FOURIER RESOLUTION OF THE HYDROMAGNETIC
ENERGY EQUATIONS

by

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SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENT FOR THE
DEGREE OF MASTER OF
SCIENCE

at the

MASSACHUSETTS INSTITUTE OF
TECHNOLOGY

September, 1968

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Department of Meteorology, August 19, 1968

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Submitted to the Department of Meteorology on August 19, 1968
in partial fulfillment of the requirement for the
degree of Master of Science

ABSTRACT

Maxwell's equations are considered along with the hydromagnetic Navier-Stoke's equation. Energy equations are formed and energy flow among magnetic, electric, and kinetic forms are traced. A Fourier resolution of the energy equations in vector form is carried out and an energy diagram showing energy flow paths for all wave numbers for magnetic, electric, and kinetic energy quantities is presented. The special cases of MHD approximation (displacement current being zero) and infinite conductivity are considered. Some cautions are stated.

The energy equations are then decomposed into cartesian components, with the appropriate energy flow being shown. The Fourier resolution of the component energy equations is then carried out. An energy flow diagram of the various energies, in component form, with all wave numbers resolved is shown.

The energy diagram of Starr and Gilman (1966) is extended by the inclusion of terms that result when the displacement current is not neglected and infinite conductivity is not assumed.

The formulation presented here can be applied to the analysis of actual data as it becomes available. That is not done here.

The Appendix sketches a derivation based directly on the MHD approximation. It exhibits the conversion terms in terms of velocity and magnetic fields.

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Acknowledgements

This research was supported in part by the United States Air Force under contract AF 19628 - 67 - C - 0229.

I wish to thank Dr. Barry Saltzman for his useful discussions at the beginning of this investigation and to express sincere appreciation to Prof. Victor P. Starr for his continuing interest and for helpful discussions.

I want to thank Miss Linda Feldman for typing the manuscript.
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I. Introduction

The Fourier resolution of the hydrodynamic energy equations (Saltzman, 1957) has provided insight into the energy flow in the atmosphere. The formalism as developed by Saltzman and applied to observational data (Saltzman, 1958, 1959; Saltzman and Fleisher, 1960a, 1960b, 1960c, 1961, 1962; Saltzman and Teweles, 1964; Teweles, 1963; Wiin-Nielsen, Brown and Drake, 1963, 1964; Murakami and Tomatsu, 1964) has shown that energy is transported from the intermediate wave numbers (cyclone scales) to both the large wave numbers and to the small wave numbers. A detailed diagram of kinetic energy transfer for 15 wave numbers (at the 500 mb level) is presented in Saltzman and Teweles, 1964.

In this paper, as a first step in elucidating the characteristics of energy flow in the solar atmosphere and in other systems a Fourier resolution of the hydromagnetic energy equations is performed. Other relevant systems might include the earth's core, the ionosphere, relativistic fluids and high frequency plasmas.

In the case of the sun, for example, the solar atmosphere consists of ionized gases. Since a magnetic field is present in the sun, Maxwell's equations must be considered as well as the hydrodynamic equations.
II. Derivation of the Energy Equations

We shall consider the conversions among magnetic, electric, and kinetic energy, not dealing with potential and other forms of energy.

The relevant equations are (in rationalized MKS units)

\[
\frac{d\vec{D}}{dt} = \nabla \times \vec{E} \tag{1}
\]

\[
\frac{d\vec{B}}{dt} = \nabla \times \vec{H} - \vec{J} \tag{2}
\]

\[
\nabla \cdot \vec{D} = \rho \tag{3}
\]

\[
\nabla \cdot \vec{B} = 0 \tag{4}
\]

\[
\vec{D} = \varepsilon \vec{E} \tag{5}
\]

\[
\vec{B} = \mu \vec{H} \tag{6}
\]

\[
\vec{J} = \sigma (\vec{E} + \nabla \times \vec{B}) \tag{7}
\]

\[
\frac{d\vec{U}}{dt} = \left[ -\vec{U} \cdot \nabla \vec{U} - \frac{1}{\rho} \nabla \rho - 2\frac{\varepsilon}{\mu} \nabla \times \nabla \times \vec{U} - \vec{J} + \rho \vec{E} \right] + \frac{1}{\mu} \vec{J} \times \vec{B} \tag{9}
\]

The symbols are:

\[
\vec{E} = \text{electric field [volt m}^{-1}\text{]}
\]

\[
\vec{D} = \text{electric displacement [coul m}^{-2}\text{]}
\]
\[ \mathbf{B} = \text{magnetic induction [ weber m}^{-2}] \]
\[ \mathbf{H} = \text{magnetic field [ ampere turns m}^{-1}] \]
\[ \mathbf{J} = \text{current [ amp m}^{-2}] \]
\[ \mathbf{F} = \text{force [ newton]} \]
\[ \mathbf{U} = \text{velocity [ m sec}^{-1}] \]
\[ q = \text{charge [ coulombs]} \]
\[ \sigma = \text{conductivity [ mho m}^{-1}] \]
\[ \varepsilon = \text{dielectric constant} \]
\[ \mathcal{M} = \text{permeability} \]

An isotropic homogeneous medium is assumed so that (5) and (6) hold.

To see the energy flow for the non-Fourier analyzed case we form the following quantities:

\[ \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{B} \cdot \mathbf{H}) = \frac{1}{2} \left[ \mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right] \]  

(10)

\[ \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{E} \cdot \mathbf{D}) = \frac{1}{2} \left[ \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{D} \cdot \frac{\partial \mathbf{E}}{\partial t} \right] \]  

(11)

\[ \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{U} \cdot \mathbf{U}) = \frac{1}{2} \left[ \mathbf{U} \cdot \frac{\partial \mathbf{U}}{\partial t} \right] \]  

(12)

It follows that

\[ \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{B} \cdot \mathbf{H}) = \frac{1}{2} \left[ -\mathbf{H} \cdot \nabla \mathbf{E} - \frac{\mathbf{B}}{\mathcal{M}} \cdot \nabla \mathbf{E} \right] = -\mathbf{H} \cdot \nabla \mathbf{E}^* = C_{HE} \]  

(13)

where \( C_{HE} \) indicates a conversion from magnetic (H) to electric (E) energy.
\[ \frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{B}) = \vec{E} \cdot \nabla \times \vec{H} - \vec{E} \cdot \vec{J} \]  \hfill (14)

\[ \frac{1}{2} \frac{\partial}{\partial t} (\vec{U} \cdot \vec{U}) = \cdots + \vec{U} \cdot \vec{J} \times \vec{B} = \cdots + C_{\nu E} \]  \hfill (15)

But

\[ \nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H} \]  \hfill (16)

And from (7)

\[ \vec{E} = \frac{\vec{J}}{\sigma} - \vec{U} \times \vec{B} \]  \hfill (17)

Substituting (16) and (7) into the second term of (14) we get:

\[ \frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{B}) = \{ \vec{H} \cdot \nabla \times \vec{E} - \nabla \cdot (\vec{E} \times \vec{H}) \} - \left\{ \frac{\vec{J}}{\sigma} - \vec{U} \times \vec{B} \right\} \cdot \vec{J} \]

\[ = \{ \vec{H} \cdot \nabla \times \vec{E} \} - \left\{ \nabla \cdot (\vec{E} \times \vec{H}) \right\} \left\{ \frac{\vec{J}}{\sigma} - \vec{U} \times \vec{B} \right\} + \{ \vec{U} \times \vec{B} \cdot \vec{J} \} \]  \hfill (18)

but

\[ \vec{U} \times \vec{B} \cdot \vec{J} = \vec{U} \cdot \vec{B} \times \vec{J} = - \vec{U} \cdot \vec{J} \times \vec{B} \]  \hfill (19)

or

\[ \frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{B}) = - \nabla \cdot (\vec{E} \times \vec{H}) - \frac{\vec{J} \cdot \vec{J}}{\sigma} + \vec{H} \cdot \nabla \times \vec{E} - \vec{U} \cdot \vec{J} \times \vec{B} \]  \hfill (20)

but

\[ C_{EH} = - C_{HE} \]

and

\[ C_{EU} = - C_{UE} \]
indicating that an interchange of energy occurs between magnetic and electric energy and also between electric and kinetic energy.

