FORCED VIBRATION IN SYSTEMS
WITH
ELASTICALLY SUPPORTED DAMPERS
by
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Jerome E. Ruzicka
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BY

JEROME E. RUZICKA

SUBMITTED TO THE DEPARTMENT OF MECHANICAL ENGINEERING ON MAY 20, 1957 IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

ABSTRACT

The dynamic response of a harmonically forced single-degree-of-freedom system with the damper elastically supported is obtained for systems containing viscous and coulomb type dampers. Exact analytical expressions for the response of the viscous damped system are developed and a linearized analysis of the coulomb damped system provides approximate solutions for the dynamic response of that system.

Analog computer systems are devised to provide exact solutions to both the viscous and coulomb damped systems. Typical response curves are given as obtained by numerical calculation of the derived equations and from the analog computer analysis. The response characteristics of the systems are compared to the classical single-degree-of-freedom systems where the damper is rigidly supported and a rather complete discussion is given on the advantages, disadvantages, and applications of the systems analysed.

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CHAPTER I - INTRODUCTION AND NOMENCLATURE

The problem to be considered is that of determining the steady-state response of a single-degree-of-freedom system that contains an elastically supported damper. Two types of damping will be considered, viscous and coulomb or slip damping. Schematic diagrams of these systems are shown in Figure I-1 and Figure I-2. This type of system is of interest because of applications in problems of turbine-blade vibration, vibration of built-up structures, and in the general field of vibration isolation.

Although the viscous damped system, being a linear system, presents no difficulty beyond solving a third order linear differential equation with constant coefficients, relatively very little information on the characteristics of the system can be found in the technical literature. This is probably a direct result of the fact that the dynamic response of
this system can be expressed as a two parameter family of functions (thus presenting difficulties in showing solutions graphically) as opposed to the system which has the damper rigidly supported which is expressible as a one parameter family of functions. Gallagher and Volterra (Reference 1) have analysed this system (which they refer to as a "relaxation type of vehicle suspension" system) for both transient and steady-state type forcing. However, the manner in which the results are presented do not show the real value of such a system. Reference to this type of system can also be found in certain books on vibration (Reference 2), but again, no convenient and clear response curves are given.

The coulomb damped system, besides having application in the design of vibration isolators, can be used to simulate built-up structures - where the engineer depends on the energy dissipation of the coulomb or slip damping to reduce the response displacement and thus the resonant stresses in the structure. Investigators, such as Goodman, Klumpp and Lazan (Reference 3, 4, 5) have done excellent work along these lines. In Reference 4, Goodman and Klumpp developed a general theory of slip damping, and investigations were made into the energy loss per cycle of oscillation. A study was also made of the damping properties of a beam made up of two identical beams held together by a clamping pressure. Results showed that for zero clamping pressure,
the energy dissipation is zero as it is for very high clamping pressure. It followed that there is always an intermediate clamping pressure at which the energy dissipation per cycle is a maximum, i.e., there is an optimum clamping pressure insofar as reduction of resonant stresses is concerned. Experiments were run where these beams were vibrated and the displacement response plotted for various values of uniform clamping pressure. Double-valued amplitudes (jump phenomenon) occurred as would be expected from a non-linear type vibration.

Studies of the structural damping of built-up beams have also been made by Pian and Hallowell (Reference 6,7) who also deal with the energy loss per cycle in such a system.

All the works cited show the hysteresis type of load deflection and damping that is characteristic of the slip damping system, and as pointed out in Reference 5, a complete theory of the dynamic behavior of mechanical systems subject to hysteresis is desirable.

In this paper, the author develops solutions to the viscous damped system and presents typical response curves for the system. Both absolute and relative steady-state response is shown in addition to a comparison of this system with the conventional single-degree-of-freedom system where the damper is rigidly supported. The full effect of varying damping from zero to infinity is shown and a method of obtaining solutions by means of an analog
computor is developed for ease in obtaining response curves.

A linearized approximate solution to the coulomb damped system is also developed which, by itself, is valid only for a specified frequency range, but when combined with exact solutions obtained for areas outside of this range, gives the steady-state response of the system with no restriction on frequency range. The effect of varying damping over a wide range of values is again shown along with graphical representations of the steady-state response. An analog computor solution is developed which shows the non-linear characteristics of the problem and response curves as obtained from the computor are given. Again, damping is made to take on a wide range of values so as to completely indicate the characteristics of the system.
NOMENCLATURE

\( m \) = Mass of Supported Body
\( k_1 \) = Spring Rate of Main Support Spring
\( k_2 \) = Spring Rate of Damper Support Spring
\( k' \) = Ratio \( k_1/k \)
\( c \) = Viscous Damping Coefficient
\( \mu F \) = Coulomb Damping Force
\( f_n \) = Normal Force Acting in Coulomb Damper
\( \mu \) = Kinetic Coefficient of Friction Between Coulomb Damper Surfaces
\( a \) = Displacement of Base
\( x_1 \) = Displacement of Damper
\( x \) = Displacement of Mass
\( s \) = \((x - a)\) = Displacement of Mass Relative to Base
\( q \) = \((x - x_1)\) = Displacement of Mass Relative to Damper
\( a_0 \) = Maximum Displacement Amplitude of Base
\( x_0 \) = Maximum Absolute Displacement Amplitude of Mass
\( s_0 \) = Maximum Displacement Amplitude of Mass Relative to Base
\( q_0 \) = Maximum Displacement Amplitude of Mass Relative to Damper
\( \phi \) = Phase Angle Between Displacements \( x \) and \( a \)
\( \Theta \) = Phase Angle Between Displacements \( s \) and \( a \)
\( \psi \) = Phase Angle Between Displacements \( q \) and \( a \)
\( \omega \) = Forcing Frequency of Base Excitation
\( \omega_n = \sqrt{\frac{k}{m}} \) = Undamped Natural Frequency
\( c_0 = 2m \omega_n \) = Critical Viscous Damping Coefficient
\( c_e \) = Equivalent Viscous Damping Coefficient for Coulomb Damping
\[ \tilde{\eta} = \frac{B}{ka_0} \] = Coulomb Damping Factor (for Constant Displacement Excitation)

\[ \eta = \frac{B}{ma_a} \] = Coulomb Damping Factor (for Constant Acceleration Excitation)

\( A_o \) = Maximum Amplitude of Base Acceleration (for Constant Acceleration Excitation)

\( (T_d)_A \) = \[ \left| \frac{x}{a_o} \right| \] = Absolute Displacement Transmissibility of Mass

\( (T_d)_R \) = \[ \left| \frac{x}{a_o} \right| \] = Relative Displacement Transmissibility of Mass

\( (T_a)_A \) = \[ \left| \frac{\ddot{x}}{a_o} \right| \] = Absolute Acceleration Transmissibility of Mass

\( P_m \) = Total Real Forces Applied to Mass

\( c_i \) = Viscous Damping Coefficient in Analog System

\( m_i \) = Mass of Coulomb Damper in Analog System

\( T_i \) = Time Constant of Integrating Component

\( \tau \) = Time Constant of Unit Lag Component

\( \superscript{*} \) = Superscript Referring to Common-Transmissibility Point

\( \prime \) = Superscript Referring to Condition Where Damper is "Locked-in"

\( \nu \) = Subscript Referring to Values of High Frequency Ratio

\( L \) = Subscript Referring to "Break Loose" or "Lock-in" Points of the Coulomb Damper
CHAPTER II - DISCUSSION OF VISCOUS DAMPED SYSTEM

The viscous damped system to be analysed as shown in Figure I-1 consists of a mass (constrained to move only in one direction) supported by a main load carrying spring which is attached to the base. The base is excited in such a manner that its displacement varies harmonically with time. The motion of the mass is damped by a viscous damper which is supported by an elastic member attached to the base.

Before embarking upon any mathematical analysis, one can draw some important conclusions about this system just by applying some physical reasoning as to the effect of elastically supporting the damper. For example, for zero viscous damping, the system is obviously undamped and infinite amplitudes would be expected at resonance. For this case, resonance occurs at the undamped natural frequency defined by the relation:
\[ \omega_n = \sqrt{\frac{k}{m}} \]  

(II-1)

On the other hand, if the viscous damping is made to approach infinity, we see that this corresponds to "locking-in" the damper which results in having an undamped system whose natural frequency would be given by:
\[ \omega_n' = \sqrt{N+1} \omega_n \]  

(II-2)

where \((N+1)k\) would be the total stiffness for this case.
Thus, the system would again experience infinite amplitudes when driven at a frequency given by equation (II-2).

Having established these two limiting conditions by simple physical reasoning, it then follows that there must be some value of viscous damping which would give a minimum peak amplitude at resonance. This is easily established if one visualizes increasing the viscous damping to a value greater than zero which would obviously decrease the resonant amplitude of the system. This process could be repeated to again reduce the maximum amplitude at resonance. But, as previously verified, if this process is continued until infinite damping is enforced, the system again becomes undamped at a higher natural frequency and infinite amplitudes occur at resonance. Thus, for the peak amplitudes to vary in a continuous manner with change in viscous damping, there must have been some particular value of viscous damping that produced a minimum resonant amplitude. This value of viscous damping will be referred to as "optimum damping".

A rather complete mathematical analysis of this system is given in Appendix A. The expression for the absolute transmissibility of the system was shown to be

\[
(T)_A = \left| \frac{x_o}{q_o} \right| = \sqrt{\frac{\frac{4 (N+1)^2 (C/c)^2 (\frac{w}{\omega_n})^2}{(1-\frac{w^2}{\omega_n^2})^2 + \frac{4}{N^2} (N+1-\frac{w^2}{\omega_n^2})(c/c)(\frac{w}{\omega_n})^2}}} {1+4 (\frac{N+1}{N}) (\frac{C/c}{c/c}) (\frac{w}{\omega_n})^2}} \tag{II-3}
\]
and the relation giving the relative transmissibility was derived as follows:

\[
(T)_R = \left| \frac{S_o}{A_o} \right| = \sqrt{\frac{(\frac{\omega}{\omega_n})^4 + \frac{4}{N^2}\left(\frac{c}{c_c}\right)^2(\frac{\omega}{\omega_n})^6}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \frac{4}{N^2}\left(N + \frac{\omega^2}{\omega_n^2}\right)\left(\frac{c}{c_c}\right)^2(\frac{\omega}{\omega_n})^2}}}
\]  

(II-4)

It is interesting to note that when \( N \to 0 \) these equations reduce to the equations which describe the undamped response of a single-degree-of-freedom system. Also, if we let \( N \to \infty \), the system reduces to the classical single-degree-of-freedom system with viscous damping and equations (II-3) and (II-4) reduce to the known equations for the absolute and relative transmissibility of that system. Moreover, if the viscous damping is allowed to approach zero, the absolute and relative transmissibility equations become:

\[
(T_A)_{c_c=0} = \frac{1}{1 - (\frac{\omega}{\omega_n})^2} 
\]  

(II-5)

\[
(T_R)_{c_c=0} = \frac{(\frac{\omega}{\omega_n})^2}{1 - (\frac{\omega}{\omega_n})^2} 
\]  

(II-6)

The corresponding equations obtained when the damping is made infinite are given by:

\[
(T_A)_{c_c=\infty} = \frac{N+1}{(N+1) - (\frac{\omega}{\omega_n})^2} 
\]  

(II-7)
Thus, equations (II-5) through (II-8) give the "undamped" response envelopes which must necessarily bound the damped solutions given by equations (II-3) and (II-4).

An interesting observation can be made if one plots the absolute values of the transmissibilities of the two undamped conditions, i.e., zero and infinite damping. As shown in Figure II-1, the response curve for zero damping intersects the response curve for infinite damping at a frequency ratio which will be defined as the common-

\[ (T_R)_{C_c=\infty} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{(N+1) - \left(\frac{\omega}{\omega_n}\right)^2} \]  

transmissibility frequency ratio because of the fact that at this point both of the undamped curves have the same transmissibility. The frequency ratio at which the two response curves intersect can be found by equating the
two transmissibility expressions (taking into account the proper sign of that portion of the response curve being considered) and solving for the frequency ratio. Proceeding in this manner, the common-transmissibility frequency ratio for absolute motion is given by:

\[
\left( \frac{\omega}{\omega_n} \right)_A^* = \sqrt{\frac{2(N+1)}{N+2}}
\]  

(II-9)

Similarly, the common-transmissibility frequency ratio for relative motion is given by:

\[
\left( \frac{\omega}{\omega_n} \right)_R^* = \sqrt{\frac{N+2}{2}}
\]  

(II-10)

To find the transmissibilities corresponding to these frequency ratios, one must substitute equation (II-9) in equation (II-3) and equation (II-10) in equation (II-4). When these substitutions are made, it is found that the damping terms disappear in each case and the same equation for transmissibility results for both absolute and relative motion. The common-transmissibility is thus given by:

\[
T_A^* = T_R^* = 1 + \frac{2}{N}
\]  

(II-11)

It is seen that this expression is independent of the damping and thus the conclusions can be drawn that all damped curves in addition to the undamped curves must
pass through the common-transmissibility point. Note that even though the common-transmissibility frequency ratio is different for absolute and relative motion, the value of the common-transmissibility itself is equal in each case.

