ldots,-a_n,d_1,\ldots d_s,-a_{n+1},\ldots,-a_p \end{matrix}\right) \quad (15)$$

which is impressive sort of in the same sense that a 500-pound canary is. G (etc.) is Meijer's G-Function, which includes HGF's as a subclass.)

The limited results of this section are intended only to indicate the potentialities of employing the HGF notation for evaluating integrals. However, a surprisingly large number of integrals which arise in applications are of the form of either Eq. (13) or Eq. (14).

In the next section a listing of Integral Representations for HGF's is given which widens the present catalogue of integral properties.

VI. Integral Representations

Simple integral representations of ${}_pF_q(.)$ exist only for ${}_1F_1(.)$ and ${}_2F_1(.)$ for general values of the parameters, excluding the elementary cases ${}_0F_0(.)$, ${}_1F_0(.)$, ${}_0F_1(.)$. For other values of p and q the two integrals of the previous section may be utilized iteratively to define any given ${}_pF_q(.)$ in terms of multiple integrals.

Therefore we emphasize the (single-) integral representations of ${}_1F_1(.)$ and ${}_2F_1(.)$ in this section.

(i) Integral Representations of ${}_1F_1(a;b;x)$:

$$\text{(a)}\quad {}_1F_1(a;b;x) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_o^1 t^{a-1}(1-t)^{b-a-1}e^{xt}dt, \quad (16)$$

$$\text{(b)}\quad = \frac{\Gamma(b)}{\Gamma(b-a)\ \Gamma(a)}\ x^{1-b}\int_o^x e^v v^{a-1}\cdot$$

$$\cdot (x-v)^{b-a-1}dv, \quad (17)$$

$$\text{(c)}\quad = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\ e^x\int_o^1 e^{-wx}\ w^{b-a-1}\ .$$

$$\cdot\ (1-w)^{a-1}\ dw, \quad (18)$$

$$\text{(d)}\quad = \frac{2^{1-b}\Gamma(b)e^{x/2}}{\Gamma(a)\Gamma(b-a)}\int_{-1}^1 e^{-xs/2}(1+s)^{b-2}\ .$$

$$\cdot\ \left(\frac{1-s}{1+s}\right)^{a-1} ds, \quad (19)$$

And, in general,

$$\text{(e)}\quad = \frac{\Gamma(b)\exp\left\{-\frac{cx}{d-c}\right\}}{\Gamma(b-a)\Gamma(a)}\ (d-c)^{1-b}\ .$$

$$\cdot\int_c^d \exp\left\{\frac{xu}{d-c}\right\}(u-c)^{a-1}\ (d-u)^{b-a-1}\ du, \quad (20)$$

where, in all cases, we assume that $Re(b) > Re(a) > 0$.

(ii) Integral Representation of ${}_2F_1(a,b;c;x)$:

$$\text{(a)}\quad {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^\infty s^{b-1} \cdot$$

$$\cdot\ (1+s)^{a-c}\ (1+sx)^{-a}\ ds\ , \tag{21}$$

$$\text{(b)}\quad {}_2F_1(a,b;c;1-x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} \cdot$$

$$\cdot\ (1-t)^{c-b-1}\ (1-xt)^{-a}\ dt\ , \tag{22}$$

$$\text{(c)}\quad {}_2F_1(a,b;c;1/x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_1^\infty (s-1)^{c-b-1} \cdot$$

$$\cdot\ s^{a-c}\ (s-z)^{-a}\ ds\ , \tag{23}$$

$$\text{(d)}\quad {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^\infty e^{-bt} \cdot$$

$$\cdot\ (1-e^{-t})^{c-b-1}\ (1-ze^{-t})^{-a}\ dt\ . \tag{24}$$

Since a and b are equivalent parameters, another set of four relations can be generated simply by interchanging a and b everywhere. As above, we require that $Re(c) > Re(b)$ (or $Re(a)$) > 0.

(iii) There are a large number of alternative Integral Representations, of course, which are obtained by transformation of the variables of integration. These have been omitted for the most part for reasons given in the subsequent remark. Also, any ${}_pF_q(.)$ can be represented in terms of a complex contour integral known as a Mellin-Barnes line integral (Slater (1), (2)). These representations have also been disregarded, at least for the present, because of the difficulty of establishing simple rules for the selection of the proper contour of integration, as well as the lack of a mechanism for performing Residue Calculus.

(iv) Throughout this Memo it has been implicitly assumed that the independent variables of the HGF are simple functions. When this is not the case, especially in applications involving integrals and differential equations, it would seem to be more efficient to first make an appropriate change of variables and then apply a relatively small amount of sorting to the result, as opposed to having to deal with a much longer list of more general forms.

For example, given

$$I = \int_0^{\pi/2} \frac{(\sin t)^{2b-1} (\cos t)^{2c-2b-1}}{(1 - z \sin^2 t)^a} dt ,$$

we set $x = \sin^2(t)$ and obtain after simplification:

$$I = \tfrac{1}{2} \int_0^1 x^{b-1} (1-x)^{c-b-1} (1-zx)^{-a} dx,$$

$$= 2 \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a,b;c;z) ,$$

according to formula (ii)(b) of this section.

This reduction is not trivial to recognize, of course. The motivation is to seek out the "most troublesome" term in the expression -- the denominator of the integrand -- and break it down into simpler terms, in the hope that the procedure does not make an even worse mess of the remaining elements of the expression. In this sense, it is difficult to define "most troublesome" in a succinct manner. For this particular case, an equivalent but less direct path for the reduction would be to let

$$y = \sin(t), \text{ then } x = y^2, \quad \text{or}$$

$$y = \cos(t), \text{ then } x = y^2,$$

assuming we "know" that $\sin^2(t) = 1 - \cos^2(t)$.

On the other hand, given an integrand which involves only trigonometric (or hyperbolic) functions of the independent variable, the substitution $y = \sin(t)$, or $\cos(t)$ ($\sinh(t)$, or $\cosh(t)$) reduces it to a purely algebraic expression; if further reductions are required, they should then be relatively easy to spot.