The term $\vec{S} = \nabla \cdot (\vec{E} \times \vec{H})$ is the Poynting vector, representing energy flow [its units are \(\text{energy x time}\)] and $\vec{J} = \frac{\vec{j} \cdot \vec{\sigma}}{\sigma}$ is the Joule heating.

III. Fourier Resolution of the Equations

Let us now consider the representation of the equations in the domain of wavenumber.

Thus

$$f(\lambda) = \sum_{n=-\infty}^{\infty} F(n) e^{i n \lambda}$$  \hfill (21)

where

$$F(n) = \frac{1}{2\pi} \int_{0}^{2\pi} f(\lambda) e^{-i n \lambda} d\lambda$$  \hfill (22)

Here $\lambda$ is longitude and $n$ is the number of waves around a latitude circle (wavenumber).

We now use the convolution theorem, namely

$$\frac{1}{2\pi} \int_{0}^{2\pi} \left[ f(\lambda) g(\lambda) \right] e^{-i n \lambda} d\lambda = \sum_{m=-\infty}^{\infty} G(m) F(m-n)$$  \hfill (23)

In the following, the Fourier transform of, say, $\overline{E}$, is represented as $\overline{E}(n)$. 
Carrying out the Fourier analysis of (1), (2), (7), (9), we have

\[
\frac{1}{2} \frac{\ddot{B}(\eta)}{\partial t} = - \nabla \times \vec{E}(\eta) \tag{24}
\]

\[
\frac{1}{2} \frac{\ddot{D}(\eta)}{\partial t} = \nabla \times \vec{H}(\eta) - \vec{J}(\eta) \tag{25}
\]

\[
\vec{J}(\eta) = \int [\vec{E}(\eta) + \sum_{\eta = -\infty}^{\eta = \infty} \vec{U}(\eta) \times \vec{B}(\eta - \eta)] \tag{26}
\]

\[
\frac{1}{2} \frac{\ddot{U}}{\partial t} = \cdots + \frac{1}{\omega} \sum_{\eta = -\infty}^{\eta = \infty} \vec{J}(\eta) \times \vec{B}(\eta - \eta) \tag{27}
\]

Forming the quantities (magnetic energy, electric energy, and kinetic energy)

\[
\frac{\mu_0}{2} \frac{\ddot{H}(\eta)}{\partial t} = \frac{1}{2} \left\{ \mu_0 \dot{H}(\eta), \frac{\partial \dot{H}(\eta)}{\partial t} + \dot{H}(\eta) \cdot \frac{\partial}{\partial t} \mu_0 \dot{H}(\eta) \right\} \tag{28}
\]

\[
\frac{\varepsilon_0}{2} \frac{\ddot{E}(\eta)}{\partial t} = \frac{1}{2} \left\{ \varepsilon_0 \dot{E}(\eta), \nabla \times \vec{E}(\eta) + \vec{H}(\eta) \cdot \nabla \vec{E}(\eta) \right\} \tag{29}
\]

\[
\frac{\varepsilon_0}{2} \frac{\ddot{E}(\eta)}{\partial t} = \frac{1}{2} \left\{ \varepsilon_0 \dot{E}(\eta), \nabla \times \vec{E}(\eta) + \vec{E}(\eta) \cdot \nabla \vec{H}(\eta) \right\} - \frac{1}{2} \left\{ \vec{E}(\eta), \vec{J}(\eta) + \vec{H}(\eta) \cdot \vec{J}(\eta) \right\} \tag{30}
\]
\[ \frac{1}{2} \sum_{\eta = \pm \infty} \left\{ \tilde{\mathcal{E}}(\eta) \cdot \ddot{\mathcal{E}}(\eta) + \mathcal{E}(\eta) \cdot \mathcal{H}(\eta) \right\} = \cdots + \frac{1}{2} \sum_{\eta = \pm \infty} \left\{ \tilde{\mathcal{E}}(\eta) \cdot \ddot{\mathcal{E}}(\eta) + \mathcal{E}(\eta) \cdot \mathcal{H}(\eta) \right\} \\
\quad + \mathcal{U}(\eta) \cdot \ddot{\mathcal{E}}(\eta) \cdot \mathcal{B}(\eta - \eta) \]
\[ \quad + \mathcal{U}(\eta) \cdot \mathcal{H}(\eta) \cdot \mathcal{B}(\eta - \eta) \]
\[ = \cdots - \frac{1}{2} \sum_{\eta = \pm \infty} \left\{ \tilde{\mathcal{F}}(\eta) \cdot \ddot{\mathcal{E}}(\eta) + \mathcal{F}(\eta) \cdot \mathcal{E}(\eta) \right\} \]
\[ + \mathcal{F}(\eta) \cdot \mathcal{H}(\eta) \cdot \mathcal{B}(\eta - \eta) \]  
(31)

but

\[ \mathcal{V} \cdot \left[ \mathcal{E}(\eta) \cdot \mathcal{H}(\eta) + \mathcal{E}(\eta) \cdot \mathcal{H}(\eta) \right] \\
= - \left[ \mathcal{E}(\eta) \cdot \mathcal{H}(\eta) + \mathcal{E}(\eta) \cdot \mathcal{H}(\eta) \right] \\
+ \left[ \mathcal{H}(\eta) \cdot \mathcal{H}(\eta) + \mathcal{H}(\eta) \cdot \mathcal{H}(\eta) \right] \\
+ \left[ \mathcal{E}(\eta) \cdot \mathcal{E}(\eta) + \mathcal{E}(\eta) \cdot \mathcal{E}(\eta) \right] \]  
(32)

and

\[ \mathcal{E}(\eta) = \frac{\tilde{\mathcal{F}}(\eta)}{\sigma} - \sum_{\eta = \pm \infty} \mathcal{U}(\eta) \cdot \mathcal{B}(\eta - \eta) \]  
(33)

Therefore:

\[ \frac{1}{2} \left\{ \mathcal{E}(\eta) \cdot \mathcal{F}(\eta) + \mathcal{F}(\eta) \cdot \mathcal{E}(\eta) \right\} \\
= \frac{\tilde{\mathcal{F}}(\eta)}{\sigma} - \frac{1}{2} \sum_{\eta = \pm \infty} \left\{ \tilde{\mathcal{F}}(\eta) \cdot \mathcal{U}(\eta) \cdot \mathcal{B}(\eta - \eta) \right\} \]
\[ + \tilde{\mathcal{F}}(\eta) \cdot \mathcal{U}(\eta) \cdot \mathcal{B}(\eta - \eta) \]  
(34)

consequently:

\[ \frac{\lambda}{2} \sup_{\eta} \left| \mathcal{H}(\eta) \right|^2 \leq -\frac{1}{2} \left\{ \mathcal{H}(\eta) \cdot \mathcal{E}(\eta) + \mathcal{H}(\eta) \cdot \mathcal{E}(\eta) \right\} \]
\[ = C_{\mathcal{H}E}^{(\eta)} \]  
(35)
\[
\xi \frac{\partial}{\partial t} |E(\nu)|^2 = -\frac{1}{2} \nabla \cdot \left\{ \vec{E}(\nu) \times \vec{H}(\nu) \right\} + \frac{1}{2} \left\{ \vec{H}(\nu) \cdot \nabla \times \vec{E}(\nu) + \vec{H}(\nu) \cdot \nabla \times \vec{E}(\nu) \right\} + \sum_{\nu' = -\infty}^{\infty} \left\{ \vec{J}(\nu') \cdot \vec{B}(\nu - \nu') + \vec{J}(\nu) \cdot \vec{B}(\nu - \nu') \right\} 
\]