The common-transmissibility frequency ratio is shown graphically in Figure II-2 which is a plot of this quantity versus the spring ratio. A limiting process applied to equation (II-9) and (II-10) indicates that for values of spring ratio approaching zero, both the absolute and relative common-transmissibility frequency ratio tend toward unity whereas for very large values of spring ratio the absolute common-transmissibility frequency ratio approaches \(\sqrt{\frac{1}{2}}\) asymptotically while the relative common-transmissibility frequency ratio increases as the square-root of one-half the spring ratio.

A graph of the common-transmissibility (for both absolute and relative motion) is given in Figure II-3. As indicated by this graph, for very small spring ratio (spring ratio approaching zero) i.e., a common-transmissibility frequency ratio approaching unity, the common-transmissibility approaches infinity. On the other hand, for very large spring ratios, the common-transmissibility approaches unity as an asymptote.
\[
\left(\frac{\omega}{\omega_n}\right)_R = \sqrt{\frac{N+2}{2}}
\]

\[
\left(\frac{\omega}{\omega_n}\right)_A = \sqrt{\frac{2(N+1)}{N+2}}
\]
\( \frac{T^*_A}{T^*_R} = 1 + \frac{2}{N} \)
Thus, for a given value of spring ratio, Figure II-2 and Figure II-3 give information as to the response of the system at a certain frequency ratio for all values of viscous damping.

Resorting again to physical reasoning, it seems evident that of all the different response curves passing through the common-transmissibility point, there must be one which has the common-transmissibility frequency ratio as its resonant frequency, i.e., its peak transmissibility corresponds to the value of the common-transmissibility. The value of damping that characterizes this special curve is then the optimum viscous damping coefficient and the response curve is the optimum damping response curve.

By inspection of equations (II-3) and II-4), the transmissibility of the system for large frequency ratio can readily be determined. This is done by requiring that the frequency ratio be very large and thus eliminate certain terms on the basis that they are small compared to other terms. The resulting expressions are given by:

\[(T_i)_A = \frac{N+1}{(\omega/\omega_n)^2}\]  \hspace{2cm} (II-12)

\[(T_i)_R = 1\]  \hspace{2cm} (II-13)
The subscript was chosen merely to indicate that these relations are associated with regions of high isolation (as opposed to magnification). By comparison with equations (II-7) and (II-8), these equations indicate that, for large frequency ratios, the transmissibility of the system approaches the undamped response of the system with the damper "locked-in". This property is very important to those involved with problems in vibration isolation and the value of this particular property will be made very evident when the response of this system is compared to the response of the classical single-degree-of-freedom system where the viscous damper is rigidly supported.

Since numerical calculation of equations (II-3) and (II-9) required considerable time and care, an analog computer system was devised to supplement the calculation of response curves. A Philbrick type analog computer was used which is a device that uses voltages to represent all variables. The analog computer system is a network of interconnected operational amplifiers where each amplifier is designed to perform basic mathematical operations as well as complex non-linear transformations. These operational amplifiers are high gain D. C. vacuum tube amplifiers having wide bandwidth, high input impedance for minimum loading, and low
output impedance in order that reasonable loads may be driven. A complete description of the analog system and its associated mathematics can be found in Appendix B. For further information on the analog computer, the reader is referred to References 9 and 10.

A typical response curve for this type of system is shown in Figure II-4 and Figure II-5. Figure II-4 shows the absolute transmissibility and Figure II-5 the relative transmissibility of a system with a spring ratio equal to 3. A spring ratio of 3 causes the undamped natural frequency of the "locked-in" system to occur at a frequency ratio of 2. These transmissibility curves vividly show the effect of varying the viscous damping from zero to infinity. As reasoned earlier, zero damping corresponds to infinite amplitudes at a frequency ratio of unity while infinite damping gives rise to an undamped system with a natural frequency defined by equation (11-2). For values of damping becoming successively larger than zero, the resonant amplitude becomes successively smaller while the resonant frequency ratio becomes greater than unity and tends toward the common-transmissibility frequency ratio. Eventually the viscous damping is increased to a critical value which corresponds to having the resonant frequency ratio coincide with the common-transmissibility frequency ratio. This value of viscous damping corresponds to "optimum
FIGURE II-4  ABSOLUTE TRANSMISSIBILITY FOR N = 3

![Graph showing absolute transmissibility as a function of frequency ratio. The graph includes curves for different values of \( \Delta c/\Delta c_p \), ranging from -2.0 to 2.0, and \( \omega/\omega_n \), ranging from 0.0 to 30.0. The graph also includes a simple mechanical system diagram with a mass, springs, and damping.](image-url)
RELATIVE TRANSMISSIBILITY

\[ \frac{\omega}{\omega_n} - \text{FREQUENCY RATIO} \]
damping'. It is this value of viscous damping that minimizes the peak amplitude response for the particular system under consideration. Further increase in viscous damping is seen to cause the resonant amplitudes to grow again while the resonant frequency ratio becomes larger than the common-transmissibility frequency ratio and tends toward the frequency ratio of the undamped 'locked-in' system. Thus, the effect of varying damping from zero to infinity on the dynamic response of the system has been demonstrated.

Figure 11-6 is a plot of the absolute transmissibility of a system whose spring ratio is 8. This corresponds to a 'locked-in' undamped natural frequency ratio of 3. This graph naturally has the same general characteristics as Figure 11-4, but comparison of the two shows the effect of increasing the spring ratio. We see that the increase of spring ratio has reduced the isolation at high frequency ratios and has had a definite effect on the value of optimum damping. As predicted, the transmissibility of the optimum damped curve has been reduced and the resonant frequency ratio of the optimum damped curve has been slightly increased. It is thus apparent that there are some counteracting influences that must be considered. One gains in isolation by reducing the spring ratio but must then accept an increase in the peak amplitude at resonance. On the other hand, if the system is designed to have a very small
FIGURE II-6 ABSOLUTE TRANSMISSIBILITY FOR N = 8
resonant amplification (e.g., optimum damping), the isolation at high frequency ratio might not be as good as desired.

The graph shown in Figure II-7 indicates the manner in which the maximum absolute transmissibility varies with the damping ratio for various values of spring ratio. The optimum damping ratio would correspond to that value which produces the minimum maximum absolute transmissibility. As indicated, these curves have a relatively flat portion in the area of optimum damping. Thus, it appears that great accuracy is not required in selecting a value for optimum damping. Four different values of the spring ratio parameter have been plotted to indicate the effect a variation in spring ratio has on the optimum damping ratio value.

For purposes of evaluating the vibration isolation properties of this system, let us now compare this system with the conventional viscous damped system with the damper rigidly supported. We shall confine our attention to the absolute motion of the mass during this comparison. As mentioned previously, the equation for the absolute transmissibility of the conventional viscous damped system is obtainable from equation (II-3) by letting the spring ratio approach infinity. When this is done, the following well known equation is obtained:
Maximum Absolute Transmissibility

![Graph showing the relationship between maximum absolute transmissibility and damping ratio](image)

- The graph plots the maximum absolute transmissibility against the damping ratio.
- The damping ratio is given as $\frac{C}{C_c}$.
- The graph includes curves for different values of $N$.
- The inset diagram illustrates a mass-spring system with spring constant $K$ and damping constant $C$, and the damping ratio $\frac{C}{C_c}$ is marked.

**Figure II-7: Absolute Transmissibility at Resonance**
By a limiting process, this equation is seen to reduce to the following equation for transmissibility at large frequency ratios:

\[
(T_A)_{N=\infty} = \sqrt{\frac{1 + \left[ 2 \left( \frac{c}{c_c} \right) \left( \frac{\omega}{\omega_n} \right) \right]^2}{\left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left[ 2 \left( \frac{\omega}{\omega_n} \right) \right]^2}}
\]  

(II-14)

Thus, by comparing equation (II-15) with equation (II-12) we see immediately the value of elastically supporting the viscous damper. For large frequency ratios, the conventional viscous damped isolation system has the property that the absolute transmissibility diminishes as the inverse of the frequency ratio. But, the system with the elastically supported damper is seen to have the property that the absolute transmissibility diminishes as the inverse square of the frequency ratio. Moreover, viscous damping is still present in the conventional system at high frequency ratios whereas the expression for transmissibility at high frequency ratio for the system with the elastically supported damper is seen to be independent of damping. Thus, by elastically supporting the viscous damper, damping is virtually eliminated at high frequency ratios which is apparently a good property of a vibration isolator.
This comparison of performance of the two systems is shown very well by Figure II-8. All curves on this graph are for a value of damping ratio \(\frac{c}{c_0} = 0.2\) which gives an absolute transmissibility of approximately 2.5 in the conventional isolation system. Several curves are shown, each for a different value of spring ratio.

The \(N=\infty\) curve corresponds to the conventional isolation system with the damper rigidly supported. The \(N=0\) curve represents the undamped response curve of the system. It is seen that for finite values of spring ratio, the resonant amplitude is slightly increased, while at high frequency ratios the isolation is enormously increased. As a quantitative measure consider, for example, the curve corresponding to a spring ratio of unity.

Comparing the response of the two systems, the system with the elastically supported damper is seen to have a resonant amplitude of 1.3 times that of the conventional isolation system. However, at a frequency ratio of 10, the isolation has increased by a factor of 2; and at a frequency ratio of 100, the isolation has increased by a factor of 20.

Thus, by accepting a relatively small increase in transmissibility at resonance, a considerable increase in isolation at high frequencies can be obtained.
CHAPTER III - DISCUSSION OF COULOMB DAMPED SYSTEM

The coulomb damped system to be analysed as shown in Figure I-2 consists of a mass (constrained to move in one direction) supported by a main load carrying spring which is attached to the base. The base is excited in such a manner that its motion varies harmonically with time. The motion of the mass is damped by a coulomb or dry friction damper which is supported by an elastic member attached to the base.

Before getting involved in any mathematical analysis of the problem, perhaps it would be best to remark about the characteristics of a coulomb damper and particularly about the effect of elastically supporting such an element. Basically, the force developed in a coulomb damper arises from the sliding between dry surfaces and depends on the normal force between them. The direction which the force acts on a given surface is opposite that of the relative velocity between the surfaces. Thus, the damping force is positive when the relative velocity across the damper is negative and it is negative when the relative velocity across the damper is positive.

It is now necessary to investigate the effect of elastically supporting a coulomb damper. This will be done by constructing the load-deflection and damping-deflection characteristics of the system.
Consider applying a load to the mass and observing the deflection of the mass and the friction force developed in the coulomb damper. Upon application of the load, the mass will displace, but since insufficient force has been developed in spring $k_1$, the damper will not move. Thus, the motion of the mass is opposed by a stiffness $(k+k_1)$. Note that for this loading the friction force developed can only equal the force developed in spring $k_1$, i.e., the damper friction force increases linearly with the displacement, the constant of proportionality being $k_1$. Thus, for this initial loading, the slope of the load deflection curve is $(k+k_1)$ while the slope of the damping characteristic curve is $k_1$. The load is now increased until a force is developed in $k_1$ which equals $\mu F$, where $F$ is the force normal to the friction surfaces and $\mu$ is the coefficient of friction. If the load is increased beyond this point, the damper will slide thus reducing the spring resistance to the motion of the mass to $k$ and producing

![Diagram](image)

**Fig. III-1**

**Fig. III-2**
a constant friction damping force of value \( \mu F \). Thus, for this portion of the load cycle, the slope of the load deflection curve is \( \alpha \) while the damping characteristic curve has zero slope. Suppose the load has been increased to some maximum value. If it now be decreased below this maximum value the damper will no longer move since the force required to move the damper, which was previously developed in spring \( k_1 \), has been diminished. Therefore, the motion of the mass will again be resisted by a total stiffness of \( (\alpha + k_1) \) and the damping force will vary linearly with the deflection of the mass.

Thus, it is seen that, if the load cycle is completed, the damper will be in motion for part of the cycle and will be at rest for the remaining part. This means that elastically supporting a coulomb damper creates hysteresis type load-deflection and damping characteristics. These characteristics for a complete load cycle are shown in Figure III-1 and Figure III-2.

