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A Useful Algebraic Property of Robinson's Unification Algorithm

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This memo presupposes some acquaintance with "A Machine Oriented Logic Based on the Resolution Principle", J.A. Robinson, JACM Jan.'65. The reader unfamiliar with this paper should be able to get a general idea of the theorem if he knows that σ_A is a post operator indicating a minimal set of substitutions (most general substitution) necessary to transform all elements of the set of formulae, A, into the same element (to "unify" A), so that when σ_A exists A σ_A is a set with one element (a "unit").

Example:

$$A = \{f(x), y, f(g(u)), f(g(z))\}$$

$$\sigma_A = \{g(u)/x, f(g(u))/y, u/z\}$$

$$A\sigma_A = \{f(g(u))\}$$

Another most general unifier of A is $\{g(z)/x, f(g(z))/y, z/u\}$.

Theorem:

$\sigma_{B\sigma_A}$ exists iff $\sigma_{A\sigma_B}$ exists. Both $\sigma_A\,\sigma_{B\sigma_A}$ and $\sigma_B\,\sigma_{A\sigma_B}$ unify the set A and at the same time unify the set B.

Proof:

ρ is a most general simultaneous unifier of A and B if $A\rho$ is a unit, $B\rho$ is a unit, and if $A\theta$ and $B\theta$ are units then there is a substitution μ such that $\theta = \rho\,\mu$.

By the Unification Theorem if θ simultaneously unifies A and B it can be written as $\sigma_A\lambda$ for some λ because it unifies A. Since θ also unifies B, $B\theta = B\,\sigma_A\lambda$ will be a unit, so that λ must unify $B\sigma_A$. $\sigma_{B\sigma_A}$ is a most general unifier of $B\sigma_A$ so there is some μ such that any unifier of $B\sigma_A$ may be written $\sigma_{B\sigma_A}\,\mu$. Therefore θ may be written as $\sigma_A\sigma_{B\sigma_A}\,\mu$ and $\sigma_A\sigma_{B\sigma_A}$ is a most general simultaneous unifier of A and B. It follows then that there exists a simultaneous unifier of A and B iff $\sigma_A\sigma_{B\sigma_A}$ exists iff σ_A and $\sigma_{B\sigma_A}$ exist. By symmetrical reasoning $\sigma_B\sigma_{A\sigma_B}$ is also a most general simultaneous unifier of A and B, and since σ_{BY} exists implies σ_B exists, $\sigma_A\sigma_B$ exists iff $\sigma_{B\sigma_A}$ exists.

Example:

$A = \{x, f(a,y), f(u,y)\}$

$B = \{y, g(u)\}$

$\sigma_A = \{f(a,y)/x, a/u\}$

$\sigma_B = \{g(u)/y\}$

$B \bar{} _B = \{x, f(a,g(u)), f(u,g(u))\}$

$B\sigma_A = \{y, g(a)\}$

$\sigma_A\sigma_B = \{f(a,g(a))/x, a/u\}$

$\sigma_B\sigma_A = \{g(a)/y\}$

$\sigma_A\sigma_B\sigma_A = \sigma_B\sigma_A\sigma_B = \{g(a)/y, f(a,g(a))/x, a/u\}$

Example:

$A = \{f(x,y), f(a,b)\}$

$B = \{x, y\}$

$\sigma_A = \{a/x, b/y\}$

$\sigma_B = \{y/x\}$

$A\sigma_B = \{f(y,y), f(a,b)\}$

$B\sigma_A = \{a, b\}$

Neither $\sigma_A\sigma_B$ nor $\sigma_B\sigma_A$ exists.

Application

This theorem shows that certain proofs in Robinson's resolution system are equivalent. Thus improvements can be made to search procedures which test equivalent proofs.

Let $A = \{A_1, A_2\}$ and $B = \{B_1, B_2\}$ such that σ_A and σ_B exist, and let $\{A_1\}$, $\{\sim A_2\ B_1\}$, and $\{\sim B_2\}$ be clauses (sets of literals) with no variables common between the sets so that $\{A_1\}\ \sigma_B = \{A_1\}$ and $\{B_2\}\ \sigma_A = \{B_2\}$. (This is the result of a preliminary step in resolution.)