VII. Differential Equations

The discussion in this section has been purposely abridged principally because an exhaustive cataloguing of the homogeneous solutions of DE's of the Hypergeometric type would tax the patience of the reader. Furthermore, these are well documented in References (1), (2), and (3). Therefore, the remarks below are principally of a general nature. However, efforts are being made at the present time to reduce this detailed classification to a less unwieldy set of fundamental procedures and operations.

The most general (and essentially only and basic) solution technique for linear DE's with polynomial coefficients is that of Frobenius (Slater (1), (2)). Other, more concise methods are nearly always limited in scope and are generally equivalent to Frobenius' in any case.

The advantage of the HGF formulation is the fact that the behavior of the solutions with respect to the nature of the singular points and other types of limit properties are compactly "encoded" into the "A" and "B" lists.

(i) The General Hypergeometric Differential Equation:

The series ${}_pF_q(\underline{A};\underline{B};x)$ satisfies

$$\left\{x\frac{d}{dx}(x\frac{d}{dx} + b_1 - 1)(x\frac{d}{dx} + b_2 - 1) \ldots (x\frac{d}{dx} + b_q - 1)\right.$$

$$\left. - x(x\frac{d}{dx} + a_1)(x\frac{d}{dx} + a_2) \ldots (x\frac{d}{dx} + a_p)\right\} y = 0 . \qquad (25)$$

The order of this DE is max(p,q+1).

For further elaboration and classification of singular points see Slater (2), pp. 42-45.

(ii) Gauss' Equation:

The series ${}_2F_1(a,b;c;x)$, known as Gauss' HGF, satisfies

$$x(1-x)\frac{d^2y}{dx^2} + \left[c - (1+a+b)x\right]\frac{dy}{dx} - aby = 0 . \tag{26}$$

See Slater (2), pp. 5-13.

(iii) Kummer's Equation:

The series ${}_1F_1(.)$ (Kummer's HGF) satisfies

$$x\frac{d^2y}{dx^2} + (b-x)\frac{dy}{dx} - ay = 0 . \tag{27}$$

See Slater (1), Ch. 1.

(iv) A Class of Second-Order Linear DE's:

Suppose we are required to find the solutions of the homogeneous equation

$$p_2(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_o(x)y = 0 , \tag{28}$$

where $p_2(x)$, $p_1(x)$, and $p_0(x)$ are real polynomials in x.

We inquire: can the solution(s) we written in terms of, say,

$$g(x) \cdot {}_2F_1(a,b;c;px + q),$$

where g(x) is an elementary function? The solutions might also involve ${}_2F_0(\cdot)$, ${}_1F_1(\cdot)$, or ${}_0F_1(\cdot)$. Then again, it may be true that the DE is not of Hypergeometric form. The details for transforming any given DE into the latter have not yet been completely worked out -- in fact, to the best of this writer's knowledge, it has never been attempted -- and there is a strong possibility that it may be simpler in general to automate Frobenius' method. On the other hand, the great majority of DE's of this form which are encountered in applications possess solutions in terms of ${}_2F_1(\cdot)$ or ${}_1F_1(\cdot)$. By testing for the location and nature of the singular points of the given DE, which is a fairly straightforward task consisting mainly of finding the roots of the $p_k(x)$, the possibilities can be cut down significantly also.

(v) General Linear DE with Polynomial Coefficients:
A great deal of additional work is required in the area of the investigation of HGF's in the DE context. This is not to say that most of the results are not known -- rather, they are numerous and widely dispersed throughout the literature and require systematization.

The best efforts to date are due to Slater((1) and (2), but there is much left to be desired, especially where the aims of MATHLAB are concerned.

Suppose we are given

$$\left\{\sum_{k=0}^{n} p_k(x)\left(\frac{d}{dx}\right)^n\right\} y = 0 , \tag{29}$$

where the $p_k(x)$ are real polynomials in x. The conditions under which this DE can be transformed into the Hypergeometric Equation are not known in general for n = 4, and only in principle for the cases of n = 4 or 3. (The latter require the solution of systems of algebraic equations of degree 4 and 3 respectively.) As before, by locating and classifying the singularities of the equation, it may be possible to determine if the given DE is not of Hypergeometric form; however, if we cannot prove that it is not Hypergeometric, it may still be or not be Hypergeometric. The Hypergeometric DE possesses at most three distinct singularities: if the independent variable is x these lie at 0, 1, and infinity. The singularity at the origin may be reular or irregular; that at 1 is regular or removable; that at infinity is regular or irregular.

VIII. Analytic Continuation

Analytic Continuation Formulas are known completely only for the HGF's ${}_2F_1(\cdot)$, ${}_1F_1(\cdot)$, and the trivial case ${}_1F_0(\cdot)$. For other ${}_pF_q(\cdot)$'s these can nearly always be generated in any particular instance, but only by resort to fairly sophisticated techniques which require a thorough background of complex variable theory, and thus lie beyond the limited scope of the present study.

(Analytic Continuation and the closely related topic of Contiguous Functions, discussed in the following section, are useful principally for numerical applications and also for the investigation of asymptotic properties. See remark (iv) below.)

(i) Trivial Case:

$$
\begin{aligned}
{}_1F_0(a;\ ;x) &= (1-x)^{-a} = e^{-i\pi a}\, x^{-a}\, (1 - 1/x)^{-a} \\
&= \frac{e^{-i\pi a}}{x^a}\, {}_1F_0(a;\ ;1/x) \ . \qquad (30)
\end{aligned}
$$

(ii) Kummer's Theorem:

$$
\left.
\begin{aligned}
&{}_1F_1(a;b;x) = e^x\, {}_1F_1(b-a;b;-x), \quad \text{or} \\
&e^x\, {}_1F_1(a;b;-x) = {}_1F_1(b-a;b;x) \ .
\end{aligned}
\right\} \qquad (31)
$$

(iii) Gauss-Euler Reductions for Gauss' HGF:

(a) $$ {}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\ {}_2F_1(c-a;c-b;c;x)\ , \tag{32}$$