Thus, the non-linear manner in which the coulomb damping enters into the vibration of the system has been demonstrated. It can be shown (see Appendix C) that the damping term \( \pm B \) necessarily enters the equations of motion, and hence, account must be taken of the non-linearity of the damping force with change in displacement as indicated by Figure III-2. It is fairly obvious that obtaining an exact
analytic solution of the equations of motion would undoubtedly be a formidable task (if not an impossible one) and thus, recourse is taken to other means of obtaining the dynamic response of the system.

A solution for the dynamic response of the system by means of an approximate energy method is given in Appendix A. This is the method of "equivalent viscous damping" where the assumption is made that equal energy is dissipated by an equivalent viscous and the coulomb damper in a cycle of vibration. Equating the cyclic energy dissipation of a viscous damper to that of a coulomb damper, the following relation is obtained:

$$C_e = \frac{4B}{\pi \dot{q}_o \omega}$$

(III-1)

In this expression $\dot{q}_o$ is the relative motion across the damper and $C_e$ is the equivalent viscous damping coefficient.

By determining the relative motion across the damper, and using equation (III-1) in conjunction with the relations for transmissibility of the viscous damped system previously derived, expressions for the transmissibility of the coulomb damped system can be obtained. The absolute displacement transmissibility is thus given by:

$$T_B = \frac{\xi}{A_o} = \sqrt{\frac{1 + \left(\frac{4}{\pi} \hat{\gamma} \right)^2 \left[\left(\frac{N+2}{N}\right)^2 - \frac{2}{(\omega_n)^2} \left(\frac{N+1}{N}\right)\right]}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2}}$$

(III-2)
The expression for the relative displacement transmissibility is as follows:

\[
(T_d)_R = \left| \frac{s_o}{a_o} \right| = \sqrt{\frac{(\frac{\omega}{\omega_n})^4 + \left( \frac{4 \omega}{\nu} \tilde{\eta} \right)^2 \left[ \frac{2}{N} (\frac{\omega}{\omega_n})^2 - (\frac{N+2}{N}) \right]}{(1 - \frac{\omega^2}{\omega_n^2})^2}} \quad (III-3)
\]

The quantity \( \tilde{\eta} \) found in both of these expressions is the coulomb damping factor for constant displacement excitation of the base and is defined by:

\[
\tilde{\eta} = \frac{B}{\nu a_o} \quad (III-4)
\]

Thus, we see the transmissibility of the coulomb damped system is dependent upon the amplitude of excitation.

An interesting observation can be made by considering the terms in the numerator of each of the transmissibility expressions. We see that if the coefficient of the damping term vanishes in each case, the transmissibility is independent of damping, and therefore, all response curves have the same transmissibility for this condition no matter how much damping is present. This is seen to correspond to the "common-transmissibility" point as defined in Chapter I. For absolute motion, setting the coefficient of the damping term equal to zero gives the following condition:

\[
\left( \frac{\omega}{\omega_n} \right)_A^* = \sqrt{\frac{2(N+1)}{N+2}} \quad (III-5)
\]

This is the exact same expression derived for the absolute common-transmissibility frequency ratio of the viscous
damped system. The condition for the relative displacement transmissibility to be independent of damping can be written:

\[
\left( \frac{\omega}{\omega_n} \right)_R^* = \sqrt{\frac{N+2}{2}}
\]  

(III-6)

Again, this corresponds to the relative common-transmissibility frequency ratio for the viscous damped system. The reason why the same relations hold for both the viscous and coulomb damped system is that the method of equivalent viscous damping was employed to derive the coulomb damped equations and there will naturally be some properties common to both systems.

The common-transmissibility for absolute and relative motion can be found by substituting equations (III-5) and (III-6) in the proper expressions for transmissibility. It is found that the common-transmissibility is the same in both cases and that it also agrees with the expression for the viscous damped system. The common-transmissibility is thus given by:

\[
T^* = 1 + \frac{2}{N}
\]  

(III-7)

Therefore, we have determined a point through which all response curves must pass no matter what value of coulomb damping is present. This is not necessarily true in reality but definitely holds for the linearized system being considered here. One should refer to Chapter I for
curves of common-transmissibility and common-transmissibility frequency ratio.

It is important to realize that the coulomb damped solutions are valid only for a certain frequency range. It seems logical that these equations be valid only when there is relative motion across the damper because the whole approximate analysis was made on an assumption of equal energy dissipation in the viscous and coulomb dampers. Therefore, if there is no motion across the coulomb damper, there obviously is no energy dissipated, and thus, the assumption of equal energy dissipation fails. This apparent limitation is not as bad as it might seem at first glance because if there is no relative motion across the coulomb damper it must necessarily be "locked-in" which means the system is undamped with a natural frequency defined by equation (II-2). Thus, for frequency ratios outside the range of validity of the coulomb damped solutions, the absolute and relative transmissibility of the system is given by the undamped solutions expressed by equations (II-7) and (II-8). The frequency ratios which determine the extremes of the range of validity of the coulomb damped solutions were derived in Appendix A and are given by:

\[
\left( \frac{\omega}{\omega_n} \right)_L = \sqrt{\frac{(\frac{4}{N} \hat{\nu})(\frac{N+1}{N})}{\frac{1}{N}(\frac{4}{\pi} \hat{\nu})^2 + 1}}
\]  

(III-8)
The "L" subscript merely indicates that the coulomb damper has either just "broken loose" or "locked-in" as would be the conditions at the extremes of the range of validity. Thus, by use of the transmissibility equations (III-2) and (III-3) and the supplementary equations (III-8), (II-7) and (II-8), the dynamic response of the linearized coulomb damped system for all values of frequency ratio can be obtained. Several plots of these equations will be discussed later.

Before discussing the transmissibility graphs, let us observe what values of the damping parameter $\tilde{\eta}$ are of practical interest. It is evident from equations (III-2) and (III-3) that for a frequency ratio of unity the transmissibility becomes infinite. Thus, to prevent such a condition from arising, the damper must be kept "locked-in" beyond this point. This obviously puts a limit on the value of the damping parameter. This limiting value can be determined from equation (III-8) by requiring the "break loose" frequency ratio to be greater than unity and solving for the damping parameter. The following restriction on the damping parameter is obtained:

$$\tilde{\eta} = \frac{B}{k\alpha_0} > \frac{\pi}{4}$$  \hspace{1cm} (III-9)

Thus, the damping parameter must be greater than $\pi/4$ to avoid having infinite transmissibility at a frequency ratio of unity.
The dynamic properties of the coulomb damped system
are shown very nicely by a series of graphs. Figure III-3
shows the absolute displacement transmissibility of a
system with a spring ratio of 3. As indicated previously,
for values of the damping parameter near \( \pi/4 \), the system
experiences very large transmissibilities in the neighbor-
hood of a frequency ratio of unity. Increase of the damping
parameter above \( \pi/4 \) decreases the resonant transmissibility
until the resonant transmissibility coincides with the
common-transmissibility for the system. The damping
parameter which is associated with this particular curve
is the "optimum" damping parameter. For the system plotted
in Figure III-3, optimum damping is seen to lie between
\( \tilde{\eta} = 1 \) and \( \tilde{\eta} = 2 \). For larger values of damping the
resonant transmissibility becomes larger and the resonant
frequency ratio tends toward the undamped frequency ratio
of the "locked-in" system. It is seen that for these
larger values of damping, the transmissibility is given
by the undamped solution for the "locked-in" system until
the damper "breaks loose" at which time the transmissibility
immediately begins to decrease. As shown, some of these
damped curves eventually intersect the undamped "locked-
in" curve and proceed along it for higher values of
frequency ratio.
Figure III-3 Absolute Displacement Transmissibility for $n = 3$
The relative transmissibility for this system is shown in Figure III-4. It is seen that all response curves do pass through the common-transmissibility point and remarks about the variation in the damping parameter made while discussing the graph of Figure III-3 also apply here.

The graph of Figure III-5 shows another example of absolute displacement transmissibility for the case where the spring ratio is 8. Again, very high transmissibilities exist for damping ratios approaching $\pi/4$. The damping is increased until the resonant transmissibility coincides with the common-transmissibility (Optimum damping). Further increase in damping causes a corresponding increase in resonant transmissibility as would be expected.

One perhaps might notice that not all the damped curves eventually become coincident with the "locked-in" undamped curve. As a matter of fact, this is true and is explainable by consideration of equation (III-8). Damping parameters for which the denominator becomes negative corresponds to those curves which represent a situation where the damper never becomes "locked-in". Thus, the damper will become "locked-in" only when the following inequality is satisfied.

$$\tilde{\eta} > \frac{\pi}{4} N$$

(III-10)
Figure III-5 Absolute Displacement Transmissibility for N = 8
Thus, damping is always present which tends to decrease isolation at high frequency ratios.

For the case where the spring ratio has a finite value, the absolute displacement transmissibility at large frequency ratios is given by the equation:

\[
s_\infty = \frac{N + 1}{\left(\frac{\omega}{\omega_n}\right)^2}
\]

This is valid for systems which satisfy equation (III-10).

Thus, it is seen that if the spring ratio and damping parameter are designed such that equation (III-10) is satisfied, the absolute displacement transmissibility at large frequency ratios will be smaller than that of the system with infinite
spring ratio by a constant amount given by the ratio of equation (III-11) and (III-12).

An interesting point can be made in favor of elastically supporting the coulomb damper when one considers the erratic properties of a rigidly supported coulomb damper that are sometimes observed at high frequency ratios. Although no investigation as to the reasons for these properties occurring will be made here, it seems reasonable that the wear problem in the coulomb damper may very well contribute to these undesirable effects. However, if one manages to design the damper to "lock-in" at high frequency ratios, there will be no motion across the damper, and thus, no associated wear problem at high frequency ratios. The real value of elastically supporting the coulomb damper is more effectively shown when acceleration transmissibility at high frequency ratios is considered.

It is certainly more sensible to talk in terms of acceleration transmissibility for systems operating in regions of high frequency ratios because, although the displacements may very well be small, the corresponding accelerations can be of considerable value. Also, certain machines operate under a constant acceleration environment as opposed to a constant displacement environment. These facts, coupled with the fact that it is the acceleration (or its associated inertia force) that causes damage to the mounted structure,
give good reason to now consider the effect that elastically supporting the coulomb damper has on the acceleration transmissibility.

The equations giving the acceleration transmissibility of the system with the elastically supported damper can be obtained from equations (III-2) and (III-3) by defining the coulomb damping parameter for constant acceleration excitation of the base as follows:

\[ \tilde{\eta} = \frac{B}{mA_0} \]  

(III-13)

where \( A_0 \) is the maximum amplitude of the sinusoidal acceleration being applied to the base. As indicated in Appendix A the equations for acceleration transmissibility are obtained by replacing \( \tilde{\eta} \) in equation (III-2) by the following quantity:

\[ \tilde{\eta} = \eta \left( \frac{\omega}{\omega_n} \right)^2 \]  

(III-14)

The expression for acceleration transmissibility is then given by:

\[ (T_A) = \left| \frac{\ddot{\alpha}}{\alpha} \right| = \sqrt{\frac{1 + \left( \frac{4}{\pi \eta} \right)^2 \left( \frac{\omega}{\omega_n} \right)^2 \left[ \left( \frac{N+2}{N} \right) \left( \frac{\omega}{\omega_n} \right)^2 - 2 \left( \frac{N+1}{N} \right) \right]}{\left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2}} \]  

(III-15)

This equation is naturally valid only for the frequency range for which there is relative motion across the
Thus, for frequency ratios outside the range specified by equation (III-16) the system is undamped and the acceleration transmissibility is given by equation (II-7). Also, it is seen by inspecting equation (III-15) that infinite acceleration transmissibility occurs for a frequency ratio of unity. Thus, to prevent such a condition from arising the damper must be designed such that it remains "locked-in" beyond a frequency ratio of unity. The limiting value placed on the damping parameter is obtained by requiring equation (III-16) to be greater than unity for the lower frequency ratio. The limit then placed on the coulomb damper is then given by:

\[
\eta = \frac{B}{mA_0} > \frac{\pi}{4}
\]

(III-17)

Inspection of the coefficient of the damping term in equation (III-15) reveals that common-transmissibility and the common-transmissibility frequency ratio are the same as for the displacement excited system. When the spring ratio becomes infinite, equation (III-15) reduces
to the known approximate equation for the acceleration transmissibility of a coulomb damped system with the coulomb damper rigidly supported, which is given by:

$$
(T_A)_{N=\infty} = \left| \frac{\ddot{x}}{\ddot{z}} \right|_{N=\infty} = \sqrt{\frac{1 + (\frac{4}{\pi} \eta)^2 (\frac{\omega}{\omega_n})^2 [(\frac{\omega}{\omega_n})^2 - 2]}{(1 - \frac{\omega^2}{\omega_n^2})^2}}
$$

(III-18)

Now, for high frequency ratios, equation (III-18) is seen to reduce to:

$$
(T_A)_{N=\infty} = \frac{4}{\pi} \eta = \frac{4B}{\pi m A_0}
$$

(III-19)

But, as previously indicated, the system with the elastically supported coulomb damper has a transmissibility at high frequency ratios as given by the equation:

$$
(T_A)_{N \neq \infty} = \frac{N+1}{(\frac{\omega}{\omega_n})^2}
$$

(III-20)

It immediately becomes obvious that there is much to gain by elastically supporting the coulomb damper. The conventional system has the property that the absolute acceleration transmissibility approaches a constant value given by equation (III-19) for large frequency ratios. This means the isolation property flattens out at high frequency ratios and a constant amount of acceleration is transmitted to the supported structure. However, one notices that by elastically supporting the coulomb damper,
the transmissibility at high frequency ratios no longer approaches a constant value but varies as the inverse square of the frequency ratio, or in other words, approaches zero as a limit. Thus, the introduction of an elastic support for the coulomb damper very effectively reduces the acceleration, and thus, the damaging inertia forces transmitted to the supported member.

As a means to more exactly ascertain the dynamic response of the system with the elastically supported damper, an analog computer system was devised from which the absolute acceleration transmissibility was obtained. The complete analog computer analysis is given in Appendix C and includes a full explanation of how the non-linear damping effects were produced. The load deflection and the hysteresis type damping characteristics were monitored on oscilloscopes during the course of the computer analysis to insure that the analog of the coulomb-damping element was properly introduced into the system. Thus, if the analog computer system is, in fact, a good analog of the mechanical system being investigated, these characteristics being monitored should correspond to Figure III-1 and Figure III-2.

These characteristics were very nicely produced in the analog system and Figure III-6 shows a picture of
the typical load deflection and the damping characteristics as photographed from the monitor oscilloscopes. Notice how closely these pictures resemble the characteristics as indicated by Figure III-1 and III-2. The top trace corresponds to the load deflection characteristic and the bottom trace corresponds to the hysteresis coulomb-damping characteristic. The curves have sharply defined corners which would correspond to the characteristics of the "idealized" mathematical model shown in Figure I-2.

For purposes of indicating the waveforms of acceleration, velocity and displacement, a photograph was taken of these quantities displayed on the monitor oscilloscope and is
shown in Figure III-7. The bottom curve is the waveform of the relative displacement; the middle curve is the waveform of the relative velocity; and the top curve shows the absolute acceleration of the mass. As shown, the displacement is very nearly sinusoidal while the velocity becomes slightly distorted and the acceleration takes on the "chopped off" sine wave shape.

Figure III-8 shows the acceleration transmissibility of a system with a spring ratio of 3 as obtained from the analog computer. The spring ratio of 3 was chosen for particular study in an attempt to provide some analytical
ABSOLUTE ACCELERATION TRANSMISSIBILITY

\[ \left( \frac{\omega^2}{\omega_m^2} \right) - \text{FREQUENCY RATIO} \]

\[ R_t^2 = \frac{B}{mA_0} \]
verification of an experimental curve given in Reference 5 for a system with an effective spring ratio of 3. There appears to be an optimum damping curve corresponding to ψ ≈ 1. For damping parameters less than optimum, a slight decrease in damping is seen to affect a very large change in transmissibility. This statement is not true for damping greater than optimum. The shape of the response curves indicate that the approximate linearized analysis gives a fairly good representation of the solution to the problem. Figure III-3 also indicates how the response curves begin out on the stiff undamped curve and eventually return to it at high frequency ratios, which bears out the results of the previous simplified analysis. These response curves definitely have a resemblance to those given in Reference 5.

The large areas of double valued response as shown in this reference clearly are not shown by the analog computer. The reason for this may lie in the fact that the experimental curves would naturally be affected by the difference in static and kinetic coefficients of friction, i.e., the system would have different "break loose" points for increasing and decreasing frequency ratios. On the other hand, the analog computer is just affected by the kinetic
coefficient of friction (because of the "idealized mathematical model" chosen) and thus, would only have one "break loose" point.

Thus, it is felt that the analog computer solution given in Appendix C makes available the non-linear response of a system with a coulomb damper elastically supported. This solution coupled with the linearized theory solution should provide sufficient information to intelligently design such a system.
CHAPTER IV - EPILOGUE

The mathematical analysis of Appendix A coupled with the analog computer analysis of Appendix B provide means of obtaining the exact solution of the dynamic response of the system with an elastically supported viscous damper. A thorough discussion and appraisal of the dynamic characteristics of this system along with graphs of the solutions can be found in Chapter II. It would be convenient if a complete series of graphs were available, and thus, a calculation program which would embody many more values of the dimensionless damping and spring ratio parameters is a suggested program for the future. Also, development of an analytical expression giving the optimum damping for a given spring ratio would be an appropriate addition to the theory developed herein.

As for the coulomb damped system, the solutions developed in Appendix A and discussed in Chapter III are approximate and should be treated as such. They certainly give much information about the characteristics of the coulomb damped system, but do not embody the non-linear characteristics which are known to exist since the solutions are linearized.

It is felt that the computer analysis
gives an accurate description of the dynamic response of the coulomb damped system. For this reason, it is suggested that further work be done in obtaining more response curves from the analog computer in addition to increasing the number of curves obtained from the approximate solution. In particular, it would be interesting to plot graphs of the approximate solutions for the acceleration transmissibility and compare these results with the computer results.

As indicated by the discussions in Chapter II and Chapter III, these systems should be very useful in problems of vibration isolation and particularly in problems where a high degree of isolation is required at high frequencies.

It should be mentioned that solutions for the problem where the force is applied to the mass are readily obtained with a small amount of additional analysis. For a force \( P = P_0 \sin \omega t \) applied to the mass as shown in Figure IV-1 and Figure IV-2, the analysis is carried out in a manner
similar to that in Appendix A. The expression giving the displacement of the viscous damped system is found to be:

\[
\frac{x_0}{P_0_e} = \sqrt{1 + \frac{4}{N^2} \left(\frac{c}{c_c}\right)^2 \left(\frac{\omega}{\omega_n}\right)^2 \left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 \left(\frac{c}{c_c}\right)^2 \left(\frac{\omega}{\omega_n}\right)^2}
\]

(IV-1)

The force transmissibility, which is defined as the ratio of the force transmitted to the base to the force applied to the mass, can be found by realizing that the force transmitted to the base is given by the sum of \(k x\) and \(k_x1\). The force transmissibility is thus given by:

\[
\frac{F_T}{P_o} = \sqrt{1 + 4\left(\frac{N+1}{N}\right)^2 \left(\frac{c}{c_c}\right)^2 \left(\frac{\omega}{\omega_n}\right)^2
\]

(IV-2)

This can be recognized as the equation for the absolute transmissibility for the case where the base is being excited. Thus, the graphs in Chapter II for the absolute transmissibility of the base excited system also give the force transmissibility of the force excited system.

* Equation not correct

J.E.R.
Similar expressions can be derived for the coulomb damped system by use of the technique of equivalent viscous damping. For example, the displacement of the coulomb damped system is given by:

\[
\frac{x_0}{P_0/f_k} = \sqrt{\frac{1 - \frac{1}{N^2} \left( \frac{4}{\pi} \frac{B}{P_0} \right)^2 \left( N + 1 - \frac{\omega^2}{\omega_n^2} \right)^2}{\left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2}}
\]  

(IV-3)

By replacing \( \eta \) by \( B/P_0 \), the force transmissibility can be shown to be the same as equation (III-15) which is the equation giving the absolute acceleration transmissibility of the base accelerated excited system. Statements analogous to those made in Chapter III concerning range of validity and good design values of \( \eta \) can easily be determined.

Thus, both analytical expressions and analog computer solutions of the forced vibrations of systems with elastically supported dampers have been developed, and it is hoped that these solutions will be a significant contribution to the field of mechanical vibrations.
# APPENDIX A

**STEADY STATE RESPONSE OF VISCOS DAMPED SYSTEM and APPROXIMATE SOLUTION OF THE COULOMB DAMPED SYSTEM**

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SECTION A-1  INTRODUCTION

The mechanical systems to be analyzed in this appendix are shown in Figure (A-1) and Figure (A-2). Figure (A-1) shows the viscous damped system with the viscous damper supported by a spring $k_1 = N k$. Figure (A-2) shows the coulomb damped system with the coulomb damper supported by a spring $k_1 = N k$. Conventional methods are used to solve the viscous damped system and the method of equivalent viscous damping is used to solve the coulomb damped system.
Applying Newton's First Law:

For Mass: \( \sum F_x = m \ddot{x} \)
\(-k(x-a) - c(\dot{x} - \dot{x}_1) = m \ddot{x} \) \hspace{1cm} (A-1)

For Damper: \( \sum F_x = 0 \)
\(-k_1(x_1-a) + c(\dot{x} - \dot{x}_1) = 0 \) \hspace{1cm} (A-2)

Substituting (A-2) in (A-1):
\(-k(x-a) - k_1(x_1-a) = m \ddot{x} \)

Thus we Obtain:
\( x_1 = -\frac{1}{k_1} (m \dddot{x} + k(x-a) - k_1 \dot{a}) \) \hspace{1cm} (A-3)

Differentiating:
\( \ddot{x}_1 = -\frac{1}{k_1} (m \dddot{x} + k(\dddot{x} - \dot{a}) - k_1 \ddot{a}) \) \hspace{1cm} (A-4)

Substitute (A-4) in (A-1):
\( \frac{mc}{k_1} \dddot{x} + m \dddot{x} + c \left( 1 + \frac{k}{k_1} \right)(\dddot{x} - \dot{a}) + k(x-a) = 0 \)

But:
\( k_1 = N k \),
\( \frac{mc}{kN} \dddot{x} + m \dddot{x} + c \left( \frac{N+1}{N} \right)(\dddot{x} - \dot{a}) + k(x-a) = 0 \) \hspace{1cm} (A-5)
Equation (A-5) is the basic differential equation of motion for the system of Figure A-1. We shall now transform this equation into forms which describe absolute and relative motion of the mass.

From (A-5):
\[ \frac{mc}{kN} \ddot{x} + m \ddot{x} + c \left( \frac{N+1}{N} \right) \dot{x} + kx = c \left( \frac{N+1}{N} \right) \dot{a} + ka \]  
(A-6)

Equation (A-6) is the differential equation for the absolute motion of the mass.

Now \( s = (x-a) \) = Motion of mass relative to base
\[ \dot{s} = (\dot{x} - \dot{a}) ; \] etc.

Substituting these identities into (A-5):
\[ \frac{mc}{kN} \ddot{s} + m \ddot{s} + c \left( \frac{N+1}{N} \right) \dot{s} + ks = -\frac{mc}{kN} \ddot{a} - ma \]  
(A-7)

Equation (A-7) is the differential equation for the motion of the mass relative to the base.

Now \( \ddot{g} = (x - x_i) \) = Motion of mass relative to the damper
\[ \ddot{g} = (\dot{x} - \dot{x_i}) ; \] etc.

From (A-2):
\[ x_i = a + \frac{c}{k_i} \dot{g} \]
From (A-1):
\[ x = a - \frac{m}{k} \ddot{x} - \frac{c}{k} \dot{g} \]

But \( \ddot{x} = \ddot{g} + \ddot{x_i} = \ddot{g} + \ddot{a} + \frac{c}{k_i} \ddot{g} \)

Subtracting:
\[ \ddot{g} = (x - x_i) = a - \frac{m}{k} \left[ \ddot{g} + \ddot{a} + \frac{c}{k_i} \ddot{g} \right] - c \left[ \frac{1}{k} + \frac{1}{k_i} \right] \ddot{g} - a \]

Noting \( k_i = Nk \),
\[ \frac{mc}{kN} \ddot{g} + m \ddot{g} + c \left( \frac{N+1}{N} \right) \dot{g} + kg = -ma \]  
(A-8)

Equation (A-8) is the differential equation for the motion of the mass relative to the damper.

**SECTION A-3 SOLUTIONS OF THE DIFFERENTIAL EQUATIONS**

The motion of the base is steady state sinusoidal motion.

\[ a = a_0 \sin \omega t \]  
(A-9, a)
\[ \dot{a} = a_0 \omega \cos \omega t \]  
(A-9, b)
\[ \ddot{a} = -a_0 \omega^2 \sin \omega t \]  
(A-9, c)
\[ \dddot{a} = -a_0 \omega^3 \cos \omega t \]  
(A-9, d)
SECTION A-3.1 ABSOLUTE MOTION OF MASS

\[
\frac{mc}{kN} \dddot{x} + m \dddot{x} + c\left(\frac{N+1}{N}\right)x + kx = c\left(\frac{N+1}{N}\right)a + ka \tag{A-6}
\]

Assume the following solution:

\[
x = x_0 \sin(\omega t + \phi) \tag{A-10,a}
\]
\[
\dot{x} = x_0 \omega \cos(\omega t + \phi) \tag{A-10,b}
\]
\[
\ddot{x} = -x_0 \omega^2 \sin(\omega t + \phi) \tag{A-10,c}
\]
\[
\dddot{x} = -x_0 \omega^3 \cos(\omega t + \phi) \tag{A-10,d}
\]

Substituting \((A-10,a, b, c, d)\) and \((A-9,a, b)\) in \((A-6)\):

\[
\frac{mc}{kN} \left(-x_0 \omega^3\right) \cos(\omega t + \phi) + m(-x_0 \omega^2 \sin(\omega t + \phi)) + c\left(\frac{N+1}{N}\right)(x_0 \omega) \cos(\omega t + \phi) \\
+ k(x_0) \sin(\omega t + \phi) = c\left(\frac{N+1}{N}\right)(a_0 \omega) \cos \omega t + k(a_0) \sin \omega t
\]

Grouping terms, we obtain:

\[
\frac{x_0}{a_0} = \frac{[k] \sin \omega t + [c \left(\frac{N+1}{N}\right) \omega] \cos \omega t}{[k - m \omega^2] \sin(\omega t + \phi) + [c \left(\frac{N+1}{N}\right) \omega - \frac{mc}{kN} \omega^2] \cos(\omega t + \phi)} \tag{A-11}
\]

Now observe the following equation:

\[
\ddot{X} = U \sin \omega t + V \cos \omega t \tag{A-12}
\]

The absolute value of \(\ddot{X}\) is given by:

\[
|\ddot{X}| = \sqrt{U^2 + V^2} \tag{A-13}
\]

Thus we may obtain the absolute value of equation \((A-11)\), and noting that \((T_D)_A = \left| \frac{x_0}{a_0} \right| = \text{absolute displacement transmissibility}:

\[
(T_D)_A = \sqrt{\frac{k^2 + (c \omega)^2 \left(\frac{N+1}{N}\right)^2}{[k - m \omega^2]^2 + [\frac{c \omega}{N}]^2 \left[\frac{N+1 - \frac{m \omega^2}{k}}{N} \right]^2}} \tag{A-14}
\]

Note the following definitions and identities:

\[
\omega_n = \sqrt{\frac{k}{m}} = \text{Undamped angular natural frequency} \tag{A-15}
\]
\[ C_c = 2m_\omega n = 2\sqrt{k/m} \]  
\[ C/c_c = \frac{c}{2m_\omega n} \]  
\[ (\frac{c}{m_\omega n}) = 2(\frac{c}{c_c}) \]  

By using Equations (A-15) and (A-18), Equation (A-14) can be shown to take the form:

\[ (T_0)_a = \left| \frac{X_0}{a_0} \right| = \sqrt{1 + 4(\frac{N+1}{N})^2(\frac{c}{c_c})^2(\frac{\omega}{\omega_n})^2} \]  
\[ \frac{1 - \omega^2}{\omega_n^2} + \frac{4}{N^2}(N+1-\omega^2)(\frac{c}{c_c})(\frac{\omega}{\omega_n})^2 \]  

Thus we have determined the absolute displacement transmissibility of the mass in terms of the dimensionless parameters \( N, c/c_c, \) and \( \omega/\omega_n \).

**SECTION A-3.2 MOTION OF MASS RELATIVE TO BASE**

\[ \frac{mc}{k_N} \dddot{s} + m \dddot{s} + c(\frac{N+1}{N}) \dddot{s} + ks = -\frac{mc}{k_N} \ddot{\alpha} - ma \]  

Assume the following solution:

\[ \dddot{s} = \dddot{S}_o \sin (\omega t + \theta) \]  
\[ \dddot{s} = \dddot{S}_o \omega \cos (\omega t + \theta) \]  
\[ \dddot{s} = -\ddot{S}_o \omega^2 \sin (\omega t + \theta) \]  
\[ \dddot{s} = -\dddot{S}_o \omega^3 \cos (\omega t + \theta) \]

Substituting \((A-20,a, b, c, d)\) and \((A-9,c, d,)\) in \((A-7)\):

\[ \frac{mc}{k_N}(-\dddot{S}_o \omega^3 \cos (\omega t + \theta) + m(-\dddot{S}_o \omega^3) \sin (\omega t + \theta) \]

\[ + c(\frac{N+1}{N})(\dddot{S}_o \omega) \cos (\omega t + \theta) + k(\dddot{S}_o) \sin (\omega t + \theta) \]

\[ = -\frac{mc}{k_N}(-S_o \omega^3) \cos \omega t - m(-S_o \omega^3) \sin \omega t \]
Grouping terms, we obtain:

$$\frac{\delta_o}{a_0} = \frac{[mw^2] \sin wt + \left[ \frac{mcw^3}{kN} \right] \cos wt}{[k-mw^2] \sin (wt+\theta) + \left[ c\left( \frac{N+1}{N} \right) w - \frac{mcw^3}{kN} \right] \cos (wt+\theta)}$$

By virtue of Equation (A-13) we have:

$$\left( \frac{T_D}{R} \right) = \left| \frac{\delta_o}{a_0} \right| = \sqrt{\frac{[mw^2]^2 + \left[ \frac{mcw^3}{kN} \right]^2}{[k-mw^2]^2 + \left[ c\left( \frac{N+1}{N} \right) w - \frac{mcw^3}{kN} \right]^2}}$$

(A-21)

Where \( \left( T_D \right)_R = \left| \frac{\delta_o}{a_0} \right| = \text{Relative displacement transmissibility of the mass (Relative to the Base)} \).

By use of Equation (A-15) and (A-18) this equation can be shown to take the form:

$$\left( \frac{T_D}{R} \right) = \left| \frac{\delta_o}{a_0} \right| = \sqrt{\frac{\left[ \frac{w}{\omega_n} \right]^4 + \frac{4}{N^2} \left( \frac{c}{c_c} \right)^2 \left( \frac{w}{\omega_n} \right)^6}{\left[ 1 - \frac{w^2}{\omega_n^2} \right]^2 + \frac{4}{N^2} \left( N+1 - \frac{w^2}{\omega_n^2} \right)^2 \left( \frac{c}{c_c} \right)^2 \left( \frac{w}{\omega_n} \right)^2}}$$

(A-22)

Thus we have determined the transmissibility of the mass relative to the base in terms of the dimensionless parameters \( N, \frac{c}{c_c} \), and \( \frac{w}{\omega_n} \).
**SECTION A-3.3 MOTION OF MASS RELATIVE TO DAMPER**

\[
\frac{mc}{kN} \ddot{y} + m \ddot{y} + c \left( \frac{N+1}{N} \right) \dot{y} + k \ddot{y} = -ma 
\]  

(A-8)

Assume the following solution:

\[ \begin{align*}
\dot{y} &= y_0 \sin (wt + \psi) \\
\ddot{y} &= y_0 \omega \cos (wt + \psi) \\
\dddot{y} &= -y_0 \omega^2 \sin (wt + \psi) \\
\ddot{y} &= -y_0 \omega^3 \cos (wt + \psi)
\end{align*} \quad \text{(A-23,a,b,c,d)}

Substituting (A-23,a, b, c, d) and (A-9,c) into (A-8):

\[
\frac{mc}{kN} (-y_0 \omega^3) \cos (wt + \psi) + m (-y_0 \omega^2) \sin (wt + \psi)
\]

\[
+ c \left( \frac{N+1}{N} \right) (y_0 \omega) \cos (wt + \psi) + k (y_0) \sin (wt + \psi)
\]

\[= -m (-y_0 \omega^2) \sin wt \]

Grouping terms, we obtain:

\[
y_0 = \frac{(m \omega^2) \sin wt}{(k-m \omega^2) \sin (wt + \psi) + \left[ c \left( \frac{N+1}{N} \right) \omega - \frac{mc}{kN} \omega^3 \right] \cos (wt + \psi)}
\]
Using equation (A-13) we have:

\[ \left| \frac{\gamma_o}{a_o} \right| = \sqrt{\frac{[m \omega^3]^2}{\left( \frac{K}{m} \right)^2 + \left[ \frac{c}{N} \right]^2 \left( N + 1 - \frac{m \omega^2}{K} \right)^2}} \]

By using equation (A-15) and (A-18):

\[ \left| \frac{\gamma_o}{a_o} \right| = \sqrt{\frac{\frac{\omega}{\omega_n}^4}{\left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \frac{4}{N^2} \left( N + 1 - \frac{\omega^2}{\omega_n^2} \right)^2 \left( \frac{c}{C_c} \right)^2 \left( \frac{\omega}{\omega_n} \right)^2}} \]  

(A-24)

Thus we have obtained the ratio of the absolute values of the displacement of the mass (relative to the damper) to the displacement of the base in terms of the dimensionless parameters \( N_0 \frac{c}{C_c} \) and \( \frac{\omega}{\omega_n} \). This expression will be useful when the problem of coulomb damping is considered (Figure A-2).

SECTION A-4 ANALYSIS OF UNDAMPED SYSTEM

If the value of \( \frac{c}{C_c} \) in equation (A-19) and equation (A-22) is made to approach zero, the classical equations for the undamped response of a single degree of freedom system are obtained. The undamped absolute transmissibility is given by:

\[ (T)_A = \frac{1}{1 - \left( \frac{\omega}{\omega_n} \right)^2} \]  

(A-25)
The undamped relative transmissibility is given by:

\[ (T)_R = \frac{(\frac{\omega}{\omega_n})^2}{1 - (\frac{\omega}{\omega_n})^2} \]  

Now as \( \frac{C}{C_c} \) is made to approach infinity, the damper becomes "locked in" and the motion of the mass is resisted by a total spring rate of \((N+1)k\). Thus, it is obvious that this undamped system has a natural frequency given by:

\[ \omega_n' = \sqrt{N+1} \omega_n \]  

Substitution of this equation into the two previous equations gives expressions for the transmissibility of system when the damper is "locked in". The absolute transmissibility of the "locked in" system is given by:

\[ (T)'_A = \frac{N+1}{(N+1) - (\frac{\omega}{\omega_n})^2} \]  

While the relative transmissibility is given by:

\[ (T)'_R = \frac{(\frac{\omega}{\omega_n})^2}{(N+1) - (\frac{\omega}{\omega_n})^2} \]  

SECTION A-5 DETERMINATION OF COMMON-TRANSMISSIBILITY POINT

If one plots the absolute values of the transmissibilities of the two undamped conditions, it is seen that the response curve for the zero damping case intersects the response curve for the infinite damping case. The frequency at which the two response curves intersect can be found by equating the two transmissibility expressions (taking in
consideration the proper sign of that portion of the
response curve being investigated) and solving for the
frequency ratio. Thus, for absolute transmissibility:

\[
\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = \frac{N+1}{(N+1) - \left(\frac{\omega}{\omega_n}\right)^2}
\]  

(A-30)

Solving for the frequency ratio and denoting it by \(\left(\frac{\omega}{\omega_n}\right)_A^\star\) - the common-transmissibility frequency ratio for absolute motion:

\[
\left(\frac{\omega}{\omega_n}\right)_A^\star = \sqrt{\frac{2(N+1)}{N+2}}
\]  

(A-31)

This expression then gives the frequency ratio at which
the absolute transmissibility of the undamped cases have
a common value of absolute transmissibility. To find the
transmissibility at this frequency ratio, substitute
equation (A-31) in the general expression for absolute
transmissibility as given by equation (A-19). When this
substitution is made, it is found that the damping terms
disappear and the transmissibility reduces to the following
equation:

\[
T_A^\star = 1 + \frac{2}{N}
\]  

(A-32)

Thus, since the viscous damping terms do not appear in this
equation, it is deduced that this point is a common-trans-
missibility point for the system - no matter what value
of viscous damping. The frequency ratio defining the
common-transmissibility point for relative motion is obtained by equating equation (A-26) to equation (A-29), which results in the following expression:

\[
\left(\frac{\omega}{\omega_n}\right)_R^* = \sqrt{\frac{N+2}{2}}
\]  
(A-33)

Substituting this relation into equation (A-22):

\[
T_R^* = 1 + \frac{2}{N}
\]  
(A-34)

Thus, it is seen that the relative transmissibility of the system at the frequency ratio defined by equation (A-33) is the same regardless what amount of viscous damping is present. Moreover, it should be noted that although the common-transmissibility frequency ratio is different for absolute and relative motion, the common-transmissibility itself is equal in each case. Thus, it has been shown that there exists a point through which all the response curves pass no matter what amount of viscous damping is represented in each curve.

SECTION A-6 TRANSMISSIBILITY FOR LARGE FREQUENCY RATIO

Of practical importance is the transmissibility of the system for large values of frequency ratio. This transmissibility can be obtained by letting the frequency ratio in equation (A-19) and equation (A-22) become very large and thus eliminate certain terms on the basis that they are small compared to other terms. Letting \( T_i \) represent the
desired transmissibility (i subscript merely indicating that large frequency ratio corresponds to "isolation" rather than a "magnification"), the following expressions are obtained:

\[(Ti)_A = \frac{N+1}{(\frac{\omega}{\omega_n})^2}\]  \hspace{1cm} (A-35)

and

\[(Ti)_R = 1\]  \hspace{1cm} (A-36)

By comparison with equation (A-28) and equation (A-29) it is seen that, for large frequency ratios, the transmissibility of the system approaches the undamped response of the system with the "locked-in" damper. Thus, the effect of viscous damping "cuts-out" at large frequency ratios thus, rendering the system essentially without damping. This property is very important to those concerned with problems in vibration isolation.

SECTION A-7 EQUIVALENT VISCOUS DAMPING FOR COULOMB DAMPING

If one assumes that the energy dissipated per cycle by a viscous system shown in Figure (A-1) is equal to that dissipated by a dry friction system shown in Figure (A-2), one can obtain an expression for the "equivalent viscous damping" of the dry friction system. (See Reference 8).

\[C_e = \frac{4B}{\pi \rho_0 \omega}\]  \hspace{1cm} (A-37)
Here \( g_0 \) must be the maximum displacement amplitude of the mass with respect to the damper. By using (A-15) and (A-16) we obtain:

\[
\left( \frac{c}{c_c} \right)_e = \frac{2B}{\pi R g_0} \left( \frac{\omega}{\omega_n} \right)
\]

(A-38)

Substituting \( g_0 \) from (A-24), squaring and collecting terms:

\[
\left( \frac{c}{c_c} \right)_e^2 = \frac{\left( \frac{2B}{\pi R g_0} \right)^2 \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2}{\left( \frac{\omega}{\omega_n} \right)^2 \left[ \left( \frac{\omega}{\omega_n} \right)^4 - \left( \frac{2B}{\pi R g_0} \right)^2 \frac{4}{N^2} \left( N+1 - \frac{\omega^2}{\omega_n^2} \right)^2 \right]}
\]

If we define

\[
\tilde{\eta} = \frac{B}{R g_0}
\]

(A-39)

Where \( \tilde{\eta} \) = dimensionless coulomb damping coefficient, we obtain:

\[
\left( \frac{c}{c_c} \right)_e^2 = \frac{\left( \frac{2\tilde{\eta}}{\pi} \right)^2 \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2}{\left( \frac{\omega}{\omega_n} \right)^2 \left[ \left( \frac{\omega}{\omega_n} \right)^4 - \left( \frac{4\tilde{\eta}}{\pi N} \right)^2 \left( N+1 - \frac{\omega^2}{\omega_n^2} \right)^2 \right]}
\]

(A-40)

Thus we have obtained an expression for the equivalent viscous damping ratio in terms of \( \tilde{\eta} \), the coulomb damping parameter.

**SECTION A-8 TRANSMISSIBILITY OF COULOMB DAMPED SYSTEM**

We may now obtain expressions for the transmissibility of the coulomb damped system shown in Figure (A-2) by use of equation (A-19), (A-22), and (A-19).
SECTION A-8.1 ABSOLUTE DISPLACEMENT TRANSMISSIBILITY

To obtain a relation for the absolute displacement transmissibility, we must replace \( \frac{c}{c_c} \) in equation (A-19) by \( \frac{c}{c_c} \) as given by equation (A-40).

\[
4 \left( \frac{N+1}{N} \right)^2 \left( \frac{c}{c_c} \right)^2 \left( \frac{w}{w_n} \right)^2 = 4 \left( \frac{N+1}{N} \right)^2 \left( \frac{w}{w_n} \right)^2 \frac{\left( \frac{2 \mu}{\pi} \right)^2 \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2}{\left( \frac{w}{w_n} \right)^2 \left[ \left( \frac{w}{w_n} \right)^4 - \left( \frac{4 \mu}{\pi N} \right)^2 \left( N+1 - \frac{\omega^2}{\omega_n^2} \right)^2 \right]}
\]

\[
= \frac{\left( \frac{N+1}{N} \right)^2 \left( \frac{4 \mu}{\pi N} \right)^2 \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2}{\left( \frac{w}{w_n} \right)^4 - \left( \frac{4 \mu}{\pi N} \right)^2 \left( N+1 - \frac{\omega^2}{\omega_n^2} \right)^2}
\]

\[
\frac{4}{N^2} \left( \frac{N+1 - \frac{\omega^2}{\omega_n^2}}{w_n^2} \right)^2 \left( \frac{c}{c_c} \right)^2 \left( \frac{w}{w_n} \right)^2 = \frac{4}{N^2} \left( \frac{N+1 - \frac{\omega^2}{\omega_n^2}}{w_n^2} \right)^2 \left( \frac{w}{w_n} \right)^2 \frac{\left( \frac{2 \mu}{\pi} \right)^2 \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2}{\left( \frac{w}{w_n} \right)^2 \left[ \left( \frac{w}{w_n} \right)^4 - \left( \frac{4 \mu}{\pi N} \right)^2 \left( N+1 - \frac{\omega^2}{\omega_n^2} \right)^2 \right]}
\]

\[
= \frac{\left( \frac{4 \mu}{\pi N} \right)^2 \left( N+1 - \frac{\omega^2}{\omega_n^2} \right)^2 \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2}{\left( \frac{w}{w_n} \right)^4 - \left( \frac{4 \mu}{\pi N} \right)^2 \left( N+1 - \frac{\omega^2}{\omega_n^2} \right)^2}
\]

Thus:

\[
(T_d)_A = \left| \frac{x_0}{a_0} \right| = \sqrt{\frac{\left( \frac{w}{w_n} \right)^4 + \left( \frac{4 \mu}{\pi N} \right)^2 \left[ \left( N+1 \right)^2 \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2 - \left( N+1 - \frac{\omega^2}{\omega_n^2} \right)^2 \right]}{\left( \frac{w}{w_n} \right)^4 \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2}}
\]

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Let $\beta = \left( \frac{w}{\omega_n} \right)$:
\[
\begin{align*}
(N+1)^2 (1-\beta^2)^2 &= (N+1)^2 (1-2\beta^2+\beta^4) \\
(N+1-\beta^2)^2 &= (N+1)^2 - 2(N+1)\beta^2 + \beta^4
\end{align*}
\]
Thus:
\[
\begin{align*}
[ (N+1)^2 (1-\beta^2)^2 - (N+1-\beta^2)^2 ] &= \beta^4 [ (N+1)^2 - N^2 ] - 2\beta^2 [ (N+1)^2 - (N+1) ] \\
&= \beta^4 [ N^2 + 2N ] - 2\beta^2 [ N+1 ] N \\
&= \beta^4 N [N+2] - 2\beta^2 N [N+1] \\
&= N [ \beta^4 (N+2) - 2\beta^2 (N+1) ] \\
&= N \beta^2 [ (N+2) \beta^2 - 2(N+1) ]
\end{align*}
\]
Substituting this identity in the relation for $(T_D)_A$:
\[
(T_D)_A = \sqrt{\frac{(\frac{w}{\omega_n})^4 + (\frac{4N}{\pi})^2 N (\frac{w}{\omega_n})^2 [ (N+2)(\frac{w}{\omega_n})^2 - 2(N+1) ]}{(\frac{w}{\omega_n})^4 (1-\frac{w^2}{\omega_n^2})^2}}
\]
Divide numerator and denominator by $(\frac{w}{\omega_n})^4$:
\[
(T_D)_A = \sqrt{\frac{1 + (\frac{4N}{\pi})^2 [ (\frac{N+2}{N}) - \frac{2}{\omega_n^2} (\frac{N+1}{N}) ]}{(1-\frac{w^2}{\omega_n^2})^2}}
\]  
(A-41)
Thus, equation (A-41) gives the absolute displacement transmissibility of the coulomb damped system in terms of the dimensionless parameters $N$, $\frac{\tilde{N}}{\pi N}$ and $(\frac{w}{\omega_n})$. It is of interest to note that when $N \to \infty$ equation (A-41) reduces to the known relation for the absolute displacement transmissibility of a coulomb damped system where the damper is rigidly supported.

SECTION A-8.2 RELATIVE DISPLACEMENT TRANSMISSIBILITY

To obtain a relation for relative displacement transmissibility, we must replace $(\%c)$ in equation (A-22) by $(\%c)_e$ as given by equation (A-40).

\[
\frac{4}{N^2} \left( \frac{c}{c_c} \right)^2 \left( \frac{w}{\omega_n} \right)^6 = \frac{\frac{4}{N^2} \left( \frac{2 \tilde{N}}{\pi} \right)^2 \left( 1 - \frac{w^2}{\omega_n^2} \right)^2 \left( \frac{w}{\omega_n} \right)^6}{\left( \frac{w}{\omega_n} \right)^2 \left[ \left( \frac{w}{\omega_n} \right)^4 - \left( \frac{4 \tilde{N}}{\pi N} \right)^2 \left( N + 1 - \frac{w^2}{\omega_n^2} \right)^2 \right]}
\]

\[
= \frac{\left( \frac{4 \tilde{N}}{\pi N} \right)^2 \left( 1 - \frac{w^2}{\omega_n^2} \right)^2 \left( \frac{w}{\omega_n} \right)^4}{\left( \frac{w}{\omega_n} \right)^4 - \left( \frac{4 \tilde{N}}{\pi N} \right)^2 \left( N + 1 - \frac{w^2}{\omega_n^2} \right)^2}
\]

As shown previously:

\[
\frac{4}{N^2} \left( N + 1 - \frac{w^2}{\omega_n^2} \right)^2 \left( \frac{c}{c_c} \right)^2 \left( \frac{w}{\omega_n} \right)^2 = \frac{\left( \frac{4 \tilde{N}}{\pi N} \right)^2 \left( N + 1 - \frac{w^2}{\omega_n^2} \right)^2 \left( 1 - \frac{w^2}{\omega_n^2} \right)^2}{\left( \frac{w}{\omega_n} \right)^4 - \left( \frac{4 \tilde{N}}{\pi N} \right)^2 \left( N + 1 - \frac{w^2}{\omega_n^2} \right)^2}
\]

Thus:

\[
(T_D)_R = \left| \frac{\delta_0}{\delta_0} \right| = \sqrt{\frac{\left( \frac{w}{\omega_n} \right)^8 + \left( \frac{4 \tilde{N}}{\pi N} \right)^2 \left( \frac{w}{\omega_n} \right)^4 \left[ \left( 1 - \frac{w^2}{\omega_n^2} \right)^2 - \left( N + 1 - \frac{w^2}{\omega_n^2} \right)^2 \right]}{\left( \frac{w}{\omega_n} \right)^4 \left( 1 - \frac{w^2}{\omega_n^2} \right)^2}
\]
Expanding the terms in the bracket, collecting terms, dividing numerator and denominator by $\left(\frac{\omega}{\omega_n}\right)^4$, we obtain:

$$
(T_0)_{\text{R}} = \left| \frac{\delta_0}{a_0} \right| = \sqrt{\frac{\left(\frac{\omega}{\omega_n}\right)^4 + \left(\frac{4}{\pi} \tilde{\gamma} \right)^2 \left[ \frac{2}{N} \left(\frac{\omega}{\omega_n}\right)^2 - \left(\frac{N+2}{N}\right) \right]}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2}}
$$  \hspace{1cm} (A-42)

Thus, equation (A-42) gives the relative displacement transmissibility of the Coulomb damped system in terms of the dimensionless parameters $N$, $\tilde{\gamma}$, and $\left(\frac{\omega}{\omega_n}\right)$. It should be noticed that when $N \to \infty$, equation (A-42) reduces to the known relation for the relative displacement transmissibility of a Coulomb damped system where the damper is rigidly supported.

