ESSAYS ON OPTIMAL ECONOMIC GROWTH

by

Peter Arthur Diamond

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Signature of Author: ........................................
Department of Economics and Social Science

Certified by: ........................................
Thesis Supervisor

Accepted by: ........................................
Chairman, Departmental Committee
on Graduate Students
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27 Carey Avenue  
Watertown, Massachusetts  

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In accordance with the requirements for graduation, I herewith submit a thesis entitled, "Essays on Optimal Economic Growth."

Respectfully yours,

Peter Arthur Diamond
ABSTRACT

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by

Peter Arthur Diamond

Submitted to the Department of Economics and Social Science on May 10, 1963 in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

The first essay is concerned with evaluating a stream of utility which extends into the infinite future. In it are derived sufficient conditions on a preference ordering to permit the definition of a continuous utility function as well as the implications of various axiom sets with respect to preferences in the timing of utility.

In the second essay, the time profiles of various economic variables in a neoclassical model are related to two indices of technical change, characterizing the rate and bias of this change. These relationships are used to discuss sufficient conditions for exponential growth and the movement over time of the factor-price frontier.

In the third essay is derived the optimal growth path for a model described by T. N. Srinivasan, which has fixed coefficients but many types of capital goods.

Thesis Advisor: Robert M. Solow

Title: Professor of Economics
Massachusetts Institute of Technology
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Peter A. Diamond
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1. Introduction

Concern with optimal economic growth (and other problems which are similar in having no natural termination date) has led to an examination of the problem of evaluating a stream, of consumption for example, which extends over an infinite future. One approach to this problem is the selection of a specific functional for the evaluation. Thus, Ramsey\(^1\) chose to maximize the integral of undiscounted utility as the criterion for optimal growth. This approach, while permitting explicit choice of timing preference, lacks generality in the nature of the evaluation and fails to define a sensitive ordering in parts of the program space.\(^2\)

This leads naturally to the approach of assuming the existence of a preference ordering over the alternative streams, and examining the implications of various axiom sets imposed on these preferences. This, then, was the approach of Koopmans\(^3\) and is the one followed in this paper. The implications derived are of two types. For some axiom sets a preference is shown for present utility over that to be enjoyed

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in the distant future. For other sets, the impossibility of treating all time periods the same is shown. Before presenting these results, sufficient conditions on preferences are derived for the existence of a utility function.

2. Notation

Throughout the paper, time will be considered in discrete units. Then, a utility stream, U, for the entire future can be denoted as 

\[ U = (u_1, u_2, \ldots, u_t, \ldots) \]

where \( u_t \) is the one period utility level enjoyed in period \( t \). The one period utility level would be the value assumed by a one-period utility function defined over one period consumption bundles, and the utility levels at different points of time could be compared in terms of commodities at different points in time. By making one period utilities, rather than one period consumption bundles, the argument of the preferences, certain types of intertemporal complementarity are ruled out. Thus, while the fact that the type of good one consumes today affects the enjoyment from consuming the same good tomorrow is ignored, the possibility of the level of today's enjoyment affecting the addition to total utility of a given level of enjoyment tomorrow is included within this context.

The one-period utility levels, \( u_t \), will be assumed to lie in the closed unit interval. This implies that the one-period utility function has a maximum and can assume it.

Then, the set, \( X \), of all utility streams will be the infinite Cartesian product of the unit interval. In order to be able to define a continuous utility function on the set of utility streams, it is necessary to define a distance function or metric and thus a topology on \( X \).
Two such metrics, d, will be considered in this paper: the sup metric:

\[ d(U, U') = \sup_t |u_t - u'_t|, \]

and the product metric:\(^4\)

\[ d(U, U') = \sum_{t=1}^{\infty} 2^{-t} |u_t - u'_t|. \]

Preferences over utility streams will be denoted by \( \succ \) and \( \sim \) for preferred and indifferent to. The vector inequality \( U \succeq U' \) will mean \( U \succeq U' \) and \( U \neq U' \). The vector with a constant one period utility level, \( u \), will be denoted by \( (u) = (u, u, \ldots) \). An infinite vector \( (u_1, u_2, \ldots, u_t, u'_1, u'_2, \ldots, u'_t, \ldots) \) will mean \( (u_1, u_2, \ldots, u_t, u'_1, u'_2, \ldots, u'_t, \ldots) \). By a utility function is meant a real, continuous, order-preserving function from \( X \) to the real line.

A natural topology for \( X \) is a topology in which the sets \( \{ U \in X \mid U \succeq U' \} \) and \( \{ U \in X \mid U' \sim U \} \) are closed for all \( U' \) in \( X \).

3. The Existence of a Utility Function

For a preference ordering over utility streams to be interesting, it must exhibit some degree of sensitivity to changes in the one period utility levels. Two different axioms expressing this sensitivity will be presented in this section. The first sensitivity axiom is that

\[(S1) \quad U \succeq U' \implies U \succeq U' \]
\[ U \succ U' \implies U \succ U'. \]

\(^4\)This is a metric for the product topology on \( X \). The product topology has the property that the reversal of the numbering of two-time periods does not alter the topology.
This axiom states that a utility stream which is greater than or equal to a second stream in every time period is preferred or indifferent to the second stream, while a utility stream greater than a second stream in every time period is strictly preferred to it.

The existence theorem will make use of a lemma of Debreu\(^5\) which will be stated first.

**Lemma.** Let \(X\) be a completely ordered set, \(Z = (Z_0, Z_1, \ldots)\) a countable subset of \(X\). If for every pair \(U, U'\) of elements of \(X\) such that \(U \succ U'\), there is an element \(Z_\frac{1}{2}\) of \(Z\) such that \(U \succeq Z_\frac{1}{2} \succ U'\), then there exists on \(X\) a real, order-preserving function, continuous in any natural topology.

The existence theorem will also use a lemma which states that every utility stream \(U\) is indifferent to some constant stream \((u_{\text{con}})\) if the preferences satisfy the sensitivity axiom (S1) and are such that the sup metric generates a natural topology.

**Lemma.** Let \(X\) be completely ordered by a preference ordering satisfying (S1) and for which the sup metric generates a natural topology. Then, for any \(U\) in \(X\) there exists a one period utility level, \(u\), such that \(U \sim (u_{\text{con}})\).

**Proof:** Let \(D\) be the set of constant utility streams, \((u_{\text{con}})\). \(D\) is connected. Assume there exists a \(U^*\) in \(X\) such that there does not exist a \((u_{\text{con}})\) indifferent to it. Define \(A = \{U \in X \mid U \succeq U^*\} \cap D,\)

\[B = \{U \in X \mid U^* \succ U\} \cap D.\]

From the assumption about \(U^*\), \(A \cap B = \emptyset\).

Since the ordering is complete, \(A \cup B = D\). By (S1), \((0_{\text{con}}) \preceq U \preceq (1_{\text{con}})\).

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for all \( U \). Therefore, \( A \neq \emptyset, B \neq \emptyset \). Since the sup metric generates a natural topology, both \( A \) and \( B \) are closed relative to \( D \). This contradicts the connectedness of \( D \).

The existence theorem can now be proved by showing that the set of constant utility streams with rational one period utility levels satisfies the conditions to be the set \( Z \) in Debreu's lemma.

**Existence Theorem:** Let \( X \) be completely ordered by a preference ordering satisfying (S1) and for which the sup metric generates a natural topology. Then there exists a utility function from \( X \) with the sup metric to the real line.

**Proof:** For any pair \( U, U' \) in \( X \) such that \( U \succ U' \), there exists a pair \( u, u' \) in the unit interval such that \( u > u', (u_{\text{con}}) \sim U \), and \( (u'_{\text{con}}) \sim U' \).

For any such pair \( u, u' \) there exists a rational number \( r \) such that \( u > r > u' \). Therefore, \( U \succ (r_{\text{con}}) \succ U' \). Thus the set \( Z = \{(r_{\text{con}}) | r \text{ rational, } 0 \leq r \leq 1\} \) satisfies the conditions of Debreu's lemma.

By the same proofs, the existence theorem could be shown to hold when the product metric is used in place of the sup metric. The theorem will also continue to be true for both metrics if the sensitivity axiom (S1) is replaced by the following sensitivity axiom which implies (S1).

\[(S2) \quad U \succeq U' \text{ implies } U \succ U'. \]

The axiom states that a utility stream that has at least as high one period utility levels in all periods as a second stream and a higher level in at least one period than the second, is preferred to the second.
4. Eventual Impatience

Preference for earlier timing of utility can be expressed by a preference for a utility stream \( U = (u_1, u_2, u_3, \ldots, u_{t-1}, u_t, u_{t+1}, \ldots) \) over a stream \( U^t = (u_t, u_2, u_3, \ldots, u_{t-1}, u_1, u_{t+1}, \ldots) \) if \( u_1 > u_t \). In other words, reversing the timing of the first and \( t \)\(^{th} \) period utility levels raises the utility of the entire stream if it places the larger one period level in the first period. This is impatience for the first over the \( t \)\(^{th} \) period. In the case where the preferences conform to a utility function \( f(U) = \sum_{t=1}^{\infty} w_t u_t \), this would mean \( w_1 > w_t \). Eventual impatience means impatience for the first period over the \( t \)\(^{th} \) period for all \( t \) sufficiently far in the future. In the example above, this would be satisfied if \( w_1 > 0 \) and \( w_t \) goes to zero as \( t \) goes to infinity.

The following two theorems examine the conditions for eventual impatience in the two cases of the product and the sup metric.

**Theorem:** Let \( X \) be completely ordered by a preference ordering satisfying \((S2)\) and for which the product metric generates a natural topology.

Then for any \( \epsilon > 0 \) there exists an \( s \) such that, for all \( U \) in \( X \), for all \( t \geq s \) for which \( |u_1 - u_t| \geq \epsilon \),

\[
U \{(\leq)\} U^t \text{ as } u_1 \{(\geq)\} u_t.
\]

**Proof:** By the existence theorem, there exists a utility function \( f \).

Given \( \epsilon > 0 \) define, for all \( U \) in \( X \), \( K_1(U) = \{ k \mid u_k \geq u_1 + \epsilon, U \succ U^k \} \).

Then \( K_1(U) \) is the set of indices \( k \), for which \((1)\) is not true when \( u_k \geq u_1 + \epsilon \).
Define

\[ K_2(U) = \{ k \mid u_k \preceq u_1 - \varepsilon, u^k \succ u \} \]

\[ K_1 = \{ k \mid \text{there exists } U \text{ in } X \text{ such that } k \text{ in } K_1(U) \} \],

and \[ K_2 = \{ k \mid \text{there exists } U \text{ in } X \text{ such that } k \text{ in } K_2(U) \} \].

It is sufficient to show, for any \( \varepsilon > 0 \), that both \( K_1 \) and \( K_2 \) have only a finite number of elements. The two cases are symmetric and it will be shown that \( K_1 \) has a finite number of elements.

Assume \( K_1 \) has infinitely many elements. For each \( k \) in \( K_1 \) select a \( U(k) \) such that \( k \) is in \( K_1(U(k)) \). Since \( X \) is compact the sequence \( \{ U(k) \mid k \in K_1 \} \) has a convergent subsequence \( (U(k_i)) \) converging to \( U^* \).

For each \( U(k_i) \) in the convergent subsequence consider \( U^{k_i}(k_i) \). This set has a convergent subsequence converging to \( U^{**} \).

By comparing the elements of the two convergent subsequences it is seen that \( u_1^{**} \geq u_1^* + \varepsilon, u_j^{**} = u_j^* \) for \( j = 2, 3, 4, \ldots \). Thus \( U^{**} \succeq U^* \) and, by (S2), \( U^{**} \prec U^* \). However for each \( k_i \)

\[ U(k_i) \succ U^{k_i}(k_i) \]. This and the continuity of \( f \) imply \( U^* \prec U^{**} \), which is a contradiction.

The case with the sup metric is complicated by the fact that infinite sequences of utility streams need not have convergent subsequences since this space is not compact. Thus to show eventual impatience for a stream \( U \) in \( X \) it is not sufficient to look at the infinite sequence \( (U^t) \) but rather, the proof involves constructing a sequence which does converge and which, without eventual impatience, would violate the continuity of the utility function. This difficulty results in a somewhat weaker statement of the theorem and a need for two additional axioms:
(NC1) For all $u, u', U, U'$ $(u, U) \succeq (u', U)$ implies $(u', U) \succeq (u', U')$.

(NC2) For all $1^u_t, 1^u'_t, U, U'$ $(1^u_t, U) \succeq (1^u'_t, U)$ implies $(1^u'_t, U') \succeq (1^u_t, U')$.

The axioms express a certain type of non-complementarity of the preferences over time or that the "preferences" over part of the time horizon are independent of the utility levels enjoyed in other times.

Theorem: Let $X$ be completely ordered by a preference ordering satisfying (S2), (NC1), and (NC2) and for which the sup metric generates a natural topology. Then for each $U$ in $X$ and $\varepsilon > 0$, there exists an $s$ such that for all $t \geq s$ for which $|u_1 - u_t| \geq \varepsilon$,

$$U \begin{cases} \gtrless \varepsilon \end{cases} u_t^t$$

Proof: By the existence theorem there exists a utility function $f$.

Given $U$ and $\varepsilon$ define $K_1 = \{k | u_k \succeq u_1 + \varepsilon, U \succeq U^k\}$

$K_2 = \{k | u_k \preceq u_1 - \varepsilon, U^k \preceq U\}$, it is sufficient to show that both sets have a finite number of elements. The two cases are symmetric and only $K_1$ will be considered.

Assume $K_1$ has infinitely many elements. Label them in order $k_i$ with $i = 1, 2, \ldots$. For $k$ in $K_1$ define:

$$U^*k = (u_1^*k, u_2^*k, \ldots) \text{ where } u_j^*k = \begin{cases} u_1 & j \text{ in } K_1 \text{ and } j \geq k \\ u_j & j \text{ in } K_1 \text{ and } j < k \text{ or } j \text{ not in } K_1 \end{cases}$$

$$U^{**k} = (u_k^*k, u_2^*k, u_3^*k, \ldots)$$

From (S2) we see that $U^{**k} > U^*k$ and $U^{*k+1} > U^{*k}$. By (NC2) $U \supset U^{k+1}$ implies $U^{*k+1} > U^{**k}$. Therefore $\lim_{k \to \infty} f(U^*k) = \lim_{k \to \infty} f(U^{**k}) = f^*$,

$$f(0^\text{con}) \leq f^* \leq f(1^\text{con})$$. For each $k$ there exists an $a_k$ such that $U^*k \sim (u_1^*k, a_k, a_k, a_k, \ldots)$. By (NC1) $U^{**k} \sim (u_k^*k, a_k, a_k, a_k, \ldots)$. 


Since \( U^*_{k_i+1} > U^*_{k_i} \), \( a^*_{k_i+1} > a^*_{k_i} \), \( \lim_{k \to \infty} a^*_k \) exists and equals \( a^* \). Then
\[
 f^* = f(u^*_1, a^*, a^*, \ldots). \]
There exists a convergent subsequence of
\[
 (u^*_{k_i} \mid k_i \in K_1) \]
converging to, say, \( u^* \). Then,
\[
 f^* = f(u^*, a^*, a^*, \ldots) \]
implying \( u^*_1 = u^* \). But \( u^*_k = u^*_1 + \epsilon \) for all \( k \) in \( K_1 \), therefore \( u^* > u^*_1 \).
This is a contradiction.

5. Equal Treatment for All Generations

A preference ordering which treats all generations equally is one which satisfies the condition:

\[(C) \quad U \sim U^t \quad \text{for all} \ U \in X \text{ and all} \ t = 1,2,\ldots .\]

This condition is satisfied, for example, by Ramsey's functional, which does not discount future utilities in the integral to be maximized. However, certain axiom sets can be shown to be inconsistent with this condition. The two axiom sets used in Section 4 to derive eventual impatience are clearly inconsistent with (C). For each of the metrics being examined in this paper a weakening of the axiom sets used in Section 4 can be permitted while preserving the inconsistency of the axiom set with (C).

Theorem: Let \( X \) be completely ordered by a preference ordering satisfying (91) and for which the product metric generates a natural topology. Then condition (C) must be violated.

Proof: Assume (C) is satisfied. By the existence theorem there exists a utility function \( f \). For \( j \geq i \) define \( a_{ij} = \underbrace{bbaa\ldots}_{\text{\( i-1 \) places}} \underbrace{bbaa\ldots}_{\text{\( j \)th place}} \)
for \( 1 \leq b > a \geq 0 \). (\( a_{ij} \) has \( i \) \( b \)'s, in the first \( i-1 \) places and in the \( j \)th place). For \( k,j \geq i \), (C) implies \( a_{ij} \sim a_{ik} \).
\[ \lim_{j \to \infty} a_{ij} = a_{i-1,i-1}. \text{ Therefore } f(a_{ij}) = f(a_{i-1,i-1}) \text{ for all } i,j. \]

\[ \lim_{j \to \infty} a_{ij} = (a_{\text{con}}). \text{ Therefore } f(a_{ij}) = f(a_{ij}) = f(a_{\text{con}}). \]

\[ \lim_{i \to \infty} a_{ii} = (b_{\text{con}}). \text{ Therefore } f(a_{ii}) = f(b_{\text{con}}) = f(a_{\text{con}}). \]

But \((b_{\text{con}}) > (a_{\text{con}}).\) This is a contradiction.

**Theorem:** Let \(X\) be completely ordered by a preference ordering satisfying (S2) and (NC1) and for which the sup metric generates a natural topology.

Then condition (C) is inconsistent with this preference ordering.

**Proof:** By the existence theorem there exists a utility function \(f.\) Assume (C) is satisfied. Define \(0_n = (11...100...)\) (\(n\)'s followed by all zeros).

\(0_{n+1} > 0_n.\) The set \((f(0_n))\) is monotonically increasing and has a limit \(f^*.\)

There exists a \(b\) such that \(f((b_{\text{con}})) = f^*,\) with \(b < 1,\) since by (C), for all \(n,(1_{\text{con}}) > (0111...) > 0_n.\) For all \(a < b\) there exists \(n\) such that \(0_{n+1} \succeq (a_{\text{con}}) \succeq 0_n.\) (C) and (NC1) imply \(0_{n+2} = (1,0_{n+1}) > (1,(a_{\text{con}})).\)

Therefore \((b_{\text{con}}) > (1,(a_{\text{con}}))\) for all \(a < b.\) But \(\lim_{a \to b} (1,(a_{\text{con}})) = (1,(b_{\text{con}}))\) and \((1,(b_{\text{con}})) > (b_{\text{con}}).\) This is a contradiction.

5. Preferences Without a Utility Function

Throughout the above sections, the preferences were assumed to be such that the existence of a utility function could be shown. This assumption shall now be dropped. Preferences over infinite horizon streams, \(U,\) will be assumed to have certain relationships with preferences over finite horizon streams, \(u_t.\) These relationships may be interpreted in
either of two ways. The interpretation may be made that the preferences, \( \succ \), over infinite streams, \( U \), are such that it is possible to construct preference orderings \( \succeq_t \) over finite sequences, \( l^U_t \), for all \( t \). Then these derived preferences, \( \succeq_t \), can be assumed to have certain properties and bear certain relationships with the basic preference ordering \( \succ \).

On the other hand, it may be interpreted to mean that persons, when facing decisions about an infinite future, do not directly know their own minds, but they do know their preferences, \( \succ_t \), over finite sequences, \( l^U_t \), and they have convictions about the relationships of preferences for the entire future and those for just parts of it.

It will be assumed that there are complete preference orderings \( \succeq \) over \( U \) and \( \succeq_t \) over \( l^U_t \), for all \( t \). \( \succeq_t \) will be assumed to be the same as \( \succeq \). Define \( (l^U_t)_{\text{rep}} \) to be the infinite stream \( (l^U_t, l^U_t, l^U_t, \ldots) \). There are three axioms that will be assumed about the relationships between the preference orderings.

(A1) For all \( t, l^U_t, l^U_t', U; \ l^U_t \succ l^U_t' \) implies \( (l^U_t, U) \succ (l^U_t', U) \)

(A2) For all \( t, l^U_t, l^U_t', l^U_t \succ l^U_t' \) implies \( (l^U_t, l^U_t') \succ (l^U_t', l^U_t') \)

(A3) There exists a \( u \) such that for all \( U, U' \)

\[ U \left\{ \begin{array}{c} \succ \\ \end{array} \right\} U' \text{ implies } (u, U) \left\{ \begin{array}{c} \succ \\ \end{array} \right\} (u, U'). \]

Axiom 1 is a noncomplementarity axiom and is equivalent to (NC2).

Axiom 2 has both noncomplementarity and persistence of preferences aspects.

Axiom 3 is a stationarity axiom, and is equivalent to the stationarity axiom of Koopmans.\(^6\) It assumes a preservation of preference orderings when the timing of all periods is moved one period into the future, while

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\(^6\)Koopmans, T.C., op. cit.
the present assumes a constant utility level for all streams.

Theorem: A set of complete preference orderings $\succ_t$, $t = 1, 2, \ldots$ satisfying (A1), (A2), and (A3) is inconsistent with (C).

Proof: Assume $u$ in (A3) is unequal to 0. (If it is equal to zero the proof will hold replacing $u$ and 0 by 0 and 1). Applying (A3) to 

\[ ((u, 0)_\text{rep}) \text{ and } (0, u)_\text{rep} \text{ we have } ((u, 0)_\text{rep}) \succ ((0, u)_\text{rep}) \text{ implies } (u, (u, 0)_\text{rep}) \succ (u, 0)_\text{rep}. \]

$u > 0$ implies $(u, (u, 0)_\text{rep}) \succ (0, u)_\text{rep}$.

Therefore $(u, 0)_\text{rep} \succ (0, u)_\text{rep}$, implying $(u, 0)_\text{rep} \succ (0, u)_\text{rep}$ and thus $(u, 0, U) \succ (0, u, U)$ contradicting (C).
CHAPTER II

TECHNICAL CHANGE AND ECONOMIC GROWTH

1. Introduction

There are two aspects of technical change that are essential for a description of the behavior of an economy over time. These are the rate of technical progress and the bias of the change. For a twice-differentiable production function, \( F(K,L,t) \), homogeneous of the first degree, with positive marginal products and a diminishing marginal rate of substitution everywhere, where \( K \) is capital; \( L \), labor; \( t \), time, these two aspects can be characterized by two indices: \(^1\)

\[
 T = \frac{F_t}{F} = \frac{KF_t + LF_t}{KF_K + LF_L}
\]

\[
 D = \frac{\partial(F_K/F_L)}{\partial t} = \frac{F_K}{F_L} - \frac{F_L}{F_L}
\]

Both indices, in general, are functions of both the capital-labor ratio, \( k = \frac{K}{L} \), and time.

The time profile of output, the real wage, and the rate of interest can be expressed in terms of these two indices and the rate of growth of the capital-labor ratio. In a two-sector model the conditions necessary to preserve equilibrium in the factor markets can be expressed in terms of these aspects of the production functions of the two sectors. These

\(^1\)These are the natural counterparts for a study of economic growth of the indices used to describe changes in unit costs in Salter, W.E.G., Productivity and Technical Change, Cambridge, 1960.
conditions can then be used to characterize those types of technical change which permit steady exponential growth. For both one and two sector models, the change, over time, in the factor-price frontier can be expressed in terms of these two indices.

2. Growth in a One Sector Model

In order to describe the time profile of various economic variables, two standard characteristics of a production function will also be used. These are the elasticity of substitution, \( \sigma \), and the share of capital \( \pi \); both of which, in general, are functions of both \( k \) and \( t \).

\[
\sigma = -\frac{d(K/L)}{d(F_K/F_L)} / \frac{K/L}{F_K/F_L} = \frac{F_K F_L}{F_K F_L}
\]

(3)

\[
\pi = \frac{F_K}{F}
\]

(4)

A dot over a variable will denote its time derivative.

The definitions of \( T \) and \( D \) can be solved for \( F_{Kt} \) and \( F_{Lt} \) giving:

\[
\frac{F_{Kt}}{F_K} = T + (1-\pi)D, \quad \frac{F_{Lt}}{F_L} = T - \pi D.
\]

(5)

From the definition of \( \sigma \) we have:

\[
\frac{\partial F_K}{\partial k} / F_K = -\frac{(1-\pi)}{\sigma_k}, \quad \frac{\partial F_L}{\partial k} / F_L = \frac{\pi}{\sigma_k}.
\]

(6)

Equations (5) and (6) can now be combined to give the rate of growth of the marginal products in terms of the two indices of technical change and the rate of growth of the capital-labor ratio.
From equation (7) can be derived the rate of growth of output and of the share of capital.

\[
\frac{\dot{F}_K}{F_K} = T + (1-\pi)D - \frac{(1-\tau)}{\sigma} \frac{k}{k}
\]

(7)

\[
\frac{\dot{F}_L}{F_L} = T - \tau D + \frac{\tau}{\sigma} \frac{k}{k}
\]

These relationships between the indices of technical change and the rates of growth of output and marginal products will hold for a one-sector model or for any sector of a multi-sector model of an economy.

It is natural now to examine the relationship between these indices of technical change and Hicks and Harrod neutral change.

3. Hicks Neutrality

Hicks neutral change is defined as the constancy over time of the ratio of marginal products at a given capital-labor ratio. Since \( D \) is the partial derivative of the log of this ratio with respect to time, Hicks neutrality is equivalent to \( D = 0 \).

Hicks neutral change is known to be equivalent to \( F(K,L,t) = A(t)G(K,L) \) or, writing \( F(K,L,t) = Lf(k,t) \), \( f(k,t) = A(t)g(k) \). This can be shown to be equivalent to \( T(k,t) \) being a function solely of time.
For this production function \( T = \frac{f_t}{f} = \frac{A'(t)}{A(t)} \).

Conversely if \( \frac{f_t}{f} = C(t) \) for some function \( C \), then \( f \) is decomposable and \( f(k,t) = A(t)g(k) \).

Note that in the case of Hicks neutrality \( T = \frac{A'}{A} \).

4. Harrod Neutrality

Harrod neutral change is defined as the constancy over time of the capital-output ratio when the marginal product of capital is constant. This is equivalent to the constancy of relative shares at a constant marginal product of capital.

This definition of neutrality can be stated in terms of \( T \) and \( D \). From (7) we see that the marginal product of capital is constant if

\[
\frac{k}{k} = \frac{\sigma T}{1-\pi} + \sigma D
\]

From (9), the constancy of relative shares is equivalent to

\[
\frac{k}{k} = -\frac{D}{1 - \frac{1}{\sigma}} \quad \text{for } \sigma \neq 1, \quad D = 0 \quad \text{for } \sigma = 1.
\]

Equating (10) and (11) gives the condition for Harrod neutral change:

\[
(1 - \frac{1}{\sigma})T + (1 - \pi)D = 0.
\]

From (12) and the equivalence of Hicks neutrality and \( D = 0 \), it is seen, as was shown by Uzawa, that both Hicks and Harrod neutrality hold if and only if \( \sigma = 1 \) for all \( k \), which is the condition for a Cobb-Douglas production function.

---

Harrod neutral change has been shown\(^2\) to be equivalent to \(f(k,t) = A(t)g\left(\frac{k}{A(t)}\right)\). The regularities of \(f\) permit the production function to be written \(f(k,t) = \phi(x,t)\) where \(x = \frac{k}{f(k,t)}\) and is the capital-output ratio. Then Harrod neutrality is equivalent to\(^2\)

\[
\phi(x,t) = A(t)\psi(x).
\]

The equivalence of (12) and (13) can be shown directly. Writing equation (12) in terms of \(F\) and its derivatives yields

\[
(14) \quad (1 - \frac{F_{KL}}{F})\left(\frac{F}{F}\right) + \frac{LF}{F}\left(\frac{KF}{F} - \frac{F_{Lt}}{F_L}\right) = 0
\]

Substituting \(f\) and its derivatives for \(F\), using \(F_{KL} = -\frac{K}{L}F_{KK}\), and simplifying yields

\[
(15) \quad (f - kf_k)f_{kt} + kf_t f_{kk} = 0
\]

This can now be written in terms of \(\phi\) and its derivatives by use of the following substitutions:

\[
\begin{align*}
f_k &= \frac{\phi}{\phi + x\phi_x} \\
f_t &= \frac{\phi_x}{\phi + x\phi_x} \\
f_{kk} &= \frac{\phi_{xx} - 2\phi_x^2}{(\phi + x\phi_x)^3}
\end{align*}
\]

\[
(16) \quad f_{kt} = \frac{x\phi_x^2 + \phi^2 \phi_{xt} + x\phi_x \phi_x - \phi \phi_x - x\phi_x \phi_x}{(\phi + x\phi_x)^3}
\]

Using these substitutions and simplifying results in:

\[
(17) \quad \frac{\phi_x t}{\phi_t} = \frac{\phi}{\phi}
\]
Integrating (17) with respect to $x$ gives:

\begin{equation}
\log \phi_t = \log \phi + c(t) \quad \text{or} \quad \frac{\phi_t}{\phi} = C(t).
\end{equation}

This implies (13).

From (13) we see that $\phi = A\psi'$, $\phi_t = A'\psi$, and $\phi_{xt} = A'\psi'$. Thus,

\begin{align*}
\frac{\phi_{xt}}{\phi_t} &= \frac{A'\psi'}{A'\psi} = \frac{\psi'}{\psi} = \frac{A\psi'}{A\psi} = \frac{x}{\phi}.
\end{align*}

Using the substitutions (16) we see that:

\begin{equation}
\frac{T}{1-\pi} = \frac{f_t}{f - kf_k} = \frac{\phi_t}{\phi}.
\end{equation}

This implies the equivalence of Harrod neutrality and $\frac{T}{1-\pi}$ being a function solely of time \(^3\) In the case of Harrod neutrality with $F(K,L,t) =$

\begin{equation}
G(K,A(t)L), \quad \frac{T}{1-\pi} = \frac{A'}{A}.
\end{equation}

5. Example

As an example consider the constant elasticity production function:

\begin{equation}
F(K,L,t) = (\beta k^{-\rho} + \alpha L^{-\rho})^{-\frac{1}{\rho}}
\end{equation}

Then

\begin{align*}
T &= -\frac{\beta k^{-\rho} + \alpha}{\rho(\beta k^{-\rho} + \alpha)} \\
D &= \frac{\beta}{\beta - \frac{\alpha}{\alpha}}
\end{align*}

\(^3\) It is not true that Harrod neutrality is equivalent to $\frac{D}{1-\sigma} = \frac{1}{\sigma}$ being a function solely of time, although Harrod neutrality implies the latter. The example in the next section reveals this, see footnote 4.
\[ \sigma = \frac{1}{1 + \rho} \]

\[ \pi = \frac{\beta k^{-\rho}}{\beta k^{-\rho} + \alpha} \]

For Hicks neutral change we have \( \frac{\beta}{\beta} = \frac{\alpha}{\alpha} \).

For Harrod neutral change (12) becomes:

\[ \beta (k^{-\rho} + \frac{\alpha}{\beta}) = 0, \]

which implies \( \beta = 0 \) for Harrod neutral change at all capital-labor ratios.

Thus the change is solely labor augmenting. \(^4\)

6. Growth in a Two Sector Model

This approach can now be used to examine the growth of a two sector economy. For the consumption good sector, let \( C = F(K_1, L_1, t) = L_1 f(k_1, t), f_k > 0, f_{kk} < 0 \). For the investment good sector,

let \( I = G(K_2, L_2, t) = L_2 g(k_2, t), g_k > 0, g_{kk} < 0 \), both \( f \) and \( g \) twice differentiable. Let \( p \) be the price of the investment good in terms of the consumption good. \(^5\)

Preservation of equilibrium in the factor markets requires that the ratios of marginal products for the two sectors remain equal. Since \[ \frac{1}{\sigma} \frac{k}{k} - D \] is the rate of change of this ratio, this condition is:

\[ \frac{4}{1 - \frac{1}{\sigma}} \]

is always constant over \( k \) for this production function while it need not be Harrod neutral. \(^4\)

Note that the equations in this section will hold for any depreciation assumption for which the rate of depreciation is independent of the use of the capital. \(^5\)
Preservation of factor market equilibrium also requires that the wage in consumption units be the same in both sectors. This implies:

\[
\frac{\dot{p}}{p} = T_1 - T_2 + D_2 (\pi_2 - \pi_1) + \frac{(\pi_2 - \pi_1)}{\sigma_2} \frac{k_2}{k_2}.
\]

From (8), (20), and (21), the remaining variables can be expressed:

\[
\frac{\dot{c}}{c} = \sigma_1 \pi_1 (D_1 - D_2) + \frac{\pi_1 \sigma_1}{\sigma_2} \frac{k_2}{k_2} + T_1 + \frac{L_1}{L_1}
\]

\[
\frac{\dot{i}}{i} = \pi_2 \frac{k_2}{k_2} + T_2 + \frac{L_2}{L_2}
\]

\[
\frac{\dot{p}}{p} = (\pi_2 - \pi_1)D_2 + \frac{\pi_1 - \pi_2}{\sigma_2} + \frac{\dot{p}}{\dot{p}} + T_1 + \frac{L_2}{L_2}
\]

For the gross saving rate, \( s = \frac{\dot{p}}{c + \dot{p}} \):

\[
\frac{s}{s} = (1-s) \left( \frac{\dot{p}}{p} + \frac{\dot{i}}{i} - \frac{\dot{c}}{c} \right) = (1-s) \left( \frac{\pi_1 (1-\sigma) - \pi_2 (1-\sigma_2)}{\sigma_2} \frac{k_2}{k_2} + (\pi_2 - \pi_1)D_2 - \pi_1 \pi_1 (D_1 - D_2) + \frac{L_2}{L_2} - \frac{L_1}{L_1} \right).
\]

Thus, the difference in the rates of technical change of the two sectors is reflected in the changing price ratio, implying that the change in the value of new investment (in consumption goods) and the change in the saving rate depend on the economy's decisions as to changes in \( k_2, L_1, \) and \( L_2 \).
and the biases of the change in the two production functions but not the difference in the rates of technical progress.

7. "Harrod" Neutrality in a Two Sector Model

In a two sector model, the natural counterpart of Harrod neutrality, which will be called "Harrod" neutrality, is the constancy of the capital-output ratio in value terms at a constant rate of interest. For the economy, the capital-output ratio depends on the capital-output ratios in the two sectors and, except when the two ratios are equal, on the relative outputs of the two sectors. Expressed in terms of the share of capital

\[ \pi = (1 - s)\pi_1 + s\pi_2 \]

where \( s \), as above, equals \( \frac{p_1}{c + p_1} \).

Thus if \( \pi_1 \) and \( \pi_2 \) are unequal, \( \pi \) is not uniquely related to \( r \), but depends on \( r \) and a decision variable, \( s \). Thus, as a definition of the character of technical change, "Harrod" neutrality cannot be applied to the entire economy, but must be applied to the two sectors separately.

From (7) and (20) the constancy of the rate of interest implies:

\[ \frac{k_1}{k_1} = \frac{\sigma_1\tau_2}{1 - \pi_2} + \sigma_1D_1 ; \quad \frac{k_2}{k_2} = \frac{\sigma_2\tau_2}{1 - \pi_2} + \sigma_2D_2 \]

From (9) the constancy of \( \pi_2 \) implies

\[ \frac{k_2}{k_2} = \frac{D_2}{1 - \frac{1}{\sigma_2}} \]

for \( \sigma_2 \neq 1 \); \( D_2 = 0 \) for \( \sigma_2 = 1 \).
Equating (27) and (28) gives the condition for "Harrod" neutrality in the investment good sector:

\[(29) \quad (1 - \frac{1}{\sigma_2})T_2 + (1 - \pi_2)D_2 = 0.\]

Since the rate of interest is the marginal product of capital in the investment good sector and since the capital-output ratio is the same in both value and physical units for this sector, equation (29) is the same as the condition for Harrod neutrality.

For the consumption good sector the price of capital appears both in the equation relating the marginal product of capital and the interest rate and in the expression for the capital-output ratio in value terms. This causes the condition for "Harrod" neutrality for this sector to depend on the nature of the change in both sectors. The constancy of \(\pi_1\) implies:

\[(30) \quad \frac{k_1}{k_1} = \frac{-D_1}{1 - \frac{1}{\sigma_1}}, \quad \text{for } \sigma_1 \neq 1, \quad D_1 = 0 \text{ for } \sigma_1 = 1.\]

Equating (27) and (30) gives the condition for "Harrod" neutrality in the consumption good sector:

\[(31) \quad (1 - \frac{1}{\sigma_1})T_2 + (1 - \pi_2)D_1 = 0.\]

As an example consider the constant elasticity production function:

\[C = (\beta_{1t} K_1^{-\rho_1} + \alpha_{1t} L_1^{-\rho_1})^{-\frac{1}{\rho_1}} - \frac{1}{\rho_1}\]

\[I = (\beta_{2t} K_2^{-\rho_2} + \alpha_{2t} L_2^{-\rho_2})^{-\frac{1}{\rho_2}} - \frac{1}{\rho_2}\]
As above, equation (29) implies $\beta_2 = 0$. This and (31) imply that

$$\frac{\beta_1}{\rho_2} \left( -\frac{\alpha_2}{\alpha_2} \right) + \frac{\beta_1}{\rho_1} - \frac{\alpha_1}{\alpha_1} = 0$$

is the other condition for "Harrod" neutrality in both sectors.

8. Kennedy's Theorem

Kennedy has shown the equivalence of Hicks neutrality in the consumption good sector and "Harrod" neutrality in that sector when there is no technical change in the investment good sector. The absence of technical change in the investment good sector implies that $T_2 = 0$. From equation (21) we see that $D_1 = 0$ is then the condition for both types of neutral change. It is also seen from (31) that, except for the Cobb-Douglas case, the presence of technical change in the investment good sector prevents the equivalence of the two types of change.

9. Exponential Growth in a One Sector Model

The equations derived above can now be used to describe the regularities of the production function which will permit equilibrium exponential growth at a constant rate. Assume that $L_t = L_0 e^{\gamma t}$, $K_t = K_0 e^{\omega t}$, $Y_t = Y_0 e^{gt}$, where $Y = F(K,L,t)$, depreciation equals $\delta K_t^\gamma$ and the rates of growth are constant. There are two equations which must be satisfied by this system, the equation relating the growth of output to the growth of inputs and the equality of savings and investment:

$$\frac{Y}{L} = T + \pi \frac{k}{k} + \frac{L}{L} \quad \text{or} \quad g = T + \pi (\rho - \gamma) + \gamma ;$$

---


7Any depreciation assumption independent of the use of the capital stock and not affecting the rate of growth of capital on an exponential path could be used here; the one-hoss shay for example.
(33) \[ K + \delta K = sY \] implying for exponential growth

\[ \frac{K}{s} = \frac{Y}{Y} \quad \text{or} \quad g = \frac{\rho - s}{s} \]

Thus, \( \frac{s}{s} \) is constant and therefore less than or equal to zero. Equating (32) and (33) gives:

(34) \[ (\rho - \gamma) = \frac{T}{1 - \pi} + \frac{s/s}{1 - \pi} \]

For growth to be feasible it is necessary for the implied savings rate to be less than or equal to one. This implies that the capital-output ratio must be less than or equal to \( \frac{1}{\rho + \delta} \). Therefore, for a given production function at \( t = 0 \), we are only interested in the technical change in the relevant part of the domain of the function. However, it will be assumed for mathematical simplicity that equation (34) holds for the entire domain of \( F \).

Consider first the case where \( s \) is constant. We shall also assume, at first, that the rate of growth of output is independent of the initial capital-labor ratio. This assumption is used in the derivation of the Golden Rule. Then, \((\rho - \gamma)\), and thus \( \frac{T}{1 - \pi} \), is not only constant along each growth path, but also for all initial capital-labor ratios. Thus, \( \frac{T}{1 - \pi} \) is independent of \( t \) and \( k \), implying that for this type of growth the production function must be Harrod neutral, \( F(K,L,t) = G(K,A(t)L) \), and that \( \frac{A'}{A} \) is constant and equal to \((\rho - \gamma)\). Thus, for exponential growth at a constant rate which is independent of the initial conditions, with a constant savings rate, we must have Harrod neutrality. This implies that along the growth path the interest rate and the capital-output ratio are constant.
While maintaining the assumption of a constant savings rate, the assumption of the independence of the growth rate from the initial conditions is dropped. As before, it is assumed that (34) holds for all initial capital-output ratios, not just those for which growth is feasible.

These assumptions can be shown to imply a production function which cannot satisfy all the usual assumptions about its derivatives for all time, except in the special case where the rate of growth is independent of the initial conditions, the case considered above.

The constancy of the growth rate on the growth path implies that \( \frac{T}{1 - \pi} \) is constant on the growth path. From (33) it is seen that the capital-output ratio, \( x \), is also constant along the path. As in section 4 define \( \phi(x,t) = f(k,t) \). Equation (19) states \( \frac{T}{1 - \pi} = \frac{\phi_t}{\phi} \). The constancy of \( \frac{\phi_t}{\phi} \) on the growth path is equivalent to:

\[
(35) \quad \frac{\partial(\phi_t/\phi)}{\partial t} = \frac{\phi_{tt} - \phi^2}{\phi^2} = 0 \quad \text{or} \quad \frac{\phi_{tt}}{\phi_t} = \frac{\phi_t}{\phi} .
\]

Integrating twice with respect to time gives:

\[
(36) \quad y = f(k,t) = \phi(x,t) = A(x)e^{tB(x)} .
\]

The rate of growth of output per laborer, \( \frac{\dot{y}}{y} \), equals \( B(x) \). The condition for growth independent of the initial conditions is the constancy of \( B \), which reduces (36) to the special case of Harrod neutrality.

The properties of this function can now be examined. Taking the partial derivatives of \( \phi(x,t) \), we have:
\[ \phi_x = e^{tB} (A' + tAB') \]
\[ \phi_t = ABe^{tB} \]

(37)

\[ \phi_{xx} = e^{tB} (A'' + tAB'' + 2tA'B' + t^2A'B'^2) \]
\[ \phi + x\phi_x = e^{tB} (A + xA' + xtAB') . \]

The partial derivatives of \( f(k,t) \) can be calculated from (16) and (37):

\[ f_k = \frac{A' + tAB'}{A + xA' + xtAB'} \]
\[ f_t = e^{tB} \frac{A^2}{A + xA' + xtAB'} \]

(38)

\[ f - kf_k = e^{tB} \frac{A^2}{(A + xA' + xtAB')} \]
\[ f_{kk} = (-e^{-tB} (AA'' + tA^2B'' - 2A'B' - 2tAA'B' - t^2A^2B'^2)). \]

For (36) to satisfy the conditions of a production function,(38) must satisfy \( f_k > 0, f_t \geq 0, f - kf_k > 0, f_{kk} < 0 \). From the case \( t = 0 \), we see that \( A \) must satisfy the conditions normally satisfied by \( \phi, A \geq 0, A' > 0, 2A'^2 > AA'' \). However, for \( B' \neq 0 \) there exists a \( t^8 \) such that \( f_k \) will equal zero, violating the conditions for a production function.

Thus, for steady exponential growth with a constant savings rate and a twice-differentiable, homogeneous of the first degree, production function with positive marginal products and diminishing rate of substitution everywhere, the rate of growth must be independent of the initial conditions.

---

8This assumes that time can assume any value on the real line, which is a natural assumption to make with exponential growth of labor, for this implies a positive labor force for all \( t \).
This can be seen directly by considering the growth paths in (log (Y_t), t) space. Each growth path is a straight line. If two paths have different rates of growth, they have non-parallel straight lines, which, therefore, must intersect for some t. Since the capital-output ratio is constant on a growth path and different for different paths, at the intersection point the production function has the same output for different capital inputs while the labor input is the same. This contradicts the positive marginal productivity of capital.

An example of a production function which permits exponential growth with a changing savings rate is \( F(K,L,t) = G(B(t)K,A(t)L) \). For this function \( T = \pi \frac{B'}{B} + (1 - \pi) \frac{A'}{A} \), \( D = (1 - \frac{1}{\sigma})(\frac{B'}{B} - \frac{A'}{A}) \). If \( s = \frac{-B'}{B} \), (34) becomes \( (\rho - \gamma) = \frac{A'}{A} - \frac{B'}{B} \). Thus both arguments of G would grow at the same rate, which is also the rate of growth of output. For this production function relative shares are constant on a growth path.

10. Exponential Growth in a Two Sector Model

For a two-sector model to be able to grow exponentially, in addition to the equation relating the growth of the capital stock and the growth of output of the investment good sector, it is necessary to satisfy equation (20) relating the rates of growth of the capital-labor ratios of the two sectors. We will first consider the case where capital and labor grow at the same rates in both sectors and then relax these assumptions.

Assume \( \frac{L_1}{L_1} = \frac{L_2}{L_2} = \gamma \), \( \frac{K_1}{K_1} = \frac{K_2}{K_2} = \rho \), and depreciation equals \( \delta \) times the capital stock.\(^9\)

\(^9\) Any depreciation function independent of the use of the capital and not affecting the rate of growth of capital on the growth path could be used here; the one-hoss shay assumption for example.
Equation (20) becomes:

\[(\rho - \gamma) = \frac{\sigma_1 (D_1 - D_2)}{1 - \frac{\sigma_1}{\sigma_2}} \quad \text{for } \sigma_1 \neq \sigma_2; \quad D_1 = D_2 \quad \text{for } \sigma_1 = \sigma_2\]  

Equation (34) becomes:

\[(\rho - \gamma) = \frac{T_2}{1 - \pi_2} \]  

As before, it is assumed that these equations hold for all initial capital-labor ratios, not just those with a sufficiently small capital-output ratio to permit growth.

Equation (40) is the same as equation (34) for the case of a constant savings ratio. Therefore the results derived there about the nature of the production function, hold here for the production function of the investment good sector.

With \((\rho - \gamma)\) constant for all initial conditions, the investment good production function is Harrod neutral \(G(K,L,t) = G(K,A(t),L)\). This permits the use of equation (29), the equation for Harrod neutrality, which may be solved for \(D_2\) in terms of \(T_2\). Equating (39) and (40), using this substitution, gives the equation for "Harrod" neutrality in the consumption good sector, equation (31). For an economy growing along such an exponential path, the interest rate is constant (as in the last section) and, from "Harrod" neutrality, relative shares are constant in both sectors.

The rate of growth of the output of consumption goods can be derived from (22):
\[
\frac{\dot{C}}{C} = T_1 + (\rho - \gamma)\tau_1 + \gamma.
\]

(For \(\frac{\dot{C}}{C}\) to equal \(\frac{I}{I}\), this implies Harrod neutrality, \(F(K,L,t) = \hat{F}(K,B(t)L)\), with \(\frac{A'}{A} = \frac{B'}{B}\).)

The change in the price ratio can be derived from (21):

\[
\frac{\dot{p}}{p} = T_1 - (1 - \pi_1)(\rho - \gamma) = T_1 - (1 - \pi_1)\frac{T_2}{1-\pi_2} = (1 - \pi_1)(\frac{T_1}{1-\pi_1} - \frac{T_2}{1-\pi_2}).
\]

Thus, the sign of the price change depends on the relative rates of technical change, \(\frac{T_1}{1-\pi_1} - \frac{T_2}{1-\pi_2}\). Thus, the amount of capital which may be purchased by the sacrifice of a unit of consumption changes over time with the changing input requirements for unit production.

This equation can also be used to derive the consumption stream which may be obtained by sacrificing one unit of consumption, at \(t = 0\), for example. One unit of consumption purchases \(\frac{1}{p_0}\) units of capital which gives a stream of \((r - \delta)\frac{1}{p_0}\) units of capital. This is worth \((p_t/p_0)(r - \delta)\) units of consumption at time \(t\) for all \(t \geq 0\). In the general case \(p_t\) changes irregularly over time. The instantaneous rate of return on a unit of consumption is \((r - \delta) + \frac{p_t}{p}\).

Equations (40) and (41) imply that the gross savings rate, \(s = \frac{p_1}{C+p}\), is constant. This constancy, and those of \(\pi_1\) and \(\pi_2\), imply constant relative shares for the entire economy.

From (7) the growth of the real wage, \(w\), satisfies:

\[
\frac{\dot{w}}{w} = T_1 + \pi_1(\rho - \gamma).
\]
Thus, the real wage rises at the same rate as the average product of labor in the consumption sector, which is also the rate of growth of consumption per head.

While still assuming \( \frac{K_1}{K_2} = \frac{L_1}{L_2} = \rho \), we shall now assume \( \frac{L_1}{L_2} = \gamma_1 \) and \( \frac{L_2}{L_1} = \gamma_2 \), with \( \gamma_1 \) and \( \gamma_2 \) not necessarily equal. Equations (39) and (40) now become:

\[
\begin{align*}
(44) \quad (\rho - \gamma_1) &= \sigma_1 (D_1 - D_2) + \frac{\sigma_1}{\sigma_2} (\rho - \gamma_2) \\
(45) \quad (\rho - \gamma_2) &= \frac{T_2}{1 - \pi_2}
\end{align*}
\]

If these two equations hold for all initial capital-labor ratios, equation (45) and the constancy of the growth rate imply Harrod neutrality in the investment good sector. For this case, the growth of the other variables can be derived.

Equating (44) and (45), using the equation for Harrod neutrality gives:

\[
(46) \quad D_1 = \frac{1}{\sigma_1} (\rho - \gamma_1) - (\rho - \gamma_2)
\]

For "Harrod" neutrality \( D_1 \) would have to equal \( \frac{1}{\sigma_1} (\rho - \gamma_2) - (\rho - \gamma_2) \).

Thus, for \( \gamma_1 \neq \gamma_2 \), there is not "Harrod" neutrality in the consumption good sector.

As before, the interest rate and relative shares in the investment good sector are constant. The share of capital in the consumption good sector satisfies:

\[
(47) \quad \frac{\pi_1}{\pi_1} = (1 - \pi_1)(\gamma_2 - \gamma_1)
\]
Thus, the sign of the change depends on the relative rates of growth of labor inputs.

The equation for the growth in the output of consumption goods becomes:

\[
\frac{\dot{C}}{C} = T_1 + (\rho - \gamma_1)\pi_1 + \gamma_1.
\]

(For \(\frac{C}{C} = \frac{I}{I}\), this implies Harrod neutrality, \(F(K, L, t) = \hat{F}(K, A(t), L)\) with \(A' + \gamma_1 = \frac{B'}{B} + \gamma_2\).)

The price change equation remains essentially the same:

\[
\frac{\dot{P}}{P} = T_1 - (1 - \pi_1)(\rho - \gamma_2) = T_1 - \frac{(1 - \pi_1)}{(1 - \pi_2)} T_2.
\]

The rate of change of the gross savings rate can be derived from (25), (48), and (49):

\[
\frac{\dot{s}}{s} = (1 - s)(1 - \pi_1)(\gamma_2 - \gamma_1).
\]

Since the price ratio reflects the differences in technical change, the sign of the change in the savings rate depends on the difference in the growth of labor inputs.

Let \(\tau\) be the share of capital for the entire economy. From (26), (47), and (50),

\[
\frac{\tau}{\pi} = (1 - s)(1 - \pi_1)(\gamma_2 - \gamma_1).
\]

Thus the change in relative shares depends on the relative rates of growth, reflecting the same dependence as \(\pi_1\) and \(s\).
The equation for the growth of the real wage becomes:

\[ \frac{w}{w} = T_1 + \pi_1 (\rho - \gamma_2). \]  

(52)

Thus, the average and marginal products in the consumption good sector no longer grow at the same rate.

If the rates of growth of the capital inputs are different for the two sectors, \( \frac{I}{I} \neq \frac{K_2}{K_2} \). Let \( \rho = \frac{I}{I} \), \( \rho_2 = \frac{K_2}{K_2} \), \( \gamma_2 = \frac{L_2}{L_2} \). Then the relation between inputs and outputs in the investment good sector is:

\[ (\rho_2 - \gamma_2) = \frac{T_2}{1 - \pi_2} + \frac{\rho_2 - \rho}{1 - \pi_2}. \]

(53)

This is similar to the expression for a one sector model with a changing savings rate, (34), with \( \rho_2 - \rho = \frac{s}{s} \). Remarks there on the nature of the production function will hold here for the investment good sector.

Summing up the results of the two sections on exponential growth, we have seen that exponential growth at a steady rate in a one-sector growth model with a constant savings rate implies a Harrod neutral production function and a rate of growth independent of the initial conditions.

Exponential growth of labor and capital inputs in both sectors of a two sector model with capital growing at the same rate in both sectors implies Harrod neutrality in the investment good sector. If the two labor inputs also grow at the same rate, "Harrod" neutrality in the consumption good sector, constant relative shares, and a constant savings rate are implied. If the labor inputs grow at different rates, there is not "Harrod" neutrality and the savings rate and relative shares are not constant.
11. The Factor-Price Frontier in a One Sector Model

The factor-price frontier (FFP) gives the locus of real wage-interest rate pairs at which equilibrium is sustainable. To determine the frontier, in a one-sector model, for a production function \( F(K,L) = Lf(k) \), \( f_k > 0 \), \( f_{kk} < 0 \), since \( f_k \) is monotone, for any \( r \) which can be achieved, \( k = f_k^{-1}(r) \) and \( w = f - kf_k \). Combining these

\[
(54) \quad w = f(f_k^{-1}(r)) - rf_k^{-1}(r)
\]

\[
(55) \quad \frac{dw}{dr} = -k
\]

\[
(56) \quad \frac{d^2w}{dr^2} = \frac{-dk}{dr} = \frac{-1}{f_{kk}}
\]

Thus the frontier, for a one-sector model, has a negative slope and is convex from below.

It is clear from the equation determining the frontier that the frontier is independent of the rate of growth of the labor force. However, the level of consumption per capita associated with a point on the frontier depends on the rate of growth of the labor force. With an increasing labor force, the stock of capital must be enlarged to preserve the constancy of the capital-labor ratio. Consumption per capita, \( c \), therefore equals \( f - \gamma k \) where \( \gamma \) is the rate of growth of the labor force. From this equation it is seen that for a given rate of population growth maximal consumption per capita occurs when the Golden Rule is satisfied, or the rate of interest equals the rate of growth. The Golden Rule also implies that consumption equals the share of labor at the optimum. Thus
the factor-price frontier can be interpreted to give the locus of maximal consumption per head for a given rate of growth of the labor force with the rate of growth of labor replacing the rate of interest and consumption per head replacing wages on the axes.

12. The FPF and Technical Change

Introducing technical change into the production function causes the factor-price frontier to shift over time. This movement can be described in terms of T and D in two ways, along lines of constant interest rate and along radii through the origin.

To examine the first, solve $F_K = 0$ for $k$, getting:

\[ \frac{k}{k} = \frac{\sigma T}{1 - \pi} + \sigma D. \]

Then the change in the real wage along this line can be determined by substituting this into the equation for $F_L$.

\[ \frac{w}{w} = \frac{T}{1 - \pi}. \]

Thus the factor-price frontier will rise proportionally if and only if $\frac{T}{1 - \pi}$ is independent of $k$, which is equivalent to Harrod neutral technical change.

To examine the movement of the frontier along a radius, the equation $\frac{F_K}{F_K} = \frac{F_L}{F_L}$ can be solved for $k$:

\[ \frac{k}{k} = \sigma D. \]

10 This interpretation has been made by C. C. von Weizsäcker, "A Note on Professor Samuelson's Factor-price Frontier," unpublished.
\[ \frac{AE}{AC} = \frac{BF}{BD} \]

HARROD NEUTRALITY
HICKS NEUTRALITY

\[
\frac{OA}{OC} = \frac{OB}{OD}
\]
Substituting this in the expression for \( \frac{F_L}{F_L} \) gives the rate of movement of the frontier in this direction

\[
\frac{\dot{L}}{\dot{w}} = \frac{w}{r} = T.
\]

Thus the factor price frontier advances proportionally along radii if and only if \( T \) is independent of \( k \) which is equivalent to Hicks neutral technical change.

12. FPF in a Two-Sector Model

To describe the frontier in a two-sector model with production functions \( C = F(K_1, L_1) = L_1 f(k_1) \) and \( I = G(K_2, L_2) = L_2 g(k_2) \), \( f_k, g_k > 0 \), \( f_{kk}, g_{kk} < 0 \), it is necessary to determine both \( k_1 \) and \( k_2 \) as functions of \( r \). Let \( \phi_1(k_1) \) be the ratio of marginal products in the two sectors,

\[
\phi_1(k_1) = \frac{f_k}{f - k_1 f_k}, \quad \phi_2(k_2) = \frac{g_k}{g - k_2 g_k}.
\]

Then the equality of marginal rates of substitution implies:

\[
(61) \quad k_1 = \phi_1^{-1}(\phi_2(g_k(r))), \quad k_2 = g_k^{-1}(r).
\]

Define \( \psi(k_1) = f(k_1) - k_1 f(k_1) = w \). Then

\[
(62) \quad w = \psi(\phi_1^{-1}(\phi_2(g_k^{-1}(r)))) .
\]

\[
(63) \quad \frac{dw}{dr} = -\frac{k_1^2 f_k^2 g}{g_k^2 f_k}
\]

\[
(64) \quad \frac{d^2 w}{dr^2} = \frac{-f_k^2}{f^2 g_k^4} \left[ \frac{f_k^2 g^2 (f - k_1 f_k)}{f_k^2 g_k} + \frac{k_1 f g^3}{g_k} + 2fg_k (k_2 - k_1) \right]
\]
Thus the frontier is always downward sloping but its convexity has not been shown. However, the FPF is convex from below when the consumption good sector is more capital intensive than the investment good sector.

As in the one-sector model the frontier is independent of the rate of growth of the labor force, while per capita consumption depends on the rate of growth. For a labor force growing at a rate \( \gamma \), the investment condition is that \( L_2 g(k_2) = \gamma (L_2 k_2 + L_1 k_1) \). Solving for per capita consumption gives:

\[
(65) \quad c = \frac{g(k_2) - \gamma k_2}{g(k_2) - \gamma k_2 + \gamma k_1} f(k_1).
\]

Maximizing per capita consumption for a given rate of population growth again gives the Golden Rule, permitting the same interpretation of the FPF in the two-sector model as in the one-sector model.

13. The FPF and Technical Change

The movement of the frontier over time along lines of constant interest rate can be determined as in the one sector model. Setting \( r = 0 \) gives:

\[
(66) \quad \frac{\dot{k}_2}{k_2} = \frac{\sigma_2 T_2}{1 - \pi_2} + \sigma_2 D_2
\]

Substituting this in equation (20) the equation for the preservation of equality of marginal rates of substitution gives:

\[
(67) \quad \frac{\dot{k}_1}{k_1} = \frac{\sigma_1 T_2}{1 - \pi_2} + \sigma_1 D_1.
\]
This implies that along a line parallel to the wage axis:

\[ \frac{w}{w} = T_1 + \frac{T_2}{1 - \pi_2} \]  

A sufficient condition for a proportional rise in the frontier is thus seen to be Hicks neutral change in the consumption good sector and Harrod neutral change in the investment good sector.

To describe the movement along radii, equation of \( \frac{F_L}{F_L} \) and \( \frac{G_K}{G_K} \) and substitution for \( k_1 \) from (20) gives:

\[ \frac{k_2}{k_1} = \frac{\sigma_2(T_2 - T_1 + D_2(1 + \pi_1 - \pi_2))}{1 + \pi_1 - \pi_2} \] 

Substituting this in the equation for \( \frac{G_K}{G_K} \) gives:

\[ \frac{r}{w} = \frac{\pi_1 T_2 + (1 - \pi_2)T_1}{1 + \pi_1 - \pi_2} \]

The form of this expression suggests that sufficient conditions for proportional movement of the frontier would involve relationships between the two production functions.
CHAPTER III

OPTIMAL GROWTH IN A MODEL OF SRINIVASAN

1. Introduction

Srinivasan has presented\(^1\) a two-sector growth model with fixed coefficients, one capital good in the investment good sector and many capital goods in the consumption good sector. This model was shown to have a "terminal path" on which only one type of capital good is used to produce consumption goods, the capital-labor ratio is constant in each sector, and per capita consumption is constant and equal to the maximum sustainable level. The optimal approach to this terminal path was defined as the approach which reached the terminal path in the shortest time. Srinivasan conjectured that the optimal approach was achieved by building at first just the capital good for the investment good sector and then just the capital good used in the consumption good sector on the terminal path. The time for the change from production of one capital good to the other was chosen so that the stock of capital goods in the investment good sector would depreciate to the level required on the terminal path, reaching that level simultaneously with the achievement of the terminal path level by the capital good for the consumption sector. It is shown below that the optimal approach takes this form or involves building more capital goods for the investment sector than will

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be needed at the time of achieving the terminal path, depending on the size of the coefficients of the model. In other words, if the capital good in the investment sector is sufficiently productive the time lost building additional capital for the investment good sector is more than recovered by the more rapid rate of construction of capital for the consumption good sector.

2. The Model and the Terminal Path

Let $\lambda_i$ be the rate of input of labor services needed to operate a machine of type $i$, $i = 0,1,2,\ldots$; $\gamma_0$, the rate of output of machines (of any type) per machine of type 0 in use (type 0, then, is the machine in the investment good sector); $\gamma_i$, the rate of output of consumer good per machine of type $i$ in use, $i = 1,2,\ldots$. It is assumed that labor grows exponentially at a rate $\theta$ from a unit level at time zero and capital depreciates exponentially at a rate $\delta$. Let $K_i(t)$, $K_i'(t)$, and $X_i(t)$ be the stock of machines in existence, the stock of machines in use, and the gross addition to the stock of machines of type $i$ at time $t$. Let $C_t$ be consumption at time $t$; $c_t$, consumption per worker at time $t$.

It is assumed that the coefficients satisfy the following:

$$\frac{\lambda_1}{\gamma_1} > \frac{\lambda_2}{\gamma_2} > \frac{\lambda_3}{\gamma_3} > \ldots > \frac{\lambda_i}{\gamma_i} > \frac{\lambda_{i+1}}{\gamma_{i+1}} > \ldots$$

$$\gamma_1 > \gamma_2 > \gamma_3 > \ldots > \gamma_i > \gamma_{i+1} > \ldots$$

$$\frac{\gamma_2 - \gamma_3}{\lambda_2 \gamma_2 - \lambda_2 \gamma_1} < \frac{\gamma_3 - \gamma_4}{\lambda_3 \gamma_3 - \lambda_3 \gamma_2} < \frac{\gamma_i - \gamma_{i+1}}{\lambda_i \gamma_i + 1 - \lambda_i \gamma_{i+1}} < \ldots$$
The terminal path is characterized by:

\[(1 + \theta)c_{t+1} = \frac{\gamma_j(\gamma_0 - \theta - \delta)}{\lambda_j(\gamma_0 - \theta - \delta) + \lambda_0(\theta + \delta)}\]

\[K^E_0(t) = K'_0(t) = \frac{(\theta + \delta)e^{\theta t}}{\lambda_j(\gamma_0 - \theta - \delta) + \lambda_0(\theta + \delta)}\]

\[K^E_j(t) = K'_j(t) = \frac{(\gamma_0 - \theta - \delta)e^{\theta t}}{\lambda_j(\gamma_0 - \theta - \delta) + \lambda_0(\theta + \delta)}\]

\[K'_i(t) = X'_i(t) = 0 \quad \text{for } i \neq 0, j\]

where \(j\) satisfies:

\[\frac{\gamma_{j-1} - \gamma_j}{\lambda_{j-1} \gamma_j - \lambda_j \gamma_{j-1}} < \frac{(\gamma_0 - \theta - \delta)}{\lambda_0(\theta + \delta)} < \frac{\gamma_j - \gamma_{j+1}}{\lambda_j \gamma_{j+1} - \lambda_{j+1} \gamma_j}.

Thus the terminal path is achieved whenever \(K^E_0(t)\) and \(K^E_j(t)\) are greater than or equal to their equilibrium levels \(K^E_0(t)\), \(K^E_j(t)\).

3. Approaches to the Terminal Path

Since only type 0 capital can be used to produce machines, the optimal approach is solely concerned with production in the investment good sector and it is assumed that there is no production in the

\[\text{It is assumed that } \lambda_j > \lambda_0 \text{ for this } j.\]
consumption good sector. It is assumed that the stocks of both \(0\) and \(j\) type machines are below equilibrium level at time zero, the time at which the optimal approach will be started.

On the approach described by Srinivasan, just type \(0\) machines are constructed from time \(0\) to time \(\hat{t}_1\), while just type \(j\) machines are constructed from \(\hat{t}_1\) to \(\hat{t}_2\) at which time the optimal path is exactly achieved. Algebraically, this can be expressed as:

\[
\begin{align*}
K_0 (t) &= K_0 (0) e^{(\gamma_0 - \delta)t} \quad 0 \leq t \leq \hat{t}_1 \\
K_j (t) &= K_j (0) e^{-\delta t} \\
K_0 (t) &= K_0 (0) e^{[(\gamma_0 - \delta)\hat{t}_1 - \delta(t - \hat{t}_1)]} \quad \hat{t}_1 \leq t \leq \hat{t}_2 \\
K_j (t) &= [\gamma_0 (t - \hat{t}_1) K_0 (0) e^{(\gamma_0 - \delta)\hat{t}_1} + K_j (0) e^{-\delta \hat{t}_1}] e^{-\delta (t - \hat{t}_1)}
\end{align*}
\]

with \(\hat{t}_1\) and \(\hat{t}_2\) determined by the equations:

\[
\begin{align*}
K_0 (\hat{t}_1) &= K_0^F (\hat{t}_2) \quad , \quad K_j (\hat{t}_2) = K_j^F (\hat{t}_2).
\end{align*}
\]

Define \(t_0\) as the solution to the equation:

\[
\lambda_0 K_0 (0) e^{(\gamma_0 - \delta)t_0} = e^{\theta t_0}
\]

\[3\text{Changes in a path made on sets of measure zero will clearly not affect the optimality of an approach, so it is assumed that any machine produced is produced in positive quantity by each machine producing it.}\]
This solution, then, is the time when, if just type 0 machines are built, there will be full employment of labor in the investment good sector.

4. The Nature of the Optimal Approach

The optimal approach will be determined by first solving the problem of achieving the terminal path level of type j machines with the capital input constraint (but not the labor constraint). This will be called the unconstrained problem. This solution will take the form of building type 0 machines until a time \( \tilde{t}_1 \) and then type j machines from \( \tilde{t}_1 \) until time \( \tilde{t}_2 \). The nature of the solution will depend on the size of \( \tilde{t}_1 \) relative to \( \hat{t}_1 \) and \( t_0 \). If \( \tilde{t}_1 \leq \hat{t}_1 \), the solution described by Srinivasan is optimal. If \( \hat{t}_1 \leq \tilde{t}_1 \leq t_0 \), the solution to the unconstrained problem is feasible and therefore the optimal approach. If \( t_0 \leq \tilde{t}_1 \), labor becomes a scarce resource at time \( t_0 \) and the solution will involve constructing just type 0 capital until \( t_0 \), both types from \( t_0 \) until \( t_3 \), and then just type j from \( t_3 \) until \( t_2 \), where \( t_3 \) is selected by a further optimization.

First the nature of the solution to both the problem and its unconstrained version will be discussed. This will limit the set of approaches from which the optimum will be chosen.

It is seen from the nature of the problem that if one approach has strictly more of both 0 and j types of capital than a second for all time between some \( t \) less than the time the second achieves the terminal path and that time, then the first is strictly better than the second. This implies that the set of approaches on which only machines of types 0 and j are produced and on which at no time are both labor and type 0 machines
unemployed contains an approach strictly better than any approach not in this set.

From the nature of the technology it can be seen that if machines of types \(i_1\) and \(i_2\), \(i_1, i_2 \neq 0\) are being produced during a finite interval of time, the total number of machines built during the period and still available at its end is independent of the relative numbers of either type or the timing in which they are constructed.\(^4\) It follows that any final stock of the two types can be reproduced by producing first just type \(i_1\) and then just type \(i_2\). It can also be seen that the stock of type \(i_1\) in the path after the retiming is always at least as large as before the retiming and strictly larger from the time at which type \(i_2\) is constructed on the original path until (but not including) the time at which the last \(i_1\) machine is built on the original path.

If one of the types in this retiming is type 0, then the retiming can be done if and only if type 0 is produced first. It follows, therefore, that for paths on which type \(j\) machines are built before type 0 machines and also before \(t_0\) (that is, while there is unemployed labor) there is a path on which type 0 machines are built first and then type \(j\) machines which is better. This holds for the unconstrained problem, without the condition about \(t_0\).

These remarks taken together imply that for any approach not in the set characterized by the construction of just type 0 machines until a time, \(t_1\), and then just type \(j\) machines until \(t_2\), the time at which the terminal paths is achieved, there is an approach in this set which is better for the unconstrained problem.

\(^4\)Changing the "timing" of construction of the machines means altering the relative outputs of the two types of machines, not the time path of total output.
5. The Unconstrained Problem

The remarks of the previous section imply that, writing \( t_2 \) as a function of \( t_1 \), the unconstrained problem can be written as:

\[
\text{Minimize } t_2(t_1) \text{ subject to:}
\]

\[
(1) \quad K_j(0)e^{-\delta t_2} + \gamma_0(t_2 - t_1)K_0(0)e^{(\gamma_0 - \delta) t_1} e^{-\delta(t_2 - t_1)} = K^E_j(0)e^{\theta t_2}.
\]

Taking the derivative of \( t_2 \) with respect to \( t_1 \) gives:

\[
\frac{dt_2}{dt_1} = \frac{\gamma_0^2 (t_2 - t_1) K_j(0)e^{\gamma_0 t_1} - \gamma_0 K_0(0)e^{\gamma_0 t_1}}{(\theta + \delta)K^E_j(0)e^{(\theta + \delta) t_2} - \gamma_0 K_0(0)e^{\gamma_0 t_1}}
\]

Equating this with zero yields the condition for the optimum \( \tilde{t} \) (which will be denoted by \( \tilde{t} \)):

\[
(3) \quad \tilde{t}_2 - \tilde{t}_1 = \frac{1}{\gamma_0}.
\]

Substituting for the optimum in equation (1) yields an implicit equation for \( \tilde{t}_1 \):

\[
(4) \quad K_j(0) + K_0(0)e^{\gamma_0 \tilde{t}_1} - K^E_j(0)e^{(\theta + \delta)(\frac{1}{\gamma_0} + \tilde{t}_1)} = 0.
\]

Define \( S_1 \) as the set \( \{t_1 | t_1 \geq \hat{t}_1, t_2(t_1) \leq \hat{t}_2\} \).

The denominator of the expression for \( \frac{dt_2}{dt_1} \) is negative for all \( t \) in \( S_1 \) since:

\[
\frac{(\theta + \delta)K^E_j(0)e^{(\theta + \delta)t_2}}{\gamma_0 K_0(0)e^{\gamma_0 t_1}} = \frac{(\gamma_0 - \delta)K_j(0)e^{(\theta + \delta) t_2}}{\gamma_0 K_0(0)e^{\gamma_0 t_1}} = \frac{\gamma_0 K_0(t_2)}{\gamma_0 K_0(t_2)} = \frac{\gamma_0 K^E_j(0)e^{(\theta + \delta)t_2}}{\gamma_0 K_0(0)e^{\gamma_0 t_1}}
\]
Evaluating the second derivative of \( t_2 \) at \( \hat{t}_1 \):

\[
\text{sign}\left(\frac{d^2 t_2}{dt_1^2}\right) = \text{sign}\left(-D \gamma_0^2 e^{\gamma_0 \hat{t}_1}\right)
\]

where \( D \) is the denominator of \( \frac{dt_2}{dt_1} \).

Thus, if there is a solution of (4) satisfying \( \hat{t}_1 \geq \hat{t}_1 \), this solution is a local minimum and is the only solution in the set \( S \).

If, also, \( \hat{t}_1 \leq t_0 \), this solution is feasible for the original problem and thus is the optimum.

If there is no solution to (4) in \( S \), the approach described by Srinivasan will be optimal if \( \frac{dt_2}{dt_1} \) is positive at \( \hat{t}_1 \) (since \( \frac{dt_2}{dt_1} \) cannot change sign in \( S \) in this case). If \( \frac{dt_2}{dt_1} \) is negative at \( \hat{t}_1 \), it would have to remain negative in the interval \((\hat{t}_1, \hat{t}_2)\) implying \( t_2(\hat{t}_2) < \hat{t}_2 \), which would be a contradiction.

6. The Third Case

It remains to consider the case where \( t_0 \leq \hat{t}_1 \). It is seen from the remarks of section 4 and the results of section 5 that for this case, for any approach that does not construct just type 0 capital until time \( t_0 \) there is a better approach which does. Thus, the optimization is reduced to the optimal behavior after \( t_0 \).

Consider the set of approaches for which the stock of type 0 machines grows at a rate \( \theta \) until time \( t_3 \) (with the remaining type 0 machines producing type \( j \) machines, thus preserving full employment of labor.
from \( t_0 \) until \( t_3 \), and just type \( j \) machines are constructed from \( t_3 \) until \( t_2 \), at which time the terminal path is achieved. Clearly the labor constraint is satisfied on all such approaches and the stock of type 0 machines at time \( t_2 \) exceeds the terminal path level. Thus the best path in this set is the solution of the following problem:

Minimize \( t_2(t_3) \) subject to:

\[
(6) \quad K_j(0) e^{-\delta t_2} + \frac{\gamma_0 - \theta - \delta}{\theta + \delta} K_0(t_0) (e^{\theta(t_3 - t_0)} - \delta(t_3 - t_0) e^{-\delta(t_2 - t_3)}) + \\
\gamma_0 K_0(t_0) e^{\theta(t_3 - t_0)} \delta(t_3 - t_2) (t_2 - t_3) = K_j(0) e^{\theta t_2}.
\]

Those paths not in this set which can be obtained by retiming the construction of the two types of machines between \( t_0 \) and \( t_3 \) by making earlier the date of construction of type 0 machines are equivalent to the paths in the set. (In other words, as long as labor is fully employed it does not matter which type of machine is standing idly by). Paths not in the set, which cannot be obtained in this fashion are inferior to paths in the set. Thus, the solution to the problem above gives the optimum for this case, but it is not unique.

Solving the problem we first compute the derivative from equation (6):

\[
(7) \quad \frac{dt_2}{dt_3} = \frac{[(t_2 - t_3)(\theta + \delta)\gamma_0 - \theta - \delta]K_0(t_0) e^{(\theta + \delta) t_3} - \theta t_0}{(\theta + \delta) K_j(0) e^{(\theta + \delta) t_2} - \gamma_0 K_0(t_0) e^{(\theta + \delta) t_3} - \theta t_0}
\]
Equating this with zero gives the optimal solution:

\[(8) \quad \tilde{t}_2 - \tilde{t}_3 = \frac{1}{\gamma_0}\]

Substituting this in (6) gives the solution for \(\tilde{t}_3\):

\[(9) \quad K_j(0) + \frac{(\gamma_0 - \theta - \delta)}{\theta + \delta} K_0(t_0)(e^{(\theta + \delta)\tilde{t}_3} - \theta t_0) - e^{\delta t_0} + K_0(t_0) e^{(\theta + \delta)\tilde{t}_3} = E \quad \frac{\theta + \delta}{\gamma_0}(\tilde{t}_3 + \frac{1}{\gamma_0})\]

Define \(S_2\) as the set \(\{t_3 \mid t_3 \geq t_0, t_2 (t_3) \leq \tilde{t}_2\}\).

The denominator of \(\frac{dt_2}{dt_3}\) is negative for \(t_3\) in \(S_2\) since:

\[
\frac{(\theta + \delta)^2 K_j(0)e^{(\theta + \delta)\tilde{t}_2}}{\gamma_0 K_0(t_0)e^{\theta(t_3-t_0)-\delta(t_2-t_3)}} = \frac{(\gamma_0 - \theta - \delta) K_0(t_2)}{\gamma_0 K_0(t_0)e^{\theta(t_3-t_0)-\delta(t_2-t_3)}}
\]

Taking the second derivative of \(t_2\) with respect to \(t_3\), it is seen that its sign is the opposite of the sign of the denominator. Since \(\frac{dt_2}{dt_3}\) at \(t_0\) does not exceed zero for this case, a solution of (9) will be in \(S_2\).
To see that the optimum may not be the path described by Srinivasan consider the case where \( K_j(0) = 0 \). Then:\(^5\)

\[
\hat{\tau}_1 = \frac{1}{\gamma_0} \left[ \frac{\theta+\delta}{\gamma_0} - \log \left( \frac{\theta+\delta}{\gamma_0} \right) - \log \frac{K_0(0)}{K_0^E(0)} \right]
\]

\[
\hat{\tau}_1 = \frac{1}{\gamma_0} \left[ \frac{\gamma_0 - \theta - \delta}{\gamma_0} + \log \frac{K_0^E(0)}{K_0(0)} \right]
\]

Thus \( \hat{\tau}_1 \) as \( \hat{\tau}_1 \) as \( \left\{ \begin{array}{l}
\gamma_0 - \theta - \delta \\
\theta + \delta
\end{array} \right\} \) e .

---

\(^5\) \( \hat{\tau}_1 \) is the solution of Srinivasan's equation (3.31).
The author, Peter A. Diamond, was born April 29, 1940 in New York, New York. He entered Yale University in 1957, whence he received, in 1960, the degree of B.A., summa cum laude. In September 1960, he entered the Massachusetts Institute of Technology. He is coauthor (with T. C. Koopmans and R. Williamson) of "Axioms for Persistent Preference" in R. Machol and P. Gray (eds.), Recent Developments in Decision and Information Processes, New York, 1962.