(b) $$ = (1-x)^{-a}\ {}_2F_1(a,c-b;x;\ \frac{x}{x-1})\ , \tag{33}$$

(c) $$ = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2F_1(a,b;a+b-c+1;1-x) + (1-x)^{c-a-b}\ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\ \cdot\ {}_2F_1(c-a,c-b;c-a-b+1;1-x)\ , \tag{34}$$

(d) $$ = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\ (-x)^{-a}\ {}_2F_1(a,1-c+a;\ 1-b+a\ ;\ 1/x) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\ (-x)^{-b}\ {}_2F_1(b,1-c+b;\ 1-a+b;\ 1/x)\ , \tag{35}$$

(e) $$ = (1-x)^{-a}\ \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\ {}_2F_1(a,c-b;a-b+1;\ \frac{1}{1-x}) + (1-x)^{-b}\ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\ {}_2F_1(b,c-a;b-a+1;\ \frac{1}{1-x})\ , \tag{36}$$

$$(f) \qquad = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, x^{-a}\ {}_2F_1(a, a-c+1; a+b-c+1; 1-\frac{1}{x})$$

$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\, (1-x)^{c-a-b}\, x^{a-c}\, \cdot$$

$$\cdot\ {}_2F_1(c-a, 1-a; c-a-b+1; 1-\frac{1}{x}) \ . \qquad (37)$$

The conditions under which these transformations are valid is usually equivalent to the requirement that the arguments on both sides of the equality are defined, and that the various Gamma Functions do not possess zero or negative integer arguments.

(iv) General Remarks:

Analytic continuation is applicable only when $p = q+1$, or $p = q$. For the case where $p > q+1$, the HGF is not analytic, and when $p < q$ the series converges uniformly everywhere (see <u>Elementary Properties</u> (i) above).

The usefulness of Analytic Continuation arises in both numerical and theoretical areas. For example, we may wish to evaluate a ${}_2F_1(a,b;c;x)$ for large values of x. Direct substitution into the series may lead to an unwieldy calculation, due to slow convergence. However, we can transform: $x \longrightarrow 1/x$ (see previous article) and the resulting series will converge relatively quickly as $x \rightarrow \infty$. On the other hand, the purely analytical advantages are well demonstrated in the example given above (V(ii)) where the ${}_2F_1(.)$ which emerged from the woodwork (line 4 p. 16) was transformed into a more readily interpretable form (line 4, p. 17).

Another valuable numerical and theoretical tool arises from the contiguous properties of HGF's, discussed in the next section.

IX. Contiguous Functions

The two HGF's ${}_pF_q(\underline{A}';\underline{B}';x)$, ${}_pF_q(\underline{A}'';\underline{B}'';x)$ are said to be contiguous when all parameters contained in their respective "A" and "B" lists are equal, except for one pair of parameters (from either $\underline{A}$ or $\underline{B}$), and these two differ by unity.

For example,

$${}_2F_0(a_1,a_2;\ ;x),\quad {}_2F_0(a_1,a_2+1;\ ;x)$$

$${}_1F_2(a_1;b_1,b_2;x)\quad {}_1F_2(a_1;b_1-1,b_2;x)$$

are contiguous pairs, but the pair

$${}_1F_1(a;b;x),\quad {}_1F_1(a-1;b+1;x)$$

is not.

However, the last pair of functions are said to be associated. In general, two ${}_pF_q(.)$'s are associated if the respective elements of their parameter lists (both $\underline{A}$ and $\underline{B}$, or either) differ by integers or <u>zero</u>.

It is not difficult to show from the differential properties (section IV) that a linear relationship exists between m+1 contiguous ${}_pF_q$'s, where m is equal to the order of the associated DE (Eq. (25)), i.e. = max (p,q+1). For example, for ${}_2F_1(.)$ or ${}_1F_1(.)$, the number of functions in the expression is <u>three</u>. Furthermore, it can be shown that a similar three-function relation exists between any three <u>associated</u> ${}_2F_1$'s or ${}_1F_1$'s.

The term "linear" in this context means that the coefficients of the various ${}_pF_q$'s in the relationship are linear functions of any specifically chosen parameter, or the independent variable. For example, one (of six) contiguous identities for ${}_1F_1(.)$ is

$$b \cdot {}_1F_1(a;b;x) = b \cdot {}_1F_1(a-1;b;x) - x \cdot {}_1F_1(a;b+1;x) \ .$$

The implication of this result is that, in the cases of ${}_2F_1(.)$ and ${}_1F_1(.)$ at least, we can compute:

(a) ${}_2F_1(a+m, b+n; c+1; x)$ from
${}_2F_1(a,b;c;x)$ and ${}_2F_1(a,b;c+1;x)$,
for example.

(b) ${}_1F_1(a+m;b+n;x)$ from
${}_2F_1(a;b;x)$ and ${}_1F_1(a;b+1;x)$,
for example,

where m, n and l are integers, positive or negative.

For other values of p and q, it is generally simpler to iterate <u>contiguous</u> relationships in order to calculate an <u>associated</u> function.

Complete listings of the <u>contiguous</u> relationships have been prepared only for ${}_2F_1(.)$ and ${}_1F_1(.)$. These are not given here for reasons of space, but may be found respectively in Slater (2), pp. 13-14, and Slater (1), p. 19.

It should be remarked that the derivation of contiguous relationships (i.e. the determination of the coefficients) for general values of p and q is by no means trivial. The calculation requires fairly careful handling of power series relationships, and there is apparently no general systematic technique.

We recall for a moment the remark made above, Section V (ii)(d), concerning "tricks": all ${}_2F_1$'s of the form appearing on that page, where k is a <u>positive even integer</u>, are <u>associated</u> functions. It follows from the results just presented that if we know the value of the integral for k = 0 and 2, say, we can generate in a simple manner the values for k = 4, 6, 8, ... 2n, ... In fact, this starting pair is easily calculated from the integral on the bottom of p. 17 by elementary methods.

X. Tabulation of Functions Representable in Hypergeometric Form

In this section we list some of the more widely known Higher Transcendental Functions which can be expressed as ${}_pF_q(.)$.

The trivial cases ${}_1F_0(.)$ and ${}_0F_1(.)$ have already been mentioned (Section III) and will not be repeated. For completeness, we should include in this category the rather obvious identity

$${}_pF_p(\underline{A};\underline{A};x) = e^x \ .$$

The reader is cautioned to be wary of notation; the one employed here is generally that of Ref. (4), since it seems to represent a reasonable compromise among the particular variations which are encountered in a cross-section of the different scientific disciplines. Also, Ref. (4) nearly always catalogues the most common variant notations for each of the functions as they are introduced.

(i) Functions Representable by ${}_1F_1(a;b;x)$:

(a) Bessel Functions

$${}_1F_1(a;2a;x) = \Gamma(a+\tfrac{1}{2})\, e^{x/2} \, (x/4)^{\frac{1}{2}-a} \, I_{a-\frac{1}{2}}(x/2) \ . \qquad (38)$$

Depending upon whether a is an integer or half of an odd integer, positive or negative, and whether x is positive, negative or imaginary, a variety of differently named members of the family of Bessel Functions can be generated. (See Ref. (4), Eqs. 13.6.1 - 13.6.7.)

(b) Polynomials

$$ {}_1F_1(-n;b;x) = \frac{n!}{(b)_n} L_n^{(b-1)}(x) . \quad \text{(LAGUERRE)} \tag{39} $$

When b = 1/2 and 3/2, we obtain respectively the even- and odd-order Hermite Polynomials:

$$ \left. \begin{aligned} {}_1F_1(-n;\tfrac{1}{2};x) &= \frac{n!}{(2n)!} (-\tfrac{1}{2})^{-n} He_{2n}(\sqrt{2x}), \\ {}_1F_1(-n;3/2;x) &= \frac{n!}{(2n+1)!} (-\tfrac{1}{2})^{-n} He_{2n+1}(\sqrt{2x}) . \end{aligned} \right\} \tag{40} $$

(HERMITE)

Another type, essentially the same as (39), is

$$ {}_1F_1(-n;-n+\alpha+1;x) = \frac{(n!)^{\frac{1}{2}} x^{n/2}}{(-n+\alpha+1)_n} \rho_n(\alpha,x) . \tag{41} $$

(POISSON-CHARLIER)

(c) Miscellaneous Functions

$$ {}_1F_1(a;a+1;-x) = ax^{-a} \gamma(a,x) . \tag{42} $$

(INCOMPLETE GAMMA)

When $a = \frac{1}{2}$, this reduces to the error function:

$$ {}_1F_1(\tfrac{1}{2};3/2;-x^2) = \frac{\sqrt{\pi}}{2x} \operatorname{erf}(x) . \quad \text{(ERROR)} \tag{43} $$

A generalization of the Hermite Polynomials, just cited (40), is

$$\left.\begin{aligned} {}_1F_1(-\nu/2;1/2;x) &= \frac{1}{\sqrt{2}}\, e^{x/2}\, E_\nu^{(0)}(\sqrt{2x}) , \\ {}_1F_1(1/2-\nu/2;\ 3/2;\ x) &= \frac{1}{2\sqrt{x}}\, e^{x/2}\, E_\nu^{(1)}(\sqrt{2x}) . \end{aligned}\right\} \quad (44)$$

(PARABOLIC CYLINDER)

$${}_1F_1\left(\frac{m+1}{2};\ n+1;\ r^2\right) = \frac{n!\,r^{-2n-m-1}}{\Gamma\left(\frac{m+1}{2}\right)}\, e^{r^2}\, T(m,n,r) . \quad (45)$$

(TORONTO)

It should be borne in mind that Kummer's Transformation (Eq.(31)) can be applied throughout, essentially doubling the size of the list.

(ii) Functions Representable by ${}_2F_1(a,b;c;x)$:

(a) Polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}\ {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) . \quad (46)$$

(JACOBI)

(Clearly, any ${}_2F_1(.)$ which reduces to a finite polynomial can be alternatively represented by a Jacobi Polynomial by means of a proper choice of n, α, and β. Furthermore, by employing the Analytic Continuation formulas of VIII(iii) at least five more representations equivalent to (46) can be obtained.)

$$C_n^{(\alpha)}(x) = \frac{\Gamma(n+2\alpha)}{n!\,\Gamma(2\alpha)}\ {}_2F_1\left(-n, n+2\alpha; \alpha+\tfrac{1}{2}; \frac{1-x}{2}\right) . \quad (47)$$

(GEGENBAUER)

$$\left.\begin{aligned} T_n(x) &= {}_2F_1(-n,n;\ \tfrac{1}{2};\ \tfrac{1-x}{2}) , \\ U_n(x) &= (n+1)\cdot {}_2F_1(-n,n+1;\ 3/2;\ \tfrac{1-x}{2}) . \end{aligned}\right\} \quad (48)$$

(TSCHEBYCHEV)

$$P_n(x) = {}_2F_1(-n,n+1;\ 1;\ \tfrac{1-x}{2}) . \quad \text{(LEGENDRE)} \quad (49)$$

The results of (47)-(49) are, of course, subject to the transformations of VIII(iii), where valid.

(b) Elementary Functions:

$${}_2F_1(1,1;2;x) = \log\ (1-x) . \quad (50)$$

$${}_2F_1(\tfrac{1}{2},1;3/2;x^2) = \frac{1}{2x} \log\left(\frac{1+x}{1-x}\right). \quad (51)$$

$${}_2F_1(\tfrac{1}{2},\tfrac{1}{2};3/2;x^2) = \frac{1}{x} \sin^{-1}(x) . \quad (52)$$

$${}_2F_1(\tfrac{1}{2},1;3/2;-x^2) = \frac{1}{x}\tan^{-1}(x) . \quad (53)$$

$${}_2F_1(a,a+\tfrac{1}{2};\tfrac{1}{2};x) = \frac{(1+x^{\frac{1}{2}})^{-2a}}{2} + \frac{(1-x^{\frac{1}{2}})^{-2a}}{2} . \quad (54)$$

$${}_2F_1(a-\tfrac{1}{2},a;2a;x) = \left[\frac{1 + (1-x)^{\frac{1}{2}}}{2}\right]^{1-2a} . \quad (55)$$

$$_2F_1(2a,a+1;a;x) = \frac{1+z}{(1-z)^{2a+1}} . \tag{56}$$

(Notice that the last three results can also be expressed in terms of simple functions of $_1F_0(\cdot)$'s.)