**SECTION A-8.3 LIMITATION OF SOLUTIONS**

Having obtained solutions for the transmissibility of the Coulomb damped system under consideration, we must be aware of the mathematical limitations which are associated with these solutions. In the case of the absolute transmissibility, the solution as given by equation (A-41) is valid only for real values of the numerator of equation (A-41). This condition can be written:

$$
\left(\frac{\omega}{\omega_n}\right)^2 \geq \frac{2 \left(\frac{4}{\pi} \tilde{\gamma}\right)^2 \left(\frac{N+1}{N}\right)}{\left(\frac{4}{\pi} \tilde{\gamma}\right)^2 \left(\frac{N+2}{N}\right) + 1} \hspace{1cm} (A-43)
$$

Thus, if equation (A-43) is not satisfied, the solution for absolute transmissibility becomes imaginary and we are
now considering a condition which is not in the range of values of \( \frac{\omega}{\omega_n} \) for which the derived relations are valid. In the case of the relative transmissibility, the condition which must be satisfied in order that we do not obtain imaginary values of relative transmissibility can be written:

\[
\left( \frac{\omega}{\omega_n} \right)^2 \geq \frac{1}{N^2} \left( \frac{4}{\pi^2} \kappa \right)^4 + \left( \frac{4}{\pi^2} \kappa \right)^2 \left( \frac{N+2}{N-1} \right) - \frac{1}{N} \left( \frac{4}{\pi^2} \kappa \right)^2 \quad \text{(A-14)}
\]

Thus, if equation (A-14) is not satisfied, then \( \frac{\omega}{\omega_n} \) is not within the range of values for which equation (A-42) is valid. Notice that when \( N \to \infty \) equation (A-43) and (A-14) reduce to the limiting conditions for the coulomb damped system in which the damper is rigidly supported.

**SECTION A-9 COMMON-TRANSMISSIBILITY POINT - COULOMB SYSTEM**

Since the solutions for transmissibility of the coulomb damped system were obtained by means of equivalent viscous damping, there will exist common-transmissibility points corresponding to those found in Section A-5 for the viscous damped system. The expression for absolute displacement transmissibility as given by equation (A-41) is seen to be independent of damping if the coefficient of the damping term vanishes. The corresponding value of the common-transmissibility frequency ratio for absolute motion is thus seen to be:

\[
\left( \frac{\omega}{\omega_n} \right)_{A}^{*} = \sqrt{\frac{2(N+1)}{N+2}}
\quad \text{(A-45)}
\]

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The same reasoning can be applied to obtain the common-transmissibility frequency ratio for relative motion. Thus from equation (A-12):

\[
\left(\frac{\omega}{\omega_n}\right)_R^* = \sqrt{\frac{N+2}{2}}
\]  

(A-46)

These equations are seen to be exactly the same as those derived for the viscous damped system. This is understandable when one considers the approximate method used in solving the coulomb damped system. These frequency ratios, when substituted in the corresponding transmissibility expressions, give the value of the common-transmissibility. As in the viscous damped system, this value is the same for both relative and absolute motion and again has the same value as that determined in the analysis of the viscous damped system. Thus the common-transmissibility is given by:

\[
\mathbf{T}_A^* = \mathbf{T}_R^* = 1 + \frac{2}{N}
\]  

(A-47)

SECTION A-10 LARGE FREQUENCY RATIO TRANSMISSIBILITY - COULOMB SYSTEM

The transmissibility at large frequency ratios can be found by letting the frequency ratio in equation (A-41) and equation (A-42) become very large and thus eliminate certain terms on the basis that they are small compared to other terms. Letting \( \mathbf{T}_i \) represent this transmissibility (the ~
symbol representing coulomb damping and displacement excited system), the following expressions are obtained:

\[
(\tilde{T}_i)_A = \frac{\sqrt{1 + \left(\frac{4}{\pi} \tilde{\eta}_i\right)^2 \left(\frac{N_i}{N}\right)^2}}{(\frac{\omega}{\omega_n})^2}
\quad (A-48)
\]

and

\[
(\tilde{T}_i)_R = 1
\quad (A-49)
\]

It should be noted that all of the expressions so far derived for the coulomb damped system are subject to conditions such as those stated in Section A-8.3. Perhaps more generally it should be stated that the equations for the coulomb system are valid only when there is relative motion in the coulomb damper. This is a necessary condition since this whole approximate analysis is based on the assumption of equal energy dissipation per cycle in the coulomb and viscous dampers. Thus, the frequency range for which there is relative motion in the coulomb damper must be determined so as to indicate when the derived solutions are valid.

SECTION A-11 FREQUENCY RANGE FOR WHICH COULOMB DAMPED SOLUTIONS ARE VALID

Since the energy method used to derive the coulomb damped expressions assumes relative motion in the damper, the frequency range for which this assumption holds true must be determined. The frequency ratios defining the ends of this frequency range will be designated by \(\left(\frac{\omega}{\omega_n}\right)_L\) where the "L" subscript refers to the point where either the
The relative motion across the damper for the viscous system is given by equation (A-2\textsubscript{14}). If the expression for equivalent viscous damping as given by equation (A-40) is substituted in equation (A-2L), the following equation results:

\[
\left| \frac{\gamma^*}{a_0} \right| = \sqrt{\frac{(\frac{\omega^*}{\omega_n})^4 - (\frac{4}{\pi N \tilde{\eta}})^2(N+1 - \frac{\omega^*}{\omega_n^2})^2}{(1 - \frac{\omega^*}{\omega_n^2})^2}} \]  \tag{A-50}

Thus, when the numerator of this expression becomes zero, there is no relative motion across the damper and the damper has either become "locked in" or has "broken loose". Equating the numerator to zero and solving for frequency ratios:

\[
\left( \frac{\omega^*}{\omega_n} \right)_L = \pm \frac{(\frac{N+1}{N})(\frac{4}{\pi \tilde{\eta}})}{1 \pm \frac{1}{N}(\frac{4}{\pi \tilde{\eta}})} \]  \tag{A-51}

Thus, the frequency range for which the coulomb damped solutions are valid is defined by:

\[
\left( \frac{\omega^*}{\omega_n} \right)_L = \sqrt{\frac{(\frac{N+1}{N})(\frac{4}{\pi \tilde{\eta}})}{\frac{1}{N}(\frac{4}{\pi \tilde{\eta}}) \pm 1}} \]  \tag{A-52}
SECTION A-12 SUMMARY OF DISPLACEMENT TRANSMISSIBILITY EXPRESSIONS

**Viscous Damped System Equations**

\[
(T_D)_A = \left| \frac{x_0}{a_0} \right| = \sqrt{1 + 4 \left( \frac{N+1}{N} \right)^2 \left( \frac{c}{c_c} \right)^2 \left( \frac{w}{w_n} \right)^2 \left( \frac{1}{1 - \frac{\omega^2}{\omega_n^2}} \right)^2 + 4 \frac{N}{N^2} (N+1 - \frac{\omega^2}{\omega_n^2})^2 \left( \frac{c}{c_c} \right)^2 \left( \frac{w}{w_n} \right)^2}
\]

\[
(T_D)_R = \left| \frac{\delta_0}{a_0} \right| = \sqrt{\left( \frac{w}{w_n} \right)^4 + 4 \frac{N}{N^2} \left( \frac{c}{c_c} \right)^2 \left( \frac{w}{w_n} \right)^6 \left( \frac{1}{1 - \frac{\omega^2}{\omega_n^2}} \right)^2 + 4 \frac{N}{N^2} (N+1 - \frac{\omega^2}{\omega_n^2})^2 \left( \frac{c}{c_c} \right)^2 \left( \frac{w}{w_n} \right)^2}
\]

**Coulomb Damped System Equations**

\[
(T_D)_A = \left| \frac{x_0}{a_0} \right| = \sqrt{1 + \left( \frac{4}{\pi} \frac{\tilde{\eta}}{\gamma} \right)^2 \left[ \left( \frac{N+2}{N} \right) - \frac{2}{\left( \frac{w}{w_n} \right)^2} \left( \frac{N+1}{N} \right) \right] \left( \frac{1}{1 - \frac{\omega^2}{\omega_n^2}} \right)^2}
\]

\[
(T_D)_R = \left| \frac{\delta_0}{a_0} \right| = \sqrt{\left( \frac{w}{w_n} \right)^4 + \left( \frac{4}{\pi} \frac{\tilde{\eta}}{\gamma} \right)^2 \left[ \frac{2}{N} \left( \frac{w}{w_n} \right)^2 - \left( \frac{N+2}{N} \right) \right] \left( \frac{1}{1 - \frac{\omega^2}{\omega_n^2}} \right)^2}
\]
The equations giving the displacement transmissibility of the viscous damped system also give the acceleration transmissibility of that system. This is a direct result of the fact that the viscous damped system is a linear system in which a ratio of characteristic displacements equals the ratio of the corresponding accelerations since acceleration differs from displacement only by a factor of $\omega^2$. Also the viscous damping factor $\left( \frac{C}{C_c} \right)$ is a function only of the mass and main spring support. Thus, the equations for displacement transmissibility require no change when acceleration transmissibility is required for the case when the base is excited with constant acceleration oscillation. However, if the acceleration transmissibility of the coulomb damped system is required, the equations derived for the displacement transmissibility cannot be used to represent acceleration transmissibility due to the fashion in which the coulomb damping factor is defined. The coulomb damping factor was defined as:

$$\tilde{\eta} = \frac{B}{ka_o} \quad (A-39)$$

But, for constant acceleration excitation of the base, the displacement $a_o$ will vary with the frequency of excitation. Thus, this factor is no longer a suitable "constant" damping factor. If the base is excited in a
manner such that \( \ddot{a} = A_0 \sin \omega t \), where \( A_0 \) is the maximum amplitude of the base acceleration, the manner in which \( A_0 \) varies with the excitation frequency is given by:

\[
A_0 = \frac{A_o}{\omega^2}
\]  

(A-53)

Thus we may write:

\[
\tilde{\gamma} = \frac{B\omega^2}{mA_o}
\]

(A-54)

Multiplying the numerator and denominator by \( m \):

\[
\tilde{\gamma} = \frac{B}{mA_o} \cdot \frac{m\omega^2}{\omega^2} = \frac{B}{mA_o} \cdot \frac{\omega^2}{\omega^2/m} = \gamma \left( \frac{\omega}{\omega_n} \right)^2
\]

(A-55)

where we define:

\[
\gamma = \frac{B}{mA_o}
\]

(A-56)

Thus, \( \gamma = \frac{B}{mA_o} \) is the coulomb damping factor for a system where the base is being excited by a constant amplitude acceleration. Since the equivalent viscous damping technique essentially assumes harmonic displacement, velocity, and acceleration functions, ratios of characteristic displacements will again equal ratios of corresponding accelerations.

Thus, the equations describing the displacement transmissibility of the coulomb damped system may be used if \( \tilde{\gamma} \) is replaced by \( \gamma \left( \frac{\omega}{\omega_n} \right)^2 \). The resulting expressions are written below:

\[
(T_A)_A = \left| \frac{\ddot{x}}{\ddot{a}} \right| = \sqrt{1 + \left( \frac{4}{m} \gamma \left( \frac{\omega}{\omega_n} \right)^2 \right) \left[ \frac{N+2}{N} \left( \frac{\omega}{\omega_n} \right)^2 - 2 \left( \frac{N+1}{N} \right) \right]} \left( 1 - \frac{\omega^2}{\omega_n^2} \right)^2
\]

(A-57)
Again there are mathematical limitations on these equations due to the fact that the numerators of these expressions cannot be allowed to become negative. These limitations can be obtained by substituting equation (A-55) in equation (A-43) and equation (A-44). Thus, we have arrived at expressions for the acceleration transmissibility of the coulomb damped system by use of the method of equivalent viscous damping. The frequency ratios defining the range of validity is obtained by substituting equation (A-55) in equation (A-52). The resulting expression is:

\[
\left(\frac{w}{w_n}\right)_L = \sqrt{\frac{(N+1)(\frac{4}{\pi} n) \pm 1}{\frac{1}{N} \left(\frac{4}{\pi} n\right)}}
\]  

(A-59)
APPENDIX B
SOLUTION OF VISCOUS SYSTEM BY ANALOG COMPUTOR

For convenience in obtaining response curves for the viscous damped system, the technique of obtaining these curves by use of the analog computer will now be investigated. The equations of motion for the viscous damped system as derived in Appendix A are given by:

\[ m \ddot{x} = -k(x-a) - c(\dot{x} - \dot{x}_1) \]  \hspace{1cm} (B-1)

and:

\[ C(\dot{x} - \dot{x}_1) = \frac{k_1}{\alpha}(x_1 - a) \]  \hspace{1cm} (B-2)

From equation (B-2) we obtain:

\[ x_1 = \frac{(\frac{c}{\alpha}) D x + a}{1 + (\frac{c}{\alpha}) D} \]  \hspace{1cm} (B-3)

where the letter "\( D \)" is the time-derivative operator.

Subtracting \( a \) from both sides of equation (B-3):

\[ (x_1 - a) = (x - a) \frac{(\frac{c}{\alpha}) D}{1 + (\frac{c}{\alpha}) D} \]  \hspace{1cm} (B-4)

Now add and subtract unity in the numerator:

\[ (x_1 - a) = (x - a) \frac{1 + (\frac{c}{\alpha}) D - 1}{1 + (\frac{c}{\alpha}) D} \]

\[ (x_1 - a) = (x - a) \left[ 1 - \frac{1}{1 + (\frac{c}{\alpha}) D} \right] \]  \hspace{1cm} (B-5)
Noting that the second term on the right hand side of equation (B-6) is the mathematical operation performed by the unit lag computer component, which embodies a characteristic time constant $\tau$, this time constant can now be defined as:

$$\tau = \left(\frac{c}{k_1}\right)$$  \hspace{1cm} (B-6)

Thus equation (B-5) can be written:

$$(x_i - a) = (x - a) \left[1 + \frac{1}{1 + \gamma D}\right]$$  \hspace{1cm} (B-7)

Substituting equations (B-7) and (B-2) in equation (B-1):

$$m\ddot{x} = -k(x - a) - k_1(x - a) \left[1 + \frac{1}{1 + \gamma D}\right]$$  \hspace{1cm} (B-8)

This equation is used to construct the functional block diagram shown in Figure B-1. This diagram describes the nature of the physical system. In contrast to Figure B-1, a diagram is shown in Figure B-2 which describes the nature of the electronic analog system which is used to represent the physical system. We see that this system contains the characteristic time constants $T_1$, $T_2$ and $\tau$. The $T_1$ and $T_2$ time constants may be included with the $C_{km}$ coefficient without affecting the gain of the loop. Now for the physical system, the undamped natural frequency is defined as:

$$\omega_n = \sqrt{\frac{k}{m}}$$  \hspace{1cm} (B-9)
FIG. B-1 PHYSICAL SYSTEM

FIG. B-2 ANALOG SYSTEM
By comparison of the two diagrams, it is seen that the corresponding quantity in the analog system is defined as:

\[ \overline{\omega_n} = \sqrt{\frac{C_{y_m} C_k}{T_1 T_2}} \]  

(B-10)

where the bar indicates analog parameters. For the loop containing the \( k \) and \( k_i \) spring constants, the dimensionless parameter \( N \) is defined as:

\[ N = \frac{k_i}{k} \]  

(B-11)

where \( N \) is the spring ratio. By considering the corresponding loop in the analog system, the following parameter can be defined:

\[ \overline{N} = \frac{C_{k_i}}{C_k} \]  

(B-12)

By considering the loop containing the unit-lag computer component, the time constant corresponding to that given by equation (B-6) for the physical system is obviously:

\[ \overline{\tau} = \frac{C}{C_{k_i}} \]  

(B-13)

Thus for a given \( C_{k_i} \), the time constant \( \tau \) and the value of viscous damping \( c \) may be varied in such a way that equation (B-13) is satisfied. Noting that critical damping is defined by:

\[ C_{c} = 2m \omega_n = \frac{2k}{\omega_n} \]  

(B-14)

The viscous damping factor \( c \) is given by:

\[ c = \frac{2(C_{c})k}{\omega_n} \]  

(B-15)
Substituting equation (B-15) in equation (B-13):

\[ \bar{\tau} = \frac{2 \left( \frac{C}{C_e} \right)}{N \omega_n} \]  

(B-16)

By use of equation (B-10), this becomes:

\[ \bar{\tau} = \frac{2 \left( \frac{C}{C_e} \right)}{N} \sqrt{\frac{T_1 T_2}{C_m C_k}} \]  

(B-17)

Thus having conveniently defined the time constant of the unit-lag component and realizing that \( f_n = \frac{\omega_n}{2\pi} \) (cps), it is easily seen that the definition of \( \bar{\tau} \) in the physical system becomes:

\[ \tau = \frac{\left( \frac{C}{C_e} \right)}{\pi N f_n} \]  

(B-18)

Since all parameters in the system have been adequately defined, the response curves for the system are available by introducing the excitation \( \alpha \) by means of a sine-wave generator and observing the absolute motion of the mass \( x \), and the relative motion of the mass \( (x - \alpha) \) at proper points in the analog circuit.
APPENDIX C

ANALOG COMPUTER SOLUTION OF COULOMB DAMPED SYSTEM

Due to difficulties involved in the solution of differential equations which involve non-linear functions (the damping force B in this case), the possibility of obtaining the response of the system by use of the analog computer will now be investigated. In the actual system under consideration, the damper is massless thereby keeping the system a single-degree-of-freedom system. But since the non-linear effects of the damping element must be incorporated into the analog system, it is convenient to temporarily include the mass of the damper \( m' \) in the analog system. The inclusion of \( m' \) in the analog
system provides a means to produce a feedback loop in the analog system which in turn is used to simulate the non-linear effects involved in the Coulomb damping element. Also, it was found that for reasons of stability of the motion of the damper, it is desirable to include some viscous damping $C_i$ between the main mass and the damper. The mechanical system that is actually setup on the analog computer is shown in Figure C-1. This system is seen to reduce to the exact system under consideration when the values of $m_i$ and $C_i$ approach zero. This is essentially how the analog computer is used to solve the problem. The analog computer system which simulates the mechanical system shown in Figure C-1 is constructed and the coefficients $m_i$ and $C_i$ are made to approach zero but keeping them as finite a value as necessary to provide stability of the analog system.

A functional block diagram is a diagram which shows the mathematical operations being made on voltage which is used to represent the variables of the problem. This diagram can be constructed by consideration of the equations of motion for the system shown in Figure C-1. The free body diagrams of the mass and damper is shown in Figure C-2 and Figure C-3. We see that dynamic equilibrium of mass $m$ gives the following equation:

$$m\ddot{x} + C_i(\dot{x} - \dot{x}_i) + k(x - a) + k_i(x - x_i) = 0$$  \hspace{1cm} (3-1)
Dynamic equilibrium of the damper requires:

\[ m_1 \dddot{x}_1 - C_1 (\dot{x}_1 - \dot{x}_2) - k_1 (x_1 - x_2) \pm B = 0 \]  \hspace{1cm} (C-2)

The damping term \( B \) must be remembered to represent the hysteresis damping curve previously discussed. It is convenient to write the equations of motion in terms of the relative variables \( S \) and \( q \), where \( S \) is the motion of the mass relative to the base and \( q \) is the motion of the mass relative to the damper. In terms of these variables, the equations of motion become:

\[ m \ddot{S} = -m \ddot{u} - P_m \]  \hspace{1cm} (C-3)

\[ m_1 \dddot{q} = m_1 \dddot{x} - C_1 \dddot{q} - k_1 q \pm B \]  \hspace{1cm} (C-4)

where we define:

\[ P_m = k S + k_1 q + C_1 \dddot{q} \]  \hspace{1cm} (C-5)
as the total real force applied to mass \( m \). Equations (C-3, 4, 5) are the equations used to construct the functional block diagram shown in Figure C-4.

The functional block diagram indicates the use of the computer components to perform the mathematical operations indicated by the equations of motion. For the most part, Figure C-4 is self-explanatory and additional remarks are required only in discussing the method used to obtain the non-linear characteristics of the damping element.

The elements of the analog system which are a part of the closed loops associated with the dynamic equilibrium of mass \( m \), are those required to produce the desired non-linear damping force. The \( \frac{1}{m} \) coefficient component and the two integrator components make available the acceleration, velocity, and displacement of the mass relative to the damper. As discussed previously, the \( C \) coefficient component is used to provide stability in this portion of the system. When the \( m \) and \( C \) coefficients approach zero, we see that the coulomb damping force \( B \) is transmitted to the vibrating mass solely by spring \( k \). Thus the force \( k, g \) in the analog computer diagram must represent the non-linear force characteristics of the coulomb damping element. The manner in which \( k, g \) is made to have the required characteristics will now be discussed.
Suppose the values of the \( m \) and \( C \) coefficient components have been allowed to approach zero. The forces applied to the massless damper are then merely \( k \) and \( B \). These forces must be equal and opposite in order to satisfy the condition that the massless damper be in equilibrium. The type of force produced by the \( B \) and \( C \) components operating together on a voltage representing the velocity is shown in Figure C-5. The velocity \( (\dot{s} - \dot{\omega}) \) is the velocity of the damper relative to the base. If this velocity be multiplied by a viscous damping coefficient \( C \), a force is produced with a value \( C(\dot{s} - \dot{\omega}) \) and is shown in Figure C-5(a).

Now if this force is operated on by a bounding component \( B \), the net result is a force which has the characteristic shown in Figure C-5(b). If the coefficient \( C \) is now increased toward infinity, the force produced is indicated by Figure C-5(c).
Thus, if the mass is vibrating in a configuration whereby insufficient force is developed in spring $k_1$ to overcome the damper force, there is no relative motion across the coulomb damper, and the feedback loop originating and terminating at adder component $A_4$ causes the force $k_{13}$ to equal force $B$. Also, since there is no relative velocity across the coulomb damper, the point representing the value of the damping force $B$ in Figure C-5(c) lies somewhere on the line $-\overrightarrow{BOB}$. Therefore, as long as the displacement of the mass $m$ relative to the damper is less than the value required to cause motion of the damper relative to the base, the force acting on mass $m$ by spring $k_1$, is merely $k_{15}$. When the displacement of mass $m$ becomes that value required to cause relative motion between damper and base, the damping force is indicated by Point $B$ or $-B$ in Figure C-5(c). The value of this damping force is $\mu F$, the maximum obtainable damping force, which is the bounding value of the $B$ component. Thus, the value of the damping force will remain at $\mu F$ for further increase in relative displacement $\gamma$. The feedback loop originating and terminating at adder $A_4$ insures that $k_{13}$ equals $\mu F$ for further increase in $\gamma$ and thus a constant damping force is transmitted to mass $m$ via spring $k_1$ for this motion. Therefore, it is seen that the analog computer by use of suitable operational amplifiers
has produced the required damping characteristics indicated by Figure C-6.

The functional block diagram of Figure C-4 also indicates the manner in which the net force applied to the mass and the net damping force are monitored on oscilloscopes. The figures displayed on these oscilloscopes are used as a guide to indicate when the $m_1$ and $C_1$ parameters should be adjusted so as to more exactly reproduce the desired force characteristics.

The following is a discussion of the manner in which the system parameters are set up on the analog computer. As shown in Appendix B, the undamped natural frequency is given by:

$$\omega_n = \sqrt{\frac{k}{m}} \quad \bar{\omega}_n = \sqrt{\frac{C_m C_k}{T_1 T_2}} \quad \text{(C-6)}$$
where the bar placed above $\omega_n$ again indicates that this quantity refers to the analog system. $T_1$ and $T_2$ are the time constants of the integrating amplifiers in the feedback loop which originates at adder $A_1$ and terminates at adder $A_2$ via adder $A_2$.

It is convenient to collect all parameters into non-dimensional groups. For example, the spring ratio $N = \frac{r_i}{r}$ would merely be defined as

$$\bar{N} = \frac{C_{k_i}}{C_k}$$

in the analog system where $C_{k_i}$ and $C_k$ are the dial settings of the $k_i$ and $k$ coefficient components.

Similarly, ratio of voltage in the analog circuit can represent certain dimensionless groups in the mechanical system. For example, the ratio of the voltage representing $P_m$ to the voltage input to adder $A_1$ from the sine-wave generator represents the ratio of the absolute acceleration of the mass to the acceleration of the base. This quantity is the absolute acceleration transmissibility of the system. This is true due to the fact that the absolute value of $P_m$ equals $m\ddot{x}$ and the sine-wave generator voltage represents $m\ddot{a}$.

Since $\ddot{a} = A_0 \sin \omega t$, it is seen that the forcing frequency $\omega$ is also a parameter whose value will be
varied. The conventional dimensionless ratio involving this quantity is \( \frac{\omega}{\omega_n} \) which is the ratio of the forcing frequency to the undamped natural frequency of the system as defined by equation (C-6).

Finally, if one divides the dial setting of the \( B \) component (which represents the maximum value of voltage that the component will pass) by the sine-wave generator voltage, the dimensionless damping parameter \( \eta = \frac{B}{mA_0} \) is obtained.

Thus, with measurements made from the analog computer, the absolute acceleration response of the system is obtained in terms of the dimensionless parameters \( \frac{\omega}{\omega_n} \), \( N \) and \( \eta \).
REFERENCES