\[
= -\nabla \cdot \vec{S}(\nu) - \vec{J}(\nu) + \sum_{\nu' = -\infty}^{\infty} \left\{ \frac{\partial}{\partial t} \nu' \vec{B}(\nu - \nu') + \vec{J}(\nu) \cdot \vec{B}(\nu - \nu') \right\} 
\]

\[
\frac{\partial}{\partial t} |\nu|^2 = -\frac{1}{2} \sum_{\nu' = -\infty}^{\infty} \left\{ \vec{J}(\nu') \cdot \vec{B}(\nu - \nu') + \vec{J}(\nu) \cdot \vec{B}(\nu - \nu') \right\} 
\]

\[
= -\sum_{\nu' = -\infty}^{\infty} \sum_{\nu = -\infty}^{\infty} \sum_{\nu'' = -\infty}^{\infty} \left\{ \vec{J}(\nu'') \cdot \vec{B}(\nu - \nu') \right\} 
\]

The terms \( \vec{C}_{EHE}^{(\nu)} \) and \( \vec{C}_{EHE}^{(\nu)} \) are equal but opposite in sign.

Thus \( \vec{C}_{EHE}^{(\nu)} = -\vec{C}_{EHE}^{(\nu)} \) is the energy conversion term between magnetic and electric energy. We see that the conversion is between the same wavenumber for the electric and magnetic energies. We notice, for example, that only electric energy of wavenumber 4 is converted to magnetic energy of wavenumber 4. There is no cross-conversion from, say, wavenumber 5 electric energy to wavenumber 4 magnetic energy.
The terms $\sum_{\gamma = -\infty}^{\infty} C_{EU}^{(\gamma,\gamma)}$ and $\sum_{\gamma = -\infty}^{\infty} C_{UC}^{(\gamma,\gamma)}$ also contain individual components which convert to one another.

Thus

$$C_{EU}^{(s,\gamma)} + C_{EU}^{(s,-\gamma)} = -\left\{ C_{UE}^{(\gamma,s)} + C_{UE}^{(\gamma,-s)} \right\}$$

To see this we look at the following

$$C_{EU}^{(s,\gamma)} + C_{EU}^{(s,-\gamma)} = \frac{1}{2} \mathcal{F}(s) \cdot \mathcal{U}(\gamma) \chi \mathcal{B}^{(-s-\gamma)} \quad (= A)$$

$$+ \frac{1}{2} \mathcal{F}(-s) \cdot \mathcal{U}(\gamma) \chi \mathcal{B}^{(s-\gamma)} \quad (= B)$$

$$+ \frac{1}{2} \mathcal{F}(s) \cdot \mathcal{U}(\gamma) \chi \mathcal{B}^{(-s-\gamma)} \quad (= C = B^*)$$

$$+ \frac{1}{2} \mathcal{F}(-s) \cdot \mathcal{U}(\gamma) \chi \mathcal{B}^{(s+\gamma)} \quad (= D = A^*) \quad (38)$$

(Where * represents complex conjugation.

Note that

$$C_{EU}^{(\gamma,s)} = C_{EU}^{(s,\gamma)} \quad C_{UE}^{(\gamma,s)} = C_{EU}^{(-s,\gamma)}$$

$$C_{UE}^{(\gamma,s)} + C_{UE}^{(\gamma,-s)} = -\left\{ \frac{1}{2} \mathcal{F}(s) \cdot \mathcal{U}(\gamma) \chi \mathcal{B}^{(-s-\gamma)} \right\} \quad (= A)$$

$$+ \frac{1}{2} \mathcal{F}(-s) \cdot \mathcal{U}(\gamma) \chi \mathcal{B}^{(s-\gamma)} \quad (= B)$$

$$+ \frac{1}{2} \mathcal{F}(s) \cdot \mathcal{U}(\gamma) \chi \mathcal{B}^{(-s+\gamma)} \quad (= C = B^*)$$

$$+ \frac{1}{2} \mathcal{F}(-s) \cdot \mathcal{U}(\gamma) \chi \mathcal{B}^{(s+\gamma)} \quad (= D = A^*) \quad (39)$$
Consequently, electric energy of wavenumber 5, for example, can convert to kinetic energy of wavenumber 3.

\[
\frac{\xi}{2} \frac{2}{\gamma^2} |E(5)|^2 = \ldots + C_{EU}^{(5,3)} + C_{EU}^{(5,3)} + \ldots \tag{40}
\]

\[
\frac{\xi}{2} \frac{2}{\gamma^2} |U(3)|^2 = \ldots + C_{VE}^{(3,5)} + C_{VE}^{(3,5)} + \ldots = \{C_{EU}^{(5,3)}, C_{EU}^{(5,3)}\} + \ldots \tag{41}
\]

IV. An Energy Flow Diagram

An energy flow diagram can be constructed as follows:

```
  H0  H1  H2  H3  \ldots
  \downarrow  \downarrow  \downarrow  \downarrow  \ldots
    E0  E1  E2  E3  \ldots

  \downarrow  \downarrow  \downarrow  \downarrow  \ldots
    U0  U1  U2  U3  \ldots
```

where the lines indicate paths for the conversion of energy, while the boxes indicate the type of energy (H = magnetic, E = electric, and U = kinetic) and the particular wavenumber (n = 0, 1, \ldots).
Magnetic energy and electric energy are interchanged only for a particular wavenumber whereas electric and kinetic energies are interchanged among all wavenumbers. There is no direct conversion for magnetic to kinetic energy or vice versa in this formulation derived from the complete Maxwell's equations. (See, however, section V, Special Cases, for the MHD approximation case.)
V. Special Cases

So far we have made a few implicit approximations in the neglect of the Hall effect, the neglect of the contribution of the convection current to \( \mathbf{j} \), and the neglect of the electric part of the body force per unit volume. In application to the ionosphere, one would want to include these effects. We have not, as yet, made the MHD or magnetohydrodynamic approximation \( \left( \frac{\partial \mathbf{B}}{\partial t} \to 0 \right) \).

Whether the displacement terms can be neglected depends upon the quantitative assumptions made.

Scale analysis indicates that for the larger size aspects of the solar atmosphere,

For example, typical velocities are

\[
\begin{align*}
\mathbf{u} & \sim 5 \times 10^6 \text{ m/sec} \\
\mathbf{u}_{\text{sound}} & \sim \sqrt{\frac{\gamma p}{\rho}} \sim \sqrt{\frac{10^5}{5 \times 10^7}} \text{ m/sec} \sim 2 \times 10^3 \text{ m/sec} \\
\mathbf{u}_{\text{thermal}} & \sim \sqrt{\frac{3 k T}{m}} = \sqrt{\frac{3 (1.3) \times 10^{22} \text{ eV}^2}{1.7 \times 10^{-27}}} \sim 10^4 \text{ m/sec} \\
\mathbf{u}_{\text{gravity}} & \sim \sqrt{g L} \sim \sqrt{\frac{1}{2} \frac{\Delta p}{\Delta L}} \sim \sqrt{\frac{10^5}{10^{-5}}} \text{ m/sec} \sim 3 \times 10^4 \text{ m/sec} \\
\mathbf{u}_{\text{Alfven}} & \sim \frac{\beta_0}{\sqrt{\mu_0 \rho_0}} \sim \frac{1 \times 10^{14}}{4 \pi \times 10^{7} \times 10^{-5}} \sim 3 \times 10^5 \text{ m/sec}.
\end{align*}
\]
since for the photosphere \( \Delta L \sim 400 \text{ km}, T \sim 6000^\circ \text{K} \)

\[ m \sim \text{mass of proton} \sim 1.7 \times 10^{-27} \text{ kg}, \quad \rho \sim 10^{-8} \text{ g/cm}^3 \]

\[ \Delta P \sim 10^5 \text{ dynes/cm}^2, \quad B_0 \sim 10^{-4} \text{ webers/m}^2 \sim 1 \text{ gauss (background field)} \]

\[ \Delta \omega \sim 10^7 \text{ gms} \quad (\text{for some sun spots, etc.).} \]

\[ \mathbf{u} = \text{convection velocity of fluid} \]

\[ u_{\text{sound}} = \text{velocity of sound} \]

\[ u_{\text{therm}} = \text{Thermal velocity of protons at 6000}^\circ \text{K} \]

\[ u_{\text{gravity}} = \text{gravity wave velocity (for long waves)} \]

\[ u_{\text{Alfven}} = \text{Alfven wave velocity} \]

