Index Theorems and Magnetic Monopoles on Asymptotically Conic Manifolds

by

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Abstract

In this thesis, I investigate the index of Callias type operators on asymptotically conic manifolds (also known as asymptotically locally Euclidean manifolds or scattering manifolds) and give an application to the moduli space of magnetic monopoles on these spaces. The index theorem originally due to C. Callias and later generalized by N. Anghel and others concerns operators of the form $D + i\Phi$, where D is a Dirac operator and Φ is a family of Hermitian invertible matrices. The first result is a pseudodifferential version of this index theorem, in the spirit of of the K-theoretic proof of the Atiyah-Singer index theorem, using the theory of scattering pseudodifferential operators. The second result is an extension to the case where Φ has constant rank nullspace bundle at infinity, using a b-to-scattering transition calculus of pseudodifferential operators. Finally I discuss magnetic monopoles, which are solutions to the Bogomolny equation $F_A = *d_A\Phi$ on an $SU(2)$ principal bundle over a complete 3-manifold, and I show how the previous results can be applied to compute the dimension of the moduli space of monopoles over asymptotically conic manifolds whose boundary is homeomorphic to a disjoint union of spheres.

Thesis Supervisor: Richard B. Melrose
Title: Simons Professor of Mathematics

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Contents

Chapter 1

Technical Background

We give a brief introduction to the important objects that we shall be considering. We discuss manifolds with boundary, the b and scattering tangent bundles and asymptotically conic manifolds first, before giving an overview of the theory of connections on such manifolds. We devote section 1.4 to Clifford modules and Dirac operators, which will be of central importance later on. Finally we give a quick overview of the b and scattering calculi of pseudodifferential operators in sections 1.5 and 1.6, respectively. A general reference for this section is [19], along with [23] for the results concerning b objects, and [21] for results concerning the scattering objects.

1.1 Manifolds with boundary

Let X be a manifold with boundary ∂X. A *boundary defining function* (hereinafter referred to as a bdf) $x \in C^\infty(X; [0, \infty))$ is a function, smooth up to the boundary, taking non-negative values, such that

$$x^{-1}(0) = \partial X, \quad \text{and} \quad dx_{|\partial X} \neq 0.$$

Such a function clearly exists locally, and by combining local bdfs via a partition of unity, exists globally as well. From the definition, two bdfs differ multiplicatively by a smooth positive function:

$$x_1 = a x_2, \quad 0 < a \in C^\infty(X) \tag{1.1}$$

Unless otherwise specified, x will typically denote a boundary defining function.

Let $\mathcal{V}(X) = C^\infty(X; TX)$ denote the Lie algebra of smooth vector fields on X, and denote by $\mathcal{V}_{\mathrm{b}}(X) = \{v \in \mathcal{V}(X) \ ; \ v_p \in T_p\partial X, \forall\, p \in \partial X\}$ the set of all vector fields tangent to the boundary. We call these *b vector fields.* In terms of local coordinates (x, y_i) where x is a bdf, $\mathcal{V}_{\mathrm{b}}(X)$ consists of vector fields of the form

$$v = a(x, y) x\partial_x + \sum_i b_i(x, y)\partial_{y_i}.$$

In such coordinates, it is easily verified that $[\mathcal{V}_{\mathrm{b}}(X), \mathcal{V}_{\mathrm{b}}(X)] \subset \mathcal{V}_{\mathrm{b}}(X)$, so $\mathcal{V}_{\mathrm{b}}(X)$ is a Lie subalgebra. Another way of stating tangency to the boundary is that

$$\mathcal{V}_{\mathrm{b}}(X) \cdot xC^\infty(X) \subset xC^\infty(X). \tag{1.2}$$

The *scattering vector fields* $\mathcal{V}_{\text{sc}}(X)$ are defined to be the set

$$\mathcal{V}_{\text{sc}}(X) = x\mathcal{V}_{\text{b}}(X) \subset \mathcal{V}(X), \quad \text{for some bdf } x.$$

From the characterization (1.1) of bdfs, it follows that this is independent of the choice of x. Locally in coordinates (x, y_i), $\mathcal{V}_{\text{sc}}(X)$ consists of vector fields of the form

$$v = a(x,y)x^2\partial_x + \sum_i b_i(x,y)x\partial_{y_i},$$

and as with b vector fields $\mathcal{V}_{\text{sc}}(X)$ is a Lie subalgebra, as we see from the following.

Lemma 1.1. *The scattering vector fields $\mathcal{V}_{\text{sc}}(X)$ form a module over $\mathcal{V}_{\text{b}}(X)$ with respect to the Lie bracket:*

$$[\mathcal{V}_{\text{b}}(X), \mathcal{V}_{\text{sc}}(X)] \subset \mathcal{V}_{\text{sc}}(X).$$

Furthermore, we have

$$[\mathcal{V}_{\text{sc}}(X), \mathcal{V}_{\text{sc}}(X)] \subset x\mathcal{V}_{\text{sc}}(X).$$

Proof. These follow easily from the tangency condition (1.2). Indeed, if $V, W \in \mathcal{V}_{\text{b}}(X)$ and x is a bdf, we have

$$[V, xW] = (V \cdot x)W = x\, a(x,y)W, \quad \text{for some } a \in C^\infty(X).$$

Similarly,

$$[xV, xW] = (V \cdot x)xW - (W \cdot x)xV = x^2 a\, W + x^2 b\, V,$$

for some $a, b \in C^\infty(X)$. □

From the Serre-Swan theorem, $\mathcal{V}_{\text{b}}(X)$ and $\mathcal{V}_{\text{sc}}(X)$ are sections of two naturally defined vector bundles: the *b tangent bundle* ${}^{\text{b}}TX$ and *scattering tangent bundle* ${}^{\text{sc}}TX$, respectively. Indeed, letting $\mathcal{I}_p = \{u \in C^\infty(X)\ ;\ u(p) = 0\}$ be the ideal of smooth functions vanishing at p, we set

$$ {}^{\text{b}}T_pX = \mathcal{V}_{\text{b}}(X)/\mathcal{I}_p \cdot \mathcal{V}_{\text{b}}(X) \qquad {}^{\text{sc}}T_pX = \mathcal{V}_{\text{sc}}(X)/\mathcal{I}_p \cdot \mathcal{V}_{\text{sc}}(X) \tag{1.3}$$