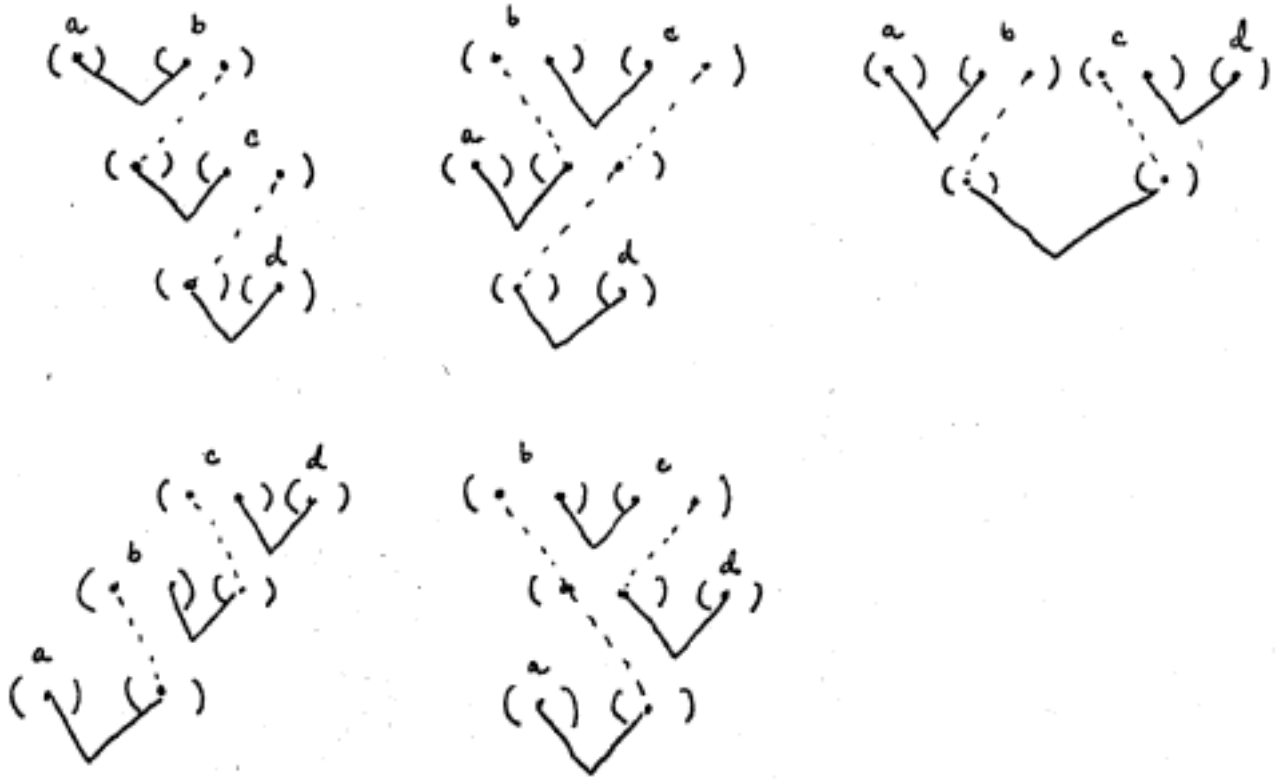
Then the proof (where $\vee$ indicates resolution)

$$\begin{array}{c} \{A_1\} \qquad \{\sim A_2, B_1\} \\ \{B_1\}\sigma_A \qquad \{\sim B_2\}\sigma_A \end{array}$$

exists, i.e. $\sigma_B\sigma_A$ exists, if and only if the proof

$$\begin{array}{c} \{\sim A_2, B_1\} \qquad \{\sim B_2\} \\ \{A_1\}\sigma_B \qquad \{\sim A_2\}\sigma_B \end{array}$$

exists, since this proof exists only if $\sigma_A\sigma_B$ exists.

By an extension of this reasoning it can be seen that any such rearrangement of a proof tree corresponding to a rearrangement of the order of unification leads to an equivalent proof. In particular there are $\frac{(2m-2)!}{m!\,(m-1)!}$ (asymptotically 4^m)

ways of developing a proof without factoring involving m-2 2-literal, terminal clauses, and 2 1-literal, terminal clauses. e.g. for m=4 there are 5 equivalent structures: