Essays on Uncertainty in Economics

by

Alp Simsek

B.S., Massachusetts Institute of Technology (2004)
M.Eng., Massachusetts Institute of Technology (2005)

Submitted to the Department of Economics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Economics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2010

© Alp Simsek. All Rights Reserved.

The author hereby grants to Massachusetts Institute of Technology permission to
reproduce and
to distribute copies of this thesis document in whole or in part.

Signature of Author .......................................................... Department of Economics

14 May 2010

Certified by .......................................................... Daron Acemoglu
Charles Kindleberger Professor of Applied Economics
Thesis Supervisor

Certified by .......................................................... Ricardo Caballero
Ford International Professor of Economics
Thesis Supervisor

Accepted by .......................................................... Esther Duflo
Abdul Latif Jameel Professor of Poverty Alleviation and Development Economics
Chairman, Department Committee on Graduate Students
Abstract

This thesis consists of four essays about "uncertainty" and how markets deal with it.

Uncertainty is about subjective beliefs, and thus it often comes with heterogeneous beliefs that may be present temporarily or even forever. The first essay analyzes the effect of uncertainty, and the associated belief heterogeneity, on financial contracts and asset prices. I assume that optimists have limited wealth and take on leverage in order to take positions in line with their beliefs. To have a significant effect on asset prices, they need to borrow from traders with moderate beliefs using loans collateralized by the asset itself. Since moderate lenders do not value the collateral as much as optimists do, they are reluctant to lend, which provides an endogenous constraint on optimists' ability to leverage and to influence asset prices. I demonstrate that optimism is asymmetrically disciplined by this constraint, in the sense that optimism concerning the likelihood of bad events has no or little effect on asset prices, while optimism concerning the relative likelihood of good events could have significant effects. This result emphasizes that what investors disagree about matters for asset prices, to a greater extent than the level of disagreement.

New financial assets are often associated with much uncertainty and belief heterogeneity, especially because they do not have a long track record. While the traditional view of financial innovation emphasizes the risk sharing role of new assets, belief heterogeneity about these assets naturally leads to speculation, which represents a powerful economic force in the opposite direction. The second essay analyzes the effect of financial innovation on the allocation of risks when both the risk sharing and the speculation forces are present. I demonstrate that speculation on new assets is amplified by the hedge-more/bet-more effect: Traders make bets on new assets which they then hedge by taking complementary positions on existing assets, which in turn enables them to place larger bets and take on greater risks. This effect suggests that, as asset markets get more complete, they become more susceptible to speculation and further financial innovation is more likely to be destabilizing.

The third essay, joint with Ricardo Caballero, concerns the sources of uncertainty. We present a model in which uncertainty suddenly and endogenously rises in response to an increase in the complexity of the economic environment. In our model, banks normally collect information about their trading partners which assures them of the soundness of these relationships. However, when acute financial distress emerges in parts of the financial network, it is not enough to be informed about these partners, as it also becomes important to learn about the health of their trading partners. As conditions continue to deteriorate, banks must
learn about the health of the trading partners of the trading partners of the trading partners, and so on. At some point, the cost of information gathering becomes too unmanageable for banks, uncertainty spikes, and they have no option but to withdraw from loan commitments and illiquid positions. A flight-to-quality ensues, and the financial crisis spreads.

The fourth essay, joint with Daron Acemoglu, analyzes the effect of uncertainty of a special kind, that involves economic agents’ private actions and anonymous market transactions, on the functioning and efficiency of competitive markets. Despite a sizeable literature, how competitive markets deal with this type of uncertainty remains unclear. A “folk theorem” originating, among others, in the work of Stiglitz maintains that competitive equilibria are always or “generically” inefficient (unless contracts directly specify consumption levels as in Prescott and Townsend, thus bypassing trading in anonymous markets). This essay critically reevaluates these claims in the context of a general equilibrium economy with moral hazard. Our results delineate a range of benchmark situations in which equilibria have very strong optimality properties. They also suggest that considerable care is necessary in invoking the folk theorem about the inefficiency of competitive equilibria with private information.

Thesis Supervisor: Daron Acemoglu
Title: Charles Kindleberger Professor of Applied Economics

Thesis Supervisor: Ricardo Caballero
Title: Ford International Professor of Economics
Contents

1 Introduction 8

2 When Optimists Need Credit: Asymmetric Disciplining of Optimism and Implications for Asset Prices 13
  2.1 Introduction ........................................... 13
  2.2 Environment and Equilibrium ........................................... 22
    2.2.1 Financial Frictions and Collateral Equilibrium ...................... 24
  2.3 Characterization of Collateral Equilibrium ......................... 26
    2.3.1 Main Result: Asymmetric Disciplining of Optimism ................. 27
    2.3.2 Asset Market Clearing and Collateral Equilibrium ................. 33
  2.4 Comparative Statics with Respect to Belief Heterogeneity ........... 35
  2.5 Collateral Equilibrium with Contingent Contracts .................. 41
    2.5.1 Definition of Equilibrium with Contingent Contracts ............. 42
    2.5.2 Asymmetric Disciplining with Contingent Contracts ............... 42
    2.5.3 Equilibrium Asset Price with Contingent Contracts ............... 46
  2.6 Collateral Equilibrium with Short Selling ........................... 46
    2.6.1 Matching of Optimists and Moderates in Debt and Short Markets ... 48
    2.6.2 Asymmetric Disciplining with Short Selling ....................... 49
    2.6.3 Equilibrium Asset Price with Short Selling ....................... 51
  2.7 Dynamic Model: Financing Speculative Bubbles ...................... 53
    2.7.1 Basic Dynamic Environment .................................. 53
    2.7.2 Speculative Bubbles without Financial Constraints .............. 55
2.7.3 Financial Frictions and Dynamic Collateral Equilibrium .......... 57
2.8 Conclusion ............................................................................ 62
2.A Appendices ........................................................................... 64
  2.A.1 Properties of Optimism Order ........................................... 64
  2.A.2 Characterization of Quasi-equilibrium ................................ 69
  2.A.3 Characterization of Collateral Equilibrium .......................... 73
  2.A.4 Comparative Statics with Respect to Belief Heterogeneity ....... 79
  2.A.5 Collateral Equilibrium with Contingent Contracts ............... 83
  2.A.6 Collateral Equilibrium with Short Selling ............................ 85
  2.A.7 Characterization of Dynamic Equilibrium ............................ 88

3 Speculation and Risk Sharing with New Financial Assets .......... 94
  3.1 Introduction ........................................................................... 94
  3.2 Environment and Equilibrium .............................................. 100
    3.2.1 Financial Innovation ..................................................... 102
    3.2.2 Definition of Equilibrium ............................................. 102
    3.2.3 Characterization of Equilibrium ..................................... 103
  3.3 Main Result: Effect of Financial Innovation on Consumption Variance ........................................................................ 104
  3.4 The Hedge-More/Bet-More Effect ....................................... 108
    3.4.1 Sketch Proof for the Main Result .................................. 111
  3.5 Comparative Statics of Consumption Variance ..................... 113
  3.6 Dynamic Environment and Equilibrium ............................... 115
    3.6.1 Dynamic Path of Consumption Variance ....................... 117
    3.6.2 Timing the Introduction of New Assets ......................... 119
  3.7 Endogenous Financial Innovation ....................................... 122
  3.8 Conclusion ........................................................................... 125
  3.A Appendix: Omitted Proofs ................................................... 127

4 Complexity and Financial Panics ........................................ 139
  4.1 Introduction ........................................................................... 139
  4.2 The Environment and a Free-Information Benchmark .......... 142
4.2.1 The Environment ................................................. 142
4.2.2 Free-Information Benchmark ................................. 149
4.3 Endogenous Complexity and the Credit Crunch .............. 154
4.4 The Collapse of the Financial System ......................... 161
4.5 Conclusion ......................................................... 166
4.A Appendices ......................................................... 167
4.A.1 The Normal Environment ....................................... 167
4.A.2 Omitted Proofs .................................................. 171

5 Moral Hazard and Efficiency in General Equilibrium with Anonymous Trading ............................... 176
5.1 Introduction ......................................................... 176
5.2 Environment and Equilibrium ................................. 182
  5.2.1 Preferences .................................................... 182
  5.2.2 Firms and Employment Contracts ........................... 184
  5.2.3 Worker's Contract Choice .................................. 185
  5.2.4 Firm's Problem ................................................. 186
  5.2.5 Definition of Equilibrium .................................... 187
  5.2.6 Firm Competition and the Indirect Problem ............. 187
  5.2.7 Existence of Equilibrium .................................... 189
5.3 Generic Inefficiency of Equilibrium ......................... 191
5.4 Efficiency of Equilibrium under Weak Separability .......... 196
5.5 Equilibrium and Efficiency with Stochastic Contracts ... 200
5.6 Approximate Efficiency of Equilibrium ....................... 203
5.7 Conclusion .......................................................... 207
5.A Appendices ........................................................ 209
  5.A.1 Omitted Results. .............................................. 209
  5.A.2 Omitted Proofs ................................................ 213
Acknowledgements

First and foremost, I am deeply indebted to my advisor, Daron Acemoglu, for my personal and professional development. Five and a half years ago, he has encouraged me to apply for the MIT economics Ph.D. program. Since then, he has provided the best advising I can imagine and he has been a superb role model. Throughout my graduate career, I have constantly learned from him as a student, as a research assistant, as a teaching assistant, and as a coauthor. This thesis has benefited enormously from his guidance and detailed comments.

I would like to also thank my advisors, Ricardo Caballero and Muhamet Yildiz, for generously sharing their time and advice throughout my graduate years. I have greatly benefited from coauthoring with them, and they have been influential role models. I owe special thanks to Muhamet for carefully reading through the first two essays of this thesis and providing detailed comments.

I would like to thank Marios Angeletos, Abhijit Banerjee, Aytek Erdil, Mike Golosov, Bengt Holmstrom, Guido Lorenzoni, Asu Ozdaglar and Ivan Werning for their guidance and support during my graduate career. Marios, Abhijit, Guido, Mike and Ivan have taken the time to listen to my vague ideas and they have helped me to sharpen them. Aytek has provided detailed comments on the first essay in this thesis. Bengt has provided excellent advice on the presentation of the essays in this thesis. Asu is my first academic mentor, and her approach to mathematical writing is ingrained in this thesis.

I have also benefited greatly from exchanging ideas with my fellow MIT students. I cannot list all with whom I had fruitful discussions, but I would like to mention Sergi Basco, Dan Cao, Fernando Duarte, Laura Feiveson, Pablo Kurlat, Jen-Jen La’o, Jean-Paul L’Huillier, Michael Peters, Mike Powell, Florian Scheuer, and Paul Schrimpf.

I would like to thank all my friends who made my graduate years enjoyable and memorable. I especially thank Gokhan Demirkan, Baris Nakiboglu, Jehanzeb Noor, and my roommate, Burak Alver, without whom the Ph.D. experience would have been quite less enjoyable.

Last but not least, I would like to thank my family for their unending love and support. I dedicate this thesis to my father, who has encouraged me to stay in Academia when I had my doubts, and to my mother, whose unconditional love has helped me smooth the ups and downs of my graduate years. I also thank my brothers, Aykut and Altug, for adding color to my life.
Chapter 1

Introduction

This thesis is a collection of four essays about "uncertainty" and how markets deal with it. Uncertainty is about subjective beliefs, and thus it often comes with heterogeneous beliefs that may be present temporarily or even forever (if we allow people to agree to disagree). The first two essays analyze the effect of uncertainty and belief heterogeneity on asset prices, financial contracts, and financial innovation. The third essay concerns the sources of uncertainty. It provides a model in which uncertainty suddenly and endogenously rises in response to an increase in the complexity of the economic environment. The fourth essay concerns the functioning and efficiency of competitive markets in economic environments with uncertainty of a special kind, that involves economic agents' private actions and anonymous market transactions.

The first essay, presented in Chapter 3, concerns the effect of uncertainty on financial contracts and asset prices. In the run-up to the recent crisis, there was considerable uncertainty, which came with associated belief heterogeneity, surrounding the housing market and the market for complex securities backed by mortgages. A number of scholars, most notably, Shiller (2005), Reinhart and Rogoff (2008) and Geanakoplos (2009), along with many other commentators, have identified belief heterogeneity and optimism as a potential cause for the increase in housing and complex security prices before the crisis. The first essay theoretically evaluates this hypothesis.

Similar to Geanakoplos (2009), I assume that optimists have limited wealth and take on leverage in order to take positions in line with their beliefs. To have a significant effect on asset
prices, they need to borrow from traders with moderate beliefs using loans collateralized by the asset itself. Since moderate lenders do not value the collateral as much as optimists do, they are reluctant to lend, providing an endogenous constraint on optimists’ ability to leverage and to influence asset prices. I demonstrate that optimism concerning the likelihood of bad events has no or little effect on asset prices because these types of optimism are disciplined by this constraint. Instead, optimism concerning the relative likelihood of good events could have significant effects on asset prices. This asymmetric disciplining of optimism is robust to allowing for state contingent loans and short selling of the asset. These results emphasize that what investors disagree about matters for asset prices, to a greater extent than the level of disagreement.

I then use a dynamic extension to show how the asymmetric disciplining result interacts with the speculative component of asset prices identified in Harrison and Kreps (1978). When optimists have limited wealth, belief heterogeneity can lead to speculative asset price “bubbles” but only if it concerns the relative likelihood of good events. The asymmetric disciplining result shows that the size of the bubble depends on the type of belief heterogeneity, and that bubbles can come to an end because of a shift in belief heterogeneity towards the likelihood of bad events.

The second essay, presented in Chapter 3, concerns the effect of uncertainty on financial innovation. New financial assets, such as the above mentioned complex securities, are often associated with much uncertainty, especially because they do not have a long track record. Belief heterogeneity comes as a natural by-product of this uncertainty and changes the implications of risk taking in such markets. While the traditional view of financial innovation emphasizes the risk sharing role of new assets, belief heterogeneity about these assets naturally leads to speculation, which represents a powerful economic force in the opposite direction. The second essay analyzes the effect of financial innovation on the allocation of risks when both the risk sharing and the speculation forces are present.

I consider this question in a standard risk-sharing model in which traders have CARA preferences, and endowments and asset payoffs are normally distributed. Financial innovation is modeled as the introduction of new financial assets into this economy. New assets provide hedging services but they are also subject to speculation as traders have heterogeneous priors.
about their payoffs. I define the average consumption variance as a measure of risk for this economy and I decompose it into two components: the *uninsurable variance*, defined as the variance that would obtain if there were no belief heterogeneity, and the *speculative variance*, defined as the residual amount of variance that results from speculative trades based on belief heterogeneity. Financial innovation always decreases the uninsurable variance because new assets increase the possibilities for risk sharing. My main result shows that financial innovation also always increases the speculative variance, regardless of the number or the type of existing assets. The intuition for this result is related to a powerful economic force that amplifies speculation, the *hedge-more/bet-more* effect: Traders make bets on new assets which they then hedge by taking complementary positions on existing assets, which in turn enables them to place larger bets and take on greater risks. I use the hedge-more/bet-more effect to show that new assets lead to a greater increase in speculative variance when they are introduced into a more complete asset market, and when they are correlated with existing assets. These results suggest that, as asset markets get more complete, they become more susceptible to speculation and further financial innovation is more likely to be destabilizing.

The third essay, which is coauthored with Ricardo Caballero and presented in Chapter 4, concerns the sources of uncertainty. Uncertainty is a natural by-product of complexity. When an economic environment becomes too complex, agents do not fully understand the consequences of their actions, which increases their perceived uncertainty. An example is offered by the recent financial crisis, at the core of which is the dramatic rise in investors’ and banks’ uncertainty. All of a sudden, a financial world that was once rife with profit opportunities for financial institutions (banks, for short), was perceived to be exceedingly complex. Although the subprime shock was small relative to the financial institutions’ capital, banks acted as if most of their counterparties were severely exposed to the shock. Confusion and uncertainty followed, triggering the worst case of flight-to-quality that we have seen in the U.S. since the Great Depression. The third essay provides a model of this phenomenon.

In our model, banks normally collect information about their trading partners which assures them of the soundness of these relationships. However, when acute financial distress emerges in parts of the financial network, it is not enough to be informed about these partners, as it also becomes important to learn about the health of their trading partners. As conditions continue
to deteriorate, banks must learn about the health of the trading partners of the trading partners of the trading partners, and so on. At some point, the cost of information gathering becomes too unmanageable for banks, uncertainty spikes, and they have no option but to withdraw from loan commitments and illiquid positions. A flight-to-quality ensues, and the financial crisis spreads.

The fourth essay, which is coauthored with Daron Acemoglu and presented in Chapter 5, concerns the effect of uncertainty of a special kind, that involves economic agents’ private actions and anonymous market transactions, on the functioning and efficiency of competitive markets. Despite a sizeable literature, how competitive markets deal with this type of uncertainty remains unclear. A “folk theorem,” originating in the work of Stiglitz and coauthors, most notably Greenwald and Stiglitz (1986), and also Arnott and Stiglitz (1988, 1991), maintains that competitive equilibria are always or “generically” inefficient (unless contracts directly specify consumption levels as in the work by Prescott and Townsend (1984), thus bypassing trading in anonymous markets). The fourth essay reevaluates these claims in the context of a general equilibrium economy with moral hazard.

We first formalize the folk theorem. Firms offer contracts to workers who choose an effort level that is private information and that affects worker productivity. To clarify the importance of trading in anonymous markets, we introduce a monitoring partition such that employment contracts can specify expenditures over subsets in the partition, but cannot regulate how this expenditure is subdivided among the commodities within a subset. We say that preferences are nonseparable (or more accurately, not weakly separable) when the marginal rate of substitution across commodities within a subset in the partition depends on the effort level, and that preferences are weakly separable when there exists no such subset. We prove that the equilibrium is always inefficient when a competitive equilibrium allocation involves less than full insurance and preferences are nonseparable. This result appears to support the conclusion of the above-mentioned folk theorem. Nevertheless, our main result highlights its limitations. Most common-used preference structures do not satisfy the nonseparability condition. We show that when preferences are weakly separable, competitive equilibria with moral hazard are constrained optimal, in the sense that a social planner who can monitor all consumption levels cannot improve over competitive allocations. Moreover, we establish ε-optimality when there
are only small deviations from weak separability. These results suggest that considerable care is necessary in invoking the folk theorem about the inefficiency of competitive equilibria with private information.
Chapter 2

When Optimists Need Credit: Asymmetric Disciplining of Optimism and Implications for Asset Prices

2.1 Introduction

Belief heterogeneity and optimism have been suggested as contributing factors to the recent financial crisis. Shiller (2005), Reinhart and Rogoff (2008) and Gorton (2008), along with many other commentators, have identified the optimism of a fraction of investors as a potential cause for the increase in prices in the housing and the complex security markets in the run-up to the crisis. As noted by Geanakoplos (2009), for the optimism of a fraction of investors to have a significant effect on asset prices, they need to leverage their investments by borrowing from less optimistic investors- from moderate lenders. Most borrowing in financial markets is collateralized, and optimists often use the asset itself as collateral (e.g., mortgages, REPOs, or asset purchases on margin). This represents a puzzle because moderate lenders do not value the collateral (the asset) as much as optimists do, which might make them reluctant to lend. Put differently, belief heterogeneity implies an endogenous constraint on optimists’ ability to
leverage and to influence asset prices.

The purpose of this essay is to understand the implications of this constraint for asset prices. I construct an equilibrium model in the asset and the loan market, and I show that certain types of optimism, specifically those concerning the likelihood of bad events, have no or little effect on asset prices because they are disciplined by the endogenous financial constraints. Instead, optimism concerning the relative likelihood of good events could have significant effects on asset prices, because these types of optimism are unchecked by these constraints.

To illustrate the effect of different types of optimism, consider a simple example in which a single risky asset is traded. There are three future states, good, normal and bad, in which the asset price will respectively be high, average and low. Moderate lenders assign an equal probability, 1/3, to each state, while optimists have a greater expected valuation of the asset. Optimists borrow from moderate lenders using loans collateralized by the asset. More specifically, the asset and collateralized loans are traded in a competitive market (as in Geanakoplos, 2009), and loans are no-recourse in the sense that payment is only enforced by the collateral pledged for the loan. Loans of different sizes (per unit collateral pledged) are available for trade, and the loan to value ratio is endogenously determined in equilibrium. For the baseline setting, suppose that loans are non-contingent, that is, they promise the same payment in all future states, and that the asset cannot be short sold. These assumptions arguably provide a good starting point, because collateralized loans (e.g., mortgages, REPOs) typically do not have many contingencies in their payoffs; and short selling of many assets other than stocks (and some stocks) is difficult and costly.¹

In this setting, there are two different ways in which optimists can be optimistic about the asset. For the first case, suppose optimists assign a probability less than 1/3 to the bad state, and equal probabilities to the normal and the good states. That is, optimists are optimistic because they think bad events are unlikely. For the second case, suppose optimists agree about the probability, 1/3, of the bad state, but they think the good state is more likely than the normal state. Moreover, construct the two cases such that optimists have the same valuation

of the asset, so that the level of the optimism is the same but the type of the optimism is different.\(^2\)

My main results, Theorem 1 and Theorem 3, show that the asset price in the first case of this example is always lower than in the second case (and strictly so for the appropriate range of parameters). In other words, optimism is *asymmetrically disciplined* by endogenous financial constraints: optimism concerning the probability of bad states is disciplined more than optimism concerning the relative likelihood of good states.

More generally, Theorem 1 considers the above setting with a continuum of states (rather than three), and shows that the asset is priced according to a mixture of moderate and optimistic beliefs: the moderate beliefs are used to assess the likelihood of default states, while the optimistic beliefs are used to assess the conditional likelihood of non-default states. More precisely, the asset price can be written as:

\[
p = \frac{1}{1+r} \left( \Pr_{\text{moderate}} [v < \bar{v}] E_{\text{moderate}} [v | v < \bar{v}] + \Pr_{\text{moderate}} [v \geq \bar{v}] E_{\text{optimistic}} [v | v \geq \bar{v}] \right),
\]

(2.1)

where \( r \) is the interest rate on a benchmark asset, the random variable \( v \) captures the future value of the asset, and \( \bar{v} \) is the endogenously determined default threshold value, that is, collateralized loans in this economy default when the asset value \( v \) falls below \( \bar{v} \). The notation \( \Pr_{\text{moderate}} [v < \bar{v}] \) captures the probability of the event \( \{v < \bar{v}\} \) with respect to the moderate beliefs, and \( E_{\text{optimistic}} [v | v \geq \bar{v}] \) captures the expected value of the asset conditional on the event \( \{v \geq \bar{v}\} \) with respect to the optimistic beliefs.

The expression in (2.1) further illustrates that optimism is asymmetrically disciplined. This asymmetric disciplining result is robust to allowing for more general collateralized loans and short selling. In particular, Sections 2.5 and 2.6 of this essay show that the asset price in these more general settings can also be represented with an expression similar to (2.1). While the details of the expressions depend on the type of the contracts available for trade, it remains true that optimism about bad states is disciplined more than optimism concerning the relative

\(^2\)For an example of case one type of optimism, consider the last quarter of 2008, when a main dimension of disagreement was whether the upcoming recession would be a depression or a garden variety recession. For an example of case two type of optimism, consider the Internet technology and the tech stocks in 1990s, when a main dimension of disagreement was how profitable the Internet technology would be.
likelihood of good states.

The intuition for the asymmetric disciplining result is related to the asymmetry in the shape of the debt contract payoffs. These contracts make the same full payment in non-default states, but they make losses in default states. Consequently, any disagreement about the probability of default states translates into a disagreement about how to value the debt contracts, which in turn tightens optimists' financial constraints. In contrast, disagreements about the relative likelihood of non-default states do not tighten the financial constraints.

More specifically, in the above example (for the relevant range of parameters) collateralized loans that are traded in equilibrium default in the bad state but not in the normal or the good states. This implies that these loans always trade at an interest rate with a spread over the benchmark rate, which compensates the lenders for expected losses in case of default. Moreover, in a competitive loan market, the spread on a loan is just enough to compensate the lenders for their expected losses according to their moderate beliefs. Nonetheless, in the first case of the example, this spread appears too high to optimists. This is because optimists assign a lower probability to the bad state, and thus they find it more likely that they will pay the spread. Therefore, optimists believe they will pay a higher expected interest rate than the benchmark rate, which discourages them from borrowing and leveraging their investments. This lowers optimists' demand for the asset and leads to an equilibrium price closer to the moderate valuation. In contrast, in the second case of the example, the spread appears fair to optimists because they agree about the probability of the bad state. This encourages optimists to borrow and leverage their investments, increases their demand for the asset, and leads to an equilibrium price closer to their valuation.

The asymmetric disciplining characterization of asset prices lends itself to a number of comparative statics results regarding the effect of a change in the level and the type of belief heterogeneity. Earlier work by Miller (1977) has suggested a link between the level of belief heterogeneity and asset prices. According to this mechanism, belief heterogeneity and limited short selling leads to an overvaluation of the asset (relative to the average valuation of the population) because the asset is held by the most optimistic investors. This mechanism has been recently emphasized and empirically tested by a growing literature in finance, e.g., Chen,
Hong and Stein (2002), Diether, Malloy and Scherbina (2002) and Ofek and Richardson (2003). In contrast to this literature, the level of belief heterogeneity in this model has ambiguous effects on the asset price. This is because, while an increase in optimists' optimism tends to increase the price, an increase in moderate lenders' pessimism tends to decrease the price through the tightening of financial constraints. This observation suggests that the Miller mechanism may not apply in markets in which optimists finance their asset purchases by borrowing from less optimistic investors.

In contrast, this model suggests that the type of the belief heterogeneity may be a more robust determinant of asset prices in these markets. To capture the effect of different types of belief heterogeneity, I formally define a notion of right-skewed (resp. left-skewed) optimism as a single-crossing condition on the hazard rates of optimists' belief distributions. In the above described example with three states, the optimism in the second case is more right-skewed than the optimism in the first case. This is because, in the second case, the optimistic hazard rate at the bad state is higher (since optimists are not optimistic about the probability of the bad state), but the hazard rate at the normal state is lower (since optimists are optimistic about the relative likelihood of the good and the normal states). Theorem 3 shows that an increase in this type of right-skewness of belief heterogeneity unambiguously increases the asset price, because a given level of optimism is disciplined less by financial constraints when it is more right-skewed. In addition, Theorem 4 shows that the level of belief heterogeneity also has an unambiguous effect on the asset price if the type of the heterogeneity is also accounted for. In particular, in response to an increase in belief heterogeneity, the asset price increases if the additional heterogeneity concerns the relative likelihood of non-default states, while it decreases if the additional heterogeneity concerns the probability of default states. These results suggest that what investors disagree about matters for asset prices, to a greater extent than the level of their disagreement.

While the baseline model with non-contingent contracts and no short selling is a good starting point, it is important to verify the robustness of the asymmetric disciplining characterization to more general settings, especially because allowing for a richer set of contracts introduces new economic forces. Theorem 5 shows that a version of the asymmetric disciplining result continues to apply in the setting in which debt contracts can be fully contingent. The optimal contingent
contract (collateralized by one unit of the asset) is such that optimistic borrowers give up the asset completely if the state realization is below a threshold level, while paying nothing if the state is above the threshold. While this threshold contract is different than a non-contingent contract, it has the same feature of making a fixed payment (namely, zero) for all relatively good states. Consequently, optimism about the relative likelihood of good states does not lead to heterogeneity in the valuation of the optimal contingent contract. It follows that these types of optimism do not tighten optimists’ financial constraints, and thus they lead to a higher asset price. In contrast, optimism about the relative likelihood of states below the threshold level tightens optimists’ financial constraints and leads to a lower asset price.

The setting with contingent contracts reveals a surprising result: the equilibrium asset price can exceed the valuation of even the most optimistic investor.\(^3\) Intuitively, the ability to fine-tune their borrowing enables optimists to take loans which they perceive to be even more favorable than borrowing at the benchmark interest rate. Optimists concentrate all of their payments at the bad states (which they find the least likely), and thus they expect to make a relatively small payment. Consequently, optimists continue to demand the asset when the price exceeds their valuation (which is calculated according to the benchmark rate), because they finance some of the purchase with contingent contracts which they perceive to be very attractive. This result creates a presumption that finer levels of financial engineering of loans can potentially have a large impact on asset prices.

Another natural question is whether the asymmetric disciplining result generalizes to the class of assets that can be short sold (e.g., the majority of stocks). While short selling reduces the general overvaluation of the asset, Theorem 6 shows that a version of asymmetric disciplining applies also in this case. The critical observation is that moderate investors that wish to short sell the asset face an endogenous borrowing constraint similar to the one faced by optimists. In particular, to short sell the asset, moderates need to borrow the asset from optimists. Moreover, since borrowing in this economy is collateralized, they need to use the riskless bond that they hold as collateral in their short contracts. But optimists value the bond (the collateral) relatively less, which might make them reluctant to lend the asset. Hence, belief heterogeneity represents

\(^3\) As discussed below, it is well known that the asset price can exceed the optimistic valuation in a dynamic setting with belief heterogeneity (through a different mechanism). However, when contingent contracts are available, the asset price can exceed the optimistic valuation even in the static setting.
an endogenous constraint on moderates' ability to short sell.

The severity of this constraint depends on the type of belief heterogeneity. If the future state is above a threshold level, then the value of the asset is greater than the value of the posted collateral, and moderates default on the short contract. Hence, a short contract effectively promises the same payment in all states above a threshold state. Consequently, if the belief heterogeneity is about the relative likelihood of states above the threshold, then moderates cannot bet on their pessimism by selling the short contract. Put differently, to bet on these types of pessimism, moderates need to choose a short contract with a higher level of collateral (that has a higher default threshold). Hence, these types of short sales are more difficult to leverage, which leads to an asset price closer to the optimistic valuation. In contrast, if the belief heterogeneity is about the probability of bad states, then moderates are able to make leveraged bets on their pessimism, which leads to an asset price closer to the moderate valuation.

While the results described so far concern a static setting, the asymmetric disciplining mechanism naturally interacts with the speculative component of asset prices identified in Harrison and Kreps (1978). I consider a dynamic extension of the baseline model to analyze this interaction. In a dynamic economy in which the identity of optimists changes over time, a speculative phenomenon obtains as the current optimists purchase the asset not only because they believe it will yield greater dividend returns, but also because they expect to make capital gains by selling the asset to future optimists. The asset price exceeds the present discounted valuation of the asset with respect to the beliefs of any trader because of the resale option value introduced by the speculative trading motive. As Scheinkman and Xiong (2003) note, this resale option value may be reasonably called a "speculative bubble." This setup is the starting point of the dynamic extension, which introduces the additional element of optimists' financial constraints. The dynamic model reveals that, when optimists need to purchase the asset by borrowing from moderate lenders, belief heterogeneity can lead to speculative asset price bubbles, but only if it concerns the relative likelihood of non-default states. When this is the case, however, the resale option value can increase the size of the speculative component of the asset price considerably because large positions can be financed by credit collateralized by the speculative asset. This is because moderate lenders' valuation, as well as optimists' valuation, features a speculative component. Put differently, in a speculative episode, moderate
lenders agree to finance optimists' purchase of the asset by extending large loans because they think, should the optimist default on the loan, they can sell the collateral (the asset) to another optimist in the next period. The asymmetric disciplining characterization shows that the size of the bubble depends on the skewness of belief heterogeneity. This result also shows that bubbles can come to an end because of a shift in belief heterogeneity towards the likelihood of default states.

The closest work to this essay is by Geanakoplos (2009), who considers the determination of leverage and asset prices in a model with two continuation states and traders with a continuum of belief types. In contrast, I consider a model with a continuum of continuation states and traders with two belief types (optimists and moderates). My assumptions are relevant for understanding a range of situations, including the effect of different types of belief disagreements on asset prices, leverage, and the default frequency of equilibrium loans. In particular, while Geanakoplos (2009) illustrates that an increase in belief heterogeneity can decrease asset prices considerably, my essay shows that an increase in the level of belief heterogeneity generally has ambiguous effects on asset prices, and identifies the skewness of belief heterogeneity as an important determinant of asset prices. In the model considered by Geanakoplos (2009), the increase in the level of heterogeneity decreases asset prices because the additional heterogeneity is concentrated on default states. An increase in the level of heterogeneity in that model would rather increase asset prices if the additional heterogeneity were concentrated on good states. Moreover, in the two state model analyzed in Geanakoplos (2009), loans that are traded in equilibrium are always fully secured with respect to the worst case scenario, i.e., there is no default. This feature makes it impossible to analyze the effect of belief heterogeneity on the default frequency and riskiness of equilibrium loans, which is one of the topics that I consider. In addition, my essay extends the model in Geanakoplos (2009) by allowing for short selling, and characterizes the effect of belief heterogeneity in this more general setting.⁴

⁴Other related papers that concern the effect of collateral constraints on leverage or asset prices include Hart and Moore (1994), Geanakoplos (1997, 2003), Kiyotaki and Moore (1997), Caballero and Krishnamurty (2001), Gromb and Vayanos (2002), Fostel and Geanakoplos (2008), Brunnermeier and Pedersen (2009), Brunnermeier and Sannikov (2009), Ashcraft, Garleanu, and Pedersen (2010), and He and Xiong (2010). My essay is also related to a large literature that concerns the endogenous determination of leverage. In addition to some of the above papers, an incomplete list includes Townsend (1979), Myers and Majluf (1984), Bernanke and Gertler (1989), Shleifer and Vishny (1992), Holmstrom and Tirole (1997), Bernanke, Gertler, and Gilchrist (1998). On
The relationship of my essay to the literatures initiated by Miller (1977) and Harrison and Kreps (1978) have already been discussed. A related literature concerns the plausibility of the heterogeneous priors assumption in financial markets. The market selection hypothesis, which goes back to Alchian (1950) and Friedman (1953), posits that investors with incorrect beliefs should be driven out of the market as they would consistently lose money. Thus, this hypothesis suggests that investors that remain in the long run should have accurate (and common) beliefs. Recent research has emphasized that the market selection hypothesis does not apply for incomplete markets, that is, traders with inaccurate beliefs may have a permanent presence when asset markets are incomplete. Of particular interest for my essay is the work by Cao (2009), who considers a similar economy in which markets are endogenously incomplete because of collateral constraints. Cao (2009) shows that belief heterogeneity in this economy remains in the long run, thus providing theoretical support for my central assumptions. Another strand of literature concerns whether investors' Bayesian learning dynamics would eventually lead to accurate, and thus common, beliefs. Recent work (e.g., by Acemoglu, Chernozhukov and Yildiz, 2009) has emphasized the limitations of Bayesian learning in generating long run agreement.

The organization of the rest of this essay is as follows. Section 2.2 introduces the baseline version of the model and defines the collateral equilibrium. Section 2.3 characterizes the collateral equilibrium and presents the asymmetric disciplining result. Section 2.4 establishes the comparative statics of the collateral equilibrium with respect to the type and the level of belief heterogeneity. Sections 2.5 and 2.6 generalize the asymmetric disciplining result to settings with respectively contingent debt contracts and short selling of the asset. Section 2.7 introduces the dynamic extension and presents the results for speculative bubbles. Section 2.8 concludes. The essay ends with several appendices that present the proofs omitted from the main text.

---


2.2 Environment and Equilibrium

Consider a two period economy with a single consumption good in which a continuum of risk neutral traders have endowments in period 0 but they need to consume in period 1. The resources can be transferred between periods by investing either in a risk-free bond, denoted by $B$, or a risky asset, denoted by $A$. Bond $B$ is supplied elastically at a normalized price 1 in period 0. Each unit of the bond yields $1 + r$ units of the consumption good in period 1. Asset $A$ is in fixed supply, which is normalized to 1. The asset pays dividend only once (in units of the consumption good), and it pays it in period 1. The dividend payment of each unit of the asset is denoted by $v(s)$. Taking the set of all possible states as $S = [s_{\text{min}}, s_{\text{max}}] \subset \mathbb{R}$, I assume that the function $v : S \to \mathbb{R}_{++}$ is strictly increasing and continuously differentiable.\footnote{Note that the state space could be equivalently defined as $v(S) = [v(s_{\text{min}}), v(s_{\text{max}})]$ over asset payoffs, so the value function $v(\cdot)$ is redundant in this section. Put differently, without loss of generality, the value function can be taken to be the identity function $v(s) = s$. I introduce the value function $v(\cdot)$ because this will considerably simplify the analysis of the dynamic model in Section 2.7, in which the value function will be endogenously determined.} I denote the price of the asset by $p$.

Traders have heterogeneous priors about the return of the asset. In particular, there are two types of traders, optimists and moderates, respectively denoted by subscript $i \in \{1, 0\}$, with corresponding prior belief about the next period state given by the probability distribution $F_i$ over $S$. Traders know each others’ priors, and thus optimists and moderates agree to disagree. I normalize the population measure of each type of traders to 1, and I let $\alpha_i$ (resp. $w_i$) denote type $i$ traders’ period 0 endowment of the asset (resp. the consumption good). The asset endowments satisfy $\alpha_0 > 0$ and $\alpha_0 + \alpha_1 = 1$. An economy is denoted by the tuple $\mathcal{E} = (S; v(\cdot); \{F_i\}_i; \{w_i\}_i; \{\alpha_i\}_i)$.

I adopt the following notion of optimism.

**Definition 1 (Optimism Order).** Consider two probability distributions $H, \tilde{H}$ over $S = [s_{\text{min}}, s_{\text{max}}]$ with density functions $h, \tilde{h}$ that are continuous and positive over $S$. The distribution $\tilde{H}$ is more optimistic than $H$, denoted by $\tilde{H} >_O H$, if $\frac{1 - \tilde{H}(s)}{1 - H(s)}$ is strictly increasing; equivalently, if the following hazard rate inequality is satisfied for all $s \in (s_{\text{min}}, s_{\text{max}})$:

$$\frac{\tilde{h}(s)}{1 - \tilde{H}(s)} < \frac{h(s)}{1 - H(s)}.$$  

\footnote{Note that the state space could be equivalently defined as $v(S) = [v(s_{\text{min}}), v(s_{\text{max}})]$ over asset payoffs, so the value function $v(\cdot)$ is redundant in this section. Put differently, without loss of generality, the value function can be taken to be the identity function $v(s) = s$. I introduce the value function $v(\cdot)$ because this will considerably simplify the analysis of the dynamic model in Section 2.7, in which the value function will be endogenously determined.}
The distribution $\bar{H}$ is weakly more optimistic than $H$, denoted by $\bar{H} \succeq_o H$, if (2.2) is satisfied as a weak inequality.

**Assumption (O).** The probability distributions $F_1$ and $F_0$ have density functions $f_1, f_0$ that are continuous and positive over $\mathcal{S}$, and they satisfy $F_1 \succ_o F_0$.

The optimism order, $\succ_o$, concerns optimists' relative probability assessment for the upper-threshold events $[s, s^{\text{max}}] \subset \mathcal{S}$, and it posits that optimists are increasingly optimistic for these events as the threshold level $s$ is increased. It captures the idea that, the "better" the event becomes, the greater the optimism is for the event. This order, also known as the hazard rate order, is related to some well known regularity conditions. It is stronger than the first order stochastic order, that is, $F_1 \succ_o F_0$ implies $F_1$ dominates $F_0$ in the first order stochastic sense. However, it is weaker than the monotone likelihood ratio property, that is, if $\frac{f_1(s)}{f_0(s)}$ is strictly increasing over $\mathcal{S}$, then $F_1 \succ_o F_0$ (cf. Appendix 2.A.1).

Let $E_i[.]$ denote the expectation operator corresponding to the belief of a type $i$ trader. Assumption (O) also implies $E_0[v(s)] < E_1[v(s)]$, that is, moderates value the asset less than optimists do. This further implies that moderates would like to short sell the asset in this economy, which is ruled out by assumption.

**Assumption (S).** Asset $A$ cannot be short sold.

This assumption will be maintained for most of the essay (until Section 2.6). In reality, many assets other than stocks, and also some stocks, are difficult and costly to short sell (see, e.g., Jones and Lamont, 2001).

Given assumption (S), if there were no financial frictions, i.e., if optimists could freely borrow and lend at the going interest rate $1 + r$, they would bid up the price of the asset to the optimistic valuation, $\frac{E_1[v(s)]}{1 + r}$. However, financial frictions may prevent optimists from increasing the asset price to this level. With financial frictions, the asset (in the baseline setting) trades at a price in the interval

$$\left[ \frac{E_0[v(s)]}{1 + r}, \frac{E_1[v(s)]}{1 + r} \right],$$

the exact location being determined by optimists' wealth and the type of the financial frictions.

The financial frictions are microfounded through a collateralized loan market.
2.2.1 Financial Frictions and Collateral Equilibrium

I make a number of institutional assumptions for the loan market. First, I assume that loans in this economy must be secured by collateral owned by the borrower, and the court system enforces the transfer of collateral to the lender in case the borrower does not pay. Second, I assume that loans are non-recourse, that is, the borrower does not get further punishment than potential loss of collateral. Third, I also assume that the loans are non-contingent, that is, they promise the same payment in all future states. These assumptions arguably provide a good starting point, because most REPO loans and some mortgages are non-recourse, and they do not have many contingencies.

Formally, a unit debt contract, denoted by $\varphi \in \mathbb{R}_+$, is a promise of $\varphi$ units of the consumption good in period 1 by the borrower, collateralized by 1 unit of the asset $A$ (which the borrower owns). In period 1, the borrower defaults on the unit debt contract $\varphi$ if and only if the asset's value is less than the promised amount. Thus, each contract $\varphi$ pays

$$\min(v(s), \varphi). \quad (2.3)$$

I analyze the loan market using a competitive equilibrium notion, collateral equilibrium, originally developed by Geanakoplos and Zame (1997, 2009). In particular, each debt contract $\varphi$ is traded in an anonymous market at a competitive price $q(\varphi)$. Note that the anonymity of the market is ensured by collateral: each lender knows that repayment is only secured by collateral,

---

9 There is a potential question of who holds the collateral throughout the term of the loan contract, i.e. should the collateral be locked in a warehouse, held by the lender, or the borrower. In reality (e.g., in mortgages or REPOS), different variants are used intuitively depending on whether the borrower or the lender benefits more from holding the contract during the loan period. A common aspect of all variants of collateralized lending relationships is that the borrower must own the asset at the time of the loan payment. This aspect is necessary because otherwise the borrower would not have any incentive to pay back the loan and collateral would not enforce payment.

In this model, traders receive no utility from holding the collateral in period 0, which implies that the different variants of collateralized lending are essentially equivalent. Therefore, without loss of generality, the borrower is required to own the collateral that she pledges.

10 The assumption that each contract pledges one unit of the asset is a normalization without loss of generality. Note also that, in principle, both the bond $B$ and the asset $A$ could be used as collateral. However, in this model, the assumption that only the asset can be used as collateral is also without loss of generality. More precisely, the equilibrium described below in Theorem 2 continues to be the essentially unique equilibrium in the more general setting in which the bond can also be used as collateral. This is because optimistic borrowers do not hold any bond in equilibrium (except for the corner case in which their wealth is more than sufficient to purchase the entire asset supply).
and that she will get the payment in (2.3) regardless of the identity of the borrower in the transaction.

I refer to a debt contract \( \varphi = v(\bar{s}) \in [v(s^{\text{min}}), v(s^{\text{max}})] \) as a \textit{loan with riskiness} \( \bar{s} \), since this contract defaults if and only if the realized state is below \( \bar{s} \). I refer to the price of the debt contract, \( q(v(\bar{s})) \), as the \textit{loan size}, since this is the amount of that the borrower receives by collateralizing one unit of the asset. In equilibrium, \( q(v(\bar{s})) \) will be increasing in \( \bar{s} \), hence \textit{larger loans} are also \textit{riskier loans}. Moreover, I define the \textit{interest rate} on the loan as the ratio of the promised interest payment to the loan size:

\[
\frac{v(\bar{s}) - q(v(\bar{s}))}{q(v(\bar{s}))}. \tag{2.4}
\]

Given these definitions, an interpretation of the model is that loans of different sizes (and thus different riskiness levels) are being traded in a competitive equilibrium at their corresponding interest rates.

To formalize traders’ portfolio choices, I assume that the price function \( q(\cdot) \) is Borel measurable. Unlike the asset, debt contracts can be short sold (but subject to a collateral constraint). In particular, taking the price function \( q(\cdot) \) as given, type \( i \) traders choose a long debt portfolio \( \mu_i^+ \in M(\mathbb{R}^+) \) and a short debt portfolio \( \mu_i^- \in M(\mathbb{R}^+) \), where \( M(\mathbb{R}^+) \) denotes the set of Borel measures over \( \mathbb{R}^+ \).\(^1\) In addition, type \( i \) traders choose their asset and bond holdings, \( x_i = (x_i^A, x_i^B) \in \mathbb{R}_+^2 \). Their budget constraint is given by:

\[
p x_i^A + x_i^B + \int_{\mathbb{R}^+} q(\varphi) \, d\mu_i^+(\varphi) - \int_{\mathbb{R}^+} q(\varphi) \, d\mu_i^-(\varphi) \leq w_i + p\alpha_i. \tag{2.5}
\]

Note that short selling debt contracts (borrowing) expands the traders’ budget and enables them to invest more on the asset or the bond (or other debt contracts). However, short selling is subject to the following collateral constraint:

\[
\int_{\mathbb{R}^+} d\mu_i^-(\varphi) \leq x_i^A. \tag{2.6}
\]

\(^1\)I define traders’ portfolios using measures, rather than measurable functions, because the optimal portfolio characterized below will be a Dirac measure (which does not correspond to a measurable function).
That is, for each unit debt contract traders sell, they need to set aside one unit of the asset they own as collateral.

Type \(i\) traders choose their portfolio to maximize their expected utility, i.e., they solve the problem:

\[
\max_{x_i \geq 0, \mu_i^+ \in M(\mathbb{R}_+)} \left[ x_i^A E_i[v(s)] + x_i^B (1 + r) + \int_{\mathbb{R}_+} E_i[\min(v(s), \varphi)] d\mu_i^+(\varphi) - \int_{\mathbb{R}_+} E_i[\min(v(s), \varphi)] d\mu_i^-(\varphi) \right]
\]

subject to (2.5) and (2.6).

Market clearing for each unit debt contract \(\varphi\) requires the sum of the long positions to be equal to the sum of the short positions, that is:

\[
\sum_{i \in \{0, 1\}} \int_C d\mu_i^+(\varphi) = \sum_{i \in \{0, 1\}} \int_C d\mu_i^-(\varphi) \text{ for each Borel set } C \subset \mathbb{R}_+.
\]

\[\text{Definition 2 (Collateral Equilibrium). Given an economy } \mathcal{E} \text{ with assumptions (O) and (S), a collateral equilibrium is a collection of prices } (p, [q(\cdot)]) \text{ and portfolios } (x_i^A, x_i^B, \mu_i^+, \mu_i^-)_{i \in \{0, 1\}} \text{ such that: the portfolio of type } i \text{ traders solves Problem (2.7) for each } i \in \{0, 1\}, \text{ the asset market clears, } \sum_{i \in \{0, 1\}} x_i^A = 1, \text{ and the debt market clears [cf. Eq. (2.8)].}\]

\[\text{2.3 Characterization of Collateral Equilibrium}\]

This section provides a characterization of collateral equilibrium and presents the main result which characterizes the effect of belief heterogeneity on the asset price. The equilibrium will intuitively have the form that moderates hold the bond and long positions on collateralized debt contracts (i.e., they lend to optimists), while optimists make leveraged investments in the asset by selling collateralized debt contracts.

To characterize the equilibrium, it is useful to define the notion of a quasi-equilibrium, which is a collection of prices \((p, [q(\cdot)])\) and portfolios \((x_i^A, x_i^B, \mu_i^+, \mu_i^-)_{i \in \{0, 1\}}\) such that markets clear and the portfolio of type \(i \in \{0, 1\}\) traders solves Problem (2.7) with the additional requirement \(\mu_i^+ = \mu_i^- = 0.12\). That is, in a quasi-equilibrium, optimists are restricted not to buy debt

\[\text{Here, } \mu_i^+ = 0 \text{ (similarly } \mu_i^- = 0) \text{ denotes the 0 measure, i.e., } \mu_i^+(C) = 0 \text{ for each Borel set } C \subset \mathbb{R}_+.\]
contracts, and moderates are restricted not to sell debt contracts. For expositional reasons, I will first construct a quasi-equilibrium. Theorem 2 below establishes that the constructed quasi-equilibrium corresponds to a collateral equilibrium with the same allocations and the same asset price (and with potentially different debt contract prices). The same theorem also establishes that the asset price in a collateral equilibrium is uniquely determined.

To construct a quasi-equilibrium, consider debt contract prices

\[ q(\varphi) = \frac{E_0[\min(v(s), \varphi)]}{1 + r} \quad \text{for each } \varphi \in \mathbb{R}_+, \tag{2.9} \]

that make moderates indifferent between holding the bond and any debt contract \( \varphi \in \mathbb{R}_+ \). Given the prices in (2.9) and the asset price \( p \geq \frac{E_0[v(s)]}{1 + r} \), moderates' optimal decision in a quasi-equilibrium is completely characterized: they are indifferent between holding the bond and any debt contract, and they always weakly prefer these options to holding the asset (and strictly so whenever \( p > \frac{E_0[v(s)]}{1 + r} \)). Moreover, these prices ensure that market clearing in debt contracts will be automatic, as moderates will absorb any supply of debt contracts from optimists.

The quasi-equilibrium asset price and allocations are then determined by optimists' portfolio choice. I next analyze optimists' problem for a given asset price \( p \), and I then combine this analysis with asset market clearing to solve for the quasi-equilibrium.

2.3.1 Main Result: Asymmetric Disciplining of Optimism

The next result, which is also the main result, characterizes optimists' portfolio choice.

**Theorem 1 (Optimal Contract Choice and Asymmetric Filtering).** Suppose assumptions (O) and (S) hold, debt prices are given by (2.9) and the asset price satisfies \( p \in \left( \frac{E_0[v(s)]}{1 + r}, \frac{E_1[v(s)]}{1 + r} \right) \). In a quasi-equilibrium:

(i) There exists \( \bar{s} \in S \) such that \( \mu^- \) is a Dirac measure that puts weight only at the contract \( \varphi = v(\bar{s}) \), i.e., optimists borrow according to a single loan with riskiness \( \bar{s} \). Optimists' collateral constraint (2.6) is binding, i.e., they borrow as much as possible according to the optimal loan. Optimists choose \( x^B = 0 \), i.e., they invest all of their leveraged wealth in the asset \( A \).

(ii) The riskiness \( \bar{s} \) of the optimal loan is characterized as the unique solution to the following
equation over $S$:

$$
p = p^{opt}(\bar{s}) \equiv \frac{1}{1+r} \left( \int_{s_{\min}}^{\bar{s}} v(s) \, dF_0 + (1-F_0(\bar{s})) \int_{\bar{s}}^{s_{\max}} v(s) \, \frac{dF_1}{1-F_1(\bar{s})} \right) \quad (2.10)$$

$$
= \frac{1}{1+r} \left( F_0(\bar{s}) E_0[v(s) \mid s < \bar{s}] + (1-F_0(\bar{s})) E_1[v(s) \mid s \geq \bar{s}] \right).
$$

The riskiness $\bar{s}$ of the optimal loan is decreasing in the price level $p$.

If instead the asset price satisfies $p = \frac{E_1[v(s)]}{1+r}$, then optimists are indifferent between making a leveraged investment by selling any safe debt contract $\varphi \leq v(s_{\min})$ or investing in the bond. \footnote{The case $p = \frac{E_0[v(s)]}{1+r}$ is omitted, since the equilibrium asset price always satisfies $p > \frac{E_0[v(s)]}{1+r}$ (cf. Theorem 2).}

I will shortly provide a sketch proof of this result along with an intuition. Before doing so, I note a couple of important aspects of the function $p^{opt}(\bar{s})$. First, the function $p^{opt}(\bar{s})$ is similar to an inverse demand function: it describes the asset price $p$ for which the riskiness level $\bar{s}$ is optimal. Assumption (O) implies $p^{opt}(\bar{s})$ is strictly decreasing and continuous (cf. Appendix 2.A.1). Since $p^{opt}(s_{\min}) = \frac{E_1[v(s)]}{1+r}$ and $p^{opt}(s_{\max}) = \frac{E_0[v(s)]}{1+r}$, this further implies that there is a unique solution to Eq. (2.10), and that the solution is strictly decreasing in $p$.

Second, note that $p^{opt}(\bar{s})$ also describes the equilibrium asset price conditional on the equilibrium loan riskiness $\bar{s}$. Hence, Theorem 1 is the main result, as it shows that optimism will be asymmetrically disciplined in equilibrium. In particular, the second line of (2.10) replicates Eq. (2.1) from the Introduction and shows that the asset is priced with a mixture of moderate and optimistic beliefs. The moderate belief is used to assess the likelihood of default states $s < \bar{s}$, along with the value of the asset conditional on these states, while the optimistic belief is used to assess the likelihood of non-default states $s > \bar{s}$. Consequently, the function $p^{opt}(\bar{s})$ will "discipline" any optimism about the probability of default states, while "incorporating" any optimism about the relative probability of states conditional on no default. The following example describes two cases that differ about the type of optimism and illustrates the asymmetric disciplining property.
Figure 2-1: The top two panels display the probability density functions for traders’ priors in the two scenarios of Example 1. The bottom panel displays the corresponding curves $p^{\text{opt}}(\hat{s})$, the inverse of which gives the optimal loan riskiness $\hat{s}$ for a given price level $p$.

Example 1 (Asymmetric Filtering of Optimism). Consider the state space $S = [1/2, 3/2]$ and the value function $v(s) = s$. As the first case, suppose moderates and optimists have the prior distributions $F_0$ and $F_{1,B}$ with density functions:

$$
    f_0(s) = 1 \text{ for each } s \in S,
$$

and

$$
    f_{1,B}(s) = \begin{cases} 
    0.4 & \text{if } s \in S_B \equiv [2/3 - 1/6, 2/3 + 1/6) \\
    1.3 & \text{if } s \in S_N \equiv [1 - 1/6, 1 + 1/6) \\
    1.3 & \text{if } s \in S_G \equiv [4/3 - 1/6, 4/3 + 1/6] 
    \end{cases}
$$

where $S_B$, $S_N$, and $S_G$ intuitively capture bad, normal and good events, respectively. In words, moderates find all states equally likely, while optimists are optimistic because they believe that a bad event, that is, a realization around the bad state $2/3$, is less likely than a normal or a good event (which they find equally likely).\(^{14}\)

Consider a second case in which moderates have the same prior, but optimists’ prior is

\(^{14}\)Note that the belief distributions of this example do not exactly satisfy the regularity assumption (O). In particular, the density functions are not continuous, and $F_{1,B}$ is only weakly more optimistic than $F_0$. These distributions are used for illustration purposes because they provide a clear intuition. The formal results use the stricter assumption (O) for analytical tractability.
changed to the distribution $F_{1,G}$ with density function

$$f_{1,G} = \begin{cases} 
1 & \text{if } s \in S_B \\
0.1 & \text{if } s \in S_N \\
1.9 & \text{if } s \in S_G 
\end{cases}$$

That is, optimists are optimistic not because they think the bad event is less likely, but because they believe the good event is more likely than the normal event. Note also that optimists are equally optimistic in both cases, i.e., $E_{1,G}[v(s)] = E_{1,B}[v(s)]$.

The bottom panel of Figure 2-1 displays the optimality curves, $p^{opt}(\bar{s})$, in both cases. Note that, for any level of loan riskiness $\bar{s}$, the asset price is higher in the second case than in the first case. Equivalently, for any price $p$, optimists choose a larger and riskier loan in the second case than in the first case.

I next provide a sketch proof of Theorem 1, which is useful to understand the intuition. The proof in Appendix 2.A.2 shows that optimists borrow using a loan with riskiness $\bar{s} \in S$ that maximizes the leveraged return:

$$R^L_1(\bar{s}) \equiv \frac{E_1[v(s)] - E_1[\min(v(s), v(\bar{s}))]}{p - \frac{1}{1+r}E_0[\min(v(s), v(\bar{s}))]}.$$ (2.11)

This expression is the expected return of optimists who buy one unit of the asset and who finance part of the purchase using a loan with riskiness $\bar{s} \in S$. The denominator is the downpayment optimists make for the leveraged investment: they pay the price $p$ of the asset but they can borrow $q(\bar{s}) = \frac{E_0[\min(v(s), v(\bar{s}))]}{1+r}$ from moderates (given the contract prices (2.9)). The numerator is optimists’ expected payoff from the leveraged investment: they expect to receive $E_1[v(s)]$ from the asset and they also expect to pay $E_1[\min(v(s), v(\bar{s}))]$ on their loan.

The relation $p = p^{opt}(\bar{s})$ is the first order optimality condition corresponding to the maximization of the leveraged return, $R^L_1(\bar{s})$. The leveraged return has a unique maximum over $S$, characterized by the first order condition, which completes the sketch proof of Theorem 3.

To understand the intuition for the theorem, it is useful to further break down the leveraged return expression (2.11) into two components. First consider the left hand side terms in the
numerator and the denominator of (2.11), which constitute the *unleveraged return*:

\[ RU = \frac{E_1[v(s)]}{p}. \]

This expression is the expected return of optimists if they buy the asset with their own wealth (without borrowing). Optimists believe the return on investing in the asset is greater than the benchmark rate, i.e., \( RU > 1 + r \), which creates a force that pushes towards leveraging. In particular, if optimists could borrow at the benchmark rate \( r \) without constraints, they would borrow infinitely to leverage this unleveraged return.

However, optimists have to borrow with a collateralized loan with riskiness \( \bar{s} \), which represents a second force that pushes towards deleveraging. This force is related to the right hand side terms in the numerator and the denominator of (2.11), which constitute optimists’ *perceived interest rate* on the loan:

\[ 1 + r_1^{\text{per}}(\bar{s}) = \frac{E_1[\min(v(s), v(\bar{s}))]}{\frac{1}{1+r}E_0[\min(v(s), v(\bar{s}))]}. \] (2.12)

Optimists borrow \( \frac{1}{1+r}E_0[\min(v(s), v(\bar{s}))] \) on the loan, but they expect to pay \( E_1[\min(v(s), v(\bar{s}))] \), which leads to the perceived interest rate \( 1 + r_1^{\text{per}}(\bar{s}) \). Assumption (O) implies that \( r_1^{\text{per}}(\bar{s}) \) is always weakly greater than the benchmark rate \( r \), and that it is increasing in \( \bar{s} \) [cf. Appendix (2.A.1)]. The intuition for this observation is two fold. First, collateralized loans always trade at a *spread* over the benchmark rate [i.e., the interest rate on the loan, (2.4), is always greater than the benchmark rate], because moderate lenders require compensation for their expected losses in case of default. In particular, since the loan market is competitive, the spread on a loan is just enough to compensate the lenders according to their moderate beliefs. Second, optimists believe that the loan will default less often than moderates believe, hence they think they will end up paying the spread more often. Consequently, optimists believe they will pay a greater interest rate than the benchmark rate, i.e., \( r_1^{\text{per}}(\bar{s}) > r \). Moreover, for greater levels of \( \bar{s} \), the scope of disagreement for default is greater, which implies that \( r_1^{\text{per}}(\bar{s}) \) is increasing in \( \bar{s} \).

It follows that, while a larger loan with a greater riskiness level \( \bar{s} \) enables optimists to leverage the unleveraged return more, it also comes at a greater perceived interest rate, \( r_1^{\text{per}}(\bar{s}) \).
Optimists’ optimal loan choice balances these two forces, as captured by the maximization of the leveraged return expression (2.11).

This breakdown of the two forces also provides an intuition for the asymmetric disciplining property of the pricing function $p^{opt}(\bar{s})$. First consider the intuition for the simpler property that $p^{opt}(\bar{s})$ is decreasing in $\bar{s}$. That is, consider why optimists choose a larger and riskier loan when the price $p$ is lower. This is because a lower asset price increases the unleveraged return, $R^U = \frac{E[v(s)]}{p}$, which tilts optimists’ trade-off towards larger loans. When the unleveraged return is greater, optimists have a greater incentive to leverage this return by taking a larger (and riskier) loan, agreeing to pay a greater expected interest rate $r_1^{per}(\bar{s})$ at the margin.

To see the intuition for the disciplining property of $p^{opt}(\bar{s})$, fix a loan with riskiness $\bar{s}$, and consider how much the price should drop (from the optimistic valuation) to entice optimists to take this particular loan. Consider this question in the context of Example 1 for a riskiness level $\bar{s} = 0.8 \in S_B$. In the first case of Example 1, optimists find the bad event $S_B$ unlikely. Hence, given a loan with riskiness $\bar{s} \in S_B$, there is disagreement about the probability of default, which implies $r_1^{per}(\bar{s}) > r$. As this loan appears expensive to optimists, the asset price should drop considerably to entice optimists to undertake a leveraged investment with this loan. Consider instead the second case of Example 1 in which optimists are optimistic because they find the good event more likely than the normal event. In this case, for a loan with riskiness $\bar{s} \in S_B$, there is no disagreement about the probability of default, which implies $r_1^{per}(\bar{s}) = r$. As the loan appears cheap to optimists, the asset price does not need to fall to entice them to take the loan (see Figure 2-1).

In other words, the asymmetric disciplining result operates through optimists’ perceived financial constraints. Disagreement about default states tightens optimists’ financial constraints (captured by a higher $r_1^{per}(\bar{s})$), which lowers their demand for the leveraged investment and leads to an asset price closer to the moderate valuation. In contrast, disagreement about non-default states does not tighten the financial constraints, and leads to an asset price closer to the optimistic valuation.
2.3.2 Asset Market Clearing and Collateral Equilibrium

Theorem 1 characterizes the riskiness $\tilde{s}$ of the optimal contract as a function of the asset price $p$. I next consider the market clearing price $p$ and solve for the equilibrium.

Suppose optimists choose to borrow using a loan with riskiness $\tilde{s}$ and consider the price that clears the asset market. This price depends on the maximum first period consumption good that optimists can obtain:

$$w_1^{\text{max}}(\tilde{s}) = w_1 + \frac{1}{1+r} E_0 [\min (v(s), v(\tilde{s}))].$$

(2.13)

Optimists are endowed with $w_1$ units of the consumption good, and if they hold the entire asset supply, they can borrow up to $\frac{1}{1+r} E_0 [\min (v(s), v(\tilde{s}))]$ units of the consumption good from moderates, leading to the expression in (2.13). The asset price depends on the comparison of $w_1^{\text{max}}(\tilde{s})$ with the value of the asset in the hands of moderates, $\alpha_0 p$, which optimists seek to purchase. In particular:

$$p = p^{\text{mc}}(\tilde{s}) = \begin{cases} \frac{E_0 [v(s)]}{1+r} & \text{if } \frac{w_1^{\text{max}}(\tilde{s})}{\alpha_0} > \frac{E_0 [v(s)]}{1+r} \quad \text{[case (i)]} \\ \frac{w_1^{\text{max}}(\tilde{s})}{\alpha_0} & \text{if } \frac{w_1^{\text{max}}(\tilde{s})}{\alpha_0} \in \left( \frac{E_0 [v(s)]}{1+r}, \frac{E_0 [v(s)]}{1+r} \right] \quad \text{[case (ii)]} \\ \frac{E_0 [v(s)]}{1+r} & \text{if } \frac{w_1^{\text{max}}(\tilde{s})}{\alpha_0} < \frac{E_0 [v(s)]}{1+r} \quad \text{[case (iii)]} \end{cases}$$

(2.14)

In case (i), optimists have access to a sufficient amount of consumption good in the first period that they purchase all of the asset in the hands of moderate lenders (and they have some consumption good left over, which they invest in the bond). In this case, optimists are marginal holders of the asset and the price is given by their valuation, $\frac{E_0 [v(s)]}{1+r}$. In case (ii), optimists still purchase all of the asset from moderate lenders, but they cannot bid up the asset price to their valuation. In this case, the market clearing price is determined by setting optimists’ consumption good equal to the value of moderates’ assets, i.e., $w_1^{\text{max}}(\tilde{s}) = p \alpha_0$. In case (iii), optimists have access to so little first period consumption good that they cannot purchase all of the asset in the hands of moderate lenders. In this case, moderate lenders hold some of the asset, and the price is given by their valuation, $\frac{E_0 [v(s)]}{1+r}$.

Note that Eq. (2.14) describes an increasing relation between the asset price and the loan riskiness $\tilde{s}$. Intuitively, when optimists take a larger and riskier loan, they have access to a
greater amount of first period consumption good, which enables them to bid up the asset price higher. Combining Theorem 1 and Eq. (2.14), the equilibrium price and loan riskiness pair, \((p, \bar{s})\), is determined as the unique intersection of the strictly decreasing function \(p^{\text{opt}}(\bar{s})\) and the weakly increasing function \(p^{\text{mc}}(\bar{s})\) (see Figure 2-2). This analysis completes the characterization of the quasi-equilibrium. The analysis in Appendix 2.A.3 establishes that this quasi-equilibrium is a collateral equilibrium with modified debt contract prices given by:\(^{15}\)

\[
q(\varphi) = \max \left( \frac{E_0 \left[ \min (v(s), \varphi) \right]}{1 + r}, \frac{E_1 \left[ \min (v(s), \varphi) \right]}{R_f^{\text{mc}}(\bar{s})} \right).
\]  

(2.15)

The following result summarizes this discussion and proves the essential uniqueness of the collateral equilibrium.

**Theorem 2 (Existence, Characterization, Essential Uniqueness).** Consider the above described economy with assumptions (O) and (S). There exists a collateral equilibrium in which contract prices are given by (2.15), moderate types are indifferent between buying bonds and lending to optimists, and optimists make leveraged investments in the asset by borrowing through a single loan with riskiness \(\bar{s} \in S\). The asset price \(p\) and riskiness \(\bar{s}\) of loans in this equilibrium are determined as the unique solution to \(p = p^{\text{opt}}(\bar{s}) = p^{\text{mc}}(\bar{s})\) over \(\bar{s} \in S\).

In any collateral equilibrium, the asset price, \(p\), and the price of the optimal debt contract, \(q(v(\bar{s}))\), are uniquely determined. Except for the corner case in which \(p = \frac{E[v(s)]}{1+r}\), traders' allocations are also uniquely determined. However, prices of the remaining debt contracts, \(q(\varphi)\) for \(\varphi \neq v(\bar{s})\), are not uniquely determined.

In other words, most of the equilibrium is uniquely determined, except for the price of debt contracts that are not traded in equilibrium. Appendix 2.A.3 establishes that, for each contract \(\varphi = v(\bar{s}) \neq v(s^*)\), there exists a continuum of prices that can support the equilibrium with no-trade in these contracts. This completes the characterization of the collateral equilibrium.

Figure 2-2 illustrates the equilibrium, and shows the effect of a decline in optimists' initial

---

\(^{15}\)Note that \(R_f^{\text{mc}}(\bar{s}^*)\) (cf. Eq. (2.11)) is optimists' expected return on capital in equilibrium. Thus, the expression \(\frac{E_1[\min(v(s),\varphi)]}{R_f^{\text{mc}}(\bar{s})}\) is optimists' valuation of the debt contract \(\varphi\) in equilibrium. Unlike in a quasi-equilibrium, optimists can demand debt contracts in a collateral equilibrium. Hence, the price of a debt contract is given by the upper-envelope of the moderate and the optimistic valuations, as captured by (2.15). The analysis in Appendix 2.A.3 establishes that optimists' and moderate lenders' allocations continue to be optimal when the prices are given by (2.15) and when the constraints \(\mu_0 = 0\) and \(\mu_1 = 0\) are relaxed.
endowment of the consumption good. When optimists’ wealth declines, the price falls towards the moderate valuation. Note also that the equilibrium loans also become larger and riskier. This is because, as the price falls, optimists see more of a bargain in the asset price which encourages them to leverage more. Hence, equilibrium leverage responds in a way to ameliorate the drop the initial wealth shock to optimists. These comparative statics are similar to the results in Geanakoplos (2009).

I next turn to the focus of this essay, and establish the comparative statics of the equilibrium with respect to the type and the level of belief heterogeneity.

2.4 Comparative Statics with Respect to Belief Heterogeneity

In addition to the equilibrium loan riskiness $\bar{s}^*$ and the asset price $p$, I consider the comparative statics of the leverage ratio for optimists’ asset purchase, denoted by $L$. Recall that optimists buy one unit of the asset by paying $p - \frac{E_0[v(s),v'(\bar{s}^*)]}{1+r}$ out of their wealth and financing the rest of the purchase by borrowing from moderates. Thus, the leverage ratio for the asset purchase is given by

Figure 2-2: The figure displays the collateral equilibrium, and the response of the equilibrium to a decline in optimists’ initial endowment of the consumption good, $w_1$. 
\[ L \equiv \frac{p}{p - E_0 \left[ \min \left( v(s), v(\bar{s}^*) \right) \right] / (1 + r)}. \]  

(2.16)

The leverage ratio has counterparts in real financial markets: the loan-to-value ratio of a mortgage loan is equal to \( 1 - \frac{1}{L} \) and the haircut on a REPO loan is equal to \( \frac{1}{L} \).

The next definition formalizes the type of belief heterogeneity that is used to state the comparative statics results.

**Definition 3 (Skewed Optimism).** Consider two probability distributions \( H, \tilde{H} \) over \( S = [s^{\min}, s^{\max}] \) with density functions \( h, \tilde{h} \) that are continuous and positive over \( S \), and consider a continuously differentiable and strictly increasing asset value function \( v : S \rightarrow \mathbb{R}^+ \). The optimism of distribution \( \tilde{H} \) about the asset is weakly more right-skewed than \( H \), denoted by \( \tilde{H} \succ_R H \), if and only if:

(a) The distributions yield the same valuation of the asset, that is, \( E[v(s) ; \tilde{H}] = E[v(s) ; H] \).

(b) There exists \( s^R \in S \) such that \( \frac{1-\tilde{H}(s)}{1-H(s)} \) is weakly decreasing over \( (s^{\min}, s^R) \) while it is weakly increasing over \( (s^R, s^{\max}) \), which is the case if and only if the hazard rates of \( \tilde{H} \) and \( H \) satisfy the (weak) single crossing condition:

\[
\begin{align*}
\frac{\tilde{h}(s)}{1-\tilde{H}(s)} &\leq \frac{h(s)}{1-H(s)} \quad \text{if } s > s^R, \\
\frac{\tilde{h}(s)}{1-\tilde{H}(s)} &\leq \frac{h(s)}{1-H(s)} \quad \text{if } s < s^R.
\end{align*}
\]

(2.17)

The optimism of distribution \( \tilde{H} \) is weakly more skewed to the right of \( \tilde{s} \in S \) than \( H \), denoted by \( \tilde{H} \succ_{R, \tilde{s}} H \), if the conditions (a)-(b) are satisfied with the additional requirement that \( s^R \geq \tilde{s} \).

To interpret this definition, note that the distributions \( \tilde{H} \) and \( H \) cannot be compared according to the optimism order in Definition 1, since their hazard rates are not ordered. In addition, these distributions lead to the same valuation of the asset, that is, they have the same “level” of optimism. Note also that \( \tilde{H} \) has a lower hazard rate than \( H \) over the region \( (s^R, s^{\max}) \). Thus, conditional on \( s \geq s^R \), \( \tilde{H} \) is more optimistic than \( H \) in the sense of Definition 1. In contrast, \( H \) has a lower hazard rate than \( \tilde{H} \) over the region \( (s^{\min}, s^R) \), and thus its optimism is concentrated more on this region. Hence, the optimism of \( \tilde{H} \) is right-skewed in the sense that it is concentrated more on relatively good states.
Figure 2-3: The top two panels display the hazard rates for traders' priors in the two cases analyzed in Example 1. The bottom panel plots the corresponding equilibria.
Note that the probability distributions $F_{1,B}$ and $F_{1,G}$ of Example 1 satisfy condition (2.17). That is, $F_{1,G}$ and $F_{1,B}$ lead to the same valuation for the asset but the optimism of $F_{1,G}$ is weakly more right skewed, as illustrated in Figure 2-3. The same figure also plots the optimality relation $p^{opt}(\bar{s})$ from Figure 2-1 together with the market clearing curve $p^{mc}(\bar{s})$, and illustrates that the equilibrium price $p$ and loan riskiness $\bar{s}^*$ are higher when optimists’ optimism is more right-skewed. The next result shows that this observation is generally true.

**Theorem 3 (Type of Heterogeneity).** Consider the collateral equilibrium characterized in Theorem 2 and let $\bar{s}^*$ denote the equilibrium loan riskiness.

(i) If optimists’ optimism becomes weakly more right-skewed, i.e., if their prior is changed to $\tilde{F}_1$ that satisfies $\tilde{F}_1 \succeq_R F_1$ and $\tilde{F}_1 \succ_O F_0$ (so that assumption (O) continues to hold), then: the asset price $p$, the loan riskiness $\bar{s}^*$, and the leverage ratio $L$ weakly increase.

(ii) If moderates’ optimism becomes weakly more skewed to the left of $\bar{s}^*$, i.e., if their prior is changed to $\tilde{F}_0$ that satisfies $F_0 \preceq_L \bar{s}^* \tilde{F}_0$ and $F_1 \succ_O \tilde{F}_0$, then: the asset price $p$ weakly increases.

I provide a sketch proof of this result, which is completed in Appendix 2.A.4. First observe that Eq. (2.10) can be written as:

$$p^{opt}(\bar{s}) - \frac{E_0[v(s)]}{1+r} = \frac{1}{1+r} (1 - F_0(\bar{s})) (E_1[v(s) \mid s \geq \bar{s}] - E_0[v(s) \mid s \geq \bar{s}]).$$

(2.18)

In view of the asymmetric disciplining result, the difference between the asset price and the moderate valuation depends on the moderate probability of no default, and traders’ valuation differences conditional on no default. For part (i), Appendix 2.A.4 shows that

$$\tilde{E}_1[v(s) \mid s \geq \bar{s}] \geq E_1[v(s) \mid s \geq \bar{s}] \text{ for each } \bar{s} \in (s_{\min}, s_{\max}),$$

(2.19)

where $\tilde{E}_i[\cdot]$ denotes the expectation operator with respect to distribution $\tilde{F}_i$. That is, when optimists’ optimism becomes more right-skewed, their valuation of the asset conditional on any upper-threshold event increases, even though their unconditional valuation is the same. It follows, by Eq. (2.19), that the optimality curve $p^{opt}(\bar{s})$ shifts up pointwise. As the market clearing curve $p^{mc}(\bar{s})$ remains constant, the equilibrium asset price $p$ and the loan riskiness $\bar{s}^*$ increase, which further implies the remaining comparative statics. Appendix 2.A.4 uses a
Figure 2.4: The left panel plots the equilibrium in the first scenario considered in Example 2: the increase in belief heterogeneity is concentrated to the left of state $\bar{s}^*$ and it decreases the asset price. The right panel plots the equilibrium in the second scenario considered in Example 2: the increase in belief heterogeneity is to the right of $\bar{s}^*$ and it increases the asset price.

A similar argument to prove part (ii).

Theorem 3 points to the importance of the skewness of belief heterogeneity for the asset price. A related question is whether the level of belief heterogeneity has similar robust predictions regarding the price of the asset. The answer is no, as illustrated in the following example.

Example 2 (Ambiguous Price Effect of Increased Belief Heterogeneity). Consider the first case of Example 1 in which optimists are optimistic because they find the bad event unlikely, i.e., they have the prior $F_{1,B}$. Suppose the moderate and the optimistic beliefs are changed to $\tilde{F}_0 = F_{0,G}$ and $\tilde{F}_1 = F_{1,BG}$ with density functions given by

$$
\tilde{f}_{0,G} = \begin{cases} 
1 & \text{if } s \in S_B \\
1 + 0.5 & \text{if } s \in S_N, \\
1 - 0.5 & \text{if } s \in S_G 
\end{cases}
$$

$$
\tilde{f}_{1,BG} = \begin{cases} 
0.4 & \text{if } s \in S_B \\
1.3 - 0.5 & \text{if } s \in S_N \\
1.3 + 0.5 & \text{if } s \in S_G 
\end{cases}
$$

39
That is, moderates’ prior probability for the normal event increases and their probability for the good event decreases, while the opposite happens to optimists’ prior. As the right panel of Figure 2-4 shows, in this case, the increase in belief heterogeneity leads to an increase in the asset price.

Consider the second case in Example 1 in which optimists are optimistic because they find the good event more likely than the normal event, i.e., they have the prior $F_{1,G}$. Suppose the moderate and the optimistic beliefs are changed to $F_0 = F_{0,B}$, $F_1 = F_{1,GB}$ with density functions given by

$$
\begin{align*}
    f_{0,B} &= \begin{cases} 
        1 + 0.5 & \text{if } s \in S_B \\
        1 - 0.25 & \text{if } s \in S_N \\
        1 - 0.25 & \text{if } s \in S_G 
    \end{cases}, \\
    f_{1,GB} &= \begin{cases} 
        1(1 - 0.5) & \text{if } s \in S_B \\
        0.1(1 + 0.25) & \text{if } s \in S_N \\
        1.9(1 + 0.25) & \text{if } s \in S_G 
    \end{cases}.
\end{align*}
$$

That is, moderates’ prior probability for the bad event increases and their relative probability for good and the normal event remains constant, while optimists’ prior probability for the bad event decreases. As Figure 2-4 shows, in this case, the increase in belief heterogeneity leads to a decrease in the asset price.

Example 2 illustrates that the increase in belief heterogeneity has no robust predictions for the asset price. In particular, the second part provides an example in which optimists become more optimistic but the asset price declines, which is in contrast with the Miller (1977) hypothesis. In view of the asymmetric disciplining property of $p^{opt}(\bar{s})$, both the optimistic and moderate beliefs play a part in the determination of the asset price. While the increase of optimists’ optimism tends to increase the asset price, the decrease in moderates’ pessimism tends to decrease it by tightening the financial constraints, and the net effect is ambiguous. This observation suggests that the Miller hypothesis may not apply in markets in which optimists finance their purchases by borrowing from less optimistic investors.

I next show that increased belief heterogeneity has robust predictions regarding the asset price if the type of the additional increase is also taken into account. In the first case of Example 2, the belief heterogeneity is concentrated to the left of the default threshold $\bar{s}^*$, and the asset price decreases. In the second case of the example, the belief heterogeneity is concentrated to the right of the default threshold $\bar{s}^*$, and the asset price increases (see also Figure 2-4). The
next result shows that these properties are general: belief heterogeneity has an unambiguous
effect on the asset price if it is concentrated to the left, or to the right, of the equilibrium default
threshold $\bar{s}$.

**Theorem 4 (Level of Heterogeneity).** Consider the collateral equilibrium characterized in
Theorem 2 and let $\bar{s}$ denote the equilibrium loan riskiness, which is also the threshold state
below which loans default. Consider a (weak) increase in belief heterogeneity, in the sense that
beliefs are changed to $F_1$ and $F_0$ that satisfy $F_1 \succeq O F_1$ and $F_0 \succeq O F_0$:

(i) Suppose the increase in belief heterogeneity is concentrated to the right of $\bar{s}$, that is, 
   suppose $1-F_1(s)$ and $1-F_0(s)$ are constant over the set $(s^{\text{min}}, \bar{s})$. Then the asset price $p$, the
   loan riskiness $\bar{s}$, and the leverage ratio $L$ weakly increase.

(ii) Suppose the increase in belief heterogeneity is concentrated to the left of $\bar{s}$, that is, 
   suppose $1-F_1(s)$ and $1-F_0(s)$ are constant over the set $(\bar{s}, s^{\text{max}})$. Then the asset price $p$ weakly
   decreases.

Taken together with the earlier results, this result demonstrates that the type of the belief
heterogeneity is a more robust determinant of asset prices than the level of belief heterogeneity.
With endogenous financial constraints, what investors disagree about matters for asset prices,
to a greater extent than the level of their disagreement.

**2.5 Collateral Equilibrium with Contingent Contracts**

The analysis in the previous sections has concerned the baseline setting in which loans are re-
stricted to be non-contingent and short selling is not allowed. While the baseline model is a good
starting point, it is important to verify the robustness of the results to more general settings,
especially because allowing for a richer set of contracts introduces new economic forces. The
analysis in this section considers an extension in which debt contracts can be fully contingent
on the continuation state $s \in \mathcal{S}$, and it establishes three results. First, the optimal contingent
contract is not a simple debt contract. Rather it is a threshold contract: optimists promise
to make a zero payment in all states above a threshold, but they promise make a payment
equal to the asset value in the states below the threshold. Second, this threshold contract is
sufficiently similar to a simple debt contract that a version of the asymmetric disciplining result
(cf. Theorem 1) also applies in this setting. Third, unlike the case with simple debt contracts, the asset price in this setting may exceed the valuation of even the most optimistic investor.

2.5.1 Definition of Equilibrium with Contingent Contracts

A unit contingent debt contract, denoted by \( \varphi : S \rightarrow \mathbb{R}_+ \), is a collection of promises of \( \varphi(s) \geq 0 \) units in each state \( s \in S \), collateralized by 1 unit of the asset.\(^{16}\) The borrower defaults on the contract if and only if the value of the asset is less than the promise on the contract. Thus, the contract pays \( \min(v(s), \varphi(s)) \) units. Let \( D \) denote the set of all unit debt contracts. As before, each debt contract \( \varphi \) is traded in an anonymous market at a competitive price \( q(\varphi) \), where \( q(\cdot) \) is a Borel measurable function over \( D \).

Let \( \mu^+_i, \mu^-_i \in M(D) \) respectively denote type \( i \) traders’ long and short debt portfolios, where \( M(D) \) denotes the set of Borel measures over \( D \). Type \( i \) traders solve an analogue of problem (2.7): they choose their portfolio \( (x^A_i, x^B_i, \mu^+_i, \mu^-_i) \) to maximize their expected payoff subject to a budget and a collateral constraint (see problem (2.A.45) in the appendix). Given this problem and the extended contract space, the equilibrium is defined similarly to Section 2.2.1.

The characterization of equilibrium closely follows the analysis in Section 2.3. In particular, consider first a quasi-equilibrium by restricting traders’ choices with the constraint \( \mu^+_0 = \mu^-_1 = 0 \). To construct a quasi-equilibrium, consider debt contract prices:

\[
q(\varphi) = \frac{E_0 [\min(v(s), \varphi(s))]}{1 + r},
\]

which make moderates indifferent between purchasing the bond and any debt contract \( \varphi \). The equilibrium will be determined by optimists’ portfolio choice given these prices.

2.5.2 Asymmetric Disciplining with Contingent Contracts

Consider optimists’ portfolio choice for a given price \( p \). The same analysis for Theorem 1 (cf. Appendix 2.A.2) shows that optimists borrow by selling a contingent debt contract,

\(^{16}\) assume that contracts must make non-negative promises in all continuation states, because a negative promise by the borrower (which is essentially a promise by the lender) would not be enforced by the court system in this economy since lenders do not set aside any collateral. This is without loss of generality, because if they wish, lenders can also make promises by selling a separate collateralized debt contract.
[φ(s) ∈ [0, v(s)])s∈S, that maximizes the leveraged return:

\[ R^{L,\text{cont}}_1(φ) = \frac{E_1[v(s)] - E_1[\min(v(s), φ(s))]}{p - \frac{1}{1+r}E_0[\min(v(s), φ(s))].} \tag{2.21} \]

The contract that maximizes this expression can be characterized under the following assumption, which is slightly stronger than assumption (O):

**Assumption (MLRP).** The probability distributions \( F_1 \) and \( F_0 \) have density functions \( f_1, f_0 \) which are continuous and positive over \( S \), and which satisfy the monotone likelihood ratio property: that is, \( \frac{f_1(s)}{f_0(s)} \) is strictly increasing over \( S \).

The analysis in Appendix 2.A.5 establishes that, under assumption (MLRP), the optimal contract is a *threshold contract*:

\[ φ_\bar{s}(s) \equiv \begin{cases} v(s) & \text{if } s < \bar{s} \\ 0 & \text{if } s \geq \bar{s}, \end{cases} \tag{2.22} \]

for a threshold state \( \bar{s} \in S \). That is, optimists make as large a promise as possible for states \( s < \bar{s} \) (they give up the asset in these states), while promising zero for states \( s \geq \bar{s} \) (they keep the asset in these states). Intuitively, optimists find bad states the least likely, and thus they concentrate all of their payments below a threshold state.

The next result, which is the analogue of Theorem 1 for contingent loans, characterizes the threshold state \( \bar{s} \in S \) of the optimal contract given price \( p \). The result also shows that, unlike the case with simple debt contracts, the maximum price at which optimists demand the asset is greater than the optimistic valuation, \( E_0[v(s)] \). This maximum price level is given by:

\[ p^{\max} = \frac{1}{1+r} \left( \int_{s_{\min}}^{s_{\cross}} v(s) \, dF_0 + \int_{s_{\cross}}^{s_{\max}} v(s) \, dF_1 \right), \tag{2.23} \]

where \( s_{\cross} \in S \) is the unique state such that \( \frac{f_0(s_{\cross})}{f_1(s_{\cross})} = 1. \)

**Theorem 5 (Asymmetric Disciplining with Contingent Contracts).** Suppose assumptions (MLRP) and (S) hold, debt prices are given by (2.20) and the asset price satisfies \( p \in \left( E_0[v(s)], \frac{1}{1+r}p^{\max} \right) \), where \( p^{\max} \) is given by (2.23). In a quasi-equilibrium:

(i) There exists \( \bar{s} \in [s_{\cross}, s_{\max}] \) such that \( μ_\bar{s} \) is a Dirac measure that puts weight only at
the threshold contract \( \varphi_\delta \) in (2.22). Optimists' collateral constraint is binding, i.e., they borrow as much as possible according to the optimal contract. Optimists choose \( x^B_1 = 0 \), i.e., they invest all of their leveraged wealth in the asset \( A \).

(ii) The threshold state \( \bar{s} \in [s^{cross}, s^{max}] \) of the optimal contract is characterized as the unique solution to:

\[
p = p_{opt, cont}(\bar{s}) \equiv \frac{1}{1 + r} \left( \int_{s_{min}}^{\bar{s}} v(s) \, dF_0 + \int_{\bar{s}}^{s_{max}} v(s) \, dF_1 \right).
\]  

If instead the asset price satisfies \( p = p^{max} \), then optimists are indifferent between making a leveraged investment in the asset by selling the safe debt contract \( \varphi_{g^{cross}} \) or investing in the bond.

Note that the function \( p_{opt, cont}(\bar{s}) \) is the analogue of the function \( p_{opt}(\bar{s}) \): it describes the asset price conditional on optimists' choice of the threshold state \( \bar{s} \). Moreover, the form of \( p_{opt, cont}(\bar{s}) \) is very similar to the form of \( p_{opt}(\bar{s}) \), which suggests that optimism is asymmetrically disciplined also in this setting. In particular, optimism about the relative likelihood of states above \( \bar{s} \) increases the asset price, while the optimism about the relative likelihood of states below \( \bar{s} \) does not increase the price. The intuition for this result can be gleaned from the shape of the optimal debt contract \( \varphi_\delta \). This threshold contract makes the same payment (namely, zero) in all states above the threshold \( \bar{s} \), while it has an increasing payment schedule in the states below the threshold \( \bar{s} \). Hence, any optimism about the relative likelihood of good states does not increase optimists' perceived interest rate, and thus these types of optimism increase the asset price. However, optimism about the relative likelihood of bad states increases optimists' perceived interest rate. Thus, these types of optimism are reflected less in the asset price.\(^{17}\)

Note also that optimists demand the asset even if the price is greater than their valuation, captured by the fact that \( p^{max} > \frac{E_1[v(s)]}{1+r} \). To see the intuition for this result, consider optimists' perceived interest rate on the contract \( \varphi_\delta \), given by:

\(^{17}\)This intuition also illustrates the limitation of the asymmetric filtering result when loans are fully contingent. Unlike regular debt contracts, a contingent debt contract, \( \varphi_\delta \), makes a lower payment in states above \( \bar{s} \) relative to states below \( \bar{s} \). Hence, if optimists' optimism is changed in a way to assign a lower probability to states below \( \bar{s} \), then the asset price increases (unlike the case with non-contingent contracts).
Figure 2-5: The top panel displays the probability densities. The solid (resp. dashed) lines in the bottom panel illustrate the equilibrium with (resp. without) contingent contracts.

\[ 1 + r_{\text{per,cont}}^{\text{per,cont}} (\bar{s}) = \frac{E_1 [\min (v (s), \varphi_{\bar{s}} (s))] - E_0 [\min (v (s), \varphi_{\bar{s}} (s))]}{1 + r} \]  

Unlike the case with non-contingent loans, \( r_{\text{per,cont}}^{\text{per,cont}} (\bar{s}) \) is not necessarily greater than \( r \). In particular, the ability to fine-tune their borrowing enables optimists to take loans which they perceive to be even more favorable than borrowing at the benchmark interest rate. Consequently, optimists invest in the asset even if the price exceeds their valuation, \( \frac{E_1 [v (s)]}{1 + r} \), because they can finance some of the purchase with these loans which they perceive to be very favorable.

A complementary intuition for this result comes from the form of \( p^{\text{max}} \) in (2.23). The availability of fully contingent loans enables optimists to split the asset in a way that each type traders hold the asset in the states which they assign a greater probability. Consequently, the maximum price at which optimists demand the asset is calculated according to an upper-envelope of the moderate and the optimistic beliefs, which exceeds the optimistic valuation. This result creates a presumption that finer levels of financial engineering of loans can potentially have a large impact on asset prices.
2.5.3 Equilibrium Asset Price with Contingent Contracts

Similar to Section 2.5, the equilibrium asset price is determined by combining optimists' optimal contract choice with asset market clearing. The market clearing condition is analogous to Eq. (2.14), and is given by:

\[
p = p^{mc, cont}(\bar{s}) = \begin{cases} 
    p_{\max} & \text{if } \frac{w_{\max, cont}(\bar{s})}{\alpha_0} > p_{\max} \\
    \frac{w_{\max, cont}(\bar{s})}{\alpha_0} & \text{if } \frac{w_{\max, cont}(\bar{s})}{\alpha_0} \in \left( \frac{E_0[v(s)]}{1+r}, p_{\max} \right] \\
    \frac{E_0[v(s)]}{1+r} & \text{if } \frac{w_{\max, cont}(\bar{s})}{\alpha_0} \leq \frac{E_0[v(s)]}{1+r}
\end{cases}
\]

Here, \( w_{\max, cont}(\bar{s}) = w_1 + \frac{1}{1+r} \int_{s_{\min}}^{\bar{s}} v(s) \, ds \) denotes optimists' maximum first period consumption good given that they choose to borrow with the contingent debt contract \( \bar{s} \). The equilibrium asset price \( p \) and the threshold level of the optimal contract \( \bar{s}^* \) are characterized by considering the intersection of the strictly decreasing curve \( p^{opt, cont}(\bar{s}) \) and the weakly increasing curve \( p^{mc, cont}(\bar{s}) \) over the range \( \bar{s} \in [s_{\text{cross}}, s_{\text{max}}] \). Figure 2-5 displays the equilibrium with contingent and non-contingent contracts. Since \( p_{\max} > \frac{E_0[v(s)]}{1+r} \), the equilibrium asset price with contingent contracts exceeds the optimistic valuation whenever the optimistic wealth is sufficiently large.

2.6 Collateral Equilibrium with Short Selling

This section considers an extension of the baseline setting in which short selling is allowed, which is relevant to understand the data for the fraction of the assets that can be short sold (e.g., for the majority of stocks). The analysis in this section establishes that a version of the asymmetric disciplining result (cf. Theorem 1) applies in this setting. I first generalize the definition of equilibrium to allow for short selling. I then characterize traders' portfolio choices for any given asset price, \( p \). I finally combine this analysis with asset market clearing to solve for the equilibrium price.

A potential short seller of an asset needs to borrow the asset from another trader. But since borrowing in this economy is collateralized, short selling also needs to be collateralized. Formally, a unit short contract, denoted by \( \psi \in \mathbb{R}_+ \), is a promise of \( v(s) \) units of the consumption good conditional on state \( s \in S \), collateralized by \( \frac{\psi}{1+r} \) units of the bond (so that \( \psi \) denotes the value of the collateral in the next period). A trader selling the unit short contract can be
interpreted as borrowing the asset from a lender, and posting \( \frac{v}{1+r} \) units of the bond as collateral in a margin account. In reality, the lender of the security will ask for a short fee.\(^{18}\) In the model, the short fee is implicitly captured by the price of the short contract, \( q^{\text{short}}(\psi) \), with the lower price corresponding to a higher short fee.

As in the baseline setting, there are also non-contingent unit debt contracts, \( \varphi \in \mathbb{R}_+ \), each of which is traded at price \( q^{\text{debt}}(\varphi) \). I also assume that only a fraction \( \gamma^{\text{short}} \in [0, 1] \) of traders can sell short contracts, while only a fraction \( \gamma^{\text{debt}} \in [0, 1] \) can sell debt contracts and leverage. These assumptions are made to simplify the analysis, but they are not unreasonable because short selling in financial markets (and to some extent, leverage) is confined to a small fraction of investors. I denote the short selling ability of a trader with \( t^{\text{short}} \in \{0, 1\} \), and the leverage ability with \( t^{\text{debt}} \in \{0, 1\} \). Taking the belief heterogeneity also into account, there are 8 types of traders, where a type is denoted by \( T = (i, t^{\text{short}}, t^{\text{debt}}) \).

Let \( \left( \mu_T^{\text{short},+}, \mu_T^{\text{short},-} \right) \) denote measures that represent type \( T \) traders' portfolio of short contracts, and define \( \left( \mu_T^{\text{debt},+}, \mu_T^{\text{debt},-} \right) \) similarly for debt contracts. The above restriction is formalized by assuming that

\[
\mu_T^{\text{short},-} = 0 \text{ for each } T = (\cdot, t^{\text{short}} = 0, \cdot), \text{ and } \mu_T^{\text{debt},-} = 0 \text{ for each } T = (\cdot, \cdot, t^{\text{debt}} = 0).
\]

The definition of equilibrium follows closely Definition 2 with minor changes that take into account the additional restriction.

As before, I first consider a quasi-equilibrium in which optimists are restricted to choose \( \mu_T^{\text{debt},+} = 0 \) (so they are not allowed to buy debt contracts) while moderates are restricted to choose \( \mu_T^{\text{short},+} = 0 \) (so they are not allowed to buy short contracts). Similar to before, these restrictions will not be binding in equilibrium and the quasi-equilibrium will correspond to a collateral equilibrium. To characterize the quasi-equilibrium, I first conjecture an equilibrium of a particular form in which traders are endogenously matched through competitive markets.

\(^{18}\)For detailed descriptions of the shorting market, see, for example, Jones and Lamont (2001), D’Avolio (2002), and Duffie, Garleanu and Pedersen (2002).
2.6.1 Matching of Optimists and Moderates in Debt and Short Markets

Under appropriate parametric restrictions there exists a quasi-equilibrium in which traders take the following positions. First, optimists that can leverage, i.e., traders with type \( T_1 \equiv (1, -, 1) \), invest all of their wealth in the asset and they leverage as much as possible given their choice of contract \( \varphi \). Second, optimists that cannot leverage, i.e., traders with type \( T_2 \equiv (1, -, 0) \), invest all of their wealth either in the asset or the short contracts sold by moderates that can short sell. Third, moderates that can short sell, i.e., traders with type \( T_3 \equiv (0, 1, -) \), invest all of their wealth in the bond and they short sell as much as possible given their choice of contract \( \psi \). Fourth, moderates that cannot short sell, i.e., traders with type \( T_4 \equiv (0, 0, -) \), invest all of their wealth either in the bond or the debt contracts sold by type \( T_1 \) traders.

In other words, type \( T_1 \) optimists borrow from type \( T_4 \) moderates that cannot short sell, while type \( T_3 \) moderates borrow the asset \( A \) from type \( T_2 \) optimists that cannot leverage. To see the intuition for this matching, note that type \( T_3 \) moderates require a greater interest rate than type \( T_4 \) moderates to part with their wealth (i.e., to lend), because, in equilibrium, they receive a greater expected return on their wealth (since they have the ability to short sell). This implies that the debt contracts sold by type \( T_1 \) optimists are purchased by type \( T_4 \) moderates. A similar reasoning shows that the short contracts sold by type \( T_3 \) moderates are bought by type \( T_2 \) optimists.

Given this matching, the characterization of the quasi-equilibrium follows closely the analysis in Section 2.3. In particular, consider debt contract prices given by (2.9), which corresponds to the valuation of type \( T_4 \) moderates. Given these prices and the asset price \( p \in \left( \frac{E_0[\psi(s)]}{1+r}, \frac{E_1[\psi(s)]}{1+r} \right) \), Theorem 1 continues to apply. That is, type \( T_1 \) optimists choose to borrow and leverage with a single loan with riskiness \( \tilde{s}_e \in S \) that solves \( p = p^\text{opt}(\tilde{s}_e) \).

Similarly, note that type \( T_2 \) optimists must be indifferent between investing in the asset and the short contracts. Type \( T_2 \) optimists’ expected return from investing in the asset is given by \( \frac{E_1[\psi(s)]}{p} \). Thus, consider short contract prices:

\[
q^{\text{short}}(\psi) = \frac{1}{\frac{E_1[\psi(s)]}{p}}E_1[\min(\psi, 1)] \text{ for each } \psi \in \mathbb{R}_+.
\]

Given the prices in (2.25), type \( T_2 \) optimists absorb any potential supply of short contracts.
from type $T_3$ moderates. Hence, the equilibrium in the short contract market is determined by type $T_3$ moderates’ optimal contract choice. I next characterize the optimal short contract and show that a version of the asymmetric disciplining result applies also in this setting.

### 2.6.2 Asymmetric Disciplining with Short Selling

Given the prices in (2.25) and the asset price $p \in \left( \frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r} \right)$, type $T_3$ moderates short sell according to a unit short contract $\psi = v(\bar{s}_{sh})$ that defaults if the realized state is above some threshold state $\bar{s}_{sh} \in S$. This is because, for sufficiently good states, the value of the promised asset exceeds the value of the posted collateral, and the short seller finds it optimal to default.

The next result, which is the counterpart of Theorem 1 for short contracts, characterizes the threshold state $\bar{s}_{sh}$ for the optimal short contract.

**Theorem 6 (Asymmetric Disciplining with Short Selling).** Suppose assumption (MLRP) holds, short contract prices are given by (2.25) and the asset price satisfies $p \in \left( \frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r} \right)$. In a quasi-equilibrium:

(i) There exists $\bar{s}_{sh} \in S$ such that $\mu_{T_3}^{short,\bar{s}_{sh}}$ is a Dirac measure that puts weight only at the contract $\psi = v(\bar{s}_{sh})$, i.e., moderates (that are able) short sell only the unit short contract $\psi = v(\bar{s}_{sh}) / 1+r$. These moderates invest all of their wealth in the bond and short sell the asset as much as possible subject to the collateral constraint.

(ii) The threshold state $\bar{s}_{sh} \in S$ of the optimal short contract is characterized as the unique solution to:

$$p = p_{short}^{\bar{s}_{sh}}(\bar{s}_{sh}) \equiv \frac{E_1[v(s)] / (1+r)}{1 + F_0(\bar{s}_{sh}) \left( \frac{f_{s_{min}}^{s_{min}} v(s) dF_1}{f_{s_{min}}^{s_{min}} v(s) dF_0} \frac{f_{s_{max}}^{s_{max}} dF_1}{f_{s_{max}}^{s_{max}} dF_0} \right)}.$$  

Note that $p_{short}^{\bar{s}_{sh}}(\bar{s}_{sh})$ describes the price for which the short contract with default threshold $\bar{s}_{sh}$ is optimal. Under assumption (MLRP), this curve is strictly decreasing, with $p_{short}^{\bar{s}_{min}} (s_{min}) = \frac{E_1[v(s)]}{1+r}$ and $p_{short}^{\bar{s}_{max}} (s_{max}) = \frac{E_0[v(s)]}{1+r}$. Thus, there is a unique solution to (2.26).

It can also be seen that the function $p_{short}^{\bar{s}_{sh}}(\bar{s}_{sh})$ features an asymmetric disciplining property. To see this, suppose the moderate belief, $F_0$, is kept constant and the optimistic belief, $F_1$, is changed in a way to keep the optimistic valuation $\frac{E_1[v(s)]}{1+r}$ constant. By Eq. (2.26), the effect
of this type of change on \( p^{\text{short}}(\bar{s}_{\text{sh}}) \) is characterized by its effect on the expression:

\[
\frac{\int_{s_{\text{min}}}^{\bar{s}_{\text{sh}}} v(s) \, dF_1}{\int_{s_{\text{min}}}^{\bar{s}_{\text{sh}}} v(s) \, dF_0} - \frac{\int_{s_{\text{min}}}^{\bar{s}_{\text{sh}}} \frac{dF_1}{dF_0}}{\int_{s_{\text{min}}}^{\bar{s}_{\text{sh}}} \frac{dF_0}{dF_1}}.
\]  

(2.27)

By assumption (MLRP), this expression is always positive. Intuitively, both terms in the expression can be thought of as an “average” of the likelihood ratios \( \left( \frac{\frac{f_1(s)}{f_0(s)}}{s \in [s_{\text{min}}, \bar{s}_{\text{sh}}]} \right) \), with the term on the left putting relatively greater weight \( v(s) \) on the higher likelihood ratios \( \frac{f_1(s)}{f_0(s)} \) (corresponding to higher \( s \)). This intuition also suggest that a shift of optimism towards the relative likelihood of states above \( \bar{s}_{\text{sh}} \) decreases the expression in (2.27). In the most extreme case, if \( \frac{f_1(s)}{f_0(s)} \) is constant over \( s \in [s_{\text{min}}, \bar{s}_{\text{sh}}] \) (so that all the optimism is concentrated on the relative likelihood of states above \( \bar{s}_{\text{sh}} \)), then the expression in (2.27) is equal to zero. By Eq. (2.26), the function \( p^{\text{short}}(\bar{s}_{\text{sh}}) \) negatively depends the expression in (2.27). It follows that a shift of optimism towards the relative likelihood of states above \( \bar{s}_{\text{sh}} \) increases \( p^{\text{short}}(\bar{s}_{\text{sh}}) \). That is, for any given level of default threshold \( \bar{s}_{\text{sh}} \) for short contracts, the asset price is higher when optimism is concentrated more on the relative likelihood of good states. This illustrates the asymmetric disciplining property of the optimal short contract.

The proof of Theorem 6 is relegated to Appendix 2.A.6. For an intuition, note that the short contract defaults above the threshold state \( \bar{s}_{\text{sh}} \). Thus, they pay the same amount \( \psi = v(\bar{s}_{\text{sh}}) \) in these states. Then, using a short contract with threshold \( \bar{s}_{\text{sh}} \), it is impossible for moderates to bet on their pessimism about the relative likelihood of states above \( \bar{s}_{\text{sh}} \). Consequently, moderates’ pessimism about the relative likelihood of good states is not reflected in the asset price, as suggested by (2.26). In contrast, moderates can bet on their pessimism about the probability of states below \( \bar{s}_{\text{sh}} \) by selling the short contract. Thus, this type of pessimism is reflected in the asset price.

Put differently, it is easier for moderates to bet on their pessimism about the probability of bad states than to bet on their pessimism for the relative likelihood of good states. To bet on the latter types of pessimism, moderates need to post a higher level of collateral \( \psi \) (equivalently, they need to choose a short contract with a high default threshold \( \bar{s}_{\text{sh}} \)). Hence, these types of short sales are more difficult to leverage, which leads to the asymmetric disciplining result with short selling. Appendix 2.A.6 provides a more complete intuition that parallels the analysis in
Section 2.3.1.

2.6.3 Equilibrium Asset Price with Short Selling

The equilibrium is characterized by type $T_1$ optimists' and type $T_3$ moderates' optimal contract choice, along with the market clearing condition for the asset, which I derive next. To simplify the analysis, suppose the parameters are such that the equilibrium asset price satisfies $p \in \left( \frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r} \right)$. Note that type $T_1$ optimists spend a total of

$$
\gamma_{le} (w_1 + p\alpha_1) \frac{p}{p - \frac{E_0[\min(v(s),v(\delta_{le}))]}{1+r}}
$$

units of the consumption good on the asset. Here, recall that $\gamma_{le}$ is the fraction of investors that are able to leverage, $w_1 + p\alpha_1$ is the total wealth of optimists, and the second term in (2.28) is the leverage ratio. Next note that type $T_2$ optimists (that make an unleveraged investment in the asset) spend a total of

$$(1 - \gamma_{le}) (w_1 + p\alpha_1) - W_{short}
$$

units of the consumption good on the asset. Here, recall that type $T_2$ optimists are indifferent between buying the asset and buying the short contracts sold by moderates. Hence, they invest in the asset all of their wealth net of $W_{short}$, which represents their expenditure on short contracts.

By market clearing in short contracts, $W_{short}$ is also equal to type $T_3$ moderates' total revenue from sales of short contracts. The analysis in the appendix shows that this expression has a similar form to the expression in (2.28), and it is given by:

$$
W_{short} = \gamma_{sh} (w_0 + p\alpha_0) \frac{p}{E_1[\min(v(s),v(\delta_{sh}))]} - \frac{E_1[v(s)]}{1+r} - p
$$

(2.30)

Here, $\gamma_{sh} (w_0 + p\alpha_0)$ denotes the wealth of type $T_3$ moderates, and the second term denotes the short leverage ratio, that is, the total value of asset short sold per unit consumption good spending. Market clearing for the asset implies that the total spending on the asset, that is, the sum of the expressions in (2.28) and (2.29), is equal to the total value of the asset, $p$. After substituting for $W_{short}$ from the expression in (2.30) and rearranging terms, the asset market
clearing condition can be written as:

\[
\gamma_{le} \left( \frac{w_1 + p\alpha_1}{p - \frac{E_0[\min(v(s), v(s_{le})]]}{1+r}} \right) + (1 - \gamma_{le}) \left( \frac{w_1 + p\alpha_1}{p} \right) = 1 + \gamma_{sh} \left( \frac{w_0 + p\alpha_0}{\frac{E_1(v(s))}{E_1[\min(v(s), v(s_{sh})]]}} \right) - p.
\]  

(2.31)

This expression shows that short selling effectively expands the supply of the asset, as captured by the second term on the right hand side.

The equilibrium tuple \((p, \bar{s}_{le}, \bar{s}_{sh})\) is characterized by the optimality conditions \(p = p^{\text{short}}(\bar{s}_{sh})\), along with the market clearing condition (2.31). Note that an increase in the fraction of short sellers, \(\gamma_{sh}\), decreases the asset price because it increases the effective supply of the asset. Conversely, an increase in the fraction of leveraged investors, \(\gamma_{le}\), increases the asset price because it increases the demand for the asset, as captured by the left hand side of Eq. (2.31).

In addition, an increase in the right-skewness of optimism increases the asset price. To illustrate this effect, consider an equilibrium, \((p, \bar{s}_{le}, \bar{s}_{sh})\), and suppose optimists’ optimism is changed to \(\tilde{F}_1\) that satisfies \(\tilde{E}_1[v(s)] = E_1[v(s)]\) and \(\tilde{f}_1(s) = f_0(s)\) for each \(s \in [0, \bar{s}_{sh}]\). That is, the distribution \(\tilde{F}_1\) is “equally” optimistic as the distribution \(F_1\), but its optimism is concentrated to the right of the current short default threshold \(\bar{s}_{sh}\). By Eqs. (2.10) and (2.26), this change in the type of optimism leads to an increase in both default thresholds, \(\bar{s}_{le}\) and \(\bar{s}_{sh}\), given the old equilibrium price \(p\). Note also that the leverage ratio in (2.28) is increasing in \(\bar{s}_{le}\), and the short leverage ratio in (2.30) is decreasing in \(\bar{s}_{sh}\). Hence, at the old equilibrium price, this change increases optimists’ leverage ratio, while it decreases moderates’ leverage ratio. Consequently, the market clearing condition (2.31) implies that the equilibrium price increases. That is, an increase in this type of right-skewness of optimism increases the asset price also in the setting with short selling.

Intuitively, when optimism is more right-skewed, optimists leverage more by choosing larger and riskier loans (captured by the increase in \(\bar{s}_{le}\)), while short sellers leverage less by posting a greater amount of collateral \(\psi = v(\bar{s}_{sh})\) for each unit short contract (captured by the increase in \(\bar{s}_{sh}\)). This increases the demand and decreases the effective supply for the asset (cf. (2.31)), which leads to a higher equilibrium price.
2.7 Dynamic Model: Financing Speculative Bubbles

The analysis so far has concerned a two-period economy. However, the asymmetric disciplining result also has dynamic implications. This section considers a dynamic extension of the baseline setting to analyze the interaction of the asymmetric disciplining mechanism with the speculative component of asset prices identified by Harrison and Kreps (1978). The analysis in this section shows that the speculative "bubbles" are also asymmetrically disciplined by endogenous financial constraints. I first describe the basic environment without financial constraints and illustrate that the asset price features a speculative component. I then characterize the dynamic equilibrium with collateral constraints, and analyze the effect of belief heterogeneity on the speculative component.

2.7.1 Basic Dynamic Environment

Consider an infinite horizon overlapping generations economy in which the periods and generations are denoted by \( n \in \{0, 1, \ldots\} \). There is a continuum of traders in each generation \( n \), who are born in period \( n \) and live in periods \( n \) and \( n + 1 \). Each trader of generation \( n \) has an endowment of the consumption good in period \( n \), and consumes only in period \( n + 1 \). The resources can be transferred between periods by investing either in the bond \( B \) or the asset \( A \). Bond \( B \) is supplied elastically at a normalized price \( 1 \) in every period. Each unit of the bond yields \( 1 + r \) units of the consumption good in the next period, and then fully depreciates (i.e., the bond pays dividend only once). Asset \( A \) is in fixed supply, which is normalized to \( 1 \). The asset yields \( a_n \) units of dividends in each period \( n \). Suppose that log dividend follows a random walk, that is, the dividend yield follows the process

\[
a_{n+1} = a_n s_{n+1}.
\] (2.32)

Here, \( s_{n+1} \) is a random variable with distribution \( F_{true} \) which has a density function that is continuous and positive over \( S = [s_{\text{min}}, s_{\text{max}}] \subset \mathbb{R}_{++} \). Suppose also that \( 1 \in S \) and that the mean of \( s_{n+1} \) is normalized to \( 1 \). In other words, the next period dividend yield fluctuates around the current dividend yield \( a_n \), with expected value equal to \( a_n \).

All young traders in period \( n \) observe all past realizations of the dividend yield and the
current realization \( a_n \), but they have heterogeneous priors about the next period realization \( a_{n+1} \). In each period \( n \), similar to the static model, there are two types of young traders, *optimists* and *moderates*, respectively with priors \( F_1 \) and \( F_0 \) about the next period state \( s_{n+1} \).

**Assumption (O_d).** Period \( n \) young traders' belief distributions \( F_1 \) and \( F_0 \) for the next period state \( s_{n+1} \) have density functions \( f_1, f_0 \) that are continuous and positive over \( S \). The moderate belief distribution is given by \( F_0 = F_{true} \) while the optimistic distribution satisfies \( F_1 >_O F_0 \). In addition, traders' beliefs for the random variables \( s_{n+k} \), for \( k \geq 2 \), are identical and given by the true distribution \( F_{true} \).

One way to interpret this assumption is that all traders know the dividend yield process described in (2.32), but in every period, some traders (optimists) become optimistic regarding the next period realization. \(^{19}\) Under assumption (O_d), optimists’ expectation for the dividend yields in any future period is given by

\[
E_{n,1}[a_{n+k}] = E_{n,1}[a_{n+1}] = E_1[a_n](1 + \epsilon).
\]

Here, the parameter

\[
\epsilon \equiv E_{n,1}[s_{n+1}] - 1 > 0
\]

controls optimists’ level of optimism (recall that the true distribution has mean equal to 1). Consequently, optimists’ present discounted value of the future dividends can be calculated as

\[
P_1^{pdv}(a_n) = \sum_{k=1}^{\infty} \frac{E_{n,1}[a_{n+k}]}{(1 + r)^k} = \frac{a_n(1 + \epsilon)}{r}.
\]

Note that the moderate present discounted value is given by \( P_0^{pdv}(a_n) = a_n/r \). Thus, optimists’ overvaluation of the asset is given by \( \epsilon/r \). Intuitively, optimists expect the next period realization for the dividend yield to be higher, and they expect future dividend yields to fluctuate around this higher (expected) level. This leads to the valuation difference \( \epsilon/r \).

\(^{19}\)There could be a number of explanations for the source of this type of optimism. As in Scheinkman and Xiong (2003), optimists may be overconfident about a signal they receive about the next period shock. Alternatively, optimists may be simply optimistic about the next period shock, thinking that the current period is special. Reinhart and Rogoff (2008) refer to this type of optimism as “this time it is different syndrome.”
Similar to the baseline setting, short selling the asset is ruled out by assumption (S). Let \( (w_{i,n})_{i \in \{1,0\}} \) denote type \( i \) traders’ endowment of the consumption good, and suppose

\[
  w_{i,n} = \omega_i a_n, \text{ where } \omega_i \in \mathbb{R}_{++}.
\]

That is, young traders’ endowments are proportional to the current dividend yield of the asset. This assumption is not essential for the economic results, but it simplifies the subsequent analysis.\(^\text{20}\) This completes the description of the basic elements of the dynamic economy. Note that the economy has a recursive structure. This is because the dividend yield process follows a random walk (cf. Eq. (2.32)), and young traders’ beliefs are formed independently of the past dividend yield realizations (cf. assumption (O_d)). This observation leads to the following lemma, which provides a sufficient statistic for the dynamic economy and simplifies the subsequent notation.

**Lemma 1.** Given any history \( (a_0, ..., a_{n-1}, a_n) \) of dividend yield realizations, the current dividend yield \( a_n \) is a sufficient statistic for the determination of the equilibrium allocations in this economy.

In view of this lemma, let \( a = a_n \in \mathbb{R}_{++} \) denote the current dividend yield, \( s = s_{n+1} \in \mathcal{S} \) denote the next period shock, and \( p(a) \) denote the current asset price.

### 2.7.2 Speculative Bubbles without Financial Constraints

As a benchmark, I first consider the asset price in an economy in which individuals can borrow and lend freely in a competitive loan market at the benchmark rate \( r \). In other words, there exists no limited liability or enforcement problems. In this case, optimists borrow and invest in the asset an infinite amount whenever the asset price is below their valuation. Hence, the equilibrium asset price is equal to the optimistic valuation:

\[
  p(a) = \frac{1}{1+r} \left( a (1 + \varepsilon) + \int_\mathcal{S} p(as) dF_1 \right), \text{ for all } a \in \mathbb{R}_{++}.
\]

\(^{20}\)I thank Ivan Werning for suggesting this simplification.
The first term on the right hand side is optimists’ expected dividend payoff from the asset, and the second term is their expected payoff from the sale of the asset. Eq. (2.34) provides a recursive characterization of the asset price which can be solved as

\[ p(a) = \frac{a(1+\varepsilon)}{r-\varepsilon}. \] (2.35)

Note that the asset price \( p(a) \) is higher than optimists’ present discounted valuation, \( p_d^1(a) = \frac{a(1+\varepsilon)}{r} \). The component of the asset price in excess of the present discounted value of the holder of the asset, \( p(a) - p_d^1(a) \), is what Scheinkman and Xiong (2003) call a speculative “bubble.” I also define

\[ \lambda = \frac{p(a) - p_d^1(a)}{p(a)} = \frac{\varepsilon}{r} \] (2.36)

as the share of the speculative component. The asset price features a speculative component because optimists hold the asset not only for the higher expected dividend gains in the next period, but also since they are planning to sell the asset to a trader who will be even more optimistic than them in the next period. In view of these expected speculative capital gains, optimists bid up the asset price higher than the present discounted value of dividends.

The expression in (2.36) also implies that the speculative component could represent a large fraction of the asset price, even for a relatively small belief disagreement \( \varepsilon \) (especially when the interest rate is low). The rationale for this observation is related to a powerful amplification effect: the dynamic multiplier. Note that optimists in the next period also expect to make speculative capital gains by selling the asset to yet more optimistic traders in the subsequent period, which increases the price in the next period. But this further increases the valuation of current optimists who are planning to sell to future optimists, increasing the current asset price further. In other words, a high asset price in the next period feeds back into the asset price today, amplifying the effect of heterogeneous beliefs and leading to a large speculative component.

I next incorporate financial constraints into this economy. With financial constraints, the asset price does not necessarily satisfy the recursion in (2.34). Rather, the asset price lies between the optimistic and the moderate valuations, and the exact recursion (and the share of the speculative component) is determined by the type of financial constraints.
2.7.3 Financial Frictions and Dynamic Collateral Equilibrium

I model financial constraints using the collateral equilibrium described in Section 2.3.1. In particular, traders in each generation trade collateralized debt contracts that mature in the next period. As in the baseline setting, debt contracts are non-recourse and non-contingent. Formally, a unit debt contract, denoted by \( \varphi \in \mathbb{R}_+ \), is a promise of \( \varphi \) units of the consumption good in the next period by the borrower collateralized by 1 unit of the asset. Given the current dividend realization \( a \), I define the value function as the payoff of the asset in the next period,

\[
v(a, s) \equiv as + p(as) \quad \text{for each } s \in S.
\] (2.37)

The debt contract \( \varphi \) defaults if and only if \( v(a, s) < \varphi \), and thus it pays \( \min(v(a, s), \varphi) \). Each debt contract \( \varphi \in \mathbb{R}_+ \) is traded in an anonymous market at a competitive price \( q(a, \varphi) \).

Let \( x_i^A(a) \), \( x_i^B(a) \) denote type \( i \) traders’ asset and debt holding, and \( \mu_i^+(a) \), \( \mu_i^-(a) \) denote their long and short debt portfolios. The traders’ problem is given by:

\[
\begin{align*}
\max_{x_i(a) \geq 0, \\ \mu_i^+(a), \mu_i^-(a)} & \quad x_i^A(a) E_i [v(a, s)] + x_i^B(a) (1 + \tau) + \\
& \quad \int_{\mathbb{R}_+} E_i [\min(v(a, s), \varphi)] d\mu_i^+(a, \varphi) - \int_{\mathbb{R}_+} E_i [\min(v(a, s), \varphi)] d\mu_i^-(a, \varphi) \\
\text{s.t.} & \quad px_i^A(a) + x_i^B(a) + \int_{\mathbb{R}_+} q(a, \varphi) d\mu_i^+(a, \varphi) - \int_{\mathbb{R}_+} q(a, \varphi) d\mu_i^-(a, \varphi) \leq \omega_i a, \\
& \quad \int_{\mathbb{R}_+} d\mu_i^-(a, \varphi) \leq x_i^A(a).
\end{align*}
\] (2.38)

**Definition 4 (Dynamic Collateral Equilibrium).** Under assumptions \((O_d)\) and \((S)\), and condition \((2.A.50)\), a dynamic collateral equilibrium is a collection of prices \( (p(a), [q(a, \varphi)]_{\varphi \in \mathbb{R}_+})_{a \in \mathbb{R}_{++}} \) and allocations \( (x_i^A(a), x_i^B(a), \mu_i^+(a), \mu_i^-(a))_{i \in \{1, 0\}} \) such that, for each dividend realization \( a \in \mathbb{R}_{++} \), the allocation of each trader \( i \in \{1, 0\} \) solves problem (2.38), and asset and unit debt markets clear.

Note that, given the value function in the next period (cf. Eq. (2.37)), the economy in the current period is very similar to the static economy analyzed earlier, with the main difference that the value function, (2.37), also depends on the price function. Hence, the dynamic equilibrium is characterized with a fixed point argument. The linear homogeneity of
endowments (cf. condition (2.33)) ensures that the price to dividend ratio and the loan riskiness are independent of the current realization of \(a\). The proof of the following theorem is relegated to Appendix 2.A.7.

**Theorem 7 (Existence and Characterization of Dynamic Equilibrium).** Under assumptions \((O_d), (S)\) and the parametric condition \((2.A.50)\) in Appendix 2.A.7, there exists a recursive collateral equilibrium in which \(p(a) = p_{da}\) and \(\bar{s}^* (a) = \bar{s}_d^*\) for each \(a \in \mathbb{R}^+\). The price to dividend ratio, \(p_d\), is the unique fixed point of the mapping \(P_d : \left[ p_d^{\min} = \frac{1}{r}, p_d^{\max} = \frac{1+\epsilon}{r-\epsilon} \right] \rightarrow \left[ p_d^{\min}, p_d^{\max} \right]\), where \(P_d(\bar{p}_d)\) is the collateral equilibrium price of the static economy.

\[
E (p_d) = \left( S; v (s \mid p_d) = s (1 + p_d); \right. \left. \{F_i\}_i; \{\omega_i = \omega\}_i; \{\alpha_1 = 0, \alpha_0 = 1\} \right). 
\] (2.39)

Note that this result reduces the characterization of the dynamic equilibrium to the characterization of the equilibrium for the static economy, \(E (p_d)\), along with a fixed point argument. Intuitively, \(P_d(\bar{p}_d)\) is the price to dividend ratio that would obtain today if the future price to dividend ratio was given by \(\bar{p}_d\). The upper limit of the fixed point interval, \(p_d^{\max} = \frac{1+\epsilon}{r-\epsilon}\), is the price to dividend ratio that would obtain if optimists always priced the asset (i.e., it is the price in the unconstrained economy). The lower limit, \(p_d^{\min} = \frac{1}{r}\), is the price to dividend ratio that would obtain if moderates always priced the asset (i.e., it is the moderate valuation of the asset). The equilibrium is in the interval \([p_d^{\min}, p_d^{\max}]\). The next example uses this characterization to illustrate the effect of financial constraints on the speculative component of the asset price.

**Example 3.** Consider the prior distributions \(F_0\) and \(F_{1,G}\) of Example 1 in which the valuation difference for the next period shock is given by \(\epsilon = E_1 [s] - E_0 [s] = 0.1\). Consider the corresponding dynamic collateral equilibrium with interest rate \(r = 0.15\) and optimistic wealth \(\omega_1 = 4\). Figure 2-6 plots the price mapping, \(P_d (\cdot), \) and shows that it intersects the 45 degree line exactly once, which corresponds to the equilibrium. The equilibrium price is lower than the unconstrained level, however it is still higher than the present discounted value according to either the moderate or optimistic priors (which are close to each other). In particular, in this example, the price has a large speculative component despite financial constraints.

The figure also illustrates optimists’ balance sheet. Optimists’ downpayment is about \(1/4\) of
Figure 2-6: The x axis is the range of possible price to dividend ratios, \([p_{min}, p_{max}]\). The lower and higher green curves respectively plot the moderate and the optimistic valuations when the future price to dividend ratio is given by the value at the x axis. The red curve (intermediate to the two green curves) plots the price mapping, \(P^d(p_d)\). The equilibrium is the intersection of the red curve with the 45 degree line (dashed blue curve).
the asset price, and they borrow the remaining amount from moderates, collateralized against one unit of the asset. In particular, moderate lenders, who correctly know the dividend yield process in (2.32), agree to finance about 3/4 of the asset purchase despite the fact that the present discounted value of the asset is less than half of its price.

The last feature of this example provides insights for how the price can feature a large speculative component when optimists are financially constrained. In this example, lenders have correct priors and they know that the asset price is considerably greater than their present discounted valuation. Nonetheless, they agree to extend large loans which are in part collateralized by the speculative component of the price. This is because lenders’ valuation of the asset (the lower green line in Figure 2-6) also contains a speculative component, and thus it is higher than their present discounted valuation. Intuitively, lenders agree to extend large loans because they think that, should the borrower default, they could always sell the collateral to another optimist in the next period.

Put differently, a marked characteristic of this speculative episode is that the bubble raises all boats: both the optimistic and the moderate valuations are greater than their present discounted valuations. Consequently, optimists’ and moderates’ valuation difference in any period (the difference between the two green lines in Figure 2-6) is relatively small. As in the unconstrained case, a large speculative bubble forms from the accumulation of small valuation differences through the dynamic multiplier. This is perhaps unfortunate, because a small valuation difference makes the financing of the asset relatively easy, opening the way for large speculative bubbles even when optimists are financially constrained.

Naturally, as the previous sections show, a small valuation difference does not guarantee that financial constraints are lax. Whether financing will actually go through, and the share of the speculative component, also depends on a number of other factors, such as optimists’ wealth level and the type of belief heterogeneity. For example, consider the equilibrium in Example 3 with the only difference that the optimistic priors are changed to $F_{1,B}$ (defined in Example 1). This prior leads to the same asset valuation, but it is more left-skewed than $F_{1,G}$. Figure 2-7 shows that, in response to this change, the speculative component shrinks by about half.

The next result shows that this is a general property, that is, an increase in the right-skewness of optimism unambiguously increases the asset price and the share of the speculative
component. To state the result, I define the overvaluation ratio $\theta_d \in (0, 1]$ as the unique solution to

$$p_d = (1 - \theta_d) \frac{E_0 [v_d (\cdot | p_d)]}{1 + r} + \theta_d \frac{E_1 [v_d (\cdot | p_d)]}{1 + r}. \quad (2.40)$$

Intuitively, $\theta_d$ captures the fraction of the optimism in prior beliefs that is reflected in the asset price. I generalize the speculative component of the asset price (cf. Eq. (2.36) to the financially constrained economy) as

$$\lambda_d = \frac{p(a) - p^{pdv}(a)}{p(a)}, \text{ where } p^{pdv}(a) = (1 - \theta_d) p_0^{pdv}(a) + \theta_d p_1^{pdv}(a). \quad (2.41)$$

Unlike the unconstrained case, the marginal holder of the asset is not necessarily an optimist, hence the relevant present discounted value is defined as an average of optimistic and moderate present discounted values, weighted by the overvaluation ratio $\theta_d$.\footnote{The share of the speculative premium is independent of the state $a \in \mathbb{R}_{++}$ because the functions $p(a), p_0^{pdv}(a), \text{ and } p_1^{pdv}(a)$ are linearly homogeneous in $a$.} The following result establishes that an increase in the right-skewness of belief heterogeneity increases the asset price and the share of the speculative component.
Theorem 8 (Effect of Type of Heterogeneity on the Speculative Component). Consider the recursive collateral equilibrium characterized in Theorem 7 and let $\bar{s}^*_d$ denote the equilibrium loan riskiness.

(i) If optimists' optimism becomes weakly more right-skewed, i.e., if their prior is changed to $\tilde{F}_1$ that satisfies $\tilde{F}_1 \succeq_R F_1$ and $\tilde{F}_1 \succ_o F_0$ (so that assumption $(O_d)$ continues to hold), then: the price to dividend ratio $p_d$, the loan riskiness $\bar{s}^*_d$, and the share of the speculative component $\lambda_d$ weakly increase.

(ii) If moderates' optimism becomes weakly more skewed to the left of $\bar{s}^*_d$, i.e., if their prior is changed to $\tilde{F}_0$ that satisfies $F_0 \succeq_R \bar{s}^*_d \tilde{F}_0$ and $F_1 \succ_o \tilde{F}_0$, then: the price to dividend ratio $p_d$ and the share of the speculative component $\lambda_d$ weakly increase.

Intuitively, if optimists' optimism becomes more right-skewed, then future optimists will perceive looser financial constraints and they will be able to bid up the asset price higher. This implies that the resale option value to future optimists is higher, which leads to a greater speculative component. Conversely, if optimists' optimism becomes more left-skewed, then the speculative component becomes smaller because the future optimists will perceive tighter financial constraints. This result shows that bubbles can come to an end because of a shift in belief heterogeneity towards the likelihood of bad events.

2.8 Conclusion

In this essay, I have theoretically analyzed the effect of belief heterogeneity on asset prices. The central feature of the model is that, to take positions in line with their beliefs, investors need to borrow from traders with different beliefs using collateralized contracts. The lenders do not value the collateral as much as the borrowers do, which represents a constraint on investors' ability to borrow and leverage their investments. I have considered the effect of this constraint on asset prices in a variety of settings that differ in the types of collateralized contracts that are available for trade. In the baseline model, I have restricted attention to non-contingent loans and disallowed short selling, and I have relaxed these restrictions in two extensions of the model. In each of these scenarios, my essay has established that optimism is asymmetrically disciplined by endogenous financial constraints. In particular, optimism about the likelihood of bad states
has a smaller effect on asset prices than optimism about the relative likelihood of good states. I have also considered a dynamic extension of the model which reveals that the speculative asset price bubbles, identified by Harrison and Kreps (1978), are also asymmetrically disciplined by optimists’ financial constraints.

Taken together, my results suggest that certain economic environments that generate uncertainty (and thus belief heterogeneity) about upside returns are conducive to asset price increases and speculative bubbles financed by credit. This prediction is in line with the observations in Kindleberger (1978), who has argued that speculative episodes typically follow a novel event (which arguably generates upside uncertainty), and that the easy availability of credit plays an important role in these episodes.

The asymmetric disciplining characterization of asset prices also emphasizes that what investors disagree about matters for asset prices, to a greater extent than the level of the disagreement. In particular, when optimists are financially constrained, an increase in the level of belief heterogeneity in general has ambiguous effects on asset prices. However, the effect can be characterized once the skewness of the increase is taken into account. Additional belief heterogeneity tends to decrease asset prices when it concerns the likelihood of bad states, but it tends to increase asset prices when it concerns the relative likelihood of good states. A growing empirical literature in finance considers the effect of the level of belief heterogeneity on asset prices and subsequent asset returns (e.g., Chen, Hong and Stein, 2001, Diether, Malloy and Scherbina, 2002, and Ofek and Richardson, 2003). My essay suggests that a fruitful future research direction may be to empirically investigate the effect of the skewness of the belief heterogeneity on asset prices.
2. A. Appendices

2. A.1 Properties of Optimism Order

This appendix establishes the properties of optimism order (cf. Definition (1)). Consider two probability distributions $H, \tilde{H}$ over $S = [s_{\text{min}}, s_{\text{max}}] \subset R$ with corresponding density functions $h, \tilde{h}$ that are continuous and positive at each $s \in S$.

I first show that $\frac{1 - \tilde{H}(s)}{1 - H(s)}$ is strictly increasing at some $s \in S$ if and only if the hazard rate inequality in (2.2) is satisfied. To see this, consider the derivative of $\frac{1 - \tilde{H}(s)}{1 - H(s)}$

$$\frac{d}{ds} \frac{1 - \tilde{H}(s)}{1 - H(s)} = \frac{-h(s)(1 - H(s)) + h(s)(1 - \tilde{H}(s))}{(1 - H(s))^2}, \text{ for each } s \in [s_{\text{min}}, s_{\text{max}}],$$

and note that this expression is positive if and only if the hazard rate inequality (2.2) holds.

I next show that the optimism order is weaker than the monotone likelihood ratio property (MLRP), that is, if $\frac{h(s)}{h(\tilde{s})}$ is strictly increasing over $S$, then $\tilde{H} \succ_0 H$. To see this, suppose (MLRP) holds and note that this implies, for each $s < s_{\text{max}}$,

$$\frac{\tilde{h}(s)}{h(s)} < \frac{\tilde{h}(\tilde{s})}{h(\tilde{s})} \text{ for all } \tilde{s} \in (s, s_{\text{max}}).$$

Integrate both sides of this equation over $(s, s_{\text{max}})$ to get

$$\frac{\tilde{h}(s)}{h(s)} (1 - H(s)) < \left(1 - \tilde{H}(s)\right),$$

which proves the hazard rate inequality (2.2) and shows that $\tilde{H} \succ_0 H$.

I next note the following result, which derives the implications of assumption (O) for the key variables used in the analysis, including the expected payoff of a loan with riskiness $\tilde{s}$, $E_i \min (v(s), v(\tilde{s}))$, the perceived interest rate, $r_i^{\text{per}}(\tilde{s})$, and the optimality curve, $p^{\text{opt}}(\cdot)$.

Lemma 2. Consider two probability distributions $F_1$ and $F_0$ that satisfy assumption (O).

(i) The expected payoff of a loan with riskiness $\tilde{s}$, $E_i \min (v(s), v(\tilde{s}))$, is strictly increasing in $\tilde{s}$.

(ii) Optimists' perceived interest rate $r_i^{\text{per}}(\tilde{s})$ (cf. Eq. (2.12)) is strictly increasing in $\tilde{s}$. In
particular, $r_{1}^{\text{per}}(\bar{s}) > r_{1}^{\text{per}}(s^{\text{min}}) = r$ for each $\bar{s} > s^{\text{min}}$.

(iii) $\rho^{\text{opt}}(\bar{s})$ is continuously differentiable and strictly decreasing, i.e., $\frac{d\rho^{\text{opt}}(\bar{s})}{d\bar{s}} < 0$.

Proof of Lemma 2. Part (i). Note that the derivative of $E_{1}[\min(v(s),v(\bar{s}))] = \int_{s^{\text{min}}}^{\bar{s}} v(s) dF_{1}(s) + v(\bar{s}) (1 - F_{1}(\bar{s}))$ is given by

$$
\frac{dE_{1}[\min(v(s),v(\bar{s}))]}{d\bar{s}} = v(\bar{s}) f_{1}(\bar{s}) + v'(\bar{s}) (1 - F_{1}(\bar{s})) - v(\bar{s}) f(\bar{s}) = v'(\bar{s}) (1 - F_{1}(\bar{s})) > 0,
$$

which completes the proof.

Part (ii). The derivative of $\frac{1 + r_{1}^{\text{per}}(\bar{s})}{1 + r}$ can be calculated as

$$
\frac{d}{d\bar{s}} \left( \frac{1 + r_{1}^{\text{per}}(\bar{s})}{1 + r} \right) = \frac{dE_{1}[\min(v(s),v(\bar{s}))]}{E_{0}[\min(v(s),v(\bar{s}))]} \frac{E_{0}[\min(v(s),v(\bar{s}))] - E_{1}[\min(v(s),v(\bar{s}))]}{d\bar{s}}
$$

$$
= \frac{E_{0}[\min(v(s),v(\bar{s}))] (1 - F_{1}(\bar{s}) - E_{1}[\min(v(s),v(\bar{s}))] (1 - F_{0}(\bar{s}))}{(E_{0}[\min(v(s),v(\bar{s}))])^{2}},
$$

where the last line uses Eq. (2.A.1).

I next claim that

$$
\frac{E_{1}[\min(v(s),v(\bar{s}))]}{E_{0}[\min(v(s),v(\bar{s}))]} < \frac{1 - F_{1}(\bar{s})}{1 - F_{0}(\bar{s})} \text{ for each } \bar{s} \in (s^{\text{min}}, s^{\text{max}}),
$$

which, in view of Eq. (2.A.2), proves that the perceived interest rate $1 + r_{1}^{\text{per}}(\bar{s})$ is strictly increasing.

To prove the claim, note that for each $\bar{s} \in (s^{\text{min}}, s^{\text{max}})$,

$$
\frac{E_{1}[\min(v(s),v(\bar{s}))]}{E_{0}[\min(v(s),v(\bar{s}))]} = \frac{\int_{s^{\text{min}}}^{\bar{s}} v(s) dF_{1}(s) + v(\bar{s}) (1 - F_{1}(\bar{s}))}{\int_{s^{\text{min}}}^{\bar{s}} v(s) dF_{0}(s) + v(\bar{s}) (1 - F_{0}(\bar{s}))}
$$

$$
< \frac{\int_{s^{\text{min}}}^{\bar{s}} v(s) \frac{1 - F_{1}(s)}{1 - F_{0}(s)} dF_{0}(s) + v(\bar{s}) (1 - F_{1}(\bar{s}))}{\int_{s^{\text{min}}}^{\bar{s}} v(s) dF_{0}(s) + v(\bar{s}) (1 - F_{0}(\bar{s}))}
$$

$$
= \frac{\int_{s^{\text{min}}}^{\bar{s}} v(s) dF_{0}(s) \frac{1 - F_{1}(s)}{1 - F_{0}(s)} + v(\bar{s}) (1 - F_{1}(\bar{s}))}{\int_{s^{\text{min}}}^{\bar{s}} v(s) dF_{0}(s) + v(\bar{s}) (1 - F_{0}(\bar{s}))} = \frac{1 - F_{1}(\bar{s})}{1 - F_{0}(\bar{s})},
$$

where the first inequality uses the hazard rate inequality (2.2) and the second inequality uses the fact that $\frac{1 - F_{1}(s)}{1 - F_{0}(s)}$ is strictly increasing. This proves the claim in (2.A.3) and completes the
proof of this part.

Part (iii). Using the definition of $p^{opt}(\bar{s})$ in Eq. (2.10), note that

$$\frac{dp^{opt}(\bar{s})}{d\bar{s}} = \frac{1}{1 + r} \left( v(\bar{s}) f_0(\bar{s}) + \left( -f_0(\bar{s}) + f_1(\bar{s}) \frac{1-F_0(\bar{s})}{1-F_1(\bar{s})} \right) \left( \frac{s_{\text{max}}}{\bar{s}} v(s) \frac{dF_1}{1-F_1(\bar{s})} \right) \right)$$

$$= -\frac{1}{1 + r} \left( \frac{f_0(\bar{s})}{1-F_0(\bar{s})} - \frac{f_1(\bar{s})}{1-F_1(\bar{s})} \right) (1 - F_0(\bar{s})) (E_1[v(s) | s \geq \bar{s}] - v(\bar{s}))$$

where the first line applies the chain rule and the second line substitutes $E_1[v(s) | s \geq \bar{s}]$ and rearranges terms. The term, $\left( \frac{f_0(\bar{s})}{1-F_0(\bar{s})} - \frac{f_1(\bar{s})}{1-F_1(\bar{s})} \right)$, in Eq. (2.14) is positive from the hazard rate inequality (2.2). Since the terms, $(1 - F_0(\bar{s}))$ and $(E_1[v(s) | s \geq \bar{s}] - v(\bar{s}))$, are also positive, it follows that $\frac{dp^{opt}(\bar{s})}{d\bar{s}} < 0$, completing the proof of the lemma.

I next present the final result of this appendix, which uses assumption (O) to derive the effects of an increase in optimists' (moderates') optimism on the curves $p^{opt}(\cdot)$ and $p^{mc}(\cdot)$.

**Lemma 3.** Consider two probability distributions $F_1$ and $F_0$ that satisfy assumption (O).

(i) Suppose optimists become weakly more optimistic, i.e., consider their beliefs are changed to $F_\tilde{1} \succeq O F_1$. Then:

(i.1) Conditional expectations increase, that is, $\bar{E}_1[v(s) | s \geq \bar{s}] \geq E_1[v(s) | s \geq \bar{s}]$ for each $\bar{s} \in [s_{\text{min}}, s_{\text{max}}]$.

(i.2) The optimality curve $p^{opt}(\bar{s})$ shifts up pointwise, that is,

$$p^{opt}(\bar{s} ; F_\tilde{1}) \geq p^{opt}(\bar{s} ; F_1) \text{ for each } \bar{s} \in [s_{\text{min}}, s_{\text{max}}].$$

(i.3) The market clearing curve changes as follows:

$$p^{mc}(\bar{s} ; F_\tilde{1}) = \begin{cases} p^{mc}(\bar{s} ; F_1) & \text{if } p^{mc}(\bar{s} ; F_1) < E_1[v(s)] \left( \frac{1}{1+r} \right) \\ \geq p^{mc}(\bar{s} ; F_1) & \text{if } p^{mc}(\bar{s} ; F_1) = E_1[v(s)] \left( \frac{1}{1+r} \right) \end{cases}$$

(ii) Suppose moderates become weakly more optimistic, i.e., consider their beliefs are changed to $F_0 \succeq O F_0$ (which also satisfies $F_1 \succeq O \tilde{F}_0$ so that assumption (O) continues to hold). Then:

\[ \text{Throughout the appendices, the notation } \bar{E}_1[s | s \geq \bar{s}] \text{ corresponds to the conditional expectation of optimists according to the belief distribution } F_\tilde{1}. \]
(ii.1) The optimality curve $p^{opt}(\bar{s})$ shifts up pointwise, that is,

$$p^{opt}(\bar{s} ; \tilde{F}_0) \geq p^{opt}(\bar{s} ; F_0) \text{ for each } \bar{s} \in [s_{\text{min}}, s_{\text{max}}].$$

(ii.2) The market clearing curve $p^{mc}(\bar{s})$ shifts up pointwise, that is,

$$p^{mc}(\bar{s} ; \tilde{F}_1) \geq p^{mc}(\bar{s} ; F_1) \text{ for each } \bar{s} \in [s_{\text{min}}, s_{\text{max}}].$$

Proof of Lemma 3. Part (i.1). Define the function $g : S \rightarrow R$ with

$$g(s) = \tilde{E}_1[v(s) | s \geq \bar{s}] - E_1[v(s) | s \geq \bar{s}]. \quad (2.A.6)$$

Note that $g(s_{\text{max}}) = 0$, and note also that the statement in the lemma is equivalent to the following claim:

$$g(\bar{s}) \geq 0 \text{ for each } \bar{s} \in [s_{\text{min}}, s_{\text{max}}]. \quad (2.A.7)$$

I will first find an upper bound for the derivative of $g(\bar{s})$ which I will then use to prove the claim in (2.A.7).

To put an upper bound on the derivative of $g(\bar{s})$, consider first the derivative of the conditional expectation $E_1[v(s) | s \geq \bar{s}]$ at some $\bar{s} \in [s_{\text{min}}, s_{\text{max}}]$. With some rearrangement, this derivative can be written as

$$\frac{d}{ds} E_1[v(s) | s \geq \bar{s}] = \frac{f_1(s)}{1 - F_1(s)} (E_1[v(s) | s \geq \bar{s}] - v(s)).$$

Using this expression, the derivative of $g(\bar{s})$ can be written as

$$g'(\bar{s}) = \frac{\tilde{f}_1(\bar{s})}{1 - \tilde{F}_1(\bar{s})} (\tilde{E}_1[v(s) | s \geq \bar{s}] - v(\bar{s})) - \frac{f_1(\bar{s})}{1 - F_1(\bar{s})} (E_1[v(s) | s \geq \bar{s}] - v(\bar{s}))$$

$$= \left( \frac{\tilde{f}_1(\bar{s})}{1 - \tilde{F}_1(\bar{s})} - \frac{f_1(\bar{s})}{1 - F_1(\bar{s})} \right) (\tilde{E}_1[v(s) | s \geq \bar{s}] - v(\bar{s})) + \frac{f_1(\bar{s})}{1 - F_1(\bar{s})} g(\bar{s}), \quad (2.A.8)$$

where the second line follows by rearranging terms and substituting the definition of $g(\bar{s})$ from Eq. (2.A.6). Note that $\tilde{E}_1[v(s) | s \geq \bar{s}] - v(\bar{s}) > g(\bar{s})$ and the first term in Eq. (2.A.8) is always non-positive (since $\tilde{F}_1 \succeq_0 F_1$), which provides the following upper bound on the derivative of
Next, to prove the claim in (2.A.7), suppose the contrary, that is, suppose there exists \( \tilde{s} < s^{\text{max}} \) such that \( g(\tilde{s}) < 0 \). Consider next

\[
\tilde{s} = \sup \{ \tilde{s} \in [\tilde{s}, s^{\text{max}}) \mid g(\tilde{s}) \leq g(s) \}.
\]

Note that \( \tilde{s} \) exists and that \( g(\tilde{s}) = g(s) < 0 \) by the continuity of the function \( g(\cdot) \). This further implies that \( \tilde{s} \neq s^{\text{max}} \) since \( g(s^{\text{max}}) = 0 \). Then, Eq. (2.A.9) applies for \( \tilde{s} \) and implies

\[
g'(\tilde{s}) \leq \frac{\tilde{f}_1(\tilde{s})}{1 - \tilde{F}_1(\tilde{s})} g(\tilde{s}) < 0.
\]

This further implies that there exists \( \tilde{s} \in (\tilde{s}, s^{\text{max}}) \) such that \( g(\tilde{s}) < g(\tilde{s}) = g(s) \), which contradicts the definition of \( \tilde{s} \). This proves the claim in (2.A.7) and completes the proof of the first part.

Part (i.2). Note, by Eq. (2.18), that the optimality curve \( p^{\text{opt}}(\tilde{s}) \) can be written as

\[
p^{\text{opt}}(\tilde{s}) = \frac{1}{1 + r} \left( E_0 [v(s)] + (1 - F_0(\tilde{s})) (E_1 [v(s) \mid s \geq \tilde{s}] - E_0 [v(s) \mid s \geq \tilde{s}]) \right).
\]

Then, using part (i.1) shows that \( p^{\text{opt}}(\tilde{s}) \) shifts up pointwise, completing the proof.

Part (i.3). Consider the definition of \( w^{\text{max}}(\tilde{s}) \) in (2.13) and note that \( w^{\text{max}}(\tilde{s}) \) does not depend on \( F_1 \), as it depends on moderates’ valuation of debt contracts. Eq. (2.A.5) then follows by the definition of \( p^{\text{mc}}(\tilde{s}) \) in (2.14). Intuitively, the change, \( \tilde{F}_1 \geq F_1 \), only affects \( p^{\text{mc}}(\tilde{s}) \) by increasing optimists’ valuation. Thus, it only shifts the \( p^{\text{mc}}(\tilde{s}) \) curve in case (ii) region of Eq. (2.14), while it leaves it constant in other cases.

Part (ii.1). Similar to part (i.1) of the lemma, define the function \( g_{\text{mix}} : S \to R \) with

\[
g_{\text{mix}}(\tilde{s}) = p^{\text{opt}}(\tilde{s}; \tilde{F}_0) - p^{\text{opt}}(\tilde{s}; F_0).
\]

Note that the statement in the lemma is equivalent to the claim:

\[
g_{\text{mix}}(\tilde{s}) \geq 0 \text{ for each } \tilde{s} \in (s^{\text{min}}, s^{\text{max}}).
\]
I will prove a stronger claim, that

\[
\frac{dg_{mix}(\bar{s})}{d\bar{s}} \geq 0 \text{ for each } \bar{s} \in (s^{\min}, s^{\max}) ,
\]

(2.A.11)

which implies the claim in (2.A.10) since \( g_{mix}(s^{\min}) = 0 \).

To prove the claim in (2.A.11), note that using Eq. (2.A.4) and rearranging terms, the derivative of \( g_{mix}(\bar{s}) \) can be written as

\[
\frac{dg_{mix}(\bar{s})}{d\bar{s}} = \frac{1}{1+r} \left[ \frac{f_0(\bar{s}) - \tilde{f}_0(\bar{s})}{1-F_1(\bar{s})} \right] \left( E_1[v(s) | s \geq \bar{s}] - v(\bar{s}) \right). 
\]

(2.A.12)

Next note that \( F_1 \geq_0 \tilde{F}_0 \geq_0 F_0 \) implies \( \frac{f_0(\bar{s})}{1-F_0(\bar{s})} \geq \frac{f_0(\bar{s})}{1-F_0(\bar{s})} \geq \frac{f_1(\bar{s})}{1-F_1(\bar{s})} \). After rearranging terms, this further implies

\[
\frac{f_0(\bar{s}) - \tilde{f}_0(\bar{s})}{\tilde{F}_0(\bar{s}) - F_0(\bar{s})} \geq \frac{\tilde{f}_0(\bar{s})}{1 - \tilde{F}_0(\bar{s})} \geq \frac{f_1(\bar{s})}{1 - F_1(\bar{s})}.
\]

Using this inequality in Eq. (2.A.12) and noting that \( E_1[v(s) | s \geq \bar{s}] - v(\bar{s}) \geq 0 \) proves the claim in (2.A.11), completing the proof of the lemma.

Part (ii.2). First note that, applying the argument in part (iii) of Lemma 2 for the distributions \( \tilde{F}_0 \geq_0 F_0 \) implies

\[
\tilde{E}_0[\min(v(s), v(\bar{s}))] \geq E_0[\min(v(s), v(\bar{s}))] \text{ for each } \bar{s} \in S.
\]

By Eq. (2.13), this further implies \( w^\max_1(\bar{s} ; \tilde{F}_0) \geq w^\max_1(\bar{s} ; F_0) \). Using this inequality and the fact that \( \tilde{E}_0[v(s)] \geq E_0[v(s)] \), Eq. (2.14) implies that \( p^mc(\bar{s}) \) shifts up pointwise, completing the proof.

2.A.2 Characterization of Quasi-equilibrium

This section completes the analysis of the quasi-equilibrium, by providing the proofs for Theorem 1 and Eq. (2.14).

**Proof of Theorem 1.** I prove the theorem in two steps. I first show that \( \varphi = v(\bar{s}) \in \text{supp}(\mu_1^-) \) only if \( \bar{s} \) maximizes the leveraged return expression in (2.11). I then show that the
problem has a unique solution characterized as the solution to Eq. (2.10). This establishes
that $\mu^-$ is a Dirac measure at the contract $v(s)$, completing the sketch proof provided after the
theorem statement.

To prove the first step, first note that optimists’ debt contract choice can be restricted
to $\varphi \in \left[v(s_{\min}), v(s_{\max})\right]$ without loss of generality, i.e., suppose $\mu^-(C) = 0$ for each $C \subset \mathbb{R}^+ \setminus \left[v(s_{\min}), v(s_{\max})\right]$. Consider the change of notation $\tilde{s} = v^{-1}(\varphi)$ and let $\eta$ denote the pushforward measure of $\mu^-$ over $S = [s_{\min}, s_{\max}]$, i.e.,

$$
\eta\left(\tilde{S}\right) = \mu^-\left(v\left(\tilde{S}\right)\right) \quad \text{for each Borel set } \tilde{S} \subset S.
$$

(2.A.13)

Using this notation, and after substituting the debt prices from Eq. (2.9), optimists’ problem
in a quasi-equilibrium can be written as:

$$
\max_{x_1^A \geq 0, \eta \in M([s_{\min}, s_{\max}])} x_1^A E_1\left[v(s)\right] - \int_{[s_{\min}, s_{\max}]} E_1\left[\min\left(v(s), v(\tilde{s})\right)\right] d\eta(\tilde{s}),
$$

s.t.

$$
p x_1^A - \int_{[s_{\min}, s_{\max}]} \frac{E_0\left[\min\left(v(s), v(\tilde{s})\right)\right]}{1 + r} d\eta(\tilde{s}) \leq w_1 + p \alpha_1,
$$

$$
\int_{[s_{\min}, s_{\max}]} d\eta(\tilde{s}) \leq x_1^A.
$$

Optimists solve a linear optimization problem. At the optimum, the budget constraint
binds. The collateral constraint also binds because (since $p > \frac{E_1[v(s)]}{1+r}$) optimists always prefer
borrowing and investing in the asset to not borrowing. Then, letting $\lambda$ denote the Lagrange
multiplier for the budget constraint and $\gamma$ the Lagrange multiplier for the collateral constraint,
the first order conditions are given by:

$$
\lambda \frac{E_0\left[\min\left(v(s), v(\tilde{s})\right)\right]}{1 + r} \leq E_1\left[\min\left(v(s), v(\tilde{s})\right)\right] + \gamma
$$

(2.A.14)

with equality if $\tilde{s} \in \text{supp}(\eta)$.

---

23 This is because any safe contract with $\varphi < v(s_{\min})$ can be replicated by the alternative safe contract $v(s_{\min})$ (which has the additional benefit of using less collateral), and any contract $\varphi > v(s_{\max})$ that defaults in all states can be replicated by the contract $v(s_{\max})$. 

---

70
Moreover, the first order condition with respect to $x_1^A$ leads to

$$\gamma = \lambda p - E_1[v(s)].$$

Plugging this expression for $\gamma$ into (2.A.14) yields the following first order condition:

$$R_1^L(\bar{s}) = \frac{E_1[v(s)] - E_1[\min(v(s), v(\bar{s}))]}{p - E_0[\min(v(s), v(\bar{s}))]/(1+r)} \leq \lambda,$$  \hspace{1cm} (2.A.15)

with strict inequality only if $\bar{s} \in supp(\eta)$.

This equation implies that any $\bar{s} \in supp(\eta)$ maximizes $R_1^L(\bar{s})$, completing the first step of the proof.

As the second step, I show that problem (2.11) has a unique solution, and I characterize the solution. To this end, consider the derivative of $R_1^L(\bar{s})$, which can be written as

$$\frac{d}{d\bar{s}} R_1^L(\bar{s}) = \frac{1}{p - E_0[\min(v(s), v(\bar{s}))]/(1+r)} \left( \frac{R_1^L(\bar{s})}{1+r} (1 - F_0(\bar{s})) - (1 - F_1(\bar{s})) \right).$$  \hspace{1cm} (2.A.16)

Note that

$$R_1^L(s_\text{min}) = \frac{E_1[v(s)] - v(s_\text{min})}{p - v(s_\text{min})/(1+r)} > 0 \quad \text{and} \quad R_1^L(s_\text{max}) = \frac{E_1[v(s)] - E_1[v(s)]}{p - E_0[v(s)]/(1+r)} = 0.$$

Thus, the derivative in (2.A.16) satisfies the boundary conditions

$$\frac{d}{d\bar{s}} R_1^L(\bar{s}) \big|_{\bar{s}=s_\text{min}} > 0 \quad \text{and} \quad \frac{d}{d\bar{s}} R_1^L(\bar{s}) \big|_{\bar{s}=s_\text{max}} < 0.$$  \hspace{1cm} (2.A.17)

Eq. (2.A.16) also leads to the first order condition

$$\frac{d}{d\bar{s}} R_1^L(\bar{s}) = 0 \quad \text{for} \quad \bar{s} \in [s_\text{min}, s_\text{max}] \iff \frac{R_1^L(\bar{s})}{1+r} = \frac{1 - F_1(\bar{s})}{1 - F_0(\bar{s})}.$$  \hspace{1cm} (2.A.18)

Plugging this first order condition into (2.11) and rearranging terms yields $p = p^{opt}(\bar{s})$. By Lemma 2, $p^{opt}(\bar{s})$ is strictly decreasing, which implies that there exists exactly one $\bar{s} \in S$ (the solution to $p = p^{opt}(\bar{s})$) that satisfies the first order condition in (2.A.18). By the boundary conditions in (2.A.17) and the continuity of $\frac{d}{d\bar{s}} R_1^L(\bar{s})$, it follows that $R_1^L(\bar{s})$ has a unique
maximum characterized as the solution to Eq. (2.10). This establishes the second step, and completes the proof of Theorem 1.

**Proof of Eq. (2.14).** Consider optimists’ budget constraint (2.5) and note that:

\[
w_1 = p(x_1^A - \alpha_1) + x_1^B + \int_\phi q(\phi) d\mu_1^- (\phi) = p(x_1^A - \alpha_1) + x_1^B + q(\bar{s}) \int_\phi d\mu_1^- (\phi) = p(x_1^A - \alpha_1) + x_1^B + q(\bar{s}) x_1^A,
\]

where the second line uses the fact that \( \mu^- \) is a Dirac measure at \( \overline{s} \), and the last line uses the fact that optimists’ collateral constraint (2.6) binds. Substituting contract prices from Eq. (2.15), the previous displayed equation implies the period 1 flow of funds constraint:

\[
p(x_1^A - \alpha_1) + x_1^B = w_1 + \frac{E_0 \left[ \min (v(s), v(\bar{s})) \right]}{1 + r} x_1^A.
\]

(2.A.19)

Next note that optimists choose \( x_1^B = 0 \) except for the corner case \( p = \frac{E_1[v(s)]}{1+r} \). In view of this observation, Eq. (2.A.19) characterizes optimists’ demand for the asset. Recall also that moderates choose \( x_1^A = 0 \) except for the corner case \( p = \frac{E_0[v(s)]}{1+r} \), which characterizes moderates’ demand for the asset. Finally, recall that the asset market clearing condition is given by \( x_1^A + x_0^A = 1 \).

There are three cases to consider. First consider case (ii), i.e., suppose the market clearing price is given by some \( p \in \left( \frac{E_0[v(s)]}{1+r} , \frac{E_1[v(s)]}{1+r} \right) \). In this case, moderates demand \( x_0^A \), which implies that optimists’ demand must be given by \( x_1^A = 1 \). Using this in Eq. (2.A.19) (along with \( x_1^B = 0 \)), the market clearing price is solved as:

\[
p = \frac{1}{\alpha_0} \left( w_1 + \frac{E_0 \left[ \min (v(s), v(\bar{s})) \right]}{1 + r} \right) = \frac{w_1^{\max}(\overline{s})}{\alpha_0}.
\]

(2.A.20)

As long as the expression \( \frac{w_1^{\max}(\overline{s})}{\alpha_0} \) lies inside \( \left( \frac{E_0[v(s)]}{1+r} , \frac{E_1[v(s)]}{1+r} \right) \), \( p \) is indeed the market clearing price, proving case (ii). If \( \frac{w_1^{\max}(\overline{s})}{\alpha_0} \geq \frac{E_1[v(s)]}{1+r} \), then the market clearing price is \( p = \frac{E_1[v(s)]}{1+r} \) (along with allocations \( x_1^A = 1 \) and \( x_1^B \geq 0 \)), proving case (iii). Finally, if \( \frac{w_1^{\max}(\overline{s})}{\alpha_0} < \frac{E_1[v(s)]}{1+r} \), then the market clearing price is \( p = \frac{E_0[v(s)]}{1+r} \) (along with allocations \( x_1^A < 1 \) and \( x_1^B = 0 \)), proving case
(i). This completes the proof of Eq. (2.14).

**Analytical Characterization of Equilibrium** I next provide an analytical characterization of the quasi-equilibrium described by \( p = p^{\text{opt}}(\bar{s}) = p^{\text{mc}}(\bar{s}) \), which will be useful for some of the subsequent proofs. Note that if optimists’ wealth is not too large, in particular, if

\[
\alpha_0 E_1 [v(s)] - v(s^{\text{min}}) < \frac{1}{1 + r},
\]

(2.A.21)

then the two curves intersect in the case (ii) region of Eq. (2.14) and the equilibrium pair \((p, \bar{s}^*)\) is characterized as follows:

\[
p = p^{\text{mc}}(\bar{s}^*) = \frac{1}{\alpha_0} \left( w_1 + \frac{1}{1 + r} E_0 [\min (v(s), v(\bar{s}^*))] \right),
\]

(2.A.22)

where \( \bar{s}^* \) is the unique solution to

\[
G(\bar{s}^*) = \alpha_0 \left( \frac{1 - F_0(\bar{s}^*)}{1 - F_1(\bar{s}^*)} \right) \int_{\bar{s}^*}^{s^{\text{max}}} (v(s) - v(\bar{s}^*)) dF_1 - \alpha_1 E_0 [\min (v(s), v(\bar{s}^*))] = w_1 (1 + r).
\]

(2.A.23)

In this case, optimists take loans with riskiness \( \bar{s}^* \in (s^{\text{min}}, s^{\text{max}}) \) and the price satisfies \( p \in \left( \frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r} \right) \). Note also that the function \( G(\cdot) \) in Eq. (2.A.23) is differentiable and strictly decreasing, which implies that there is a unique solution to Eq. (2.A.23).

If the opposite of condition (2.A.21) holds, then the two curves intersect in the case (i) region of Eq. (2.14). In this case, optimists’ financial constraints are not binding, they borrow with a safe loan (with riskiness \( \bar{s}^* = s^{\text{min}} \)) and they bid up the asset price to the optimistic valuation, i.e., \( p = \frac{E_1[v(s)]}{1+r} \). This analysis also verifies that the two curves never intersect in case (iii) region of Eq. (2.14), which implies that the equilibrium price satisfies \( p > \frac{E_0[v(s)]}{1+r} \). This completes the analytical characterization of equilibrium.

**2.A.3 Characterization of Collateral Equilibrium**

This section completes the characterization of the collateral equilibrium by providing the proof of Theorem 2.
Proof of Theorem 2. As the first step, I show that the prices and allocations in Theorem 2 constitute a collateral equilibrium. I next prove the essential uniqueness of the collateral equilibrium.

Existence of the Collateral Equilibrium. I claim that the allocation in Theorem 2 constitutes a collateral equilibrium. The analysis for the corner price \( p = \frac{E_1[v(s)]}{1+r} \) is straightforward. Therefore, suppose that the asset price satisfies \( p \in \left( \frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r} \right) \). Eq. (2.14) ensures that the asset market clearing condition is satisfied. Hence, all that remains to check is that loan market is in equilibrium. This amounts to checking that debt contract choices are optimal for traders after relaxing the restrictions \( \mu_0^- = 0 \) and \( \mu_1^+ = 0 \), and that debt contract prices clear the market.

I next establish an easy-to-check condition for equilibrium in the loan market. Recall that moderates' rate of return on capital is given by \( 1 + r \), while optimists' rate of return on capital is given by \( R_f(s^*) > 1 + r \) (cf. Eq. (2.11)), where the inequality follows since \( p < \frac{E_1[v(s)]}{1+r} \). Given the rates of return \( 1 + r \) and \( R_f(s^*) \), consider traders' bid prices for each debt contract \( \varphi \in R_+ \), defined as:

\[
q_0^{bid}(\varphi) = \frac{E_0[\min(v(s), \varphi)]}{1+r} \quad \text{and} \quad q_1^{bid}(\varphi) = \frac{E_1[\min(v(s), \varphi)]}{R_f(s^*)}.
\]  

(2.A.24)

Note that these are the prices that would make moderates (resp. optimists) indifferent between holding a debt contract \( \varphi \) and holding their equilibrium portfolio.

Similarly, consider the ask prices for a debt contract \( \varphi \) that would make the traders indifferent between selling the debt contract \( \varphi \) and holding their equilibrium portfolio. There is a slight complication because, to be able to short sell the contract \( \varphi \), the trader must also hold 1 unit of the asset. Hence, consider the cross investment strategy of short selling one unit of contract \( \varphi \) and buying one unit of the asset. Let \( q_i^{ask}(\varphi) \) denote the price that makes type \( i \) traders indifferent between pursuing this strategy and holding their equilibrium portfolio. Note that \( q_0^{ask}(\varphi) \) and \( q_1^{ask}(\varphi) \) are respectively defined as the solutions to:

\[
\frac{E_0[v(s)] - E_0[\min(v(s), \varphi)]}{p - q_0^{ask}(\varphi)} = 1 + r \quad \text{and} \quad \frac{E_1[v(s)] - E_1[\min(v(s), \varphi)]}{p - q_1^{ask}(\varphi)} = R_f(s^*). \quad (2.A.25)
\]
The bid and ask prices in (2.A.24) and (2.A.25) can also be used to define the aggregate bid and ask price for the contract \( \varphi \), given by:

\[
q^{\text{bid}}(\varphi) = \max_i q^{\text{bid}}_i(\varphi) \quad \text{and} \quad q^{\text{ask}}(\varphi) = \min_i q^{\text{ask}}_i(\varphi).
\]

Note that, if the price of a contract \( \varphi \) is below \( q^{\text{bid}}(\varphi) \), a trader would demand infinite units of the contract, which would violate market clearing. Similarly, if the price is above \( q^{\text{ask}}(\varphi) \), a trader would sell infinite units, which would again violate market clearing. Moreover, non-zero trade in a contract requires at least one type of trader to buy the contract and another type of trader to sell, which can happen only if \( q^{\text{bid}}(\varphi) = q^{\text{ask}}(\varphi) \). It follows that the loan market is in equilibrium if and only if debt contract prices and allocations satisfy the following condition:

\[
\begin{align*}
q^{\text{bid}}(\varphi) &\leq q(\varphi) \leq q^{\text{ask}}(\varphi), \quad \text{and} \\
q(\varphi) &= q^{\text{bid}}(\varphi) = q^{\text{ask}}(\varphi) \quad \text{whenever} \quad \varphi \in \text{supp} \left( \mu_i^{-} \right) \quad \text{for some} \quad i.
\end{align*}
\]  

(2.A.26)

I next show that the loan market allocation of Theorem 2 satisfies the loan market equilibrium condition (2.A.26). In particular, I claim:

\[
q^{\text{bid}}(\varphi) \leq q^{\text{ask}}(\varphi) \quad \text{with equality iff} \quad \varphi = v(\bar{s}^*).
\]  

(2.A.27)

Note that the debt contract prices of Theorem 2 (cf. Eq. (2.15)) are chosen such that \( q(\varphi) = q^{\text{bid}}(\varphi) \). Moreover, the allocations are such that there is trade only for contract \( \varphi = v(\bar{s}^*) \). Hence, the claim in (2.A.27) implies (2.A.26), which ensures that the loan market is indeed in equilibrium.

Note that the claim in (2.A.27) is true for all \( \varphi \in R_{++} \), if it is true for all \( \varphi \in [v(\bar{s}^{\text{min}}), v(\bar{s}^{\text{max}})] \). To prove the claim for the relevant set of debt contracts, \( \varphi = v(\bar{s}) \) for some \( \bar{s} \in S \), first note that

\[
q^{\text{bid}}_i(\varphi(\bar{s})) < q^{\text{ask}}_i(\varphi(\bar{s})) \quad \text{for each} \quad \bar{s} \in S \quad \text{and} \quad i.
\]  

(2.A.28)

which is straightforward to check. There is a wedge between each type traders' bid and ask prices, intuitively because buying the debt contract has no collateral requirements while selling
the debt contract requires the trader to pledge collateral (and thus, the traders’ ask price to sell a contract is higher).

Second, note that \( \tilde{s}^* \) is the unique solution to problem (2.7) by definition, and thus

\[
R_L^* (\tilde{s}^*) = \frac{E_1 [v (s)] - E_1 [\min (v (s), v (\tilde{s}^*))]}{p - E_0 [\min (v (s), v (\tilde{s}^*))]} / (1 + r) > \frac{E_1 [v (s)] - E_1 [\min (v (s), v (\tilde{s}^*))]}{p - E_0 [\min (v (s), v (\tilde{s}))]} / (1 + r) \text{ for each } \tilde{s} \neq \tilde{s}^*.
\]

Using this inequality and the definition of \( q_1^{ask} (v (\tilde{s})) \) in (2.25) shows

\[
q_1^{ask} (v (\tilde{s})) \geq \frac{E_0 [\min (v (s), v (\tilde{s}))]}{1 + r} = q_0^{bid} (v (\tilde{s})) \text{ with equality iff } \tilde{s} = \tilde{s}^*.
\]

Third, recall that \( s = s_{\min} \) is equal to 1 for \( \tilde{s} = s_{\min} \), and is strictly increasing in \( \tilde{s} \). By Eq. (2.24), it follows that \( \frac{q_1^{bid} (v (s_{\min}))}{q_0^{bid} (v (s_{\min}))} = \frac{1 + r}{R_L (\tilde{s})} < 1 \), and that \( \frac{q_1^{bid} (v (\tilde{s}))}{q_0^{bid} (v (\tilde{s}))} \) is strictly increasing in \( \tilde{s} \). Then, there are two cases to consider. As the first case, \( \frac{q_1^{bid} (v (\tilde{s}))}{q_0^{bid} (v (\tilde{s}))} \) may never exceed 1, that is, it may be the case that

\[
q_1^{bid} (v (\tilde{s})) < q_0^{bid} (v (\tilde{s})) \text{ for each } \tilde{s} \in S.
\]

In this case, combining Eqs. (2.28),(2.29) and (2.30) proves the claim in (2.27). The left panel of Figure 2-8 plots the bid and ask prices in this first case. The figure illustrates that, in this case, the quasi-equilibrium debt prices in (2.9) and the collateral equilibrium debt prices in (2.9) are identical.

As the second case, \( \frac{q_1^{bid} (v (\tilde{s}))}{q_0^{bid} (v (\tilde{s}))} \) may exceed 1 for sufficiently large \( \tilde{s} \). That is, it may be the case that there exists \( \tilde{s}_{\cross} \) such that

\[
\begin{align*}
q_1^{bid} (v (\tilde{s})) &< q_0^{bid} (v (\tilde{s})) \text{ for all } \tilde{s} < \tilde{s}_{\cross}, \\
q_1^{bid} (v (\tilde{s})) &\geq q_0^{bid} (v (\tilde{s})) \text{ for all } \tilde{s} \geq \tilde{s}_{\cross}.
\end{align*}
\]

Note that, in this case, \( \tilde{s}_{\cross} \) is uniquely defined as the solution to

\[
\frac{E_1 [\min (v (s), v (\tilde{s}_{\cross}))]}{E_0 [\min (v (s), v (\tilde{s}_{\cross}))]} = \frac{R_L (\tilde{s}^*)}{1 + r}.
\]
Figure 2-8: The left panel displays the bid and ask prices for the case in which the inequality in (2.A.30) holds, and the right panel displays the case in which the inequality in (2.A.30) fails. The shaded areas display the set of all possible equilibrium debt contract prices in each case.

Moreover, it can be checked that $E_1[v(s), v(\bar{s})] < \frac{R_0(\bar{s})}{1+r}$, which implies $\bar{s}^{cross} > \bar{s}$. It can also be seen that$^{25}$

$$q_0^{ask}(v(\bar{s})) \geq q_1^{ask}(v(\bar{s})) \text{ for each } \bar{s} \geq \bar{s}^{cross}. \quad (2.A.33)$$

$^{24}$To see this, consider the leveraged return expression (2.11), which can be rewritten as

$$p - \frac{E_0[v(s), v(\bar{s})]}{1+r} = \frac{E_1[s]}{R_1^L(\bar{s})} - \frac{E_1[v(s), v(\bar{s})]}{R_1^L(\bar{s})}.$$

Note that $R_1^L(\bar{s}) > \frac{E_1[v(s)]}{p}$ because optimists always have the option of buying the asset without borrowing. Hence, the previous inequality implies $E_1[v(s), v(\bar{s})] < \frac{E_0[v(s), v(\bar{s})]}{1+r}$, which can be rewritten as $E_0[v(s), v(\bar{s})] < \frac{E_1[v(s), v(\bar{s})]}{R_1^L(\bar{s})}$.

$^{25}$Note that from the definition of $q_1^{ask}(v(\bar{s}))$ in (2.A.25), the inequality in (2.A.33) is equivalent to

$$\frac{E_0[v(s)] - E_0[v(s), v(\bar{s})]}{1+r} \leq \frac{E_1[v(s)] - E_1[v(s), v(\bar{s})]}{R_1^L(\bar{s})} \text{ for each } \bar{s} \geq \bar{s}^{cross}. \quad (2.A.32)$$

Recall that $\frac{E_1[v(s)]}{E_0[v(s)]} \geq \frac{E_1[v(s), v(\bar{s})]}{E_0[v(s), v(\bar{s})]}$, which implies that

$$\frac{E_1[s] - E_1[v(s), v(\bar{s})]}{E_0[s] - E_0[v(s), v(\bar{s})]} \geq \frac{E_1[v(s)] - E_1[v(s), v(\bar{s})]}{E_0[v(s)] - E_0[v(s), v(\bar{s})]} \geq \frac{R_1^L(\bar{s})}{1+r},$$

where the last inequality holds for each $\bar{s} \geq \bar{s}^{cross}$ in view of the definition of $\bar{s}^{cross}$. This proves the inequality in (2.A.33), which in turn shows the inequality in (2.A.33).
Then, using Eqs. (2.4.28), (2.4.29) and (2.4.31), it follows that $q^\text{bid}(v(\bar{s})) \leq q^\text{ask}(v(\bar{s}))$ for each $\bar{s} \leq \bar{s}^{\text{cross}}$, with equality iff $\bar{s} = \bar{s}^*$. Moreover, using Eqs. (2.4.28), (2.4.29), (2.4.33) and (2.4.31), it also follows that $q^\text{bid}(\bar{s}) < q^\text{ask}(\bar{s})$ for each $\bar{s} \geq \bar{s}^{\text{cross}}$. This completes the proof of claim (2.4.27), and establishes that the allocation characterized in Theorem 2 is indeed a collateral equilibrium. The right panel of Figure 2-8 plots the bid and ask prices in this second case. This figure illustrates that, in this case, the quasi-equilibrium debt prices in (2.9) and the collateral equilibrium debt prices in (2.9) are not the same, but the difference in prices does not overturn the optimality of the debt contract $\bar{s}^*$.

Figure 2-8 also illustrates that the debt contract prices are not uniquely determined in equilibrium (except for the price of the optimal contract $v(\bar{s}^*)$, which is uniquely determined). In particular, any price function $q(\cdot)$ such that $q(v(\bar{s})) \in \left[q^\text{bid}(v(\bar{s})), q^\text{ask}(v(\bar{s}))\right]$ can support the equilibrium allocation in equilibrium. However, the equilibrium allocations in the loan market and the equilibrium asset price $p$ is uniquely determined, as I next prove.

**Essential Uniqueness of Collateral Equilibrium.** I first prove that the equilibrium asset price $p$ is uniquely determined. Let $R_0$ and $R_1$ denote traders’ equilibrium rates of return on capital (in the above equilibrium, $R_0 = 1 + r$ and $R_1 = R^L_1(\bar{s})$). Since traders always have the option to invest in the bond, $R_0$ and $R_1$ are always weakly greater than $1 + r$. Moreover, in equilibrium some investors must agree to hold the bond, which implies that either $R_0$ or $R_1$ must be equal to $1 + r$. Since optimists have a greater valuation of the asset, the equilibrium rates of return satisfy $R_1 \geq R_0 = 1 + r$.

I next claim that, for any given price $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$, optimists’ rate of return is uniquely determined as $R_1 = R^L_1(\bar{s})$, and the loan market allocations are uniquely determined by Theorem 1. To prove this claim, consider optimists’ bid and ask prices in (2.4.24) and (2.4.25) for an arbitrary price level $p \in \left(\frac{E_0[v(s)]}{1+r}, \frac{E_1[v(s)]}{1+r}\right)$ and an arbitrary required rate of return $R_1$ (i.e., replace $R^L_1(\bar{s}^*)$ in these expressions with $R_1$). Eq. (2.4.25) shows that $q^\text{ask}_1(v(\bar{s}))$ increases in the required rate of return (and Eq. (2.4.24) shows $q^\text{bid}_1(v(\bar{s}))$ decreases in the required rate of return). It follows that the loan market is at equilibrium for a unique required rate of return such that $q^\text{ask}_1(v(\bar{s})) \geq q^\text{bid}_0(v(\bar{s}))$ for all $\bar{s}$ with equality for exactly one state $\bar{s}$.
Then, this rate of return $R_1$ satisfies:

$$
\frac{E_1[v(s)] - E_1[\min(v(s), v(\tilde{s}))]}{p - q_0^{bid}(v(\tilde{s}))} = R_1 = \frac{E_1[v(s)] - E_1[\min(v(s), v(\tilde{s}))]}{p - q_0^{ask}(v(\tilde{s}))} \geq \frac{E_1[v(s)] - E_1[\min(v(s), v(\tilde{s}))]}{p - q_0^{bid}(v(\tilde{s}))}, \tag{2.A.34}
$$

where the first equality uses the definition of $q_0^{ask}(v(\tilde{s}))$ and the fact that $q_1^{ask}(v(\tilde{s})) = q_0^{bid}(v(\tilde{s}))$, the second equality uses the definition of $q_1^{ask}(v(\tilde{s}))$, and the last inequality follows from the inequality $q_1^{ask}(v(\tilde{s})) \geq q_0^{bid}(v(\tilde{s}))$. The comparison between the first and the last terms in (2.A.34) shows that $\tilde{s} \in S$ solves problem (2.11). In particular, for any price $p \in \left(\frac{E_0[v(\tilde{s})]}{1+r}, \frac{E_1[v(\tilde{s})]}{1+r}\right)$, the unique loan market allocation is the same as the quasi-equilibrium loan market allocation characterized by Theorem 1, and the unique rate of return that equilibrates the loan market is given by $R_1 = R_1^T(\tilde{s})$.

I next prove the uniqueness of the collateral equilibrium price $p$. For any price $p \in \left(\frac{E_0[v(\tilde{s})]}{1+r}, \frac{E_1[v(\tilde{s})]}{1+r}\right)$, optimists loan market allocation is uniquely determined. This implies that optimists’ leveraged investment on the asset (i.e., their demand for the asset) is uniquely determined. Combining this with the asset market clearing condition, Eq. (2.14), the equilibrium price $p$ is also uniquely determined.

The above analysis also establishes that the price of the optimal contract, $q(v(\tilde{s}))$, and the equilibrium allocations are uniquely determined for any price $p \in \left(\frac{E_0[v(\tilde{s})]}{1+r}, \frac{E_1[v(\tilde{s})]}{1+r}\right)$. For the corner price $p = \frac{E_0[v(\tilde{s})]}{1+r}$ (which corresponds to the optimal contract $\tilde{s}^* = s^{min}$), the price of the optimal contract $q(v(s^{min}))$ is still uniquely determined. However, in this case, the equilibrium allocations are not necessarily unique since optimists may be indifferent among some of the safe contracts $\varphi \leq q(s^{min})$. This establishes the essential uniqueness of the collateral equilibrium and completes the proof of Theorem 2.

**2.A.4 Comparative Statics with Respect to Belief Heterogeneity**

**Proof of Theorem 3.** Part (i). Similar to the proof of Lemma 3, define the function $g : S \rightarrow R$ with

$$
g(\tilde{s}) = E_1[v(s) \mid s \geq \tilde{s}] - E_1[v(s) \mid s \geq \tilde{s}].
$$
Note that \( g(s_{\text{min}}) = 0 \) since \( \tilde{E}_1 \[v(s)\] = E_1 \[v(s)\] \), and note also that \( g(s_{\text{max}}) = 0 \). I claim that

\[
g(\tilde{s}) \geq 0 \text{ for all } \tilde{s} \in (s_{\text{min}}, s_{\text{max}}),
\]

which implies Eq. (2.19) in the main text. Most of the proof then follows by the argument provided after Theorem 3. For the comparative statics of the leverage ratio, substitute Eq. (2.A.22) into (2.16) to get:

\[
L = \frac{p}{p - (p\alpha_0 - w_1)} = \frac{1}{1 - \alpha_0 + \frac{w_1}{p}}.
\]

Since \( p \) weakly increases, the leverage ratio also weakly increases, completing the proof conditional on the claim in (2.A.35).

To prove the claim in (2.A.35), first note that Eq. (2.A.8) applies also in this setting. Since \( \frac{f_1(\tilde{s})}{1-F_1(\tilde{s})} \leq \frac{f_1(s)}{1-F_1(s)} \) over the range \( \tilde{s} \in (s^R, s_{\text{max}}) \), the same argument used in the proof of part (i) of Lemma 3 shows that

\[
g(\tilde{s}) \geq 0 \text{ for all } \tilde{s} \in [s^R, s_{\text{max}}].
\]

Second, suppose, to reach a contradiction, that there exists \( \tilde{s} \in [s_{\text{min}}, s^R] \) with \( g(\tilde{s}) < 0 \). Since \( g(s_{\text{min}}) = 0 \), this further implies that there exists \( \tilde{s} \in [s_{\text{min}}, \tilde{s}] \) such that \( g(\tilde{s}) = 0 \) and \( g'(\tilde{s}) < 0 \), since otherwise, the differentiable function \( g(\cdot) \) could not become negative over the range \( [s_{\text{min}}, \tilde{s}] \). Considering Eq. (2.A.8) for \( \tilde{s} = \tilde{s} \) and using \( g(\tilde{s}) = 0 \) implies

\[
g'(\tilde{s}) = \left( \frac{\tilde{F}_1(\tilde{s})}{1 - \tilde{F}_1(\tilde{s})} - \frac{f_1(\tilde{s})}{1-F_1(\tilde{s})} \right) \left( \tilde{E}_1 \[v(s) \mid s \geq \tilde{s}\] - v(\tilde{s}) \right) \geq 0,
\]

where the inequality follows since \( \frac{\tilde{F}_1(\tilde{s})}{1 - \tilde{F}_1(\tilde{s})} \geq \frac{f_1(\tilde{s})}{1-F_1(\tilde{s})} \) (as \( \tilde{s} < s^R \)). Since \( g'(\tilde{s}) < 0 \) by the choice of \( \tilde{s} \), the previous displayed inequality yields a contradiction, completing the proof.

Part (ii). Applying the proof of part (i) for distributions \( F_0 \succeq_R \tilde{F}_0 \) shows that

\[
E_0 \[v(s) \mid s \geq \tilde{s}\] \geq \tilde{E}_0 \[v(s) \mid s \geq \tilde{s}\] \text{ for each } \tilde{s} \in \mathcal{S}.
\]

Note also that \( F_0 \succeq_R \tilde{F}_0 \) implies \( \frac{1 - \tilde{F}_0(\tilde{s})}{1 - F_0(\tilde{s})} \) is weakly increasing for \( \tilde{s} \in (s_{\text{min}}, \tilde{s}^*) \), which further
implies \( \tilde{F}_0 (\bar{s}) \leq F_0 (\bar{s}) \) over this range. In view of this observation and Eq. (2.A.37), Eq. (2.18) implies that

\[
p^{opt} (\bar{s}) \text{ weakly increases for each } \bar{s} \in (s^{min}, \bar{s}^*) \tag{2.A.38}
\]

Next consider the effect on the market clearing curve \( p^{mc} (\bar{s}) \). Note that since \( F_0 \succeq_{R, \bar{s}^*} \tilde{F}_0 \),

\[
\frac{\tilde{F}_0 (\bar{s})}{1 - \tilde{F}_0 (\bar{s})} \leq \frac{F_0 (\bar{s})}{1 - F_0 (\bar{s})} \text{ for each } \bar{s} \in (s^{min}, \bar{s}^*)
\]

Then, the same steps as in the proof of part (ii.2) of Lemma 3 applies in this case and shows

\[
p^{mc} (\bar{s}) \text{ weakly increases for each } \bar{s} \in (s^{min}, \bar{s}^*) \tag{2.A.39}
\]

Using Eqs. (2.A.38) and (2.A.39) along with the fact that \( p^{opt} (\bar{s}) \) is a decreasing relation and \( p^{mc} (\bar{s}) \) is a weakly increasing relation, it follows that \( p \) is weakly greater at the new intersection point, completing the proof. 26

**Proof of Theorem 4.** Part (i). The fact that \( \tilde{F}_0 \) and \( F_0 \) are equally optimistic over \( (s^{min}, \bar{s}^*) \) implies that

\[
\frac{1 - \tilde{F}_0 (\bar{s})}{1 - F_0 (\bar{s})} = \frac{1 - \tilde{F}_0 (s^{min})}{1 - F_0 (s^{min})} = 1, \text{ that is, } \tilde{F}_0 (\bar{s}) = F_0 (\bar{s}) \text{ for each } \bar{s} \in (s^{min}, \bar{s}^*)
\]

Then, Eq. (2.10) implies

\[
p^{opt} \left( \bar{s} ; \tilde{F}_0, \tilde{F}_1 \right) = p^{opt} \left( \bar{s} ; F_0, F_1 \right) \text{ for each } \bar{s} \in (s^{min}, \bar{s}^*)
\]

By part (i) of Lemma 3, \( \tilde{F}_1 \succeq \tilde{F}_1 \) implies \( p^{opt} \left( \bar{s} ; F_0, \tilde{F}_1 \right) \geq p^{opt} \left( \bar{s} ; F_0, F_1 \right) \). Using this along with the previous displayed equation shows Eq. (2.A.38), that is, \( p^{opt} (\bar{s}) \) weakly increases for each \( \bar{s} \in (s^{min}, \bar{s}^*) \).

Note also that \( \tilde{E}_0 [v (s)] \leq E_0 [v (s)] \). Combining this with the fact that \( \tilde{F}_0 (\bar{s}) = F_0 (\bar{s}) \) for each \( \bar{s} \in (s^{min}, \bar{s}^*) \) implies that \( p^{mc} (\bar{s}) \) curve remains constant over \( (s^{min}, \bar{s}^*) \) except for the

---

26 On the other hand, the effect on \( \bar{s}^* \) is ambiguous. The effect on \( \bar{s}^* \) depends on how much the \( p^{mc} (\bar{s}) \) curve shifts up. If the effect on the \( p^{mc} (\bar{s}) \) curve is strong, perhaps because \( \alpha_0 \) is small, then \( \bar{s}^* \) may decrease. To see the intuition for this result, suppose many units of the asset are already endowed to optimists, i.e. \( \alpha_1 \) is high. Then the increase in lenders' valuation of debt contracts acts similar to a positive wealth shock to optimists (cf. Eq. (2.13)), because optimists can borrow more against the units they already own. This effect tends to lower the leverage ratio, and if sufficiently strong, it can overcome the effect from the shift of the \( p^{opt} (\bar{s}) \) curve which (similar to part (i)) tends to increase the leverage ratio.
fact that its lower bound "extends," that is,

\[ p^{mc}(\bar{s} ; \tilde{F}_0, \tilde{F}_1) \begin{cases} = p^{mc}(\bar{s}) & \text{if } p^{mc}(\bar{s}) > \frac{E_0[v(s)]}{1+r} \\ \leq p^{mc}(\bar{s}) & \text{if } p^{mc}(\bar{s}) = \frac{E_0[v(s)]}{1+r} \end{cases} \text{ for each } \bar{s} \in (s^{\min}, \bar{s}^*). \]

Note also that the proof of Theorem 2 establishes that \( p = p^{mc}(\bar{s}^*) > \frac{E_0[v(s)]}{1+r} \). Using this in the previous displayed equation implies that there exists \( \varepsilon > 0 \) such that \( p^{mc}(\bar{s}) \) remains constant in a neighborhood \( \bar{s} \in (\bar{s}^* - \varepsilon, \bar{s}^*) \). Combining this with Eq. (2.A.38) and using the facts that \( p^{opt}(\cdot) \) is a decreasing curve and \( p^{mc}(\cdot) \) is an increasing curve shows that the new intersection point of these curves is weakly to the right of \( \bar{s}^* \), which further implies \( p \) and \( \bar{s}^* \) weakly increase.

The comparative statics for the leverage ratio follows by the same argument as in part (i) of Theorem 3.

Part (ii). First, I claim that \( F_1 \) and \( \tilde{F}_1 \) have the same distribution conditional on any upper-threshold event \([\bar{s}, s^{\max}]\) with \( \bar{s} > \bar{s}^* \). That is, for any \( \bar{s} \in (\bar{s}^*, s^{\max}) \),

\[ \frac{f_1(s)}{1 - F_1(s)} = \frac{\tilde{f}_1(s)}{1 - \tilde{F}_1(s)} \text{ for each } s \in [\bar{s}, s^{\max}). \] (2.A.40)

To see this, note that by assumption

\[ \frac{1 - \tilde{F}_1(s)}{1 - F_1(s)} = \frac{1 - \tilde{F}_1(\bar{s})}{1 - F_1(\bar{s})} \text{ for each } s \in [\bar{s}, s^{\max}). \] (2.A.41)

Moreover, taking the derivative of this equation with respect to \( s \) implies

\[ \frac{f_1(s)}{1 - F_1(s)} = \frac{\tilde{f}_1(s)}{1 - \tilde{F}_1(s)} \text{ for each } s \in [\bar{s}, s^{\max}). \] (2.A.42)

Using Eqs. (2.A.41) and (2.A.42), it follows that, for each \( s \in (\bar{s}, s^{\max}) \),

\[ \frac{f_1(s)}{1 - F_1(s)} = \frac{f_1(s)}{1 - F_1(s)} \frac{1 - F_1(s)}{1 - F_1(s)} = \frac{\tilde{f}_1(s)}{1 - \tilde{F}_1(s)} \frac{1 - \tilde{F}_1(s)}{1 - \tilde{F}_1(\bar{s})} = \frac{\tilde{f}_1(s)}{1 - \tilde{F}_1(s)}, \]

proving Eq. (2.A.40).

Next consider the case in which condition (2.A.21) is violated so that \( \bar{s}^* = s^{\min} \). Using Eq. (2.A.40) for \( \bar{s} = \bar{s}^* + \varepsilon \) and taking \( \varepsilon \to 0 \) shows that, in this case, \( F_1 \) and \( \tilde{F}_1 \) are the same
distribution. By the same argument, \( F_0 \) and \( \tilde{F}_0 \) are also the same distribution. It follows that the asset price \( p \) remains constant for this case.

Consider next the case in which condition (2.A.21) holds so that \( s^* > s^{\min} \). Consider respectively the shift in the \( p^{\text{opt}}(\bar{s}) \) and \( p^{\text{mc}}(\bar{s}) \) curves. First consider the optimality curve \( p^{\text{opt}}(\bar{s}) \) and note that Eq. (2.A.40) in Eq. (2.10) implies

\[
p^{\text{opt}}(\bar{s}; \tilde{F}_0, \tilde{F}_1) = p^{\text{opt}}(\bar{s}; F_0, F_1) \quad \text{for each } \bar{s} \in (s^*, s^{\max}).
\]

By part (ii) of Lemma 3, \( F_0 \succeq \tilde{F}_0 \) implies \( p^{\text{opt}}(\bar{s}; \tilde{F}_0, F_1) \leq p^{\text{opt}}(\bar{s}; F_0, F_1) \). Using this in the previous displayed equation shows

\[
p^{\text{opt}}(\bar{s}) \text{ weakly decreases for each } \bar{s} \in (s^*, s^{\max}). \tag{2.A.43}
\]

Next consider the market clearing curve \( p^{\text{mc}}(\bar{s}) \). Note that \( s^* > s^{\min} \) implies \( p^{\text{opt}}(\bar{s}^*) = p^{\text{mc}}(\bar{s}^*) < \frac{E_i[v(s)]}{1+r} \), thus by Eq. (2.4.5) (cf. part (ii) of Lemma 3), the increase in optimism of \( F_1 \) leaves \( p^{\text{mc}}(\bar{s}) \) unchanged in a neighborhood \( (\bar{s}^*, \bar{s}^* + \epsilon) \). Note also that the decrease in optimism of \( F_0 \) weakly decreases \( p^{\text{mc}}(\bar{s}) \) downwards pointwise. It follows that

\[
p^{\text{mc}}(\bar{s}) \text{ weakly decreases for each } \bar{s} \in (\bar{s}^*, \bar{s}^* + \epsilon). \tag{2.A.44}
\]

Combining Eqs. (2.A.43) and (2.A.44) and using the fact that \( p^{\text{opt}}(\bar{s}) \) is a decreasing curve while \( p^{\text{mc}}(\bar{s}) \) is an increasing curve, the asset price \( p \) is weakly lower at the new intersection point. This completes the proof of Theorem 4.

2.A.5 Collateral Equilibrium with Contingent Contracts

With contingent loans, traders solve the following analogue of problem (2.7):

\[
\max_{x_i \geq 0; \mu_i^+, \mu_i^- \in M(D)} \left[ x_i^A E_i[v(s)] + x_i^B (1 + r) + \int_D E_i[\min(v(s), \varphi(s))] \mu_i^+(\varphi) - \int_D E_i[\min(v(s), \varphi(s))] \mu_i^-(\varphi) \right] \tag{2.A.45}
\]

s.t.

\[
px_i^A + x_i^B + \int_D q(\varphi) \mu_i^+(\varphi) - \int_D q(\varphi) \mu_i^-(\varphi) \leq w_i + p\alpha_i.
\]

\[
\int_D \mu_i^-(\varphi) \leq x_i^A.
\]
The same analysis as in the proof of Theorem 1 applies and shows that optimists choose a debt contract, \([\varphi(s) \in [0, v(s)])_{s \in S}\), that maximizes the leveraged return (2.21).

**Proof of Theorem 5.** I first claim that the leveraged return in (2.21) is maximized by the solution to equation \( p = p^{\text{opt,\,cont}}(\bar{s}) \). Second, I claim that the optimal leveraged return, \( R_1^{L,\text{cont}}(\varphi_{\bar{s}} \mid p) \), is greater than \( 1 + r \) if and only if \( p < p^{\text{max}} \). This implies that optimists make a leveraged investment in the asset if and only if \( p < p^{\text{max}} \), which completes the proof of the theorem.

To prove the first claim, note that the debt contracts can be restricted to the set such that \( \varphi(s) \in [0, v(s)] \) for each \( s \in S \). The same steps in the proof of Theorem 1 show that \( \varphi \in \text{supp}(\mu_\pi) \) if and only if \( \varphi \) solves Problem (2.21). To characterize the solution to this problem, note that the derivative of \( R_1^{L,\text{general}}(\varphi) \) with respect to \( \varphi(s) \) is given by:

\[
\frac{\partial R_1^{L,\text{cont}}(\varphi)}{\partial \varphi(s)} = \frac{-f_0(s)}{p - \frac{1}{1+r} \int_{s_{\text{min}}}^{s_{\text{max}}} \varphi(s) \, dF_0} \left( \frac{f_1(s)}{f_0(s)} - \frac{R_1^{L,\text{cont}}(\varphi)}{1 + r} \right).
\]

From this expression and assumption (MLRP), it follows that the derivative \( \frac{\partial R_1^{L,\text{cont}}(\varphi)}{\partial \varphi(s)} \) satisfies a cutoff property. In particular, there exists a threshold state \( \bar{s} \in S \) such that

\[
\frac{\partial R_1^{L,\text{cont}}(\varphi)}{\partial \varphi(s)} \begin{cases} > 0 & \text{for each } s < \bar{s} \\ < 0 & \text{for each } s > \bar{s} \end{cases}
\]

Consequently, the optimal level of promise for each state \( s \in S \setminus \{\bar{s}\} \) has a corner solution. Hence, the solution \( \varphi \) to Problem (2.21) has the form in Eq. (2.22), except potentially a Lebesgue measure zero of states. In particular, the contract specified in Eq. (2.22) is one optimal solution to Problem (2.21). Moreover, the threshold state \( \bar{s} \in S \) is characterized as the solution to \( \frac{f_1(s)}{f_0(s)} = \frac{R_1^{L,\text{cont}}(\varphi)}{1 + r} \), which after using the form in Eq. (2.22) can be written as

\[
\frac{\frac{1}{1+r} \int_{s_{\text{min}}}^{s_{\text{max}}} v(s) \, dF_1}{p - \frac{1}{1+r} \int_{s_{\text{min}}}^{s_{\text{max}}} v(s) \, dF_0} = \frac{f_1(\bar{s})}{f_0(\bar{s})}.
\]

Rearranging this expression shows that the threshold state is characterized as the unique solution to \( p = p^{\text{opt,\,cont}}(\bar{s}) \), completing the proof of the first claim.
To prove the second claim, fix a price level $p$, consider the corresponding optimal threshold $\bar{s}(p)$, and note that the optimal leveraged return is given by:

$$R_{1,\text{cont}}^{L}(\varphi_{\bar{s}(p)} \mid p) = \frac{E_1[v(s)] - \int_{s_{\min}}^{\bar{s}(p)} v(s) \, dF_1}{p_{\text{opt},\text{cont}}(\bar{s}(p))} = (1 + r) \frac{f_1(\bar{s}(p))}{f_0(\bar{s}(p))}. \tag{2.A.46}$$

Here, the first equality uses the fact that $p = p_{\text{opt},\text{cont}}(\bar{s}(p))$ (since $\bar{s}(p)$ is optimal), and the second equality uses Eq. (2.24). Next, note that $p_{\text{max}} = p_{\text{opt},\text{cont}}(s_{\text{cross}})$ (cf. Eqs. (2.23) and (2.24)), and thus $s_{\text{cross}}$ is the optimal threshold corresponding to price level $p_{\text{max}}$. Using Eq. (2.A.46), this further implies that $R_{1,\text{cont}}^{L}(\varphi_{s_{\text{cross}}} \mid p_{\text{max}}) = 1 + r$. Since $R_{1,\text{cont}}^{L}(\varphi_{s(p)} \mid p)$ is decreasing in $p$, it follows that the optimal leveraged return is greater than $1 + r$ if and only if $p < p_{\text{max}}$. Hence, optimists borrow and invest in the asset if $p < p_{\text{max}}$, but they are indifferent between investing in the asset and the bond if $p > p_{\text{max}}$. This completes the proof of the theorem.

2.A.6 Collateral Equilibrium with Short Selling

This appendix provides a proof of Theorem 6 in Section 2.6. It then uses the proof to provide an intuition for the asymmetric disciplining property (which is more complete than the intuition provided in the main text). The appendix ends by deriving the total expenditure on short sales, Eq. (2.30), which is used in the main text.

Similar steps as in the derivation of Theorem 1 show that the default threshold $\bar{s}_{\text{sh}}$ for short contracts maximizes:

$$R_0^{\text{short}}(\bar{s}) = \frac{v(\bar{s}) - E_0[\min(v(s), v(\bar{s}))]}{\frac{v(\bar{s})}{1+r} - E_1[\min(v(s), v(\bar{s}))] / \frac{E_1[v(s)]}{p}}. \tag{2.A.47}$$

Eq. (2.26) corresponds to the first order condition for this maximization problem. Under assumption (MLRP), the unique solution to this equation maximizes the return in (2.A.47), completing the sketch proof of Theorem 6.

To interpret problem (2.A.47), note that $R_0^{\text{short}}(\bar{s})$ is the return of short sellers from selling
one unit of the short contract \( \psi = v(\tilde{s}) \). More specifically, short sellers receive

\[
q_{\text{short}}(v(\tilde{s})) = E_1 \left[ \min (v(s), v(\tilde{s})) \right] / E_1 \left[ v(s) \right] \tag{2.A.48}
\]

from the sale of this contract, and they use this amount towards meeting the collateral requirement. However, they need to post a total of \( \frac{v(\tilde{s})}{1+r} \) units of the consumption good as collateral. Thus, they pay the difference (the denominator of (2.A.47)) out of their wealth. In the next period, short sellers receive \( v(\tilde{s}) \) from the collateral that they have posted, and they expect to pay \( E_0 \left[ \min (v(s), v(\tilde{s})) \right] \) on the promises they have made. This is because, short sellers return the asset if the realized state is below \( \tilde{s} \), but they default on the short contract if the realized state is above \( \tilde{s} \). In the latter scenario, short sellers lose only the collateral that they have posted, which is worth \( v(\tilde{s}) \). Hence, short sellers’ expected payment is given by \( E_0 \left[ \min (v(s), v(\tilde{s})) \right] \).

Problem (2.A.47) captures the essential trade-off that short sellers are facing. Note that moderates’ perceived interest rate on a short contract is given by:

\[
1 + r_0^{\text{per}}(\tilde{s}) \equiv \frac{E_0 \left[ \min (v(s), v(\tilde{s})) \right]}{E_1 \left[ \min (v(s), v(\tilde{s})) \right] / E_1 \left[ v(s) \right]} = \frac{E_0 \left[ \min (v(s), v(\tilde{s})) \right]}{E_1 \left[ \min (v(s), v(\tilde{s})) \right]} \frac{E_1 \left[ v(s) \right]}{E_0 \left[ \min (v(s), v(\tilde{s})) \right] / E_1 \left[ v(s) \right]} \frac{E_1 \left[ v(s) \right]}{E_0 \left[ \min (v(s), v(\tilde{s})) \right] / E_1 \left[ v(s) \right]} \frac{E_1 \left[ v(s) \right]}{p}. \tag{2.A.49}
\]

This expression further implies that \( r_0^{\text{per}}(\tilde{s}) < r \) for the equilibrium short contract \( \tilde{s} = \tilde{s}_{sh} \). Intuitively, short sellers expect to make a net positive return, \( r - r_0^{\text{per}}(\tilde{s}_{sh}) \), by selling the short contract and buying the bond with the proceeds. Moreover, under assumption (MLRP), this return is increasing in the short threshold \( \tilde{s} \). This is because, the higher \( \tilde{s} \), the less often the short contract defaults, and the greater portion of the asset the short sellers effectively sell. On the other hand, problem (2.A.47) shows that a higher threshold \( \tilde{s} \) requires short sellers to post a greater amount of collateral, \( \frac{v(\tilde{s})}{1+r} \). This restricts short sellers’ ability to leverage the net return \( r - r_0^{\text{per}}(\tilde{s}_{sh}) \). It follows that, when choosing \( \tilde{s}_{sh} \), short sellers trade off greater leverage against a lower net return. This trade-off is resolved by problem (2.A.47), and leads to the optimal short contract characterized by (2.26).

I next provide the intuition for why the function \( p^{\text{short}}(\tilde{s}_{sh}) \) is decreasing in the default threshold \( \tilde{s}_{sh} \), and why it has the asymmetric disciplining property. Consider first the former statement, which is equal to saying that the default threshold \( \tilde{s}_{sh} \) for the optimal short contract is decreasing in the asset price. Note that, by Eq. (2.A.49), a higher price \( p \) increases the wedge
\( r - \tau^\text{per}_0 (\bar{s}_{sh}) \) that short sellers expect to make. This incentivizes short sellers to leverage more, by choosing a lower default threshold \( \bar{s}_{sh} \). Intuitively, as prices are higher, short sellers see a greater bargain in short selling and they leverage their short sales more.

Consider next the intuition for the asymmetric disciplining property of \( p^\text{short} (\bar{s}_{sh}) \). To understand this property, suppose the equilibrium default threshold is given by \( \bar{s}_{sh} \), and consider how high the asset price should be (relative to the moderate valuation) to entice moderates to choose a short contract with this default threshold. If the belief heterogeneity is concentrated on states below \( \bar{s}_{sh} \), then Eq. (2.A.49) reveals that the return wedge \( r - \tau^\text{per}_0 (\bar{s}_{sh}) \) perceived by moderates is higher. Thus, the asset price does not need to be too high to entice moderates to choose the short contract with threshold \( \bar{s}_{sh} \). Consequently, with this type of belief heterogeneity, the asset price is closer to the moderate valuation. In contrast, suppose the belief heterogeneity is concentrated more on the relative likelihood of states above \( \bar{s}_{sh} \). In this case, Eq. (2.A.49) implies that the perceived return wedge \( r - \tau^\text{per}_0 (\bar{s}_{sh}) \) is lower. Then, moderates are enticed to choose the threshold level \( \bar{s}_{sh} \) only if prices are sufficiently higher than the moderate valuation. Hence, optimism about the probability of states below \( \bar{s}_{sh} \) is disciplined more than optimism about the relative likelihood of states above \( \bar{s}_{sh} \), as suggested by the form of \( p^\text{short} (\bar{s}_{sh}) \).

Finally, consider the derivation of the total expenditure on short sales, denoted by \( W^\text{short} \). Note that \( \frac{v(\bar{s}_{sh})}{1+r} - q^\text{short} \left( \frac{v(\bar{s})}{1+r} \right) \) is the amount of wealth moderates need to allocate to sell one unit of the short contract \( \psi = \frac{v(\bar{s}_{sh})}{1+r} \). Type \( T_3 \) moderates (that are able to short sell) have a total wealth of \( \gamma_{sh} (w_0 + p\alpha_0) \). Thus, the total number of short contracts \( \frac{v(\bar{s}_{sh})}{1+r} \) sold by moderates is given by \( \frac{\gamma_{sh} (w_0 + p\alpha_0)}{v(\bar{s}_{sh})/1+r - q^\text{short} \left( \frac{v(\bar{s})}{1+r} \right)} \). The total expenditure on short sales is then given by:

\[
W^\text{short} = \frac{\gamma_{sh} (w_0 + p\alpha_0)}{v(\bar{s}_{sh})/1+r - q^\text{short} \left( \frac{v(\bar{s})}{1+r} \right)} \cdot v(\bar{s}) \left( \frac{1}{1+r} \right).
\]

Substituting for \( q^\text{short} \left( \frac{v(\bar{s})}{1+r} \right) \) from Eq. (2.A.48), and rearranging terms yields the expression (2.30) for \( W^\text{short} \).
2.A.7 Characterization of Dynamic Equilibrium

This section completes the characterization of the dynamic equilibrium analyzed in Section 2.7, by providing a proof of Theorem 7. I first note a preliminary lemma which is necessary for the proof of the theorem.

Note that the problem of traders in the dynamic economy (cf. (2.38)) is similar to their problem in the static economy (cf. (2.7)), with the only difference that the asset is not endowed to the current young generation. The next lemma uses this observation to show that a recursive collateral equilibrium can be constructed based on the analysis in Section 2.3. The result requires the condition

$$\omega_0 \geq \frac{1 + \varepsilon}{r - \varepsilon},$$

(2.A.50)

which ensures that young traders' endowment is sufficient to purchase the entire asset supply. Recall that a loan with riskiness $\bar{s}$ is a debt contract $\varphi = v(a, \bar{s})$ that defaults if and only if the next period state is below the threshold level $\bar{s} \in S$.

**Lemma 4.** Consider a dynamic economy with condition (2.A.50), and suppose there exists a collection of price and loan riskiness pairs, $(p(a) \in \mathbb{R}_+, \bar{s}^* (a) \in S)_{a \in \mathbb{R}_{++}}$, such that for each $a \in R_{++}$, the pair $(p(a), \bar{s}^*(a))$ corresponds to the collateral equilibrium characterized in Theorem 2 for the static economy

$$(S; v(a, \cdot); \{F_t\}_t; \{w_i \equiv \omega_i a\}_i; \{\alpha_1 = 0, \alpha_0 = 1\}). \quad (2.A.51)$$

Then, there exists a recursive collateral equilibrium in which, for each $a \in R_{++}$, optimists make leveraged investments in the asset by borrowing through a single loan with riskiness $\bar{s}^*(a)$ and the asset price is $p(a)$.

**Proof of Lemma 4.** Let the tuple $\left( p(a), [q(a, \varphi)]_{\varphi \in \mathbb{R}}, (x^{A}_i (a), x^{B}_i (a), \mu^+_i (a), \mu^-_i (a))_{i \in \{1, 0\}} \right)_{a \in \mathbb{R}_{++}}$ be such that, the prices and allocations for each $a$ correspond to the collateral equilibrium of the static economy in (2.A.51). I claim that this tuple corresponds to a dynamic equilibrium with a modified bond allocation for moderates, $\bar{\tilde{x}_0^B} (a)$.

Note that optimists' problem (2.38) is equivalent to their problem (2.7) in this static econ-
omy, given prices
\[ p = p(a) \text{ and } q(\varphi) = q(a, \varphi) \text{ for each } \varphi \in \mathbb{R}_+. \]  
(2.A.52)

Hence optimists’ allocations are also optimal in the dynamic economy. Moderates’ problem is slightly different since \( \alpha_0 = 1 \) in the static economy whereas \( \alpha_0 = 0 \) in the dynamic economy (as the asset is held by the old generation). Because of this difference, the allocation \((x_0^A(a), x_0^B(a), \mu_0^+(a), \mu_0^-(a))\) violates the budget constraint of moderates in the dynamic economy by an amount \( p(a) \). Consider instead the modified bond allocation
\[ \tilde{x}_0^B(a) \equiv x_0^B(a) - p(a) \geq 0 \text{ for each } a \in \mathbb{R}_{++}. \]  
(2.A.53)

Note that the allocation \((x_0^A(a), \tilde{x}_0^B(a), \mu_0^+(a), \mu_0^-(a))\) satisfies moderates’ budget constraint. When this is the case, it can also be seen that this allocation solves Problem (2.38) given the prices in (2.A.52).\(^{27}\)

Hence, if the inequality in (2.A.53) is satisfied, then the conjectured tuple with the modified \( \tilde{x}_0^B(a) \) constitutes an equilibrium of the dynamic economy. To verify the inequality in (2.A.53), consider moderates’ budget constraint in the static economy
\[ x_0^B(a) + \int_{\mathbb{R}_+} q(a, \varphi) d\mu_0^+(a, \varphi) = \omega_0 a + p(a), \]  
(2.A.54)

where the equality holds since \( \alpha_0 = 1 \) and \( x_0^A(a) = 0 \). Next note that
\[
\int_{\mathbb{R}_+} q(a, \varphi) d\mu_0^+(a, \varphi) \leq p(a) \int_{\mathbb{R}_+} d\mu_0^+(a, \varphi) = p(a) \int_{\mathbb{R}_+} d\mu_1^-(a, \varphi) \leq p(a) x_1^A(a) = p(a),
\]

where the first line follows from the inequality \( q(a, \varphi) \leq p(a) \) (which follows from Eq. (2.9)), the second line uses the debt market clearing condition (2.8), the third line uses the collateral

\(^{27}\)To see this, note that Eq. (2.A.53) implies \( x_0^B(a) > p(a) > 0 \) for the static economy in (2.A.51), which further implies that moderate traders are indifferent between holding bonds and debt contracts. As their budgets and bond holdings are reduced by the same amount \( p(a) \), the allocation \((x_0^A(a), \tilde{x}_0^B(a), \mu_0^+(a), \mu_0^-(a))\) is optimal for moderates in the dynamic economy.
constraint (2.6), and the last line uses the asset market clearing condition $x_1^d(a) = 1$. Using the last displayed inequality, the budget constraint (2.A.54) implies

$$x_B^a(a) \geq \omega_0 a \geq \frac{1 + \varepsilon}{r - \varepsilon} a \geq p(a),$$

where the second inequality follows from condition (2.A.50), and the third inequality follows from the fact that $p(a)$ is weakly less than the unconstrained level $\frac{1 + \varepsilon}{r - \varepsilon} a$. It follows that $x_B^a(a)$ in (2.A.53) is positive, completing the proof of Lemma 4.

Lemma 4 reduces the characterization of the dynamic equilibrium to the static case, along with a fixed point argument (since the value function $v(a, \cdot)$ depends on the price function). I next use this characterization to prove Theorem 7.

**Proof of Theorem 7.** Plugging the conjecture, $p(a) = pd a$ and $\bar{s}^*(a) = \bar{s}^*_d \in S$, into (2.37) implies that the value function is also linearly homogeneous. In particular, $v(a, s) = v_d(s \mid pd) a$, where

$$v_d(s \mid pd) = s(1 + pd).$$

(2.A.55)

Next note that using the conjecture in the characterization of the static equilibrium (cf. Eqs. (2.10) and (2.14)) and using linear homogeneity in $a$, the constants $pd$, $\bar{s}^*_d$ are characterized as the collateral equilibrium of the static economy $E(pd)$ in (2.39). In particular, $(pd, \bar{s}^*_d)$ is the unique solution to the following equations:

$$pd = p^{opt}(\bar{s}_d \mid v_d(\cdot \mid pd)) = p^{mc}(\bar{s}_d \mid v_d(\cdot \mid pd)),
\quad (2.A.56)$$

where the notation $p^{opt}(\cdot \mid v_d(\cdot \mid pd))$ denotes the function $p^{opt}(\cdot)$ evaluated with the value function $v_d(\cdot \mid pd)$. Given a pair, $(pd, \bar{s}_d^*)$, that solves (2.A.56), Lemma 4 implies that the conjectured allocation is an equilibrium.

The remaining step is to characterize the solution to the fixed point equation (2.A.56). To this end, let $(P_d(\bar{pd}), S_d(\bar{pd}))$ denote the solution to (2.A.56) when the value function is given by $v_d(\cdot \mid \bar{pd})$ (i.e., when the future price to dividend ratio is given by $\bar{pd}$). Then, the solution to

---

28The notations $p^{opt}(\cdot \mid v), p^{mc}(\cdot \mid v)$ respectively denote the functions $p^{opt}(\cdot), p^{mc}(\cdot)$ evaluated with the particular value function $v(\cdot)$.
(2.A.56) is a fixed point of the mapping \( P_d(\cdot) \) over the interval \( [p_d^{\text{min}}, p_d^{\text{max}}] \). I next claim that \( P_d(\cdot) \) is strictly increasing over this interval, and it satisfies the boundary conditions

\[
P_d(p_d^{\text{min}}) > p_d^{\text{min}} \quad \text{and} \quad P_d(p_d^{\text{max}}) \leq p_d^{\text{max}}.
\]  

This claim implies that that \( P_d(\cdot) \) has a unique fixed point \( p_d \in (p_d^{\text{min}}, p_d^{\text{max}}] \), which characterizes the dynamic equilibrium.

To prove the claim, I first show that the loan riskiness \( S_d(p_d) \in [s^{\text{min}}, s^{\text{max}}] \) is weakly increasing in \( p_d \). There are two cases depending on condition (2.A.21). Using the value function

\[
v_d(s | p_d) = s(1 + p_d) \quad \text{(cf. Eq. (2.A.55))}, \quad \alpha_0 = 1, \quad \text{and} \quad E_1[v(s)] = 1 + \varepsilon,
\]

condition (2.A.21) can be written as

\[
\omega_1 < (1 + p_d) \frac{1 + \varepsilon - s^{\text{min}}}{1 + r}.
\]

First suppose \( p_d \) is sufficiently large that this condition is violated. In this case, by the characterization in the proof of Theorem 2, the loan riskiness \( S_d(p_d) = s^{\text{min}} \) is constant. Second, suppose condition (2.A.58) is satisfied, and thus \( S_d(p_d) \in (s^{\text{min}}, s^{\text{max}}) \) is determined as the unique solution to Eq. (2.A.23). This equation can be simplified to

\[
\frac{1 - F_0(\bar{s})}{1 - F_1(\bar{s})} \int_{\bar{s}}^{s^{\text{max}}} (v(s) - v(\bar{s})) dF_1 = w_1 \frac{1 + r}{1 + p_d}.
\]

The proof of Theorem 2 shows that the left hand side of this expression is a strictly decreasing function of \( \bar{s} \). Since the right hand side is decreasing in \( p_d \), it follows that in this case \( S_d(p_d) \) is increasing. Combining the two cases, \( S_d(p_d) \) is weakly increasing in \( p_d \).

Next, to show that \( P_d(p_d) \) is strictly increasing in \( p_d \), note that

\[
P_d(p_d) = p^{\text{mc}}(S_d(p_d) \mid v_d(\cdot | p_d))
\]

\[
= \min \left( \frac{E_1[v_d(s | p_d)]}{1 + r}, \omega_1 + \frac{E_0\left[ \min\left( v_d(s | p_d), v_d(S_d(p_d) | p_d) \right) \right]}{1 + r} \right)
\]

where the second equality combines cases (i) and (ii) of Eq. (2.14) and uses \( \alpha_0 = 1 \). Substituting the value function \( v_d(s | p_d) = s(1 + p_d) \) (cf. Eq. (2.A.55)) and using \( E_1[s] = 1 + \varepsilon \), the previous
displayed equation can be written as

\[
P_d(\tilde{p}_d) = \min \left( (1 + \tilde{p}_d) \frac{1 + \varepsilon}{1 + r}, \omega_1 + (1 + \tilde{p}_d) \frac{E_0 \left[ \min (s, S_d(\tilde{p}_d)) \right]}{1 + r} \right).
\]

Since \( S_d(\tilde{p}_d) \) is weakly increasing in \( \tilde{p}_d \), this equation implies that \( P_d(\tilde{p}_d) \) is strictly increasing in \( \tilde{p}_d \).

Finally, to show that \( P_d(\tilde{p}_d) \) satisfies the boundary conditions in (2.A.57), note that Eq. (2.10) implies

\[
P_d(\tilde{p}_d) = p^{opt}(S_d(\tilde{p}_d) ; v_d(\cdot, |\tilde{p}_d)).
\]

Using the definition of \( p^{opt} (\cdot) \) from Eq. (2.10) and substituting \( v_d (s | \tilde{p}_d) = s (1 + \tilde{p}_d) \) (cf. Eq. (2.A.55)), the previous displayed equation can be written as

\[
P_d(\tilde{p}_d) = \frac{1 + \tilde{p}_d}{1 + r} \left( \int_{s_{min}}^{S_d(\tilde{p}_d)} F_0 \left( S_d(\tilde{p}_d) \right) \int_{s_{max}}^{s_{max}} s \right) F_1 \left( S_d(\tilde{p}_d) \right).
\]

Next, consider this expression for \( \tilde{p}_d = \frac{1}{r} \) and note that

\[
P_d \left( \frac{1}{r} \right) = \frac{1 + \frac{1}{r}}{1 + \frac{r}{1 + r}} \left( \int_{s_{min}}^{S_d \left( \frac{1}{r} \right)} F_0 \left( S_d \left( \frac{1}{r} \right) \right) \int_{s_{max}}^{s_{max}} s \right) F_1 \left( S_d \left( \frac{1}{r} \right) \right) = \frac{1 + \frac{1}{r}}{1 + \frac{r}{1 + r}} E_0 \left[ s \right] = \frac{1}{r}.
\]

Here, the second line replaces \( S_d \left( \frac{1}{r} \right) \) in the first line with \( s_{max} > S_d \left( \frac{1}{r} \right) \), and the inequality follows since the expression in the first line is a decreasing function of \( S_d \left( \frac{1}{r} \right) \). Similarly,

\[
P_d \left( \frac{1 + \varepsilon}{r - \varepsilon} \right) = \frac{1 + \frac{1 + \varepsilon}{r - \varepsilon}}{1 + \frac{r - \varepsilon}{1 + r}} \left( \int_{s_{min}}^{S_d \left( \frac{1 + \varepsilon}{r - \varepsilon} \right)} F_0 \left( S_d \left( \frac{1 + \varepsilon}{r - \varepsilon} \right) \right) \int_{s_{max}}^{s_{max}} s \right) F_1 \left( S_d \left( \frac{1 + \varepsilon}{r - \varepsilon} \right) \right) \leq \frac{1}{r - \varepsilon} \left( \int_{s_{min}}^{s_{max}} s \right) F_1 \left( S_d \left( \frac{1 + \varepsilon}{r - \varepsilon} \right) \right) = \frac{1}{r - \varepsilon} E_1 \left[ s \right] = \frac{1 + \varepsilon}{r - \varepsilon},
\]

where the second line replaces \( S_d \left( \frac{1 + \varepsilon}{r - \varepsilon} \right) \) in the first line with \( s_{min} \leq S_d \left( \frac{1 + \varepsilon}{r - \varepsilon} \right) \). It follows that \( P_d(\tilde{p}_d) \) satisfies the boundary conditions in (2.A.57), completing the proof of Theorem 7.
Proof of Theorem 8. Part (i). Since \((P_d(p_d), S_d(p_d))\) is the static equilibrium for the economy \(E(p_d)\) and since optimists’ optimism becomes weakly more right-skewed, Theorem 3 applies and shows that \(P_d(p_d)\) and \(S_d(p_d)\) weakly increase for each \(p_d\). Since \(p_d\) is the fixed point of the strictly increasing mapping \(P_d(\cdot)\) and since \(P_d(\cdot)\) shifts up, it follows that \(p_d\) weakly increases.

Next note that \(S_d(\cdot)\) is a weakly increasing function (by the proof of Theorem 7) and that the equilibrium price to dividend ratio, \(p_d\), weakly increases. Since \(S_d(p_d)\) also weakly increases for each \(p_d\), it follows that the equilibrium loan riskiness, \(s^*_d = S_d(p_d)\), weakly increases.

Next consider the share of the speculative component, \(\lambda_d\). Plugging in the value function, \(v_d(\cdot | p_d) = s(1 + p_d)\) (cf. Eq. (2.A.55)), and using \(E_0[s] = 1\) and \(E_1[s] = 1 + \varepsilon\), Eq. (2.40) can be rewritten as

\[
\frac{p_d}{1 + p_d} = (1 - \theta_d) \frac{1}{1 + r} + \theta_d \frac{1 + \varepsilon}{1 + r}.
\]

Note also that Eq. (2.41) can be written as

\[
1 - \lambda_d = \frac{p^{dd}(a)}{p(a)} = \frac{1}{p_d} \left( (1 - \theta_d) \frac{1}{r} + \theta_d \frac{1 + \varepsilon}{r} \right).
\]

Combining the last two displayed equalities, the share of the speculative component is given by:

\[
\lambda_d = 1 - \frac{1 + 1/r}{1 + p_d}
\]

Since \(p_d\) weakly increases, \(\lambda_d\) also weakly increases, completing the proof for the first part.

Part (ii). Since \((P_d(p_d), S_d(p_d))\) is the static equilibrium for the economy \(E(p_d)\) and since optimists’ optimism becomes weakly more skewed to the left of \(s^*_d\), Theorem 3 applies and shows that \(P_d(p_d)\) and \(S_d(p_d)\) weakly increase for each \(p_d\). Since \(p_d\) is the fixed point of the strictly increasing mapping \(P_d(\cdot)\) and since \(P_d(\cdot)\) shifts up, it follows that \(p_d\) weakly increases. The same steps for part (i) show that \(\lambda_d\) also weakly increases, completing the proof of Theorem 8.
Chapter 3

Speculation and Risk Sharing with New Financial Assets

3.1 Introduction

According to the traditional view, financial innovation typically facilitates the diversification and the sharing of risks among market participants. However, this view does not take into account that new financial assets are often associated with much uncertainty, especially because they do not have a long track record. Belief heterogeneity comes as a natural by-product of this uncertainty and changes the implications of risk taking in such markets. While the traditional view emphasizes the risk sharing role of new assets, belief heterogeneity about these assets naturally leads to speculation, which represents a powerful economic force in the opposite direction. The purpose of this essay is to understand the effect of financial innovation on the allocation of risks when both the risk sharing and the speculation forces are present.

I consider this question in the context of a standard CARA-Normal framework in which a finite and large number of traders with CARA utilities have risky and normally distributed endowments. Traders can hedge part of their endowment risks by trading a finite number of risky financial assets with normally distributed payoffs. Financial innovation is captured by the introduction of new assets into this economy. The central feature of the model is that traders

---

1Cochrane (2005) summarizes this view as follows: “Better risk sharing is much of the force behind financial innovation. Many successful new securities can be understood as devices to more widely share risks.”
have heterogeneous prior beliefs about asset returns, which implies that new assets provide their risk sharing services while being subject to speculation. I define a trader’s consumption variance as the ex-ante variance of her ex-post wealth (resulting from her portfolio returns) according to her own beliefs. I define the average consumption variance as the average of the consumption variance across all traders weighted by traders’ absolute risk aversion coefficients. I then decompose the average consumption variance into two components: an uninsurable variance, defined as the variance that would obtain if there were no belief heterogeneity, and a speculative variance, defined as the residual amount of variance that results from speculative trades based on belief heterogeneity.

My main result characterizes the effect of financial innovation, i.e., the introduction of new assets, on each component of the average consumption variance. In line with the traditional view, financial innovation always decreases the uninsurable variance because new assets increase the possibilities for risk sharing. Theorem 9 shows that financial innovation also always increases the speculative variance. Moreover, there exist economies in which this increase in the speculative variance dominates the decrease in the risk sharing variance, and financial innovation increases the average consumption variance by an arbitrary amount (cf. Proposition 2).

It is relatively straightforward to see why the introduction of one new asset into an economy with no assets would increase the speculative variance (the new asset will generate some speculative trades, while there was no speculation before). But Theorem 9 shows that the introduction of a new asset increases the speculative variance regardless of the number or the type of existing assets. For a simple illustration of the generality of this result, consider an example in which a new asset $B$ is introduced into an economy in which there is one existing asset $A$. Suppose there are two traders, denoted by 1 and 2, who have no endowment risks, which implies that all trades in this economy are driven by speculation considerations. Suppose trader 1 is more optimistic than trader 2 about both assets, which implies that she takes positive speculative positions on both assets (and trader 2 takes the opposite positions). Suppose also that the returns of assets $A$ and $B$ are negatively correlated, which implies that trader 1’s (and similarly, trader 2’s) speculative positions on the two assets provide a natural hedge to each other. Then, a naive intuition would suggest that the introduction of the new asset
would reduce the speculative variance in this economy (compared to the case in which only the asset \( A \) is available). However, Theorem 9 shows that this naive intuition is wrong. The introduction of the new asset \( B \) increases the speculative variance in this economy, as well as any other CARA-Normal economy.

The intuition for the main result is related to a powerful economic force, the *hedge-more/bet-more* effect, which amplifies the amount of speculation on new assets. To understand this effect, first note that assets have risky returns, which implies that traders who take speculative positions on assets bear additional risks. Since traders dislike risk, the riskiness of new assets acts as a natural buffer which tends to limit speculation on these assets. However, if new assets are correlated with existing assets, then traders hedge their speculative positions on new assets by taking complementary positions on existing assets. This ability to *hedge more* weakens the natural buffer against speculation, which in turn enables traders to *bet more* on new assets, that is, to place larger speculative positions and take on greater speculative risks. In the above example, the fact that trader 1’s (and similarly, trader 2’s) bets provide a natural hedge against each other makes the joint bet on both assets more attractive than betting on each asset in isolation. Consequently, traders amplify their joint bets, which leads to a greater speculative variance than the case in which only the existing asset \( A \) is available. The above mentioned naive intuition is incorrect because it ignores the amplification of the speculative positions by the hedge-more/bet-more effect. Theorem 9 shows that this amplification is always sufficiently strong to ensure that financial innovation increases the speculative variance.

Theorem 9 unambiguously characterizes the effect of financial innovation on components of variance, but it is silent on the relative strength of these effects. Understanding the strength of these effects is important to identify the environments in which financial innovation is likely to have a net positive effect on average consumption variance. I address this question by analyzing a number of comparative statics for the components of variance. Theorem 10 uses the hedge-more/bet-more effect to show that, keeping all else equal, new assets increase the speculative variance more when they are introduced into a relatively more complete asset market. In more complete markets, traders have more opportunities to hedge their speculative positions. This induces them to engage in more hedge-more/bet-more trades and to take on greater speculative risks. Using a similar reasoning, Theorem 11 shows that new assets increase the
speculative variance more when they are correlated with existing assets, e.g., when they are derivatives of existing assets. These results suggest that, as asset markets get more complete, they become more susceptible to speculation and further financial innovation is more likely to be destabilizing.

These results naturally raise the question of whether, and how, trade in new assets should be regulated. The equilibrium in this model is Pareto efficient despite the fact that new assets may increase the average consumption variance. This is because, in view of belief heterogeneity, each trader perceives a large expected return from her trades that justifies the additional risks that she is taking. Despite the Pareto efficiency of equilibrium, it is important to analyze the potential policy interventions for new assets, for two main reasons. First, this essay considers the standard CARA-Normal framework which does not feature any externalities, but a similar trade-off between speculation and risk sharing will also be present in more general models that may feature externalities. For example, if the traders are financial intermediaries that do not fully internalize the social costs of their losses (or bankruptcies), then speculation by these intermediaries may have social costs. Second, the notion of Pareto efficiency with heterogeneous priors is somewhat unsatisfactory. This is because while all traders perceive a large expected return, at most one of these expectations can be correct. As noted by Stiglitz (1989), “there are real difficulties in interpreting the welfare losses associated with impeding trades based on incorrect expectations.”

For both reasons, I consider the minimization of the average consumption variance as an alternative policy objective. The analysis of a number of potential policies, e.g., delaying trade in new assets, necessitates a dynamic model. I take a first step in this direction by considering a simple dynamic extension of the model with the central feature that traders update their priors as they observe the returns of new assets. As traders learn about the returns, belief heterogeneity gradually disappears. Thus, in the long run, new assets provide the risk sharing opportunities without being subject to speculation. Using these features, Theorem 12 shows that a policy of delaying the introduction of new assets into markets, if correctly timed, decreases the average consumption variance at all times.²

²However, this result relies in part on the assumption that traders continue to learn about the payoffs of newly innovated assets, even if the assets are not traded. It may be more reasonable to assume that traders learn more about the assets which they actively trade. This assumption implies a separate trade-off between speculation
Another natural question concerns whether new assets that lead to a greater average consumption variance will be endogenously innovated in this economy. I address this question by introducing a profit seeking market maker that innovates new assets for which it subsequently serves as the intermediary. The market maker’s expected profits depend on the volume of trade in new assets. Thus, traders’ speculative trading motive, as well as their risk sharing motive, creates innovation incentives for the market maker. Theorem 14 shows that assets that will increase the average consumption variance may be endogenously innovated, and that this is more likely when the market maker is more short-sighted (i.e., discounts the profits further in the future at a greater rate). This is because speculation considerations generate the most amount of trade in the short run (before traders’ beliefs have converged), while the risk sharing considerations generate roughly the same amount of trade in the short and the long run.

My essay is related to a sizeable literature on security design and financial innovation (see, for example, Van Horne, 1985, Miller, 1986, Duffie and Jackson, 1989, Allen and Gale, 1994, Duffie and Rahi, 1994 and Tufano, 2003). To my knowledge, this literature has not explored the implications of heterogenous beliefs for security design. For example, in their survey of the literature, Duffie and Rahi (1994) note that “one theme of the literature, going back at least to Working (1953) and evident in the Milgrom and Stokey (1982) no-trade theorem, is that an exchange would rarely find it attractive to introduce a security whose sole justification is the opportunity for speculation.” While this observation is true if the source of belief heterogeneity is purely informational, it does not apply if traders have heterogeneous priors, as demonstrated by Theorem 14. The observation also does not apply if the source of belief heterogeneity is informational but asset prices do not reveal information fully due to the presence of noise traders. The analogues of my results can be derived for this alternative setting. The important economic ingredient is that traders continue to have some disagreement after observing asset prices.

The hedge-more/bet-more effect also appears in Brock, Hommes and Wagener (2009). They consider a reinforcement learning model in which traders with heterogeneous beliefs trade a single risky asset. Traders' beliefs are endogenous because at each date traders choose from a

and learning, which I plan to explore in future work.
finite number of forecasting tools according to a fitness measure, such as past profits made by
the tool. Because of this reinforcement learning feature, the steady-state corresponding to the
"fundamental" asset price can be unstable. Brock et al. (2009) use the hedge-more/bet-more
effect to show that the introduction of new Arrow securities increases the range of parameters
for which the steady-state becomes unstable. When new Arrow securities are available, traders
can hedge their bets on the asset better, which implies that a trader with a given forecast tool
bets more on her belief. Consequently, if the forecast tool turns out to be correct, it will yield
a greater profit, and it will be chosen by a greater number of traders in the next period. This
in turn implies that the steady-state will be unstable for a greater range of parameters. In
contrast to Brock et al. (2009), I consider the static CARA-Normal framework and I use the
hedge-more/bet-more effect to show that financial innovation always increases the speculative
variance in this standard model. I also use the hedge-more/bet-more effect to establish the
comparative statics of the speculative variance with respect to the completeness of the asset
market and the correlation of the new and existing assets.

Another strand of literature studies the implications of heterogenous beliefs for asset prices,
but it does not analyze the effect on aggregate risk sharing and the variance of allocations. My model is closest to Lintner (1969), who generalizes CAPM to a model in which beliefs and
risk aversion coefficients are heterogeneous. However, Lintner (1969) does not analyze the risk
sharing implications of this theory. The idea that speculative trading based on belief differences
may create financial instability appears also in Stiglitz (1989), Summers and Summers (1991),
and Stout (1995). However, these analyses are mostly informal and they do not derive any
results similar to my theorems. The idea that financial innovation may increase financial insta-
bility also appears in Rajan (2005) and Calomiris(2008). These papers emphasize the effect of
financial innovation on agency problems rather than speculation.

The rest of the essay is organized as follows. Section 3.2 introduces the model, defines the
equilibrium, and provides a characterization of the unique equilibrium. Section 3.3 decomposes
the average consumption variance into uninsurable and speculative components and presents

---

and Xiong (2003), Geanakoplos (2009), Cao (2009), Simsek (2010a).
the main result, which shows that financial innovation always increases the speculative variance. Section 3.4 introduces the hedge-more/bet-more effect and uses this effect to provide the intuition for the main result. Section 3.5 shows that new assets increase the speculative variance more when they are introduced into a relatively more complete market, or when they are correlated with existing assets. Section 3.6 introduces a simple dynamic extension of the model in which agents Bayesian update their beliefs for the asset payoffs. This section characterizes the dynamic evolution of the average consumption variance, and analyzes a policy of timing the introduction of new assets. Section 3.7 endogenizes financial innovation and provides sufficient conditions under which new assets that increase the average consumption variance will be designed by a profit seeking market maker. Section 3.8 concludes. Appendix 3.A contains proofs of all the results stated in the text.

3.2 Environment and Equilibrium

I consider a generalization of the standard CARA-Normal framework which allows traders to have heterogeneous prior beliefs for asset returns, along the lines of Lintner (1969). This section describes the model and characterizes the unique equilibrium.

Consider a one period endowment economy with a single consumption good. There are a finite and large number of traders denoted by $i \in I = \{1, ..., |I|\}$, and a finite number of financial assets denoted by $j \in J = \{1, ..., |J|\}$. Traders receive random endowments of the consumption good and assets make a single random dividend payment in terms of the consumption good. Each asset can be costlessly short sold, and it is in fixed supply that is normalized to 0 units.\(^4\) Traders are endowed with 0 units of each asset, and they trade assets before the realization of their endowments, potentially to share and diversify their endowment risks. After the asset markets clear, the uncertainty (about endowments and asset payoffs) is resolved and each trader consumes her end-of-period wealth, denoted by the random variable $C_i$. Trader $i$'s preference over consumption is represented by the CARA utility function, $-\exp(-\theta_i C_i)$, where $\theta_i$ is her coefficient of absolute risk aversion.

---

\(^4\)This normalization is without loss of generality because my focus is on the allocation of risks in this economy (rather than the level of asset prices) and traders have CARA utility functions.
The uncertainty in the economy is characterized by the independent and normally distributed random variables, \( \{V, U^1, \ldots, U^{|J|}\} \). The random variable \( V \sim N(0, \mathbf{I}) \) is \( m \) dimensional standard normal, and it captures traders’ endowment risks. In particular, the endowment of trader \( i \) is given by \( \mathbf{w}_i' V \), where \( \mathbf{w}_i \in \mathbb{R}^m \) is a vector. Each random variable \( U^j \sim N(\mu^j_{\text{true}}, 1/\tau^j_{\text{true}}) \) is one dimensional, and it represents the residual uncertainty of asset \( j \) that is independent of traders’ endowments. In particular, the payoff of one unit of asset \( j \) is given by:

\[
(k^j)' V + U^j,
\]

(3.1)

where \( k^j \in \mathbb{R}^m \) captures the covariance of the asset payoff with traders’ endowments, and \( U^j \) captures an additional source of uncertainty specific to asset \( j \). Note that the payoff vector of all assets has a multivariate normal distribution \( N(\mu_{\text{true}}, \Lambda_{\text{true}}) \), where \( \mu_{\text{true}} \equiv (\mu^j)_{j \in J} \) denotes the mean and \( \Lambda_{\text{true}} \equiv k'k + \text{Diag} \left( \frac{1}{\tau_{\text{true}}} \right) \) denotes the variance.

The central aspect of the model is traders’ belief heterogeneity about asset payoffs. I capture this aspect by assuming that traders know and agree about the distribution of the endowment uncertainty, \( V \), and the correlation vectors, \( \{k^j\}_j \), but they have potentially heterogeneous prior beliefs about the distributions of the asset specific random variables, \( \{U^j\}_j \). In particular, trader \( i \) knows that \( U^j \) is independent and normally distributed with precision \( \tau^j_{\text{true}} \), but she does not know the true mean, \( \mu^j_{\text{true}} \). Trader \( i \) believes that \( \mu^j_{\text{true}} \) is normally distributed with mean \( \mu^i_{\text{true}} \) and precision \( \tau^j_{\text{true}} \). I simplify the analysis by assuming that traders’ beliefs have the same precision, that is, there exists \( \tau^j \) such that \( \tau^j_{\text{true}} = \tau^j \) for each \( j \). This implies that the marginal distribution of \( U^j \) with respect to trader \( i \)’s beliefs is \( N(\mu^j_{\text{true}}/\tau^j, \frac{1}{\tau^j_{\text{true}}} + \frac{1}{\tau^j}) \). This further implies that the marginal distribution of asset payoffs with respect to trader \( i \)’s beliefs is also multivariate normal:

\[
N \left( \mu_i \equiv (\mu^j_{\text{true}})_{j \in J}, \Lambda \equiv k'k + \text{Diag} \left( \frac{1}{\tau^j} + \frac{1}{\tau^j_{\text{true}}} \right) \right).
\]

(3.2)

Remark 1 (The Role of the Payoff and Belief Structure). The above described payoff and belief structure is more restrictive than necessary for the main result, Theorem 9. This structure
essentially ensures that traders' beliefs for the asset payoffs are given by \( \{ N(\mu_i, \Lambda) \}_i \), where the covariance matrix, \( \Lambda \), is common across traders. This property is all that is necessary for the analysis until Section 3.5, and in particular, for Theorem 9. The payoff structure in (3.1) (and restricting the belief heterogeneity to the asset specific random variables, \( \{ U_j \}_j \)) simplifies the Bayesian learning dynamics and facilitates the analysis of the dynamic equilibrium in Section 3.6.

3.2.1 Financial Innovation

Let \( \mathcal{E}(J) = (J; \{ w_i \}_i; \{ k^i_j \}_j; \{ \mu^i_j, \tau^i_j \}_j; \{ \mu^i_j, \tau^i_j \}_i,j) \) denote the economy described so far. Given an economy \( \mathcal{E}(J) \) and a subset \( J_E \subset J \) of assets, I also define the sub-economy \( \mathcal{E}(J_E) \), which is identical to the economy \( \mathcal{E}(J) \) except for the fact that the set of available assets is \( J_E \).

I model financial innovation as the introduction of a set of new assets, \( J_N \), into an economy with an existing set of (old) assets, \( J_O \). Given the set of all assets, \( J = J_O \cup J_N \), analyzing the effect of financial innovation amounts to a comparison of the economy, \( \mathcal{E}(J) \), with the sub-economy, \( \mathcal{E}(J_E) \).

3.2.2 Definition of Equilibrium

Consider a sub-economy \( \mathcal{E}(J_E) \) and let \( \mathbf{p}(J_E) = (p^j)_{j \in J_E} \) denote the price vector for the available assets in terms of the consumption good. Given the price vector \( \mathbf{p}(J_E) \), each trader \( i \) chooses a portfolio \( \mathbf{d}_i(J_E) \in \mathbb{R}^{|J_E|} \) of asset holdings that solves:

\[
\max_{\mathbf{d}_i \in \mathbb{R}^{|J_E|}} E_i [\exp(-C_i)]
\]

such that

\[
C_i = \left( w^i_{J_E} \right)' V^{J_E} \mathbf{d}_i + \hat{\mu}_i' \left( (k^i)' V^{J_E} + U^{J_E} \right) - \hat{\mu}_i' \mathbf{p}(J_E) . \tag{3.3}
\]

Note that the trader maximizes her expected utility from consumption, where the expectation is taken with respect to her beliefs in (3.2). The first term in the consumption expression (3.3) is the trader’s random endowment. The second term is the random payment she gets from her

\[6\] An alternative interpretation of the economy, \( \mathcal{E}(J_E) \), is that all of the assets in \( J \) are innovated, but only a fraction, \( J_E \), are exchanged (hence the notation \( J_E \)).
portfolio of risky assets, and the last term is her cash holding (which is allowed to be negative, i.e., traders can costlessly borrow).

**Definition 5.** Consider an economy $\mathcal{E}(J)$ and a subset $J_E \subset J$ of assets. An equilibrium of the sub-economy $\mathcal{E}(J_E)$ is a collection of portfolio allocations $\{d_i(J_E) \in \mathbb{R}^{|J_E|}\}_i$ and a market price vector $p(J_E) \in \mathbb{R}^{|J_E|}$ such that the portfolio allocation of each trader $i$ solves problem (3.3) and the markets for traded assets clear, i.e., $\sum_i d_i(J_E) = 0$.

**3.2.3 Characterization of Equilibrium**

This section characterizes the equilibrium in a sub-economy $\mathcal{E}(J_E)$ and establishes its uniqueness. Consider first the demand by an individual $i$ given any price vector $p \in \mathbb{R}^{|J_E|}$. To simplify the notation, let $\lambda_j^i = (k^i)' w_i$ denote the covariance of trader $i$’s endowment with the payment of asset $j$, and let $\lambda^J_i = \left(\lambda_j^i\right)_{j \in J_E}$ denote the corresponding column vector. For any portfolio choice $\hat{d}_i \in \mathbb{R}^{|J_E|}$, trader $i$’s consumption in (3.3) is normally distributed with respect to her beliefs. Consequently, the trader effectively maximizes the certainty equivalent consumption:

$$\max_{\hat{d}_i \in \mathbb{R}^{|J_E|}} \left(\hat{d}_i\right)' \left(\mu_i^J - p\right) - \frac{\theta_i}{2} \left(\left(\lambda_j^i\right)' w_i^J + \hat{d}_i' \Lambda |J_E| \hat{d}_i + 2 \hat{d}_i' \lambda_i^J \right).$$

From the first order conditions for this problem, the trader’s optimal portfolio is given by:

$$d_i(J_E) = (\Lambda |J_E|)^{-1} \left(\mu_i^J - p - \lambda_i^J \right).$$

Note that a trader tends to hold a long position on an asset either because the asset is negatively correlated with her returns (captured the $\lambda_j^i$ term), or because she believes that the asset’s mean return is greater than its price (captured by the $\mu_i^J - p$ term). These two effects represent the main economic forces in this model. The first effect implies that assets have an insurance role because they enable traders to hedge their endowment risks. The second effect implies that assets also have a speculative role, because they potentially enable traders to bet on their optimistic (or pessimistic) views.

Next consider the determination of the equilibrium price vector, $p(J_E)$. Given the characterization of demand in (3.5), the market clearing yields a closed form solution for the price
vector

\[ p(J_E) = \frac{1}{|I|} \sum_{i \in I} \left( \frac{\bar{\vartheta}_i \mu_i^J - \bar{\vartheta}_i \lambda_i^J}{\theta_i} \right), \] (3.6)

where \( \bar{\vartheta} = (\sum_{i \in I} \theta^{-1}/|I|)^{-1} \) is the Harmonic mean of traders' absolute risk aversion coefficients. Intuitively, the price of an asset is high either if the asset is negatively correlated with traders' endowments (captured by the \( \lambda_i^J \) term) or if traders on average believe that the asset will yield a high dividend payment (captured by the \( \mu_i^J \) term). The beliefs of more risk averse traders have a relatively smaller effect on the price since they bet relatively less on their opinions.

Using the price expression (3.6), trader \( i \)'s portfolio in (3.5) can also be solved in closed form. Moreover, trader \( i \)'s portfolio can be decomposed into two components, \( d_i(J_E) = d_i^R(J_E) + d_i^S(J_E) \), where

\[ d_i^R(J_E) = - (\Lambda |J_E|)^{-1} \bar{\lambda}_i^J \text{ and } d_i^S(J_E) = (\Lambda |J_E|)^{-1} \bar{\mu}_i^J \] (3.7)

Here, \( \bar{\mu}_i = \mu_i - \frac{1}{|I|} \sum_{i \in I} \frac{\bar{\vartheta}_i}{\theta_i} \mu_i \) denotes the optimism of trader \( i \) relative to a weighted average optimism, and \( \bar{\lambda}_i = \lambda_i - \frac{\bar{\vartheta}_i}{\theta_i} \sum_{i \in I} \lambda_i \) denotes the covariance of trader \( i \)'s payments with asset payments relative to a weighted average covariance. The decomposition in (3.7) emphasizes that the two motives for trade in this economy are risk sharing and speculative betting [reinforcing the message of Eq. (3.5)]. I refer to \( d_i^R(J_E) \) as the risk sharing component of the trader's portfolio and \( d_i^S(J_E) \) as the speculative component. The following proposition summarizes the discussion in this section and establishes the uniqueness of equilibrium.

**Proposition 1.** Consider an economy \( E(J) \) and a subset \( J_E \subset J \) of assets. The sub-economy \( E(J_E) \) has a unique equilibrium. The equilibrium price vector is given by Eq. (3.6) and the equilibrium portfolio demand of a trader is the sum of the two components in (3.7).

### 3.3 Main Result: Effect of Financial Innovation on Consumption Variance

This section presents the main result which characterizes the effect of financial innovation, i.e., the introduction of the new assets, \( J_N \), on the riskiness of equilibrium portfolios. To state the result, I first define the appropriate measure of riskiness for this economy.
Definition 6. Given a sub-economy $\mathcal{E}(J_E)$, the average consumption variance corresponding to a portfolio allocation $\{\tilde{d}_i \in \mathbb{R}^{|J_E|} \}$ is:

$$\Omega \left( J_E \mid \{\tilde{d}_i \} \right) = \frac{1}{|I|} \sum_{i \in I} \frac{\theta_i}{\bar{\varrho}} \text{var}_i(C_i) = \frac{1}{|I|} \sum_{i \in I} \frac{\theta_i}{\bar{\varrho}} \left( (\tilde{w}_i^{J_E})' \tilde{w}_i^{J_E} + \tilde{d}_i J_E \tilde{d}_i + 2\tilde{d}_i \lambda_i^{J_E} \right). \quad (3.1)$$

The equilibrium variance, denoted by $\Omega(J_E)$, is the average consumption variance corresponding to the equilibrium allocation $\{d_i(J_M)\}_i$.

Note that the average consumption variance, $\Omega \left( J_E \mid \{\tilde{d}_i \} \right)$, is a weighted average of traders’ consumption variances, where each trader’s variance is calculated with respect to her own beliefs. Traders with a greater coefficient of risk aversion are given a greater weight: that is, $\Omega \left( \{\tilde{d}_i \} \mid J_E \right)$ is a measure of riskiness that takes into account the heterogeneity in risk aversion. The equilibrium variance, $\Omega(J_E)$, provides a measure of riskiness for the equilibrium portfolio allocation.

I next decompose the equilibrium variance into two components, which is necessary to state the main result. The decomposition of variance relies on the following lemma, which establishes that the risk sharing components of portfolios, $\{d_i^R(J_E)\}_i$, minimize the average consumption variance among all feasible portfolio allocations.

Lemma 5. Consider the unique equilibrium of the sub-economy $\mathcal{E}(J_E)$ characterized in Proposition 1. The risk sharing components of traders’ portfolios, $\{d_i^R(J_E)\}_i$, solve the following problem:

$$\min_{\{\tilde{d}_i \in \mathbb{R}^{|J_E|} \}} \Omega \left( J_E \mid \{\tilde{d}_i \} \right) \quad (3.2)$$

subject to $\sum_i \tilde{d}_i^{J_E} = 0$.

The optimum value of the problem is given by:

$$\Omega^R(J_E) = \frac{1}{|I|} \sum_{i \in I} \frac{\theta_i}{\bar{\varrho}} \left( (\tilde{w}_i^{J_E})' \tilde{w}_i^{J_E} - (\tilde{\lambda}_i^{J_E})' (\Lambda | J_E)^{-1} \tilde{\lambda}_i^{J_E} \right). \quad (3.3)$$
The intuition for this result can be gleaned from the portfolio demand decomposition in (3.7). Note that, if traders have no belief heterogeneity, i.e., if \( \mu_i^{\mathcal{I}} = \mu_i^{\mathcal{J}} \) for each \( i, i \in I \) and \( j \in J \), then portfolio allocations are determined purely by risk sharing considerations, i.e., \( d_i^{\mathcal{R}}(J_E) = d_i(J_E) \). In this case, Lemma 5 formalizes the traditional view of financial innovation by showing that the equilibrium portfolio allocation diversifies and distributes the risks “effectively.” More generally, Lemma 5 partially generalizes the traditional view to the case with belief heterogeneity, by showing that the risk sharing components of trades always minimize the average consumption variance.

I refer to \( \Omega^R(J_E) \) as the \textit{uninsurable variance}, since this is the minimum possible average consumption variance among all feasible portfolio allocations. Lemma 5 implies that \( \Omega(J_E) \) will be greater than \( \Omega^R(J_E) \) to the extent that the equilibrium portfolios, \( \{d_i(J_E) = d_i^{\mathcal{R}}(J_E) + d_i^{\mathcal{S}}(J_E)\}_{i} \), differ from the risk sharing components, \( \{d_i^{\mathcal{R}}(J_E)\}_{i} \). In particular, the residual variance, \( \Omega^S(J_E) = \Omega(J_E) - \Omega^R(J_E) \), can be purely attributed to the speculative components of portfolios, which are driven by belief heterogeneity. Thus, I refer to \( \Omega^S(J_E) \) as the \textit{speculative variance} in this economy. Using Eqs. (3.1) and (3.3), the speculative variance can also be calculated as (cf. Appendix 3.A):

\[
\Omega^S(J_E) = \frac{1}{|I|} \sum_{i \in I} \frac{\theta_i}{\tilde{\theta}_i} \left( \frac{\tilde{\mu}_i^{J_E}}{\tilde{\theta}_i} \right) \left( A|J_E \right)^{-1} \frac{\tilde{\mu}_i^{J_E}}{\tilde{\theta}_i}. (3.4)
\]

Eqs. (3.3) and (3.4) establish the decomposition of the average consumption variance into two components, \( \Omega(J_E) = \Omega^R(J_E) + \Omega^S(J_E) \). I next present the main result, which shows that financial innovation always decreases the uninsurable variance, while it also always increases the speculative variance.

\textbf{Theorem 9.} Consider an economy \( \mathcal{E}(J) \), and let \( J_O \) and \( J_N \) respectively denote the set of old and new assets. Consider the average consumption variance and its components respectively for the sub-economy \( \mathcal{E}(J_O) \) without any new assets and for the sub-economy \( \mathcal{E}(J_O \cup J_N) \) with new assets.

(i) Financial innovation always reduces the uninsurable variance, that is:

\[
\Omega^R(J_O \cup J_N) \leq \Omega^R(J_O).
\]
The inequality is strict if and only if \( d_i^{R_{JN}} (J_O \cup J_N) \neq 0 \) for some trader \( i \in I \).

(ii) Financial innovation always increases the speculative variance, that is

\[
\Omega^S (J_O \cup J_N) \geq \Omega^S (J_O).
\]

The inequality is strict if and only if \( d_i^{S_{JN}} (J_O \cup J_N) \neq 0 \) for some trader \( i \in I \).

Consequently, financial innovation reduces the average consumption variance, i.e., \( \Omega (J_O \cup J_N) \leq \Omega (J_O) \), if and only if it reduces the uninsurable variance more than it increases the speculative variance, i.e., \( \Omega^R (J_O) - \Omega^R (J_O \cup J_N) \geq \Omega^S (J_O \cup J_N) - \Omega^S (J_O) \).

The first part of this theorem is a corollary of Lemma 5, and it shows that financial innovation always provides some risk sharing benefits. But the second part of the theorem identifies a second force which always operates in the opposite direction. In particular, when there is belief heterogeneity, financial innovation also always increases the speculative variance. Hence, the net effect of financial innovation on average consumption variance is ambiguous, and it depends on the relative strength of the two forces. Most of the literature on financial innovation considers the special case without belief heterogeneity. Theorem 9 shows that the assumption of no belief heterogeneity is restrictive, as it shuts down an important economic channel by which financial innovation always has a positive effect on the average consumption variance.

It is also worth emphasizing the generality of the second part of Theorem 9. The result applies for all sets of existing and new assets, \( J_O \) and \( J_N \), with no restrictions on their joint distribution \( N (\mu_{true}, \Lambda_{true}) \) and no restrictions on traders’ beliefs \( \left\{ N (\mu_i^{j}, 1/\tau) \right\}_{i,j} \) for their mean returns. For example, Theorem 9 shows that financial innovation increases the speculative variance even if there is no belief disagreement about new assets, i.e., even if \( \mu_i^{j} = \mu^j \) for all \( i, i \in I \) and \( j \in J_N \). The intuition behind the generality of this result depends on a powerful economic force: the hedge-more/bet-more effect. In the next section, I introduce this effect and I use it to provide a sketch proof for Theorem 9. Before doing so, I note the following result which complements Theorem 9 by showing that there exists economies in which financial innovation increases the average consumption variance by an arbitrary amount.

**Proposition 2.** Consider an economy \( \mathcal{E} (J) \), and suppose that there exists \( i, i \in I \) and \( j \in J_N \) such that \( \mu_i^{j} \neq \mu_i^{\hat{i}} \) (that is, there are some belief differences about new assets). Let \( \mathcal{E}_K (J) \) denote
the economy which is identical to the economy $\mathcal{E}(J)$ except for the fact that traders’ beliefs about new assets are given by $\mu_{i,K}^{J_N} = K \mu_{i}^{J_N}$ for each $i \in I$. Denote the average consumption variance in the sub-economy $\mathcal{E}_K (J_E)$ with $\Omega_K (J_E)$. Then, for any $\eta \in \mathbb{R}_{++}$, there exists $K_{\eta} > 0$ such that for each $K > K_{\eta}$, financial innovation in the economy $\mathcal{E}_K (J)$ increases the average variance by more than $\eta$, i.e.,

$$\Omega_K (J_O \cup J_N) > \Omega_K (J_O) + \eta.$$ 

### 3.4 The Hedge-More/Bet-More Effect

This section introduces the hedge-more/bet-more effect and provides a sketch proof of the main result, Theorem 9. The hedge-more/bet-more effect is best understood in the context of a simple example, which I introduce next. I then provide a sketch proof of Theorem 9, and use the hedge-more/bet-more effect to provide the intuition for the result.

Let $I = \{1, 2\}$, $\theta_1 = \theta_2 = \theta$, and $w_1 = w_2 = 0$, which ensures that traders have constant endowments. This further implies that the return of each asset is uncorrelated with traders’ endowments, i.e., $\lambda_i^j = 0$ for each $i \in I$. Thus, the portfolio allocations in this economy will be determined purely by speculation considerations. Consider a set of two assets, $J = \{A, B\}$, where $A$ is the old asset and $B$ is the new asset, i.e., $J_O = \{A\}$ and $J_N = \{B\}$. Suppose traders’ beliefs for the means of the asset payments are given by $\mu_1^j = \bar{\mu}_1^j \equiv \bar{\mu}^j \geq 0$ and $\mu_2^j = \bar{\mu}_2^j = -\bar{\mu}^j$ for each $j \in J$. That is, trader 1 is relatively optimistic for both assets. Let traders’ common belief matrix be given by $\Lambda = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$ [cf. Eq. (3.2)].

First consider the equilibrium of the sub-economy $\mathcal{E}(\{A\})$ in which only the old asset, $A$, is available. Using Eq. (3.7), the speculative component of trader 1’s portfolio can be calculated as

$$d_{1}^S (\{A\}) = \frac{1}{\theta} \frac{1}{x} \bar{\mu}^A.$$  

(3.1)

The speculative variance [cf. Eq. (3.4)] is given by:

$$\Omega^S (\{A\}) = \frac{1}{\theta^2} \frac{(\bar{\mu}^A)^2}{x}.$$  

(3.2)
Next consider the equilibrium of the economy $E \{(A, B)\}$ in which both assets are available. After inverting the matrix $\Lambda$, the speculative components of portfolios can be calculated as

$$d_1^S \{(A, B)\} = -d_2^S \{(A, B)\} = \frac{1}{\theta} \frac{1}{x^2 - y^2} \left[ \bar{\mu}^A x - \bar{\mu}^B y \right].$$

(3.3)

The speculative variance is given by:

$$\Omega^S \{(A, B)\} = \frac{1}{\theta^2} \frac{x \left( (\bar{\mu}^A)^2 + (\bar{\mu}^B)^2 \right) - 2y\bar{\mu}^A\bar{\mu}^B}{x^2 - y^2}.$$

(3.4)

In the context of this example, Theorem 9 implies that the speculative variance expression in (3.4) is greater than the speculative variance in (3.2) for any set of parameters $\mathcal{P} \equiv (\bar{\mu}^A, \bar{\mu}^B, x, y)$. To illustrate the generality of this result, I consider a particular parameterization

$$\mathcal{P}(1) = (\bar{\mu}^A = \bar{\mu} > 0, \bar{\mu}^B = \bar{\mu} > 0, x > 0, y < 0),$$

(3.5)

so that trader 1 is equally optimistic about the assets and the assets are negatively correlated. When both assets are available, trader 1 holds positive speculative positions on each one of them because she is equally optimistic about each asset [cf. Eq. (3.3)]. Then, given the negative correlation of assets, trader 1’s two bets happen to provide a natural hedge. In this case, a naive intuition would predict that the speculative variance should be lower than the case in which only asset $A$ is available. However, a closer look reveals that this is not the case. In particular, calculating the expressions in (3.2) and (3.4) for the parameterization in $\mathcal{P}(1)$ shows that

$$\Omega^S \{(A)\} = \frac{1}{\theta^2} \frac{x}{x} < \frac{1}{\theta^2} \frac{2\mu^2}{x + y} = \Omega^S \{(A, B)\}. $$

(3.6)

Hence, financial innovation increases the speculative variance also for the economy with parameterization $\mathcal{P}(1)$. In fact, the inequality (3.6) reveals that financial innovation increases the speculative variance more in the case $y < 0$ than in the case $y > 0$, which is in stark contrast

---

7Note that the parameters $x$ and $y$ can be freely chosen subject only to the restriction that $\Lambda$ is a positive definite matrix. In particular, Eq. (3.2) implies that any positive definite matrix $\Lambda$ can be obtained by varying the exogenous matrix $k$ and the exogenous precision parameters $\{r_{true}^j \in \mathbb{R}^+\}_j$ and $\{r^j \in \mathbb{R}^+\}_j$. 109
with the naive intuition. The naive intuition is misleading because it does not account for the hedge-more/bet-more effect, which I introduce next.

To illustrate the hedge-more/bet-more effect, it is useful to consider an alternative parameterization of the example

$$\mathcal{P}(2) = (\mu^A > 0, \mu^B = 0, \ x, \ y = x - \varepsilon), \quad (3.7)$$

where $\varepsilon > 0$ is a small positive scalar. That is, trader 1 is only optimistic about the existing asset $A$, and the new and the existing assets are highly correlated. Consider the bets of trader 1 on asset $A$ respectively in the sub-economies $\mathcal{E}(\{A\})$ and $\mathcal{E}(\{A, B\})$. If the new asset $B$ is not available, then the trader’s speculative position on $A$ is given by $d_1^S(\{A\}) = \frac{1}{\theta} \mu^A$ [cf. Eq. (3.1)]. If instead asset $B$ is available, then her speculative position on $A$ is given by

$$d_1^{S,A}(\{A, B\}) = \frac{1}{\theta} \frac{\mu^A x}{x^2 - y^2} = \frac{1}{\theta} \frac{\mu^A x}{x^2 - (x - \varepsilon)^2} \quad (3.8)$$

[cf. Eq. (3.3)]. Comparing these expressions, note that the trader makes a much larger bet on the existing asset $A$ when she can also invest in a new asset $B$ which is highly correlated with $A$ [in particular, note that $\lim_{\varepsilon \to 0} d_1^S(\{A, B\}) = \infty$]. To see the intuition for this observation, note also that the trader’s position on asset $B$ is given by:

$$d_1^{S,B}(\{A, B\}) = -\frac{1}{\theta} \frac{\mu^A y}{x^2 - y^2}. \quad (3.9)$$

That is, the trader takes an opposite position (of similar size) on the new asset $B$ to hedge the risks from her bets on asset $A$. Moreover, as $\varepsilon \to 0$, the assets are nearly perfectly correlated and the trader is able to hedge almost any risk she takes. That is, the trader is expecting almost riskless returns (almost arbitrage) from her bet on asset $A$. This makes betting on $A$ very attractive to the trader, which increases her positions on both assets as captured by the portfolio demands in Eqs. (3.8) and (3.9).

More generally, when there are multiple correlated assets, the trader is able to hedge her speculative positions on the assets with disagreement. Because the trader can hedge more, the speculative trades look more attractive to her, which in turn enables her to bet more on these
Consider next the role of the hedge-more/bet-more effect in the case with parameterization \( P(1) \) in (3.5). Consider trader 1’s speculative positions on \( A \) with and without the new asset \( B \), which are given by:

\[
\mathbf{d}_1^{S,A}(\{A, B\}) = \frac{1}{\theta x + y} > \mathbf{d}_1^{S}(\{A\}) = \frac{1}{\theta x}
\]

(where the inequality follows since \( y < 0 \)). That is, in view of the hedge-more/bet-more effect, the trader makes an amplified bet on the first asset when a second asset which is negatively correlated with the first is available. This amplification of speculative positions increases the speculative variance, countering the naive intuition noted above. The naive intuition is misleading because it only concerns the signs of trader’s bets and misses the fact that the levels of trader’s speculative positions also respond to the correlation structure.

### 3.4.1 Sketch Proof for the Main Result

This subsection provides a sketch proof for the second part of Theorem 9, and uses the hedge-more/bet-more effect to provide the intuition. It is useful to note the following lemma which simplifies the problem of analyzing the speculative variance to the problem of analyzing the average consumption variance in a hypothetical economy in which all trade is driven by speculative considerations. The proof of this lemma directly follows from the expressions (3.6) and (3.7) (cf. Appendix 3.A).

**Lemma 6.** Given an economy \( \mathcal{E}(J) \), consider the hypothetical economy \( \mathcal{E}(J; H) \) that is identical except for the fact that traders have no endowment risk, i.e., \( \mathbf{w}_i = 0 \) for all \( i \in I \). Denote the price vector and portfolio allocations in the sub-economy \( \mathcal{E}(J_E; H) \) with \( \mathbf{p}(J_E; H) \) and \( \{\mathbf{d}_i(J_E; H)\}_i \).

(i) The speculative components of portfolios in the sub-economy \( \mathcal{E}(J_E) \) are identical to the portfolios in the economy \( \mathcal{E}(J_E; H) \), that is, \( \mathbf{d}_i^S(J_E) = \mathbf{d}_i(J_E; H) \) for each \( i \). Moreover, these trades solve the following problem:

\[
\max_{\mathbf{d}_i \in \mathbb{R}^{|J_E|}} \left( \tilde{\mu}_1^{J_E} \right)' \mathbf{d}_i - \frac{\theta_i}{2} \mathbf{d}_i' \Lambda_{J_E} \mathbf{d}_i,
\]  

(3.10)
which is the certainty equivalent problem (3.4) for trader $i$ in the hypothetical economy $\mathcal{E}(J_E)$.

(ii) The speculative variance in the sub-economy $\mathcal{E}(J_E)$ is equal to the average consumption variance in the hypothetical economy $\mathcal{E}(J_E ; H)$, which is given by:

$$
\Omega^S(J_E) = \Omega(J_E ; H) = \frac{1}{|I|} \sum_{i} \frac{\theta_i}{\theta} d_i(J_E ; H)' A_{J_E} d_i(J_E ; H).
$$

In view of this lemma, it suffices to show that financial innovation always increases the average consumption variance in the hypothetical economy $\mathcal{E}(J ; H)$. To show this, note that the constraint set of the maximization problem (3.2) for the sub-economy $\mathcal{E}(J_O ; H)$ is a subset of the constraint set of the same problem for the economy $\mathcal{E}(J_O \cup J_N ; H)$. Hence, the maximum value of problem (3.10) is always greater for the economy $\mathcal{E}(J_O \cup J_N ; H)$ than for the economy $\mathcal{E}(J_O ; H)$. Moreover, since problem (3.10) is quadratic, its maximum value is an increasing linear function of the quadratic cost term in the objective function [evaluated at the optimum, $d_i(J_E ; H)$]. More specifically, the maximum for this problem is given by:

$$
\frac{1}{2\theta_i} d_i(J_E ; H)' A_{J_E} d_i(J_E ; H).
$$

Since this expression is greater for the economy $\mathcal{E}(J_O \cup J_N)$ compared to the economy $\mathcal{E}(J_O)$, the expression in (3.11) [which is a weighted sum of the quadratic terms in (3.12)] is also greater for the economy $\mathcal{E}(J_O \cup J_N)$. This implies that the speculative variance satisfies $\Omega^S(J_O \cup J_N) \leq \Omega^S(J_N)$, which completes the sketch proof of Theorem 9. The proof in Appendix 3.A establishes that the inequality is strict whenever $d_i^{S,J_N}(J_O \cup J_N) \neq 0$, that is, whenever new assets are not redundant for speculative trading purposes.

I next use this sketch proof and the hedge-more/bet-more effect to provide the intuition for the second part of Theorem 9. In view of Lemma 6, it is sufficient to understand the average consumption variance in the hypothetical economy $\mathcal{E}(J ; H)$ in which all trades are driven by speculative considerations.

Problem (3.10) shows that the portfolio allocations in this economy are determined by two forces. On the one hand, trader $i$ makes positive expected returns by betting on assets for which her belief is different than the average belief (captured by the first term, $(\hat{\mu}_i^{J_E})' \hat{d}_i$). On the other hand, since assets are risky and the trader is risk averse, she makes certainty
equivalent losses from her bets (captured by the quadratic cost term, $\frac{\theta_i}{2} \mathbf{d}_i' \Lambda |_{J_E} \mathbf{d}_i$). Hence, the equilibrium portfolio allocation of trader $i$ balances the benefits of speculative bets with the costs of additional variance generated by these bets. Moreover, in view of the quadratic nature of problem (3.10), traders’ net benefits are proportional to her certainty equivalent losses. To understand this point better, consider an exogenous change in traders’ beliefs, $\{\mu^E_i\}_i$, which increases the net benefit of a trader $i$ [for example, by making the trader relatively more optimistic]. This type of change will also necessarily increase trader $i$’s certainty equivalent losses, $\frac{\theta_i}{2} \mathbf{d}_i (J_E ; H)' \Lambda |_{J_E} \mathbf{d}_i (J_E ; H)$. Intuitively, a trader that expects to make a greater return must be taking greater speculative positions, which implies that she must also be taking greater risks.

Next note that, in view of the hedge-more/bet-more effect, new assets make speculative trades more attractive for the traders. This is captured by the fact that each trader’s perceived net returns [i.e., the solution to problem (3.10)] increases after the introduction of new assets. However, the trader is able to attain high returns only by betting more and incurring high costs of variance. Hence, financial innovation always increases the average consumption variance in the hypothetical economy $\mathcal{E} (J ; H)$, and thus, the speculative variance in the original economy $\mathcal{E} (J)$.

### 3.5 Comparative Statics of Consumption Variance

Theorem 9 characterizes the effect of financial innovation on the components of average consumption variance. The result also shows that the net effect on average consumption variance depends on the difference expressions:

$$\Omega^R (J_O) - \Omega^R (J_O \cup J_N) \text{ and } \Omega^S (J_O \cup J_N) - \Omega^S (J_O).$$

These expressions respectively capture the “benefits” of financial innovation in terms of the reduction of the uninsurable variance and its “costs” in terms of the increase in speculative variance. This section establishes a number of comparative statics of these expressions, which are useful to identify the economic environments in which financial innovation is more likely to increase the average consumption variance. The following assumption simplifies the analysis in
Assumption (A2). Suppose $\bar{\mu}_i^{Jo} = 0$ for all $i \in I$ so the traders agree on the payoffs of existing assets.

This assumption is somewhat extreme, but it captures the intuitive idea that there is relatively more disagreement about new assets than about old assets (the past returns of which have been observed by traders). Moreover, this assumption is further justified in Section 3.6 in which traders’ prior beliefs are endogenized through learning. The next result establishes that, under assumption (A2), new assets increase the speculative variance more when they are introduced into a relatively more complete asset market.

**Theorem 10.** Consider an economy $\mathcal{E}(J)$, and let $J_O$ and $J_N$ respectively denote the set of old and new assets. Let $\hat{J}_O \subset J_O$ denote a strict subset of old assets, and consider the sub-economy $\mathcal{E}(\hat{J}_O \cup J_N)$ which is identical to the original economy except for the fact that it has fewer old assets. Then, the introduction of the same set of new assets, $J_N$, has a greater effect on the speculative variance in the economy $\mathcal{E}(J_O \cup J_N)$ than in the sub-economy $\mathcal{E}(\hat{J}_O \cup J_N)$, that is:

$$\Omega^S(\hat{J}_O \cup J_N) - \Omega^S(J_O) \leq \Omega^S(J_O \cup J_N) - \Omega^S(J_O).$$

The intuition for Theorem 10 is the hedge-more/bet-more effect. As illustrated by the example described in Section 3.4, traders take greater speculative positions on new assets when there are existing assets with correlated payments. This is because traders can hedge their bets on the new assets using the existing assets, which enable them to bet more on the new assets. When there are more existing assets, there are more opportunities for hedging the bets on new assets, which suggests that the hedge more bet more effect is stronger. Theorem 10 formalizes this intuition and shows that new assets generate more speculative variance in more complete asset markets.

The next result uses the same reasoning to show that new assets that are correlated with existing assets generate relatively more speculative variance. Intuitively, the hedge more bet more effect is operational only if new assets are correlated with existing assets. This result creates a presumption that new derivative assets may increase the speculative variance more (relative to new non-derivative assets) because the bets on these assets are relatively easy to
hedge using the existing underlying assets.

**Theorem 11.** Consider an economy $E(J)$, and let $J_N = \{j_1, j_2\}$ denote the set of new assets. Assume that assets $j_1$ and $j_2$ are identical except for the fact that $j_1$ is correlated with at least one existing asset $j \in J_O$ while $j_2$ is not correlated with any of the assets in $J_O$.\(^8\) Consider the speculative variance in the sub-economies $E(J_O \cup \{j_1\})$ and $E(J_O \cup \{j_2\})$ in which only one of the new assets is available. The introduction of the new asset that is correlated with existing assets increases the speculative variance more than the introduction of the new asset that is uncorrelated with existing assets, that is:

$$\Omega^S(J_O \cup \{j_1\}) - \Omega^S(J_O) > \Omega^S(J_O \cup \{j_2\}) - \Omega^S(J_O).$$

### 3.6 Dynamic Environment and Equilibrium

This section introduces a simple dynamic extension of the model for two purposes. First, the dynamic model enables the analysis of the short and the long run effects of financial innovation in a unified framework. The main result in this section shows that, while financial innovation may increase the average consumption variance in the short run, it always reduces the variance in the long run. Second, the dynamic model is also useful to analyze a policy of timing the introduction of new assets to exchange. This analysis shows that delaying the introduction of new assets reduces the average consumption variance at all dates.

Consider a dynamic model in which the above described static economy is repeated at each date $t \in \{0, 1, \ldots, \infty\}$. In particular, trader $i$'s endowment at date $t$ is given by the random variable $w_i V(t)$, where $V(t)$ is standard normal and i.i.d. over time. At each date $t$, traders can exchange a given set of assets, potentially to share and diversify their risks. After the Walrasian markets for assets clear, traders' endowments and the asset returns are realized.

\(^8\)More specifically, suppose the unknown components of these traders, $U^{j_1}, U^{j_2}$, have the same distribution and traders have the same beliefs for these components. Suppose also that known components $k^{j_1}, k^{j_2}$ are such that

$$\begin{align*}
(k^{j_1})' k^{j_0} &\neq 0, \\
(k^{j_2})' k^{j_0} &= 0 \\
\text{and } (k^{j_1})' w_i &= (k^{j_2})' w_i \text{ for all } i \in I.
\end{align*}$$
Traders consume their net income and the economy moves to the next period (i.e., traders do not carry asset positions from one period to the other, which simplifies the analysis).

There are a total of $J^D$ assets, created gradually at dates $t_C = \{t^j_C\}_{j \in J^D}$. In particular, the set of available assets at date $t$ is given by:

$$J(t) = \{ j \in J^D \mid t^j_C \leq t \}.$$  \hspace{1cm} (3.1)

The payoff of each available asset $j \in J(t)$ is given by the random variable $k^j \mathbf{V}(t) + U^j(t)$ [cf. Eq. (3.1)], where the asset specific random variable $U^j(t)$ has distribution $N(\mu^j_{true}, \tau^j_{true})$ and is i.i.d. over time.

The central feature of the dynamic model is the evolution of traders' beliefs for asset payments. Similar to the static model, I assume that traders have prior beliefs for the means of the unknown components $\{\mu^j_{true}\}_{j \in J^D}$ given by $\{N(\mu^j_{true}, \tau^j_{true})\}_{i \in I, j \in J^D}$. However, different than the static model, traders also update these priors as they observe the realizations of asset returns [more specifically, the returns of the asset specific components $\{U^j\}_{j}$]. In particular, trader $i$ observes payoff realizations of asset $j$ starting at the creation date $t^j_C$. Thus, the trader's Bayesian posterior for $\mu^j$ is also normally distributed with mean

$$\mu^j_i(t) = \frac{\tau^j \mu^j_{true} + (t - t^j_C) \tau^j_{true} w^j(t^j_C, t)}{\tau^j + (t - t^j_C) \tau^j_{true}}$$ \hspace{1cm} (3.2)

and precision $\tau^j_i(t) = \tau^j + (t - t^j_C) \tau^j_{true}$. Here, $w^j(t^j_C, t) = \{w^j(t^j_C), ..., w^j(t - 1)\}$. This further implies that the marginal distribution of asset returns with respect to trader $i$'s beliefs is also multivariate normal, given by:

$$N(\mu_i(t) \equiv (\mu^j_i(t))_{j \in J}, \Lambda_i(t) \equiv (k^j(t))' k^j(t) + \text{Diag}(t) \left( \frac{1}{\tau^j + (t - t^j_C) \tau^j_{true}} + \frac{1}{\tau^j_{true}} \right)).$$ \hspace{1cm} (3.3)

Note that $\mu_i(t)$ is a random variable that depends on history while $\Lambda_i(t)$ evolves deterministically.

To analyze the asset introduction policy, I allow for the possibility that assets may be
introduced to exchange at a later date than \( t_c^j \). In particular, let \( t_E = \{ t_E^j \geq t_C^j \}_{j \in J_D} \) denote the times at which assets are introduced to exchange. I refer to \( t_E \) as the asset exchange policy. Given the policy \( t_E \), the set of exchanged assets at date \( t \) is given by

\[
J_E(t) = \left\{ j \in J \mid t_E^j \leq t \right\} \subset J(t).
\] (3.4)

Let \( E^D(t_E) \) denote the above described dynamic economy, which is a collection of the static economies

\[
E(t, J_E(t)) = \left( J_E(t) \ ; \ \{ w_i \}_i ; \ \{ k^j \}_j ; \ \{ \mu^j_{true}, \tau^j_{true} \}_j ; \ (\mu_i(t), \Lambda_i(t))_{i,j} \right).
\] (3.5)

Let \( p(t), \{ d_i(t, J_E(t)) \}_i \) denote the equilibrium price vector and allocations for the economy \( E(t, J_E(t)) \). Also let \( \Omega(t, J_E(t)) \) denote the equilibrium variance for the economy \( E(t, J_E(t)) \). A dynamic equilibrium can be described as follows.

**Definition 7.** Consider a dynamic economy \( E^D(t_E) \) with an asset introduction policy \( t_E \). A dynamic equilibrium is a collection of stochastic processes for marginal beliefs \( \{ N(\mu_i(t), \Lambda_i(t)) \}_{i \in I} \) for asset returns, the price vector \( p(t) \) and asset allocations \( \{ d_i(t, J_E(t)) \}_i \) such that the marginal beliefs after every history are formed with Bayes’ rule (cf. Eqs. (3.2) – (3.3)) and \( (p(t), \{ d_i(t, J_E(t)) \}_i) \) constitute the unique equilibrium of the static economy \( E(t, J_E(t)) \) in (3.5).

Proposition 1 implies that a dynamic equilibrium also exists and is unique. The next subsection characterizes the dynamic equilibrium and the evolution of average consumption variance.

### 3.6.1 Dynamic Path of Consumption Variance

First note that the traders’ beliefs in (3.6) implies

\[
\hat{\mu}^j_i(t) = \frac{\tau^j}{\tau^j + (t - t_c^j) \tau^j_{true}} \hat{\mu}^j_i.
\] (3.6)
In particular, even though the levels of traders’ beliefs can be stochastic, their belief differences evolve deterministically. This is because all traders update their priors from the same set of observations, \( \{u^j(t), \ldots, u^j(t-1)\} \) (and they are assumed to have the same initial precision, i.e., \( \tau^j_i = \tau^j \) for each \( j \)). Moreover, note that \( \tilde{\mu}_i(t) \) declines monotonically and limits to 0 as \( t \to \infty \). Intuitively, as traders learn by observing asset returns, their prior beliefs converge and their belief disagreements disappear in the long run.

Second, note that Eq. (3.6) also implies that traders’ portfolio allocations in (3.7) evolve deterministically. This further implies that the risk sharing and the speculative components of the average consumption variance evolve deterministically. Using the expressions in (3.3) and (3.4) for the static economy \( \mathcal{E}(t, J_E(t)) \), the components of variance evolve according to:

\[
\Omega^R(t, J_E(t)) = \frac{1}{|I|} \sum_{i \in I} \frac{\theta_i}{\bar{\theta}} \left(w_i'w_i - (\hat{\lambda}_i^{J_E(t)})'(\Lambda(t)|J_E(t))^{-1}\hat{\lambda}_i^{J_E(t)}\right), \quad (3.7)
\]

\[
\Omega^S(t, J_E(t)) = \frac{1}{|I|} \sum_{i \in I} \frac{\theta_i}{\bar{\theta}} \left(\tilde{\mu}_i^{J_E(t)}(t)\right)'(\Lambda(t)|J_E(t))^{-1}\tilde{\mu}_i^{J_E(t)}(t).
\]

The next result characterizes the dynamic path of \( \Omega^R(t, J_E(t)) \) and \( \Omega^S(t, J_E(t)) \).

**Theorem 12.** Consider a dynamic economy \( \mathcal{E}^D(t_E) \) with an asset introduction policy \( t_E \). Then:

(i) The uninsurable variance always decreases over time, that is, \( \Omega^R(t+1, J_E(t+1)) < \Omega^R(t, J_E(t)) \) for all \( t+1 \).

(ii) If there is no financial innovation at date \( t+1 \), i.e., if \( J_E(t+1) = J_E(t) \), then the speculative variance may increase or decrease at date \( t+1 \). If instead there is financial innovation at date \( t+1 \), then the speculative variance decreases at date \( t+1 \), that is, \( \Omega^S(t+1, J_E(t+1)) \leq \Omega^S(t, J_E(t)) \).

Moreover, the asymptotic limit of the speculative variance is zero, that is, \( \lim_{t \to \infty} \Omega^S(t, J_E(t)) = 0 \).

The first part of this result follows from the first part of Theorem 9 along with the observation that the covariance matrix \( \Lambda(t)|J_E(t) \) becomes smaller over time (in positive definite order) as traders’ uncertainty about asset payoffs decreases. The second part of the result similarly follows from the second part of Theorem 9. At dates of no financial innovation, the specula-
Figure 3-1: The figure plots the dynamic path of average consumption variance in two example economies. In the top panel, financial innovation (at dates 0 and 10) reduces the average consumption variance at all dates. In the bottom panel, financial innovation increases the average consumption variance in the short run, and decreases it in the long run.

tive variance declines because traders' belief disagreements become smaller as they learn about asset payoffs. Asymptotically, belief disagreements disappear completely and the speculative variance limits to zero. Figure 3-1 demonstrates the dynamic evolution of average consumption variance, which is the sum of the two components analyzed in Proposition 12. The average consumption variance $\Omega (t, J_E (t))$ typically declines over time, but it spikes up, and may increase, at dates of financial innovation. Whether financial innovation increases or decreases the average consumption variance depends on the strength of the risk sharing and the speculative motives for trade, as analyzed in Section 3.5.

3.6.2 Timing the Introduction of New Assets

Theorem 12 illustrates a short run and long run trade-off regarding the effect of financial innovation on the average consumption variance. On the one hand, financial innovation always decreases the average consumption variance in the long run (asymptotically), because new assets provide some risk sharing services. On the other hand, financial innovation may temporarily
increase the average consumption variance in the short run, because new assets also generate some speculation. In view of these observations, a natural conjecture is that a policy of slowing down financial innovation, by delaying the introduction of a newly innovated asset, may decrease the average consumption variance at all dates. The next result formalizes this conjecture.

The first part of the result shows that delaying the introduction of new assets always entails a trade-off since it has opposing effects on the two components of the average consumption variance. The second part shows that, whenever financial innovation increases the average consumption variance in the short run, there is a delayed introduction policy that reduces the average consumption variance at all dates.

**Theorem 13.** Consider a dynamic economy $E^D (t_E = t_C)$ in which all assets are introduced as soon as they are created. Suppose $\Omega (1, J_C (1)) > \Omega (1, J_C (0))$, that is, financial innovation at date 1 has a positive net effect on the average consumption variance. Consider a delayed introduction policy $t_D$ which is identical to $t_C$ except that the new assets $J_C (1) \setminus J_C (0)$ are introduced at date $T + 1$ instead of date 1. Then, there exists a unique $T \geq 1$ such that:

$$\Omega (t, J_D (t)) < \Omega (t, J_C (t))$$

for all $t$ with strict inequality for $t = 1$, if $T \in [1, T]$, and

$$\Omega (t, J_D (t)) > \Omega (t, J_C (t))$$

for some $t$, if $T > T$.

(3.8)

That is, a delayed introduction policy reduces the average consumption variance at all dates, if and only if the delay $T$ is less than a threshold $T$.

To see the intuition for this result, first note that Theorem 9 implies that any delayed introduction policy satisfies:

$$\Omega^R (t, J_D (t)) \geq \Omega^R (t, J_C (t)) \quad \text{and} \quad \Omega^S (t, J_D (t)) \leq \Omega^S (t, J_C (t))$$

for all dates $t \geq 1$.

That is, delaying the introduction of new assets always leads to a greater risk sharing variance (due to the reduction of risk sharing opportunities), but it also always leads to a lower speculative variance (due to the reduction of speculation opportunities). Moreover, the assumption in Section 3.6 is that traders learn about the payoffs of new assets regardless of whether they are introduced to exchange or not. Under this assumption, the reduction in the speculative variance, $\Omega^S (t, J_C (t)) - \Omega^S (t, J_D (t))$, becomes smaller over time, as traders’ belief disagree-
Figure 3-2: The top left panel illustrates the effect of a well-timed delayed introduction policy on the average consumption variance, and the top right panel breaks down this effect into the two components of variance. The bottom panel illustrates the effect of delaying the introduction of new assets for too long.

Intuitively, delaying the introduction of new assets ensures that traders observe and become familiar with new assets before they actually trade these assets. If timed well, this type of delayed introduction can reduce the average consumption variance at all dates. Note, however, that this result is driven by the assumption that traders continue to learn about the returns of newly innovated assets, even if these assets are not introduced to exchange. It may be more
reasonable to assume that traders learn faster about assets which they actively trade. In that case, an alternative to the delayed introduction policy of Theorem 13 may be to reduce the amount of trade in new assets in the short run (rather than completely eliminate), e.g., through taxes that phase out over time.

3.7 Endogenous Financial Innovation

This section endogenizes the innovation of new assets. The main result in this section shows that new assets that increase the average consumption variance may be endogenously innovated by profit seeking market makers.

To analyze endogenous financial innovation, consider a simple extension of the dynamic model of Section 3.6 in which new assets are innovated by market makers. Suppose, for simplicity, that there is a single market maker with capacity to market $M$ assets where $M$ is a fixed positive integer. Suppose that the market maker can choose from a finite set of assets $\mathcal{J}$ (with at least $M$ elements) which can be roughly interpreted as the technology for financial innovation. The market maker invests in an $M$ element subset $J^D$ of the assets in $\mathcal{J}$, creates these assets at date 0, and makes the market for these assets at all dates. For simplicity, consider the introduction policy $t_E = \left(t^D_c = 0\right)_{j \in J}$ so that all innovated assets are immediately introduced to exchange.

The market maker unilaterally sets a fixed membership fee $\pi_i(t, J^D)$ for each trader $i$ and each date $t$. If the trader agrees to pay the fixed fee, then she becomes a member of the market and she can trade the assets, $J^D$, at equilibrium prices. Otherwise, she remains outside the market and she consumes her endowment. This assumption is extreme since it gives all the bargaining power to the market maker (and since it allows the market maker to price differentiate perfectly). I adopt this assumption for two reasons. First, it simplifies the analysis considerably while capturing the essential economic trade-offs facing the market maker. Second, given that my focus is on understanding the market maker’s choice among the various subsets of $\mathcal{J}$, the assumption is appropriate since it does not necessarily bias this choice towards particular subsets of assets.\footnote{In contrast, the assumption would be inappropriate if my objective was to analyze the innovation/no-}
The equilibrium of this economy can be characterized in two steps. I first characterize the dynamic equilibrium for any given choice of assets, $J^D \subset J$. I next calculate the present discounted profits of the market maker from each choice of $J^D \subset J$, which characterizes the optimal innovation decision.

Given the choice, $J^D$, of assets, the market maker sets the fixed fees $\{\pi_i (t, J^D)\}_{i,t}$ such that each trader chooses to participate at each date. Note that the fixed fees that traders pay are independent of their portfolio decisions. Thus, the optimal portfolio decision of a trader is the same as the previous sections, which implies that the characterization of the dynamic equilibrium remains unchanged. In particular, the equilibrium price vector and portfolio allocations at each date $t$ are respectively given by Eqs. (3.6) and (3.7) [corresponding to the date $t$ static economy $E (t, J^D)$].

Next consider the fixed fee that the market maker sets, $\pi_i (t, J)$. Since the market maker has all the bargaining power, it extracts the full surplus from each trader at each date. Note that the trader is just willing to pay the difference of her certainty equivalent payoff from being a member of the market [which is the expression in (3.4)] and her certainty equivalent payoff from consuming her endowment [which is equal to $-\frac{\theta_i}{2} w_i' w_i$]. Substituting for the price vector and trader demands from Eqs. (3.6) and (3.7), the expression in (3.4) can be simplified and the membership price can be solved as:

$$
\pi_i (t, J^D) = \frac{\theta_i}{2} \left( \frac{\bar{\mu}_i^{J^D} (t) \bar{\lambda}_i^{J^D}}{\theta_i} \right)' \Lambda (t) \left[ \frac{\bar{\mu}_i^{J^D} (t)}{\theta_i} - \bar{\lambda}_i^{J^D} \right].
$$

(3.1)

This expression illustrates the two motives for innovation in this economy. The market maker’s profits are larger for assets that are either correlated with traders’ endowment [assets with large $\bar{\lambda}_i^{J^D}$], or for assets that generate greater belief disagreements [assets with large $\bar{\mu}_i^{J^D} (t)$]. In the first case, the market maker caters to traders that use the assets to hedge their endowments. In the second case, the market maker caters to traders that use the assets to speculative on their differing opinions.

Consider next the market maker’s innovation decision at date 0. The market maker chooses the set of assets, $J^D$, which maximizes the present discounted profits from all traders. That is,
$J^D$ solves:

$$
\max_{J^D \subset J, \ |J^D|=M} \sum_{t=0}^{\infty} \gamma^t \sum_{i \in I} \pi_i \left(t, J^D\right),
$$

where $\gamma < 1$ denotes the discount factor of the market maker. The next result characterizes the optimal innovation decision in two extreme cases.

**Theorem 14.** Consider the above described dynamic economy with endogenous innovation.

(i) There exists $\bar{\gamma} \in (0, 1)$ such that if $\gamma > \bar{\gamma}$, then the set of innovated assets, $J^D$, solves the problem:

$$
\min_{J^D \subset J, \ |J^D|=M} \left( \lim_{t \to \infty} \Omega^R \left(t, J^D\right) \right).
$$

That is, the market maker innovates assets that minimize the asymptotic uninsurable variance.

(ii) Suppose traders disagree on some of assets in $J$, that is, there exists $i, i \in I$ and $j \in J$ such that $\mu_i^j \neq \mu_i^j$. Suppose also that traders’ beliefs are parameterized by $K \in \mathbb{R}_+$, with $\mu_{i,K}^j = K \mu_i^j$ for all $i$. Given any discount factor $\gamma < 1$, there exists $\tilde{K}$ such that if $K > \tilde{K}$ then the set of innovated assets, $J^D$, solves the problem:

$$
\max_{J^D \subset J, \ |J^D|=M} \sum_{t=0}^{\infty} \gamma^t \Omega^S \left(t, J^D\right).
$$

That is, the market maker innovates assets that maximize a discounted sum of speculative variance.

To see the intuition for the first part, recall that new assets asymptotically provide only risk sharing services [cf. Theorem 12]. Consequently, a sufficiently forward looking market maker innovates the set of assets that provides the best risk sharing services, by minimizing the uninsurable variance. In contrast, when there is sufficient disagreement about asset payoffs, a sufficiently impatient market maker innovates the set of assets that generates the most amount of speculation. In this case, the market maker completely disregards the risk sharing aspect in innovating new securities.

---

10 A low discount factor, $\gamma$, may capture in reduced form the potential future competition from the entry of other market makers since financial products are not protected by patents. It may also capture the fact that the assets are designed by managers at the firm who may have shorter horizons than the shareholders.
3.8 Conclusion

In this essay, I have theoretically analyzed the effect of financial innovation on the allocation of risks in a CARA-Normal economy in which both the speculation and risk sharing forces are present. Financial innovation increases the opportunities for risk sharing and diversification because new assets are correlated with traders’ endowment risks. However, financial innovation also increases the opportunities for speculation because traders have heterogeneous prior beliefs about asset payoffs. I have defined the average consumption variance as a measure of portfolio risk for this economy, and I have decomposed it into two components: the uninsurable variance, defined as the variance that would obtain if there were no belief heterogeneity, and the speculative variance, defined as the residual amount of variance that results from speculative trades based on belief heterogeneity. My main result has established that financial innovation always increases the speculative variance in a CARA-Normal economy. This result suggests that the traditional analyses of financial innovation have been restrictive, because they have considered a special case (homogeneous priors) and shut down an important economic channel through which financial innovation has a positive effect on the average consumption variance.

My analysis has also uncovered an economic force that amplifies speculation, the hedge-more/bet-more effect: Traders make bets on new assets which they then hedge by taking complementary positions on existing assets, which in turn enables them to place larger bets and take on greater risks. Using the hedge-more/bet-more effect, I have shown that new assets increase the risks more when they are introduced into more complete asset markets, and when they are correlated with existing assets. These results suggest that as asset markets get more complete, they become more susceptible to speculation and further financial innovation is more likely to be destabilizing. A fruitful future research direction may be to investigate in more detail these and other implications of the hedge-more/bet-more effect.

My analysis naturally raises the question of whether, and how, financial innovation should be regulated. I have made a first pass at this question by considering a dynamic extension of the model in which traders observe and learn about the payoffs of new assets. My dynamic model suggests that a policy of delaying the introduction of new assets, if well timed, reduces the average consumption variance at all dates. However, this result relies on the assumption that traders continue to learn about the payoffs of newly innovated assets, even if the assets
are not introduced to exchange. Relaxing this assumption introduces an additional trade-off between speculation and learning, which I plan to analyze in future work.
3.A Appendix: Omitted Proofs

Proof of Lemma 5. Note that the objective function for Problem (3.2) is given by

\[ \Omega \left( \{ \hat{d}_i \} _i \right) = \frac{1}{|I|} \sum _i \frac{\theta _i}{\theta} \left( w'_i w_i + \left( \hat{d}_i \right)' \Lambda |J_E \hat{d}_i + 2 \left( \hat{d}_i \right)' \lambda _i ^{JE} \right) . \]

The first order conditions give

\[ \Lambda |J_E \hat{d}_i + \lambda _i = \gamma \frac{\theta}{\theta_i} \text{ for each } i \in I, \]

where \( \gamma \in \mathbb{R}_{|J_E|} \) is a vector of Lagrange multipliers. It can be checked that \( \hat{d}_i ^R (J_E) = - (\Lambda |J_E|)^{-1} \hat{\lambda}_i ^{JE} \) satisfies these first order conditions given the vector \( \gamma = \left( \sum _{i \in J} \lambda _i ^{JE} \right) / |I| \), showing that \( \{ \hat{d}_i ^R (J_E) \} _i \) is the unique solution to Problem (3.2). Plugging in the optimal solution, the objective function can be solved as

\[
\frac{1}{|I|} \sum _i \frac{\theta _i}{\theta} \left( w'_i w_i + \left( \hat{\lambda}_i ^{JE} \right)' (\Lambda |J_E|)^{-1} \hat{\lambda}_i ^{JE} - 2 \left( \hat{\lambda}_i ^{JE} \right)' (\Lambda |J_E|)^{-1} \lambda _i ^{JE} \right) = \\
\frac{1}{|I|} \sum _i \frac{\theta _i}{\theta} \left( w'_i w_i + \left( \hat{\lambda}_i ^{JE} \right)' (\Lambda |J_E|)^{-1} \hat{\lambda}_i ^{JE} - 2 \left( \hat{\lambda}_i ^{JE} \right)' (\Lambda |J_E|)^{-1} \lambda _i ^{JE} \right) = \Omega ^R (J_E) ,
\]

where the second line replaces \( \left( \hat{\lambda}_i ^{JE} \right)' (\Lambda |J_E|)^{-1} \lambda _i ^{JE} \) with \( \left( \hat{\lambda}_i ^{JE} \right)' (\Lambda |J_E|)^{-1} \lambda _i ^{JE} \) in view of the identity \( \sum _i x_i y_i = \sum _i x_i (y_i - \bar{y}) \) which holds when \( \sum _i x_i = 0 \). This completes the proof of Lemma 5.

Derivation of Eq. (3.4). To derive Eq. (3.4), first consider the expression

\[ |I| \left( \Omega (J_E) - \frac{1}{|I|} \sum _{i \in I} \frac{\theta _i}{\theta} \left( w'_i ^{JE} \right)' w_i ^{JE} \right) . \]

Using the definition of the average consumption variance in (3.1) and the portfolio demand in (3.7), this expression can be written as:
where the last line follows by simple algebra. Next, using the fact that the last line equals
\[\sum_{i \in I} \frac{\theta_i}{\theta} \left( \left( \frac{\mu_i^{JE}}{\theta} - \tilde{\chi}_i^{JE} \right) (\Lambda |J_E|)^{-1} \left( \frac{\mu_i^{JE}}{\theta} - \tilde{\chi}_i^{JE} \right) + 2 \left( \frac{\mu_i^{JE}}{\theta} - \tilde{\chi}_i^{JE} \right) (\Lambda |J_E|)^{-1} \chi_i^{JE} \right),\]
the average consumption variance can be solved as:
\[\Omega(J_E) = \frac{1}{|I|} \sum_{i \in I} \frac{\theta_i}{\theta} \left( \left( \frac{\mu_i^{JE}}{\theta} \right)^{\prime} \left( \frac{\mu_i^{JE}}{\theta} \right) - \left( \frac{\mu_i^{JE}}{\theta} \right)^{\prime} \left( \frac{\chi_i^{JE}}{\theta} \right) (\Lambda |J_E|)^{-1} \chi_i^{JE} \right) + \frac{1}{|I|} \sum_{i \in I} \frac{\theta_i}{\theta} \left( \frac{\mu_i^{JE}}{\theta} \right)^{\prime} (\Lambda |J_E|)^{-1} \tilde{\mu}_i^{JE}.\]

Using the definition of \(\Omega^R(J_E)\) in (3.3), it follows that the speculative variance, \(\Omega^S(J_E) = \Omega(J_E) - \Omega^R(J_E)\), is also given by the expression in (3.4).

**Proof of Lemma 6.** Consider any sub-economy \(\mathcal{E}(J_E ; H)\) of the hypothetical economy, and note that this economy features \(\chi_i^{JE} = 0\) for all \(i \in I\). By Eq. (3.7), this implies that \(d_i^R(J_E ; H) = 0\) [equivalently, \(d_i(J_E ; H) = d_i^S(J_E ; H)\)] for all \(i\). Intuitively, the hypothetical sub-economy has no need for risk sharing, which implies that the portfolio allocations are driven purely by speculation considerations. Note that a trader's portfolio in this hypothetical economy, which is equal to \(d_i^S(J_E ; H)\), solves the certainty equivalent problem (3.4) in the hypothetical economy. But note that the hypothetical economy features \(w_i^{JE} = 0\) and \(\lambda_i^{JE} = 0\), and also \(\lambda(J_E) = \frac{1}{|I|} \sum_{i \in I} \frac{\theta_i}{\theta} \mu_i^{JE}\) [by Eq. (3.6)]. Using these expressions, it follows that the certainty equivalent problem (3.4) for the hypothetical economy is equivalent to problem (3.10). This shows that \(d_i^S(J_E ; H)\) solves problem (3.10). Finally, by Eq. (3.7), note that the speculative trades in the original sub-economy \(J_E\) are identical to the speculative trades in the hypothetical economy, i.e., \(d_i^S(J_E) = d_i^S(J_E ; H)\). This completes the proof for the first part of the lemma. The proof for the second part follows by the expression for the average consumption variance in (3.1) after noting that \(w_i^{JE} = 0\) and \(\lambda_i^{JE} = 0\) for the hypothetical.
Proof of Theorem 9. Part (i). Consider the economy $\mathcal{E}(J)$ with the alternative risk-sharing portfolio $\hat{d}_i^R = \begin{bmatrix} d_i^R(J_O) \\ 0 \end{bmatrix}$ for each $i$. Note that

$$\Omega^R(J) \leq \Omega \left( J \mid \left\{ \begin{bmatrix} d_i^R(J_O) \\ 0 \end{bmatrix} \right\}_i \right) = \Omega^R(J_O). \tag{3.A.1}$$

Here the first inequality follows by invoking Lemma 5 for $J_E = J$ and the second equality follows by using the definition of $\hat{d}_i^R$ and invoking the same lemma for $J_E = J_O$. Note also that problem (3.2) in Lemma 5 is strictly concave, which implies that the inequality in (3.A.1) is strict if and only if $\hat{d}_i^R \neq d_i^R(J)$ for some $i$. This implies that, if $d_i^{R,JN}(J) \neq 0$ for some $i \in I$, then the inequality in (3.A.1) is strict. Moreover, if $d_i^{R,JN}(J) = 0$ for each $i$, then it can be seen that $d_i^{R,JO}(J) = d_i^R(J_O)$, which in turn implies that $\hat{d}_i^R = d_i^R(J)$ for each $i$. Hence, in this case, the inequality in Eq. (3.A.1) is satisfied with equality. This establishes that $\Omega^R(J) \leq \Omega^R(J_O)$ with equality if and only if $d_i^{R,JN}(J) = 0$ for each $i$, completing the proof of the first part.

Part (ii). The proof formalizes and completes the sketch proof given in Section 3.4.1. Note that, using Eqs. (3.10), (3.11) and (3.12), the speculative variance $\Omega^S(J_E)$ can be written as

$$\Omega^S(J_E) = \frac{2}{\theta} \frac{1}{|I|} \sum_{i \in I} \max_{\hat{d}_i \in \mathbb{R}^{|J_E|}} \left( \left( \hat{\mu}_i^{J_E} \right)' \hat{d}_i - \frac{\theta_i}{2} \hat{d}_i' \Lambda_{J_E} \hat{d}_i \right), \tag{3.A.2}$$

for each sub-economy $\mathcal{E}(J_E)$. Note also that the maximization problems in (3.A.2) are satisfied with equality if and only if $\hat{d}_i = d_i^S(J_E)$. In view of these observations, consider the economy $\mathcal{E}(J)$ and consider the alternative speculative portfolio $\hat{d}_i^S \neq \begin{bmatrix} d_i^S(J_O) \\ 0 \end{bmatrix}$, and note that:
\[
\Omega^S(J) = \frac{2}{\theta} \sum_{i \in I} \left( (\bar{\mu}_i)' d_i^S(J) - \frac{\theta_i}{2} \tilde{d}_i^S(J)' \Lambda \bar{d}_i^S(J) \right)
\]

(3.A.3)

\[
\geq \frac{2}{\theta} \sum_{i \in I} \left( (\bar{\mu}_i)' \tilde{d}_i^S - \frac{\theta_i}{2} (\tilde{d}_i^S)' \Lambda \tilde{d}_i^S \right)
\]

\[
= \frac{2}{\theta} \sum_{i \in I} \left( (\bar{\mu}_i^J)' d_i^S(J_i) - \frac{\theta_i}{2} (d_i^S(J_i))' \Lambda|J_i| d_i^S(J_i) \right) = \Omega^S(J_O).
\]

Here, the inequality in the second line follows by applying Eq. (3.A.2) for \( J_E = J \), and the last line follows by using the definition of \( \tilde{d}_i^S \) and applying Eq. (3.A.2) for \( J_E = J_O \). Note also that the optimization problems in Eq. (3.A.2) are strictly concave, which implies that the inequality in (3.A.3) is strict if and only if \( \tilde{d}_i^S \neq d_i^S(J) \) for some \( i \). This implies that, if \( d_i^{S,J_N}(J) \neq 0 \) for some \( i \in I \), then the inequality in (3.A.3) is strict. Moreover, if \( d_i^{S,J_N}(J) = 0 \) for each \( i \), then it can be seen that \( d_i^{S,J_O}(J) = d_i^S(J_O) \), which in turn implies that \( \tilde{d}_i^S = d_i^S(J) \) for each \( i \). Hence, in this case, the inequality in Eq. (3.A.3) is satisfied with equality. This establishes that \( \Omega^S(J) \geq \Omega^S(J_O) \) with equality if and only if \( d_i^{S,J_N}(J) = 0 \) for each \( i \), completing the proof of the second part of Theorem 9.

**An Alternative Proof of Theorem 9.** The proof presented above is based on the optimization problems provided in Lemmas 5 and 6. These results can also be shown using matrix algebra. To demonstrate this approach, I include an alternative proof for the second part of the theorem. First note that the definition of \( \Omega^S(J_E) \) in (3.4) implies that

\[
\Omega^S(J) - \Omega^S(J_O) = \frac{1}{|I|} \sum_{i \in I} \frac{\theta_i}{\bar{\theta}} \left( (\bar{\mu}_i) - \frac{\theta_i}{\bar{\theta}} (\bar{\mu}_i^J) \right)' \Lambda^{-1} \frac{\theta_i}{\bar{\theta}} (\bar{\mu}_i^J)
\]

(3.A.4)

I next claim that the matrix in parenthesis,

\[
(\Lambda|J_L)^{-1} - \begin{bmatrix} (\Lambda|J_C)^{-1} & 0 \\ 0 & 0 \end{bmatrix}
\]

(3.A.5)
is a positive semidefinite matrix. In view of this claim, Eq. (3.A.4) implies $\Omega^S (J) \geq \Omega^S (J_O)$, providing an alternative proof of the second part.

The claim in (3.A.5) follows from Lemma 5.16 in Horn-Johnson (2007). This lemma considers a positive definite matrix partitioned into submatrices of arbitrary dimension $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$, and shows that the matrix $A^{-1}$ is weakly greater than the matrix

$\begin{bmatrix} (A_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix}$

in positive semidefinite order. More specifically, this lemma shows that the matrix $A^{-1} - \begin{bmatrix} (A_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ is positive semidefinite with rank equal to the size of $A_{22}$. Invoking this lemma for $A = \Lambda$ and $A_{11} = \Lambda_{|J_O}$ shows that the matrix in Eq. (3.A.5) is positive semidefinite, completing the alternative proof.

**Proof of Proposition 2.** Eq. (3.3) implies that, for each policy $J_E$, the beliefs do not affect the uninsurable variance $\Omega^R (J_E)$, that is, $\Omega^R (J_E)$ is independent of $K$. Similarly, the decomposition in (3.4) shows that the beliefs about the new assets $J_N$ do not affect the speculative component for the sub-economy $E (J_O)$, that is, $\Omega^K (J_O)$ is independent of $K$. Consequently, the result follows from the claim:

$$\lim_{K \to \infty} \Omega^S_K (J) = \infty. \quad (3.A.6)$$

To show this claim, first note that $\tilde{\mu}_{i,K}^{J_N} (t) = K \tilde{\mu}_{i,K}^{J_N} (t)$. Second, note that Eq. (3.4) implies

$$\Omega^S_K (J) = \frac{1}{|I|} \sum_{i \in I} \frac{\theta_i}{\theta} \left( \frac{1}{\theta_i} \left[ \begin{array}{c} \tilde{\mu}_{i,K}^{J_O} \\ G_{i,K} \tilde{\mu}_{i,K}^{J_N} \end{array} \right] \right)' \left( \Lambda |J \right)^{-1} \left( \frac{1}{\theta_i} \left[ \begin{array}{c} \tilde{\mu}_{i,K}^{J_O} \\ G_{i,K} \tilde{\mu}_{i,K}^{J_N} \end{array} \right] \right)$$

$$= \frac{K^2}{|I|} \sum_{i \in I} \frac{\theta_i}{\theta} \left( \frac{1}{\theta_i} \left[ \begin{array}{c} \tilde{\mu}_{i,K}^{J_O} / K \\ G_{i,K} \tilde{\mu}_{i,K}^{J_N} \end{array} \right] \right)' \left( \Lambda |J \right)^{-1} \left( \frac{1}{\theta_i} \left[ \begin{array}{c} \tilde{\mu}_{i,K}^{J_O} / K \\ G_{i,K} \tilde{\mu}_{i,K}^{J_N} \end{array} \right] \right). \quad (3.A.7)$$

Consider next the limit of the summand for each $i$, which implies

$$\lim_{K \to \infty} \left( \frac{1}{\theta_i} \left[ \begin{array}{c} \tilde{\mu}_{i,K}^{J_O} / K \\ \tilde{\mu}_{i,K}^{J_N} \end{array} \right] \right)' \left( \Lambda |J \right)^{-1} \left( \frac{1}{\theta_i} \left[ \begin{array}{c} \tilde{\mu}_{i,K}^{J_O} / K \\ \tilde{\mu}_{i,K}^{J_N} \end{array} \right] \right) = \lim_{K \to \infty} \left( \tilde{\mu}_{i,K}^{J_N} / \theta_i \right)' \left( \Lambda |J \right)^{-1} \left( \tilde{\mu}_{i,K}^{J_N} / \theta_i \right) > 0,$$

131
where the last line follows since the matrix \((A|j)^{-1}|j_N\) is positive definite (because \((A|j)^{-1}\) is positive definite). Combining this with Eq. (3.A.7) establishes the claim in Eq. (3.A.6), completing the proof of Proposition 2.

**Proof of Theorem 10.** Consider an alternative economy in which \(\hat{J}_O \cup J_N\) is the set of old assets and \(\hat{J}_N = (J_O \cup J_N) \setminus (\hat{J}_O \cup J_N)\) is the set of new assets. Applying the second part of Theorem 9 to this alternative economy implies \(\Omega^S(\hat{J}_O \cup J_N) \leq \Omega^S(J_O \cup J_N)\). Note also that Assumption (A2) implies \(\Omega^S(\hat{J}_O) = \Omega^S(J_O) = 0\), since there is no belief heterogeneity about the payoffs of old assets. Putting these observations together proves Theorem 10.

**Proof of Theorem 11.** Note that the assets \(\{j_1, j_2\}\) are identical except for the correlation structure with the existing assets (in particular, \(\tilde{\mu}_i^{J_O \cup \{j_1\}} = \tilde{\mu}_i^{J_O \cup \{j_2\}}\)). Thus, consider the expression for the speculative variance in (3.4) respectively for the sub-economies \(\mathcal{E}(J_O \cup \{j_1\})\) and \(\mathcal{E}(J_O \cup \{j_2\})\), which implies:

\[
\Omega^S(J_O \cup \{j_1\}) - \Omega^S(J_O \cup \{j_2\}) = \frac{1}{|I|} \sum_{i \in I} \theta_i \left( \frac{\tilde{\mu}_i^{J_O \cup \{j_1\}}}{\theta_i} \right)' \left[ (A|j_O \cup \{j_1\})^{-1} - (A|j_O \cup \{j_2\})^{-1} \right] \frac{\tilde{\mu}_i^{J_O \cup \{j_1\}}}{\theta_i}.
\]

\[
= \frac{1}{|I|} \sum_{i \in I} \theta_i \left( \frac{\tilde{\mu}_i^{j_1}}{\theta_i} \right)' \left[ \left( (A|j_O \cup \{j_1\})^{-1} \right)^{j_1,j_1} - \left( (A|j_O \cup \{j_2\})^{-1} \right)^{j_2,j_2} \right] \frac{\tilde{\mu}_i^{j_1}}{\theta_i}, \tag{3.A.8}
\]

where the last line uses the assumption that there is no disagreement about the assets in \(J_O\).

Hence, the result follows if the bracketed term in Eq. (3.A.8) is positive. To see this, note that by the assumption about the correlation of \(j_1\) and \(j_2\) with new assets, the covariance matrices are given by

\[
A|j_O \cup \{j_1\} = \begin{pmatrix} X & y' \\ y & x \end{pmatrix} \quad \text{and} \quad A|j_O \cup \{j_2\} = \begin{pmatrix} X & 0 \\ 0 & x \end{pmatrix},
\]

where \(X\) is a positive definite matrix, \(y\) is a vector, and \(x\) is a scalar. Since \(A|j_O \cup \{j_2\}\) is a block diagonal matrix, the scalar \(x\) also solves:

\[
\left( (A|j_O \cup \{j_2\})^{-1} \right)^{j_2,j_2} = x^{-1}. \tag{3.A.9}
\]

132
Note also that $\Lambda_{J_0 \cup \{j_1\}} = \begin{bmatrix} X & y' \\ y & x \end{bmatrix}$ is a positive definite and symmetric matrix. Using the Schur complement identity, its inverse is given by 

$$(\Lambda_{J_0 \cup \{j_1\}})^{-1} = \begin{bmatrix} * & * \\ * & (x - y'X^{-1}y)^{-1} \end{bmatrix},$$

where *'s represent conformable matrices or vectors. In particular,

$$(\Lambda_{J_0 \cup \{j_1\}})^{-1})^{j_1j_1} = (x - y'X^{-1}y)^{-1}. \quad (3.10)$$

Finally, note that $x > y'X^{-1}y > 0$, which implies

$$\frac{1}{x - y'X^{-1}y} > \frac{1}{x}.$$

In view of this inequality, Eqs. (3.9) and (3.10) imply that the bracketed term in (3.8) is positive, which completes the proof of Theorem 11.

**Proof of Theorem 12.** Part (i). Note that Eq. (3.3) implies

$$\Lambda(t)|_{J_E(t)} > \Lambda(t + 1)|_{J_E(t)} > 0,$$

where $X \succ Y$ represents that the matrix $X$ is strictly greater than matrix $Y$ in positive definite order. Using the expression for the risk-sharing variance in (3.7), this further implies $\Omega^R(t + 1, J_E(t)) < \Omega^R(t, J_E(t))$. Finally, note also that Theorem 9 implies $\Omega^R(t + 1, J_E(t + 1)) \leq \Omega^R(t + 1, J_E(t))$ since $J_E(t) \subset J_E(t + 1)$. Combining these two inequalities prove the first part of the theorem.

**Part (ii).** Suppose no new assets are introduced for trade at time $t$, that is, $J_E(t + 1) = J_E(t)$. Using the expression (3.3), note that the traders' covariance matrix can be written as

$$\Lambda(t)|_{J_E(t)} = \Lambda_{true}|_{J_E(t)} + F(t), \text{ where } F(t) = Diag^{J_E(t)} \left( \frac{1}{\tau^j + \left( \frac{1}{\tau_{truc}^j} \right)} \right). \quad (3.11)$$

Consider the expressions for $\Omega^S(t + 1, J_E(t + 1))$ and $\Omega^S(t, J_E(t))$ from Eq. (3.7). Using the fact that $J_E(t) = J_E(t + 1)$ and substituting for the covariance matrix $\Lambda(t)|_{J_E(t)}$ from Eq.
(3.A.11), the difference between speculative variances can be written as

\[
\left[ \begin{array}{c} \Omega^S(t + 1, J_E(t + 1)) \\ -\Omega^S(t, J_E(t)) \end{array} \right] = \frac{1}{|I|} \sum_{i \in I} \frac{\theta_i}{\theta} \left[ \begin{array}{c} \left( \frac{\hat{\mu}_i^{J_E(t)}(t+1)}{\theta} \right) \left( \Lambda_{\text{true}}|J_E(t) + F(t + 1) \right)^{-1} \frac{\hat{\mu}_i^{J_E(t)}(t+1)}{\theta} \\ - \left( \frac{\hat{\mu}_i^{J_E(t)}(t)}{\theta} \right) \left( \Lambda_{\text{true}}|J_E(t) + F(t) \right)^{-1} \frac{\hat{\mu}_i^{J_E(t)}(t)}{\theta} \end{array} \right].
\]

(3.A.12)

Next note that Eq. (3.6) implies

\[
\tilde{\mu}_i^j(t + 1) = \frac{\tau^j + (t - t_c^j)}{\tau^j + (t + 1 - t_c^j)} \tilde{\mu}_i^j(t) \text{ for each } j \in J_E(t).
\]

(3.A.13)

This can be written in matrix notation as \( \tilde{\mu}_i^{J_E(t)}(t + 1) = G(t + 1) \tilde{\mu}_i^{J_E(t)}(t) \), where

\[
G(t + 1) = \text{Diag}^{J_E(t)} \left( \frac{\tau^j + (t - t_c^j)}{\tau^j + (t + 1 - t_c^j)} \right).
\]

(3.A.14)

Substituting for \( \tilde{\mu}_i^{J_E(t)}(t + 1) = G(t + 1) \tilde{\mu}_i^{J_E(t)}(t) \) in Eq. (3.A.12) yields:

\[
\Omega^S(t + 1, J_E(t + 1)) - \Omega^S(t, J_E(t)) = \frac{1}{|I|} \sum_{i \in I} \frac{\theta_i}{\theta} \left[ G(t + 1) \left( (\Lambda_{\text{true}})|J_E(t) + F(t + 1) \right)^{-1} G(t + 1) \right] \frac{\hat{\mu}_i^{J_E(t)}(t)}{\theta_i}. \]

Hence, the result follows if the matrix in the brackets on the right hand side is positive definite, which is true if and only if the following matrix inequality is satisfied:

\[
G(t + 1) \left( (\Lambda_{\text{true}})|J_E(t) + F(t + 1) \right)^{-1} G(t + 1) < \left( (\Lambda_{\text{true}})|J_E(t) + F(t) \right)^{-1}.
\]

Taking the inverse of each side, this statement is equivalent to

\[
G(t + 1)^{-1} \left( (\Lambda_{\text{true}})|J_E(t) + F(t + 1) \right) G(t + 1)^{-1} > (\Lambda_{\text{true}})|J_E(t) + F(t) \).
\]

(3.A.15)

To show the claim in (3.A.15), first consider the matrix \( G(t + 1)^{-1} F(t + 1) G(t + 1)^{-1} \), which is also diagonal. Using the expressions in (3.A.11) and (3.A.14), the \( j^{th} \) diagonal entry
of this matrix is given by:

\[
\frac{F^{ij} (t+1)}{G^{ij} (t+1)^2} = \frac{1}{\tau^j + \left(t + 1 - t_c^j\right) \tau_{true}^j} \left(\frac{\tau^j + \left(t + 1 - t_c^j\right) \tau_{true}^j}{\tau^j + \left(t - t_c^j\right) \tau_{true}^j}\right)^2
\]

where the inequality follows since \( t + 1 > t \) and the last equality uses the definition of the matrix \( F (t) \) in (3.11). Note that the last chain of inequalities apply for any \( j \in J_E (t) \), which further implies:

\[
G (t + 1)^{-1} F (t + 1) G (t + 1)^{-1} \succ F (t) .
\]

Moreover, note also that

\[
G (t + 1)^{-1} \Lambda_{true} |_{J_E (t)} G (t + 1)^{-1} \succ \Lambda_{true} |_{J_E (t)},
\]

since \( G(t + 1)^{-1} \) is a diagonal matrix with diagonal entries greater than 1 [cf. Eq. (3.14)]. Combining the last two displayed inequalities shows the claim in Eq. (3.15) and completes the proof of the second part.

**Part (iii).** Note that the belief dynamics in (3.3) implies \( \lim_{t \to \infty} \mu^j_i (t) = 0 \) for each \( t \). Using the expression for the speculative variance in (3.7), this further implies that \( \lim_{t \to \infty} \Omega^S (t, J_E (t)) = 0 \), completing the proof of the theorem.

**Proof of Theorem 13.** For each \( t \geq 1 \), define the difference the net effect of the assets \( J_C (1) \backslash J_C (0) \) on the average consumption variance as \( \eta (t) = \Omega (t, J_C (t)) - \Omega (t, J_D (t)) \) (given that the exchange set, \( J_D (t) \), corresponds to the policy of delaying the introduction of the assets, \( J_C (1) \backslash J_C (0) \), for at least \( t \) dates). Note that \( \eta (1) > 0 \) by assumption. Note also that \( \lim_{t \to \infty} \eta (t) < 0 \), by part (iii) of Theorem 12. It follows that there exists a unique \( T \geq 1 \) such that

\[
\eta (T) > 0 \text{ for all } T \in \{1, ..., T\} \text{ and } \eta (T + 1) < 0 .
\]

135
Then, for each \( T \in \{1, \ldots, T\} \), the average consumption variance satisfies

\[
\begin{align*}
\Omega(t, J_C(t)) - \Omega(t, J_D(t)) &= \eta(t) > 0 \text{ for all } t \in \{1, \ldots, T\} \text{ and } \\
\Omega(t, J_C(t)) - \Omega(t, J_D(t)) &= 0 \text{ for all } t \geq T + 1.
\end{align*}
\]

This establishes Eq. (3.8) for \( T \in \{1, \ldots, T\} \). Note also that, for \( T \geq T + 1 \), the average consumption variance satisfies

\[
\Omega(T + 1, J_C(T + 1)) - \Omega(T + 1, J_D(T + 1)) = \eta(T + 1) < 0.
\]

This establishes Eq. (3.8) for \( T \geq T + 1 \), which completes the proof of the theorem.

**Proof of Theorem 14.** Part (i). First note that objective function of Problem (3.3) can be written as

\[
f\left(J^D\right) \equiv \lim_{t \to \infty} \Omega^R(t, J^D) = \frac{1}{|I|} \sum_{i \in I} \theta_i \left( w_i^j w_i - \left( \lambda_i^{jD}\right)' \Lambda_{true} |J^D|^{-1} \lambda_i^{jD} \right).
\]

Next consider a set \( J^D \subset J \) with \( |J^D| = M \), which is not a solution to Problem (3.3). I will first show that there exists \( \gamma_{J^D} \in (0, 1) \) such that \( J^D \) is not the optimal selection of assets for any \( \gamma \in (\gamma_{J^D}, 1) \). To show this, first note that there exists some \( J^D \subset J \) with \( |J^D| = M \) such that \( f\left(J^D\right) > f\left(J^D\right) \). Let \( \delta = \left(f\left(J^D\right) - f\left(J^D\right)\right)/2 > 0 \). I claim that there exists \( \gamma_{\bar{J}^D} \) such that the market maker prefers \( \bar{J}^D \) to \( J^D \) for each \( \gamma \in (\gamma_{\bar{J}^D}, 1) \).

To prove this claim, first note that the asymptotic profits from choosing the set \( J^D \) are given by

\[
\lim_{t \to \infty} \sum_{i \in I} \pi_i(t, J^D) = \lim_{t \to \infty} \sum_{i \in I} \theta_i \left( \frac{\mu_i^{jD}(t) - \lambda_i^{J^D}}{\theta_i} \right) \left( \Lambda(t) |J^D|^{-1} \left( \frac{\mu_i^{jD}(t) - \lambda_i^{J^D}}{\theta_i} \right) \right) = f\left(J^D\right),
\]

where the last equality follows since \( \lim_{t \to \infty} \mu_i^{J^D}(t) = 0 \) and \( \lim_{t \to \infty} \Lambda(t) |J^D| = \Lambda_{true} |J^D| \). This further implies that

\[
\lim_{\gamma \to 1} (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t \sum_{i \in I} \pi_i(t, J^D) = f\left(J^D\right),
\]

136
since the left hand side limit is equal to \( \lim_{t \to \infty} \sum_{i \in I} \pi_i(t, J^D) \). By the previous limit equation, there exists \( \gamma_1 \in (0, 1) \) such that

\[
(1 - \gamma) \sum_{t=0}^{\infty} \gamma^t \sum_{i \in I} \pi_i(t, J^D) > f(J^D) - \delta \quad \text{for each} \quad \gamma \in (\gamma_1, 1). \tag{3.A.16}
\]

Repeating the same analysis for the set \( J^D \) implies that there exists \( \gamma_2 \in (0, 1) \) such that

\[
(1 - \gamma) \sum_{t=0}^{\infty} \gamma^t \sum_{i \in I} \pi_i(t, J^D) < f(J^D) + \delta \quad \text{for each} \quad \gamma \in (\gamma_2, 1) \tag{3.A.17}
\]

Let \( \gamma_{J^D} = \max(\gamma_1, \gamma_2) \in (0, 1) \). Then, combining Eqs. (3.A.16) and (3.A.17) and using

\[
\delta = \left( f(J^D) - f(J^D) \right) / 2
\]

implies

\[
\sum_{t=0}^{\infty} \gamma^t \sum_{i \in I} \pi_i(t, J^D) > \sum_{t=0}^{\infty} \gamma^t \sum_{i \in I} \pi_i(t, J^D) \quad \text{for each} \quad \gamma \in (\gamma_{J^D}, 1).
\]

This shows that the market maker prefers \( J^D \) to \( J^D \), which in turn establishes the claim that \( J^D \) is not the optimal selection of assets for any \( \gamma \in (\gamma_{J^D}, 1) \).

Next, fix some threshold \( \gamma_{J^D} \in (0, 1) \) for each set \( J^D \subset J \) with \( |J^D| = M \) that does not solve Problem (3.3), and let \( \bar{\gamma} \) denote the maximum of these thresholds (which exists and lies in \( (0, 1) \) since there are finitely sets, \( J^D \)). Consider \( \gamma \in (\bar{\gamma}, 1) \) and note that each selection \( J^D \) (which does not solve Problem (3.3)) is dominated by some other selection \( \bar{J} \). It follows that, for each \( \gamma \in (\bar{\gamma}, 1) \), the optimal selection for the market maker solves Problem (3.3), completing the proof for the first part.

Part (ii). Define the monopolist’s present discounted profits for parameter \( K \) and the
selection \( \tilde{J}^D \) with \( \Pi_K ( \tilde{J}^D ) \). First note that

\[
\lim_{K \to \infty} \frac{\Pi_K ( \tilde{J}^D )}{K^2} = \sum_{t=0}^{\infty} \gamma^t \sum_{i \in I} \frac{\theta_i}{2} \left( \frac{\bar{\mu}_i^{jD}(t)}{\theta_i} - \lim_{K \to \infty} \frac{\bar{\lambda}_i^{jD}}{K} \right) (\Lambda(t) | \tilde{J}^D)^{-1} \left( \frac{\bar{m}_i^{jD}(t)}{\theta_i} - \lim_{K \to \infty} \frac{\bar{\lambda}_i^{jD}}{K} \right)
\]

\[
= \sum_{t=0}^{\infty} \gamma^t \sum_{i \in I} \frac{\theta_i}{2} (\Lambda(t) | \tilde{J}^D)^{-1} \frac{\bar{m}_i^{jD}(t)}{\theta_i}
\]

\[
= \frac{\bar{\theta}}{2} \sum_{t=0}^{\infty} \gamma^t \Omega^S(t, \tilde{J}^D), \tag{3.A.18}
\]

where the limit and the sum are interchanged using the Dominated Convergence Theorem. Next consider \( \tilde{J}^D \subset \mathcal{J} \) with \( |\tilde{J}^D| = M \), which does not solve Problem (3.4). Eq. (3.A.18) implies that there exists \( K_{\tilde{J}^D} \) such that, if \( K > \tilde{J}^D \), then the set \( \tilde{J}^D \) is not optimal for the market maker since it is dominated by some \( \tilde{J}^D \) that solves Problem (3.4). Let \( \bar{K} \) denote the maximum of such \( K_{\tilde{J}^D} \) (of which there are finitely many). It follows that, for each \( K > \bar{K} \), the optimal selection for the market maker solves Problem (3.4), completing the proof of the theorem.
Chapter 4

Complexity and Financial Panics

*Joint with Ricardo Caballero, MIT.

4.1 Introduction

The dramatic rise in investors’ and banks’ perceived uncertainty is at the core of the 2007-2009 U.S. financial crisis. All of a sudden, a financial world that was once rife with profit opportunities for financial institutions (banks, for short), was perceived to be exceedingly complex. Although the subprime shock was small relative to the financial institutions’ capital, banks acted as if most of their counterparties were severely exposed to the shock (see Figure 4-1). Confusion and uncertainty followed, triggering the worst case of flight-to-quality that we have seen in the U.S. since the Great Depression.

In this essay we present a model of the sudden rise in complexity, followed by widespread panic in the financial sector. In the model, banks normally collect information about their direct trading partners which serves to assure them of the soundness of these relationships. However, when acute financial distress emerges in parts of the financial network, it is not enough to be informed about these partners, but it also becomes important for the banks to learn about the health of their trading partners. And as conditions continue to deteriorate, banks must learn about the health of the trading partners of the trading partners, of their trading partners, and so on. At some point, the cost of information gathering becomes too large and banks, now facing enormous uncertainty, choose to withdraw from loan commitments and illiquid positions.
The line corresponds to the TED spread in basis points (source: Bloomberg), the interest rate difference between the interbank loans (3 month LIBOR) and the US government debt (3 month Treasury bills). An increase in the TED spread typically reflects a higher perceived risk of default on interbank loans, that is, an increase in the banks’ perceptions of counterparty risk.

A flight-to-quality ensues, and the financial crisis spreads.

The starting point of our framework is a standard liquidity model where banks (representing financial institutions more broadly) have bilateral linkages in order to insure against local liquidity shocks. The whole financial system is a complex network of linkages which functions smoothly in the environments that it is designed to handle, even though no bank knows with certainty all the many possible connections within the network (that is, each bank knows the identities of the other banks but not their exposures). However, these linkages may also be the source of contagion when an unexpected event of financial distress arises somewhere in the network. Our point of departure with the literature is that we use this contagion mechanism not as the main object of study but as the source of confusion and financial panic. During normal times, banks only need to understand the financial health of their neighbors, which they can learn at low cost. In contrast, when a significant problem arises in parts of the network and the possibility of cascades arises, the number of nodes to be audited by each bank rises since it is possible that the shock may spread to the bank’s counterparties. Eventually the problem becomes too complex for them to fully figure out, which means that banks now face significant uncertainty and they react to it by retrenching into liquidity-conservation mode.
This essay is related to several strands of literature. There is an extensive literature that highlights the possibility of network failures and contagion in financial markets. An incomplete list includes Allen and Gale (2000), Lagunoff and Schreft (2000), Rochet and Tirole (1996), Freixas, Parigi and Rochet (2000), Leitner (2005), Eisenberg and Noe (2001), Cifuentes, Ferucci and Shin (2005) (see Allen and Babus (2008) for a recent survey). These papers focus mainly on the mechanisms by which solvency and liquidity shocks may cascade through the financial network. In contrast, we take these phenomena as the reason for the rise in the complexity of the environment in which banks make their decisions, and focus on the effect of this complexity on banks' prudential actions. In this sense, our essay is related to the literature on flight-to-quality and Knightian uncertainty in financial markets, as in Caballero and Krishnamurthy (2008), Routledge and Zin (2004) and Easley and O'Hara (2005); and also to the related literature that investigates the effect of new events and innovations in financial markets, e.g. Liu, Pan, and Wang (2005), Brock and Manski (2008) and Simsek (2010b). Our contribution relative to this literature is in endogenizing the rise in uncertainty from the behavior of the financial network itself. More broadly, this essay belongs to an extensive literature on flight-to-quality and financial crises that highlights the connection between panics and a decline in the financial system's ability to channel resources to the real economy (see, e.g., Caballero and Kurlat (2008), for a survey).

We build our argument in several steps. In Section 2 we first characterize the financial network and describe a rare event as a perturbation to the structure of banks' shocks. Specifically, one bank suffers an unfamiliar liquidity shock for which it was unprepared. We next show that if banks can costlessly gather information about the network structure, the spreading of this shock into precautionary responses by other banks is typically contained. This scenario with no network uncertainty is the benchmark for our main results and is similar (although with an interior equilibrium) to Allen and Gale (2000).

Our main contribution is in Section 3, where we make information gathering costly. In this context, if the cascade is small, either because the liquidity shock is limited or because banks' buffers are significant, banks are able to gather the information they need about their indirect exposure to the liquidity shock and we are back to the full information results of Section 2. However, once cascades are large enough, banks are unable to collect the information they need
to rule out a severe indirect hit. Their response to this uncertainty is to hoard liquidity and to retrench on their lending, which triggers a credit crunch. In Section 4 we show that under certain conditions, the response in Section 3 can be so extreme, that the entire financial system can collapse as a result of the flight to quality. The essay concludes with a final remarks section and several appendices.

4.2 The Environment and a Free-Information Benchmark

In this section we first introduce the environment and the characteristics of the financial network along with a shock which was unanticipated at the network formation stage (i.e. the financial network was not designed to deal with this shock). We next characterize the equilibrium for a benchmark case in which information gathering is free so that the market participants know the financial network.

4.2.1 The Environment

There are three dates \{0, 1, 2\}. There is a single good (one dollar) that serves as numeraire, which can be kept in liquid reserves or it can be loaned to production firms. If kept in liquid reserves, a unit of the good yields one unit in the next date. Instead, if a unit is loaned to firms at date 0, it then yields \( R > 1 \) units at date 2 if it is not unloaded before this date. At date 1, the lender can unload the loan (e.g. by settling it with the borrower at a discount) and receive \( r < 1 \) units. To simplify the notation, we assume \( r \approx 0 \) throughout this essay.

Banks and Their Liquidity Needs

The economy has \( 2n \) continuums of banks denoted by \( \{b^j\}_{j=1}^{2n} \). Each of these continuums is composed of identical banks and, for simplicity, we refer to each continuum \( b^j \) as bank \( b^j \), which is our unit of analysis.\(^1\) Each bank \( b^j \) has initial assets which consist of \( y \) units of liquid reserves set aside for liquidity payments, \( \bar{y}_0 \leq 1 - y \) units of flexible reserves set aside for making new loans at date 0 (but which can also be hoarded as liquid reserves) and \( 1 - y - \bar{y}_0 \) units of loans. The bank’s liabilities consist of a measure one of demand deposit contracts. A demand deposit

\(^1\)The only reason for the continuum is for banks to take other banks' decisions as given.
contract pays $l_1 > 1$ at date 1 if the depositor is hit by a liquidity shock and $l_2 > l_1$ at date 2 if the depositor is not hit by a liquidity shock. Let $\omega^j \in [0,1]$ be the measure of liquidity-driven depositors of bank $b^j$ (i.e. the size of the liquidity shock experienced by the bank), which takes one of the three values in $\{\bar{\omega}, \omega_L, \omega_H\}$ with $\omega_H > \omega_L$ and $\bar{\omega} \equiv (\omega_H + \omega_L)/2$, and suppose

$$y = l_1 \omega \quad \text{and} \quad (1 - y) R = l_2 \bar{\omega}.$$ 

Note that, if the size of the liquidity shock is $\omega$, the bank that loans all of its flexible reserves $\bar{y}_0$ at date 0 has assets just enough to pay $l_1$ (resp. $l_2$) to early (resp. late) depositors. The central trade-off in this economy will be whether the bank will loan its flexible reserves $\bar{y}_0$ (which it set aside for this purpose) or whether it will hoard some of this liquidity as a precautionary response to a rare event that we describe below.

**The Financial Network**

The liquidity needs at date 1 may not be evenly distributed among banks, which highlights one of the (many) reasons for an interlinked financial network. Moreover, the main source of complexity later on will be confusion about the linkages between different banks. To capture this possibility we let $i \in \{1, \ldots, 2n\}$ denote slots in a financial network and consider a permutation $\rho : \{1, \ldots, 2n\} \rightarrow \{1, \ldots, 2n\}$ that assigns bank $b^{\rho(i)}$ to slot $i$. We consider a financial network denoted by:

$$b(\rho) = \left( b^{\rho(1)} \rightarrow b^{\rho(2)} \rightarrow b^{\rho(3)} \rightarrow \ldots \rightarrow b^{\rho(2n)} \rightarrow b^{\rho(1)} \right), \quad (4.1)$$

where the arc $\rightarrow$ denotes that the bank in slot $i$ (i.e., bank $b^{\rho(i)}$) has a demand deposit in the bank in the subsequent slot $i + 1$ (i.e., bank $b^{\rho(i+1)}$) equal to

$$z = (\bar{\omega} - \omega_L), \quad (4.2)$$
where we use modulo $2n$ arithmetic for the slot index $i$. We refer to bank $b^{\rho(i+1)}$ as the forward neighbor of bank $b^{\rho(i)}$ (and similarly, to bank $b^{\rho(i)}$ as the backward neighbor of bank $b^{\rho(i+1)}$).

The possibility of confusion arises later on from banks knowing the identity of other banks but not their particular linkages (i.e., the actual permutation $\rho$).

As we formally describe in Appendix 4.A.1 (and similar to Allen-Gale (2000)), in the normal environment, the financial network facilitates liquidity insurance and enables liquidity to flow from banks that experience a low liquidity shock ($\omega_L$) to the banks that experience a high liquidity shock ($\omega_H$), even when the financial network $b(\rho)$ is unknown to the banks. Our focus is on the effect of the financial interlinkages in case of an unanticipated shock for which the financial network is not necessarily designed for, which we describe next.

A Rare Event

At date 0 the banks learn that all banks will experience the average liquidity shock $\bar{\omega}$ at date 1, however, they also learn that one bank, $b^j$, becomes distressed and loses $\theta \leq y$ of its liquid assets. As we formally demonstrate below, the losses in the distressed bank $b^j$ might spill over to the other banks via the financial network $b(\rho)$, thus the banks' knowledge of the financial network is potentially payoff relevant. In particular, this knowledge influences whether the banks use the flexible reserves $\bar{y}_0$ to make new loans or to hoard liquidity. We are thus lead to describe the central feature of our model: the banks' uncertainty about the financial network $b(\rho)$.

Network Uncertainty and Auditing Technology

We let

$$B = \{b(\rho) \mid \rho : \{1, \ldots, 2n\} \rightarrow \{1, \ldots, 2n\} \text{ is a permutation}\},$$

(4.3)

denote the set of possible financial networks. Each bank $b^j$ observes its slot $i = \rho^{-1}(j)$ and the identities of the banks in its neighboring slots $i-1$ and $i+1$. This information narrows down

---

\(^2\)In particular, $i$ represents the slot with index $i' \in \{1, \ldots, 2n\}$ that is the modulo $2n$ equivalent of integer $i$. For example, $i = 2n + 1$ represents the slot with index 1.
the potential networks to the set:

\[ B^i_j(\rho) = \left\{ b(\tilde{\rho}) \in B \mid \begin{cases} \tilde{\rho}(i - 1) = \rho(i - 1) \\ \tilde{\rho}(i) = \rho(i) \\ \tilde{\rho}(i + 1) = \rho(i + 1) \end{cases}, \text{ where } i = \rho^{-1}(j) \right\}. \]

Note that the bank \( b^j \) does not know how the remaining banks \( (b^j_j \neq \{\rho(i - 1), \rho(i), \rho(i + 1)\}) \) are assigned to the remaining slots (see Figure 4-2). In particular, each bank \( b^j \neq b^j \) knows that the bank \( b^j \) is distressed, but it does not necessarily know the slot

\[ i = \rho^{-1}(j) \]

of the distressed bank. This is key, since it means that a bank \( b^j \neq b^j \) does not necessarily know how far removed it is from the distressed bank.

Each bank \( b^j \) can acquire more information about the financial network through an auditing technology. At date 0, a bank \( b^j \) in slot \( i \) (i.e. with \( j = \rho(i) \)) can exert effort to audit its forward neighbor \( b^{\rho(i+1)} \) in order to learn the identity of this bank’s forward neighbor \( b^{\rho(i+2)} \). Continuing this way, a bank \( b^j \) that audits a number, \( a^j \), of balance sheets learns the identity of its \( a^j + 1 \) forward neighbors and narrows the set of potential financial networks to:

\[ B^i_j(\rho \mid a^j) = \left\{ b(\tilde{\rho}) \in B \mid \begin{cases} \tilde{\rho}(i - 1) = \rho(i - 1) \\ \vdots \\ \tilde{\rho}(i + a^j + 1) = \rho(i + a^j + 1) \end{cases}, \text{ where } i = \rho^{-1}(j) \right\}. \]

We denote the posterior beliefs of bank \( b^j \) with \( f^j(. \mid \rho, a^j) \) which has support equal to \( B^i_j(\rho \mid a^j) \) given assumption:

**Assumption (FS).** Each bank has a prior belief \( f^j(.) \) over \( B \) with full support.

In the example illustrated in Figure 4-2, if bank \( b^1 \) audits one balance sheet, then it would learn that bank \( b^3 \) is assigned to slot 3 and it would narrow down the set of networks to the two boxes at the left hand side of the bottom row in Figure 4-2.
Figure 4-2: The financial network and uncertainty. The bottom-left box displays the actual financial network. Each circle corresponds to a slot in the financial network, and in this realization of the network, each slot $i$ contains bank $b^i$ (i.e. $\rho(i) = i$). The remaining boxes show the other networks that bank $b^1$ finds plausible after observing its neighbors (i.e. the set $B^1(\rho)$). Bank $b^1$ cannot tell how the banks $\{b^3, b^4, b^5\}$ are ordered in slots $\{3, 4, 5\}$. 
Bank Preferences and Equilibrium

Consider a bank \( b^j \) and denote the bank's actual payments to early and late depositors by \( q_1^j \) and \( q_2^j \) (which may in principle be different than the contracted values \( l_1 \) and \( l_2 \)). Because banks are infinitesimal, they make decisions taking the payments of the other banks as given. The bank makes the audit and liquidity hoarding decisions, \( a^j \in \{0, 1, \ldots, 2n - 3\} \) and \( y_0^j \in [0, \bar{y}_0] \), at date 0 (equivalently, \( \bar{y}_0 - y_0^j \in [0, \bar{y}_0] \) denotes the number of new loans the bank makes at date 0). At date 1, the bank chooses to withdraw some of its deposits in the neighbor bank, which we denote by \( z^j \in [0, z] \), and it may also unload some of its outstanding loans. The bank makes these decisions to maximize \( q_1^j \) until it can meet its liquidity obligations to depositors, that is, until \( q_1^j = l_1 \). Increasing \( q_1^j \) beyond \( l_1 \) has no benefit for the bank, thus once it satisfies its liquidity obligations, it then tries to maximize the return to the late depositors \( q_2^j \).

We capture this behavior with the following objective function

\[
v \left( 1 \left\{ q_1^j \leq l_1 \right\} q_1^j + 1 \left\{ q_1^j \geq l_1 \right\} q_2^j \right) - d \left( a^j \right), \tag{4.4}\]

where \( v : \mathbb{R}_+ \rightarrow \mathbb{R}_{++} \) is a strictly concave and strictly increasing function and \( d \left( . \right) \) is an increasing and convex function which captures the bank's non-monetary disutility from auditing. When the bank \( b^j \) is making a decision that would lead to an uncertain outcome for \( \left( q_1^j, q_2^j \right) \) (which will be the case in Section 4.3), then it maximizes the expectation of the expression in (4.4) given its posterior beliefs \( f^j \left( . \mid \rho, a^j \right) \).

Suppose that the depositors' early/late liquidity shocks are observable, and a bank which is able to pay its late depositors at least \( l_1 \) at date 2 can refuse to pay the late depositors if they arrive early.\(^3\) With this assumption, the continuation equilibrium for bank \( b^j \) at date 1 takes one of two forms. Either there is a no-liquidation equilibrium in which the bank is solvent and pays

\[
q_1^j = l_1, q_2^j \geq l_1, \tag{4.5}\]

while the late depositors withdraw at date 2; or there is a liquidation equilibrium in which the

\(^3\)Without this assumption, there could be multiple equilibria for late depositors' early/late withdrawal decisions. In cases with multiple equilibria, this assumption selects the equilibrium in which no late depositor withdraws.
Banks’ Initial Balance Sheets

<table>
<thead>
<tr>
<th>Assets:</th>
<th>Liabilities:</th>
</tr>
</thead>
<tbody>
<tr>
<td>y liquid reserves.</td>
<td>Date 1 payment to early depositors: ( \omega_1 = y ).</td>
</tr>
<tr>
<td>( g_0 ) flexible reserves.</td>
<td>Date 2 payment to late depositors: ( \omega_2 = (1-y)R ).</td>
</tr>
<tr>
<td>( 1-y-g_0 ) loans.</td>
<td>( z ) demand deposits held by backward neighbor bank.</td>
</tr>
<tr>
<td>( z ) demand deposits in forward neighbor bank.</td>
<td></td>
</tr>
</tbody>
</table>

Date 0

Banks learn that:
- Each will have shock \( \omega \) (reflected on above balance sheets).
- Bank \( b^j \) becomes distressed and loses \( \theta \) liquid reserves.
- Banks make the audit decision \( a_j^j \in \{0,1, \ldots, 2n-3\} \).
- Banks make the liquidity hoarding decision \( y_0^j \in [0, g_0] \).
  (Equivalently bank extend \( g_0 - y_0^j \) new loans.)

Date 1

Banks make the deposit withdrawal decision \( z^j \in [0, z] \).
- Early depositors demand their deposits.
- Late depositors demand their deposits if and only if the bank cannot promise \( q_1^j \geq l_1 \).
- Insolvent banks (that pay \( q_1^j < l_1 \)) unload all of their outstanding loans.

Date 2

Banks pay \( q_1^j \) to late depositors.

Figure 4-3: Timeline of events.

Bank is insolvent, unloads all outstanding loans, and pays

\[
q_1^j < l_1, q_2^j = 0,
\]

while all depositors (including the late depositors) draw their deposits at date 1.

Figure 4-3 recaps the timeline of events in this economy. We formally define the equilibrium as follows.

Definition 8. The equilibrium is a collection of bank auditing, liquidity hoarding, deposit withdrawal, and payment decisions \( \{ a^j (\rho), y_0^j (\rho), z^j (\rho), q_1^j (\rho), q_2^j (\rho) \}_{j \in b(\rho) \in B} \) such that, given the realization of the financial network \( b (\rho) \) and the rare event, each bank \( b^j \) maximizes expected utility in (4.4) according to its prior belief \( f^j (\cdot) \) over \( B \), the insolvent banks (with \( q_1^j (\rho) < l_1 \)) unload all of their outstanding loans at date 1 and the late depositors withdraw deposits early if and only if \( q_2^j (\rho) < l_1 \) (cf. Eqs. (4.5) and (4.6)).

We next turn to the characterization of equilibrium. Note that for each financial network
b (ρ) and for each bank b^j, there exists a unique \( k \in \{0, \ldots, 2n - 1\} \) such that

\[ j = \rho (i - k), \]

which we define as the distance of bank b^j from the distressed bank. As we will see, the distance k will be the payoff relevant information for a bank b^j that decides how much liquidity to hoard at date 0 since it will determine whether or not the crisis that originates at the distressed bank b^j will cascade to bank b^j. The banks b^{\rho(i-1)}, b^{\rho(i)}, b^{\rho(i+1)}, respectively with distances 1, 0 and 2n − 1, know their distances, but the remaining banks (with distances \( k \in \{2, \ldots, 2n - 2\} \)) do not have this information a priori and they assign a positive probability to each \( k \in \{2, \ldots, 2n - 2\} \) (they rule out \( k \in \{1, 2n - 1\} \) by observing their forward and backwards neighbors). Note, however, that the bank b^j can use the auditing technology to learn about the financial network and, in particular, about its distance from the distressed bank. A bank b^{\rho(i-k)} (with distance k) that audits \( a^j \geq 1 \) banks either learns its distance k (if \( k \leq a^j + 1 \)) or it learns that \( k \geq a^j + 2 \).

In the remaining half of this section, we characterize the equilibrium in a benchmark case in which auditing is free so each bank learns its distance from the distressed bank. In subsequent sections, we characterize the equilibrium with costly audit and compare it with the free-information benchmark.

### 4.2.2 Free-Information Benchmark

We first describe a benchmark case in which auditing is free so each bank b^{\rho(i-k)} chooses full auditing \( a^{\rho(i-k)} = 2n - 3 \). In this context banks learn the whole financial network b (ρ) and, in particular, their distances.

At date 0 all banks anticipate receiving a liquidity shock, \( \bar{\omega} \), at date 1 and have liquid reserves equal to \( y = \bar{\omega}l_1 \) (plus \( \bar{y}_0 \) of flexible reserves), except for bank b^j = b^{\rho(i)} which has liquid reserves \( y - \theta \). At date 1, the distressed bank b^{\rho(i)} withdraws its deposits from the forward neighbor bank. As we show in Appendix 4.A.2, this triggers further withdrawals until, in equilibrium, all cross deposits are withdrawn. That is

\[ z^j = z \quad \forall \ j \in \{1, \ldots, 2n\}. \]
In particular, bank \( b^{(i)} \) tries, but cannot, obtain any net liquidity through cross withdrawals. The bank also cannot obtain any liquidity by unloading the loans at date 1, since each unit of unloaded loan yields \( r \approx 0 \). Anticipating that it will not be able to obtain additional liquidity at date 1, the distressed bank \( b^{(i)} \) hoards some of its flexible reserves \( \bar{y}_0 \) by cutting new loans at date 0 in order to meet its liquidity demand at date 1.

In order to promise late depositors at least \( l_1 \), a bank with no liquid reserves left at the end of date 1 must have at least

\[
1 - y - \bar{y}_0^n = \frac{(1 - \varpi) l_1}{R}
\]

units of loans. The level \( \bar{y}_0^n \) is a natural limit on a bank’s liquidity hoarding (which plans to deplete all of its liquidity at date 1) since any choice above this would make the bank necessarily insolvent. If the amount of flexible reserves \( \bar{y}_0 \) is greater than \( \bar{y}_0^n \), then the bank can hoard at most \( \bar{y}_0^n \) of the flexible reserves while remaining solvent; or else it can hoard all of the flexible reserves \( \bar{y}_0 \). Combining the two cases, a bank’s buffer is given by

\[
\beta = \min \{ \bar{y}_0, \bar{y}_0^n \}.
\]

A bank can accommodate losses in liquid reserves up to the buffer \( \beta \), but becomes insolvent when losses are beyond \( \beta \). It follows that the distressed bank \( b^{(i)} \) will be insolvent whenever

\[
\theta > \beta,
\]

that is, whenever its losses in liquid reserves are greater than its buffer. Suppose this is the case so bank \( b^{(i)} \) is insolvent. Anticipating insolvency, this bank will hoard as much liquidity as it can \( y_0^{(i)} = \bar{y}_0 \) (since it maximizes \( q_1^{(i)} \)) and unloads all remaining loans at date 1. Since the bank is insolvent, all depositors (including late depositors) arrive early and the bank pays

\[
q_1^{(i)} = \frac{y + \bar{y}_0 - \theta + z q_1^{(i+1)}}{1 + z} < l_1,
\]

where recall that \( q_1^{(i+1)} \) denotes bank \( b^{(i+1)} \)'s payment to early depositors (which is equal to \( l_1 \) if bank \( b^{(i+1)} \) is solvent).
Partial Cascades. Since bank $b^{\rho(i)}$ is insolvent, its backward neighbor bank $b^{\rho(i-1)}$ will experience losses in its cross deposit holdings, which, if severe enough, may cause bank $b^{\rho(i-1)}$’s insolvency. Once the crisis cascades to bank $b^{\rho(i-1)}$, it may then similarly cascade to bank $b^{\rho(i-2)}$, continuing its cascade through the network in this fashion.

We conjecture that, under appropriate parametric conditions, there exists a threshold $K \in \{1, \ldots, 2n-2\}$ such that all banks with distance $k \leq K - 1$ are insolvent (there are $K$ such banks) while the banks with distance $k \geq K$ remain solvent. In other words, the crisis will partially cascade through the network but will be contained after $K \leq 2n-2$ banks have failed. We refer to $K$ as the cascade size.

Under this conjecture, bank $b^{\rho(i+1)}$, which has a distance $2n-1$, is solvent. Therefore $q_1^{\rho(i+1)} = l_1$ and $q_1^{\rho(i)}$ in Eq. (4.10) can be calculated explicitly. Consider now the bank $b^{\rho(i-1)}$ with distance 1 from the distressed bank. To remain solvent, this bank needs to pay $l_1$ on its deposits to bank $b^{\rho(i-2)}$ but it receives only $q_1^{\rho(i)} < l_1$ on its deposits from the distressed bank $b^{\rho(i)}$, so it loses $z(l_1 - q_1^{\rho(i)})$ in cross-deposits. Hence, bank $b^{\rho(i-1)}$ will also go bankrupt if and only if its losses from cross-deposits are greater than its buffer, $z(l_1 - q_1^{\rho(i)}) > \beta$, which can be rewritten as

$$q_1^{\rho(i)} < l_1 - \frac{\beta}{z}. \quad (4.11)$$

If this condition fails, then the only insolvent bank is the original distressed bank and the cascade size is $K = 1$. If this condition holds, then bank $b^{\rho(i-1)}$ anticipates insolvency, it will hoard as much liquidity as it can, i.e. $y_0^{\rho(i-1)} = \bar{y}$ and it will pay all depositors

$$q_1^{\rho(i-1)} = f(q_1^{\rho(i)}) = \frac{y + \bar{y} + zq_1^{\rho(i)}}{1 + z}. \quad (4.12)$$

From this point onwards, a pattern emerges. The payment by an insolvent bank $b^{\rho(i-k)}$ (with $k \geq 1$) is given by

$$q_1^{\rho(i-k)} = f(q_1^{\rho(i-(k-1))})$$

and this bank’s backward neighbor $b^{\rho(i-(k+1))}$ is also insolvent if and only if $q_1^{\rho(i-k)} < l_1 - \frac{\beta}{z}$. Hence, the payments of the insolvent banks converge to the fixed point of the function $f(.)$.
given by \( y + \bar{y}_0 \), and if\(^4\)
\[
y + \bar{y}_0 > l_1 - \frac{\beta}{z},
\]
then (under Eq. (4.11)) there exists a unique \( K \geq 2 \) such that
\[
q_{1}^{\rho(i-k)} < l_1 - \frac{\beta}{z} \quad \text{for each } k \in \{0, \ldots, K-2\}
\]
and
\[
q_{1}^{\rho(i-(K-1))} \geq l_1 - \frac{\beta}{z}.
\]

If \( 2n - 2 \) is greater than the solution, \( K \), to this equation, i.e. if
\[
2n - 2 \geq K,
\]
then, Eq. (4.14) shows that (in addition to the trigger-distressed bank \( b^\rho(i) \)) all banks \( b^\rho(i-k) \) with distance \( k \in \{1, \ldots, K-1\} \) are insolvent since their losses from cross deposits are greater than their corresponding buffers. In contrast, bank \( b^\rho(i-K) \) (that receives \( q_{1}^{\rho(i-(K-1))} \) from its forward neighbor) is solvent, since it can meet its losses from cross deposits by hoarding some liquidity while still promising the late depositors at least \( l_1 \) (i.e. \( q_2^{\rho(i-K)} \geq l_1 \)). Since bank \( b^\rho(i-K) \) is solvent, all banks \( b^\rho(i-k) \) with distance \( k \in \{K + 1, \ldots, 2n - 1\} \) are also solvent since they do not incur losses in cross-deposits. Hence these banks do not hoard any liquidity, \( y_0^{\rho(i-k)} = 0 \), and they pay \( \left(q_{1}^{\rho(i-k)} = l_1, q_2^{\rho(i-k)} = l_2\right) \), verifying our conjecture for a partial cascade of size \( K \) under conditions (4.13) and (4.15).

Since our goal is to study the role of network uncertainty in generating a credit crunch, we take the partial cascades as the benchmark. The next proposition summarizes the above discussion and also characterizes the aggregate level of liquidity hoarding, which we use as a benchmark in subsequent sections.

**Proposition 3.** Suppose the financial network is realized as \( b(\rho) \), auditing is free, and conditions (4.9), (4.13) and (4.15) hold. For a given financial network \( b(\rho) \), let \( i = \rho^{-1}(j) \) denote the slot of the distressed bank.

(i) For the continuation equilibrium (at date 1): The banks’ equilibrium payments

\(^4\)If condition (4.13) fails, then the sequence \( \left(q_{1}^{\rho(i-k)} = f \left(q_1^{\rho(i-(k-1))}\right)\right)_k \) always remains below \( l_1 - \frac{\beta}{z} \), and it can be checked that there is a full cascade, i.e. all banks are insolvent.
Figure 4-4: The free-information benchmark. The top figure plots the cascade size $K$ as a function of the losses in the originating bank $\theta$, for different levels of the flexible reserves $\tilde{y}_0$. The bottom figure plots the aggregate level of liquidity hoarding, $\mathcal{F}$, for the same set of $\{\tilde{y}_0\}$.

$\left(q_{1}^{(\bar{i}-k)}, q_{2}^{(\bar{i}-k)}\right)$ are (weakly) increasing with respect to their distance $k$ from the distressed bank and there is a partial cascade of size $K \leq 2n - 2$ where $K$ is defined by Eq. (4.14). In particular, banks $\{b^{(\bar{i}-k)}\}_{k=0}^{K-1}$ (with distance from the distressed bank $k \leq K-1$) are insolvent while the remaining banks $\{b^{(\bar{i}-k)}\}_{k=K}^{2n-1}$ (with distance $k \geq K$) are solvent.

(ii) For the ex-ante equilibrium (at date 0): Banks $\{b^{(\bar{i}-k)}\}_{k=0}^{K-1}$ hoard as much liquidity as they can and unload all of their existing loans at date 1, while banks $\{b^{(\bar{i}-k)}\}_{k=K}^{2n-1}$ do not hoard any liquidity or unload any loans. Bank $b^{(\bar{i}-K)}$ hoards a level of liquidity $y_0^{(\bar{i}-K)} = z \left(l_1 - q_1^{(\bar{i}-(K-1))}\right)$ which is just enough to meet its losses from cross deposits (and does not unload any loans).

The aggregate level of liquidity hoarding is:

$$\mathcal{F} \equiv \sum_j y_0^j = K\tilde{y}_0 + y_0^{(\bar{i}-K)}. \quad (4.16)$$

Discussion. Proposition 3 shows that, under appropriate parametric conditions, the equilibrium features a partial cascade and the aggregate level of liquidity hoarding, $\mathcal{F}$, is roughly linear in the size of the cascade $K$ (and is roughly continuous in $\theta$). Figure 4-4 demonstrates
this result for particular parameterization of the model.

The top panel of the figure plots the cascade size $K$ as a function of the losses in the originating bank $\theta$ for different levels of the flexible reserves $\bar{\gamma}_0$. This plot shows that the cascade size is increasing in the level of losses $\theta$ and decreasing in the level of flexible reserves $\bar{\gamma}_0$. Intuitively, with a higher $\theta$ and a lower $\bar{\gamma}_0$, there are more losses to be contained and the banks have less emergency reserves to counter these losses, thus increasing the spread of insolvency.

The bottom panel plots the aggregate level of liquidity hoarding $F$, which is a measure of the severity of the credit crunch, as a function of $\theta$. This plot shows that $F$ also increases with $\theta$ and falls with $\bar{\gamma}_0$. This is an intuitive result: In the free-information benchmark only the insolvent banks (and one transition bank) hoard liquidity, thus the more banks are insolvent (i.e. the greater $K$) the more liquidity is hoarded in the aggregate. Note also that $F$ increases "smoothly" with $\theta$.

These results offer a benchmark for the next sections. There we show that once auditing becomes costly, both $K$ and $F$ may be non-monotonic in $\bar{\gamma}_0$ and, more importantly, can jump with small increases in $\theta$.

### 4.3 Endogenous Complexity and the Credit Crunch

We have now laid out the foundation for our main result. In this section we add the realistic assumption that auditing is costly and demonstrate that a massive credit crunch can arise in response to an endogenous increase in complexity once a bank in the network is sufficiently distressed. In other words, when $K$ is large, it becomes too costly for banks to figure out their indirect exposure. This means that their perceived uncertainty rises and they eventually respond by hoarding liquidity as a precautionary measure (i.e., $F$ spikes).

Note that, unlike in Section 4.2.2, we cannot simplify the analysis by solving the equilibrium for a particular financial network $b(\rho)$ in isolation, since, even when the realization of the financial network is $b(\rho)$, each bank also assigns a positive probability to other financial networks $b(\tilde{\rho}) \in B$. As such, for a consistent analysis we must describe the equilibrium for any realization of the financial network $b(\rho) \in B$ (cf. Definition 8).
Solving this problem in full generality is cumbersome but we make assumptions on the form of the adjustment cost function, the banks’ objective function, and on the level of flexible reserves, that help simplify the exposition. First, we consider a convex and increasing cost function \( d(.) \) that satisfies

\[
d(1) = 0 \quad \text{and} \quad d(2) > v(l_1 + l_2) - v(0).
\] (4.1)

This means that banks can audit one balance sheet for free but it is very costly to audit the second balance sheet. In particular, given the bank’s preferences in (4.4), the bank will never choose to audit the second balance sheet and thus each bank audits exactly one balance sheet, \( \{a^j(\rho) = 1\} \) \( \forall j \in B \). Given these audit decisions and the actual financial network \( b(\rho) \), a bank \( b^j \) has a posterior belief \( f^j(., |\rho, 1) \) with support \( B^j(\rho, 1) \), which is the set of financial networks in which the bank \( j \) knows the identities of its neighbors and its second forward neighbor. In particular, the bank \( b^{\rho(\tilde{\rho} - 2)} \) learns its distance from the distressed bank \( b^{\rho(i)} \) (in addition to banks \( b^{\rho(\tilde{\rho} - 1)}, b^{\rho(i)}, b^{\rho(i + 1)} \) which already have this information from the outset). We denote the set of banks that know the slot of the distressed bank (and thus their distance from this bank) by

\[
B^{\text{know}}(\rho) = \{ b^{\rho(\tilde{\rho} - 2)}, b^{\rho(\tilde{\rho} - 1)}, b^{\rho(i)}, b^{\rho(i + 1)} \}.
\]

On the other hand, each bank \( b^{\rho(\tilde{\rho} - k)} \) with \( k \in \{3, ..., 2n - 2\} \) learns that its distance is at least 3 (i.e. \( \tilde{\rho} \geq 3 \)), but otherwise assigns a probability in \((0, 1)\) to all distances \( \tilde{\rho} \in \{3, ..., 2n - 2\} \). We denote the set of banks that are uncertain about their distance by

\[
B^{\text{uncertain}}(\rho) = \{ b^{\rho(\tilde{\rho} - 3)}, b^{\rho(\tilde{\rho} - 4)}, ..., b^{\rho(\tilde{\rho} - (2n - 2))} \}.
\]

Second, we assume that the preference function \( v(.) \) in (4.4) is Leontieff \( v(x) = (x^{1-\sigma} - 1) / (1 - \sigma) \) with \( \sigma \to \infty \), so that the bank’s objective is:

\[
\min_{b(\rho) \in \tilde{B}(\rho, 1)} \left( \mathbb{1} \left\{ q_1^j(\tilde{\rho}) \leq l_1 \right\} q_1^j(\tilde{\rho}) + \mathbb{1} \left\{ q_1^j(\tilde{\rho}) \geq l_1 \right\} q_2^j(\tilde{\rho}) - d(a^j(\rho)) \right). \quad (4.2)
\]

This means that banks evaluate their decisions according to the worst possible network realization, \( b(\tilde{\rho}) \), which they find plausible.
The third and last assumption is that

\[ \bar{y}_0 \leq \bar{y}_0^n. \]  

(4.3)

That is, the bank has less flexible reserves than the natural limit on liquidity hoarding defined in Eq. (4.8) (which also implies that the buffer is given by \( \beta = \bar{y}_0 \)). This condition ensures that, in the continuation equilibrium at date 1, the banks that have enough liquidity are also solvent (since, no matter how much of their flexible reserves they hoard, they have enough loans to pay the late depositors at least \( l_1 \) at date 2). We drop this condition in the next section.

We next turn to the characterization of the equilibrium under these simplifying assumptions. The banks make their liquidity hoarding decision at date 0 and deposit withdrawal decision at date 1 under uncertainty (before their date 1 losses from cross-deposits are realized). At date 1 the distressed bank \( b_{\bar{v}(0)} \) withdraws its deposits from the forward neighbor which leads to the withdrawal of all cross deposits (see Eq. (4.7) and Appendix 4.A.2) as in the free-information benchmark. Thus, for any distressed bank, the only way to obtain additional liquidity at date 1 is through hoarding liquidity at date 0, which we characterize next.

**A Sufficient Statistic for Liquidity Hoarding.** Consider a bank \( b_{\bar{v}(i-k)} \) other than the original distressed bank (i.e., \( k > 0 \)). A sufficient statistic for this bank to make the liquidity hoarding decision is \( q_1^{\rho(i-(k-1))}(\bar{\rho}) \leq l_1 \), which is the amount it receives in equilibrium from its forward neighbor. In other words, to decide how much of its flexible reserves to hoard, this bank only needs to know whether (and how much) it will lose in cross-deposits. For example, if it knows with certainty that \( q_1^{\rho(i-(k-1))}(\bar{\rho}) = l_1 \) (i.e. its forward neighbor is solvent), then it hoards no liquidity, i.e. \( y_0^{\rho(i-k)} = 0 \). If it knows with certainty that \( q_1^{\rho(i-(k-1))}(\bar{\rho}) < l_1 - \beta/z \) (i.e. its forward neighbor will pay so little that this bank will also be insolvent), then it hoards as much liquidity as it can, i.e. \( y_0^{\rho(i-k)} = \bar{y}_0 \). More generally, if the bank \( b_{\bar{v}(i-k)} \) hoards some \( y_0 \in [0, \bar{y}_0] \) at date 0 and its forward neighbor pays \( x = q_1^{\rho(i-(k-1))}(\bar{\rho}) \) at date 1, then this bank’s payment can be written as

\[ q_1^{\rho(i-k)}(\bar{\rho}) = q_1[y_0, x] \quad \text{and} \quad q_2^{\rho(i-k)}(\bar{\rho}) = q_2[y_0, x], \]  

(4.4)
where the functions $q_1[y'_0,x]$ and $q_2[y'_0,x]$ are characterized in Eqs. (4.A.1) and (4.A.2) in Appendix 4.A.2. At date 0, the bank does not necessarily know $x = q_1^{\rho(\bar{i}-(k-1))}(\bar{\rho})$ and it has to choose the level of liquidity hoarding under uncertainty.

The characterization in Appendix 4.A.2 also shows that $q_1[y'_0,x]$ and $q_2[y'_0,x]$ are (weakly) increasing in $x$ for any given $y'_0$. That is, the bank’s payment is increasing in the amount it receives from its forward neighbor regardless of the ex-ante liquidity hoarding decision. Using this observation along with Eq. (4.4), the bank’s objective value in (4.2) can be simplified and its optimization problem can be written as

$$\max_{y'_0 \in \{0,\ldots,y_{0,0}\}} \left(1 \cdot \{q_1[y'_0,x^m] \geq l_1\} \cdot q_1[y'_0,x^m] + 1 \cdot \{q_1[y'_0,x^m] \geq l_1\} \cdot q_2[y'_0,x^m] \right),$$

s.t. $x^m = \min \left\{ x \mid x = q_1^{\rho(\bar{i}-(k-1))}(\bar{\rho}), b(\bar{\rho}) \in B(\rho,1) \right\}.$

In words, a bank $b^{\rho(\bar{i}-k)}$ (with $k > 0$) hoards liquidity as if it will receive from its forward neighbor the lowest possible payment $x^m$.

**Distance Based and Monotonic Equilibrium.** Next we define two equilibrium allocation notions that are useful for further characterization. First, we say that the equilibrium allocation is *distance based* if the bank’s equilibrium payment can be written only as a function of its distance $k$ from the distressed bank, that is, if there exists payment functions $Q_1, Q_2 : \{0,\ldots,2n-1\} \rightarrow \mathbb{R}$ such that

$$\left( q_1^{\rho(\bar{i}-(k-1))}(\rho), q_2^{\rho(\bar{i}-(k-1))}(\rho) \right) = (Q_1[k], Q_2[k])$$

for all $b(\rho) \in B$ and $k \in \{0,\ldots,2n-1\}$. Second, we say that a distance based equilibrium is *monotonic* if the payment functions $Q_1[k], Q_2[k]$ are (weakly) increasing in $k$. In words, in a distance based and monotonic equilibrium, the banks that are further away from the distressed bank yield (weakly) higher payments.

We next conjecture that the equilibrium is distance based and monotonic (which we verify below). Then, a bank $b^{\rho(\bar{i}-k)}$’s uncertainty about the forward neighbor’s payment $x = q_1^{\rho(\bar{i}-(k-1))}(\bar{\rho}) = Q_1[k-1]$ reduces to its uncertainty about the forward neighbor’s distance $k - 1$, which is equal to one less than its own distance $k$. Hence, the problem in (4.5)

157
can further be simplified by substituting $q^\rho_{i-(k-1)}(\tilde{\rho}) = Q_1[k-1]$. In particular, since a bank $b_{p(i-k)} \in B^{\text{know}}(\rho)$ (for $k > 0$) knows its distance $k$, it solves problem (4.5) with $x^m = Q_1[k-1]$.

On the other hand, a bank $b_{p(i-k)} \in B^{\text{uncertain}}(\rho)$ assigns a positive probability to all distances $\tilde{k} \in \{3, \ldots, 2n-2\}$. Moreover, since the equilibrium is monotonic, its forward neighbor’s payment $Q_1[\tilde{k}-1]$ is minimal for the distance $\tilde{k} = 3$, hence a bank $b_{p(i)} \in B^{\text{uncertain}}(\rho)$ solves problem (4.5) with $x^m = Q_1[2]$.

We are now in a position to state the main result of this section, which shows that all banks that are uncertain about their distances to the distressed bank hoard liquidity as if they are closer to the distressed bank than they actually are.

More specifically, all banks in $B^{\text{uncertain}}(\rho)$ hoard the level of liquidity that the bank with distance $\tilde{k} = 3$ would hoard in the free-information benchmark. When the cascade size is sufficiently large (i.e. $K \geq 3$) so that the bank with distance $\tilde{k} = 3$ in the free-information benchmark would hoard extensive liquidity, all banks in $B^{\text{uncertain}}(\rho)$ with actual distances $k > K$ also hoard large amounts of, even though they end up not needing it.

To state the result, we let $(y_{0,\text{free}}(\rho), q_{1,\text{free}}(\rho), q_{2,\text{free}}(\rho))_j$ denote the liquidity hoarding decisions and payments of banks in the free-information benchmark for each financial network $b(\rho) \in B$ (characterized in Proposition 3).

**Proposition 4.** Suppose assumptions (FS), (4.1), and (4.2) are satisfied and conditions (4.9), (4.13), (4.15), and (4.3) hold. For a given financial network $b(\rho)$, let $i = \rho^{-1}(j)$ denote the slot of the distressed bank.

(i) For the continuation equilibrium (at date 1): The equilibrium allocation is distance based and monotonic. The cascade size in the continuation equilibrium is the same as in the free-information benchmark, that is, at date 1, banks $\{b_{p(i-k)} \}^{K-1}_{k=0}$ are insolvent while banks $\{b_{p(i-k)} \}^{2n-1}_{k=K}$ are solvent where $K$ is defined in Eq. (4.14).

(ii) For the ex-ante equilibrium (at date 0): Each bank $b^j \in B^{\text{know}}(\rho)$ hoards the same level of liquidity $y_{0}^j(\rho) = y_{0,\text{free}}^j(\rho)$ as in the free-information benchmark, while each bank $b^j \in B^{\text{uncertain}}(\rho)$ hoards $y_{0}^j(\rho) = y_{0,\text{free}}^j(\rho)$, which is the level of liquidity bank $b_{p(i-3)}$ would hoard in the free-information benchmark.

For the aggregate level of liquidity hoarding, there are three cases depending on the cascade size $K$: 158
If $K \leq 2$, then the crisis in the free-information benchmark would not cascade to bank $b^{(i-3)}$, which would hoard no liquidity, i.e. $y_{0,\text{free}}^{\rho(i-3)}(\rho) = 0$. Thus, each bank $b^i \in B^{\text{uncertain}}(\rho)$ hoards no liquidity and the aggregate level of liquidity hoarding is equal to the benchmark Eq. (4.16).

If $K = 3$, then the crisis in the free-information benchmark would cascade to and stop at bank $b^{(i-3)}$, which would hoard an intermediate level of liquidity $y_{0,\text{free}}^{\rho(i-3)}(\rho) \in [0, \bar{y}_0]$. Thus, each bank $b^i \in B^{\text{uncertain}}(\rho)$ hoards $y_{0,\text{free}}^{\rho(i-3)}(\rho)$ and the aggregate level of liquidity hoarding is:

$$F = \sum_j y_0^j = 3\bar{y}_0 + (2n - 4) y_{0,\text{free}}^{\rho(i-3)}.$$  

(4.6)

If $K \geq 4$, then in the free-information benchmark bank $b^{(i-3)}$ would be insolvent and would hoard as much liquidity as it can, i.e. $y_{0,\text{free}}^{\rho(i-3)}(\rho) = \bar{y}_0$. Thus, each bank $b^i \in B^{\text{uncertain}}(\rho)$ hoards as much liquidity as it can and the aggregate level of liquidity hoarding is:

$$F = \sum_j y_0^j = (2n - 1) \bar{y}_0.$$  

(4.7)

The proof of this result is relegated to Appendix 4.A.2 since most of the intuition is provided in the discussion preceding the proposition. Among other features, the proof verifies that the equilibrium allocation at date 1 is distance based and monotonic, and that the cascade size is the same as in the free-information benchmark. The date 0 liquidity hoarding decisions are characterized as in part (ii) since the payments $Q_1[k-1]$ for $k \in \{1, 2, 2n - 1\}$ (that a bank $b^{(k-1)} \in B^{\text{know}}(\rho)$ with $k > 0$ expects to receive) and the payment $Q_1[2]$ (that the banks in $B^{\text{uncertain}}(\rho)$ effectively expect to receive) are the same as their counterparts in the free-information benchmark.

**Discussion.** The plots in Figure 4-5 are the equivalent to those in the free-information case portrayed in Figure 4-4. The top panel plots the cascade size $K$ as a function of the losses in the originating bank $\theta$. The parameters satisfy condition (4.3) so that the cascade size in this case is the same as the cascade size in the free-information benchmark characterized in Proposition 3, and both figures coincide.

The key differences are in the bottom panel, which plots the aggregate level of liquidity
hoarding $F$ as a function of $\theta$. The solid lines correspond to the costly audit equilibrium characterized in Proposition 4, while the dashed lines reproduce the free-information benchmark also plotted in Figure 4-4. These plots demonstrate that, for low levels of $K$ (i.e. for $K < 3$), the aggregate level of liquidity hoarding with costly-auditing is the same as the free-information benchmark, in particular, it increases roughly continuously with $\theta$. As $K$ switches from below 3 to above 3, the liquidity hoarding in the costly audit equilibrium make a very large and discontinuous jump. That is, when the losses (measuring the severity of the initial shock) are beyond a threshold, the cascade size becomes so large that banks are unable to tell whether they are connected to the distressed bank. All uncertain banks act as if they are closer to the distressed bank than they actually are, hoarding much more liquidity than in the free-information benchmark and leading to a severe credit crunch episode. This is our main result.

Note also that the aggregate level of liquidity hoarding (and the severity of the credit crunch) is not necessarily monotonic in the level of flexible reserves $\bar{y}_0$. For example, when $\theta = 0.5$, Figure 4-5 shows that providing more flexibility to the banks by increasing $\bar{y}_0$ actually
increases the level of aggregate liquidity hoarding. That is, at low levels of \( \theta \), an increase in flexibility stabilizes the system but the opposite may take place when the shock is sufficiently large. Intuitively, if the increase in flexibility is not sufficient to contain the financial panic (by reducing the cascade size to manageable levels), more flexibility backfires since it enables banks to hoard more liquidity and therefore exacerbate the credit crunch.

### 4.4 The Collapse of the Financial System

Until now, the uncertainty that arises from endogenous complexity affects the extent of the credit crunch but not the number of banks that are insolvent, \( K \). In this section we show that if banks have “too much” flexibility, in the sense that condition (4.3) no longer holds and

\[
\bar{y}_0 \in (\bar{y}_0^0, 1 - y)
\]  

(4.1)

(which also implies \( \beta = \bar{y}_0^0 \)), then the rise in uncertainty itself can increase the number of insolvent banks.

The reason is that a large precautionary liquidity hoarding compromises banks’ long run profitability by giving up high return \( R \) for low return \( 1 \). In this context, even if the worst outcome anticipated by a bank does not materialize, it may still become insolvent if sufficiently close (but farther than \( K \)) from the distressed bank. In other words, a bank’s large precautionary reaction improves its liquidation outcome when very close to the distressed bank but it does so at the cost of raising its vulnerability with respect to more benign scenarios. Since ex-post a large number of banks may find themselves in the latter situation, there can be a significant rise in the number of insolvencies as a result of the additional flexibility.

The analysis is very similar to that in the previous section. In particular, a bank’s payment still depends on its choice \( y'_0 \in [0, \bar{y}_0] \) at date 0 and its forward neighbor’s payment \( x = q_1^{r=(i-k)}(\bar{\rho}) \) at date 1. That is:

\[
q_1^{r=(i-k)}(\bar{\rho}) = q_1[y'_0, x] \text{ and } q_2^{r=(i-k)}(\bar{\rho}) = q_2[y'_0, x]
\]

for some functions \( q_1[y'_0, x] \) and \( q_2[y'_0, x] \). However, the characterization of the piecewise func-
tions \( q_1[y_0, x] \) and \( q_2[y_0, x] \) changes a little when condition (4.3) is not satisfied. In particular, these functions are identical to those in (4.A.1) and (4.A.2) in Appendix 4.A.2 (as in Section 4.3) but now there is an additional insolvency region:

\[
y_0' > y_0^u [(l_1 - x) z].
\]

The critical new element is the bound \( y_0^u [(l_1 - x) z] \). This is a function of the losses from cross-deposits and is calculated as the level of liquidity hoarding above which the bank’s loans and liquid reserves (net of losses) would not be sufficient to pay the late depositors at least \( l_1 \). That is, \( y_0^u [(l_1 - x) z] \) is the solution to

\[
R(1 - y - y_0^u [(l_1 - x) z]) + y_0^u [(l_1 - x) z] - (l_1 - x) z = l_1 (1 - \omega).
\]

We refer to scenarios where \( y_0' > y_0^u [(l_1 - x) z] \) as, for lack of a better jargon, scenarios of precautionary insolvency.

The functions \( q_1[y_0, x] \) and \( q_2[y_0, x] \) remain (weakly) increasing in \( x \). Moreover, we conjecture as before that the equilibrium is monotonic and distance based (which we verify below), so the banks’ liquidity hoarding decisions still solve problem (4.5). It can be verified that all banks (except potentially bank \( b_0^{(t+1)} \)) hoard the level of liquidity characterized in part (ii) of Proposition 4. In particular, all banks \( b_i \in B^{\text{uncertain}}(\rho) \) hoard the level of liquidity that the bank with distance \( k = 3 \) would hoard in the free-information benchmark. However, part (i) of the proposition, which characterizes the equilibrium at date 1, changes once \( y_0' \) exceeds \( y_0^u \).

We divide the cases by the cascade size: \( K \leq 2, K = 3, \) and \( K \geq 4 \). In the first two of these cases there is no additional panic relative to the case where banks’ flexibility is limited. If \( K \leq 2 \), each bank \( b_i \in B^{\text{uncertain}}(\rho) \) hoards \( y_0^d = 0 \). The date 1 equilibrium in this case is as described in part (i) of Proposition 4, in particular, there are no precautionary insolvencies and the cascade size is equal to \( K \). Similarly, if \( K = 3 \), each bank \( b_i \in B^{\text{uncertain}}(\rho) \) hoards \( y_0^d = y_{0, \text{free}}^{\rho(t-3)} \leq y_0^u \) (where the inequality follows since the transition bank \( b_{3-3}^{\rho(t-3)} \) is solvent in the free-information benchmark). Since \( y_0^d \leq y_0^u \), it can be seen that \( y_0^d \leq y_0^u [(l_1 - x) z] \), so the
banks in $B_{\text{uncertain}}(\rho)$ are solvent.\footnote{To see this, first note that $y_0^i \leq \bar{y}_0^i$, which implies $(R - 1) y_0^i \leq R\bar{y}_0^i - y_0^i = (1 - y) R - (1 - \omega) l_1 - y_0^i,$ where the equality follows from Eq. (4.8). Combining this inequality with the inequality $y_0^i \geq (l_1 - x) z$ (since $K \leq 3$, the banks in $B_{\text{uncertain}}(\rho)$ have sufficient liquid reserves at date 1) leads to $y_0^i \leq \frac{(1 - y) R - (1 - \omega) l_1 - (l_1 - x) z}{R - 1} = y_0^i [(l_1 - x) z]$. Note also that condition (4.3) implies $y_0^i \leq \bar{y}_0 \leq \bar{y}_0^i$ and thus rules out precautionary insolvencies by the above steps.} It follows that there are no precautionary insolvencies and the equilibrium is again as described in part (i) of Proposition 4, with a cascade size equal to $K = 3$.

The new scenarios arise when $K \geq 4$. In this case, each bank $b^i \in B_{\text{uncertain}}(\rho)$ hoards $y_0^i = \bar{y}_0 > \bar{y}_0^i$, and may experience a precautionary insolvency depending on its losses from cross-deposits. To analyze this case, first note that the bound $y_0^i [(l_1 - x) z]$ is decreasing in $(l_1 - x) z$, and thus increasing in $x$. That is, the more a bank receives from its forward neighbor, the higher the bound above which it will experience a precautionary insolvency. Second, note the inequality, $y_0^i [0] < 1 - y$ (which follows from some algebra and using $l_2/l_1 < R$). Then, there are two subcases to consider depending on whether or not the level of flexible reserves $\bar{y}_0$ is greater than $y_0^i [0]$ (which is the highest value of the bound $y_0^i [(l_1 - x) z]$).

**Subcase 1.** If $\bar{y}_0$ is in the interval $(y_0^i [0], 1 - y)$, then $\bar{y}_0$ is always greater than the upper bound $y_0^i [(l_1 - x) z]$ and a bank $b^i$ experiences a precautionary insolvency regardless of the amount $x$ it receives from its forward neighbor. In particular, all banks in $\{b^{i(1)}, \ldots, b^{i(2n-2)}\}$ are insolvent. It can be verified that the informed bank $b^{i(1)}$ averts insolvency by hoarding some $y_0^{i(1)} \leq \bar{y}_0$ (see Appendix 4.A.2).

**Subcase 2.** If $\bar{y}_0 \in (y_0^i [0], y_0^i [0])$, then there exists a unique $x [\bar{y}_0] \in (l_1 - \beta/z, l_1)$ that solves

$$y_0^i [(l_1 - x [\bar{y}_0]) z] = \bar{y}_0.$$  

In this case, a bank $b^i$ that hoards $\bar{y}_0$ of liquidity is insolvent if and only if it receives from its forward neighbor $x < x [\bar{y}_0]$ (so that $y_0^i [(l_1 - x) z] < \bar{y}_0$). By a similar analysis to that in Section 4.2.2 for the partial cascades (which we carry out in Appendix 4.A.2), it can be checked that there exists $K \in [K, 2n - 1]$ such that the banks $\{b^{i(1)}, \ldots, b^{i(2n-2)}\}$ are insolvent while
the banks \( \{ b^{(i-k)}, \ldots, b^{(i-(2n-1))} \} \) are solvent. In other words, there is a partial cascade which is at least as large as (and potentially greater than) the partial cascade in the free-information benchmark.

We summarize our findings in the following proposition.\(^6\)

**Proposition 5.** Suppose assumptions \((FS), (4.1)\) and \((4.2)\) are satisfied and conditions \((4.9), (4.13), (4.15)\) hold. Suppose also that condition \((4.1)\) (which is the opposite of condition \((4.3)\)) holds. For a given financial network \( b(\rho) \), let \( \tau = \rho^{-1}(j) \) denote the slot of the distressed bank.

(i) For the ex-ante equilibrium (at date 0): Each bank \( b^j \in \{ b^{(i)}, b^{(i-1)}, b^{(i-2)} \} \subset B^{\text{know}}(\rho) \) hoards the same level of liquidity \( y_0^j(\rho) = y_{0,\text{free}}^j(\rho) \) that it would hoard in the free-information benchmark, while each bank \( b^j \in B^{\text{uncertain}}(\rho) \) hoards \( y_0^j(\rho) = y_{0,\text{free}}^j(\rho) \), which is the level of liquidity the bank \( b^{(i-3)} \) would hoard in the free-information benchmark. The bank \( b^j \in b^{(i+1)} \) hoards \( y_0^{i+1}(\rho) \leq \bar{y}_0^j \) which is just enough to avert insolvency.

(ii) For the continuation equilibrium (at date 1): The equilibrium allocation is distance based and monotonic. There exists a unique \( \hat{K} \in [K, 2n-1] \) such that banks \( \{ b^{(i)}, \ldots, b^{(i-(K-1))} \} \) are insolvent while banks \( \{ b^{(i-K)}, \ldots, b^{(i-(2n-1))} \} \) are solvent. The cascade size \( \hat{K} \) is potentially larger than the cascade size \( K \) in the free-information benchmark. In particular, there are two cases:

If \( K \leq 3 \), then each bank \( b^j \in B^{\text{uncertain}}(\rho) \) hoards some \( y_0^j(\rho) \leq \bar{y}_0^j \), and avoids a precautionary insolvency. The cascade size in this case is identical to the free-information benchmark, i.e. \( \hat{K} = K \).

If \( K \geq 4 \), then each bank \( b^j \in B^{\text{uncertain}}(\rho) \) hoards \( y_0^j(\rho) = \bar{y}_0 > \bar{y}_0^j \), which may lead to a precautionary insolvency. There are two sub-cases:

---

\(^6\)Given the possibility of precautionary insolvencies, one may also wonder whether there could be multiple equilibria due to banks' coordination failures. Suppose, for example, \( K = 3 \), so that the crisis is contained after 3 banks fail. Could there also be a bad equilibrium in which all banks hoard the maximum level of liquidity, and their liquidity hoarding decisions are justified since their forward neighbors also hoard the maximum level of liquidity and experience a precautionary insolvency (thus paying a small \( q_j \))?

This kind of coordination failure is not possible in our setup, precisely because of conditions \((4.13)\) and \((4.15)\). These conditions ensure that bank \( b^{(i+1)} \) is always solvent, even if all other banks hoard the maximum level of liquidity and experience precautionary insolvencies. To see this, note that the losses from cross-deposits decrease as we move away from the distressed bank and eventually \( q^{(i+2)} \geq l_1 - \beta/z \). Since bank \( b^{(i+1)} \) expects to receive at least \( l_1 - \beta/z \) from its forward neighbor, it can avoid insolvency by hoarding an intermediate level of liquidity. Hence, it is never optimal for bank \( b^{(i+1)} \) to undergo a precautionary insolvency. But once we fix \( q^{(i+1)} = l_1 \), the rest of the equilibrium is uniquely determined as described above, that is, there is no coordination failure among banks.

164
Figure 4-6: The costly-audit equilibrium with precautionary insolvencies. Each one of the four panels plots the cascade size $\hat{K}$ as a function of $\theta$ for a different level of the flexible reserves $\bar{y}_0$ (in increasing order of $\bar{y}_0$ from top to bottom). The dashed lines plot the cascade size $K$ in the free-information benchmark.

If $\bar{y}_0 \in (\bar{y}_0^u [0], 1 - y]$, all banks $b^i \in B^{\text{uncertain}}(\rho)$ are insolvent and the cascade size is given by $\hat{K} = 2n - 1 > K \geq 4$.

If $\bar{y}_0 \in (\bar{y}_0^b, \bar{y}_0^u [0])$, there exists a unique $x [\bar{y}_0] \in (l_1 - \beta/z, l_1)$ characterized by Eq. (4.2) such that bank $b^i \in B^{\text{uncertain}}(\rho)$ is insolvent if and only if its forward neighbor’s payment is below $x [\bar{y}_0]$. The cascade size is an intermediate level $\hat{K} \in [K, 2n - 1]$.

Discussion. Figure 4-6 plots the cascade size $\hat{K}$ as a function of $\theta$, for different levels of the flexible reserves $\bar{y}_0$. For comparison, the dashed lines plot the cascade size $K$ in the free-information benchmark for the same parameters. The top panel corresponds to the case in which $\bar{y}_0 \leq \bar{y}_0^u$, i.e. when condition (4.3) holds. By Proposition 4, in this case there are no precautionary insolvencies and the cascade size is the same as the cascade size in the free-information benchmark. The second panel corresponds to a higher level of $\bar{y}_0$ that satisfies $\bar{y}_0 > \bar{y}_0^u$. In this case, precautionary insolvencies are possible, and for sufficiently large $\theta$ more banks are insolvent in the costly audit benchmark than in the free-information benchmark, i.e. $\hat{K} > K$. The third panel shows that, as we increase $\bar{y}_0$, a sufficiently large shock $\theta$ may trigger
a collapse of the whole financial system (i.e., $\tilde{K} = 2n - 1$).

The bottom panel in Figure 4-6 shows that as $\bar{\gamma}_0$ continues to rise, then at some point the amplification disappears and again $\tilde{K} = K$. That is, the effect of the flexible reserves $\bar{\gamma}_0$ on the size of the cascade $\tilde{K}$ is non-monotonic: The whole financial system collapses with an intermediate level of $\bar{\gamma}_0$, but the health of the financial system is restored (and, in fact, is stronger) with sufficiently high levels of $\bar{\gamma}_0$. The intuition for this non-monotonicity is the same as the intuition for the non-monotonic effect of $\bar{\gamma}_0$ on $\mathcal{F}$. Increasing the level of flexible reserves $\bar{\gamma}_0$ reduces the cascade size $K$ in the free-information benchmark. If this increase in flexibility is not sufficiently large, $K$ does not fall to manageable levels and the financial panic remains. As long as there is a financial panic, the increase in $\bar{\gamma}_0$ backfires and, in the current case, it also amplifies the cascade by generating more precautionary insolvencies. However, if the increase in $\bar{\gamma}_0$ is sufficiently large, it may end the financial panic and restore the health of the financial system.

### 4.5 Conclusion

Our model captures what appears to be a central feature of financial panics: During severe financial crises the complexity of the environment rises dramatically, and this in itself causes confusion and financial retrenchment. The perception of counterparty risk arises even in transactions among apparently sound financial institutions engaged in long term relationships. All of a sudden, economic agents are faced with massive uncertainty as things are no longer business-as-usual. The collapse of Lehman Brothers during the current financial crisis is one such instance, which froze essentially all private credit markets and triggered massive run downs of credit lines and withdrawals even from the safest money market funds.

In the model we capture the complexity of the environment with the size of the partial cascades. When these cascades are small, banks only need to understand the financial health of their immediate neighbors to make their decisions. In contrast, when financial conditions worsen and cascades grow, banks need to understand and be informed about a larger share of the network. At some point, this is simply too costly and banks withdraw from intermediation rather than risk exposure to enormous uncertainty, which triggers a flight to quality.
We also showed that banks’ flexibility, defined as their ability to hoard liquidity by not extending new loans or by selling illiquid assets while in distress, makes it harder for large cascades to develop, but if they do develop they can trigger more severe credit crunches and even a collapse in the financial system. Intuitively, a gain in flexibility is very useful if it succeeds in containing panic, but it can be counterproductive if it does not as it facilitates banks’ withdrawal from intermediation.

An aspect we did not explore in this essay but one which we are currently pursuing in a related work, is that of secondary markets for loans at date 0. Our preliminary findings point to yet another amplification aspect of the mechanism we highlight in this essay: With full information, the distant banks (i.e., the banks with \( k > K \)) are the natural buyers of the loans sold by the distressed banks. However, once distant banks face uncertainty and become worried that they may be too close to the distressed bank, they cease to buy loans from these banks as they would rather hoard their liquidity, which exacerbates the network’s distress.

There are some obvious policy conclusions that emerge from our framework. For example, there is clearly scope for having banks hold a larger buffer than they would be privately inclined to do. Also, transparency measures, by reducing the cost of gathering information, increase the resilience of the system to a lengthening in potential cascades. There is even an argument to limit banks’ flexibility to cut loan commitments. However, we are interested in going beyond these observations, and in particular in exploring the impact of policies that modify the structure of the network. For example, there is an emerging consensus that the prevalence of bilateral OTC markets for CDS transactions compounded the confusion and complexity of the current financial crisis, and that it is imperative to organize these transactions in well capitalized exchanges to prevent a recurrence. Our framework can help with the formal analysis of this type of policy considerations. We leave this analysis for future research.

4.A Appendices

4.A.1 The Normal Environment

The analysis in the text characterizes the equilibrium following a rare event for which the financial network is not prepared. In this Appendix, we describe the functioning of the financial
network in the normal environment. In particular, we show that the financial network enables
the banks to insure against heterogenous liquidity shocks and facilitates the flow of liquidity
across banks, even if the banks are uncertain about the network structure.

In the normal environment, there are three aggregate states of the world, denoted by \( s(0) \),
\( s(r) \) and \( s(g) \), revealed at date 0. In state \( s(0) \) all banks expect to receive at date 1 the same
liquidity shock \( \bar{\omega} \). The states \( s(r) \) and \( s(g) \) are realized with equal probability and the liquidity
shocks in these states are heterogeneous across banks. More specifically, the banks are divided
half and half between two types: red and green. In state \( s(r) \) (resp. \( s(g) \)), the banks with
red type (resp. green type) expect to receive a high liquidity shock, \( \omega_H \), and the other banks
expect to receive a low liquidity shock, \( \omega_L \). This means that in states \( s(r) \) and \( s(g) \) there is
enough aggregate liquidity but there is a need to transfer liquidity across banks.

To transfer liquidity in states \( s(r) \) and \( s(g) \), the banks form the financial network of bilateral
deposits in (4.1). We say that the financial network is consistent if all odd slots (resp. all even
slots) contain banks of the same type, which means that red and green type banks alternate
around the financial circle. We restrict the set of feasible networks to consistent ones (as opposed
to, for example, any circular network in which banks may be arbitrarily ordered around the
circle), since these networks ensure that each bank that needs liquidity has deposits on a bank
with excess liquidity, facilitating bilateral liquidity insurance (see below) with the minimally
required level of cross-deposits \( z \) in (4.2).

Banks' types and the financial network are realized as follows: First the types of banks are
realized at random (half of the banks become red type and the other half green type); then a
particular consistent financial network \( b(\rho) \) (with respect to these types) is realized. Banks' 
types are their private information, thus the set \( B \) in (4.3) represents the set of consistent
financial networks from an ex-ante point of view (i.e. before the types of the banks are realized).
This shock structure ensures that the actual realization of the financial network is always
consistent while the banks' uncertainty about the financial network is still described as in the
rare event case. In particular, a bank \( b^j \) observes the slots of its neighbors (and since the
network is consistent, it also indirectly learns the types of its neighbors), but it does not know
the slots (or the types) of the remaining banks \( \{b^j\}_{j \notin \{\rho(i-1),\rho(i),\rho(i+1)\}} \) (see Figure 4-2). By
auditing \( a^j \) balance sheets, the bank can further narrow down the set of possible networks to
Banks' Initial Balance Sheets

Assets:  
- $y$ liquid reserves.
- $\bar{y}_0$ flexible reserves.
- $1 - y - \bar{y}_0$ loans.

Liabilities:  
- Measure 1 of demand deposits that pay $l_1$ at date 1 or $l_2$ at date 2.
- Liquid reserves.

<table>
<thead>
<tr>
<th>Date 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Banks' types are realized at random.</td>
</tr>
<tr>
<td>A consistent financial network is realized.</td>
</tr>
<tr>
<td>NORMAL ENV: State in ${s(0), s(r), s(g)}$ is realized.</td>
</tr>
<tr>
<td>RARE EVENT: State $s'(0)$ is realized.</td>
</tr>
<tr>
<td>Banks make the audit decision $a^j \in {0, 1, \ldots, 2n - 3}$.</td>
</tr>
<tr>
<td>Banks make the liquidation hoarding decision $y_0^j \in [0, \bar{y}_0]$. (Equivalently banks extend $\bar{y}_0 - \bar{y}_0$ new loans.)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Date 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Banks make the deposit withdrawal decision $z^j \in [0, z^j]$.</td>
</tr>
<tr>
<td>Early depositors demand their deposits.</td>
</tr>
<tr>
<td>Late depositors demand their deposits: if and only if the bank cannot promise $q_1^j \geq l_1$.</td>
</tr>
<tr>
<td>Insolvent banks (that pay $q_1^j &lt; l_1$) unload all of their outstanding loans.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Date 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Banks pay $q_2^j$ to late depositors.</td>
</tr>
</tbody>
</table>

Figure 4-7: Timeline of events, for both the normal environment and the rare event.

$B^j (\rho | a^j)$.

Figure 4-7 recaps the timeline of events in this economy both for the normal environment and the rare event analyzed in the main text. Note that the rare event analyzed in the main text is characterized by an unanticipated aggregate state $s^j (0)$ which is very similar to one of the states in the normal environment (i.e. the state $s (0)$) except for the fact that one bank, $b^j$, becomes distressed and loses $\theta$ of its liquid reserves.

The equilibrium in the normal environment is a collection of bank auditing, liquidity hoarding, deposit withdrawal, and payment decisions $\left\{ a^j (\rho), y_0^j (\rho), z^j (\rho), q_1^j (\rho), q_2^j (\rho) \right\}_j$ such that, for each realization of the financial network $b (\rho)$ and the aggregate state in $\{s (0), s (r), s (g)\}$, each bank $b^j$ makes decisions that maximize the preferences in (4.4), and the late depositors withdraw deposits early if and only if $q_2^j < l_1$ (cf. Eqs. (4.5) and (4.6)). We next characterize this equilibrium.

The Normal Functioning of the Financial Network. We claim that, in the normal environment, the financial network facilitates liquidity flow and enables each bank $b^j$ to pay the contracted values $\left( q_1^j = l_1, q_2^j = l_2 \right)$ in each state of the world. Suppose that a consistent financial network, $b (\rho)$ and state $s (r)$ is realized, and suppose without loss of generality that...
red type banks are assigned to odd slots (the case in which red type banks are assigned to even slots is symmetric). It suffices to prove the statement for this case since the case in which \( s(g) \) is realized is symmetric to the \( s(r) \) case, and the case in which \( s(0) \) is realized is trivial.

We conjecture (and verify below) that each bank \( b^j \) chooses not to audit (for any positive audit costs \( d(.) > 0 \)) and hoards no liquidity, i.e. \( a^j = 0 \) and \( y_0^j = 0 \). Consider the equilibrium at date 1. A red type bank, \( b^{(2i-1)} \), (which is assigned to an odd slot by assumption) needs liquidity so it draws its deposits from the forward neighbor bank, i.e. chooses \( z^{p(2i-1)} = z \). For each green type bank, \( b^{(2i)} \), regardless of the financial network in \( B^{p(2i)}(\rho) \), drawing \( z^{p(2i)} \in [0, z] \) deposits leads to the payments \( q^{p(2i)} = l_1 \) and

\[
q_2^{p(2i)} = \frac{(1 - y) R + z^{p(2i)} l_1 + (z - z^{p(2i)}) l_2}{1 - \omega_L}.
\]

Since \( l_2 > l_1 \) and the preferences are given by (4.4), the green type banks do not draw their deposits regardless of their beliefs \( f^{p(2i)}(., \rho) \), i.e. they choose \( z^{p(2i)} = 0 \). It follows that liquidity flows through the network at date 1 even though each bank is uncertain about the network structure. In particular, each bank \( b^j \) pays the contracted values \( (q_1^j = l_1, q_2^j = l_2) \) in state \( s(r) \) (and similarly in states \( s(g) \) and \( s(0) \)).

We next consider the equilibrium at date 0 and verify our conjecture that the banks choose not to audit and not to hoard any liquidity. First note that a bank \( b^{(i)} \) in need of liquidity at date 1 is able to obtain it by withdrawing its deposits in the forward neighbor at a cost of \( l_2/l_1 \) units at date 2 for each unit of liquidity. The bank could also obtain liquidity by hoarding flexible reserves at date 0 but this would cost \( R > l_2/l_1 \) units for each unit of liquidity (since \( R > l_2 > l_1 > 1 \)). Therefore, each bank \( b^{(i)} \) optimally chooses not to hoard any liquidity at date 0. Second note that a bank's, \( b^{(i)} \), optimal actions (for liquidity hoarding at date 0 and deposit withdrawal at date 1) only depend on its slot \( i \) (and only on its parity), and in particular, it is independent of the financial network in \( B^{p(i)}(\rho) \). Thus the bank does not benefit from auditing and optimally chooses not to audit, \( a^{p(i)} = 0 \) (whenever \( d(.) > 0 \)), thus verifying our conjecture.

This completes the proof of our claim that, in the normal environment, the financial network facilitates liquidity flow across banks and enables each bank \( b^j \) to pay the contracted values \( (q_1^j = l_1, q_2^j = l_2) \) in each aggregate state.
4.A.2 Omitted Proofs

Proof of Eq. (4.7) for Sections 4.2.2 and 4.3. We claim that all cross-deposits are fully withdrawn, i.e. Eq. (4.7) holds, in both the free-information benchmark analyzed in Section 4.2.2 and the costly audit model analyzed in Section 4.3. By condition (4.9), the original distressed bank, \( b^{(t)} \), is insolvent thus it withdraws all of its deposits, i.e. \( z^{(t)} = z \). Suppose that, for some \( k \in \{0, \ldots, 2n - 1\} \), bank \( b^{(t-(k+1))} \) withdraws all of its deposits in bank \( b^{(t-k)} \). We claim that bank \( b^{(t-k)} \) also withdraws deposits, i.e. \( z^{(t-k)} = z \), which proves Eq. (4.7) by induction.

To prove the claim, we first consider the free-information benchmark and analyze two cases in turn. As the first case, suppose that the forward neighbor of bank \( b^{(t-k)} \) is insolvent (i.e. it pays \( q^{(t-(k-1))}_1 < l_1 \) and \( q^{(t-(k-1))}_2 = 0 \)). Recall that bank \( b^{(t-k)} \) is small and takes the payment of its forward neighbor as given (see footnote 1), in particular, it cannot potentially bail out its forward neighbor by withdrawing less than \( z \). This further implies that the bank withdraws all of its deposits from its forward neighbor, i.e. \( z^{(t-k)} = z \). As the second case, suppose that the forward neighbor bank, \( b^{(t-(k-1))} \), is solvent, i.e. \( q^{(t-(k-1))}_1 = l_1 \). In this case, bank \( b^{(t-k)} \) needs liquidity \( z \) (to pay its backward neighbor) and it can obtain this liquidity either by withdrawing its deposits, which costs \( l_2/l_1 \) units at date 2 per unit of liquidity, or by hoarding flexible reserves at date 0, which costs \( R > l_2/l_1 \) units per unit of liquidity. Since the former is a cheaper way to obtain liquidity, bank \( b^{(t-k)} \) withdraws all of its deposits from its forward neighbor, proving our claim that \( z^{(t-k)} = z \).

Next consider the costly audit model of Section 4.3. Recall that bank \( b^{(t-k)} \) makes the deposit withdrawal decision before the resolution of uncertainty for cross-deposits (see Figure 4-3). As the first case, suppose that bank \( b^{(t-k)} \) assigns a positive probability to a network structure \( b(\bar{p}) \) such that \( q^{(t-(k-1))}_1(\bar{p}) < l_1 \) (that is, suppose the bank assigns a positive probability that its forward neighbor will be insolvent). Since the bank takes the payment of its forward neighbor as given and its preferences are given by the Leontieff form in (4.2), in this case the bank necessarily withdraws all of its deposits, i.e. \( z^{(t-k)} = z \). Next suppose bank \( b^{(t-k)} \) believes that \( q^{(t-(k-1))}_1(\bar{p}) = l_1 \) with probability 1 (that is, the bank knows that its forward neighbor is solvent). In this case, as in the free-information benchmark, the bank withdraws \( z^{(t-k)} = z \) to meet its liquidity obligations to its backward neighbor. This completes
the proof of the claim and proves Eq. (4.7) by induction.

Proof of Proposition 3. Contained in the discussion preceding the proposition.

Characterization of Banks’ Payment Functions $q_1 [y'_0, x]$ and $q_2 [y'_0, x]$ in Section 4.3.
If bank $b^{p(i-k)}$ hoards a level of liquidity $y'_0 \in [0, \bar{y}_0]$ at date 0, and its forward neighbor pays $x = q'_1 (p) (i-(k-1))$ at date 1 (and if condition (4.3) holds), then this bank’s payment is given by functions $q_1 [y'_0, x]$ and $q_2 [y'_0, x]$ which are characterized as follows:

Case 1. If $x \in [l_1 - \beta/z, l_1]$ and $y'_0 \geq (l_1 - x) z$, then

$$q_1 = l_1 \text{ and } q_2 = \frac{y'_0 - (l_1 - x) z + (1 - y - y'_0) R}{1 - \omega} \geq l_1. \tag{4.A.1}$$

Case 2. If $x < l_1 - \beta/z$ or $y'_0 < (l_1 - x) z$, then

$$q_1 = \frac{y + y'_0 + zx}{1 + z} \leq l_1 \text{ and } q_2 = 0. \tag{4.A.2}$$

The first case characterizes the payment when the bank’s losses from cross-deposits do not exceed its buffer and the bank has hoarded enough flexible reserves to counter these losses. In this case, the bank is solvent and pays according to (4.A.1). The second case characterizes the payment when the bank’s losses from cross-deposits exceed its buffer, or when the losses do not exceed the buffer but the bank has not hoarded enough flexible reserves to counter these losses. In this case, the bank is insolvent and pays according to (4.A.2).

Proof of Proposition 4. First consider part (i) taking as given the characterization of the liquidity hoarding decisions in part (ii). Note that the liquidity hoarding decision of each bank depends only on its distance from the distressed bank, which implies that the payments of banks in the continuation equilibrium can be written as a function of their distances, i.e. that the equilibrium is distance based. The characterization in part (ii) shows that each bank $b^{p(i-k)} \in \{ b^{p(i)}, b^{p(i-1)}, \ldots, b^{p(i-(K-1))} \}$ that would be insolvent in the free-information benchmark hoards $y'_0 = \bar{y}_0$, and thus it pays the same allocation it would pay in the free-information
benchmark:

\[ Q_1[k] \equiv q_{1,\text{free}}^{\rho(i-k)}(\rho) \leq l_1 \text{ and } Q_2[k] \equiv q_{2,\text{free}}^{\rho(i-k)}(\rho) = 0 \text{ for } k \in \{0, \ldots, K - 1\}. \tag{4.A.3} \]

The bank \( b^{\rho(i-K)} \) hoards at least as much liquidity as it would hoard in the free-information benchmark, thus it is solvent given condition (4.3) (which ensures that hoarding too much liquidity does not cause insolvency) and pays (cf. Eq. (4.A.1)):

\[ Q_1[K] = l_1 \text{ and } Q_2[K] \geq l_1. \tag{4.A.4} \]

The banks \( b^{\rho(i-k)} \in \{b^{\rho(i-(K+1))}, b^{\rho(i-(K+2))}, \ldots, b^{\rho(i-(2n-1))}\} \) are solvent and thus pay (cf. Eq. (4.A.1)):

\[ Q_1[k] = l_1 \text{ and } Q_2[k] = \frac{y_0^{\rho(i-k)} + \left(1 - y - y_0^{\rho(i-k)}\right) R}{1 - \bar{\omega}} \geq l_1, \text{ for } k \in \{K + 1, \ldots, 2n - 1\}. \tag{4.A.5} \]

In particular, the size of the cascade is \( K \) as it is in the free-information benchmark. Since \( q_{1,\text{free}}^{\rho(i-k)}(\rho) \) is increasing in \( k \) (see Proposition 3) and \( y_0^{\rho(i-k)} \) is decreasing in \( k \) (given the liquidity hoarding decisions in part (ii)), the characterization in (4.A.3) through (4.A.5) also implies that the payments, \( Q_1[k] \) and \( Q_2[k] \), are increasing in \( k \) and proves that the distance based equilibrium is monotonic.

We next turn to the liquidity hoarding decisions at date 0 and prove that the choices prescribed in part (ii) are optimal. Consider first the banks in \( B^{\text{know}}(\rho) \). Comparing the characterization of the continuation equilibrium in (4.A.3) through (4.A.5) to the characterization in Proposition 3, each bank \( b^i \in B^{\text{know}}(\rho) \) expects to receive the same payment from its forward neighbor compared to what it would receive in the free-information benchmark (i.e. each bank \( b^i \in B^{\text{know}}(\rho) \) solves problem (4.5) with \( x^m = q_{1,\text{free}}^{\rho(i-(k-1))} \)). Thus it also hoards the same level of liquidity that it would hoard in the free-information benchmark.

Next we consider a bank \( b^i \in B^{\text{uncertain}}(\rho) \) which solves problem (4.5) with \( x^m = Q_1[2] \). We claim that \( Q_1[2] \) characterized in Eqs. (4.A.3) through (4.A.5) is equal to \( q_{1,\text{free}}^{\rho(2)} \) (the payment of the forward neighbor of bank \( b^{\rho(i-3)} \) in the free-information benchmark), which in turn proves that the bank \( b^i \) hoards the same level of liquidity \( y_0^{\rho(i-3)} \) that \( b^{\rho(i-3)} \) would hoard
in the free-information benchmark. To prove the claim that \( Q_1 [2] = q^{(2)}_{1, \text{free}} \), first suppose that \( K \leq 2 \). Note that in this case \( Q_1 [2] \) is given by Eq. (4.A.4) or Eq. (4.A.5) and in either case \( Q_1 [2] = l_1 \). Note that by Proposition 3, \( q^{(2)}_{1, \text{free}} = l_1 \) when \( K \leq 2 \), proving the claim in this case. Next suppose \( K \geq 3 \) and note that in this case \( Q_1 [2] \) is given by Eq. (4.A.3) which shows \( Q_1 [2] = q^{(l-2)}_{1, \text{free}} (\rho) \), completing the proof of part (ii).

The characterization for the aggregate level of liquidity hoarding for the cases \( K \leq 2, K = 3 \) and \( K \geq 4 \) then trivially follow from part (ii) and Proposition 3, thus completing the proof.

**Proof of Proposition 5.** Most of the proof is contained in the discussion preceding the proposition. Here, we consider in turn the subcases 1 and 2 (for case \( K \geq 4 \)) and we verify the claims in the main text. We also verify the conjecture that the equilibrium is distance based and monotonic.

**Subcase 1.** If \( \bar{y}_0 \in (y_0^u [0], 1 - y] \), then

\[
\bar{y}_0 > y_0^u [0] \geq y_0^u [(l_1 - x) z]
\]

for any \( x \in [0, l_1] \). This implies that all banks in \( \{b^{(i)}, \ldots, b^{(l-(2n-2))}\} \) are insolvent since they hoard a level of liquidity greater than their corresponding upper limits. These banks’ payments are characterized by the system of equations

\[
q^{(l-k)} f (q^{(l-(k-1))}) = f (q^{(l-(k-1))}) \quad \text{for each } k \in \{1, \ldots, 2n - 2\}, \tag{4.A.6}
\]

where \( f (.) \) is defined in Eq. (4.12) and the initial condition \( q^{(i)} \) is given by Eq. (4.10) (after plugging in \( q^{(l+1)} = l_1 \)).

By condition (4.13), the solution to the above system is increasing (and converges to the fixed point \( y + \bar{y}_0 \leq 1 < l_1 \)), verifying our conjecture that the equilibrium is distance based and monotonic. By condition (4.15), we have \( K \leq 2n - 2 \), which implies \( q^{(l-(2n-2))} > q^{(l-(K-1))} = q^{(l-(K-1))} \). Then, since bank \( b^{(l-K)} \) in the free-information benchmark is able to avert insolvency by hoarding the level \( y^{(l-K)}_{0, \text{free}} \leq \bar{y}_0 \), the informed bank \( b^{(l+1)} \) in this case can also avert insolvency by hoarding \( y^{(l+1)}_{0, \text{free}} \leq \bar{y}_0 \). It follows that the cascade size is \( \hat{K} = 2n - 1 \), which is greater than the free-information cascade size \( K \) (under
condition (4.15)), completing the characterization of the date 1 equilibrium in this case.

**Subcase 2.** If \( \bar{y}_0 \in (\bar{y}_0^n, y_0^n [0]) \), there exists a unique \( x [\bar{y}_0] \in (l_1 - \bar{y}_0^n / z, l_1) \), characterized in Eq. (4.2) and increasing in \( \bar{y}_0 \), such that a bank \( b^j \in B^{uncertain}(\rho) \) is insolvent if and only if receives from its forward neighbor \( x < x [\bar{y}_0] \). Using the conjecture that the equilibrium is distance based and monotonic, we further conjecture that the banks \( \{b^{(i)}, ..., b^{(i-(K-1))}\} \) are insolvent while the banks \( \{b^{(i-K)}, ..., b^{(i-(2n-1))}\} \) are solvent. The payments of the banks in \( \{b^{(i)}, ..., b^{(i-(K-1))}\} \) are characterized by

\[
q_1^{(i-k)} = f \left( q_1^{(i-(k-1))} \right) \quad \text{for each} \quad k \in \{1, ..., K - 1\},
\]

which is an increasing sequence (by condition (4.13)). Then, either \( q_1^{(i-(2n-2))} < x [\bar{y}_0] \) and we are back to subcase 1 (i.e. \( \hat{K} = 2n - 1 \)), or there exists a unique \( \hat{K} \in [K, 2n - 1] \) such that

\[
q_1^{(i-(\hat{K}-2))} < x [\bar{y}_0] \leq q_1^{(i-(\hat{K}-1))}.
\]

In the latter case, the banks in \( \{b^{(i-K)}, ..., b^{(i-(\hat{K}-1))}\} \subset B^{uncertain}(\rho) \) are insolvent (since they receive less than \( x [\bar{y}_0] \) from their forward neighbor) but the bank \( b^{(i-\hat{K})} \) is solvent since it receives at least \( x [\bar{y}_0] \) from its forward neighbor. The banks in \( \{b^{(i-(\hat{K}+1))}, ..., b^{(i-(2n-2))}\} \) are also solvent since they receive \( l_1 \geq x [\bar{y}_0] \) from their forward neighbors. The informed bank \( b^{(i+1)} \) is also solvent as in subcase 1. Finally, this analysis implies that the equilibrium is distance based and monotonic, completing the characterization of the date 1 equilibrium in this case.
Chapter 5

Moral Hazard and Efficiency in General Equilibrium with Anonymous Trading

*Joint with Daron Acemoglu, MIT.

5.1 Introduction

A central question for economic theory is the efficiency of competitive markets. In economies with complete markets, this question is conclusively answered by the celebrated First and Second Welfare Theorems, which show that, under some regularity conditions, competitive equilibria are Pareto optimal and every Pareto optimal allocation can be decentralized as a competitive equilibrium. Nevertheless, the complete market benchmark does not cover many empirically-relevant economies where missing markets are ubiquitous. Arguably the most important reason for missing markets in practice is private information. Individual agents know more about their preferences, risks and actions than the market can observe. Despite a sizable literature on this topic, efficiency properties of economies with private information are not yet fully understood. In this essay, we investigate the efficiency of competitive equilibria in a subclass of economies with private information, those with moral hazard, where individuals take privately-observed actions affecting their endowments (and/or production).
One approach to the study of efficiency in moral hazard economies has been pioneered by Prescott and Townsend (1984a, 1984b). Prescott and Townsend propose the important idea of considering insurance contracts as commodities that should also be priced in equilibrium. Prescott and Townsend show that competitive equilibria with moral hazard are (constrained) Pareto optimal under two key assumptions: exclusivity and full monitoring. The first implies that individuals can sign exclusive contracts and is a good starting point for the study of employment contracts. We focus on exclusive contracts throughout the essay. The second assumption, full monitoring, is more problematic. Under full monitoring, contracts specify complete consumption bundles for individuals in different states of nature. This essentially implies that firms or some other outside agency can fully monitor individual consumptions. This assumption is not only unrealistic but also goes against the spirit of “competitive markets.” Competitive markets should allow anonymous trading, so that individuals are able to buy at least a subset of commodities in anonymous markets without a central agency keeping track of their exact transactions.

A systematic analysis of the structure and efficiency of competitive equilibria with anonymous trading is not available, but a series of papers by Stiglitz and coauthors, most notably, Greenwald and Stiglitz (1986), and also Arnott and Stiglitz (1986, 1990, 1991), claim that competitive equilibria under these circumstances are always or “generically” Pareto suboptimal. These claims are supported by local analysis of first-order conditions, though without a rigorous proof that this type of local analysis is valid or economically important. Hence one may say that the inefficiency of competitive equilibria with anonymous trading has emerged as a folk theorem. This folk theorem is not only of theoretical interest but has been very influential

---

1Exclusivity may be a less satisfactory assumption for insurance contracts, in particular, when informal insurance is also possible; see, e.g., Arnott and Stiglitz (1991) and Bisin and Guaitoli (2003). Real-world insurance and financial contracts often explicitly regulate what other contracts individuals can sign for the same risks, or whether they can pledge the revenues of the same business. Exclusivity is much more natural in the context of employment contracts we focus on in this essay.

2This is particularly concerning for three reasons. First and more importantly, this local analysis assumes that a range of Lagrange multipliers exist and are strictly positive, though there is no mathematical or economic reason for them to be so. One of our main results will establish the (constrained) optimality of competitive equilibria under certain conditions explained below, thus invalidating this line of analysis. Second, the local analysis makes use of differentiability assumptions and the first-order approach, which do not generally apply in these environments (see Grossman and Hart, 1983, Rogerson, 1985, Jewitt, 1988 on the first-order approach). Third, as we will show, it may well be that even if efficiency is “nongeneric,” small deviations from this efficiency benchmark might still lead to allocations that are approximately (c-) efficient and many inefficiencies identified via this method may not be first order.
in applied work. It is often invoked to argue that decentralized allocations in insurance, labor and credit markets are inefficient and necessitate government intervention (or to provide the intuition for specific models in which this is the case).

In this essay, we consider a general equilibrium environment where the structure and efficiency of competitive equilibria with anonymous trading can be studied. The economy consists of a large number of firms and risk-averse individuals. Individuals accept employment contracts from firms and choose an effort level, which determines the probability distribution over a vector of production. Individual effort is private information. Commodities in this economy are partitioned, such that expenditures over subsets in a given monitoring partition of commodities are observable (for example, how much an individual spends on vacation can be determined but not how this spending is distributed across different activities in the vacation resort). Employment contracts specify payments to workers and expenditure levels over the subsets in the monitoring partitions as a function of the realization of the state of nature. The Prescott-Townsend economy is a special case where each subset in the monitoring partition is a singleton.\(^3\) After all uncertainty is resolved (the underlying states of the world are realized), individuals allocate the contractually-specified expenditures within the subsets in the partition at given market prices.

We establish the existence of a competitive equilibrium and an indirect maximization problem that characterizes equilibrium allocations (Theorem 15 and Proposition 6). We then formalize the above-mentioned folk theorem. We say that there is no full insurance at an equilibrium if the marginal rate of substitution of some good between any two states is not one. can we say that preferences are nonseparable (or more accurately, “not weakly separable”), if there is a subset in the monitoring partition such that the marginal rate of substitution between the goods in the subset change if the effort level is modified. Conversely, we say that preferences are weakly separable when there exists no such subset (this is significantly weaker than the standard separability assumptions often adopted in theoretical and applied work). Our first result (contained in Theorem 16) shows that when preferences are nonseparable and there is no full insurance at an equilibrium, then this equilibrium is constrained suboptimal (inefficient), in

\(^3\) Another special case is one in which the partition consists of a number of singleton elements, which correspond to “monitored goods,” and the remainder, which comprises “nonmonitored goods.”
the sense that a social planner who is constrained by the same moral hazard problems (but who is allowed to monitor expenditures on all goods) can improve over the equilibrium allocation.\(^4\) Note that this theorem is silent on whether a social planner who is also constrained by the same monitoring technology can implement such a Pareto improvement, and we show that this is not necessarily the case.\(^5\)

While Theorem 16 appears to give some support to the folk theorem, the rest of our analysis sheds doubt on its general validity and applicability. Theorem 17 shows that competitive equilibria are constrained Pareto optimal when preferences are weakly separable (see also Theorem 20 for the case in which contracts with randomization are allowed). This is an important result for two reasons. First, most preferences used in applied work satisfy this weak separability condition.\(^6\) Second, it establishes that the equilibrium is constrained optimal relative to a very strong notion in which the social planner has access to more instruments than the market (she can monitor and specify expenditures for all goods, whereas contracts can only do so for goods within a subset in the partition). This result suggests that, at least in most of environments considered in applied work, the inefficiencies emphasized by the folk theorem do not arise.

Finally, in Theorem 21 we show that when there are only small deviations from this benchmark environment with weak separability, competitive equilibria remain approximately efficient (or are \(\varepsilon\)-efficient for the right choice of \(\varepsilon\)). This result highlights another important conceptual point, that a large set of economically relevant environments may not feature meaningful inefficiencies even if there is a "generic" inefficiency result.

Overall, although our results do not imply that competitive equilibria are always efficient in private information economies, they delineate a range of benchmark situations in which equilibria have very strong optimality properties. They also show that when such benchmark situations are a good approximation to the actual environment, efficiency will hold approxi-

\(^4\) Prescott and Townsend (1984a,b) emphasized the importance of contracts that allow for randomization for efficiency in their analysis. Such randomization is important for existence in their environment, but is not central for the baseline efficiency results in our or their framework. To highlight this and to simplify the exposition, we start with an environment that does not allow for such randomization. We then establish the equivalent results when randomization is allowed (e.g., Theorem 19 generalizes Theorem 16, etc.).

\(^5\) Nevertheless, the notion of efficiency we use for most of the analysis may be more relevant than this latter weaker notion, since in a production economy the social planner can often achieve the allocations under full monitoring by using taxes and subsidies. To simplify notation, we do not explicitly study such tax policies.

\(^6\) See, for example, Atkeson and Lucas (1992), Golosov and Tsyvinski (2006), Golosov, Kocherlakota and Tsyvinski (2003).
mately. Suppose for example that the worker’s preferences are defined at two levels: the worker has preferences over a number of needs (such as food, entertainment, procrastination, vacation, health care and so on) and each one of these needs is satisfied by consumption of various goods in the economy. Suppose also that there are only a few needs that interfere with the worker’s effort choice such as vacation, procrastination and health care. In particular, a higher effort choice might make the worker enjoy vacations and procrastination less relative to other needs (since she spends most of her time working) and enjoy health care more relative to other needs (since the cost of effort decreases with level of health), but the worker’s preferences for other needs may not depend on effort level. Suppose also that employers can monitor the consumption levels for the few needs that interfere with effort choice (but not necessarily the consumption of the particular goods that make up these needs). This amounts to assuming that firms can enforce how long a vacation the worker takes, how much time she spends in the office, and how good a health care she receives. Under these assumptions, our Theorem 17 applies and shows that the equilibrium will be constrained optimal. This scenario thus constitutes a counterexample to the conjectured suboptimality of competitive equilibria under private information and anonymous trading, and suggests considerable caution in appealing to the above-mentioned folk theorem. Moreover, in this example, the folk theorem would suggest taxing the goods that provide procrastination services (such as TVs), but our results indicate that this might be the wrong policy recommendation since firms already monitor the consumption of TVs (and other procrastination goods) by making the workers come to the office during work hours.

As the above discussion clarifies, our essay is related to a number of literatures. The relationship of our essay to Prescott and Townsend (1984a, 1984b) and to Greenwald and Stiglitz (1986) has already been discussed. Another set of closely related papers are by Geanakopolos and Polemarchakis (1986, 2008) and Citanna, Kajii and Villanacci (1998), which establish the generic inefficiency of competitive equilibria in economies with (exogenously-given) incomplete markets. Our work extends the Geanakoplos-Polemarchakis results to environments with endogenously incomplete markets (because of moral hazard). It also highlights that in such environments constrained optimality may result if the appropriate subsets of goods are monitored. Similar issues arise in other economies with price externalities due to endogenously incomplete markets. For example, Kehoe and Levine (1993) provide results similar to ours for an econ-
omy with participation constraints. Our essay is also related to Citanna and Villanacci (2000), which establishes generic inefficiency of equilibria for a moral hazard economy with exclusive contractual relationships in which the principal has all the bargaining power. Our work shows that their inefficiency result crucially depends on the assumption that the principal has the bargaining power. Given the separability assumptions they impose on preferences, our results imply that the equilibrium would be efficient in the polar opposite case in which the agent has all the “bargaining power,” i.e. when insurance contracts are exclusive and the insurance market is competitive.

In addition to these works, the paper most closely related to our essay is Lisboa (2001), which establishes the Pareto optimality of competitive equilibria in the context of an economy with moral hazard and fully separable utility. A number of key differences between our work and Lisboa are worth emphasizing. First, Lisboa considers a special case of the model studied here, in which utility functions are fully separable across all goods and effort and there is no monitoring of consumption in any subset of goods. Second, Lisboa’s analysis relies on the first-order approach applying everywhere, which is restrictive and not used in our analysis. Third, Lisboa’s analysis contains neither our characterization results on inefficiency of comparative equilibria nor our result on approximate efficiency.

There is also a large literature on various different aspects of moral hazard in general equilibrium. Bennardo and Chiappori (2003) and Gottardi and Jerez (2007) discuss the problems that arise in general equilibrium economies with moral hazard because of potential non-transferability of utility. They show how Bertrand competition might lead to equilibria with positive profits. This is an issue that also arises in our model and we provide sufficient conditions (that are not very restrictive) for Bertrand competition to lead to zero profits. Bisin, Geanakoplos, Gottardi, Minelli and Polemarchakis (2007) consider an alternative approach to moral hazard in general equilibrium, where, in contrast to our setup, contracts are not necessarily individualized. This introduces natural externalities across the actions of different individuals signing the same type of contract.7

Finally, some of the same issues we emphasize in the context of general equilibrium also

---

7 Also related are recent papers considering adverse selection in general equilibrium, for example, Bisin and Gottardi (1999, 2006) and Jerez (2003).
arise in the public finance and mechanism design literatures. See, for example, Hammond (1987), Allen (1985), Atkeson and Lucas (1992), Guesnerie (1998), Cole and Kochelekota (2001), Werning (2001), Kochelekota (2004), Golosov and Tsyvinski (2006), and Doepke and Townsend (2006). None of these studies derive results similar to our main theorems in this essay.

The rest of the essay is organized as follows. Section 5.2 presents the environment, defines a competitive equilibrium, and establishes the existence of equilibrium. Section 5.3 introduces the notion of constrained optimality and provides sufficient conditions under which the equilibrium is not constrained optimal. Section 5.4 introduces the notion of weak separability and presents our main result, which shows that the equilibrium is constrained optimal when the preferences are weakly separable. Section 5.5 introduces the environment with stochastic contracts and generalizes the efficiency results to this setting. Section 5.6 presents our second main result, which shows that the equilibrium is approximately constrained optimal when the preferences are approximately weakly separable. Section 5.7 concludes. Appendix 5.A.1 discusses additional results omitted from the main text, and Appendix 5.A.2 contains the proofs of all the results stated in the text.

5.2 Environment and Equilibrium

5.2.1 Preferences

We consider a static production economy with a finite set of goods denoted by $G$ and a finite set of (individual-specific) states of nature denoted by $S$. We use $g \in G$ and $s \in S$ to index goods and states, and use $|G|$ and $|S|$ to denote the cardinality of these sets. There is a continuum of individual workers, denoted by $\mathcal{N}$, with measure normalized to 1. To simplify the analysis and the exposition, we assume that all workers have identical utility and identical production technology.\(^8\) In particular, each worker chooses an effort level $e \in E$, where $E = \{e_1, \ldots, e_{|E|}\} \subset \mathbb{R}$ is a finite set. The effort choice of the worker induces a probability distribution over an endowment (production) vector $y \in \mathbb{R}^{G}_{+}$. We represent this probability distribution by the

\(^8\)The results generalize to multiple types straightforwardly provided that worker type is observable and contractible.
function \( q \), whereby \( q_s(e) \in [0, 1] \) is the probability of state \( s \in S \) for the worker in question when she exerts effort \( e \) (naturally with \( \sum_{s \in S} q_s(e) = 1 \) for all \( e \in E \)). Each state \( s \in S \) is, in turn, associated with a production vector \( y_s \in \mathbb{R}^{G|} \). For each \( g \), there exists \( s \in S \) such that \( y^g_s \neq 0 \), which ensures that each good \( g \) is in positive supply in some states. We also assume throughout that the realization of states in \( S \) (conditional on effort) is independent across individuals. Thus, with a law of large numbers type argument there is no aggregate uncertainty.\(^9\)

We assume that each worker has VNM preferences over consumption of goods and effort choice represented by

\[
U(x, e) = \sum_{s \in S} q_s(e) u(x^s, e),
\]

where \( x^s = (x^1_s, \ldots, x^{|G|}_s) \in \mathbb{R}^{G|} \) is the vector of consumption in state \( s \) and \( u(\cdot) \) denotes the state utility function. Throughout, we use the notation \( x_s \equiv (x^g_s)_{g \in G} \) to designate vectors, and \( x = (x^g_s)|_{s \in S, g \in G} \) to designate matrices. We refer to the pair \((x, e)\) as an allocation, and let \( A \equiv \mathbb{R}^{S \times G} \times E \) denote the set of allocations. We make the following standard assumption on the utility function:

**Assumption A1 (Preferences)** The state utility function \( u(\cdot) \) is twice continuously differentiable in \( x^s \), strictly increasing in each \( x^g_s \), and strictly concave in \( x^s \).

Throughout, the effort choice of the worker is her private information, so there is a moral hazard problem and employment contracts cannot be conditioned on effort choices. The realized production vector is publicly observable and employment contracts can condition on these realizations. Motivated by the discussion in the introduction, we consider a partition \( G = \{G_1, \ldots, G_{|G|}\} \) of the set of goods (i.e., a collection of disjoint subsets of goods the union of which is equal to the set of all goods, \( G \)). The employment contracts can specify the worker’s expenditure \( w^m_s \in \mathbb{R}_+ \) on the monitoring subset \( G_m \), for each \( m \in M \equiv \{1, \ldots, |G|\} \). The goods within each monitoring subset \( G_m \) are traded in spot markets that operate after all production vectors are realized (and at market clearing prices as described below). We use

\(^9\)See, for example, Bewley (1986) or Malinvaud (1973). Nevertheless, some care is necessary in defining the right notion of integral in applying such a law of large numbers. The simplest approach, proposed by Uhlig (1996), is sufficient here.
\( w = \{w^m_s\}_{s \in S, m \in M} \) to denote the matrix where each element denotes the individual's expenditure on each monitoring subset at a given state. We denote the vector of prices by \( p \in \mathbb{R}^{|G|}_+ \). We choose good 1 as the numeraire, i.e., \( p^1 = 1 \). For any subset of commodities \( G' \subset G \), we denote by \( p^{G'} \) the corresponding price sub-vector, and by \( x^{G'} \) the corresponding consumption sub-matrix.

### 5.2.2 Firms and Employment Contracts

A large finite number of firms can sign employment contracts with the workers. We denote the set of firms by \( J = \{1, 2, \ldots, |J|\} \). Firms are owned by the workers and maximize the expected profits. Since, as we will see shortly, in equilibrium firms will make zero profits, we do not introduce additional notation to specify the allocation of their profits.

Throughout we impose exclusivity and assume that each worker can only contract with a single firm. An employment contract between a firm and a worker gives the property rights over the worker's production, \( y_s \), to the firm and specifies worker's expenditures, \( w = \{w^m_s\}_{s \in S, m \in M} \), and prescribes consumption levels for goods, \( x = (x^G_g)_{s \in S, g \in G} \), and effort choice, \( e \in E \). Note that at this point we are not allowing stochastic contracts, which would instead determine a probability distribution over outcomes. We return to stochastic contracts in Section 5.5.

We denote a contract by the tuple \( c = (w, x, e) \), and we denote the set of contracts by \( C = \mathbb{R}^{S \times |M|}_+ \times A \) (where recall that \( A = \mathbb{R}^{S \times |G|}_+ \times E \) denotes the set of allocations). Notice that \( C \) is a subset of a finite dimensional space and its elements, denoted by \( c \in C \), are simply vectors. An incentive compatible contract is \( c = (w, x, e) \in C \) such that the effort choice and the level of consumption of goods are incentive compatible given prices \( p \) and wage schedule \( w \). More formally, this means that for a given market price vector \( p > 0 \),

\[
(x, e) \in \arg \max_{(\tilde{x}, \tilde{e}) \in A} U(\tilde{x}, \tilde{e}) \tag{5.1}
\]

subject to \( \tilde{x}^{G_m}_s p^{G_m}_s \leq w^m_s \) for each \( s \) and \( m \).

We denote the set of incentive compatible contracts by \( C^I(p) \).

The maximization problem in (5.1) can be conceptually divided into two parts. Given an
effort level \( \tilde{e} \), the consumption choice at each state \( s \) is uniquely determined as the solution to:

\[
x_s \in \arg \max_{\tilde{x}_s \in \mathbb{R}^{[G]}} u(\tilde{x}_s, \tilde{e}) \\
\text{subject to } \tilde{x}_s^{G_m} p^{G_m} \leq w^m_s \text{ for each } m.
\]

We denote the consumption choice at state \( s \) by the function \( x_s(w_s, p, \tilde{e}) \), and the consumption choice at all states by the function \( x(w, p, \tilde{e}) \). By Berge’s Maximum Theorem and the strict concavity of \( u(\cdot) \) in \( x \), the function \( x(\cdot) \) is continuous in its arguments. The incentive compatible effort choice is then a solution to the following problem:

\[
e \in \arg \max_{\tilde{e} \in E} U(x(w, p, \tilde{e}), \tilde{e}). \quad (5.2)
\]

Since \( E \) is finite, this problem always has a solution. The solution is represented by a correspondence \( e(w, p) \). We also define the indirect utility function

\[
V(c, p) = \max_{\tilde{e} \in E} U(x(w, p, \tilde{e}), \tilde{e}) \quad (5.3)
\]

for each \( p > 0 \) and \( c \in C^I(p) \). By Berge’s Maximum Theorem, the correspondence \( e(w, p) \) is upper hemicontinuous, and the indirect utility function is continuous in its arguments (here recall that \( c \in C^I(p) \) is simply a vector).

### 5.2.3 Worker’s Contract Choice

Each individual worker \( \nu \) faces a menu of incentive compatible contracts, one from each firm, \( \{c(\nu, j)\}_{j \in J} \), and she chooses the contract that maximizes her utility. The worker can also reject all contract offers. In this case, she generates an alternative production vector \( \tilde{y}_s \leq y_s \) in each \( s \) (since she will be less productive without the firm). Since, as we will see, in equilibrium the worker will never reject all contract offers, there is no loss of generality in assuming that \( \tilde{y}_s = y_s \), and we adopt this assumption to simplify the notation. So when the worker rejects all
contract offers, she solves

\[
\max_{(\hat{z}, \hat{e}) \in A} U(\hat{z}, \hat{e}),
\]
\[
\text{subject to } \hat{x}_s p \leq y_s p \text{ for each } s.
\]

Let \((x, e)\) be a solution to the preceding problem, and define the contract \(c(\nu, 0 \mid p) \equiv (\{w^m \equiv xG^m pG^m\}_{m \in M}, x, e)\) as the outside option of the worker. Rejecting every contract is equivalent for the worker to accepting the contract \(c(\nu, 0 \mid p)\).

Hence, a strategy for the worker \(\nu\) is a function \(J_\nu : C^I(p)^{[J]} \rightarrow J \cup \{0\}\) that specifies the index of the firm she chooses or 0 if she chooses her outside option:

\[
J_\nu \left[ \{c(\nu, j)\}_{j \in J} \right] \in \arg \max_{j \in J \cup \{0\}} V(c(\nu, j), p).
\] (5.5)

### 5.2.4 Firm’s Problem

We assume that firm \(j\) offers a continuum of incentive compatible contracts, one for each worker, taking the price vector \(p\), the contracts offered by other firms \(\{c(\nu, j)\}_{\nu \in N, j \in J \setminus \{j\}}\), and the worker strategies \((J_\nu)_{\nu \in N}\) as given.\(^{10}\) Formally, the contract offer of firm \(j\) is a Lebesgue measurable function \(c(\cdot, j) : [0, 1] \rightarrow C^I(p)\). Let \(\mathcal{L}\left(C^I(p)^{[0,1]}\right)\) denote the set of Lebesgue measurable functions from \([0, 1]\) to \(C^I(p)\), and let \(\pi(c, p)\) denote the profit of the firm from an accepted contract \(c = (w, x, e)\), given by:

\[
\pi((w, x, e), p) = \sum_{s \in S} q_s(e)(y_s p - \sum_{m \in M} w^m_s).
\]

Note that the firm’s profit from a contract that is not accepted is equal to 0. Hence, firm \(j\) solves the following problem:

\[
\max_{c(\cdot, j) \in \mathcal{L}(C^I(p)^{[0,1]})} \int_{\{\nu \in N \mid J_\nu(\{c(\nu, j)\}_{j \in J}) = j\}} \pi(c(\nu, j), p) \, d\nu.
\] (5.6)

\(^{10}\)To justify the assumption that firms take the price of goods as given, we could put an exogenous limit on the measure of contracts a firm can sign (a capacity constraint). If the capacity constraint is sufficiently small and the number of firms is sufficiently large, each firm will be too “small” to influence equilibrium prices while there will be sufficiently many firms to provide each worker with employment contracts. This extension is straightforward, but we present the model without capacity constraints to simplify the exposition.
5.2.5 Definition of Equilibrium

Let us refer to the economy described in the previous section with $E$. In this section, we define a competitive equilibrium for economy $E$ and show that such an equilibrium exists.

**Definition 9.** A competitive equilibrium in economy $E$ is a collection of contract offers $[c(\nu, j)]_{j \in J, \nu \in \mathcal{N}}$ by the firms, a collection of strategies for the workers $(J_\nu)_{\nu \in \mathcal{N}}$, a price vector $p$, and accepted contracts, $[w(\nu), x(\nu), e(\nu)]_{\nu \in \mathcal{N}}$, such that

1. Workers’ contract choice is optimal, i.e., for each $\nu \in \mathcal{N}$, the strategy $J_\nu$ satisfies (5.5).

2. Firms maximize expected profits, i.e., for each $j \in J$, the contract offer $c(\cdot, j)$ solves problem (5.6).

3. Goods markets clear, i.e., for each $g \in G$,

$$\int_{\mathcal{N}} \sum_{s \in S} q_s(e(\nu))(y_s^g - x_s^g(\nu))d\nu \geq 0, \text{ with equality if } p^g > 0. \quad (5.7)$$

5.2.6 Firm Competition and the Indirect Problem

To facilitate the characterization of equilibrium, we also impose the following assumptions.

**Assumption A2 (Probability function)** The probability function $q_s$ is strictly positive, that is, $q_s(e) > 0$ for each $s \in S$ and $e \in E$.

**Assumption A3 (Local Transferability)** There exists a monitoring subset $G_1$ and functions $u^{G_1}(\cdot)$ and $u^{G \setminus G_1}(\cdot)$ such that

$$u(x_s, e) = u^{G_1}(x_s^{G_1}) + u^{G \setminus G_1}(x_s^{G \setminus G_1}, e). \quad (5.8)$$

In addition, $u^{G_1}(\cdot)$ satisfies:

$$\lim_{\|x_s^{G_1}\| \to 0} \frac{\partial u^{G_1}(x_s^{G_1})}{\partial x_s^{G_1}} = \infty. \quad (5.9)$$
Assumption A2 is standard. Assumption A3 is less standard, but relatively weak. The first part of the assumption, for example, will hold if there is one good which is completely separable from effort choice, and consumption of which can be monitored by the firm. The second part of Assumption A3 is a relatively weak form of the standard Inada condition (in particular, it implies a minimum consumption requirement on the consumption vector, \( x_{s}^{G_1} \), as opposed to separate requirements on the consumption of each good, \( x_{s}^{g} \)). This requirement, along with Assumption A2, ensures that the worker’s expenditure on the goods in \( G_1 \) is strictly positive at each state \( s \), i.e., \( w_{s}^{1} > 0 \). The first part of Assumption A3 ensures that the firm can slightly increase (or decrease) the worker’s expenditure on the goods in \( G_1 \) while keeping the incentive compatible effort level the same. Consequently, the assumption allows for at least a small amount of utility transfer between the worker and the firm, while respecting the incentive compatibility constraints (see Lemma 8 in Appendix 5.A.2 for a formalization). Without a condition that allows for this type of utility transfer, Bertrand competition may not drive profits to zero (see Bennardo and Chiappori, 2003). Since this problem is already well understood and is orthogonal to our main concerns, Assumption A4 enables us to focus on the questions of interest for us.

The following proposition provides an equivalent characterization for the equilibrium. In particular, a collection of accepted contracts is part of an equilibrium if and only if all but a measure zero of them maximize the utility of the worker subject to the incentive compatibility constraint and a non-negative profit constraint for the firm. As with all the other proofs in this essay, the proof of this Proposition is in Appendix 5.A.2.

**Proposition 6. (The Indirect Problem)** Consider an economy \( \mathcal{E} \) that satisfies Assumptions A1-A3. Then, the prices and accepted contracts, \((p, c(\nu) = [w(\nu), x(\nu), e(\nu)]_{\nu \in \mathcal{N}})\), are part of an equilibrium if and only if:

1. The contract \( c(\nu) \) is a solution to

\[
\max_{\tilde{c} \in C^1(p)} V(\tilde{c}, p) \quad \text{(5.10)}
\]

subject to

\[
\pi(\tilde{c}, p) \geq 0, \quad \text{(5.11)}
\]

for all but a measure zero of \( \nu \in \mathcal{N} \).
2. The goods markets clear [cf. Eq. (5.7)].

Moreover, at the solution to problem (5.10), the constraint (5.11) binds, that is, each firm makes zero profits in equilibrium.

5.2.7 Existence of Equilibrium

The assumptions we made so far do not guarantee the existence of an equilibrium. If the solution set to problem (5.10), denoted by \( S(p) \), is not upper hemicontinuous in the price vector \( p \), then the equilibrium may not exist. In particular, the correspondence \( S(p) \) could fail to be upper hemicontinuous if the constraint set, \( C^I(p) \), is discontinuous. In this setting, the worker’s incentive compatible effort choice can be discontinuous in the price vector \( p \). Nevertheless, because the elements of \( C^I(p) \) are contracts, \( c = (w, x, e) \) (and not simply the effort choice, \( e \)), the correspondence \( C^I(p) \), which lies in the larger contract space \( C \), could be continuous even when effort choice is discontinuous in \( p \). The following assumption is sufficient to establish the continuity of the constraint set, \( C^I(p) \) (see Lemma 9 in Appendix 5.A.2).

**Assumption A4 (Effort Targeting)** For each \( e \in E \), there exists a vector of utility transfers \( t \in \mathbb{R}^{|S|} \) such that

\[
\sum_{s \in S} q_s(e) t_s > \sum_{s \in S} q_s(\hat{e}) t_s \quad \text{for each } \hat{e} \in E \setminus \{e\}. \tag{5.12}
\]

This assumption is natural and quite weak. It loosely corresponds to the requirement that firms should be able to induce (target) any level of effort if they wish to do so. It is also not difficult to satisfy. For example, it holds when \( |E| \leq |S| \) and the probability vectors, \( (q_s(e))_{s \in S} \) for each \( e \in E \), are linearly independent. In the commonly studied case of two effort levels, this assumption holds when there are at least two states and the efforts do not lead to identical success probabilities. Assumption A4 ensures the continuity of \( C^I(p) \), which in turn implies that the solution correspondence to problem (5.10), \( S(p) \), is upper hemicontinuous (by a version of Berge’s Maximum Theorem). The existence of equilibrium then follows from standard arguments.

Remark 2. (The Role of Assumptions A3 and A4) An alternative to imposing Assumptions A3 and A4 would be to assume continuous effort choice, e.g., $E = [0, 1]$, and to make sufficiently strong assumptions to ensure that the effort choice changes continuously in the price vector $p$. The following set of sufficient conditions are typically adopted in general equilibrium analyses of moral hazard economies with continuous effort (see, for example, Arnott and Stiglitz, 1991, 1993, or Lisboa, 2001): (1) The state utility function is fully separable between consumption and effort choice, i.e., $u(x, e) = v(x) - c(e)$ for some function $v(\cdot)$ and a cost function $c(\cdot)$. (2) There are only two individual states, i.e., $S = \{h, l\}$ (corresponding to high output and low output). (3) The probability function $q_h(e)$ is strictly increasing and strictly concave in $e$, and the cost function $c(e)$ is strictly increasing and convex in $e$. Under these conditions, it can be shown that the first-order approach is valid, which in turn implies that the worker's effort choice is continuous in $p$. However, these conditions are too restrictive for our purposes since they rule out nonseparable state utility functions (defined in Definition 11 below), which play a crucial role in our main efficiency results in Sections 5.3 and 5.4. Our Assumptions A3 and A4 are considerably weaker than the standard assumptions, and as such, they enable us to establish the existence of equilibrium for nonseparable as well as separable state utility functions. The analytical difficulties that emerge in the setting with nonseparable state utility functions have been emphasized by Arnott and Stiglitz (1988b, 1993). In this light, Theorem 15 could also be viewed as a methodological contribution to general equilibrium analyses of moral hazard economies.

It is also worth noting that allowing for stochastic contracts as in Prescott and Townsend (1984a) does not bypass the need to impose Assumptions A3 and A4 (see Section 5.5). Intuitively, allowing for stochastic contracts convexifies the incentive compatibility set, $C^f(p)$, which simplifies the analysis in some dimensions. However, convexifying a discontinuous correspondence does not necessarily make it continuous. Hence, the essential difficulty for establishing the existence of equilibrium remains in the setting with stochastic contracts. This difficulty does not arise in the Prescott and Townsend (1984a) economy, because contracts directly prescribe consumption levels for each good and thus endogenous prices do not affect the set of incentive
compatibility contracts.

5.3 Generic Inefficiency of Equilibrium

This section formalizes the folk theorem for imperfect information economies, by providing sufficient conditions under which the equilibrium is inefficient. We start by describing the notion of efficiency used in our analysis.

An allocation in our setting, \((x, e) \in A\), is effort-incentive compatible if the effort choice is optimal given the level of consumption prescribed, that is, if

\[ U(x, e) \geq U(x, \tilde{e}) \text{ for each } \tilde{e} \in E. \]

We denote the set of effort-incentive compatible allocations with \(A^f\). An economy-wide allocation \([x(v), e(v)]_{v \in \mathcal{N}}\) is effort-incentive compatible and feasible if \((x(v), e(v)) \in A^f\) for each \(v \in \mathcal{N}\), and the resource constraints hold, that is,

\[ \int_{\mathcal{N}} \sum_{s \in S} q_s(e(v))(y^g_s - x^g_s(v)) \, dv \geq 0, \text{ for each } g. \quad (5.1) \]

**Definition 10.** An economy-wide allocation \([x(v), e(v)]_{v \in \mathcal{N}}\) is constrained (Pareto) optimal if it is effort-incentive compatible and feasible, and there does not exist another effort-incentive compatible and feasible economy-wide allocation \([\hat{x}(v), \hat{e}(v)]_{v \in \mathcal{N}}\) such that \(U(\hat{x}(v), \hat{e}(v)) \geq U(x(v), e(v))\) for all \(v \in \mathcal{N}\), with strict inequality for a positive measure of \(v \in \mathcal{N}\).

Consider an equilibrium of the economy \(\mathcal{E}\) with price vector and accepted contracts, \((p, [w(v), x(v), e(v)]_{v \in \mathcal{N}})\). We say that the equilibrium is constrained optimal if the economy-wide allocation \([x(v), e(v)]_{v \in \mathcal{N}}\) is constrained optimal.

**Remark 3.** (Full Monitoring by the Social Planner) Our notion of efficiency provides the social planner with the same informational constraints as the firms but with better contracting (monitoring) technology. In particular, the planner cannot observe the effort choice of the worker, but can specify the consumption of all goods in the employment contract. This notion of optimality is arguably strong. Nevertheless, it is natural for us for two reasons. First,
this notion helps us to develop our main point more succinctly. For example, our main result
(Theorem 17) delineates a range of benchmark situations in which the equilibrium is efficient in
this strong sense. Second, the social planner can approximate a constrained optimal outcome
according to our definition using more limited instruments, i.e., a tax and transfer system.
For example, the government can reduce the consumption of a particular good by levying a
linear tax on that good (though a tax and transfer system is not equivalent to full monitoring:
Appendix 5.A.1 considers a weaker notion of optimality and provides an example economy that
is constrained suboptimal according to the strong optimality notion, but not according to this
weaker optimality notion).

The main result in this section establishes sufficient conditions under which the equilibrium
is constrained suboptimal. These conditions are related to the notions of nonseparability and
no full insurance, which we define next.\textsuperscript{11}

\textbf{Definition 11.} Consider an allocation \((x, e)\). The state utility function is \textit{nonseparable} (not
\textit{weakly separable}) at \((x, e)\) if there exists a state \(s\), a monitoring subset \(G_m\), and two goods \(g_1, g_2 \in G_m\) such that the marginal rate of substitution between \(g_1\) and \(g_2\) at state \(s\) changes when
\(\text{effort level is modified, that is}
\[
\frac{\partial u(x, e)}{\partial x^{g_1}} \neq \frac{\partial u(x, \hat{e})}{\partial x^{g_1}} \quad \text{for any } \hat{e} \in E \setminus \{e\}.
\]

There is \textit{no full insurance} at \((x, e)\), if there exists a good \(g \in G\) and two states \(s_1, s_2 \in S\)
such that the marginal rate of substitution for good \(g\) between states \(s_1, s_2\) is not equal to 1, that
\[
\frac{\partial u(x, e)}{\partial x^{g_1}} \neq 1.
\]

The next theorem is our main inefficiency result. It shows that the equilibrium is constrained
suboptimal whenever the state utility function is nonseparable and there is no full insurance at
the equilibrium allocation for a positive measure of workers.

\textsuperscript{11}As Definition 12 below makes it clear, "nonseparable" preferences are the converse of "weakly separable"
preferences. We use the terminology "nonseparable" since it's easier to use than "not weakly separable" or
"weakly nonseparable".
Theorem 16. (Nonseparability and Inefficiency) Consider an economy $E$ that satisfies Assumptions A1-A4. Let $[p, (w(v), x(v), e(v))_{v \in \mathcal{V}}]$ denote the prices and accepted contracts in an equilibrium. Suppose that there is a positive measure set $\mathcal{N}^* \subseteq \mathcal{N}$ such that for each $v \in \mathcal{N}^*$, the state utility function is nonseparable and there is no full insurance at the equilibrium allocation $(x(v), e(v))$. Then, the equilibrium is not constrained optimal.

The intuition for the result is closely related to double deviations by the worker, that is, deviations in which a worker switches to a different effort level and reoptimizes her consumption of nonmonitored goods for the new effort level. When the preferences are nonseparable and there is no full insurance at the equilibrium allocation, double deviations bind in the incentive compatibility constraints. That is, they prevent firms from providing more insurance to the workers. A social planner who can also prescribe the consumption of nonmonitored goods is not constrained by double deviations, and can therefore provide better insurance without violating the incentive compatibility constraints.

It is also worth noting that Theorem 16 is a generic inefficiency result, since nonseparable utility functions are open and dense (or weakly separable utility functions, as defined in Definition 12, are nowhere dense) in the set of all continuous utility function (with the sup norm). Put differently, if $u$ is weakly separable and $\tilde{u}$ is nonseparable, then $\varepsilon u + (1 - \varepsilon) \tilde{u}$ is nonseparable for any $\varepsilon \in (0, 1)$.

The next example illustrates the intuition of Theorem 16.

Example 4. Suppose that there are two states, $S = \{h, l\}$, corresponding to high output and low output, and two goods, $G = \{1, 2\}$. The monitoring partition is given by $G = \{G\}$, which implies that the firm can only specify wages and otherwise cannot monitor worker’s consumption. For simplicity, consider a partial equilibrium setting in which the relative price of the goods is fixed and is equal to 1, i.e., suppose $p^1 = p^2 = 1$. Suppose there are two effort levels, i.e., $E = \{0, 1\}$, which respectively correspond to shirking and working. Assume the firm

---

12 The partial equilibrium setting is a special case of our model in which there is a linear production technology (that operates without moral hazard considerations) which can convert the two goods to each other. To simplify the notation, we present our main results in an environment without this type of production technology. Appendix B, which is available on request, shows that all of our results continue to hold when we introduce a general (potentially nonlinear) production technology that is not subject to moral hazard. Also, again to simplify the exposition, in this example we use functional forms that do not satisfy Assumptions A2 and A3.
makes zero profits in the low output state and positive profits in the high output state, that is, \( \pi_l = y_l p = 0 \), and \( \pi_h = y_h p > 0 \). Assume \( q_h (e = 1) = 1/2 \) and \( q_h (e = 0) = 0 \). Assume also that good 2 is relatively more complementary to leisure than good 1. More specifically, the worker’s state utility function is given by:

\[
\begin{align*}
    u (x_s, e) &= \ln \left( x_s^1 + \left( 1 + (1 - e) x_s^2 \right) \right) . 
\end{align*}
\]

Note that the worker enjoys good 2 relatively more (in comparison to good 1) when she does not work, and relatively less when she works. Consider a social planner that chooses the allocation \( (\hat{x}_h^1, \hat{x}_h^2, \hat{x}_l^1, \hat{x}_l^2, \hat{e}) \), subject to effort-incentive compatibility and resource constraints. It can be seen that the social planner implements \( \hat{e} = 1 \) and provides the worker with the consumption of only good 1, i.e., she chooses \( \hat{x}_h^2 = \hat{x}_l^2 = 0 \). Moreover, given that the worker does not consume good 2, the state utility function in (5.2) implies that there is no incentive problem. Thus, the social planner provides the worker with full insurance. That is, the worker’s consumption of the first good is given by:

\[
\hat{x}_h^1 = \frac{\pi_h}{2} ,
\]

and the worker’s utility is given by

\[
U (\hat{x}, \hat{e}) = \frac{1}{2} \ln \left( \frac{\pi_h}{2} \right) + \frac{1}{2} \ln \left( \frac{\pi_h}{2} \right) = \ln \left( \frac{\pi_h}{2} \right) .
\]

Next consider a firm that offers a wage contract \( (w_h, w_l) \). It can be seen that the social planner’s full insurance solution is not incentive compatible. That is, given the wages just enough to consume the bundle (5.3), the worker would instead not work and consume a different bundle. Since \( \ln \left( \frac{\pi_h}{2} \right) < \ln \left( \frac{3 \pi_h}{2} \right) \), the worker can increase her utility with a double deviation in which she changes her effort choice and reoptimizes her consumption for the new effort decision. The firm will instead offer the wage contract \( (w_h, w_l) \) that is the solution to the following equations:

\[
\frac{\hat{w}_h + \hat{w}_l}{2} = \frac{\pi_h}{2} \quad \text{(Budget constraint),}
\]

\[
\frac{1}{2} \ln (\hat{w}_h) + \frac{1}{2} \ln (\hat{w}_l) \geq \ln \left( \frac{3}{2} \frac{\pi_h}{2} \right) \quad \text{(Incentive compatibility)}.
\]
The equilibrium wages and consumption are given by:

\[ w_h = \frac{9}{13} \pi_h, \quad x_h^1 = w_h, \quad x_h^2 = 0, \text{and } w_l = \frac{4}{13} \pi_h, x_l^1 = w_l, \quad x_l^2 = 0. \] (5.5)

The equilibrium utility is given by:

\[ U(x, e) = \frac{1}{2} \ln \left( \frac{9}{13} \pi_h \right) + \frac{1}{2} \ln \left( \frac{4}{13} \pi_h \right) = \ln \left( \frac{6}{13} \pi_h \right). \] (5.6)

Comparing Eqs. (5.5)-(5.6) with (5.3)-(5.4), note that the firm is only partially insuring the worker, and the contract offered by the firm is strictly worse for the worker than the allocation offered by the social planner.

Theorem 16 establishes the inefficiency of equilibrium under conditions on the equilibrium allocation. A natural question is whether there are any economies for which these conditions hold. To address this question, Theorem 22 in Appendix 5.A.1 characterizes a class of economies in which any equilibrium is constrained optimal. The result essentially provides conditions on the preferences and the technology such that Theorem 16 applies at any equilibrium. To ensure that the equilibrium allocation of a worker always features less than full insurance, we assume that there exists a shirking effort level which is always preferred by the worker under full insurance, and which yields the firm (almost) zero profits. To ensure that every equilibrium allocation satisfies the nonseparability property, we assume that there exists a monitoring subset \( G_m \) and two goods \( g_1, g_2 \in G_m \) such that the marginal rate of substitution between \( g_1 \) and \( g_2 \) always change monotonically in the effort level.

Theorems 16 and 22 provide some support for the folk theorem for the inefficiency of the equilibrium. Recall, however, that these theorems rely on a strong notion of optimality which essentially provides the social planner with a better monitoring technology than the firms (cf. Remark 3). Theorems 16 and 22 are silent on whether a social planner who is also constrained by the same monitoring technology can implement such a Pareto improvement. To address this issue, Appendix 5.A.1 introduces a weaker notion of optimality which constrains the social planner with the same monitoring technology as the firms. This appendix also provides an example economy with nonseparable utility, which is constrained suboptimal as implied by Theorem 22, but is weakly constrained optimal. The example shows that the inefficiency of
equilibrium established in Theorems 16 and 22 in part stems from the strong notion of optimality which gives the social planner a technological advantage in monitoring. This suggests that care must be taken in invoking these theorems.

5.4 Efficiency of Equilibrium under Weak Separability

We next provide our main result, which is the converse of Theorem 16. In particular, the result shows that the equilibrium is constrained optimal when worker preferences are weakly separable. The next definition formalizes the notion of weak separability.

**Definition 12.** The state utility function \( u(\cdot) \) is **weakly separable** if, for any monitoring subset \( G_m \) and two goods \( g_1, g_2 \in G_m \), the marginal rate of substitution between \( g_1 \) and \( g_2 \) is independent of effort level. That is,

\[
\frac{\partial u(x_s, e)}{\partial x_s^{g_1}} / \frac{\partial x_s^{g_1}}{\partial x_s^{g_2}} = \frac{\partial u(x_s, e)}{\partial x_s^{g_2}} / \frac{\partial x_s^{g_2}}{\partial x_s^{g_1}} \quad \text{for each } g_1, g_2 \in G_m, \ x_s \in \mathbb{R}_+^{|G|} \text{ and } e, e' \in E. \tag{5.1}
\]

Note that the state utility function is weakly separable if and only if it is not nonseparable at any allocation \( (x, e) \in A \). Our next result shows that weak separability is sufficient for the competitive equilibrium to be constrained Pareto optimal.

**Theorem 17.** (Efficiency under Weak Separability) Consider an economy \( \mathcal{E} \) that satisfies Assumptions A1-A4. Assume also that the state utility function \( u(\cdot) \) is weakly separable. Then, any equilibrium of the economy \( \mathcal{E} \) is constrained optimal.

The intuition of this result is that, under weak separability, the social planner chooses for the worker the consumption bundle which the worker would have chosen by herself in the anonymous trading market. Hence, there is no benefit to additional monitoring, and competition among firms leads to the allocations that the social planner would have chosen. A complementary intuition is that double deviations in which the worker changes effort level and reoptimizes her consumption bundle accordingly are not valuable. This further implies that the relative price changes caused by other contracts in the economy does not change the insurance-incentive trade-off for a worker, rendering pecuniary externalities ineffective. We demonstrate this theorem with a simple example in which preferences are fully separable.
Example 5. Consider the same setup as in Example 4 with two differences. First, to emphasize that price externalities do not create inefficiencies in this setting, we consider more general relative prices than in Example 4, that is, we consider the price vector \((p^1 = 1, p^2 \in \mathbb{R}_+)\) (we continue to assume that the prices are fixed). Second, we assume that the worker’s state utility function is given by

\[ u(x_s, e) = u^{sep}(v(x^1_s, x^2_s), e), \]

where \(u^{sep}(v, e)\) is a scalar valued function that is strictly increasing and strictly concave in \(v\) and decreasing in \(e\), and \(v(x^1_s, x^2_s)\) is a scalar valued function that is strictly increasing and strictly concave in both arguments (a particular instance is \(u^{sep}(v, e) = \ln(v) - e\) and \(v(x^1_s, x^2_s) = \left(\left(x^1_s\right)^{\frac{\varepsilon-1}{\varepsilon}} + \left(x^2_s\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}\) for some \(\varepsilon > 0\)). Here, the inner function \(v(x^1_s, x^2_s)\) can be thought of as the consumption level of a particular “need,” which is satisfied by consuming goods 1 and 2. The outer function \(u^{sep}(v, e)\) gives the worker’s utility corresponding the level of the need, \(v\), and the effort choice, \(e\). Note that the state utility function, \(u(\cdot)\), is weakly separable, since

\[ \frac{\partial u(x_s, e)}{\partial x^1_s} = \frac{\partial v(x^1_s, x^2_s)}{\partial x^1_s}, \]

\[ \frac{\partial u(x_s, e)}{\partial x^2_s} = \frac{\partial v(x^1_s, x^2_s)}{\partial x^2_s}, \]

which is independent of the effort choice.

Consider first a planner who determines an effort-incentive compatible allocation \(((x^1_h, x^1_l, x^2_h, x^2_l), e)\) for each worker. Suppose, for simplicity, that the planner offers every worker the same employment allocation (the result in Theorem 17 is more general and does not rely on this assumption). The planner maximizes the representative worker’s utility subject to effort-incentive compatibility and resource constraints, that is:

\[
\max_{(\bar{x}, \bar{e}) \in \mathcal{A}} q_h(\bar{e}) u^{sep}(v(x^1_h, x^2_h), \bar{e}) + (1 - q_h(\bar{e})) u^{sep}(v(x^1_l, x^2_l), \bar{e})
\]

subject to

\[
\sum_{s \in S} q_s(\bar{e})(\bar{x}^1_s + p^2 \bar{x}^2_s) = \sum_{s \in S} q_s(\bar{e}) \pi_s,
\]

\[ U(\bar{x}, \bar{e}) \geq U(\bar{x}, \bar{e}) \text{ for all } \bar{e} \in E. \]

To solve this problem, the planner computes the value from implementing any \(\bar{e} \in E\). The planner then implements the effort level that yields the highest utility to the worker. Similar to the analysis in Grossman and Hart (1983), the social planner’s problem is simplified in view
of the weak separability of the state utility function. In particular, the planner's problem is equivalent to first deciding how much of the need to provide in each state, \( \bar{v}_s \), and then deciding the optimal consumption bundle that provides this level of the need. Hence, the optimal utility from implementing \( \bar{e} \in E \) is given by

\[
\max_{\bar{v}_h \in \mathbb{R}, \bar{v}_i \in \mathbb{R}} \sum_{s \in S} q_s (\bar{e}) \left( \bar{x}_s^1 + p^2 \bar{x}_s^2 \right) \geq \sum_{s \in S} q_s (\bar{e}) u^{sep} (\bar{v}_h, \bar{e}) \text{ for all } \bar{e} \in E,
\]

and two additional constraints \( v (\bar{x}_h^1, \bar{x}_h^2) = \bar{v}_h \) and \( v (\bar{x}_l^1, \bar{x}_l^2) = \bar{v}_l \).

Note that, given \( \bar{v}_s \), the planner would like to minimize the cost of providing this level of the need. That is, for each \( s \in \{ h, l \} \), the vector \( x_s \) is the solution to:

\[
\min_{\bar{x}_s \in \mathbb{R}} \bar{x}_s^1 + p^2 \bar{x}_s^2 \text{ subject to } v (\bar{x}_s^1, \bar{x}_s^2) = \bar{v}_s.
\]

This is a strictly convex minimization problem that provides a one-to-one relationship between \( w_{pl} (\cdot) \) and the optimum choice of the consumption vector \( x_s \). By duality, the optimum vector \( x_s \) also maximizes \( v (x_s^1, x_s^2) \) subject to the expenditure being not greater than \( w_{pl} (\bar{v}_s) \). Using these observations, problem (5.2) can be rewritten as

\[
\max_{w_{pl}(\bar{v}_h) \in \mathbb{R}^+, w_{pl}(\bar{v}_l) \in \mathbb{R}} \sum_{s \in S} q_s (\bar{e}) \left( \max_{\bar{x}_s \in \mathbb{R}^+ \mid \bar{x}_s^1 + p^2 \bar{x}_s^2 \leq w_{pl}(\bar{v}_s)} u^{sep} (v (\bar{x}_s^1, \bar{x}_s^2), \bar{e}) \right)
\]

subject to

\[
\sum_{s \in S} q_s (\bar{e}) w_{pl} (\bar{v}_s) = \sum_{s \in S} q_s (\bar{e}) \pi_s
\]

and

\[
\sum_{s \in S} q_s (\bar{e}) \left( \max_{\bar{x}_s \in \mathbb{R}^+ \mid \bar{x}_s^1 + p^2 \bar{x}_s^2 \leq w_{pl}(\bar{v}_s)} u^{sep} (v (\bar{x}_s^1, \bar{x}_s^2), \bar{e}) \right) \geq \sum_{s \in S} q_s (\bar{e}) \left( \max_{\bar{x}_s \in \mathbb{R}^+ \mid \bar{x}_s^1 + p^2 \bar{x}_s^2 \leq w_{pl}(\bar{v}_s)} u^{sep} (v (\bar{x}_s^1, \bar{x}_s^2), \bar{e}) \right) \text{ for each } \bar{e} \in E.
\]

Next consider the problem of a firm offering a wage contract \( \{ w_g, w_b \} \). To implement effort
level \( \tilde{e} \in E \), the firm will solve

\[
\max_{w_s \in \mathbb{R}^+, w_t \in \mathbb{R}^+} \sum_{s \in S} q_s(\tilde{e}) \left( \max_{\tilde{z}_s \in \mathbb{R}_{+}^{\left| G_s \right|}} \left( \frac{\max_{\tilde{y}^1 + p^2 \tilde{y}^2 \leq w_s} u^{sep}(v(\tilde{x}_s^1, \tilde{x}_s^2), \tilde{e})}{\sum_{s \in S} q_s(\tilde{e}) w_s} \right) \right)
\]

subject to

\[
\sum_{s \in S} q_s(\tilde{e}) w_s = \sum_{s \in S} q_s(\tilde{e}) \pi_s \]

\[
\sum_{s \in S} q_s(\tilde{e}) \left( \max_{\tilde{z}_s \in \mathbb{R}_{+}^{\left| G_s \right|}} \left( \frac{\max_{\tilde{y}^1 + p^2 \tilde{y}^2 \leq w_s} u^{sep}(v(\tilde{x}_s^1, \tilde{x}_s^2), \tilde{e})}{\sum_{s \in S} q_s(\tilde{e}) w_s} \right) \right) \geq \sum_{s \in S} q_s(\tilde{e}) \left( \max_{\tilde{z}_s \in \mathbb{R}_{+}^{\left| G_s \right|}} \left( \frac{\max_{\tilde{y}^1 + p^2 \tilde{y}^2 \leq w_s} u^{sep}(v(\tilde{x}_s^1, \tilde{x}_s^2), \tilde{e})}{\sum_{s \in S} q_s(\tilde{e}) w_s} \right) \right)
\]

for each \( \tilde{e} \in E \).

A comparison of problems (5.3) and (5.4) shows that they are equivalent. Consequently, the social planner and the firm will choose to implement the same effort level, \( e \), and the worker will receive the same utility in each case. Hence, in this example, monitoring is not valuable, and a firm that can only offer a wage contract provides incentives as well as a social planner who can monitor worker’s consumption.

The analysis in this example also suggests a general class of preferences that satisfies the weak separability condition. Suppose that the state utility function can be written as

\[
u(x_s, e) = u^{sep}(v^1(x_s^{G_1}), v^2(x_s^{G_2}), ..., v^{\left| G \right|}(x_s^{G_{\left| G \right|}}), e),
\]

where \( u^{sep}(v^1, v^2, ..., v^{\left| G \right|}, e) \) is a scalar valued function which is strictly increasing and strictly concave in its first \( \left| G \right| \) arguments, and for each \( m \), \( v^m : \mathbb{R}_{+}^{\left| G \right|} \rightarrow \mathbb{R} \) is a scalar valued function which is increasing and concave in its arguments. This functional form is intuitive, as it implies that the worker receives utility over a number of higher order needs (such as food, entertainment, procrastination, vacation, health care, education etc.) each of which is provided by the consumption of a distinct set of goods. Clearly, the state utility function in (5.5) is weakly separable. Thus, Theorem 17 implies that equilibria of economies in which preferences can be represented as in (5.8) are constrained optimal. This analysis suggests that, if firms can monitor the consumption of broad aggregates that interfere with effort choice, then the equilibrium will be efficient even though firms are unable to monitor workers’ exact consumption choices.
5.5 Equilibrium and Efficiency with Stochastic Contracts

This section extends our setup to allow for contracts with random outcomes, briefly “stochastic contracts.” Allowing for randomization can be beneficial in this economy because of the non-convexity of problem (5.2) in Section 5.2 (e.g., Prescott and Townsend, 1984a, 1984b, Arnott and Stiglitz, 1988a). Our analysis in this section shows that allowing for stochastic contracts does not change the efficiency properties of this economy. In particular, analogues of our main results, Theorems 16 and 17, apply for the economy with stochastic contracts.

Given a finite dimensional space \( Z \), we let \( \mathcal{B}(Z) \) denote the Borel \( \sigma \)-algebra of \( Z \) and \( \mathcal{P}(Z) \) denote the set of probability measures over \((Z, \mathcal{B}(Z))\). We endow \( \mathcal{P}(Z) \) with the weak* topology. A *stochastic allocation* in our setting is a probability measure \( \eta \) over the allocation space \( \left( A = \mathbb{R}_+^{[S] \times [G]} \times E, \mathcal{B}(A) \right) \). The worker’s utility from a stochastic allocation \( \eta \) is denoted by \( U^R(\eta) \), and it is equal to \( \int_{(x,e) \in A} U(x,e) \, d\eta \). Similarly, a *stochastic contract* is a probability measure, \( \mu \in \mathcal{P}(C) \), over the contract space \( \left( C = \mathbb{R}_+^{[S] \times [M]} \times A, \mathcal{B}(C) \right) \). Given a stochastic contract \( \mu \), we let \( \mu|_{(x,e)} \in \mathcal{P}(A) \) denote the marginal measure over the allocation space \( A \).

Depending on when the contract uncertainty is resolved, a stochastic contract \( \mu \) can be interpreted as having either ex-ante or ex-post randomization. With ex-post randomization, the contract uncertainty is resolved after the effort decision is made, that is, the worker chooses an effort level and then learns which contract she will receive. With ex-ante randomization, the contract uncertainty is resolved before the effort decision is made, i.e., the worker learns her contract before she chooses an effort level (see, Arnott and Stiglitz, 1988a, for further discussion and the role of each type of randomization). For analytical and notational convenience, we analyze the case with only ex-ante randomization. All of the results in this section generalize to the case with both ex-ante and ex-post randomization.\(^{13}\)

Given the assumption of only ex-ante randomization, a stochastic contract \( \mu \) is *incentive compatible* if and only if its support lies in the set of deterministic incentive compatible contracts, that is, \( \text{supp}(\mu) \subset C^I(p) \). Equivalently, an incentive compatible stochastic contract is a probability measure \( \mu \in \mathcal{P}(C^I(p)) \). The expected utility of the worker and the expected

\(^{13}\)It is well known that ex-post randomization in this setting could be useful to provide the worker with additional incentives (see Arnott and Stiglitz, 1988a). However, as noted by Bennardo and Chiappori (2003), this additional incentive provision does not interfere with the existence or efficiency properties of equilibrium.
profits of the firm are respectively given by \( \int_{C^I(p)} V(c, p) \, d\mu \) and \( \int_{C^I(p)} \pi(c, p) \, d\mu \). As in the deterministic case, each worker \( \nu \) faces a menu of incentive compatible contracts \( \{\mu_{(\nu,j)}\}_{j \in J} \), and chooses the contract that maximizes her utility. Each firm \( j \) offers a continuum of contracts to maximize its expected profit, taking the workers' strategies and other firms' contract offers as given. We denote the economy with stochastic contracts with \( E^R \), and we define the equilibrium for this economy as follows.

**Definition 13.** A **competitive equilibrium** in economy \( E^R \) is a collection of incentive compatible contract offers \( \{\mu_{(\nu,j)}\}_{j \in J, \nu \in \mathcal{N}} \) by the firms, a collection of strategies for workers, \( \{J^R_{\nu}\}_{\nu \in \mathcal{N}} \), a price vector \( p \), allocations \( \{x_{\nu}\}_{\nu \in \mathcal{N}} \), such that: workers' contract choice is optimal, firms maximize expected profits, and goods markets clear, that is, for each \( g \):

\[
\int_{\mathcal{N}} \int_{(w, x, e) \in C^I(p)} \sum_{s \in S} q_s(e)(y_{\nu}^g - x_{\nu}^g) \, d\mu_{\nu} \, dv \geq 0, \text{ with equality for } p^g > 0. \tag{5.1}
\]

The following result is the counterpart of Proposition 6 and Theorem 15 with stochastic contracts.

**Theorem 18.** *(Existence of Equilibrium with Stochastic Contracts)* Consider an economy \( E^R \) that satisfies Assumptions A1-A4. There exists a competitive equilibrium for the economy \( E^R \). The prices and accepted contracts, \( (p, \{\mu_{\nu}\}_{\nu \in \mathcal{N}}) \), are part of an equilibrium if and only if:

1. For all but a measure zero of workers \( \nu \in \mathcal{N} \), the contract \( \mu_{\nu} \) is a solution to

\[
\max_{\tilde{\mu} \in \mathcal{P}(C^I(p))} \int_{C^I(p)} V(c, p) \, d\tilde{\mu} \tag{5.2}
\]

subject to

\[
\int_{C^I(p)} \pi(c, p) \, d\tilde{\mu} \geq 0. \tag{5.3}
\]

2. The goods markets clear [cf. Eq. (5.1)].

Moreover, at the solution to Problem (5.2), the profit constraint (5.3) binds, that is, every firm makes zero profits in equilibrium.

Analysis of the efficiency of the equilibrium closely parallels the case with deterministic
contracts. First, we provide the analogous definition of constrained optimality with stochastic contracts.

Given the assumption of only ex-ante randomization, a stochastic allocation $\eta$ is effort-incentive compatible if and only if the support of $\eta$ lies in the set of deterministic effort-incentive compatible allocations, that is, $\eta \in \mathcal{P}(A')$. An economy-wide stochastic allocation $[\eta_\nu]_{\nu \in \mathcal{N}}$ is effort-incentive compatible and feasible if and only if $\eta_\nu \in \mathcal{P}(A')$ for each $\nu$ and the resource constraints hold:

$$\int_\mathcal{N} \int_{(x,e) \in A'} \sum_{s \in S} q_s(e) (y_s^g - x_s^g) d\eta_\nu d\nu \geq 0 \text{ for each } g. \tag{5.4}$$

**Definition 14.** An economy-wide stochastic allocation $[\eta_\nu]_{\nu \in \mathcal{N}}$ is **constrained optimal** if it is effort-incentive compatible and feasible, and there does not exist another effort-incentive compatible and feasible economy-wide allocation $[\tilde{\eta}_\nu]_{\nu \in \mathcal{N}}$ such that $U^R(\tilde{\eta}_\nu) \geq U^R(\eta_\nu)$ for all $\nu \in \mathcal{N}$ with strict inequality for a positive measure of $\nu$.

Consider an equilibrium of the economy $\mathcal{E}^R$ with the prices and accepted contracts, $(p, [\mu_\nu]_{\nu \in \mathcal{N}})$. We say that the equilibrium is constrained optimal if the economy-wide allocation $[\mu_\nu(\cdot,e)]_{\nu \in \mathcal{N}}$ is constrained optimal.

Our next result is the analogue of Theorem 16 for stochastic contracts, and establishes sufficient conditions under which the equilibrium is not constrained optimal. To state the result, we generalize the notions of nonseparability and no full insurance to stochastic allocations.

**Definition 15.** Consider a stochastic allocation $\eta$ and a compact set $A^* \subset A$ such that $\eta(A^*) > 0$. The state utility function $u(\cdot)$ is **nonseparable** at $(\eta, A^*)$, if there exists a state $s$, a monitoring subset $G_m$, and two goods $g_1, g_2 \in G_m$ such that the marginal rate of substitution between $g_1$ and $g_2$ at state $s$ changes when the effort level is modified, that is,

$$\frac{\partial u(x_s,e)}{\partial x_s^{g_1}} / \frac{\partial u(x_s,e)}{\partial x_s^{g_2}} \neq \frac{\partial u(x_s,\hat{e})}{\partial x_s^{g_1}} / \frac{\partial u(x_s,\hat{e})}{\partial x_s^{g_2}} \text{ for each } (x,e) \in A^* \text{ and } \hat{e} \in E \setminus \{e\}.$$  

There is no full insurance at $(\eta, A^*)$ if there exists a good $g \in G$ and two states $s_1, s_2 \in S$ such that

$$\frac{\partial u(x_{s_1},e)}{\partial x_{s_1}^g} \neq \frac{1}{\partial u(x_{s_2},e)} / \frac{\partial x_{s_2}^g} {\partial x_{s_2}^g} \text{ for each } (x,e) \subset A^*.$$
The following is our main inefficiency result for the economy with stochastic contracts.

**Theorem 19. (Inefficiency under Stochastic Contracts)** Consider an economy $\mathcal{E}^R$ that satisfies Assumptions A1-A4. Let $(p, \mu_{\nu})_{\nu \in \mathcal{N}}$ denote the prices and accepted contracts in an equilibrium. Suppose that there is a positive measure set $N^* \subset \mathcal{N}$ such that for each $\nu \in N^*$, there exists a compact set $A^* \subset A$ with $\mu_{\nu}|_{\{x, e\}}(A^*) > 0$ such that the state utility function is nonseparable and there is no full insurance at $(\mu_{\nu}|_{\{x, e\}}, A^*)$. Then, the equilibrium is **not constrained optimal**.

We next provide the analogue of Theorem 17 for stochastic contracts. The following result shows that, when the utility function is weakly separable, any equilibrium with stochastic contracts is constrained optimal.

**Theorem 20. (Efficiency under Stochastic Contracts)** Consider economy $\mathcal{E}^R$ that satisfies Assumptions A1-A4. Assume also that the state utility function $u(\cdot)$ is weakly separable. Then, any equilibrium of the economy $\mathcal{E}^R$ is **constrained optimal**.

### 5.6 Approximate Efficiency of Equilibrium

As noted in Section 5.3, weak separability is a nongeneric property (in the sense that any weakly separable utility function will become nonseparable after a small perturbation). However, there is also a sense in which such genericity results may not be relevant for understanding the economic importance of certain types of inefficiencies. In particular, such genericity results do not preclude the possibility that in most economically relevant situations equilibria might be "approximately" efficient. In the present context, weakly separable utility functions might be a fairly good approximation to most economic situations (including almost all cases considered in applied work). Thus, what might be relevant is whether there are significant inefficiencies when there are only small deviations from such weak separability. In this section, we investigate this question. We introduce a notion of approximate efficiency ($\varepsilon$-constrained optimality) and show that when the economy is "close to" an alternative economy with weakly separable utility functions, its equilibrium will be approximately constrained optimal.

**Definition 16.** For each $\varepsilon \geq 0$, an economy-wide allocation $(\eta_{\nu})_{\nu \in \mathcal{N}}$ is **\(\varepsilon\)-constrained optimal** if it is effort-incentive compatible and feasible, and there does not exist another effort-incentive
compatible and feasible economy-wide allocation \([\tilde{\eta}_v]_{v \in \mathcal{N}}\) such that \(U^R(\tilde{\eta}_v) \geq U^R(\eta_v) + \varepsilon\) for all \(v \in \mathcal{N}\), with strict inequality for a positive measure of \(v \in \mathcal{N}\).

Consider an equilibrium of the economy \(E^R\) with the prices and accepted contracts, \((p, [\mu_v]_{v \in \mathcal{N}})\). We say that the equilibrium is \(\varepsilon\)-constrained optimal if the economy-wide allocation \([\mu_v(x,e)]_{v \in \mathcal{N}}\) is \(\varepsilon\)-constrained optimal.

The notion of \(\varepsilon\)-constrained optimality is a generalization of the notion of constrained optimality (cf. Definition 10) since the two notions are equivalent for \(\varepsilon = 0\). We next introduce the notion of a perturbation of the state utility function. Recall that, by Assumption A3, the state utility function has the representation

\[
 u(x_s,e) = u^{G_1}(x^{G_1}_s) + u^{G\setminus G_1}(x^{G\setminus G_1}_s, e),
\]

for some functions \(u^{G_1}(\cdot)\) and \(u^{G\setminus G_1}(\cdot)\). For simplicity, we keep the transferable component, \(u^{G_1}(\cdot)\), unchanged and we focus on perturbations of \(u^{G\setminus G_1}(\cdot)\) (this is without loss of any generality). Let \(u^{G_1}(\cdot)\) denote a continuously differentiable, strictly increasing, strictly concave and bounded function, and consider the set of state utility functions:

\[
 \mathcal{U} = \left\{ \begin{pmatrix} u^{G_1} \\ u^{G\setminus G_1} : \mathbb{R}^+\times E \rightarrow \mathbb{R} \end{pmatrix} \middle| \text{u satisfies Assumption A1 and is bounded.} \right\}.
\]

Note that any state utility function \(u \in \mathcal{U}\) satisfies Assumptions A1 and A3. We endow the set \(\mathcal{U}\) with the sup norm, which makes \(\mathcal{U}\) into a normed vector space, and thus a metric space.\(^{14}\)

For a state utility function \(\bar{u} \in \mathcal{U}\), we let \(B(\bar{u}, \delta) = \{u \in \mathcal{U} \mid \|u - \bar{u}\| \leq \delta\}\) denote the \(\delta\) neighborhood of \(\bar{u}\). When \(\delta > 0\), the functions in \(B(\bar{u}, \delta)\) can be thought of as small perturbations of the function \(\bar{u}\). If \(\bar{u}\) is weakly separable, then the state utility functions in \(B(\bar{u}, \delta)\) are close to being weakly separable. Our next result shows that equilibria of economies with utility functions in \(B(\bar{u}, \delta)\) are approximately efficient (\(\varepsilon\)-constrained optimal).

\(^{14}\)The assumption that \(u\) is bounded is without loss of generality, because each worker’s production of each good \(g\) is bounded above by \(\max_{s \in S} y^g_s\), which implies that the utility function can be restricted to a bounded consumption set. More specifically, for any unbounded state utility function, \(\tilde{u}\), there exists a bounded state utility function, \(u\), which agrees with \(\tilde{u}\) on the relevant consumption set and which leads to the same equilibrium set.
Theorem 21. (Approximate Efficiency) Let $[\mathcal{E}^R(u)]_{u \in \mathcal{U}}$ denote the class of economies that satisfy Assumptions A1-A4 and that differ only in the state utility function, $u \in \mathcal{U}$. Consider a weakly separable state utility function $\tilde{u} \in \mathcal{U}$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $u \in B(\tilde{u}, \delta)$, then any equilibrium of the economy $\mathcal{E}^R(u)$ is $\varepsilon$-constrained optimal.

We provide a sketch proof of this result, which is completed in Appendix 5.A.2. We first present a social planner's problem for this economy, which is useful to characterize the efficiency properties of the equilibrium. Consider the problem of maximizing an (equal) weighted utility of the workers subject to effort-incentive compatibility and resource constraints:

\[
\max_{\tilde{\eta} \in \mathcal{P}(A^I)} \int_N \int_{(x,e) \in A^I} U(x, e) \, d\tilde{\eta}_u \, d\nu \tag{5.1}
\]

subject to

\[
\int_N \int_{(x,e) \in A^I} \sum_{s \in S} q_s(e)(y^g_s - x^g_s) d\tilde{\eta}_u d\nu \geq 0.
\]

Note that problem (5.1) is a linear optimization problem and the constraint set, $\mathcal{P}(A^I)$, is convex. Hence, problem (5.1) has the same optimal value as the following simpler problem:

\[
\max_{\tilde{\eta} \in \mathcal{P}(A^I)} \int_{(x,e) \in A^I} U(x, e) \, d\tilde{\eta} \tag{5.2}
\]

subject to

\[
\int_{(x,e) \in A^I} \sum_{s \in S} q_s(e)(y^g_s - x^g_s) d\tilde{\eta} \geq 0,
\]

where $\tilde{\eta} \in \mathcal{P}(A^I)$ is the average measure defined by:

\[
\tilde{\eta}(\tilde{A}) = \int_N \eta_u(\tilde{A}) \, d\nu \text{ for each } \tilde{A} \in B(A^I). \tag{5.3}
\]

In other words, the social planner can be thought of as choosing a single stochastic allocation to maximize a worker's expected utility subject to resource constraints. Let $U^{R\,\text{planner}}(u)$ denote the optimal value of problem (5.2) for the economy with the state utility function $u \in \mathcal{U}$.

The next lemma characterizes approximate constraint optimality of the equilibrium by comparing the optimal value of the social planner's problem (5.2) with the optimal value of the equilibrium problem (5.2). To state the result, let $U^{R\,\text{eq}}(p, u)$ denote the value of problem (5.2) when the price vector is given by $p$ and the state utility function is $u \in \mathcal{U}$.

205
Lemma 7. Consider an equilibrium of the economy $\mathcal{E}^R$ with the price vector $p$. For any $\varepsilon \geq 0$, the equilibrium is $\varepsilon$-constrained optimal if and only if $U^R_{\text{planner}}(u) \in [U^R_{eq}(p,u), U^R_{eq}(p,u) + \varepsilon]$. In particular, the equilibrium is constrained optimal if and only if $U^R_{\text{planner}}(u) = U^R_{eq}(p,u)$.

Given this lemma, Theorem 21 intuitively follows from Theorem 20 and the continuity properties of problems (5.2) and (5.2). To see this, first note that Theorem 20 implies that any equilibrium of the economy $\mathcal{E}(\bar{u})$ is constrained optimal. By Lemma 7, this implies

$$U^R_{\text{planner}}(\bar{u}) = U^R_{eq}(p,\bar{u}), \quad (5.4)$$

for any equilibrium price vector $p$ of the economy $\mathcal{E}(\bar{u})$. Next, the analysis in Appendix 5.A.2 establishes that a version of Berge's Maximum Theorem applies to problem (5.2), and thus $U^R_{\text{planner}}(u)$ is a continuous function of $u$. Similarly, $U^S_{eq}(p,u)$ is a continuous function of $(p,u)$. Moreover, it can be seen that the equilibrium price correspondence,

$$P(u) = \left\{ p \in \mathbb{R}^{|S|}_+ \mid p \text{ is an equilibrium price vector of } \mathcal{E}^R(u) \right\}, \quad (5.5)$$

is upper hemicontinuous in $u$. These observations, along with Eq. (5.4), imply that $U^R_{\text{planner}}(u)$ and $U^R_{eq}(p,u)$ are close to each other for a state utility function, $u$, that lies in a neighborhood of $\bar{u}$. Appendix 5.A.2 formalizes and completes this argument, establishing a proof of Theorem 21.

Theorem 21 also highlights why the notion of generic inefficiency is not always economically useful. Take the set of economies corresponding to utility functions in the set $B(\bar{u},\delta)$ for some $\delta > 0$. With the same argument as above, weakly separable utility functions are nowhere dense within the set $B(\bar{u},\delta)$. But our result shows that all of the corresponding economies have equilibria that are approximately efficient.

Remark 4. (The Role of Stochastic Contracts in Approximate Efficiency) Note that Theorem 21 concerns the economy $\mathcal{E}^R$ with stochastic contracts, and we do not have an analogous approximate efficiency result for an economy $\mathcal{E}$ with deterministic contracts. We conjecture that the result generalizes to economy $\mathcal{E}$, but this conjecture is not straightforward to prove.
The proof of Theorem 21 does not generalize to economy $E$, mainly because there appears to be no analogue of Lemma 7 in that setting. In particular, due to the nonconvexity of the set of incentive compatible contracts, $C^I(p)$, the Pareto frontier of economy $E$ is not necessarily characterized as the solution to a weighted social planner's problem. One can construct example economies in which the equilibrium is constrained optimal; but neither problem (5.1) nor problem (5.2) (nor any other weighted social planner's problem) yields the worker the same utility as the equilibrium. Since our proof of Theorem 21 exploits the continuity properties of the social planner's problem, this proof does not generalize to economy $E$. Stochastic contracts are useful in this context because they convexity the set of incentive compatible contracts, which in turn enables us to characterize constrained optimality using a social planner's problem (as formalized by Lemma 7).

5.7 Conclusion

This essay investigated the efficiency of competitive equilibria in environments with private information. Prescott and Townsend (1984a, 1984b) establish the constrained optimality of competitive equilibrium in such environments when (insurance, employment or credit) contracts can fully specify consumption bundles. Though important, these results are not applicable to situations in which individuals are allowed to trade in anonymous markets. We view such anonymous trading to be an essential feature of competitive equilibria. Less is known about the structure and efficiency of competitive equilibria in the presence of such anonymous trading.

A "folk theorem" originating in the work of Stiglitz and coauthors maintains that competitive equilibria are always or "generically" inefficient in such environments. This folk theorem has widespread applicability in both applied models and in policy discussions, though it has not been formally investigated. This essay critically reevaluates this folk theorem in the context of a general equilibrium economy with moral hazard. In our economy, firms offer contracts to workers who choose an effort level that is private information and affects the probability distribution of endowment and production vectors. We establish the existence of a competitive equilibrium and characterize some of its properties.

To investigate the efficiency properties of competitive equilibrium, we introduce a monitoring
partition such that employment contracts can specify expenditures over subsets in the partition but cannot regulate how this expenditure is subdivided among the commodities within a subset. We say that preferences are nonseparable (or not weakly separable) when the marginal rate of substitution across commodities within a subset in the partition depends on the effort level. We prove that the equilibrium is always inefficient when a competitive equilibrium allocation involves less than full insurance and preferences are nonseparable. While this result is consistent with the folk theorem on the inefficiency of competitive equilibrium, our main result shows why such inefficiency does not always arise and can be mitigated by partial monitoring. We show that when there is weak separability in preferences, a condition satisfied by preference is used in most applied theory work, competitive equilibria with moral hazard are constrained optimal, in the sense that a social planner who can regulate and monitor all consumption levels cannot improve over these competitive allocations. We also show that equilibria in economies that have utility functions that are approximately weakly separable will be approximately efficient. These results imply that the strong suboptimality claims of the folk theorem for competitive equilibria in private information economies may be somewhat exaggerated. At the very least, considerable care is necessary in concluding that competitive equilibria are inefficient and government intervention is necessary without knowing the details of preference and information structure.

Our results also emphasize that the efficiency properties of competitive equilibria depend on the monitoring partition, which raises the question of how the monitoring partition is determined in practice. In related work, we develop a framework for the analysis of competitive equilibria in which firms pay a cost to choose which subsets of commodities and actions to monitor. Among other things, this framework shows that endogenous monitoring will create another force towards efficiency. In particular, additional welfare loss in equilibrium compared to the constrained efficient allocation is bounded above by the cost of monitoring a particular partition (which depends on the worker’s preferences). This further implies that, when the cost of monitoring this partition is sufficiently small, the competitive equilibrium is approximately constrained optimal, despite the costs of monitoring that it incurs (relative to the social planner who does not incur them). This result reinforces our point that considerable care is necessary in invoking the folk theorem about the inefficiency of competitive equilibria with
private information.

5.A Appendices

5.A.1 Omitted Results.

The appendix presents results omitted from the main text.

Class of economies in which any equilibrium is inefficient. Theorem 16 in the main text established sufficient conditions under which the equilibrium allocation is constrained suboptimal. The next result identifies a class of economies in which any equilibrium satisfies the conditions of Theorem 16, and thus, is inefficient. To ensure that the equilibrium always features less than full insurance, we assume that there exists a shirking effort level which is always preferred by the worker under full insurance, and which yields the firm (almost) zero profits. To ensure that every equilibrium has the nonseparability property, we assume that there exists two goods within the same monitoring partition such that the marginal rate of substitution between the goods change monotonically in the effort level. The result then follows from Theorem 16 (proof omitted).

Theorem 22. (Sufficient Conditions For Inefficiency) Consider a class of economies \([\mathcal{E}(\theta)]_{\theta \in (0,1)}\) which differ only in the parameter \(\theta\) defined below. Suppose that

1. There exists an effort level, \(e_{\text{shirk}} \in E\), such that \(u(x, e_{\text{shirk}}) > u(x, e)\) for all \(e \in E \setminus \{e_{\text{shirk}}\}\) and \(x \in \mathbb{R}^{|S| \times |G|}_+\). There also exists a state \(s_{\text{low}} \in S\) such that \(y_{s_{\text{low}}} = 0\) and \(q_{s_{\text{low}}} (e_{\text{shirk}}) = 1 - \theta\). Moreover, \(U(0, e_{\text{shirk}}) < \max_{e \in E} U(y, e)\), i.e., the allocation \((x = 0, e = e_{\text{shirk}})\) is less desirable than the allocation that offers no insurance (and lets the worker choose the effort level).

2. There exists a monitoring subset \(G_m\) and two goods \(g_1, g_2 \in G_m\) such that \(\frac{\partial u(x, e)}{\partial x_{g_1}} \) is strictly increasing in \(e \in E \subset \mathbb{R}\) for any \(x \in \mathbb{R}^{|S| \times |G|}_+\).

There exists \(\bar{\theta} \in (0,1)\) such that, for each \(\theta \leq \bar{\theta}\), Assumptions A1-A4 hold, \(\mathcal{E}(\theta)\) has at least one competitive equilibrium, and any equilibrium of the economy \(\mathcal{E}(\theta)\) is not constrained optimal.
Alternative notions of efficiency. We next consider a weaker notion of optimality than studied in the main text. This notion of optimality constrains the social planner with the same monitoring technology as the firms.

Definition 17. A price and contract allocation pair, \((p, [c(\nu) = (w(\nu), x(\nu), e(\nu)]_{\nu \in \mathcal{N}})\), is market feasible if the contracts are incentive compatible given the price vector, i.e., \(c(\nu) \in C^I(p)\) for each \(\nu \in \mathcal{N}\), and the resource constraints in (5.1) hold. The pair is weakly constrained optimal if it is market feasible and there does not exist another market feasible pair, \((\hat{p}, \hat{c}(\nu) = (\hat{w}(\nu), \hat{x}(\nu), \hat{e}(\nu)]_{\nu \in \mathcal{N}})\), such that \(U(\hat{x}(\nu), \hat{e}(\nu)) > U(x(\nu), e(\nu))\) for all \(\nu \in \mathcal{N}\) with strict inequality for a positive measure of \(\nu \in \mathcal{N}\).

We next provide an example with nonseparable state utility function in which the equilibrium is not constrained optimal, as implied by Theorem 16, but which is weakly constrained optimal. The example shows that the inefficiency of equilibrium established in Theorems 16 and 22 in part stems from the strong notion of optimality which gives the social planner a technological advantage in monitoring.