In local coordinates,

$$ {}^{\text{b}}T_pX = \text{span}_{\mathbb{R}}\{x\partial_x, \partial_{y_i}\} \qquad {}^{\text{sc}}T_pX = \text{span}_{\mathbb{R}}\{x^2\partial x, x\partial_{y_i}\} \tag{1.4}$$

Of course in the interior we have $T_pX \cong {}^{\text{b}}T_pX \cong {}^{\text{sc}}T_pX$, $p \notin \partial X$, but the bundles are not naturally identified over ∂X. However, from the inclusions $\mathcal{V}_{\text{sc}}(X) \subset \mathcal{V}_{\text{b}}(X) \subset \mathcal{V}(X)$, we do have natural maps

$$ {}^{\text{sc}}TX \longrightarrow {}^{\text{b}}TX \longrightarrow TX. \tag{1.5}$$

The *b normal bundle* to ∂X is intrinsically defined as the kernel of the second map above over ∂X; in fact the sequence

$$0 \longrightarrow {}^{\text{b}}N\partial X \longrightarrow {}^{\text{b}}T_{|\partial X}X \longrightarrow T\partial X \longrightarrow 0 \tag{1.6}$$

is exact. Locally, we have

$$ {}^{\text{b}}N_p\partial X = \text{span}_{\mathbb{R}}\{x\partial_x\}.$$

The *scattering normal bundle* is defined using a choice of bdf x; locally,

$$^{\mathrm{sc}}N_p\partial X = \mathrm{span}_{\mathbb{R}}\left\{x^2\partial_x\right\}.$$

Both are trivial real line bundles over ∂X.

The *b and scattering cotangent bundles*, ${}^{\mathrm{b}}T^*X$ and ${}^{\mathrm{sc}}T^*X$, are respective dual bundles to ${}^{\mathrm{b}}TX$ and ${}^{\mathrm{sc}}TX$. Locally,

$$^{\mathrm{b}}T_p^*X = \mathrm{span}_{\mathbb{R}}\left\{\frac{dx}{x}, dy_i\right\} \qquad {}^{\mathrm{sc}}T_p^*X = \mathrm{span}_{\mathbb{R}}\left\{\frac{dx}{x^2}, \frac{dy_i}{x}\right\} \tag{1.7}$$

We use the shorthand ${}^{\mathrm{b}}\Lambda^k$ to denote the bundle $\bigwedge^k {}^{\mathrm{b}}T^*X \cong \left(\bigwedge^k {}^{\mathrm{b}}TX\right)^*$, and similarly ${}^{\mathrm{sc}}\Lambda^k \equiv \bigwedge^k {}^{\mathrm{sc}}T^*X$. Sections $u \in C^\infty(X; {}^{\mathrm{b}}\Lambda^k)$ (resp. $u \in C^\infty(X; {}^{\mathrm{sc}}\Lambda^k)$) of these bundles are the *b (resp. scattering) differential forms.* From (1.5), we have natural maps

$$T^*X \longrightarrow {}^{\mathrm{b}}T^*X \longrightarrow {}^{\mathrm{sc}}T^*X.$$

In particular,

$$C^\infty(X; {}^{\mathrm{b}}\Lambda^k) \subset C^\infty(X; \Lambda^k), \quad \text{and} \tag{1.8}$$
$$C^\infty(X; {}^{\mathrm{sc}}\Lambda^k) = x^k C^\infty(X; {}^{\mathrm{b}}\Lambda^k) \subset C^\infty(X; {}^{\mathrm{b}}\Lambda^k). \tag{1.9}$$

1.2 Asymptotically conic manifolds

A smooth, nondegenerate section $g \in C^\infty\left(X; \mathrm{Sym}^2({}^{\mathrm{sc}}T^*X)\right)$ is called a *scattering metric* on X; it restricts to a complete metric on the interior of X. In the special case that g has the form

$$g = \frac{dx^2}{x^4} + \frac{h(x,y,dy)}{x^2}, \quad \text{with } h_{|\partial X} \text{ a metric on } \partial X \tag{1.10}$$

(i.e. no cross terms, and the section $x^2\partial_x \in C^\infty(\partial X; {}^{\mathrm{sc}}TX)$ has magnitude 1), we call g an *exact scattering*, or *asymptotically conic* metric.

The label "asymptotically conic" is appropriate considering the form of the metric on the interior of X, $g = dr^2 + r^2h(1/r, y, dy)$ in terms of the "radial" function $r = 1/x$.

Definition. *An* asymptotically conic manifold *is a pair (X, g), where X is a manifold with boundary and g is an asymptotically conic metric, i.e. of the form (1.10).*

In particular, we note that an asymptotically conic manifold comes equipped with a preferred class of boundary defining functions; namely those x such that $x^2\partial_x$ has magnitude 1, or equivalently, those x for which g takes the form (1.10).

The primary example of such a manifold is the radial compactification $\overline{\mathbb{R}^n}$ of Euclidean n-space, with $x = 1/r$; in this case $\partial X = S^{n-1}$, and $g = \frac{dx^2}{x^4} + \frac{h_{S^{n-1}}}{x^2}$. For this reason, asymptotically conic manifolds are often referred to as "asymptotically locally Euclidean" (ALE), with the term "asymptotically Euclidean" reserved for the case when $(\partial X, h) = (S^{n-1}, h_{S^{n-1}})$.

Analogously, a nondegenerate section of $C^\infty\left(X; \mathrm{Sym}^2({}^{\mathrm{b}}T^*X)\right)$ is called a *b metric*; again it restricts to a complete metric on the interior of X. It is an *exact b metric* if it has the form

$$\frac{dx^2}{x^2} + h, \quad h_{|\partial X} \text{ a metric.}$$

Such a metric is sometimes called "asymptotically cylindrical," as it has the form $dt^2 + h$ in the interior of X, where $t = \log(r) = \log(1/x) \in (-\infty, \infty)$.

Any asymptotically conic metric g on X is conformal to an exact b metric. We denote this *conformally related b metric* by g_{b}, where

$$g_{\mathrm{b}} = x^2 g = \frac{dx^2}{x^2} + h \tag{1.11}$$

Note that g_{b} depends on a choice of bdf x.

1.3 Connections and covariant derivatives

We will need to consider connections on principal bundles over asymptotically conic manifolds. Recall that a connection on a principal G-bundle $P \longrightarrow X$ is an equivariant choice of splitting of the exact sequence

$$0 \longrightarrow V_pP \longrightarrow T_pP \xrightarrow{\pi_*} T_{\pi(p)}X \longrightarrow 0$$

where $V_pP = \ker((\pi_p)_*) \cong \mathfrak{g}$ is the vertical tangent bundle. This data is determined in a number of equivalent ways:

- in terms of a *horizontal bundle* $HP \subset TP$ with $TX \xrightarrow{\cong} HP$, and such that $TP = HP \oplus VP$, and $g_*(H_pP) = H_{gp}P$,
- in terms of a *connection form* $\omega \in C^\infty(P; \Lambda^1 \otimes \mathfrak{g})$ such that $H_pP = \ker(\omega_p)$ and $V_pP \cong \mathfrak{g} \ni v \implies \omega(v) = v$, and $g^*(\omega_{gp}) = \omega_p$, and finally
- in terms of a *covariant derivative* $\nabla : C^\infty(X; \mathrm{ad}(P)) \longrightarrow C^\infty(X; \Lambda^1 \otimes \mathrm{ad}(P))$, which satisfies
$$\nabla(f\, s) = df \otimes s + f\, \nabla s, \quad f \in C^\infty(X),\ s \in C^\infty(X; \mathrm{ad}(P)) \tag{1.12}$$

We refer to the above as *true connections.*

On a principal G-bundle $P \longrightarrow X$ where X is a manifold with boundary, a *scattering connection* is an equivariant splitting of the exact sequence

$$0 \longrightarrow V_pP \longrightarrow {}^{\mathrm{sc}}T_pP \xrightarrow{\pi_*} {}^{\mathrm{sc}}T_{\pi(p)}X \longrightarrow 0$$

and a *b connection* is a splitting of

$$0 \longrightarrow V_pP \longrightarrow {}^{\mathrm{b}}T_pP \xrightarrow{\pi_*} {}^{\mathrm{b}}T_{\pi(p)}X \longrightarrow 0.$$

Again, such connections are determined by covariant derivatives

$$\nabla : C^\infty(X; \mathrm{ad}(P)) \longrightarrow C^\infty(X; {}^{\mathrm{b/sc}}\Lambda^1 \otimes \mathrm{ad}(P)).$$

As in the true case, on any local trivialization, a connection has the form

$$\nabla = d + A, \quad A \in C^\infty(X; {}^{\mathrm{b/sc}}\Lambda^1 \otimes \mathrm{ad}(P)).$$

Given any representation $\pi : G \longrightarrow \mathrm{Aut}(F)$, where F is a vector space, we can form the *associated bundle*

$$V = P \otimes_\pi F \longrightarrow X$$

which is a vector bundle with fiber F. A connection on P descends to the quotient, and we again have a covariant derivative $\nabla : C^\infty(X;V) \longrightarrow C^\infty(X; {}^{(\mathrm{b/sc})}\Lambda^1 \otimes V)$ satisfying the product rule (1.12. Thus a connection on P determines a unique connection on any associated bundle. Conversely, any connection on a vector bundle $V \longrightarrow X$ determines a connection on the *frame bundle*

$$\mathrm{GL}(V) \longrightarrow X,$$

a principal $\mathrm{GL}(F)$-bundle whose fiber at $p \in X$ is the set of isomorphisms $g : V_p \cong F$. If the connection respects an inner product on V, the group can be taken to be $O(F)$ or $U(F)$ instead of $\mathrm{GL}(F)$.

We denote by $\mathcal{A}(X;P)$, ${}^{\mathrm{b}}\mathcal{A}(X;P)$, and ${}^{\mathrm{sc}}\mathcal{A}(X;P)$ the spaces of smooth true, b, and scattering connections, respectively on P. On a vector bundle $V \longrightarrow X$ associated to P, we will also use the notation $\mathcal{A}(X;V)$, ${}^{\mathrm{b}}\mathcal{A}(X;V)$, and ${}^{\mathrm{sc}}\mathcal{A}(X;V)$ for connections on V. Each is an affine space modeled on the space of smooth true/b/scattering $\mathrm{ad}(P)$-valued one-forms.

Note that the natural maps $T^*X \longrightarrow {}^{\mathrm{b}}T^*X \longrightarrow {}^{\mathrm{sc}}T^*X$ induce maps

$$C^\infty(X;\Lambda^1) \longrightarrow C^\infty(X;{}^{\mathrm{b}}\Lambda^1) \longrightarrow C^\infty(X;{}^{\mathrm{sc}}\Lambda^1)$$

and hence we have maps

$$\mathcal{A}(X;P) \longrightarrow {}^{\mathrm{b}}\mathcal{A}(X;P) \longrightarrow {}^{\mathrm{sc}}\mathcal{A}(X;P).$$

We say a scattering connection is the *lift* of a b (respectively true) connection if it is in the image of the appropriate map. In coordinates, this is simply the following phenomenon. If we have a true covariant derivative, say ∇, then it induces a scattering connection by linearity:

$$\nabla_{x^2\partial_x} s = x^2 \nabla_{\partial_x} s, \qquad \nabla_{x\partial_y} = x\nabla_{\partial_y}.$$