But

\[ \frac{\partial D}{\partial t} \sim \frac{\xi E}{\gamma} \]

\[ \nabla \times H \sim \frac{H}{L} \]

But \( \frac{\partial B}{\partial t} = -\nabla \times E \) says that

\[ \frac{B}{\gamma} \sim \frac{H L}{\gamma} \sim \frac{E}{L} \]

\[ \gamma \sim \frac{H L}{\gamma} \]

\[ E \sim \frac{H L}{\gamma} \]

Therefore

\[ \frac{\partial D}{\partial t} \sim \frac{\xi E}{\gamma} \sim \frac{E}{H L} \sim \frac{E}{H L} \sim \xi \frac{(L^2)}{E^2} \sim \left( \frac{H}{L} \right)^2 \]
where \( c = \) velocity of light, and \( \mathcal{U} \) is a characteristic velocity.

For high frequency oscillations of an ionized gas in an electromagnetic field, or for a relativistic fluid or gas, the displacement term can be important. Also for low conductivities \( \frac{1}{\kappa} \ll \frac{\varepsilon}{\sigma} \) may not be negligible relative to \( \frac{1}{2} \frac{\varepsilon}{\sigma c^2} \langle \vec{B}, \vec{E} \rangle \) since in that case

\[
\vec{D} \cdot \vec{E} = \varepsilon E^2 = \varepsilon \vec{E} \cdot \vec{E}
\]

But \( \vec{E} = \frac{j}{\varepsilon} - \mathcal{U} \chi \vec{B} \) so that

\[
\varepsilon \vec{E} \cdot \vec{E} = \varepsilon j \cdot j + \varepsilon (\mathcal{U} \chi \vec{B}) \cdot (\mathcal{U} \chi \vec{B}) - \frac{2\varepsilon}{c^2} j \cdot \chi \vec{B}
\]

\[
\frac{\varepsilon E^2}{\mathcal{A}^2 H^2} \approx \varepsilon \frac{1}{\mathcal{A}^2} \left( \frac{j}{\mathcal{A}} \right)^2 + \left( \frac{\chi}{c} \right)^2 - 2 \left( \frac{\chi}{c} \right) \frac{1}{\mathcal{A}} \left( \frac{j}{\mathcal{A}} \right)
\]

for \( \frac{j}{\mathcal{A}} \approx \) constant, \( \mathcal{A} \rightarrow 0 \Rightarrow \frac{1}{\mathcal{A}^2} \rightarrow \infty \)

The general equations derived previously can be specialized without any problem. If we make MHD or magnetohydrodynamic approximations, we have

\[
\frac{1}{2} \frac{\partial}{\partial t} \langle \vec{E}, \vec{D} \rangle = \frac{1}{2} \left[ \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{D} \cdot \frac{\partial \vec{E}}{\partial t} \right] = 0
\]

so that

\[
\frac{1}{2} \frac{\partial}{\partial t} \langle \vec{E}, \vec{H} \rangle = \mathcal{CH}_{E}
\]

(A)
Using (B), (A) may be written as

\[ \frac{1}{2} \frac{\partial}{\partial t} (\vec{u} \cdot \vec{u}) = - \vec{\nabla}^2 \psi - \mathcal{F} - C_{\nu E} \] (D)

With (C), we then trace the energy flow

Following the same procedure for the Fourier representation we have
Similar diagrams would result in sections VIII. and X. if we were to make the boxes part of the connecting lines.

VI. Cautions

It should be noted that we are dealing with a system of equations of the sort

\[ \frac{\partial A}{\partial t} = C_{A} \beta \]  (6.1)
\[ \frac{\partial \beta}{\partial t} = C_{B} \gamma + C_{C} \lambda \]  (6.2)
\[ \frac{\partial \lambda}{\partial t} = C_{C} \beta \]  (6.3)

where

\[ C_{B} \gamma = X - C_{A} \beta \]
\[ C_{C} \lambda = Y - C_{C} \beta \]

The correspondence to our equations is

\[ C_{A} \beta = C_{HE} \]
\[ X = -\nabla \cdot \Sigma \]
\[ Y = -\bar{\Sigma} \]
\[ C_{C} \beta = C_{Ce} \]
\[ A = \frac{1}{4} |H|^2 \]
\[ B = \frac{1}{2} |E|^2 \]
\[ C = \frac{1}{2} |U|^2 \]
so that we can have

\[ \frac{\partial A}{\partial t} = X - C_{BA} \]  
(6.1 A)

\[ \frac{\partial B}{\partial t} = C_{BA} + Y - C_{BC} \]  
(6.2 A)

\[ \frac{\partial C}{\partial t} = C_{BC} \]  
(6.3 A)

However, we can equally well write

\[ \frac{\partial A}{\partial t} = X - C_{BA} \]  
(6.1 B)

\[ \frac{\partial B}{\partial t} = C_{BA} + C_{BC} \]  
(6.2 B)

\[ \frac{\partial C}{\partial t} = Y - C_{BC} \]  
(6.3 B)

or

\[ \frac{\partial A}{\partial t} = C_{AB} \]  
(6.1 C)

\[ \frac{\partial B}{\partial t} = X - C_{AB} + C_{BC} \]  
(6.2 C)

\[ \frac{\partial C}{\partial t} = Y - C_{BC} \]  
(6.3 C)

or

\[ \frac{\partial A}{\partial t} = C_{AB} \]  
(6.1 D)

\[ \frac{\partial B}{\partial t} = X - C_{AB} + Y - C_{AB} \]  
(6.2 D)

\[ \frac{\partial C}{\partial t} = C_{BC} \]  
(6.3 D)
Consequently there is a certain arbitrariness in where $X$ and $Y$ are attached to the system. In our diagrams $-\nabla \cdot \vec{S}$ and $\vec{J}$ have been omitted.

For the system $(\zeta)$ and $(\psi)$ of section V, there is a similar arbitrariness.

It should be noted that in some cases such as the ionosphere we would have to deal with a multi-species fluid to achieve a proper description.

The solar atmosphere, ionosphere, etc. may not be isotropic and homogenous enough with respect to the electrical properties so that the assumption of $\vec{D} = \varepsilon \vec{E}$, $\vec{B} = \mu \vec{H}$ can be used. The present treatment could be modified by using a tensor dielectric constant and a tensor permeability.

The non-linear interactions of the electromagnetic fields with, for example, the solar plasma might introduce other energy generation, dissipation or conversion terms of the sort that have not been considered here.
VII. Component Form of Energy Equations

We can decompose the velocity field, magnetic field, and electric field into $x$, $y$, $z$ components. This will allow various combinations of the electric and magnetic field, for example, into toroidal and poloidal components.