(c) Other Functions:

$$\left.\begin{aligned} {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;x^2) &= \frac{2}{\pi} K(x) , \\ {}_2F_1(-\tfrac{1}{2},\tfrac{1}{2};1;x^2) &= \frac{2}{\pi} E(x) . \end{aligned}\right\} \tag{57}$$

(COMPLETE ELLIPTIC INTERVALS)

$$_2F_1(p,1-q;p+1;x) = px^{-p} B_x (p,q) . \tag{58}$$

(INCOMPLETE BETA)

$$_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} E(a,b;c;-\frac{1}{x}) . \tag{59}$$

(MACROBERT'S E-)

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} G^{21}_{22}\left(-\frac{1}{x}\middle| \begin{matrix} 1,c \\ a,b \end{matrix}\right). \tag{60}$$

(MEITER'S G-)

$(1 + x)^n$ to N terms:

$$= 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{N}x^N$$

$$= {}_1F_0(-n; ;-x) \text{ to N terms}$$

$$= (-n)_N \frac{x^N}{N!} \; {}_2F_1(-N,1; 1+n-N; -\frac{1}{x}) \; . \qquad (61)$$

(TRUNCATED BINOMIAL SERIES)

(See generalization in next section.)

(iii) Other Representations:

For ${}_pF_q$'s other than the cases already listed, the reader is referred to Slater (2), pp. 46-7, and Bateman (4), Higher Transcendental Functions, Vol. II, under "Generalized Hypergeometric Series".

One result which is of interest, much like that of Eq. (61) above, is the following:

e^x to N terms

$$= 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^N}{N!}$$

$$= {}_0F_0(; ;x) \text{ to N terms}$$

$$= \frac{x^N}{N!} \; {}_2F_0(-N,1; ;- \frac{1}{x}) \; . \qquad (62)$$

(TRUNCATED EXPONENTIAL SERIES)

(Notice that although it was stated previously that if $p > q + 1$ the associated ${}_pF_q(\cdot)$ is undefined in general, this restriction does not apply if one or more members of the "A" list are nonpositive integers, in which case the series terminates in a finite number of terms, as in the present example.)

All other truncated series of circular and hyperbolic functions can be represented in a like manner. For example,

cosh x to N terms

$= \frac{1}{2}(e^x + e^{-x})$ to N terms

$$= \frac{1}{2} \cdot \frac{x^N}{N!}\left({}_2F_0(-N,1;\ ;-\frac{1}{x}) + (-1)^N\ {}_2F_0(-N,1;\ ;\frac{1}{x})\right) .$$

XI. Miscellaneous Results and Random Facts

This section lists some useful and powerful properties which do not fit neatly into any of the above categorizations.

(i) Truncation of Series

Suppose that we are given a power series whose individual terms are of the form of those of a ${}_pF_q(\cdot)$, but which is terminated after N terms, as in Eqs. (61) and (62) above. We state the following theorem:

$$\left[{}_pF_q(\underline{A};\underline{B};x)\right]_N$$

$$\triangleq \sum_{k=0}^{N} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_p)_k} \frac{x^k}{k!}$$

$$= \frac{(a1)_N(a_2)_N\cdots(a_p)_N}{(b_1)_N(b_2)_N\cdots(b_q)_N} \frac{x^N}{N!} \cdot {}_{q+2}F_p(1-N-\underline{B},\ -N,\ 1;$$

$$1-N-\underline{A};\ \frac{(-1)^{q-p+1}}{x}\ ), \qquad (63)$$

where $1-N-\underline{B} \leftrightarrow (1-N-b_1),\ (1-N-b_2),\ \ldots,\ (1-N-b_q),$

$1-N-\underline{A} \leftrightarrow (1-N-a_1),\ (1-N-a_2),\ \ldots,\ (1-N-a_p)$.

Eq. (63) is a generalization of the aforementioned examples.

(ii) Reversal of Series

A closely related result applies to <u>terminating</u> ${}_pF_q$'s, i.e. those which possess 0 or a negative integer in their "A" list. The series can be expressed in terms of powers of $(1/x)$ as follows:

$${}_pF_q(a_1,a_2,\ldots a_{p-1},-m;\ b_1,b_2,\ldots b_q;\ x)$$

$$= \frac{(a1)_m(a_2)_m\cdots(a_{p-1})_m}{(b_1)_m(b_2)_m\cdots(b_q)_m} (-x)^m \cdot {}_{q+1}F_{p-1}(1-\underline{B},\ 1-m;$$

$$1-\underline{A};\ \frac{(-1)^{p+q+1}}{x}\ ) . \qquad (64)$$

(iii) Particular Values of the Argument

For certain ${}_pF_q$'s, especially when $p = q+1$, the series is summable in closed form when the independent variable takes on special values. Some of these are listed below.

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} . \tag{65} $$

$$ {}_2F_1(a,b;\ 1+a-b;\ -1) = 2^{-a}\ \frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1-b+a/2)} . \tag{66} $$

$$ {}_2F_1(a,1-a;\ c;\ \tfrac{1}{2}) = \frac{\Gamma(c/2)\Gamma(\frac{1+a+b}{2})}{\Gamma(\frac{a+c}{2})\Gamma(\frac{1+c-a}{2})} . \tag{67} $$

A few others for ${}_2F_1(\cdot)$ can be generated by means of the Analytic Continuation formulas given above. For other ${}_pF_q$'s of the form ${}_{p+1}F_p(\cdot)$, with particular arguments, see Slater (2), pp. 48-84. (The underlying theory of the latter is quite deep, and is of purely abstract interest for the most part, in any case.)

(iv) Confluence

The adjective "confluent" is often encountered in the literature of Hypergeometric Functions. The word means, literally, "running together". In the present context, the term denotes the "running together" of the singular points of a ${}_pF_q(\cdot)$ at one and infinity. To fully appreciate the manner in which this is carried off would require a lengthy and complicated discussion of the DE theory of HGF's. For example, if we make the change of variables $x \longrightarrow x/b$ in Gauss' Equation (26), and then let $b \longrightarrow \infty$ (carefully) we obtain Kummer's Equation (27) (with b replaced by c).

Roughly speaking, ${}_2F_1(a,b;c;\text{arg})$ possesses a regular singularity when arg = 1, as well as the singularities at arg = 0 and ∞; thus, if arg = x/b, the "middle" singular point lies at b, and thus goes to infinity as $b \rightarrow \infty$ (to join the one already there).

All of this is easier to visualize in terms of power series; suppose we want to "do a confluence" on a given ${}_pF_q(\cdot)$ with respect to a member of the "A" list, say a_p (the last member). We can represent the operation symbolically as

$$\text{CONFLUENCE} \quad ({}_pF_q(\underline{A};\underline{B};x)) \text{ wrt } a_p$$

$$= \lim_{a_p \rightarrow \infty} \left({}_pF_q(a_1,a_2,\ldots a_{p-1},a_p;\ \underline{B};\ \frac{x}{a_p})\right)$$

$$= {}_{p-1}F_q(a_1,a_2,\ldots a_{p-1};\ \underline{B};\ x). \qquad (68)$$

This result is fairly obvious if one recalls Eq. (1): the only terms in the summand which involve a_p constitute the ratio:

$$\frac{(a_p)_n}{a_p^n} = \frac{(a_p(a_{p+1})\cdots(a_{p+n-1})}{a_p \cdot a_p \ \cdots \ a_p}$$

$$= 1(1 + 1/a_p)(1 + 2/a_p)\cdots(1 + \frac{n-1}{a_p}) ,$$

which clearly goes to unity as $a_p \longrightarrow \infty$, thus wiping out all trace of a_p and leaving the independent variable unscathed.

Thus, briefly, to obtain a confluent form of a ${}_pF_q(\cdot)$, one member of the "A" list is deleted and p is reduced by one.

By way of justification of all of this verbiage, which may on the surface appear to be describing a fairly trivial property, let it be noted that Slater (1), (2) employs confluence nearly everywhere to demonstrate theorems regarding ${}_1F_1$'s by "doing a confluence" throughout the analogous theorems for ${}_2F_1$. (The overall theory of the latter is richer and more versatile.) In the same vein, but in reverse, this writer has on several occasions dealt with a recalcitrant ${}_1F_1(\cdot)$ (say, in an integrand) as follows:

Let

$${}_1F_1(a;c;x) \longrightarrow {}_2F_1(a,b;c;x/b) ,$$

play ${}_2F_1$-type games, get an answer (if possible), and then after all the dust has settled let $b \longrightarrow \infty$. If the result makes sense at all, it is probably correct.

(v) Remarks

There are a large number of other isolated properties of HGF's which are not described here in detail for reasons of space and present applicability. The most important of these concern: <u>Addition Theorems</u>, i.e. the expansion of ${}_pF_q(\underline{A};\underline{B};x+y)$ in terms of ${}_pF_q$'s involving x or y singly; similarly, <u>Multiplication Theorems</u> for ${}_pF_q(\underline{A};\underline{B};xy)$; expansions of ${}_pF_q(\cdot)$ in terms of HGF's with smaller values of p and/or q. (As an elementary example of the latter, recall that $I_\nu(x)$ can be expressed alternatively as ${}_0F_1(\cdot)$ or ${}_1F_1(\cdot)$.)

For further documentation, the reader is referred to Slater (1), (2), Bateman (3), or the Bureau of Standards Table (4).

XII. Important Omissions

At several junctures in this report this writer has passed over or detoured subject matter which should properly have been included, but for the fact that the text would have at least doubled in size. Considerable pains have been taken to sift through the body of theory in order to simultaneously maintain readability, pertinence and brevity, and wherever gaps or incomplete listings have arisen, a specific reference has generally been provided.

However, at least two significant topics have received inadequate attention.

(i) Differential Equtions:

As indicated in the closing remarks of Section VII above, the theory of HGF's as applied to linear ordinary DE's with polynomial coefficients requires extensive study. A tabulation of the known results was not included because, at least to the insight of this author, no "pattern" seemed to emerge, except at the elementary "Frobenius Method" level.

(ii) Associated Kummer Functions:

Associated Kummer Functions constitute a class of solutions of Kummer's Equation (27) for which "${}_1F_1(.)$" is not valid. In many cases, these functions (usually denoted by U(a;b;x)) can be written in terms of ${}_1F_1$'s; when they cannot, they are nearly always characterized by logarithmic singularities or other types of branch points near the origin.

The principal reason that these were not mentioned is that the associated theory parallels that of ${}_1F_1(.)$ except for a number of nagging exceptions which are just sufficient to create confusion (very much like attempting to match up trig-function and hyperbolic function identities*). Eventually, of course, these functions will

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* $\cosh^2 x + \sinh^2 x = ?$ (quickly, now)

have to be included, but it was felt in the course of the preparation of this report that the penalty in terms of length and reader attention did not justify the relatively minor virtue of a more complete presentation.

A listing of functions representable by $U(a;b;x)$ is given in Ref. (4), p. 510. It can be seen by comparison to the list on p. 509 (for ${}_1F_1(a;b;x)$) that the degree of nonoverlap is not serious.

XIII. LISP Representation

R. Fateman has suggested that the following format be adopted for LISP representation of hypergeometric functions:

$$ {}_pF_q(\underline{A};\underline{B};x) $$

$$ \longleftrightarrow \quad ((\text{MFHYP P Q})(A_1\ A_2\ \ldots\ A_p)(B_1\ B_2\ \ldots\ B_q)x) \qquad (69) $$

The car of the right side identifies the function as being a ${}_pF_q(\cdot)$; the cadr and caddr are the "A" and "B" lists, respectively, and the cadddr is the independent variable. To be slightly more general, the last element should probably be replaced by ('FUNCTION x) or a quoted lambda expression, in the event that the independent variable is not x itself.