In particular, a scattering connection the lift of a b connection if

$$\nabla_V : C^\infty(X;\mathrm{ad}(P)) \longrightarrow x\, C^\infty(X;\mathrm{ad}(P)), \quad \text{for all } V \in \mathcal{V}_{\mathrm{sc}}(X),$$

since in this case, we can define its action on b vector fields by

$$\mathcal{V}_{\mathrm{b}}(X) \ni W \longmapsto \frac{1}{x}\nabla_{xW}.$$

Similarly, a b connection is the lift of a true connection if

$$\nabla_V : C^\infty(X;\mathrm{ad}(P)) \longrightarrow x\, C^\infty(X;\mathrm{ad}(P)), \quad \text{for all } V \in C^\infty(X;{}^{\mathrm{b}}NX).$$

Equivalently, if $P \longrightarrow X$ is trivial in a neighborhood U of ∂X, so that $\nabla = d + A$, $A \in C^\infty(U;{}^{\mathrm{sc}}\Lambda^1 \otimes \mathrm{ad}(P))$, then ∇ is the lift of a b connection if

$$A \in x\, C^\infty(U;{}^{\mathrm{sc}}\Lambda^1 \otimes \mathrm{ad}(P))$$

and ∇ is the lift of a true connection if

$$A(x^2\partial_x) \in x^2 C^\infty(U; \mathrm{ad}(P)), \text{ and } A(x\partial_y) \in x C^\infty(U; \mathrm{ad}(P)).$$

In the presence of an asymptotically conic metric, we have a canonical product structure near ∂X; that is, a neighborhood of ∂X has the form $\partial X \times [0,1)_x$ where x is a boundary defining function for which the metric g has the form (1.10). With this product structure, we have a canonical way to extend vector fields on ∂X to a neighborhood in X, and thus a true connection ∇ induces a connection $\nabla_{|\partial X}$ on the boundary in the presence of such a metric. The same is true for a b connection since, from (1.6), ${}^{\mathrm{b}}T_{\partial X}X/{}^{\mathrm{b}}N\partial X = T\partial X$.

On the other hand, a scattering connection doesn't generally induce a connection on the boundary since ${}^{\mathrm{sc}}T_{\partial X}X/{}^{\mathrm{sc}}N\partial X = {}^{\mathrm{sc}}T\partial X \neq T\partial X$. Only in the case that it is the lift of a true or b connection can we naturally define $\nabla_{|\partial X}$.

For later reference, we will need a local description of the Levi-Civita connection on forms induced by the asymptotically conic metric (1.10), on ∂X.

Proposition 1.1. *Let $\left(X, g = \frac{dx^2}{x^4} + \frac{h}{x^2}\right)$ be an asymptotically conic manifold. Let $p \in \partial X$, and let $\{e_i\}$ be an orthonormal frame for $(T^*\partial X, h(x,y,dy))$ near p, and thus $\{\frac{dx}{x^2}, \frac{e_i}{x}\}$ is an orthonormal frame for ${}^{\mathrm{sc}}T^*X$ near p, and wedge products of these form a local trivialization of $C^\infty(U; {}^{\mathrm{sc}}\Lambda^k)$ for all k, on a neighborhood $U \ni p$. Let $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$, for $I = \{i_1 < i_2 < \cdots < i_k\}$, $|I| = k$. Let $n = \dim X$, and $\{\hat{e}_i\}$ be the dual frame for $T\partial X$*

Then in this trivialization, the Levi-Civita connection $\nabla^{\mathrm{LC}(g)}$ acts by

$$\nabla^{\mathrm{LC}(g)}_{x^2\partial_x}\left(\alpha \frac{dx}{x^2} \wedge \frac{e_I}{x^{|I|}}\right) = \left(x^2\partial_x\alpha - x\,|I|\,\alpha\right)\frac{dx}{x^2} \wedge \frac{e_I}{x^{|I|}} \tag{1.13}$$

$$\nabla^{\mathrm{LC}(g)}_{x^2\partial_x}\left(\alpha \frac{e_I}{x^{|I|}}\right) = \left(x^2\partial_x\alpha - x(n-1-|I|)\alpha\right)\frac{e_I}{x^{|I|}} \tag{1.14}$$

$$\nabla^{\mathrm{LC}(g)}_{x\hat{e}_i}\left(\alpha \frac{dx}{x^2} \wedge \frac{e_I}{x^{|I|}}\right) = x\hat{e}_i\left(\alpha\right)\frac{dx}{x^2} \wedge \frac{e_I}{x} \tag{1.15}$$

$$\nabla^{\mathrm{LC}(g)}_{x\hat{e}_i}\left(\alpha \frac{e_I}{x^{|I|}}\right) = x\hat{e}_i\left(\alpha\right)\frac{e_I}{x^{|I|}}. \tag{1.16}$$

In particular, $\nabla^{\mathrm{LC}(g)}$ is the lift of a b connection, and as such induces a connection on ∂X which coincides with $\nabla^{\mathrm{LC}(h)}$, the Levi-Civita connection for h, (using the identification $x^k : {}^{\mathrm{sc}}\Lambda^k\partial X \xrightarrow{\cong} \Lambda^k\partial X$).

Remark. Note that $\nabla^{\mathrm{LC}(g)}$ is *not* the lift of $\nabla^{\mathrm{LC}(g_{\mathrm{b}})}$, as it differs by the lower order terms appearing in (1.13) and (1.14), though tangentially, they agree. Another way to state this is that, in a neighborhood of ∂X,

$$\nabla^{\mathrm{LC}(g)} = \nabla^{\mathrm{LC}(g_{\mathrm{b}})} - A, \qquad A \in C^\infty(X; {}^{\mathrm{b}}T^*X \otimes \mathrm{End}({}^{\mathrm{b}}\Lambda^{\cdot}))$$

where A is the connection-type one form

$$A = N \otimes \frac{dx}{x}, \qquad \mathrm{End}({}^{\mathrm{b}}\Lambda^{\cdot}) \ni N = \begin{cases} (n-1-k) & \text{on } {}^{\mathrm{b}}\Lambda^k\partial X \\ k & \text{on } \frac{dx}{x} \wedge {}^{\mathrm{b}}\Lambda^k\partial X \end{cases} \tag{1.17}$$

Proof. Though it may seem a bit circular, from our point of view, the most efficient way to compute the action of $\nabla^{\mathrm{LC}(g)}$ on forms is to utilize the Dirac structure of the operator

$d+d^*$ on forms, which is associated with the Clifford multiplication $c\ell(\xi) = \xi\wedge\cdot - \xi\lrcorner\cdot$ on the total form bundle.

Recall that $*^2 = (-1)^{k(n-k)}$ on k forms, where $*$ is the Hodge star operator, and that $d^* = (-1)^{nk+n+1} * d *$. We will also use the fact that

$$*v\wedge* = (-1)^{nk+n} v\lrcorner \tag{1.18}$$

on k forms, where v is a one-form.

Consider then the action of d:

$$d\left(\alpha(x,y)\frac{dx}{x^2}\wedge\frac{e_I}{x^{|I|}}\right) = \sum_i x\hat{e}_i(\alpha)\frac{e_i}{x}\wedge\left(\frac{dx}{x^2}\wedge\frac{e_I}{x^{|I|}}\right)$$

and

$$d\left(\alpha(x,y)\frac{e_I}{x^{|I|}}\right) = \left(x^2\partial_x\alpha - x\,|I|\,\alpha\right)\frac{dx}{x^2}\wedge\left(\frac{e_I}{x^{|I|}}\right) + \sum_i x\hat{e}_i(\alpha)\frac{e_i}{x}\wedge\left(\frac{e_I}{x^{|I|}}\right).$$

On the other hand, we have d^* acting by

$$\begin{aligned}(-1)^{n(|I|+1)+n+1} * d * &\left(\alpha(x,y)\frac{dx}{x^2}\wedge\frac{e_I}{x^{|I|}}\right)\\ &= -\left((x^2\partial_x\alpha - x\,(n-1-|I|)\alpha)\frac{e_I}{x^{|I|}}\right) - \sum_i x\hat{e}_i(\alpha)\frac{e_i}{x}\lrcorner\left(\frac{dx}{x^2}\wedge\frac{e_I}{x^{|I|}}\right)\end{aligned}$$

using (1.18), and

$$(-1)^{n|I|+n+1} * d * \left(\alpha(x,y)\frac{e_I}{x^{|I|}}\right) = -\sum_i x\hat{e}_i(\alpha)\frac{e_i}{x}\lrcorner\left(\frac{dx}{x^2}\wedge\frac{e_I}{x^{|I|}}\right).$$

We therefore conclude that $d+d^*$ acts via

$$d+d^* = \sum_{i=0}^{n-1}(v_i\wedge - v_i\lrcorner)\nabla^{\mathrm{LC}(g)}_{\hat{v}_i}$$

where $\{v_0, v_1, \ldots, v_{n-1}\} = \{\frac{dx}{x^2}, \frac{e_1}{x}, \ldots, \frac{e_{n-1}}{x}\}$ and $\nabla^{\mathrm{LC}(g)}$ acts as in (1.13)-(1.16). □

1.4 Clifford modules and Dirac operators

As in the case of compact manifolds, a scattering metric allows us to construct the (complex) Clifford bundle

$$\mathbb{C}\ell(X,g) \longrightarrow X, \quad \text{where } \mathbb{C}\ell(X,g)_p = \mathbb{C}\ell({}^{\mathrm{sc}}T^*_pX, g(p)),$$

We follow [18] for our sign convention, so for $\{e_i\}$ an orthonormal basis of ${}^{\mathrm{sc}}T^*_pX$, we have

$$\mathbb{C}\ell({}^{\mathrm{b}}T^*_pX, g) \ni e_i\cdot e_j = \begin{cases} 0 & \text{for } i\neq j,\\ -|e_i|^2 & \text{for } i=j.\end{cases}$$

The Levi-Civita connection on scattering forms extends to the Clifford bundle, so we have

$$\nabla^{\mathrm{LC}(g)} : C^\infty(X; \mathbb{C}\ell(X)) \longrightarrow C^\infty(X; {}^{\mathrm{sc}}\Lambda^1 \otimes \mathbb{C}\ell(X)))$$

A complex vector bundle $V \longrightarrow X$ is a *scattering Clifford module* if it admits an action

$$c\ell : \mathbb{C}\ell(X, g) \longrightarrow \mathrm{End}(V).$$

If $V \longrightarrow X$ is equipped with a Hermitian metric, we assume Clifford multiplication by unit length vectors is unitary, which implies that Clifford multiplication in general is anti-Hermitian:

$$\langle c\ell(e_i)u, v\rangle = -\langle u, c\ell(e_i)v\rangle \tag{1.19}$$

by our convention that $e_i^2 = -|e_i|$.

A connection ∇ on V is a *Clifford connection* if

$$\nabla(c\ell(\alpha)\, u) = c\ell(\nabla^{\mathrm{LC}(g)}\alpha)\, u + c\ell(\alpha)\, \nabla u. \tag{1.20}$$

With (1.19) and (1.20), we have a canonical *scattering Dirac operator*

$$\not{D} : C^\infty(X; V) \longrightarrow C^\infty(X; V)$$

defined at any $p \in X$ by

$$\not{D} = \sum_i c\ell(e_i)\nabla_{\hat{e}_i} \tag{1.21}$$

where $\{e_i\}$ is an orthonormal basis of ${}^{\mathrm{sc}}T_p^*X$, with dual basis $\{\hat{e}_i\}$ of ${}^{\mathrm{sc}}T_pX$. The usual proof (see [18], Proposition 5.3 and Theorem 5.7) shows

Theorem 1.1. *$\not{D}$ is an elliptic differential operator, with symbols*

$$\sigma_{\mathrm{int}}(\not{D})(\xi) = ic\ell(\xi), \quad \xi \in {}^{\mathrm{sc}}T^*X, \qquad \sigma_{\mathrm{sc}}(\not{D}) = ic\ell(\xi), \quad \xi \in {}^{\mathrm{sc}}T^*_{\partial X}X.$$

Furthermore, $\not{D}$ is essentially self-adjoint on the space $L^2(X; V)$ induced by the Hermitian product on V and the metric g restricted to $\mathring{X}$, (equivalently, $\not{D}$ is essentially self-adjoint on the space of square integrable scattering half densities (see A) $L^2(X; V \otimes \Omega_{\mathrm{sc}}^{1/2})$) and $\not{D}^2$ is a Laplacian operator, with scalar principal symbol

$$\sigma_{\mathrm{int}}(\not{D}^2)(\xi) = |\xi|^2$$

Remark. $\not{D} \in \mathrm{Diff}^1_{\mathrm{sc}}(X; V)$ is a scattering differential operator as defined in section 1.6, though it is *not* fully elliptic, since $\sigma_{\mathrm{sc}}(\not{D}) = ic\ell(\xi)$ is of course never invertible at $0 \in {}^{\mathrm{sc}}T_p^*X$, $p \in \partial X$.

We now examine how the Dirac operator (1.21) is related to a natural Dirac operator on ∂X. It will now be important to assume the metric is asymptotically conic, i.e. of the form (1.10).

Recall the standard isomorphism

$$\mathbb{C}\ell(\mathbb{R}^{n-1}) \cong \mathbb{C}\ell^0(\mathbb{R}^n)$$

which is given by extending the map $e_i \longmapsto e_ie_n$ on generators. By choosing a bdf x so that $\left|\frac{dx}{x^2}\right|_{|\partial X} \equiv 1$, whose existence is guaranteed by the form of g, we obtain a unit normal

section
$$e_n = \frac{dx}{x^2} : \partial X \longrightarrow {}^{\mathrm{sc}}N^*\partial X,$$
giving an isomorphism of Clifford bundles
$$\mathbb{C}\ell({}^{\mathrm{sc}}T^*\partial X, g_{|\partial X}) \cong \mathbb{C}\ell^0(X, g).$$
Furthermore, using x again, we identify
$$x : \mathbb{C}\ell({}^{\mathrm{sc}}T^*\partial X, g_{|\partial X}) \xrightarrow{\cong} \mathbb{C}\ell(\partial X, h) \tag{1.22}$$
by extending the map $e_i/x \longmapsto e_i$ for an orthonormal frame $\{e_i\}$ on $T^*\partial X$. This leads to the following construction.

Lemma 1.2. *Let $V_{|\partial X} = V^0 \oplus V^1$ be the splitting into $\pm i$ eigenbundles of $\mathrm{c}\ell(e_n)$ where $e_n : \partial X \longrightarrow {}^{\mathrm{sc}}N^*\partial X$ is a chosen unit normal section. Then $V_{|\partial X}$ has the structure of a* graded *Clifford module over the algebra $\mathbb{C}\ell(\partial X, h)$ with respect to the representation*
$$\mathrm{c}\ell_0 : \mathbb{C}\ell(\partial X, h) \longrightarrow \mathrm{End}_{\mathrm{gr}}(V^0 \oplus V^1), \quad \mathrm{c}\ell_0(e_i) = \mathrm{c}\ell(e_i e_n)$$
where we are identifying $\mathbb{C}\ell(\partial X, h)$ and $\mathbb{C}\ell({}^{\mathrm{sc}}T^*\partial X, g_{|\partial X})$.

Remark. In coordinates, this induced representation has the form
$$\mathrm{c}\ell_0(dy_i) = \mathrm{c}\ell\left(\frac{dy_i}{x}\frac{dx}{x^2}\right).$$

Proof. That the representation extends to the full Clifford algebra $\mathbb{C}\ell(\partial X, h)$ is clear since $e_i \longmapsto e_i e_n$ extends to a Clifford algebra homomorphism as mentioned above. That the representation is graded follows from the anti-commutativity of degree 1 elements in the algebra. To wit,
$$\mathrm{c}\ell(e_n)\mathrm{c}\ell_0(e_i) = \mathrm{c}\ell(e_n e_i e_n)v = -\mathrm{c}\ell(e_i e_n e_n) = -\mathrm{c}\ell_0(e_i)\mathrm{c}\ell(e_n)$$
so $\mathrm{c}\ell_0(e_i) : V^0 \longrightarrow V^1$ and vice versa. □

As for the Clifford connection, we have

Lemma 1.3. *Let ∇ be a $\mathbb{C}\ell(X, g)$ Clifford connection on an asymptotically conic manifold which is the lift of a b connection $\nabla \in {}^{\mathrm{b}}\mathcal{A}(X; V)$. Then $\nabla_{|\partial X}$ is a $\mathbb{C}\ell(\partial X, h)$ Clifford connection under the above identifications, and furthermore $\nabla_{|\partial X}$ preserves the splitting $V_{|\partial X} = V^0 \oplus V^1$.*

Proof. Since $\nabla \in {}^{\mathrm{b}}\mathcal{A}(X; V)$ and the manifold is asymptotically conic, ∇ induces a connection on the boundary by
$$\nabla_V = \frac{1}{x}\nabla_{xV}, \quad V \in \mathcal{V}(\partial X)$$
Identifying $\mathbb{C}\ell(\partial X, h)$ and $\mathbb{C}\ell({}^{\mathrm{sc}}T^*\partial X, g_{|\partial X})$ by (1.22), we verify
$$\nabla(\mathrm{c}\ell_0(\alpha)\, u) = \nabla(\mathrm{c}\ell(\alpha e_n)\, u) = \mathrm{c}\ell\left((\nabla^{\mathrm{LC}(h)}\alpha)e_n\right) u + \mathrm{c}\ell(\alpha e_n)\,\nabla u = \mathrm{c}\ell_0(\nabla^{\mathrm{LC}(h)}\alpha)\, u + \mathrm{c}\ell_0(\alpha)\,\nabla u$$

since ${}^{\mathrm{b}}\mathcal{A}(X;\mathbb{C}\ell(X)) \ni \nabla^{\mathrm{LC}(g)}_{|\partial X} = \nabla^{\mathrm{LC}(h)}$ as proved in Proposition 1.1, and since $\nabla^{\mathrm{LC}(g)}_{\partial_y} e_n = \frac{1}{x}\nabla^{\mathrm{LC}(g)}_{x\partial_y}\frac{dx}{x^2} = 0$ for $\partial_y \in T\partial X$, by the form of the metric (that is to say, $\frac{dx}{x^2}$ is covariant constant tangentially).