The decomposition is straightforward.

Equations (1), (2), (7), and (9) become (in Cartesian coordinates)

\[
\frac{\partial B_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \quad (1'A)
\]

\[
\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \quad (1'B)
\]

\[
\frac{\partial B_z}{\partial t} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \quad (1'C)
\]

\[
\frac{\partial D_x}{\partial t} = \frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y} - J_x \quad (2'A)
\]

\[
\frac{\partial D_y}{\partial t} = \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} - J_y \quad (2'B)
\]

\[
\frac{\partial D_z}{\partial t} = \frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} - J_z \quad (2'C)
\]
\[ J_x = \sigma (E_x + V B_z - W B_y) \]  
\[ J_y = \sigma (E_y + W B_x - U B_z) \]  
\[ J_z = \sigma (E_z + U B_y - V B_x) \]

\[ \oint \frac{dU}{d\tau} = \cdots + \left[ J_y B_z - J_z B_y \right] \]  
\[ \oint \frac{dV}{d\tau} = \cdots + \left[ J_z B_x - J_x B_z \right] \]  
\[ \oint \frac{dW}{d\tau} = \cdots + \left[ J_x B_y - J_y B_x \right] \]
The energy equations are:

\[ \frac{M}{2} \frac{\partial^2 (H_x)}{\partial t^2} = \frac{1}{2} \left[ H_x \frac{\partial E_x}{\partial y} - H_x \frac{\partial E_y}{\partial z} \right] = \left( x^2 \right) C_{HE} + \left( x y \right) C_{HE} \]  
(13'A)

\[ \frac{M}{2} \frac{\partial^2 (H_y)}{\partial t^2} = \frac{1}{2} \left[ H_y \frac{\partial E_y}{\partial z} - H_y \frac{\partial E_y}{\partial x} \right] = \left( y^2 \right) C_{HE} + \left( y z \right) C_{HE} \]  
(13'B)

\[ \frac{M}{2} \frac{\partial^2 (H_z)}{\partial t^2} = \frac{1}{2} \left[ H_z \frac{\partial E_y}{\partial x} - H_z \frac{\partial E_x}{\partial y} \right] = \left( z^2 \right) C_{HE} + \left( z x \right) C_{HE} \]  
(13'C)
\[
\frac{\xi}{2} \frac{\partial}{\partial t} (E_x)^2 = \frac{1}{2} \left[ E_x \frac{\partial H_y}{\partial x} - E_x \frac{\partial H_z}{\partial y} - E_x J_x \right]
\]

\[
= \frac{1}{2} \left[ \frac{\partial}{\partial x} (E_x H_y) - \frac{\partial}{\partial y} (E_x H_z) - H_y \frac{\partial E_x}{\partial x} + H_z \frac{\partial E_x}{\partial y} - E_x J_x \right]
\]

\[
= \frac{1}{2} \left[ \frac{\partial}{\partial x} (E_x H_y) - \frac{\partial}{\partial y} (E_x H_z) - \frac{1}{2} [E_x J_x] - \frac{\partial}{\partial x} (E_x J_x) - \frac{\partial}{\partial y} (E_x J_x) \right]
\]

\[
= \frac{1}{2} \left[ \frac{\partial}{\partial x} (E_x H_y) - \frac{\partial}{\partial y} (E_x H_z) - \frac{1}{2} [E_x J_x] - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} (E_x J_x) \right) - \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} (E_x J_x) \right) \right]
\]

\[
(14'A)
\]

\[
\frac{\xi}{2} \frac{\partial}{\partial t} (E_y)^2 = \frac{1}{2} \left[ E_y \frac{\partial H_x}{\partial y} - E_y \frac{\partial H_z}{\partial x} - E_y J_y \right]
\]

\[
= \frac{1}{2} \left[ \frac{\partial}{\partial y} (E_y H_x) - \frac{\partial}{\partial x} (E_y H_z) - H_x \frac{\partial E_y}{\partial y} + H_z \frac{\partial E_y}{\partial x} - E_y J_y \right]
\]

\[
= \frac{1}{2} \left[ \frac{\partial}{\partial y} (E_y H_x) - \frac{\partial}{\partial x} (E_y H_z) - \frac{1}{2} [E_y J_y] - \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} (E_y J_y) \right) - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} (E_y J_y) \right) \right]
\]

\[
= \frac{1}{2} \left[ \frac{\partial}{\partial y} (E_y H_x) - \frac{\partial}{\partial x} (E_y H_z) - \frac{1}{2} [E_y J_y] - \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} (E_y J_y) \right) - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} (E_y J_y) \right) \right]
\]

\[
(14'B)
\]

\[
\frac{\xi}{2} \frac{\partial}{\partial t} (E_z)^2 = \frac{1}{2} \left[ E_z \frac{\partial H_x}{\partial z} - E_z \frac{\partial H_y}{\partial x} - E_z J_z \right]
\]

\[
= \frac{1}{2} \left[ \frac{\partial}{\partial z} (E_z H_x) - \frac{\partial}{\partial x} (E_z H_y) - H_x \frac{\partial E_z}{\partial z} + H_y \frac{\partial E_z}{\partial x} - E_z J_z \right]
\]

\[
= \frac{1}{2} \left[ \frac{\partial}{\partial z} (E_z H_x) - \frac{\partial}{\partial x} (E_z H_y) - \frac{1}{2} [E_z J_z] - \frac{\partial}{\partial z} \left( \frac{\partial}{\partial x} (E_z J_z) \right) - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial z} (E_z J_z) \right) \right]
\]

\[
= \frac{1}{2} \left[ \frac{\partial}{\partial z} (E_z H_x) - \frac{\partial}{\partial x} (E_z H_y) - \frac{1}{2} [E_z J_z] - \frac{\partial}{\partial z} \left( \frac{\partial}{\partial x} (E_z J_z) \right) - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial z} (E_z J_z) \right) \right]
\]

\[
(14')
\]
\[ \frac{\rho}{2} \frac{\partial (w^2)}{\partial t} = \frac{1}{2} \left[ U J_y B_x - U J_x B_y \right] \]
\[ = \left( \rho \gamma_2 \right) c_{UE} + \left( \rho \gamma_1 \right) c_{UE} \]  
\hspace{1cm} (15'A)

\[ \frac{\rho}{2} \frac{\partial (l^2)}{\partial t} = \frac{1}{2} \left[ V J_x B_x - V J_y B_y \right] \]
\[ = \left( \gamma_2 \gamma_3 \right) c_{UE} + \left( \gamma_1 \gamma_3 \right) c_{UE} \]  
\hspace{1cm} (15'B)

\[ \frac{\rho}{2} \frac{\partial (w^2)}{\partial t} = \frac{1}{2} \left[ W J_x B_y - W J_y B_x \right] \]
\[ = \left( \gamma_2 \gamma_1 \right) c_{UE} + \left( \gamma_1 \gamma_3 \right) c_{UE} \]  
\hspace{1cm} (15'C)

We rearrange (7'):

\[ E_X = \frac{J_x}{\sigma} + W B_y - V B_z \]  
\hspace{1cm} (17'A)

\[ E_Y = \frac{J_y}{\sigma} + U B_z - W B_x \]  
\hspace{1cm} (17'B)

\[ E_Z = \frac{J_z}{\sigma} + V B_x - U B_y \]  
\hspace{1cm} (17'C)
forming \(-E_x J_x\), etc., we have

\[
- E_x J_x = - \left\{ \frac{J_x^2}{\sigma} \right\} - W J_x B_y + V J_x B_z
\]

\[
= - \frac{J_x^2}{\sigma} - \left( \frac{x^2}{C_{HE}} \right) - \left( \frac{y^2}{C_{UE}} \right) \quad (42A)
\]

\[
- E_y J_y = - \frac{J_y^2}{\sigma} - U J_y B_z + W J_y B_x
\]

\[
= \frac{J_y^2}{\sigma} - \left( \frac{y^2}{C_{UE}} \right) - \left( \frac{z^2}{C_{UE}} \right) \quad (42B)
\]

\[
- E_z J_z = - \frac{J_z^2}{\sigma} - V J_z B_x + U J_z B_y
\]

\[
= \frac{J_z^2}{\sigma} - \left( \frac{z^2}{C_{UE}} \right) - \left( \frac{x^2}{C_{UE}} \right) \quad (42C)
\]

Consequently, we can trace the energy flow connections. In (13'A) there is a term \(\frac{x^2}{C_{HE}}\) indicating a conversion from magnetic \(x\)-component energy to electric \(z\)-component energy. The same term occurs in (14'A) with the opposite sign. Similarly, for the other terms in equations (13'), (14'), (15'), and (42).
VIII. Component Energy Flow Diagram

From the above we arrive at a modified energy diagram:
IX. Fourier Resolution of the Component Energy Equations

We can proceed to the Fourier resolution of the component equations. After writing (24), (25), (26), and (27) in component form we arrive at:

\[
\frac{\partial B_X(\eta)}{\partial \tau} = \frac{\partial E_y(\eta)}{\partial z} - \frac{\partial E_z(\eta)}{\partial y} \tag{1''A}
\]

\[
\frac{\partial B_y(\eta)}{\partial \tau} = \zeta \eta E_z(\eta) - \frac{\partial E_X(\eta)}{\partial z} \tag{1''B}
\]

\[
\frac{\partial B_z(\eta)}{\partial \tau} = \frac{\partial E_X(\eta)}{\partial y} - \zeta \eta E_y(\eta) \tag{1''C}
\]

\[
\frac{\partial D^c_1(\eta)}{\partial \tau} = \frac{\partial H_z(\eta)}{\partial y} - \frac{\partial H_y(\eta)}{\partial z} - J_X(\eta) \tag{2''A}
\]

\[
\frac{\partial D^c_2(\eta)}{\partial \tau} = \frac{\partial H_X(\eta)}{\partial z} - \zeta \eta H_z(\eta) - J_y(\eta) \tag{2''B}
\]

\[
\frac{\partial D^c_2(\eta)}{\partial \tau} = \zeta \eta H_y(\eta) - \frac{\partial H_X(\eta)}{\partial y} - J_z(\eta) \tag{2''C}
\]
\[ J_x(n) = \sigma \left[ E_x(n) + \sum_{m=-\infty}^{\infty} \left[ V(m) B_z(n-m) - W(m) B_y(n-m) \right] \right] \]  
(7"A)

\[ J_y(n) = \sigma \left[ E_y(n) + \sum_{m=-\infty}^{\infty} \left[ W(m) B_x(n-m) - U(m) B_z(n-m) \right] \right] \]  
(7"B)

\[ J_z(n) = \sigma \left[ E_z(n) + \sum_{m=-\infty}^{\infty} \left[ U(m) B_y(n-m) - V(m) B_x(n-m) \right] \right] \]  
(7"C)

\[ \frac{dU(n)}{dt} = \sum_{m=-\infty}^{\infty} \left[ J_y(n) B_z(n-m) - J_z(n) B_y(n-m) \right] \]  
(9"A)

\[ \frac{dV(n)}{dt} = \sum_{m=-\infty}^{\infty} \left[ J_x(n) B_y(n-m) - J_y(n) B_x(n-m) \right] \]  
(9"B)

\[ \frac{dW(n)}{dt} = \sum_{m=-\infty}^{\infty} \left[ J_x(n) B_y(n-m) - J_y(n) B_x(n-m) \right] \]  
(9"C)
Now we form energy quantities in the usual way, as, for example:

\[
\frac{1}{2} \frac{\partial}{\partial t} \left| \mathbf{H}_X(\eta) \right|^2 = \frac{1}{2} \left[ \mathbf{H}_X(-\eta) \frac{\partial \mathbf{H}_X(\eta)}{\partial t} + \mathbf{H}_X(\eta) \frac{\partial \mathbf{H}_X(-\eta)}{\partial t} \right]
\]

Then:

\[
\frac{1}{2} \frac{\partial}{\partial t} \left| \mathbf{H}_X(\eta) \right|^2 = \frac{1}{2} \left[ \mathbf{H}_X(-\eta) \frac{\partial \mathbf{E}_y(\eta)}{\partial t} + \mathbf{H}_X(\eta) \frac{\partial \mathbf{E}_y(-\eta)}{\partial t} \right]
\]

\[
= \left[ \begin{array}{c} (\mathbf{x},\mathbf{y}) \end{array} \right] \begin{bmatrix} \mathbf{C}_{HE} \end{bmatrix} + \left[ \begin{array}{c} (\mathbf{x},\mathbf{y}) \end{array} \right] \begin{bmatrix} \mathbf{C}_{HE} \end{bmatrix}
\]

(13''A)

\[
\frac{1}{2} \frac{\partial}{\partial t} \left| \mathbf{H}_y(\eta) \right|^2 = \frac{1}{2} \left[ \mathbf{H}_y(-\eta) \frac{\partial \mathbf{E}_x(\eta)}{\partial t} + \mathbf{H}_y(\eta) \frac{\partial \mathbf{E}_x(-\eta)}{\partial t} \right]
\]

\[
= \left[ \begin{array}{c} (\mathbf{y},\mathbf{y}) \end{array} \right] \begin{bmatrix} \mathbf{C}_{HE} \end{bmatrix} + \left[ \begin{array}{c} (\mathbf{y},\mathbf{y}) \end{array} \right] \begin{bmatrix} \mathbf{C}_{HE} \end{bmatrix}
\]

(13''C)

\[
\frac{1}{2} \frac{\partial}{\partial t} \left| \mathbf{H}_z(\eta) \right|^2 = \frac{1}{2} \left[ \mathbf{H}_z(-\eta) \frac{\partial \mathbf{E}_y(\eta)}{\partial t} + \mathbf{H}_z(\eta) \frac{\partial \mathbf{E}_y(-\eta)}{\partial t} \right]
\]

\[
= \left[ \begin{array}{c} (\mathbf{z},\mathbf{y}) \end{array} \right] \begin{bmatrix} \mathbf{C}_{HE} \end{bmatrix} + \left[ \begin{array}{c} (\mathbf{z},\mathbf{y}) \end{array} \right] \begin{bmatrix} \mathbf{C}_{HE} \end{bmatrix}
\]

(13''C)
Similarly, for electric energy we have:

\[
\frac{\mathcal{E}}{2} \frac{\partial}{\partial t} \left| E_x(\mathbf{r}) \right|^2 = \frac{1}{2} \left\{ -E_x(\mathbf{r}) \frac{\partial H_y(\mathbf{r})}{\partial z} - E_y(\mathbf{r}) \frac{\partial H_x(\mathbf{r})}{\partial z} + E_x(\mathbf{r}) \frac{\partial H_z(\mathbf{r})}{\partial y} - E_z(\mathbf{r}) J_x(\mathbf{r}) - E_x(\mathbf{r}) J_y(\mathbf{r}) \right\}
\]

\[
= \frac{1}{2} \left\{ \frac{\partial}{\partial y} \left[ E_y(\mathbf{r}) H_x(\mathbf{r}) + E_z(\mathbf{r}) H_y(\mathbf{r}) \right] - \frac{\partial}{\partial z} \left[ E_x(\mathbf{r}) H_z(\mathbf{r}) + E_y(\mathbf{r}) H_y(\mathbf{r}) \right] - E_y(\mathbf{r}) J_x(\mathbf{r}) - E_x(\mathbf{r}) J_y(\mathbf{r}) \right\}
\]