The inclusion of the parameters P and Q in the car is admittedly redundant, inasmuch as their values are specified by the lengths of the "A" and "B" lists; however, it appears at this time that it is more convenient in the long run to have P and Q available at a fairly high level in the form of atoms rather than cluttering up the landscape with expressions like (LENGTH (CADR Y)) and so forth. Furthermore, the decision paths which determine how a particular ${}_pF_q(\cdot)$ is to be manipulated branch at a very early stage, depending upon the values of P and Q, somewhat as follows.

After reduction (i.e. cancellation of common members of the "A" and "B" lists), we examine P and Q to see if ${}_0F_0(\cdot)$, ${}_0F_1(\cdot)$ or ${}_1F_0(\cdot)$ has emerged; if not, then test for ${}_1F_1(\cdot)$; if T, then branch to a recognition package for ${}_1F_1(\cdot)$; if not, test for ${}_2F_1(\cdot)$; if T, then branch to a recognition package for ${}_2F_1(\cdot)$; otherwise:

(a) If "A" list contains 0 or negative integers, and "B" list does not, the ${}_pF_q(\cdot)$ is a finite polynomial.

(b) If "B" list contains 0 or negative integers, and "A" list does not, function is undefined.

(c) If both "A" and "B" lists contain 0 or negative integers, and all such members of the "A" list are greater than all such members of the "B" list, then function is a polynomial; otherwise undefined.

(d) Otherwise (i.e. none of "A" and/or "B" lists are 0 or negative integers) function is defined in terms of an infinite series if $p \leqslant q+1$, undefined otherwise.

A flow chart for this logic is shown in Fig. 1, by way of example. Other packages for integrals, derivatives, and differential equations have also been flowcharted. (See remarks in Conclusions, below.)

The overall input is assumed to be a HGF which has emerged from lowest-level operations in "unrefined" form, i.e. the package is presented with a $_pF_q(.)$ with no *a priori* knowledge. We call this input G, and the output of the reduction routine F. The notations "T" and "NIL" on the branches denote "yes" and "no"; a_i and b_j denote members of the "A" and "B" lists.

(i) HYPREDUCE

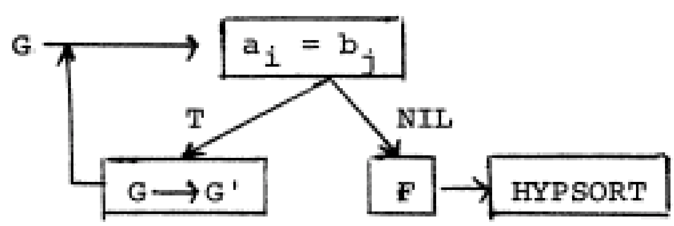

Fig. 1a. (G' denotes the reduced form of G which omits a_i and b_j, if they are equal and reduces both p and q by unity.)

(ii) HYPSORT

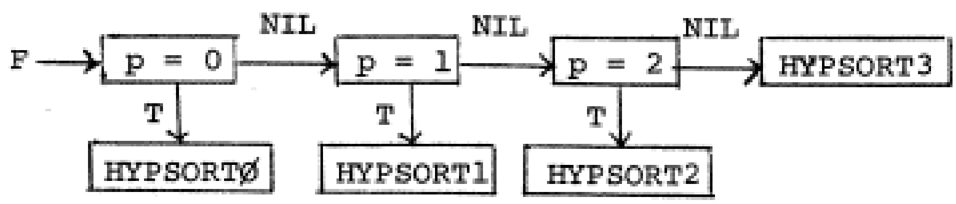

Fig. 1b.

(iii) HYPSORTØ

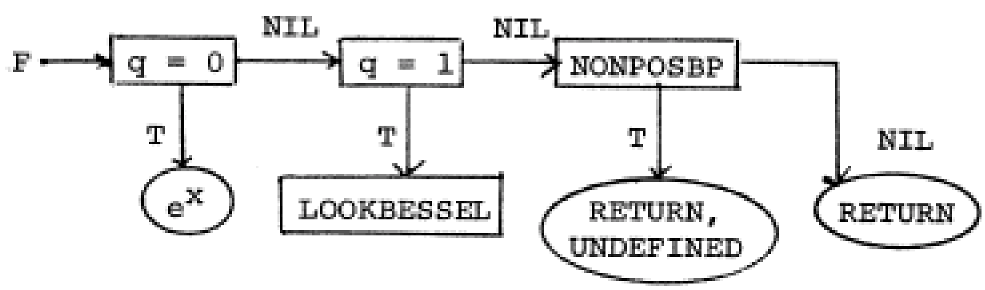

Fig. 1c. LOOKBESSEL looks at F (which is, by now, a ${}_0F_1(.)$, and therefore a Bessel Function) and determines which particular Bessel Function best fits the parameters.

NONPOSBP (predicate) examines the "B" list; T is returned if none of the members are 0 or a negative integer; otherwise NIL.

(iv) HYPSORT1

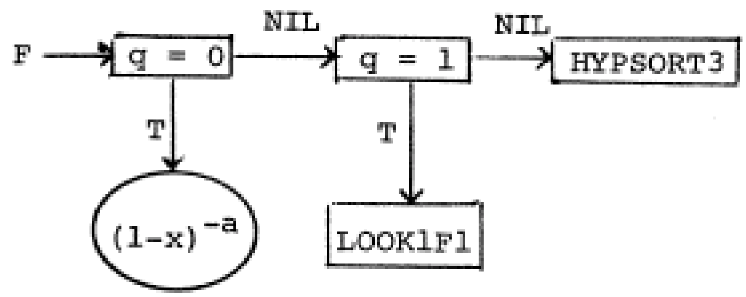

Fig. 1d. LOOK1F1 is a recognition package for ${}_1F_1(\cdot)$ which attempts to identify F in a list of known representations (see X(i), above).