To see that $\nabla_{|\partial X}$ preserves the splitting, we again use that, for $\partial_y \in T\partial X$,

$$\nabla_{\partial_y} c\ell(e_n) = c\ell\left(\frac{1}{x}\nabla^{\mathrm{LC}(g)}_{x\partial_y} e_n\right) + c\ell(e_n)\nabla_{\partial y} = c\ell(e_n)\nabla_{\partial y}$$

so $\nabla_{|\partial X}$ commutes with $c\ell(e_n)$, and must preserve its eigenvalues. □

Given a $\mathbb{C}\ell(X,g)$ module $V \longrightarrow X$, a lifted b Clifford connection $\nabla \in {}^{\mathrm{b}}\mathcal{A}(X;V)$ and the associated Dirac operator $\not{D}$, we define the *induced boundary Dirac operator* at $p \in \partial X$ by

$$\not{\partial} = \sum_i c\ell_0(e_i)\nabla_{\hat{e}_i} = \begin{pmatrix} 0 & \not{\partial}^- \\ \not{\partial}^+ & 0 \end{pmatrix}, \tag{1.23}$$

where $\{e_i\}$ is an orthonormal frame for $T^*_p\partial X$ with dual frame $\{\hat{e}_i\}$ for $T_p\partial X$. It must have the form on the right hand side of (1.23) with respect to $V_{|\partial X} = V^0 \oplus V^1$ since ∇ preserves the splitting and $c\ell_0(e_i) : V^i \longrightarrow V^{i\pm 1}$.

Note that since $\not{\partial}$ is self-adjoint, $\not{\partial}^+$ and $\not{\partial}^-$ are mutual adjoints of one another.

Proposition 1.2. *If $\not{D}$ is a scattering Dirac operator associated to a lifted b Clifford connection $\nabla \in {}^{\mathrm{b}}\mathcal{A}(X;V)$, then*

$$\not{D} = x\not{D}',$$

where

$$\not{D}'_{|\partial X} = c\ell(e_n)\left[\nabla_{x\partial_x} + \not{\partial}\right]$$

Remark. Written out in full, we have

$$\not{D}'_{|\partial X} = \begin{pmatrix} i\nabla_{x\partial_x} & 0 \\ 0 & -i\nabla_{x\partial_x} \end{pmatrix} + \begin{pmatrix} 0 & i\not{\partial}^- \\ -i\not{\partial}^+ & 0 \end{pmatrix}$$

Proof. This just comes from factoring out x and $c\ell(e_n)$. Indeed, $c\ell(e_i) = -c\ell(e_ie_ne_n) = c\ell(e_n)c\ell(e_ie_n) = c\ell(e_n)c\ell_0(e_i)$, and with $\hat{e}_n = x^2\partial_x$ we have

$$\not{D} = c\ell(e_n)\left(\nabla_{x^2\partial_x} + \sum_{i=1}^{n-1} c\ell_0(e_i)\nabla_{x\hat{e}_i}\right) = xc\ell(e_n)\left(\nabla_{x\partial_x} + \sum_i c\ell_0(e_i)\nabla_{\hat{e}_i}\right)$$

which gives the result. □

Remark. Completely analogous constructions are available in the case of a b metric, and produce b Dirac operators. As we shall not require them here, we will only give a brief description of this theory.

As above, we can form the b Clifford bundle

$$\mathbb{C}\ell(X,g_b)_p = \mathbb{C}\ell({}^{\mathrm{b}}T^*_pX, g_b(p))$$

to which the Levi-Civita connection extends:

$$\nabla^{\mathrm{LC}(g_{\mathrm{b}})} : C^\infty(X;\mathbb{C}\ell(X,g_{\mathrm{b}})) \longrightarrow C^\infty(X;{}^{\mathrm{b}}\Lambda^1 \otimes \mathbb{C}\ell(X,g_{\mathrm{b}})).$$

Given a skew Hermitian $\mathbb{C}\ell(X, g_{\mathrm{b}})$ module $V \longrightarrow X$ and a Clifford connection satisfying the analogue of (1.20), we have a canonical *b Dirac operator*

$$\eth\!\!\!/ = \sum_i c\ell(e_i)\nabla_{\hat{e}_i}, \qquad \{e_i\} \text{ an orthonormal frame for } {}^{\mathrm{b}}T^*X$$

which is essentially self-adjoint on $L^2(X; V; g_b)$ (equivalently, on b half densities $L^2(X; V \otimes \Omega_{\mathrm{b}}^{1/2})$).

In contrast to the scattering case, a normal section $x\partial_x \in C^\infty(\partial X; {}^{\mathrm{b}}N\partial X)$ exists *canonically*; also since ${}^{\mathrm{b}}TX/{}^{\mathrm{b}}N\partial X \cong T\partial X$, the identification

$$\mathbb{C}\ell(T\partial X, (g_{\mathrm{b}})_{|\partial X}) \cong \mathbb{C}\ell(\partial X, h)$$

is automatic in this case, and $\nabla_{|\partial X}$ is *always* a $\mathbb{C}\ell(\partial X, h)$ Clifford connection on ∂X if the metric is exact. Hence in the b case, the induced boundary Dirac operator

$$\partial\!\!\!/ = \sum_i c\ell_0(e_i)\nabla_{e_i}, \qquad \{e_i\} \text{ an orthonormal frame for } T^*\partial X$$

does not depend on choices.

1.5 b Differential and Pseudodifferential operators

We briefly recall the main results and definitions of Melrose's b-calculus of pseudodifferential operators.