\[
= \frac{1}{2} \left\{ \frac{\partial}{\partial y} \left[ E_y(\mathbf{r}) H_x(\mathbf{r}) + E_z(\mathbf{r}) H_y(\mathbf{r}) \right] - E_y(\mathbf{r}) J_x(\mathbf{r}) - E_x(\mathbf{r}) J_y(\mathbf{r}) \right\}
\]

\[
= \frac{1}{2} \left\{ \frac{\partial}{\partial z} \left[ E_x(\mathbf{r}) H_z(\mathbf{r}) + E_y(\mathbf{r}) H_y(\mathbf{r}) \right] - E_x(\mathbf{r}) J_y(\mathbf{r}) - E_y(\mathbf{r}) J_x(\mathbf{r}) \right\}
\]

\[
= \frac{1}{2} \left\{ \frac{\partial}{\partial z} \left[ E_x(\mathbf{r}) H_z(\mathbf{r}) + E_y(\mathbf{r}) H_y(\mathbf{r}) \right] - E_x(\mathbf{r}) J_y(\mathbf{r}) - E_y(\mathbf{r}) J_x(\mathbf{r}) \right\}
\]
For the component kinetic energies we have the following quadratic forms:

\[
\frac{p}{2} \frac{\partial}{\partial t} |U(\gamma)|^2 = \sum_{\eta_2 = -\infty}^{\infty} \left[ U(\eta) J_y (\eta) B_2 (\eta - \eta_2) + U(\eta) J_2 (\eta) B_2 (-\eta - \eta_2) \right] \\
- \left[ U(\eta) J_2 (\eta) B_2 (\eta - \eta_2) + U(\eta) J_2 (\eta) B_2 (-\eta - \eta_2) \right] \\
= \sum_{\eta_2 = -\infty}^{\infty} \left\{ \begin{pmatrix} x \ y \end{pmatrix} C_{u,v}^{(\eta_2, \eta_2)} \right\} + \sum_{\eta_2 = -\infty}^{\infty} \left\{ \begin{pmatrix} y \ x \end{pmatrix} C_{u,v}^{(\eta_2, \eta_2)} \right\} \tag{15''A}
\]

\[
\frac{p}{2} \frac{\partial}{\partial t} |V(\gamma)|^2 = \sum_{\eta_2 = -\infty}^{\infty} \left[ V(\eta) J_x (\eta) B_2 (\eta - \eta_2) + V(\eta) J_2 (\eta) B_2 (-\eta - \eta_2) \right] \\
- \left[ V(\eta) J_2 (\eta) B_2 (\eta - \eta_2) + V(\eta) J_2 (\eta) B_2 (-\eta - \eta_2) \right] \\
= \sum_{\eta_2 = -\infty}^{\infty} \left\{ \begin{pmatrix} x \ y \end{pmatrix} C_{u,v}^{(\eta_2, \eta_2)} \right\} + \sum_{\eta_2 = -\infty}^{\infty} \left\{ \begin{pmatrix} y \ x \end{pmatrix} C_{u,v}^{(\eta_2, \eta_2)} \right\} \tag{15''B}
\]

\[
\frac{p}{2} \frac{\partial}{\partial t} |W(\gamma)|^2 = \sum_{\eta_2 = -\infty}^{\infty} \left[ W(\eta) J_y (\eta) B_2 (\eta - \eta_2) + W(\eta) J_2 (\eta) B_2 (-\eta - \eta_2) \right] \\
- \left[ W(\eta) J_2 (\eta) B_2 (\eta - \eta_2) + W(\eta) J_2 (\eta) B_2 (-\eta - \eta_2) \right] \\
= \sum_{\eta_2 = -\infty}^{\infty} \left\{ \begin{pmatrix} x \ y \end{pmatrix} C_{u,v}^{(\eta_2, \eta_2)} \right\} + \sum_{\eta_2 = -\infty}^{\infty} \left\{ \begin{pmatrix} y \ x \end{pmatrix} C_{u,v}^{(\eta_2, \eta_2)} \right\} \tag{15''C}
\]
We can arrange (7'') in the form:

\[ E_x(\eta) = \frac{J_x(\eta)}{\sigma} - \sum_{\eta_2 = -\infty}^{\infty} \left[ V(\eta) B_x(\eta_2, \eta_2) - W(\eta) B_y(\eta_2, \eta_2) \right] \] \hspace{1cm} (17''A)

\[ E_y(\eta) = \frac{J_y(\eta)}{\sigma} - \sum_{\eta_2 = -\infty}^{\infty} \left[ W(\eta) B_x(\eta_2, \eta_2) - U(\eta) B_y(\eta_2, \eta_2) \right] \] \hspace{1cm} (17''B)

\[ E_\perp(\eta) = \frac{J_\perp(\eta)}{\sigma} - \sum_{\eta_2 = -\infty}^{\infty} \left[ U(\eta) B_y(\eta_2, \eta_2) - V(\eta) B_x(\eta_2, \eta_2) \right] \] \hspace{1cm} (17''C)

We now calculate the \( \vec{E} \cdot \vec{J} \) terms in (14''):

\[-E_x(\eta) J_x(-\eta) - E_x(-\eta) J_x(\eta) = -2 \frac{J_x(\eta) J_x(-\eta)}{\sigma} +
\]

\[ + \sum_{\eta_2 = -\infty}^{\infty} \left\{ \left[ J_x(\eta) V(\eta) B_x(-\eta_2, \eta_2) + J_x(-\eta) V(\eta) B_x(-\eta_2, \eta_2) \right] - \left[ J_x(\eta) W(\eta) B_y(-\eta_2, \eta_2) + J_x(-\eta) W(\eta) B_y(-\eta_2, \eta_2) \right] \right\} \]

\[ = -2 \frac{|J_x(\eta)|^2}{\sigma} + \sum_{\eta_2 = -\infty}^{\infty} (x)_{\sigma_1} (x)_{\sigma_2} C_{E\sigma} + \sum_{\eta_2 = -\infty}^{\infty} (x)_{\sigma_1} C_{E\sigma} \]

\[ (42''A) \]
\[-E_y(\eta)J_y(-\eta) - E_y(-\eta)J_y(\eta)\]

\[= -2 \frac{J_y(\eta)}{\sigma} \sum_{\eta = -\infty}^{\infty} \left[ J_y(\eta) U(\mu) B_y(-\eta - \mu) + J_y(-\eta) W(\mu) B_y(\eta - \mu) \right] \]

\[-\left[ J_y(\eta) U(\mu) B_y(-\eta - \mu) - J_y(-\eta) U(\mu) B_y(\eta - \mu) \right] \]

\[= -2 \frac{J_y(\eta)}{\sigma} \sum_{\eta = -\infty}^{\infty} C_E U(\eta, \eta) + \sum_{\eta = -\infty}^{\infty} C_E U(\eta, \eta) \quad (42'B)\]

\[-E_x(\eta) J_x(-\eta) - E_x(-\eta) J_x(\eta)\]

\[= -2 \frac{J_x(\eta)}{\sigma} \sum_{\eta = -\infty}^{\infty} \left[ J_x(\eta) U(\mu) B_x(-\eta - \mu) + J_x(-\eta) W(\mu) B_x(\eta - \mu) \right] \]

\[-\left[ J_x(\eta) U(\mu) B_x(-\eta - \mu) - J_x(-\eta) U(\mu) B_x(\eta - \mu) \right] \]

\[= -2 \frac{J_x(\eta)}{\sigma} \sum_{\eta = -\infty}^{\infty} C_E U(\eta, \eta) + \sum_{\eta = -\infty}^{\infty} C_E U(\eta, \eta) \quad (42'C)\]
But

\[ (\xi y)_{(n,y,5)} + (\xi y)_{(n,y,5)} \subset C_{E_U} \]

as in (38) and (39). In the next section the conversion paths are summarized in a diagram.
As before, we can trace the energy flow by looking at the conversion terms. We find:

Magnetic energy of wavenumber 0, x-component can convert to electric energy wavenumber 0, y- and z-components, and vice versa. Electric energy of wave number 0, x-component can convert to kinetic energy, wavenumbers 0, 1, 2, 3, ..., y- and z-components and vice versa. Kinetic energy of wavenumber 0, x-component, can convert to electric energy wavenumbers 0, 1, 2, 3, ..., y- and z-components and vice versa. Similarly, one can trace the energy flow of all the other components, for a particular wavenumber.
From Saltzman, 1957, we have added the bottom connections, showing that all the kinetic energy wavenumbers, all components, are connected. Compare this part of our diagram with the diagram in Saltzman and Teweles, 1964.