(v) HYPSORT2

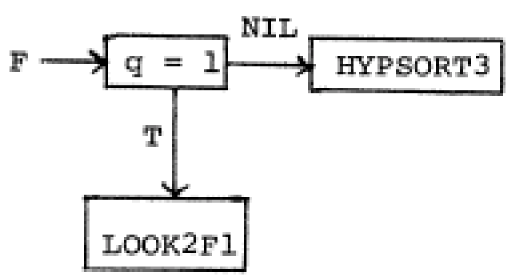

Fig. 1e. LOOK2F1: like LOOK1F1, but for ${}_2F_1(\cdot)$ (see X(ii), above).

(vi) HYPSORT3

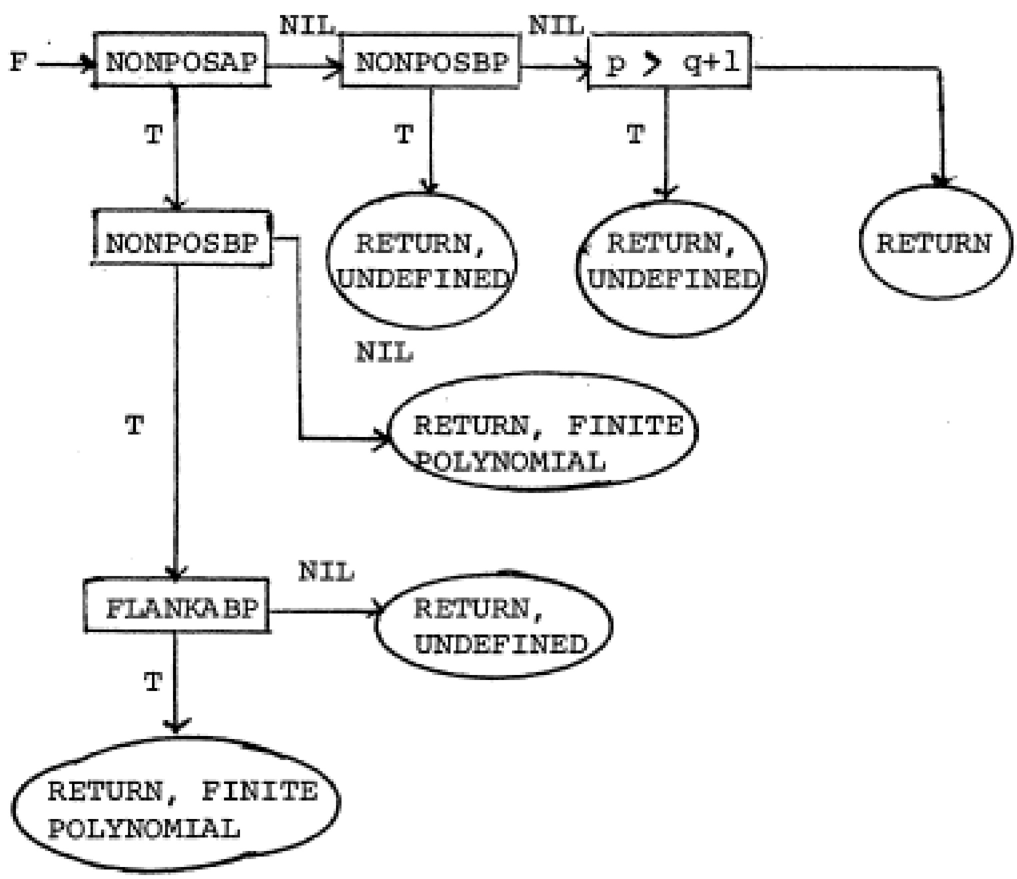

Fig. 1f. NONPOSAP (predicate) is identical to NONPOSBP, except operates on "A" list.

FLANKABP (predicate) flags members of both "A" and "B" lists which are 0 or a negative integer; returns T if all such members of "A" list are greater than all such members of "B" list; otherwise NIL.

Some fragments of this package and of others have been coded and appear to work satisfactorily. The most difficult problems in the programming area are anticipated to lie in the interfacing with the present MATHLAB, particularly in the "simplification" procedures, and in designing conventions for I/O which are efficient from the point of view of both the user and the system.

Most of these problems have not yet been considered in detail, although none of them appear insurmountable at present. At the moment, other more urgent problems are at hand, so that this work will have to be suspended temporarily.

XIV. Conclusions

The purpose of this report has been to introduce the reader to Hypergeometric Functions as painlessly as possible, and to give some indication of the strengths and weaknesses of the Hypergeometric Formulation.

Its power arises from its generality and the fact that it enables one to deal with a much wider class of functions than can be handled at present, and to do so in a unified manner. The several more-or-less detailed examples which are scattered throughout the text were chosen to support this contention.

On the other hand, implementing these techniques presents a large number of systems and programming problems, and it would be difficult at the present time to predict how much time and effort would be required to solve them. Furthermore, such a package would probably be costly in terms of core and running time, but that is of secondary importance.

It is this writer's firm belief that if MATHLAB is in the future to appeal to a wide variety of users, from freshman calculus student to theoretical physicist, it will have to eventually incorporate some or all of the methods outlined in this report. There is literally no other sensible way of dealing with Higher Transcendental Functions efficiently within the confines of a finite system.

Perhaps this study is premature, inasmuch as there are bugs in the present system which must be eliminated, as well as some new foundation blocks to be incorporated before attempting to build in the Hypergeometric Function methods. At least, these have now been documented for future reference and a promising direction for research has been provided.

References

1. L.J. Slater, Confluent Hypergeometric Functions, Cambridge University Press, 1960.

2. L.J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, 1966.

3. A. Erdelyi, et al., Bateman Manuscript Project: Tables of Integral Transforms, Vols. I and II; Higher Transcendental Functions, Vols. I, II, and III, McGraw-Hill, 1954-57.

4. National Bureau of Standards, Handbook of Mathematical Functions, Applied Mathematics Series, No. 55, 1964.