SNOWFLAKE AGGREGATION-A NUMERICAL MODEL

by

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ABSTRACT

Computations have been made to investigate the evolution of particle-size spectra in snow. Initial conditions assume a volume of atmosphere containing unaggregated crystals of uniform size and type. The volume is sufficiently large so that edge effects may be neglected. Aggregation results from:

1) Random collisions between both individual crystals and aggregates. The probability of a random collision involving a snowflake of a given size is assumed proportional to the concentration of the snowflakes.

2) Ordered collisions when aggregates overtake individual crystals or smaller aggregates. Assumed values of terminal fall velocities and collision cross sections of snowflakes are based on values reported by various observers.

Effects of crystal concentration, random collision frequency, and crystal type are investigated by varying the initial assumptions.

Some of the computed spectra approach those observed in natural snow within a realistic time period.

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I. INTRODUCTION

During the last two decades considerable attention has been given to the problem of the size distribution of raindrops. Much of the impetus for such measurements has stemmed from the importance of the knowledge of drop size for radar measurements of precipitation, but the size spectra are also of interest in studies concerned with the development of hydrometeors. In temperate latitudes, where clouds may extend above the melting level, much of the rain that falls originates as snow. Ice crystals, produced well above the melting level, grow by sublimation in the super-cooled region of the cloud. As they approach a level where the temperature is just below 0°C the crystals begin to aggregate into snowflakes. Indeed, radar measurements suggest that rapid aggregation of snowflakes occurs just above the melting level. As the snowflakes fall below the melting level, they melt into liquid drops which may grow further by condensation and coalescence before they reach the ground as rain. Fujiwara (1957) has shown that once a fairly broad size distribution has been established further aggregation will generally proceed at a faster rate than in the case of the coalescence of raindrops because of the relatively large collision cross sections of snowflakes as compared with water drops.

It is, therefore, an acknowledged fact that the process of snowflake aggregation is a major factor affecting the ultimate raindrop-size distribution. This process has, however, received little attention
as compared with the development of warm rain. Some numerical studies of coalescence mechanisms in liquid clouds have achieved realistic distributions in realistic time periods. For example, Twomey (1966), and Berry (1967) have shown that drops can grow to raindrop size within a reasonable length of time provided that statistical effects of random collisions are taken into account.

It is, however, difficult to set up a realistic and general model for the initiation of ice crystal aggregation because a number of unknown factors are involved. For example, ice crystals occur in a bewildering variety of shapes and sizes (Nakaya (1954)), observations of crystal number or ice content per cubic meter in snow made up of unaggregated crystals are sparse, and the influence of electrical charges on snow crystal collisions are not understood.

The purpose of this study is to investigate the process of ice crystal aggregation in characteristic stratiform-type precipitation. The simple model presented here assumes an ensemble of ice crystals of uniform size and type. Aggregation is initiated through "random" collisions, possibly due to erratic fall paths, electrical attractions, or other, unspecified, very small-scale effects. Aggregation proceeds by further "random" collisions and also through "ordered" collisions as snowflakes overtake crystals or aggregates with smaller fall velocities. An intended result of this study is to determine the effects of changes in the assumptions of initial size, number, and type of crystals, and the frequency of "random" collisions. It is also desired to see which combinations of these assumptions, if any, will produce realistic size spectra. If realistic size spectra are produced, the time necessary for their evolution is considered.
II. OBSERVED SNOWFLAKE SPECTRA

Data on snowflake size distributions are relatively sparse. Measurements by Gunn and Marshall (1958) satisfied the relationship:

\[ N_D = N_0 e^{-\Lambda D} \]

where \( N_D dD \) is the number of aggregates with equivalent drop diameter between \( D \) and \( D + dD \) per unit volume. Both \( N_0 \) and \( \Lambda \) are functions of the precipitation rate, \( R \) (mm/hr):

\[ N_0 = 3.8 \times 10^3 R^{-0.87} \text{ (m}^{-3} \text{ mm}^{-1}) \]

and

\[ \Lambda = 25.5 R^{-0.48} \text{ (cm}^{-1}) \]

Fig. 1 shows snowflake spectra measured by Gunn and Marshall for equivalent rainfall rates of 1.5 and 2.5 mm/hr.

Some measurements have been made by Japanese observers, Fujiwara (1957), Imai (1955), and Magono (1954). Their agreement with the results of Gunn and Marshall is fair. The spectra computed from the model are compared with those computed by Gunn and Marshall.
Fig. 1. Snowflake spectra from Gunn and Marshall (1958). Precipitation rates: 1.5 mm/hr and 2.5 mm/hr. (after Lougher (1966)).
III. THE MODEL

The basic model and some preliminary computations have been described by Lougher (1966). The two collision types are expressed numerically in the following pages. A snowflake containing \( p \) crystals is designated by \( S_p \) and the number of these flakes per unit volume is \( N_p \).

Ordered collisions occur because of different terminal fall velocities for snowflakes of different masses. On the average, all crystals of the same type have the same terminal fall velocity and are not involved in ordered collisions among themselves. Ordered collisions between crystals, \( S_1 \), and snowflakes, \( S_p \) \((p>1)\), and among snowflakes of different sizes \((\text{different values of } p)\) will occur.

The number of ordered collisions between \( S_p \)'s and \( S_q \)'s which occur in a unit volume of atmosphere in a unit time interval is:

\[
\dot{f}_{pq} = N_p N_q A_{pq} \left| V_p - V_q \right| E_{pq}
\]  

(1)

where 

\( V_i \) = terminal fall velocity

\( A_{ij} \) = effective collision cross section

\( E_{pq} \) = collection efficiency

In the computations the collection efficiency is assumed to be unity.

The fall velocities can be expressed in terms of the mass after Nakaya (1954) for individual crystals and Langleben (1954) for aggregates. Specific relations are dependent on the crystal type.

The effective collision cross section, also expressed in terms of the mass, is assumed to be a circle with diameter equal to the sum of the maximum dimensions of the two colliding flakes.

The horizontal diameters of the snowflakes and crystals are expressed
in terms of the equivalent melted diameter. The diameter factor, \( k \),
is defined as the ratio of the horizontal diameter of the snowflake
to that of the equivalent melted drop. Therefore:

\[
A_{pq} = \frac{H}{4} (k_p d_p + k_q d_q)^2
\]

where \( d_1 \) (meters) = \( 1.24 \times 10^{-1} \) \((m)\)^{1/3} (kg.), and \( m_i \) represents the mass
of a flake containing \( i \) crystals.

For individual crystals, values of \( k \) are obtained from Nakaya's
(1954) relations between dimension and mass for various crystal types.
It is assumed that the ice crystals are always horizontally oriented
in space.

There are very few observations of snowflake dimensions because of
the technical difficulties involved in such measurements. The shape and
size of snowflakes are extremely variable since they depend on the mode
of attachment, number and type of component ice crystals, and on microscopic
forces.

Lougher (1966) used a value for \( k \) of 5.5 for all aggregates, after
Fujiwara (1957), based on the following assumptions:

1) Snowflakes maintain an oblate spheroid shape; the ratio of the
major to minor axis is 3.2 regardless of mass.

2) Each flake preserves its orientation in space and the horizontal
cross section is accordingly circular.

3) A density value for dry flakes is given by Magono (1954) as \( .0087 \)
g/cm\(^3\). This value is assumed applicable.

Referring, now, back to equation (1), we may define a probability
function, \( o^P_{pq} \), for ordered collisions between aggregates containing \( p \)
crystals, \( S_p \), and those containing \( q \) crystals, \( S_q \):

\[
o^p_{pq} = f_{pq}/N_{pq} = \rho A_{pq} \mid V_p - V_q \mid
\]  

(2)

Since \( A_{pq} \) and the fall velocities have been represented in terms of the mass of the snowflakes, and since the mass of an \( S_p \) flake is simply \( \rho m_1 \), \( o^p_{pq} \) is a function of \( p \) and \( q \) only, for a particular computation, but depends on the initial assumptions regarding mass and type of the individual crystals. When \( p \) and \( q \) are equal, the fall velocities will be identical. Therefore, \( o^p_{pp} = 0 \).

All the other collisions are assumed to be "random". This collision process, operating in a volume filled with uniform particles, is assumed to produce collisions between two particles at regular, discrete intervals of time in direct proportion to the particle density. Therefore, the random collision probability function will be a constant:

\[
R^p_{pq} = C
\]  

(3)

Since we have insufficient knowledge of the physical processes involved in the "random" collisions, the value of \( C \) is dependent upon the assumed initial concentration and random collision rate.

The probability of a random collision of two \( S_p \)'s is proportional to \( N_p^{2/2} \), because two flakes of the same type are involved in each collision. Since \( N_p \) is decreased by two in this case, it is convenient to define the probability function as above so that \( R^p_{pp} = C \).

Equations for computing the development of a size spectrum are set up by considering the change in any \( N_p \) during a time interval \( \Delta t \):

(1) \( N_p \) is decreased by one through any random collision of an \( S_p \) with a flake of any other size and by two through a random collision with another \( S_p \). The
total decrease in $N_p$ through random collisions is:

$$\sum_{i=1}^{\infty} N_i N_i R_{p,i}^P \Delta t$$  \hfill (4)

(2) $N_p$ is decreased by one in each collision where a larger flake overtakes an $S_p$ and also where an $S_p$ overtakes a smaller one. The net decrease in $N_p$ due to these two effects is:

$$\sum_{i=1}^{\infty} N_i N_i o_{p,i} \Delta t$$  \hfill (5)

(3) $N_p$ is increased by one through each collision involving an $S_i$ and an $S_{p-i}$. Collisions of this type add:

$$1/2 \sum_{i=1}^{p-1} N_i N_{p-i} R_{i,p-i}^P \Delta t + 1/2 \sum_{i=1}^{p-1} N_i N_{p-i} o_{i,p-i} \Delta t$$  \hfill (6)

The factors of 1/2 are necessary since the summation for $i$ going all the way from 1 to $p-1$ causes each collision to be counted twice.

The net value of $\Delta N_p/\Delta t$ is obtained by subtracting expressions (4) and (5) from (6) and dividing by $\Delta t$:

$$\frac{\Delta N_p}{\Delta t} = 1/2 \sum_{i=1}^{p-1} N_i N_{p-i} R_{i,p-i}^P + 1/2 \sum_{i=1}^{p-1} N_i N_{p-i} o_{i,p-i}$$

$$- \sum_{i=1}^{\infty} N_i N_i R_{p,i}^P - \sum_{i=1}^{\infty} N_i N_i o_{p,i}$$  \hfill (7)

The snowflake size distributions are obtained through numerical integration of (7) for all values of $p$ which may be involved.

It is assumed that the whole process takes place in a layer sufficiently deep so that there is no significant sorting because of the changing fall velocities.
IV. COMPUTATIONS

For the computations, plane dendritic crystals were chosen because this crystal type is commonly observed in New England snow storms, and also because its relatively large dimensions and slow fall velocity make it most likely to form aggregates. Average values for dimensions, mass, and fall velocities as observed by Nakaya (1954) are assumed:

\[ d = 4 \times 10^{-3} \text{ meters; } m_1 = 6.55 \times 10^{-8} \text{ kg; } V_1 = 3 \times 10^{-1} \text{ m sec}^{-1}; k_1 = 8.3 \]

For these crystals a value of \( N_1 = 10^4 \text{ m}^{-3} \) would give a liquid water content of about 0.6 g m\(^{-3}\), giving a precipitation rate close to 1 mm hr\(^{-1}\). For the aggregates of plane dendrites, the fall velocity used is from Langleben (1954):

\[ V_1 = 3.70(m_1)^{1/9} \]

where \( V_1 \) is in m sec\(^{-1}\) and \( m_1 \) is in kg.

Preliminary computations by Lougher (1966) with similar initial conditions showed that an assumption of one random collision per second in each cubic meter containing \( 10^4 \) particles would result in an unrealistic distribution with relatively few aggregates and almost equal numbers in each size category. She found that assuming 10 collisions per second seemed to be leading toward a distribution similar to those observed by Gunn and Marshall (1958). A value of \( N_1 \) equal to \( 10^4 \) and a random collision rate of 10 collisions per second \( (R_{ij}^P = 2 \times 10^{-7}) \) were chosen for the first set of machine computations. In order to see how the developing spectra differ for varying initial concentrations and random collision frequencies, distributions were also computed using the following initial conditions:
The machine program, with numerical values appropriate for the last case, is included in the Appendix.

Values of $N_p$ are computed to eight decimal places. It was decided, however, to disregard all values of $N_p$ which were less than $10^{-3}$ in order to save computer time. It is recognized that there exists a small probability that for large values of $p$ an $S_p$ flake may appear at any given time having a value of $N_p$ less than $10^{-3}$. It is expected, however, that the omission of such flakes from the distribution will not affect it appreciably.

The results obtained through the numerical integration of equation (7) are also subject to errors involving the length of the time interval, $\Delta t$. This time interval should be chosen so that it is somewhat shorter than the average time for one aggregate to pick up a crystal. The probable number of collisions per particle taking place within $\Delta t$ takes the form of a Poisson distribution. If the probability of multiple collisions within $\Delta t$ is small enough, this time interval is suitable for use in the computations. If the chosen time interval is too large the model will operate unrealistically in that it will severely limit the amount of aggregation taking place. In this case, the number of collisions per aggregate is still limited to one although more collisions are likely to occur in reality.

An unrealistically large $\Delta t$ will cause an unstable situation to develop
within the computations, leading to negative values of \( N_p \), a physical impossibility. There should not be much of a difference in \( \Delta N/\Delta t \) at the beginning and at the end of each time interval. It has been shown by Lougher (1966) that when \( N_1 = 10^4 \text{ m}^{-3} \) it takes a single \( S_2 \) flake about 12 seconds to overtake an ice crystal, so that within about one second an \( S_2 \) flake will collide with a crystal when \( N_2 = 10 \text{ m}^{-3} \). Therefore, it is realistic to set \( \Delta t \) equal to 1 second when \( N_1 = 10^4 \text{ m}^{-3} \). For the computations of \( N_1 = 3 \times 10^4 \text{ m}^{-3} \) and \( N_1 = 10^5 \text{ m}^{-3} \), it was necessary to use a smaller time interval.

As time passes the amount of aggregation during each time interval decreases because the total number of particles is always decreasing. It is possible, then, to increase \( \Delta t \) since the change per time interval becomes very small. This is a welcome effect, for the amount of computer time used may be decreased immensely.
V. DISCUSSION OF RESULTS

A modified version of the distributions after Gunn and Marshall (1958) is shown in Fig. 2. The units were changed to conform with those used in this study, i.e. \( N_p \ (m^{-3} \cdot 0.065 \text{ mg}^{-1}) \) vs. \( p \) (number of crystals in an \( S_p \) flake) in order to facilitate comparison with the computed distributions. The computed distributions are shown in Figs. 3-6, except for the case where \( N_1 = 10^5 \text{ m}^{-3} \). The distribution for \( N_1 = 10^5 \text{ m}^{-3} \), and 100 random collisions per second per cubic meter was computed for only 5 seconds in order to compare it with the case in Fig. 3, which has one tenth the number of initial crystals, and one tenth the random collision rate. As expected, the distributions for small \( t \) were nearly identical, except for \( N_1 \), but were achieved in one tenth of the time than those in Fig. 3. Hereafter, the distributions in Figs. 3-6 will be referred to as A, B, C, and D respectively.

There are certain characteristics common to A, B, C, and D. First, it may be seen that as time progresses, larger flakes are formed and the distributions become less vertical. For values of \( p \) greater than about 25, \( N_p \) increases with time, although some slowing in the rate of increase is apparent. For values of \( p \) less than about 25, \( N_p \) increases, increases at a slower rate, and then remains almost constant or begins to decrease with time. Note that the rate of change of the distribution for C and D is about three times as fast as that for A and B. This is due to the threefold increase in the value of \( N_1 \).

The development of the spectra is initiated by random collisions among ice crystals. The newly formed snowflakes then grow further by ordered
Fig. 2. Modified distribution after Gunn and Marshall (1958).
Fig. 3. Distribution for $N_1 = 10^4$, $R_{ij}^p = 2 \times 10^{-7}$. 
Fig. 4. Distribution for $N_1 = 10^4$, $R^p_{ij} = 2 \times 10^{-6}$. 
Fig. 5. Distribution for $N_1 = 3 \times 10^4$, $R_{ij} = 0.67 \times 10^{-6}$. 

$N_p (m^{-3} \cdot 0.06 \text{mg}^{-1})$ 

$p (0.06 \text{mg})$
Fig. 6. Distribution for $N_1 = 3 \times 10^4$, $R_{ij}^p = 0.22 \times 10^{-6}$. 
collisions with ice crystals. Some aggregates become sufficiently numerous to enable random collisions with crystals to become important. In the final stages of the spectral growth, ordered collisions among the aggregates account for further changes.

The extent to which any one or more of these processes dominate is dependent upon the initial values of $N_1, R_{ij}$ that are assumed, and on the stage of the development of the spectrum. For instance, the importance of random collisions among crystals decreases as the supply of crystals is depleted. Initially, this process will be most important for the distribution assuming the largest initial values of $N_1$ and $R_{ij}$. As time progresses, however, the large $R_{ij}$ causes the crystals to be more rapidly depleted. Therefore, the choice of a lower $R_{ij}$ value enables this process to remain important for a longer period of time.

Each of the four distributions to be discussed was chosen with a different combination of $N_1$ and $R_{ij}$. An attempt was made to vary one of the parameters for each case.

A comparison of A with B, and C with D shows that if $N_1$ is held fixed, an increase in the random collision rate not only causes the distribution to develop faster, but also changes the appearance of the distribution. Distributions B and C show a much greater decrease in $N_1$ for low $p$ values and a great deal of slowing down in the rate of change of the distribution. This signifies a well developed stage of the distribution, perhaps in an unrealistically short period of time.

Figs. 7-10 show plots of $N_1$ vs. time for various values of $p$. Figs. 7-10 correspond with A, B, C, and D respectively, as did Figs. 3-6.
ll-13, respectively, compare the total number of flakes (aggregates plus crystals), the number of aggregates, and the largest value of $p$ present for A, B, C, and D as a function of time. Largest $p$ values should only be used to compare the different cases in a relative manner since the absolute values are functions of the cut-off point of $N_p$ used in the program.

A study of Figs. 7-10 shows how slowly distribution A changes as compared to the other cases, especially C, which shows the greatest rate of change. The relatively quick development of C may be explained by the effects of the large value of $N_1$ and the high random collision rate combining to produce many aggregates in a shorter period of time (Fig.11). This situation reinforces itself in time since the early presence of a large number of aggregates enables them to become involved in more ordered collisions, sooner. As a result, the total number of flakes begins to decrease rapidly as more and more aggregates combine, and the crystals disappear still further (Fig. 12).

The rate of change of the distribution slows down as the aggregates become less numerous. This can be seen in the way the lines remain closer together as time goes on (Figs. 3-6). This feature is obviously due to the dependence of $\Delta N/\Delta t$ on $N_p$, as seen in equation (7). Distribution C slows down within a short time because of the large reduction in the total number of flakes.

Comparison of A, B, C, and D with the measurements of Gunn and Marshall (G-M) in Fig. 2 lead to some interesting observations. Distribution A shows fair agreement only for those flakes somewhere between $S_{10}$ and $S_{20}$. 
Fig. 7. Value of $N_p$ vs time for values of $p$ as indicated.

$N_1 = 10^4$, $P_{ij} = 2 \times 10^{-7}$
Fig. 8. Value of $N_p$ vs. time for values of $p$ as indicated.

$N_1 = 10^4$, $P_{ij} = 2 \times 10^{-6}$
Fig. 9. Value of $N_p$ vs. time for values of $p$ as indicated.

$N_1 = 3 \times 10^4$, $P_{1j} = 0.67 \times 10^{-6}$
Fig. 10. Value of $N_p$ vs. time for values of $p$ as indicated.

$N_1 = 3 \times 10^4, R_{ij} = 0.22 \times 10^{-6}$
Fig. 11. Total number of flakes (aggregates plus crystals) vs. time for each case: A, B, C, and D (see text).
Fig. 12. Number of aggregates vs. time for each case: A, B, C, and D (see text).
Fig. 13. Largest value of $p$ present vs. time for each case: A, B, C, and D (see text).
Other $S_p$ flakes are fewer in number than observed by G-M. Estimates for later time show that the distribution is not progressing towards a closer agreement with G-M.

Distribution B, after about 3 minutes, was in better agreement with G-M than would have been possible with A. The broad convex maximum of A doesn't appear, the $N_p$ values are quite realistic, but the supply of crystals and small snowflakes is rapidly decreasing. The computed precipitation rate at 172 seconds is $2 \text{ mm hr}^{-1}$.

In distribution C, most values of $N_p$ for flakes larger than $S_{10}$ are greater than the G-M values. However, in addition to this, crystals and small flakes are low in number and being depleted rapidly, as in B.

Distribution D shows the problem of rapid loss of crystals and small aggregates apparently solved. Values of $N_p$ for flakes around the size of $S_{20}$ are lower than in C, but proceeding in the correct direction. $N_p$ values for large flakes are about the same, which is close to the G-M values.

The depletion of crystals and small aggregates may indicate a departure of this model from reality in that no allowance is made for the replenishment of crystals. If the values given by Gunn and Marshall are representative, then this appears to be the reason for the consistently small computed values of $N_p$ for flakes between the sizes of $S_1$ and $S_{10}$. It is unrealistic to assume much higher values of $N_1$ for the model since the liquid water content represented by such a large number of crystals would be prohibitively high.
VI. PHYSICAL INTERPRETATION

The basis of this numerical model is that collisions between snowflakes take place because of random motions and of overtaking due to differences in terminal fall velocities. It is assumed that upon collision coalescence results. According to the model two flake aggregates, $S_2$'s, are initially created at a uniform rate by random collisions among crystals, and a progression to larger flakes is accomplished mainly through ordered collisions between aggregates and individual crystals, or between aggregates and other aggregates. The model applies to a stratiform-type situation because of the nature of the assumptions of the crystal sizes and types.

Realistic descriptions of crystals and flakes are used. Ordered collisions are realistically described in terms of crystal and snowflake masses, and values of terminal fall velocities are based on actual observations. The problem of how many collisions result in a coalescence, or, on the contrary, in disintegration, is not yet solved. In this model it was assumed that flakes fall with a horizontal orientation and straight line collision.

"Random" collisions, however, are invoked to include all effects which are not understood well enough to be described mathematically. Complexities in the shape and motions of the flakes, for example, are included in the "random" collision rate.

Experiments by Hosler (1957) concerning the effects of temperature and humidity on aggregation shed light on these problems. Hosler saw that the higher the temperature the greater the aggregation regardless of the degree of saturation, and that at ice supersaturation aggregation is greatly
increased above that observed with ice saturation at all temperatures. Small fluctuations in the vapor pressure in ice clouds can become extremely important in determining whether coalescence will occur. Supersaturation will promote the rapid growth of cloud elements to precipitation size by the aggregation of ice crystals.

The fact that a realistic distribution does not develop from a very low "random" collision rate suggests a threshold in conditions, possibly related to the quantities Hosler observed, where the probability of crystals colliding and sticking increases rather sharply. Once this threshold is reached, aggregation may begin, and once initiated would be expected to progress rather rapidly.

Other than the consideration of the collision mechanisms, there is no apparent reason why the computed distributions should agree with the experimental observations of Gunn and Marshall. The equation derived by Gunn and Marshall represents observations averaged over a number of different storms. Any storm, taken individually, may have a considerably different distribution. The numerical computations represent, in effect, a single storm and no reason exists to label it "average".

Some attention should be focused on the layer of atmosphere involved in the model. During the time it takes for a fairly broad spectrum to develop, it is likely that the crystals have fallen on the order of only one hundred meters. It is, therefore, evident that observations would be more likely to encounter either slight aggregation or a well developed spectrum, not at some intermediate stage.

More observations of snowflake spectra combined with a knowledge of
the state of the atmosphere above the observations are needed in order to relate changing patterns of temperature, vapor pressure, and wind velocities to the shape of the distributions.

Reliable data on the aggregation, and possible breakup, of snowflakes in the vicinity of the melting layer is also needed.

It is desirable to experimentally clarify the details of snowflake collisions, and define the state of the environment where the collisions occur, before further computations are undertaken. The difficulties involved in making observations necessary for such precise measurements are recognized by the author. If it is possible to observe snowflake aggregation and snowflake spectra, then parameterizations of different forms of the random collision function and variations of the equation with time may be made. Some of the complexities previously discussed may have large effects on the nature of spectral development, some may not. A parameterization of each effect, inserted in the formula, may yield results capable of distinguishing the significant from the inconsequential.
VII. CONCLUSION

From these computations certain conclusions can be drawn about the way the initial assumptions affect the development of the spectrum. Initially, \( S_2 \)'s are created at a uniform rate by random collisions. A progression to higher \( p \) values is accomplished by ordered collisions with individual crystals. After a while, the smaller flakes begin to be depleted and an inflection point appears. If a realistic distribution is to develop there must be enough crystals initially so that large flakes will develop before the smaller ones are depleted and the inflection appears. Also, the initial random collision rate must be large enough so that aggregates do not progress to larger \( p \) values as rapidly as small aggregates are created, thus leading to a very flat distribution.

Initial values of \( N_1 \) between \( 10^4 \) and \( 10^5 \) crystals per cubic meter appear reasonable on the basis of precipitation rates and observed ice nuclei concentrations. Something on the order of 100 collisions per second are then needed initially to produce a realistic distribution.

The fact that a realistic distribution does not develop from a very low "random" collision rate suggests some sort of threshold in conditions (such as temperature, vapor pressure, or size of crystals) where the probability of crystals colliding and sticking increases rather sharply. Once this threshold is reached, aggregation commences and proceeds rapidly for several minutes. As the aggregates grow, the total number of particles decreases and the probability of any type of collision becomes progressively smaller. It is difficult to suggest a physical basis for such a threshold. Many more observations are needed of the extent of aggregation in natural snow, and of the conditions under which different degrees of aggregation are found. Further exploration with
the numerical model might also be pursued. One possibility is to define a time dependence for the collection efficiency and the random collision rate, thus denoting changes in the atmospheric conditions as the process occurs over a specified height range. Additional modifications may be made by introducing a continuous supply of crystals, or adding more parameters representing other physical realities related to the non-uniformity of the environment.
VIII. APPENDIX

The Appendix consists of the program used for computing the growth of a spectrum up to 5 seconds, at .2 second intervals. Initial values are:

\[ N_1 = 3 \times 10^4; \quad P_{ij} = 0.22 \times 10^{-6} \]

which corresponds to distribution D (Fig. 6 and 10).
SNOWFLAKE COLLISIONS PROGRAM

DIMENSION PN(400), AN(400), AD(4), AV(400)
WRITE(6, 5)

5 FORMAT IH, 50X, 'SNOWFLAKE COLLISIONS'// IX, 'SEC', 2X, 'CRYSTALS',
   XIX, 'DELTA 1', 13X, 'DELTA 2', 13X, 'DELTA 3', 13X, 'DELTA 4', 11X,
   X 'AGGREGATES', 9X, //

C INITIALIZATION

LINES = 5
MAX = 400
AN = 6.55E-08**.6666666
AV(1) = .30
AV(2) = .58
AV(3) = .45
AV(4) = .51
AV(5) = .56
AV(6) = .61
AV(7) = .66
AV(8) = .70
AV(9) = .74
DU 8 1 = 10, MAX
AI = 1
8 AV(1) = 3.7*(AI*6.55E-08)**111111
PN(1) = 30000
DU 10 1 = 2, MAX
10 PN(1) = 0.
C CALCULATIONS

DU 50 IT = 1, 25
1XERLS = 0
CRYSTALS = 0.
FLAKES = 0.

11 DU 40 IP = 1, MAX
12 AG(1) = 0.
11 IF(IP.EQ.1) GO TO 18
C DELTA 1

AS = 0.
LIM = IP - 1
DU 14 1 = 1, LIM
14 AS = AS + PN(IP) * PN(IP-1)
AG(1) = .1111111E-06 * AS
C DELTA 2
12 IF(IP.EQ.2) GO TO 16
16 LIM = IP/2
APM1 = IP - 1
AS = PN(1) * PN(IP-1) * (AV(IP-1) - AV(1)) * (5.5*APM1**.3333333 + 6.3)**2

39
IF(IP LE.4)GO TO 17
DO 16 I=2,LIM
A1=1
APM=IP-1
AS=AS+PN(1)*PN(IP-1)*(AV(IP-1)-AV(I))*(5.5*APM**.333333+X5.5*A1**.333333)**2
16 CONTINUE
AD(2)=1.21E-02*AM*AS
C DELTA 3
AS=0.
JZEROS=0
DU 20 I=1,MAX
IF(ABS(PN(I)).GT.1.E-07)GO TO 19
JZEROS=JZEROS+1
19 JZEROS=0
20 AS=AS+PN(I)
AD(3)=-.22222222E-06*PN(IP)*AS
C DELTA 4
AP=IP
AS=PN(I)*ABS(AV(IP)-AV(I))*(AKP*AP**.333333+8.3)**2
DO 24 I=2,MAX
IF(PN(I).LT.1.E-10)GO TO 25
A1=1
AS=AS+PN(I)*ABS(AV(IP)-AV(I))*(AKP*AP**.333333+X5.5*A1**.333333)**2
24 CONTINUE
25 AKP=5.5
AD(4)=-1.21E-02*PN(IP)*AM*AS
C SUMMARY
26 AS=0.
DU 28 I=1,4
AD(I)=AD(I)*.2
26 AS=AS+AD(I)
AN(IP)=PN(IP)+AS
T=IT
I=I+.2*000001
P=IP
CRYS=CRYS+AN(IP)*P
FLAKES=FLAKES+AN(IP)
IF(LINES.LT.59)GO TO 30
WRITE(6,100)
LINES=3
30 WRITE(6,110)T,P,AD,AN(IP)
IF(AN(IP).LT.-1.E-16)GO TO 90
LINES=LINES+1
34 IF(ABS(AN(IP)).GT.1.E-03)GO TO 36
JZEROS=JZEROS+1
IF (IZEROS=3) 40, 40, 42
36  IZEROS=0
40  CONTINUE
41  IF (AN(MAX).GT.1.) GO TO 90
42  DO 44 I=1,IP
44  PK(I)=AN(I)
45  WRITE (6, 130) CRYSTALFLAKES
46  LINES=LINES+3
47  CONTINUE
90  STOP
100  FORMAT (1H1, 'SEC', 2X, 'CRYSTAL/AGG', 11X, 'DELTA 1', 13X, 'DELTA 2', 13X,
110         'DELTA 3', 13X, 'DELTA 4', 11X, 'AGGREGATES' )
120  FORMAT (F13. 6)
130  FORMAT (20X, 'CRYSTALS =', F10.2, 20X, 'FLAKES =', F10.2)
140  FORMAT (F14. 6)
END