XI. An Extension of the Starr-Gilman Energy Diagram

We can arrange our energy diagram in a form parallel to that of Starr and Gilman (1966) by noting the following correspondences after taking the volume integral of the energies involved so that

\[
\begin{align*}
\text{potential energy} &= \text{PE} \\
\text{internal energy} &= \text{IE} \\
\text{mean zonal kinetic energy} &= \text{ZKE} = U_0 \\
\text{mean zonal, or toroidal, magnetic energy} &= \text{ZME} = \frac{\nabla \times \phi}{2} \\
\text{mean meridional kinetic energy} &= \text{MKE} = (\nabla \phi) + (\omega \phi) \\
\text{mean meridional, or poloidal, magnetic energy} &= \text{MME} = (\frac{\nabla}{\nabla} \phi) + (\nabla \phi) \\
\text{eddy kinetic energy} &= \text{EKE} = (u_1 + u_2) + \ldots \\
&+ (v_1 + v_2) + \ldots \\
&+ (w_1 + w_2) + \ldots
\end{align*}
\]

potential energy = PE
internal energy = IE
mean zonal kinetic energy = ZKE = U_0
mean zonal, or toroidal, magnetic energy = ZME = \(\frac{\nabla \times \phi}{2}\)
mean meridional kinetic energy = MKE = (\nabla \phi) + (\omega \phi)
mean meridional, or poloidal, magnetic energy = MME = (\frac{\nabla}{\nabla} \phi) + (\nabla \phi)
eddy kinetic energy = EKE = (u_1 + u_2) + \ldots 
+ (v_1 + v_2) + \ldots 
+ (w_1 + w_2) + \ldots
eddy magnetic energy = EME = \( H_x^1 + H_x^2 + \ldots \) 
+ \( H_y^1 + H_y^2 + \ldots \) 
+ \( H_z^1 + H_z^2 + \ldots \)

We introduce:

mean zonal, or toroidal, electric energy = ZEE = \( E_x^0 \)
mean meridional, poloidal, electric energy = MEE = \( E_y^0 \)
eddy electric energy = EEE = \( E_{x1} t E_{x2} + \ldots \) 
+ \( E_{y1} t E_{y2} + \ldots \) 
+ \( E_{z1} + E_{z2} + \ldots \)

We arrive at:
XII. A Remark about Applications to Data

The analysis presented here can be applied to actual observational data of such systems as the sun in the same manner in which Saltzman's equations were applied by himself and others to the earth's atmosphere (see introduction). Hopefully the appropriate data will become available so that this can be done. The direction and amount of energy flow along the possible channels could then be determined.
We can neglect the displacement current term at the outset.

From
\[ \frac{\partial \vec{E}}{\partial t} = -\nabla \times \vec{E} \]
\[ \nabla \times \vec{H} = \vec{j} \]
\[ \frac{\partial \vec{B}}{\partial t} = \ldots + (\vec{j} \times \vec{B}) \]

we have
\[ \frac{\partial \vec{B}}{\partial t} = \mu \vec{H} \times \vec{H} = -\vec{H} \cdot \nabla \cdot \vec{H} + \vec{H} \cdot (\nabla \cdot \vec{H}) \vec{H} - (\vec{H} \cdot \vec{H}) (\nabla \cdot \vec{H}) \]
\[ = -\nabla \cdot \left( \vec{j} \times \vec{H} \right) + \nabla \cdot \left( \vec{j} \times \vec{H} \right) \vec{j} + \nabla \cdot \left( \vec{j} \times \vec{H} \right) \vec{j} - \nabla \cdot \left( \vec{j} \times \vec{H} \right) \vec{j} \]
\[ = -\nabla \cdot \left( \vec{j} \times \vec{H} \right) + \nabla \cdot \left( \vec{j} \times \vec{H} \right) \vec{j} - \nabla \cdot \left( \vec{j} \times \vec{H} \right) \vec{j} \]
\[ = -\nabla \cdot \left( \vec{j} \times \vec{H} \right) + \nabla \cdot \left( \vec{j} \times \vec{H} \right) \vec{j} - \nabla \cdot \left( \vec{j} \times \vec{H} \right) \vec{j} \]

But
\[ \varepsilon_{\kappa \mu \nu \lambda} = \varepsilon_{\lambda \mu \nu \kappa} \]
and
\[ \varepsilon_{\mu \nu \kappa \lambda} \delta_{\kappa \lambda} - \delta_{\mu \nu \kappa} \delta_{\kappa \lambda} \]
so that
\[ \frac{\partial \vec{B}}{\partial t} = \varepsilon_{\kappa \mu \nu \lambda} \frac{\partial \vec{H}}{\partial x_{\nu}} \vec{H}_{\lambda} - \vec{j} \cdot \frac{\partial \vec{H}}{\partial x_{\nu}} \vec{H}_{\lambda} + \ldots \]
However

\[ u_k \frac{H_j}{\partial x_j} \frac{\partial H_k}{\partial x_j} = \frac{\partial}{\partial x_j} \left( u_k \frac{H_j}{\partial x_j} \right) - u_k \frac{H_j}{\partial x_j} \frac{\partial H_k}{\partial x_j} - H_j \frac{\partial u_k}{\partial x_j} \]

\[ = R_1 - o - c \]

Now \[ \frac{\partial H_j}{\partial x_j} = 0 \] since \[ \nabla \cdot H = 0 \]

\[ - u_j H_k \frac{\partial H_k}{\partial x_j} = \frac{\partial}{\partial x_j} \left( u_j \frac{H_k}{\partial x_j} \right) + H_k \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial H_k}{\partial x_j} \]

\[ = R_2 + c_2 + c_3 \]

Consequently

\[ \frac{1}{2} \frac{\partial^2 r}{\partial t^2} - c_3 + c_1 - c_2 = c_1 - c_2 - c_3 \]

\[ \frac{1}{2} \frac{\partial^2 u_j}{\partial t^2} = \ldots + R_1 + R_2 - c_1 + c_2 + c_3 \]

Fourier resolution of these equations gives conversion terms like

\[ u_j \frac{H_k}{\partial x_j} \rightarrow \sum_{\ell = -\infty}^{\infty} \left( u_j(\ell) \frac{H_k(\ell - \eta)}{\partial x_j} + u_j(-\ell) \frac{H_k(\ell - \eta)}{\partial x_j} \right) \]

\[ e \& c. \]

and we arrive at a diagram like the one in section V., or the component form, as in section X., with the boxes suppressed.

The conversion terms in the above forms resemble the type of term responsible for the transfer of kinetic energy into the mean zonal motions of the earth's atmosphere (cf., Starr, 1953). They are of a form: stress x gradient.
BIBLIOGRAPHY


Starr, V. P., 1953: Note concerning the nature of the large-scale eddies in the atmosphere, Tellus, 5, 494-498.