Example 6 (Nonseparable Preferences, Weak Optimality). Consider the same setup as in Example 4 but assume, additionally, that workers are heterogeneous in their preferences. In particular, there are two types of workers, denoted by \(T_1\) and \(T_2\), of equal measure, 1/2. The state utility functions of type \(T_1\) and \(T_2\) workers are respectively given by:

\[
\begin{align*}
    u(x, e; T_1) &= \ln \left( x_1 + \left( \frac{1}{2} + (1 - e) \right) x_2^2 \right) \quad \text{and} \\
    u(x, e; T_2) &= \ln \left( \left( \frac{1}{2} + (1 - e) \right) x_1^1 + x_2^2 \right).
\end{align*}
\]

Let \(p^1 = 1, p^2 = 1, (c[T_i] = w[T_i], x[T_i], e[T_i])_{i \in \{1, 2\}}\) denote an equilibrium price and contract allocation pair in which workers of the same type accept the same contract. The equilibrium allocation for type \(T_1\) workers is characterized exactly as in Example 4 [cf. Eq. (5.5)], while the equilibrium allocation for type \(T_2\) workers is the mirror image allocation in which the consumption of goods 1 and 2 are reversed. We claim that this equilibrium is weakly constrained optimal.

First consider the Pareto improving allocation in Example 4 (the allocation chosen by a
social planner that can fully monitor consumption), given by $\hat{\epsilon} [T_i] = 1$ for each $i \in \{1, 2\}$, and the consumption allocations:

\begin{align*}
\text{For type } T_1 : & \quad \hat{x}_h^1 [T_1] = \hat{x}_l^1 [T_1] = \frac{\pi_h}{2} \quad \text{and} \quad \hat{x}_h^2 [T_1] = \hat{x}_l^2 [T_1] = 0, \\
\text{For type } T_2 : & \quad \hat{x}_h^1 [T_2] = \hat{x}_l^1 [T_2] = 0 \quad \text{and} \quad \hat{x}_h^2 [T_2] = \hat{x}_l^2 [T_2] = \frac{\pi_h}{2}.
\end{align*}

We claim that the corresponding contract allocation, $\{\hat{\epsilon} [T_i] = (\hat{w} [T_i] = \hat{x} [T_i], \hat{p}, \hat{x} [T_i], \hat{\epsilon} [T_i])\}_{i \in \{1, 2\}}$, is not market feasible for any relative price level $\hat{p}^2$ (recall that $\hat{p}^1$ is normalized to 1). Given the state utility function in (5.A.1) (and $\hat{\epsilon} [T_1] = 1$), the allocation in (5.A.2) is incentive compatible for the type $T_1$ workers only if $\hat{p}^2 \leq \frac{1}{2}$. However, similarly, the allocation in (5.A.3) is incentive compatible for the type $T_2$ workers only if $\hat{p}^2 \geq 2$. Hence, there is no relative price level that can make the contract allocation $\{\hat{\epsilon} [T_1], \hat{\epsilon} [T_2]\}$ market feasible, verifying the claim.

It remains to show that there is no other market feasible price and contract allocation pair which is a Pareto improvement over the equilibrium. Suppose, to reach a contradiction, that there is such a price and allocation pair, denoted by $\left(\hat{p}^1 = 1, \hat{p}^2, (\hat{w} [T_i] = \hat{x} [T_i], \hat{p}, \hat{x} [T_i], \hat{\epsilon} [T_i])\}_{i \in \{1, 2\}}\right)$. It can be seen that $\hat{\epsilon} [T_i] = 1$ for each $i \in \{1, 2\}$.\(^{15}\) In view of the conversion technology, the resource constraints can be reduced to a single constraint:

$$
\sum_{i \in \{1, 2\}} \frac{1}{2} \sum_{s \in \{l, h\}} \frac{1}{2} \sum_{g \in \{1, 2\}} x_g^s [T_i] \leq \frac{\pi_h}{2}.
$$

First consider the case $\hat{p}^2 \in \left(\frac{1}{2}, \frac{1}{2}\right)$, so that type $T_1$ workers always choose to consume good 1 and type $T_2$ workers always choose to consume good 2, i.e., $x_g^s [T_i] = 0$ for $i \neq g$. Then, the incentive compatibility constraints can be written as:

\begin{align*}
\frac{1}{2} \ln (\hat{x}_h^1 [T_1]) + \frac{1}{2} \ln (\hat{x}_l^1 [T_1]) & \geq \ln \left(\frac{3}{2} \frac{\hat{x}_h^1 [T_1]}{\hat{p}_2}\right) \\
\frac{1}{2} \ln (\hat{x}_h^2 [T_2]) + \frac{1}{2} \ln (\hat{x}_l^2 [T_2]) & \geq \ln \left(\frac{3}{2} \frac{\hat{x}_h^2 [T_2]}{\hat{p}_2}\right).
\end{align*}

\(^{15}\)More specifically, for the case in which only one type works, it can be seen that it is not possible to attain a Pareto improvement without violating the resource constraints.
These constraints can be simplified to

\[
\frac{\bar{x}_h^1 [T_1]}{\bar{x}_h^1 [T_1]} \geq \frac{9}{4} \frac{1}{(\hat{p}_2)^2} \quad \text{and} \quad \frac{\bar{x}_h^2 [T_2]}{\bar{x}_h^2 [T_2]} \geq \frac{9}{4} (\hat{p}_2)^2.
\] (5.4.5)

Note also that this allocation is a Pareto improvement over the equilibrium allocation (cf. Eqs. (5.6) – (5.5)) if only if:

\[
\frac{1}{2} \ln (\bar{x}_h^i [T_i]) + \frac{1}{2} \ln (\bar{x}_h^i [T_i]) \geq \ln \left( \frac{6}{13} \pi_h \right) \quad \text{for each } i \in \{1, 2\},
\] (5.4.6)

with strict inequality for some \(i \in \{1, 2\}\). It can be seen that conditions (5.4.5) and (5.4.6) establish a lower bound on the amount of resources that need to be spent on each type workers, that is:

\[
\frac{\bar{x}_h^1 [T_1]}{\bar{x}_h^1 [T_1]} + \frac{\bar{x}_h^2 [T_2]}{\bar{x}_h^2 [T_2]} \geq \left( \frac{3}{2\hat{p}_2} + \frac{2\hat{p}_2}{3} \right) \frac{6}{13} \pi_h \quad \text{and}
\]

\[
\frac{\bar{x}_h^2 [T_2]}{\bar{x}_h^2 [T_2]} + \frac{\bar{x}_h^2 [T_2]}{\bar{x}_h^2 [T_2]} \geq \left( \frac{3\hat{p}_2}{2} + \frac{2}{3\hat{p}_2} \right) \frac{6}{13} \pi_h,
\]

with strict inequality for some \(i \in \{1, 2\}\). Combining the last two inequalities with the resource constraint (5.4.4) (along with the fact that \(\bar{x}_h^g [T_i] = 0 \) for \(i \neq g\)), we have:

\[
\frac{3}{2\hat{p}_2} + \frac{3\hat{p}_2}{3} + \frac{3\hat{p}_2}{2} + \frac{2}{3\hat{p}_2} < \frac{13}{3}.
\] (5.4.7)

On the other hand, using the arithmetic-mean geometric-mean inequality, we have:

\[
\frac{3}{2\hat{p}_2} + \frac{3\hat{p}_2}{3} \geq 2 \sqrt{\frac{3}{2\hat{p}_2} \cdot \frac{3\hat{p}_2}{3}} = 3 \quad \text{and}
\]

\[
\frac{2\hat{p}_2}{3} + \frac{2}{3\hat{p}_2} \geq 2 \sqrt{\frac{2\hat{p}_2}{3} \cdot \frac{2}{3\hat{p}_2}} = \frac{4}{3}.
\]

Combining these inequalities with Eq. (5.4.7) yields a contradiction. A similar contradiction can be obtained for the case in which \(\hat{p}_2 \notin \left[ \frac{1}{2}, 2 \right]\), proving that the equilibrium is weakly constrained optimal.

The key intuition behind this result is captured by the incentive compatibility constraints,
Note that a high level of the relative price \( \hat{p}_2 \) relaxes the incentive compatibility constraints for the type \( T_1 \) workers (who have a greater temptation to shirk when they consume good 2), but it also simultaneously tightens the incentive compatibility constraints for the type \( T_2 \) workers (who have a greater temptation to shirk when they consume good 1). Hence, the Pareto improving allocation constructed in Example 4 cannot be decentralized with any price vector, and the equilibrium is weakly constrained optimal despite the fact that it is not constrained optimal.

5.A.2 Omitted Proofs

This appendix presents proofs of the results in the main text.

Proofs for Section 5.2

We first show that Assumptions A2 and A3 imply the following lemma, which we use in the subsequent analysis.

Lemma 8. (Local Transferability) Suppose Assumptions A1-A3 hold and consider an incentive compatible contract, \( c = (w, x, e) \in C'(p) \). Then, for each \( \varepsilon > 0 \), there exists contracts \( c_+, c_- \in C'(p) \) such that

\[
V(c_-, p) > V(c) - \varepsilon, \quad \pi(c_-, p) > \pi(c, p),
\]

(5.A.8)

and

\[
V(c_+, p) > V(c, p), \quad \pi(c_+, p) > \pi(c, p) - \varepsilon.
\]

(5.A.9)

Proof of Lemma 8. We first define a transfer function that will be useful in this and some of the subsequent proofs. For each non-zero consumption vector \( x_{G_1} \), price vector \( p \), and transfer level \( t_s \in \mathbb{R} \), we define \( w^1_s(x_{G_1}, p, t_s) \in \mathbb{R} \) as the optimum value and \( \tilde{x}^G_{G_1}(x_{G_1}, p, t_s) \) as the solution of the following strictly convex optimization problem:

\[
w^1_s(x_{G_1}, p, t_s) = \min_{x_{G_1} \geq 0} \tilde{x}^G_{G_1} p_{G_1}
\]

subject to \( u^{G_1}(\tilde{x}^G_{G_1}) = u^{G_1}(x_{G_1}^G) + t_s \).
That is, \( w_s^1(x_s^{G_1}, p, t_s) \) describes the minimum wage level necessary to increase the state \( s \) utility obtained from the goods in \( G_1 \) by \( t_s \), given that the current consumption is \( x_s^{G_1} \) and the current price vector is \( p \). Note that, since \( x_s^{G_1} \) is non-zero, there exists a sufficiently small interval \( (-\delta, \delta) \) such that \( w_s^1(x_s^{G_1}, p, t_s) \) is well defined for each \( t_s \in (-\delta, \delta) \). Note also that, over this interval, \( w_s^1(x_s^{G_1}, p, t_s) \) is continuous and strictly increasing in \( t_s \). Given the transfer vector \( t = (t_s)_{s \in S} \), we also define \( w^1(x^{G_1}, p, t) = (w_s^1(x_s^{G_1}, p, t_s))_{s \in S} \) and \( \tilde{x}^{G_1}(x^{G_1}, p, t) = (\tilde{x}_s^{G_1}(x_s^{G_1}, p, t_s))_{s \in S} \) as the corresponding transfer functions over all states.

We next prove Lemma 8. Consider an incentive compatible contract \( c = (w, x, e) \in C^I(p) \) and a positive scalar \( \varepsilon > 0 \). We first construct a contract \( c_+ \) that satisfies Eq. (5.A.9). Note that we have \( w_s^1(x_s^{G_1}, p, 0) = w_s^1 \) since \( c \in C^I(p) \). Then, from the continuity of \( w_s^1(\cdot) \), there exists \( \delta > 0 \) such that

\[
\begin{align*}
\text{for each } s, \quad w_s^1(x_s^{G_1}, p, \delta) &\in (w_s^1, w_s^1 + \varepsilon) \\
\end{align*}
\]  

(5.A.11)

By definition of \( w_s^1(\cdot) \) (and incentive compatibility of \( c \)) we also have

\[
\begin{align*}
\text{for each } s \in S, \quad u^{G_1}(\tilde{x}_s^{G_1}(x_s^{G_1}, p, \delta)) &= u^{G_1}(x_s^{G_1}) + \delta, \\
\end{align*}
\]  

(5.A.12)

Define the transfer vector \( t(\delta) = (t_s = \delta) \) for each \( s \). Since \( c \) is incentive compatible and \( u(\cdot) \) satisfies condition (5.8), Eq. (5.A.12) implies that \( e \) remains incentive compatible after the utility transfer \( t(\delta) \). It follows that \( c_+ = \left( (w^1(x^{G_1}, p, t(\delta)), w^{M \setminus \{1\}}), (\tilde{x}^{G_1}(x^{G}, p, t(\delta)), x^{G \setminus G_1}), e \right) \) is incentive compatible. By Eq. (5.A.11), this contract costs the firm at most \( \varepsilon \) more than the contract \( c \). Thus, \( c_+ \) satisfies Eq. (5.A.9). Next recall that Assumptions A2 and A3 ensure \( w_s^1 > 0 \) for each \( s \). Hence, a similar argument establishes the existence of a contract \( c_- \) that satisfies Eq. (5.A.8). This completes the proof of the lemma.

Proof of Proposition 6. First consider the only if part of the proposition. Let \( (p, [w(\nu), x(\nu), e(\nu)]_{\nu \in \mathcal{N}}) \) denote an equilibrium price and contract allocation pair. The second claim holds holds by definition of equilibrium. To prove the first claim, first note that firms can always guarantee themselves 0 profits (by offering contracts that will not be accepted). This implies \( \pi(c(\nu), p) \geq 0 \), that is, the contract \( c(\nu) \) is in the constraint set of problem (5.10) for all
\(\nu\) (except potentially a measure zero set). We claim that \(c(\nu)\) solves problem (5.10) for all but a measure zero set of workers. Suppose, to reach a contradiction, that there exists a positive measure set \(N^* \subset N\) and contracts \((\hat{c}(\nu))_{\nu \in N^*}\) such that \(c(\nu)\) and \(\hat{c}(\nu)\) are both in the constraint set of problem (5.10), but \(V(\hat{c}(\nu), p) > V(c(\nu), p)\). Since \(N^* \subset N\) is of positive measure and the set of firms is finite, there exists a firm \(j\) such that the set \(N^j = \{\nu \in N^* \mid J(\nu) = j\}\) is of positive measure. Consider a contract \(c(\nu)\), with \(\nu \in N^j\). Applying Lemma 8 to this contract for \(\epsilon = V(\hat{c}(\nu), p) - V(c, p) > 0\), there exists another incentive compatible contract \(\hat{c}_-(\nu)\) such that

\[V(\hat{c}_-(\nu), p) > V(\hat{c}(\nu), p) \quad \text{and} \quad \pi(\hat{c}_-(\nu), p) > \pi(\hat{c}(\nu), p) \geq 0. \tag{5.A.13}\]

Then, we claim that another firm \(j' \neq j\) can strictly increase its profits by changing its contract offers to the workers in \(N^j\) to \((\hat{c}_-(\nu))_{\nu \in N^j}\). Note that, after this change, all the workers \(\nu \in N^j\) switch to firm \(j'\). Since each contract \(\hat{c}_-(\nu)\) makes the firm positive profits (cf. (5.A.13)), the expected profits of firm \(j'\) strictly increase after this deviation, proving our claim. This contradicts the fact that firm \(j'\) maximizes profits in equilibrium, completing the proof for the only if part of the proposition.

We next prove the claim in the proposition that the constraint \(\pi((w, x, e), p) \geq 0\) binds for any solution \((w, x, e)\) to problem (5.10). Suppose, to reach a contradiction, that the contract \(c = (w, x, e)\) is a solution to problem (5.10) and satisfies \(\pi(c, p) > 0\). Then, by Lemma 8, there exists another incentive compatible contract \(c_+\) such that \(\pi(c_+, p) \geq 0\) and \(V(c_+, p) > V(c, p)\). This contradicts the fact that \(c\) is a solution to problem (5.10), showing that the profit constraint binds.

Next consider the if part of the proposition. Let \(p\) and \([w(\nu), x(\nu), e(\nu)]_{\nu \in N}\) be a price and allocation system that satisfies the two claims of the proposition. We conjecture a symmetric equilibrium in which every firm offers every worker the same contract, i.e., \((c(\nu, j) = [w(\nu), x(\nu), e(\nu)])_{\nu \in N, j \in J}\). Given these offers, any worker strategy is optimal. Moreover, the goods market clearing condition is satisfied by assumption. Hence, we are only left with by proving that firms’ contract offers are optimal. Suppose, to reach a contradiction, that there exists a firm \(j'\) that can make strictly positive profits by offering a collection of incentive compatible contracts \([\hat{c}(\nu, j')]_{\nu \in N}\). Then, there exists \(N^* \subset N\) with positive measure...
such that for each $\nu \in \mathcal{N}^*$,

$$J(\nu) = j' \text{ after the deviation by firm } j', \text{ and}$$

$$\pi(\hat{c}(\nu, j'), p) > \pi(c(\nu), p) = 0. \quad (5.A.14)$$

Since the worker $\nu \in \mathcal{N}^*$ prefers the contract offered by $j'$ over the contract $c(\nu)$, we have

$$V(\hat{c}(\nu, j'), p) \geq V(c(\nu), p). \quad (5.A.15)$$

By Lemma 8 and equations (5.A.14) and (5.A.15), there exists an incentive compatible contract $\hat{c}_+(\nu, j')$ such that

$$\pi(\hat{c}_+(\nu, j'), p) > 0, \text{ and } V(\hat{c}_+(\nu, j'), p) > V(c(\nu), p).$$

It follows that $c(\nu)$ is not a solution to problem (5.10) for $\nu \in \mathcal{N}^*$, which is a contradiction. This shows that the price vector and contract allocations constructed above constitute an equilibrium, completing the proof for the if part of the proposition.

We next show that Assumptions A3 and A4 imply the following lemma, which we need to establish the existence of equilibrium.

**Lemma 9. (Continuity of Incentive Compatible Contracts)** Suppose Assumptions A1-A4 hold. Then, the correspondence $C^I(p)$ is lower hemicontinuous in $p$. That is, for any incentive compatible contract $c \in C^I(p)$ and any sequence $p[n] \to p$, there exists a sequence of contracts $c[n] \in C^I(p[n])$ such that $c[n] \to c$.

Since $C^I(p)$ is also upper hemicontinuous, Lemma 9 establishes the continuity of $C^I(p)$ in $p$.

**Proof of Lemma 9.** Consider the price vector $p$, an incentive compatible contract $c \equiv (w, x, e)$, and a sequence $\{p[n]\}_{n=1}^{\infty} \to p$. We claim that, for any $\varepsilon > 0$, there exists an index $n$ and a contract $c[n] \equiv (x[n], e, w[n])$ such that

$$c[n] \in C^I(p[n]) \text{ and } w[n] \in B(w, \varepsilon). \quad (5.A.16)$$
Given this claim, a sequence \( \{c[n]\}_{n=0}^{\infty} \) can be constructed such that \( c[n] \in C'(p[n]) \) and \( c[n] \to c \). It thus follows that \( C'(p) \) is lower hemicontinuous.

To prove the claim in (5.A.16), first note that Assumption A4 implies that there exists a transfer vector \( t \in \mathbb{R}^{|S|} \) such that Eq. (5.12) holds for the effort level \( e \) and the vector \( t \). Given this vector \( t \), note that Eq. (5.12) also holds for the transfer vector \( \eta t \), where \( \eta > 0 \) is an arbitrary positive scalar.

Next consider the function \( w^1(\cdot) \) defined in (5.A.10). Since \( w^1(\cdot) \) is continuous in \( p \) and \( t \), and since \( w^1(x^{G_1}, 0, p) = w^1 > 0 \) (by Assumptions A2 and A3), there exists \( \delta > 0 \) such that \( w^1(x^{G_1}, \bar{p}, \bar{t}) \in B(w^1, \varepsilon) \cap \mathbb{R}_{++} \) for each \( \bar{p} \) and \( \bar{t} \) that satisfies \( ||\bar{t}|| \leq \delta \) and \( ||\bar{p} - p|| \leq \delta \). Let \( \eta \) be sufficiently small so that \( ||\eta t|| \leq \delta \) and define

\[
\bar{\varepsilon} = \sum_{s \in S} \eta t_s q_s(e) - \max_{\hat{e} \in E \setminus \{e\}} \sum_{s \in S} \eta t_s q_s(\hat{e}), \tag{5.A.17}
\]

which is strictly positive in view of Eq. (5.12). Since the indirect utility function \( V(c, p) \) and the function \( U(x(w, p, e), e) \) are continuous in \( p \), there exists a sufficiently large \( n \) such that \( ||p[n] - p|| \leq \delta \) and

\[
V(c, p[n]) < V(c, p) + \varepsilon/2, \text{ and } \quad U(x(w, p[n], e), e) > U(x(w, p, e), e) - \varepsilon/2.
\]

Since \( V(c, p) = U(x(w, p, e), e) \), these two inequalities jointly imply

\[
U(x(w, p[n], e), e) < U(x(w, p[n], e), e) + \varepsilon \text{ for all } \hat{e} \in E \setminus \{e\}. \tag{5.A.18}
\]

Given the constructed \( \eta t \) and \( p[n] \), we define the contract \( c[n] = (x[n] = (\bar{x}^{G_1} (x^{G_1}, p[n], \eta t), x^{G \setminus G_1}), e) \) and we claim that \( c[n] \) satisfies (5.A.16). By construction of \( x[n] \) and \( w[n] \) (and \( \delta \)), we have that \( w[n] \in B(w^1, \varepsilon) \). Moreover, \( x[n] \) is incentive compatible given the wages \( w[n] \) and the effort level \( e \). Hence, we only need to show that the effort level \( e \) is incentive compatible. Suppose, to reach a contradiction, that there
exists \( \hat{e} \neq e \) such that

\[
U(\mathbf{x}(w[n], p[n], \hat{e}), \hat{e}) \geq U(\mathbf{x}(w[n], p[n], e), e).
\] (5.A.19)

Note that the construction of \( w[n] \) differs from \( w \) only for the wages for the monitoring subset \( G_1 \). Note also that, from condition (5.8), the consumption choice for goods within \( G_1 \) does not affect the consumption choice for the goods within other monitoring subsets. This implies

\[
\mathbf{x}^{G \setminus G_1}(w[n], p[n], \hat{e}) = \mathbf{x}^{G \setminus G_1}(w, p[n], \hat{e})
\]
for any effort level \( \hat{e} \). These observations, along with the definition of the function \( w^1(\cdot) \) (cf. Eq. (5.A.10)), further imply that:

\[
U(\mathbf{x}(w[n], p[n], \hat{e}), \hat{e}) = \sum_{s \in S} \eta_t q_s(\hat{e}) + U(\mathbf{x}(w, p[n], \hat{e}), \hat{e}).
\]

Considering this equality for \( \hat{e} \in \{e, \hat{e}\} \), and plugging in the inequality (5.A.19) implies:

\[
\sum_{s \in S} \eta_t q_s(\hat{e}) + U(\mathbf{x}(w, p[n], \hat{e}), \hat{e}) \geq \sum_{s \in S} \eta_t q_s(e) + U(\mathbf{x}(w, p[n], e), e).
\]

Combining this inequality with the inequality in (5.A.18), we have

\[
\sum_{s \in S} \eta_t q_s(\hat{e}) > \sum_{s \in S} \eta_t q_s(e) - \tilde{e}.
\]

This inequality yields a contradiction to the definition of \( \tilde{e} \) in (5.A.17), proving that the effort level \( e \) is incentive compatible. This shows that \( c[n] \) satisfies the claim in (5.A.16) and completes the proof of the lemma.

**Proof of Theorem 15.** The key step in establishing the existence of the equilibrium is to show that the solution correspondence for the problem (5.10), denoted by \( S(p) \), is upper hemicontinuous in \( p \). We establish this using the continuity of \( C^l(p) \) (cf. Lemma 9) along with the local transferability condition (cf. Lemma 8).

Let \( \{(w_n, x_n, e_n), p_n\}_{n=1}^{\infty} \) denote a sequence such that \((w_n, x_n, e_n) \in S(p_n) \) for each \( n \) and

\[
\lim_{n \to \infty} (w_n, x_n, e_n) = (w, x, e) \quad \text{and} \quad \lim_{n \to \infty} p_n = p.
\]

218
Note that \((w, x, e)\) satisfies the constraints of problem (5.10) for the price vector \(p\). We claim that \((w, x, e) \in S(p)\). Suppose, to reach a contradiction, that there exists \((\hat{w}, \hat{x}, \hat{e})\) that satisfies the constraints of problem (5.10) and that yields the worker \(V((\hat{w}, \hat{x}, \hat{e}), p) > V((w, x, e), p)\). By Lemma 8, there exists another incentive compatible contract \((\overline{w}, \overline{x}, \overline{e})\) which satisfies the non-zero profit constraint strictly,

\[
\pi((\overline{w}, \overline{x}, \overline{e}), p) > 0, \tag{5.A.20}
\]

and which yields a strictly greater utility for the worker,

\[
V((\overline{w}, \overline{x}, \overline{e}), p) > V((w, x, e), p). \tag{5.A.21}
\]

Since \((\overline{w}, \overline{x}, \overline{e}) \in C^I(p)\), by Lemma 9, there exists \((\overline{w}_n, \overline{x}_n, \overline{e}_n) \to (\overline{w}, \overline{x}, \overline{e})\) such that \((\overline{w}_n, \overline{x}_n, \overline{e}_n) \in C^I(p_n)\) for each \(n\). Since \(\pi(\cdot)\) and \(V(\cdot)\) are continuous functions, Eqs. (5.A.20) and (5.A.21) imply that there exists a sufficiently large \(n\) such that

\[
\pi((\overline{w}_n, \overline{x}_n, \overline{e}_n), p_n) > 0 \text{ and } V((\overline{w}_n, \overline{x}_n, \overline{e}_n), p_n) > V((w_n, x_n, e_n), p_n).
\]

But since \((\overline{w}_n, \overline{x}_n, \overline{e}_n) \in C^I(p_n)\), these inequalities contradict the fact that \((w_n, x_n, e_n)\) is a solution to problem (5.10) given the price vector \(p_n\). It follows that \((w, x, e) \in S(p)\), which implies that \(S(p)\) is upper hemicontinuous.

The rest of the proof follows standard arguments. Consider the excess demand correspondence \(D : \{1\} \times \mathbb{R}^{[G]}_{\geq 1} \to \mathbb{R}^{[G]}\) (recall that the price of good 1 is normalized to 1), given by:

\[
D(p) = \left\{ \int_{\mathcal{N}} \sum_{s \in S} \left(\bar{x}_s(\nu) - y_s\right) q_s(\bar{e}(\nu)) d\nu \in \mathbb{R}^{[G]} \mid (\bar{w}(\nu), \bar{x}(\nu), \bar{e}(\nu)) \in S(p) \text{ for each } \nu \in \mathcal{N} \right\}. \tag{5.A.22}
\]

Since there is a continuum of workers, the demand correspondence \(D(p)\) is convex valued. Since \(S(p)\) is upper hemicontinuous, the correspondence \(D(p)\) is also upper hemicontinuous. We have, \(\lim_{p^g \to 0} D^g(p) > 0\), since the supply of good \(g \neq 1\) is bounded from above (since \(y_s < \infty\) for each \(s\)) while the demand for good \(g\) tends to infinity as \(p^g \to 0\). Similarly, \(\lim_{p^g \to \infty} D^g(p) < 0\) since the supply of good \(g\) is bounded away from zero, while the demand
for good \( g \) tends to zero as \( p^g \to \infty \). Therefore, Kakutani’s Fixed Point Theorem applies to this economy, which implies that there exists a price vector \( p \) such that \( 0 \in D(p) \). By definition of \( D(p) \), there exists \((w(\nu), x(\nu), e(\nu))_{\nu \in \mathcal{N}}\) such that \((w(\nu), x(\nu), e(\nu)) \in S(p)\) for each \( \nu \in \mathcal{N} \) and the goods markets clear. By Proposition 6, the price vector \( p \) and the contract allocations \((w(\nu), x(\nu), e(\nu))_{\nu \in \mathcal{N}}\) correspond to an equilibrium, completing the proof of the theorem.

**Proofs for Section 5.3**

**Proof of Theorem 16.** Consider a worker \( \nu \in \mathcal{N}^* \) and denote her allocation by \( c \equiv (w, x, e) \). Since \((x, e)\) does not feature full insurance, there exists \( s_1, s_2 \in S \) and \( g \in G \) such that the MRS for good \( g \) between states \( s_1 \) and \( s_2 \) is not equal to 1. We will reallocate the consumption of good \( g \) across states so that the worker receives better insurance while her equilibrium effort choice \( e \) remains effort-incentive compatible. Formally, we claim that there exists \( \hat{x}^g \in \mathbb{R}^{|S|}_+ \) such that

\[
\sum_{s \in S} q_s(e) \hat{x}^g_s = \sum_{s \in S} q_s(e) x^g_s, \tag{5.23}
\]

\[
ee \in \text{arg max}_{e \in E} U\left(x^{G\backslash\{g\}}, \hat{x}^g, e \right), \tag{5.24}
\]

and

\[
U\left(x^{G\backslash\{g\}}, \hat{x}^g, e \right) > U(x, e). \tag{5.25}
\]

Once we prove this claim, it follows that there is an effort-incentive compatible and Pareto improving deviation for each worker \( \nu \in \mathcal{N}^* \), which implies that the equilibrium allocation is not constrained optimal.

To prove the claim, we first show that there exists a deviation \( \hat{x}^g \) that satisfies (5.23) and (5.25). Recall that \( \frac{\partial u(x_{s_1}, e)}{\partial x^g_{s_1}} / \frac{\partial u(x_{s_2}, e)}{\partial x^g_{s_2}} \neq 1 \). Suppose, without loss of generality, that

\[
\frac{\partial u(x_{s_1}, e)}{\partial x^g_{s_1}} > \frac{\partial u(x_{s_2}, e)}{\partial x^g_{s_2}}. \tag{5.26}
\]

For any \( \varepsilon > 0 \), consider the deviation vector \( v[\varepsilon] \in \mathbb{R}^{|S|} \) defined by \( v_s[\varepsilon] = 0 \) for any \( s \notin \{s_1, s_2\} \) and

\[
v_{s_1}[\varepsilon] = \frac{\varepsilon K}{q_{s_1}(e)} \quad \text{and} \quad v_{s_2}[\varepsilon] = -\frac{\varepsilon K}{q_{s_2}(e)}, \tag{5.27}
\]

220
where the constant $K = \frac{\min(q_1(e), q_2(e))}{2} > 0$ ensures that $\|v[e]\| \leq \varepsilon$ for any $\varepsilon > 0$. Note that, by construction, the vector $\hat{x}^g = x^g + v[e]$ satisfies the resource constraints in (5.A.23). Note also that

$$U\left(x^{G\setminus\{g\}}, x^g + v[e], e\right) = \sum_{s \in S} q_s(e) u\left(x^{G\setminus\{g\}}, x^g + v_s[e], e\right)$$

$$= \left(\sum_{s \in S} q_s(e) u(x_s, e) + q_{s_1}(e) \frac{\partial u(x_{s_1}, e)}{\partial x_{s_1}} \frac{\varepsilon K}{q_{s_1}(e)} - q_{s_2}(e) \frac{\partial u(x_{s_2}, e)}{\partial x_{s_2}} \frac{\varepsilon K}{q_{s_2}(e)}\right) + o(\varepsilon)$$

$$= U(x, e) + \varepsilon K \left(\frac{\partial u(x_{s_1}, e)}{\partial x_{s_1}} - \frac{\partial u(x_{s_2}, e)}{\partial x_{s_2}}\right) + o(\varepsilon) (5.A.28)$$

where the second line considers a first order Taylor expansion for the functions $u(x_{s_1}, e)$ and $u(x_{s_2}, e)$, and the notation $o(\varepsilon)$ captures the residual which satisfies $\lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon} = 0$. From Eqs. (5.A.28) and the inequality in (5.A.26), it follows that there exists $\varepsilon > 0$ such that $U\left(x^{G\setminus\{g\}}, x^g + v[e], e\right) > U(x, e)$ for each $\varepsilon \in (0, \varepsilon)$. This proves our claim that there exists a deviation $\hat{x}^g$ that satisfies (5.A.23) and (5.A.25).

We next consider a sequence of vectors $\{v[e[n]]\}_{n=1}^{\infty}$, where $\{e[n]\}_n$ is a sequence of scalars such that $e[n] \in (0, \varepsilon)$ for each $n$ and $e[n] \to 0$. We claim that there exists $n$ such that $\hat{x}^g = x^g + v[e[n]]$ also satisfies the effort-incentive compatibility constraint in (5.A.24). Suppose, to reach a contradiction, that there is no such $n$. Then, for each vector $v[e[n]]$, there exists $e[n] \in E \setminus \{e\}$ such that

$$U\left(x^{G\setminus\{g\}}, x^g + v[e[n]], e[n]\right) > U(x, e). (5.A.29)$$

Since the space $E \setminus \{e\}$ is finite (and thus compact), the sequence $\{e[n]\}_{n=1}^{\infty}$ has a convergent subsequence. Let $\hat{e} \in E \setminus \{e\}$ be a limit point of this sequence. Since $\lim_{n \to \infty} v[e[n]] = 0$, Eq. (5.A.29) implies

$$U(x, \hat{e}) \geq U(x, e). (5.A.30)$$

Note that, since preferences are nonseparable at $(x, e)$, there exists $s, m$ and $g_1, g_2 \in G_m$ such
that
\[
\frac{\partial u(x_s, \hat{e})}{\partial x_s^1} / \frac{\partial x_s^1}{\partial u(x_s, \hat{e})} \neq \frac{\partial u(x_s, e)}{\partial x_s^2} / \frac{\partial x_s^2}{\partial u(x_s, e)} = \frac{p^u_1}{p^u_2},
\]
where the last equality follows since \( c \in C^I(p) \). The last equation further implies that \( x_s^{\{g_1,g_2\}}(w_s, p, \hat{e}) \neq x_s^{\{g_1,g_2\}} \). This further implies
\[
U(x(w, p, \hat{e}), \hat{e}) > U(x, \hat{e}) \geq U(x, e),
\]
where the last inequality uses (5.A.30). This yields a contradiction to of the fact that the contract \((w, x, e)\) is incentive compatible. It follows that there exists a vector \( v[\epsilon[n]] \) such that \( \hat{x}^g = x^g + v[\epsilon[n]] \) satisfies equations (5.A.23) – (5.A.25), which completes the proof of the theorem. □

Proofs for Section 5.4

The proof of Theorem 17 requires a preliminary step. It is intuitively clear that, if preferences satisfy the separability condition (5.1), then the indifference curves between any two goods in the same monitoring partition \( m \) should be independent of the effort choice \( e \). The next lemma formalizes this observation.

**Lemma 10.** Suppose the separability condition (5.1) in Definition 12 holds for the monitoring subset \( G_m \). Consider a vector \((x_s, e) \in \mathbb{R}_+^{[G]} \times E\), and let \( \hat{x}_s^{G_m} \) denote a reallocation of the goods in \( G_m \) such that the level of the utility is kept constant, i.e., suppose:
\[
u(\hat{x}_s^{G_m} \cup G_m, \hat{e}) = u(x_s, e).
\]
Then, this reallocation keeps the level of utility constant also for any other effort level, i.e.:
\[
u(\hat{x}_s^{G_m} \cup G_m, \hat{e}) = u(x_s, \hat{e}) \text{ for any } \hat{e} \in E. \tag{5.A.31}
\]
**Proof of Lemma 10.** We first claim that, under the separability condition (5.1), the partial derivative of the utility admits the following representation:

\[
\frac{\partial u(\tilde{x}_s, \tilde{e})}{\partial \tilde{x}^G_m} = h(\tilde{x}_s, \tilde{e}) F(\tilde{x}_s),
\]  

(5.32)

where \( F : \mathbb{R}_+^{|G|} \rightarrow \mathbb{R}^{G_m} \) is a vector valued function and \( h : \mathbb{R}_+^{G_m} \times E \rightarrow \mathbb{R}_+ \) is a strictly positive scalar valued function. To prove this claim, fix some good \( g \in G_m \) and define the scalar valued function

\[
h(\tilde{x}_s, \tilde{e}) = \frac{\partial u(\tilde{x}_s, \tilde{e})}{\partial \tilde{x}^g}
\]

which is strictly positive. Applying condition (5.1) for all pairs \((\tilde{g} \in G_m, g)\), we have that the ratio,

\[
\frac{\partial u(\tilde{x}_s, \tilde{e}) / \partial \tilde{x}^G_m}{\partial u(\tilde{x}_s, \tilde{e}) / \partial \tilde{x}^g} = \frac{\partial u(\tilde{x}_s, \tilde{e}) / \partial \tilde{x}^G_m}{h(\tilde{x}_s, \tilde{e})},
\]

is independent of \( \tilde{e} \). Hence, it can be denoted by a vector valued function \( F(\tilde{x}_s) \). This completes the proof of the claim in (5.32).

Next, to prove the lemma, note that there exists a continuously differentiable function \( \tilde{x}_s^{G_m} : [0, 1] \rightarrow \mathbb{R}_+^{G_m} \) which satisfies \( \tilde{x}_s^{G_m}(0) = x_s^{G_m}, \tilde{x}_s^{G_m}(1) = \tilde{x}_s^{G_m}, \) and

\[
u(\tilde{x}_s^{G_m}(t) : x_s^{G_m}G \in E \! = \! u(x_s, e) \text{ for each } t \in [0, 1].
\]

(5.33)

Note that \( \tilde{x}_s^{G_m}(t) \) represents a curve that lies inside the indifference surface (which is the higher dimensional analogue of an indifference curve). Totally differentiating Eq. (5.33) with respect to \( t \), we have

\[
\frac{\partial u(\tilde{x}_s^{G_m}(t), x_s^{G_m}G \in E \! = \! u(x_s, e) \text{ for each } t \in [0, 1].
\]

(5.34)

Plugging in the representation in (5.32), the previous equality implies:

\[
h(\tilde{x}_s^{G_m}(t), x_s^{G_m}G \in E \! = \! u(x_s, e) \text{ for each } t \in [0, 1].
\]

(5.34)

Since \( h(\cdot) \) is a strictly positive scalar valued function, the previous equality implies that
This further implies that

\[ \frac{d}{dt}S_g(t) = 0 \]

for each \( t \in [0, 1] \),

which is the same as Eq. (5.A.34), except for the fact that the effort level, \( e \), is replaced by an arbitrary \( \bar{e} \in E \). Using the representation in (5.A.32) one more time, the previous equality implies

\[ \frac{\partial u}{\partial S_g(t)} \frac{dS_g(t)}{dt} = 0 \]

for each \( t \in [0, 1] \).

Integrating this equation and using the fundamental theorem of calculus, we establish Eq. (5.A.31), completing the proof of the lemma.

We next use this lemma to provide a proof of Theorem 17.

**Proof of Theorem 17.** By the equilibrium conditions, \([x(v), e(v)]_{v \in \mathcal{N}}\) is incentive feasible. Assume, to reach a contradiction, that \([x(v), e(v)]_{v \in \mathcal{N}}\) is constrained suboptimal. Then, there exists an incentive feasible allocation \([\bar{x}(v), \bar{e}(v)]_{v \in \mathcal{N}}\) such that

\[ U(\bar{x}(v), \bar{e}(v)) > U(x(v), e(v)) \]  

(5.A.35)

with strict inequality for a positive measure of \( v \in \mathcal{N} \).

Consider \( v \in \mathcal{N} \) such that the inequality (5.A.35) holds strictly, and drop the \( v \)'s from the notation for convenience. We will show that the allocation \((\bar{x}, \bar{e})\) violates the profit constraint of the indirect problem (5.10), that is, we claim

\[ \sum_{s \in S} q_s(\bar{e})(y_s - \bar{x}_s)p < 0. \]  

(5.A.36)

And similarly, we claim that the same inequality holds weakly whenever the inequality in (5.A.35) holds weakly. Once we establish the claim in (5.A.36), the result follows from the standard proof of the first welfare theorem. In particular, integrating Eq. (5.A.36) over all \( v \in \mathcal{N} \) yields a contradiction to the fact that \([\bar{x}(v), \bar{e}(v)]_{v \in \mathcal{N}}\) satisfies the resource constraints, which proves that the equilibrium is constrained optimal.
To prove the claim in (5.A.36), we will construct a contract \( \hat{c} = (\hat{w}, \hat{x}, \hat{e}) \) which satisfies the following three properties:

(P1) \( \hat{c} \) is incentive compatible.

(P2) \( \hat{c} \) yields the worker the same utility than the allocation \((\bar{x}, \bar{e})\).

(P3) \( \hat{c} \) costs the firm less than the allocation \((\bar{x}, \bar{e})\), i.e.,

\[
\sum_{s \in S} q_s (\hat{e}) \hat{x}_s p \leq \sum_{s \in S} q_s (\bar{e}) \bar{x}_s p. \tag{5.A.37}
\]

Once we construct a contract \( \hat{c} \) with these properties, we have the following implications. By (P1), the contract \( \hat{c} \) satisfies all the constraints of problem (5.10) except for the profit constraint (5.11). By (P2), the contract \( \hat{c} \) yields the worker greater utility than the equilibrium contract. Since the equilibrium contract solves problem (5.10), the contract \( \hat{c} \) must violate the remaining constraint of problem (5.10), that is, it violates the profit constraint. By (P3), the contract \( \hat{c} \) costs the firm less than the allocation \((\bar{x}, \bar{e})\), which in turn implies that \((\bar{x}, \bar{e})\) also violates the profit constraint. This shows the claim in Eq. (5.A.36), completing the proof of the theorem.

Hence, we are left with constructing a contract \( \hat{c} = (\hat{w}, \hat{x}, \hat{e}) \) that satisfies the properties (P1)-(P3). We will construct the wage and allocation pair, \((\hat{w}, \hat{x})\) as the limit of a sequence \( \{\hat{w}[n], \hat{x}[n]\}_{n=0}^{\infty} \), where the sequence will be constructed recursively. In particular, let \((\hat{w}[0], \hat{x}[0]) = (\hat{w} \equiv \bar{x} p, \bar{x})\) denote the initial vector. Suppose \(\{(\hat{w}[0], \hat{x}[0]), \ldots, (\hat{w}[n-1], \hat{x}[n-1])\}\) is constructed for \(n \geq 1\), and consider the construction of \((\hat{w}[n], \hat{x}[n])\). Let \(m \in \{1, \ldots, |M|\}\) denote the modulo \(|M|\) value of \(n\). For each \(s\), define \(\hat{x}[n]_s = (\hat{x}^G_m, \hat{x}[n-1]_{s \setminus G_m})\) and \(\hat{w}[n]_s = (\hat{w}^m, \hat{w}[n-1]_{s \setminus M(m)})\), where \(\hat{w}^m_s\) is the minimum value and \(\hat{x}^G_m\) is the unique solution to the following strictly convex optimization problem:

\[
\min_{\hat{x}^G_m \geq 0} \hat{x}^G_m p^{G_m} \tag{5.A.38}
\]

subject to

\[
u (\hat{x}^G_m, \hat{x}[n-1]_{s \setminus G_m}, \hat{e}) = u (\hat{x}[n-1]_s, \hat{e}). \tag{5.A.39}
\]

That is, at each step, the vector, \((\hat{w}[n]_s, \hat{x}[n]_s)\), is constructed by reallocating the goods within one partition, \(G_m\), in a way to minimize the costs while providing the worker with the same utility as before. The partitions are subject to this operation one at a time and in an order.
Once we operate over all partitions, we start the process over (which is formally captured above by taking the modulo $|M|$ value of $n$). We claim that the sequence $\{\hat{w}[n], \hat{x}[n]\}$ converges to a vector $(\hat{w}, \hat{x})$, and that the contract $\hat{c} = (\hat{w}, \hat{x}, \hat{e})$ satisfies (P1)-(P3).

We first show that the sequence $\{\hat{w}[n], \hat{x}[n]\}$ converges. Note that the allocations and the wage for each partition, $(\hat{w}[n]^m_s, \hat{x}[n]^G_m)$, is updated once every $|M|$ turns. Moreover, problem (5.A.38) shows that the wage corresponding to each partition, $\hat{w}[n]^m_s$, weakly decreases each time an updating occurs (and it is constant if an updating does not occur). In particular, $\{\hat{w}[n]^m_s\}_{n=0}^\infty$ is a decreasing sequence. Moreover, it is bounded below by 0. This means that, for each $m$ and $s$, $\hat{w}[n]^m_s$ has a unique limit point which we denote by $\hat{w}^m_s$. This further implies that $\hat{w}[n] \to \hat{w}$. Next, note that, the solution $\hat{x}[n]^G_m$ to problem (5.A.38) satisfies the first order conditions:

$$ p^g \geq \lambda[n]^m_s \frac{\partial u(\hat{x}[n]^m_s, e_0)}{\partial \hat{x}^g_s} \text{ for each } s, m, \text{ and } g \in G_m, $$

with equality if $\hat{x}^G_m > 0$,

where $\lambda[n]^m_s$ is a strictly positive Lagrange multiplier for each $s$ and $m$. Hence, given the wages $\hat{w}[n]$, the solutions $\hat{x}[n]$ are uniquely characterized by the conditions in (5.A.40) along with the equations

$$ \hat{w}[n]^m_s = \hat{x}[n]^G_m p^G_m \text{ for each } s \text{ and } m, $$

which hold since the minimum of the problem (5.A.38) is attained. Since the equations in (5.A.40)–(5.A.41) are continuous in $\hat{w}[n]$, and since $\hat{w}[n] \to \hat{w}$, the solutions $\hat{x}[n]$ also converge to a vector $\hat{x}$. The limiting vector, $\hat{x}$, is the solution to Eqs. (5.A.40)–(5.A.41) corresponding to the limiting wages $\hat{w}$. This establishes that the sequence we have constructed converges to a vector, $(\hat{w}, \hat{x})$.

We next show that the contract $\hat{c} = (\hat{w}, \hat{x}, \hat{e})$ satisfies (P1), that is, $\hat{c}$ is incentive compatible. We break this argument into two steps. First, we claim that the allocation $(\hat{x}, \hat{e})$ is effort-incentive compatible, that is,

$$ \sum_{s \in S} q_s(\hat{e}) u(\hat{x}_s, \hat{e}) \geq \sum_{s \in S} q_s(e) u(\hat{x}_s, e) \text{ for each } \hat{e} \in E. $$

(5.A.42)
We will then establish that the contract \( \hat{c} \) satisfies the stronger incentive compatibility condition (5.4). The proof of the claim in (5.A.42) relies on Lemma 10. In particular, note that Eq. (5.A.39) implies that the utility is preserved at each step of the above construction, that is:

\[
u(\hat{x}[n]_s, \hat{e}) = \nu(\hat{x}[n-1]_s, \hat{e}) \quad \text{for each } s.
\]

Moreover, note that, at each step, the above operation reallocates the goods within exactly one monitoring subset \( G_m \). That is, the allocations \( \hat{x}[n]_s \) and \( \hat{x}[n-1]_s \) are the same except (potentially) for the allocations \( \hat{x}[n]_s^{G_m} \). Hence, Lemma 10 applies and shows that

\[
u(\hat{x}[n]_s, \hat{e}) = \nu(\hat{x}[n-1]_s, \hat{e}) \quad \text{for each } s \text{ and } \hat{e} \in E.
\]

This further implies that the limiting utility is equal to the initial utility for any effort level \( \hat{e} \), that is:

\[
u(\hat{x}_s, \hat{e}) = \nu(\hat{x}_s, \hat{e}) \quad \text{for each } s \text{ and } \hat{e} \in E. \tag{5.A.43}
\]

Since the allocation \( (\hat{x}, \hat{e}) \) is effort-incentive compatible, Eq. (5.A.43) implies that the allocation \( (\hat{x}, \hat{e}) \) is also effort-incentive compatible, proving the claim in (5.A.42).

We next prove that the contract \( \hat{c} \) is incentive compatible. Given wages \( \hat{w} \) and some effort choice \( \hat{e} \in E \), the worker’s consumption choice satisfies the following first order conditions:

\[
\frac{\partial u(\hat{x}_s, \hat{e})}{\partial x_s^g} \leq \gamma_s^m p^g \quad \text{for each } s, m, \ g \in G^m, \tag{5.A.44}
\]

with equality if \( \hat{x}_s^{G_m} > 0 \),

where \( \gamma_s^m \) is a positive Lagrange multiplier for each \( s \) and \( m \). In particular, the workers’ consumption choice is uniquely determined by the conditions in (5.A.44) along with the budget constraints:

\[
\hat{w}_s^m = \hat{x}_s^{G_m} p^{G_m} \quad \text{for each } s \text{ and } m. \tag{5.A.45}
\]

Note that the first order conditions in (5.A.44) are identical to the first order conditions in (5.A.40), and Eq. (5.A.41) is identical to Eq. (5.A.45). Recall also that \( \hat{x} \) is the unique solution to Eqs. (5.A.40) – (5.A.41). Hence, the worker’s consumption choice is independent of her
effort choice $\bar{e}$, and is equal to $\bar{x}$. This implies that the effort-incentive compatibility condition (5.A.42) implies the incentive compatibility of the contract $\bar{c}$, establishing (P1).

We next show that $\bar{c}$ satisfies (P2) and (P3). Note that, by construction, $\bar{c}$ yields the same utility to the worker as the allocation $(\bar{x}, \bar{e})$, establishing (P2). Recall also that $\{\hat{w}_n\}_{n=0}^{\infty}$ is a decreasing sequence, which implies that $\hat{w} = \hat{x}p \leq \hat{w}[0] = \bar{v}$. Hence, contract $\bar{c}$ costs the firm less than the allocation $(\bar{x}, \bar{e})$ and Eq. (5.A.37) holds, establishing (P3). This completes the proof of Theorem 17. ■

Proofs for Section 5.5

**Proof of Theorem 18.** We first claim that the pair, $(p, [\mu])_{\nu \in \mathcal{N}}$, is part of an equilibrium if and only if conditions 1-2 of the theorem are satisfied. Since the space, $C^I[p]$, satisfies the transferability condition in 8, the space, $\mathcal{P}(C^I[p])$, also satisfies the analogous transferability condition. Given this result, the argument in the proof of Proposition 5.10 applies unchanged with stochastic contracts, proving the claim.

We next show that an equilibrium exists. First, it can be seen that the constraint set of problem (5.2) is compact. Since the objective function is linear (and thus continuous) in $\bar{p}$, a solution to problem (5.2) always exists. Next, we claim that the solution set to problem (5.2), denoted by $S^R[p] \subset \mathcal{P}(C^I[p])$, is upper hemicontinuous in $p$ (recall that $\mathcal{P}(C)$ is endowed with weak* topology). By Lemma 9, $C^I[p]$ is a continuous correspondence in $p$. It follows that $\mathcal{P}(C^I[p]) : \mathbb{R}^{[S]} \Rightarrow \mathcal{P}(C)$ is also a continuous correspondence in $p$. Then, a similar argument to Theorem 15 establishes that $S^R[p]$ is upper hemicontinuous. Recall also that $S^R[p]$ is convex valued since problem (5.10) is linear.

The rest of the proof follows standard arguments. For a given price $p$, define the excess demand correspondence $D : \mathbb{R}^{[S]} \Rightarrow \mathbb{R}^{[S]}$ by

$$D(p) = \left\{ \int_N \int_{[w(v), x(v), e(v)]} \sum_{s \in S} (x_s(v) - y_s)q_s(e(v))d\bar{\mu}_\nu dv \right\}_{\bar{\mu}_\nu \in S^R[p], \nu \in \mathcal{N}}$$

(5.A.46)

Note that, for any collection of allocations, $[\bar{\mu}_\nu \in S^R[p]]_\nu$, the excess demand in (5.A.47) is
equivalent to
\[ \int_N \int_{(w,x,e)} \sum_{s \in S} (x_s - y_s)q_s(e(\nu))d\bar{\mu}, \]
where \( \bar{\mu} \in \mathcal{P}(C) \) is the average measure defined by
\[ \bar{\mu}(C) = \int \bar{\mu}_\nu(C) d\nu \text{ for each } C \in \mathcal{B}(C). \]
Note that \( \bar{\mu} \in S^R(p) \), since \( \bar{\mu}_\nu \in S^R(p) \) for each \( \nu \) and \( S^R(p) \) is convex valued. Hence, problem (5.A.46) can equivalently be written as:
\[ D(p) = \left\{ \int_{(w,x,e)} \sum_{s \in S} (x_s - y_s)q_s(e)d\bar{\mu} \mid \bar{\mu} \in S^R(p) \right\}. \quad (5.A.47) \]
Since \( S^R(p) \) is upper hemicontinuous in \( p \), \( D(p) \) is also upper hemicontinuous in \( p \). Then, the same arguments in the proof of Theorem 15 show that an equilibrium exists, completing the proof of Theorem 18. 

**Proof of Theorem 19.** As in the proof of Theorem 16, we will show that there is an incentive compatible allocation, \( \hat{\eta} \), that satisfies the resource constraints and improves the utility of the worker over the equilibrium allocation, \( \eta \equiv \mu_{\nucaled}(x,e) \).

Let \( A^{**} = A^* \cap \text{supp}(\eta) \), and note that \( A^{**} \) is compact (since \( A^* \) is compact and \( \text{supp}(\eta) \) is closed) and that \( \eta(A^{**}) = \eta(A^*) > 0 \). Let \( (x,e) \in A^{**} \) and note that Definition 15 implies that preferences are nonseparable at the deterministic allocation \( (x,e) \). Then, consider the deviation constructed in the proof of Theorem 16. In particular, for each \( \varepsilon > 0 \), let \( v[\varepsilon \mid (x,e)] \in \mathbb{R}^{|S|} \) denote the vector constructed in (5.A.27). Note that \( \|v[\varepsilon \mid (x,e)]\| \leq \varepsilon \), and that \( v[\varepsilon \mid (x,e)] \) is continuous in \( (x,e) \). Recall also that, by the proof of Theorem 16, there exists \( \bar{\varepsilon} \) such that
\[ U(x^{g \setminus \{s\}}, x^g + v[\varepsilon \mid (x,e)]), e) > U(x,e) \quad (5.A.48) \]
for each \( \varepsilon < \bar{\varepsilon} \) and each \( (x,e) \in A^{**} \). For each \( \varepsilon \in (0,\bar{\varepsilon}) \), define the function \( \zeta(\cdot,e) : A \to A \) with:
\[ \zeta((x,e),\varepsilon) = \begin{cases} 
(x,e) & \text{if } (x,e) \in A \setminus A^{**}, \\
(x^{g \setminus \{s\}}, x^g + v[\varepsilon \mid (x,e)]), e) & \text{if } (x,e) \in A^{**}.
\end{cases} \]
Note that $\zeta(\cdot, \varepsilon)$ is a measurable function from $(A, B(A))$ to $(A, B(A))$, because the perturbation function $v[\varepsilon | (x, e)]$ is continuous in $(x, e)$. Hence, given the probability measure $\eta$, the function $\zeta(\cdot, \varepsilon)$ induces another probability measure over $(A, B(A))$ (the push-forward measure), defined by:

$$\eta[\varepsilon](A) = \eta(\zeta^{-1}(A))$$

for each $A \in B(A)$.

Note that $\eta[\varepsilon]$ is a stochastic allocation. Moreover, by equation (5.A.48) and by the definition of $\eta[\varepsilon]$, we have that $U^R(\eta[\varepsilon]) > U^R(\eta)$ and that $\eta[\varepsilon]$ satisfies the resource constraints (5.4).

We next consider the sequence of stochastic allocations $\{\eta[\varepsilon[n]]\}_{n=1}^{\infty}$, where $\{\varepsilon[n]\}_{n=1}^{\infty}$ is a sequence of scalars such that $\varepsilon[n] \in (0, \bar{\varepsilon})$ for each $n$ and $\varepsilon[n] \to 0$. We claim that there exists $n$ such that $\eta[\varepsilon[n]]$ is also incentive compatible. Suppose, to reach a contradiction, that for each $n$, $\text{supp}(\eta[\varepsilon[n]])$ is not a subset of the set of deterministic incentive compatible allocations, $A^I$. By the construction of $\eta[\varepsilon[n]]$, this implies that there exists a vector $(x[n], \varepsilon[n]) \in A^{**}$ such that the perturbed vector $(x[n] \cap \{\varrho\}, x[n] \cap \{\varrho\} + v[\varepsilon[n] | (x[n], \varepsilon[n])], \varepsilon[n])$ is not an element of $A^I$. That is, there exists $\hat{e}[n] \in E \setminus \{e[n]\}$ such that

$$U(\bar{x}[n] \cap \{\varrho\}, x[n] \cap \{\varrho\} + v[\varepsilon[n] | (x[n], \varepsilon[n])], \hat{e}[n]) > U(x[n], e[n]).$$

(5.A.49)

Since the space $A^{**} \times E$ is compact, the sequence of vectors, $\{(x[n], \varepsilon[n]), \hat{e}[n]\}_{n=1}^{\infty}$, has a convergent subsequence. Let $(\bar{x}, \bar{e}, \hat{e}) \in A^{**} \times E$ be a limit point of this sequence and note that $\hat{e} \neq e$ (since $\hat{e}[n] \neq e[n]$ for each $n$). Note also that Eq. (5.A.49) implies

$$U(x, \hat{e}) > U(x, e).$$

Since preferences are nonseparable at the deterministic allocation $(x, e) \in A^{**}$, by the proof of Theorem 16, we have

$$U(x(w \equiv xp, \hat{e}), \hat{e}) > U(x, \hat{e}) \geq U(x, e).$$

This yields a contradiction to the fact that the contract, $c = (w \equiv xp, x, e)$ is incentive compatible (the contract $c$ is incentive compatible since the allocation, $(x, e) \in A^{**}$, is in the support of the equilibrium stochastic allocation, $\eta = \mu_{\nu}(x,e)$). This completes the proof of Theorem 19.
Proof of Theorem 20. Suppose to obtain a contradiction that \((\eta_\nu)_{\nu \in \mathcal{N}}\) is not constrained optimal. Then, there exists an effort-incentive compatible and feasible stochastic allocation \((\bar{\eta}_\nu)_{\nu \in \mathcal{N}}\) such that

\[
U^R(\bar{\eta}_\nu) \geq U^R(\eta_\nu)
\]

(5.A.50)

with strict inequality for a positive measure of \(\nu \in \mathcal{N}\).

Consider an allocation \(\bar{\eta}_\nu \in \mathcal{N}\) for which the inequality in (5.A.50) is satisfied strictly, and drop the \(\nu\)'s from the notation for convenience. We will show that the allocation \(\bar{\eta}\) violates the profit constraint of the indirect problem (5.2), that is, we claim

\[
\int_{(w,x,e) \in \mathcal{C}} \left( \sum_{s \in S} q_s(e) (y_s - x_s)p \right) d\bar{\eta} < 0.
\]

(5.A.51)

And similarly, we claim that the same inequality holds weakly whenever the inequality in (5.A.51) holds weakly. Once we establish the claim in (5.A.51), the result follows from the standard proof of the first welfare theorem. In particular, integrating Eq. (5.A.51) over all \(\nu \in \mathcal{N}\) yields a contradiction to the fact that \((\bar{\eta}_\nu)_{\nu \in \mathcal{N}}\) satisfies the resource constraint (5.4), which proves that the equilibrium is constrained optimal.

Consider \((\bar{x}, \bar{e}) \in \text{supp}(\bar{\eta})\) and note that \((\bar{x}, \bar{e}) \subset A^I\) since \(\bar{\eta}\) is incentive compatible. By the proof of Theorem 17, there exists a contract \(\hat{c} = (\hat{w}, \hat{x}, \hat{e})\) that satisfies properties (P1)-(P3), where recall that \(\text{(P1)} \iff \hat{c} \in \mathcal{C}^I(p), \text{(P2)} \iff U(\hat{x}, \hat{e}) = U(\bar{x}, \hat{e}),\) and

\[
\text{(P3)} \iff \sum_{s \in S} q_s(\hat{e}) \hat{x}_s p \leq \sum_{s \in S} q_s(\hat{e}) \bar{x}_s p.
\]

(5.A.52)

From the construction in Theorem 17, it can also be seen that the contract \(\hat{c}[\bar{x}, \bar{e}] \equiv (\hat{w}[\bar{x}, \bar{e}], \hat{x}[\bar{x}, \bar{e}], \bar{e})\) is a continuous function of \((\bar{x}, \bar{e})\), for each \((\bar{x}, \bar{e}) \in \text{supp}(\bar{\eta})\). Extend \(\hat{c}[\cdot]\) to a measurable function over all of \(A\) by defining \(\hat{c}[\bar{x}, \bar{e}] = (\hat{w} \equiv \hat{x}p, \hat{x}, \bar{e})\) for each \((\hat{x}, \bar{e}) \notin \text{supp}(\bar{\eta})\). Note that \(\hat{c}[\cdot]\) is a measurable mapping from \((A, B(A))\) to \((C, B(C))\). Hence, given the probability measure \(\bar{\eta}\) over \((A, B(A))\), the mapping \(\hat{c}[\cdot]\) induces a probability measure over \((C, B(C))\) (push-forward measure), defined by:

\[
\hat{\mu}(\tilde{C}) = \bar{\eta}(\hat{e}^{-1}(\tilde{C})) \text{ for each } \tilde{C} \in B(\tilde{C}).
\]
Note that the support of $\hat{\mu}$ is in $C^I(p)$, since property (P1) implies that $\hat{\epsilon} [\hat{x}, \hat{e}] \in C^I(p)$ for each $(\hat{x}, \hat{e}) \in supp(\hat{\eta})$. Hence, the stochastic contract $\hat{\mu}$ is incentive compatible. Moreover, property (P2) and the fact that (5.A.50) is satisfied strictly implies that $\hat{\mu}$ yields the worker strictly greater utility than the equilibrium allocation $\mu$. Since the equilibrium allocation $\mu$ solves problem (5.A.50), it follows that $\hat{\mu}$ violates the remaining constraint of (5.A.50), that is:

$$\int_{(x,e)} \left( \sum_{s \in S} q_s(e)(y_s - x_s)p \right) d\hat{\mu}|_{(x,e)} < 0.$$ 

By property (P3) and the definition of $\hat{\mu}$, we also have that the stochastic allocation $\hat{\mu}|_{(x,e)}$ costs the firm more than the allocation $\bar{\eta}$. This implies the inequality (5.A.51), completing the proof of Theorem 20. ■

Proofs for Section 5.6

Proof of Lemma 7. We first claim that $U^R_{\text{planner}}(u) \geq U^R_{eq}(p, u)$ holds for all $u \in U$ and equilibrium price vector $p \in P(u)$. Note that the solution to problem (5.1) is always weakly greater than the solution to problem (5.2) (because the equilibrium allocation $[\mu_\nu|_{(x,e)}]_{\nu \in \mathcal{N}}$ is always in the constraint set of problem (5.1)). Since problems (5.1) and (5.2) are equivalent, it follows that $U^R_{\text{planner}}(u) \geq U^R_{eq}(p, u)$, proving the claim.

We next prove the lemma. First consider the only if part, that is, suppose the equilibrium allocation, $[\mu_\nu|_{(x,e)}]_{\nu \in \mathcal{N}}$, is $\varepsilon$-constrained optimal. Suppose, to reach a contradiction, that $U^R_{\text{planner}}(u) \notin [U^R_{eq}(p, u), U^R_{eq}(p, u) + \varepsilon]$. Since $U^R_{\text{planner}}(u) \geq U^R_{eq}(p, u)$, we have $U^R_{\text{planner}}(u) > U^R_{eq}(p, u) + \varepsilon$. Let $\hat{\eta}$ denote the solution to problem (5.2) and consider the allocation $[\hat{\eta}_\nu]_{\nu \in \mathcal{N}}$, defined by $\hat{\eta}_\nu = \bar{\eta}$ for each $\nu$. The allocation $[\hat{\eta}_\nu]_{\nu \in \mathcal{N}}$ is effort-incentive compatible and feasible, and it improves every worker's utility by more than $\varepsilon$. This contradicts the fact that the equilibrium allocation $[\mu_\nu|_{(x,e)}]_{\nu \in \mathcal{N}}$ is constrained optimal, proving the only if part of the lemma.

Next consider the if part, that is, suppose $U^R_{\text{planner}}(u) \in [U^R_{eq}(p, u), U^R_{eq}(p, u) + \varepsilon]$. Suppose, to reach a contradiction, that the equilibrium is not constrained optimal. Then, there exists an allocation, $[\hat{\eta}_\nu]_{\nu \in \mathcal{N}}$, which is effort-incentive compatible and feasible, and which improves the utility of all workers by $\varepsilon$ (and strictly so for a positive measure workers). Consider the
average allocation \( \hat{\eta} \) defined in Eq. (5.3), which satisfies the resource constraints, and which is incentive compatible since \( \mathcal{P}(A^I) \) is a convex set. Hence, \( \hat{\eta} \) is in the constraint set of problem (5.2). Moreover, since \([\hat{\eta}_u]_u\) is a Pareto improvement, the average allocation \( \hat{\eta} \) improves the utility of all workers by strictly more than \( \varepsilon \), that is, \( U_R(\hat{\eta}) > U_R^R(p, u) + \varepsilon \). It follows that \( U^R_{\text{planner}}(u) > U^R_{eq}(p, u) + \varepsilon \), which yields a contradiction, completing the proof of the lemma.

Proof of Theorem 21. We first claim that \( U^R_{\text{planner}}(u) \) is continuous in \( u \in \mathcal{U} \) (where recall \( \mathcal{U} \) is a metric space with the sup norm). To show this, first consider the set of deterministic incentive compatible allocations, \( A^I(u) \):

\[
A^I(u) = \left\{ (x, e) \in A \mid \sum_a q_a(e) u(x, e) \geq \sum_a q_a(\bar{e}) u(x, \bar{e}) \quad \text{for each } \bar{e} \in E \right\}.
\]

Note that \( A^I(u) \) is an upper hemicontinuous correspondence of \( u \). Next, since each \( u \in \mathcal{U} \) satisfies Assumption A3 for the same function \( u^{G1}(\cdot) \), an argument similar to the proof of Lemma 9 shows that \( A^I(u) \) is a lower hemicontinuous correspondence of \( u \). It follows that \( A^I(u) \) is a continuous correspondence of \( u \). This further implies that \( \mathcal{P}(A^I(u)) \) is also a continuous correspondence of \( u \) (when viewed as a correspondence from \( \mathcal{U} \) to \( \mathcal{P}(A) \)). Then, an argument similar to the proof of Proposition 6 shows that the solution to problem (5.2) is upper hemicontinuous, and the optimal value is continuous (i.e., a version of the Maximum Theorem applies to problem (5.2)). This shows that \( U^R_{\text{planner}}(u) \) is continuous in \( u \).

Consider next the correspondence \( \overline{U}^R_{eq} : \mathcal{U} \rightarrow \mathbb{R} \) defined by

\[
\overline{U}^R_{eq}(u) = \left\{ U^R_{eq}(p, u) \mid p \in \mathcal{P}(u) \right\},
\]

where recall that \( \mathcal{P}(u) \) is the equilibrium price correspondence defined in (5.5). We claim that the correspondence \( \overline{U}^R_{eq}(u) \) is upper hemicontinuous in \( u \). To see this, let \( S^R(p, u) \subset \mathcal{P}(A^I) \) denote the solution to problem (5.2), and recall that \( U^R_{eq}(p, u) \) denotes the optimal value of the same problem. A similar argument to the previous paragraph establishes that \( U^R_{eq}(p, u) \) is a continuous function and \( S(p, u) \) is a continuous correspondence of \((p, u)\). By the same argument
as in the proof of Theorem 18, the excess demand correspondence is given by

\[ D(p, u) = \left\{ \int_{(w, x, e)} \sum_{s \in S} (x_s - y_s) q_s(e) d\tilde{\mu} \mid \tilde{\mu} \in S^R(p, u) \right\}. \]

Since \( S^R(p, u) \) is upper hemicontinuous, \( D(p, u) \) is also upper hemicontinuous in \((p, u)\). Moreover, note that the equilibrium price correspondence satisfies:

\[ P(u) = \left\{ p \in \mathbb{R}^{|G|}_+ \mid 0 \in D(p, u) \right\}. \]

Since \( D(p, u) \) is upper hemicontinuous in \((p, u)\), it has a closed graph, which implies that \( P(u) \) is an upper hemicontinuous correspondence of \( u \). Since \( P(u) \) is upper hemicontinuous and \( U_{eq}^R(p, u) \) is continuous, Eq. (5.A.53) implies that \( \overline{U}_{eq}^R(u) \) is upper hemicontinuous, proving the claim.

We have thus established that \( U_{planner}^R(u) \) is a continuous function and \( \overline{U}_{eq}^R(u) \) is a continuous correspondence of \( u \in \mathcal{U} \). Next consider the values of these expressions for \( u = \bar{u} \). Recall that any equilibrium of the economy \( \mathcal{E}(\bar{u}) \) is constrained optimal by Theorem 20. Then, by Lemma 7, it follows that \( \overline{U}_{eq}^R(\bar{u}) \) is a singleton and it is equal to \( U_{planner}^R(\bar{u}) \), that is,

\[ \overline{U}_{eq}^R(\bar{u}) = \{ U_{planner}^R(\bar{u}) \} \quad (5.54) \]

Fix some \( \varepsilon > 0 \). From the continuity of \( U_{planner}^R(\cdot) \), there exists \( \delta_{planner} > 0 \) such that, if \( ||u - \bar{u}|| < \delta_{planner} \), then

\[ |U_{planner}^R(u) - U_{planner}^R(\bar{u})| \leq \varepsilon/2. \quad (5.55) \]

Similarly, from the upper hemicontinuity of \( \overline{U}_{eq}^R(\cdot) \) and Eq. (5.54), there exists \( \delta_{eq} \) such that if \( ||u - \bar{u}|| < \delta_{planner} \), then

\[ |\hat{U}_{eq} - U_{planner}^R(\bar{u})| \leq \varepsilon/2 \text{ for any } \hat{U}_{eq} \in \overline{U}_{eq}^R(u). \quad (5.56) \]

Let \( \delta = \min(\delta_{planner}, \delta_{eq}) \) and note that Eqs. (5.55) and (5.56) imply \( |U_{planner}^R(u) - \hat{U}_{eq}| \leq \varepsilon \) for any \( u \in B(\bar{u}, \delta) \) and any \( \hat{U}_{eq} \in \overline{U}_{eq}^R(u) \). By Lemma 7, it follows that any equilibrium of any economy \( \mathcal{E}^R(u) \), with \( u \in B(\bar{u}, \delta) \), is \( \varepsilon \)-constrained optimal. This completes the proof. \( \blacksquare \)
Bibliography


Chicago and Princeton University.


SIMSEK, A. (2010a): “When Optimists Need Credit: Asymmetric Disciplining of Optimism and Implications for Asset Prices,” working Paper, MIT.