To start, we have the *b differential operators*, $\mathrm{Diff}_{\mathrm{b}}^k(X) : \dot{C}^\infty(X) \longrightarrow \dot{C}^\infty(X)$, which is a filtered (by $k \in \mathbb{N}$) algebra of operators defined as the k-fold composition of b vector fields acting on $\dot{C}^\infty(X)$, with $\mathrm{Diff}_{\mathrm{b}}^0(X) = C^\infty(X)$ acting by multiplication. In coordinates, elements look like

$$\mathrm{Diff}_{\mathrm{b}}^k(X) \ni D = \sum_{j+|\alpha| \leq k} a_{j,\alpha}(x,y)\,(x\partial_x)^j\,\partial_y^\alpha, \quad a_{j,\alpha} \in C^\infty(X), \tag{1.24}$$

where $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ is a multi-index. We obtain b differential operators acting on sections of a vector bundle $V \longrightarrow X$ by setting

$$\mathrm{Diff}_{\mathrm{b}}^k(X; V) = \mathrm{End}(V) \otimes_{C^\infty(X)} \mathrm{Diff}_{\mathrm{b}}^k(X)$$

(considering $\mathrm{Diff}_{\mathrm{b}}^k(X)$ as a *left* $C^\infty(X)$ module). This corresponds locally to replacing $a_{j,\alpha}$ in (1.24) by sections of $C^\infty(X; \mathrm{End}(V))$.

We have the usual *principal symbol*, which in coordinates has the form

$$C^\infty({}^{\mathrm{b}}T^*X; \mathrm{End}(\pi^*V)) \ni \sigma_{\mathrm{int}}(D) = \sum_{j+|\alpha|=k} a_{j,\alpha}(x,y)\xi^j\eta^\alpha, \quad a_{j,\alpha}(x,y) \in \mathrm{End}(V), \tag{1.25}$$

where $(\xi, \eta) = \xi\frac{dx}{x} + \sum_i \eta_i dy_i$, though principal symbol invertibility is not enough to guarantee that D admits a suitable Fredholm extension.

The other important object is the *b normal operator*, obtained as follows. The evaluation of elements of $\mathcal{V}_{\mathrm{b}}(X)$ at points $p \in \partial X$ is valued in the Lie algebra $\mathcal{V}_{\mathrm{b}}(X)/x \cdot \mathcal{V}_{\mathrm{b}}(X)$, which makes sense as $x\mathcal{V}_{\mathrm{b}}(X)$ is a Lie ideal from Lemma 1.1. From the fact that $v_i \in$

$C^\infty(X; {}^{\mathrm{b}}N\partial X), i = 1, 2$ satisfy $[v_1, v_2] \in x\mathcal{V}_{\mathrm{b}}(X)$ (which follows from (1.2)), it follows that b normal sections form a *trivial* subalgebra upon restriction. Along with the restriction of $C^\infty(X)$ to ∂X, we get the *normal operator homomorphism*

$$N_{\mathrm{b}} : \mathrm{Diff}^k_{\mathrm{b}}(X; V) \longrightarrow \mathrm{Diff}^k_{\mathrm{I}}({}^{\mathrm{b}}N_+\partial X; V), \tag{1.26}$$

where the range space consists of b differential operators on the inward pointing b normal bundle ${}^{\mathrm{b}}N_+\partial X$ (so a space diffeomorphic to $\partial X \times (0, \infty)$) which are invariant with respect to the natural $\mathbb{R}_+$ action on the fibers of ${}^{\mathrm{b}}N_+\partial X$. In coordinates, this amounts to

$$D = \sum_{j+|\alpha|\le k} a_{j,\alpha}(x, y)\,(x\partial_x)^j\,\partial_y^\alpha \longmapsto N_{\mathrm{b}}(D) = \sum_{j+|\alpha|\le k} a_{j,\alpha}(0, y)\,(s\partial_s)^j\,\partial_y^\alpha \tag{1.27}$$

where $s \in (0, \infty)$ is a coordinate on the fiber $({}^{\mathrm{b}}N_+\partial X)_{(0,y)}$. By compactifying ${}^{\mathrm{b}}N_+\partial X$ to $\overline{{}^{\mathrm{b}}N_+\partial X}$ by adding the points $s = 0$ and $1/s = 0$, we can alternatively view the normal operator as being again a b differential operator $N_{\mathrm{b}}(D) \in \mathrm{Diff}_{\mathrm{b,I}}(\overline{{}^{\mathrm{b}}N_+\partial X}; V)$.

By choosing a trivialization of ${}^{\mathrm{b}}N_+\partial X$, we can take the fiberwise Mellin transform of the normal operator (which has the effect of mapping $s\partial_s$ to $i\lambda$) to obtain the *indicial family*

$$I(D, \lambda) \in \mathrm{Diff}^k(\partial X; V), \quad \text{for all } \lambda \in \mathbb{C}, \tag{1.28}$$

which extends to an entire map

$$\mathbb{C} \ni \lambda \longmapsto I(D, \lambda) \in \mathcal{B}(H^{k+l}(\partial X; V), H^l(\partial X; V))$$

from $\mathbb{C}$ into the space of bounded linear operators on the L^2 based Sobolev spaces on ∂X.

This construction is "microlocalized" to produce a calculus of pseudodifferential operators $\Psi^{m,\mathcal{E}}_{\mathrm{b}}(X; V \otimes \Omega^{1/2}_{\mathrm{b}})$ by considering operator kernels on the space obtained by blowing up[1] $\partial X \times \partial X \subset X^2$:

$$X^2_{\mathrm{b}} = [X^2; \partial X^2]. \tag{1.29}$$

This blown up space has boundary hypersurfaces lb and rb, corresponding to the lifts of $\partial X \times X$ and $X \times \partial X$, and ff, the new hypersurface corresponding to the lift of ∂X^2. The *b pseudodifferential operators* are defined by

$$\Psi^{m,\mathcal{E}}_{\mathrm{b}}(X; V \otimes \Omega^{1/2}_{\mathrm{b}}) = \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}} I^m(X^2_{\mathrm{b}}, \Delta; \mathrm{End}(V) \otimes \Omega^{1/2}_{\mathrm{b}}(X^2_{\mathrm{b}})) \tag{1.30}$$

where $I^m(X^2_{\mathrm{b}}, \Delta)$ is the space (as defined by Hörmander) of distributions conormal to the lifted diagonal Δ with order $m \in \mathbb{R}$, $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}$ requires that the kernels have polyhomogeneous (classical) conormal expansions at the boundary hypersurfaces in terms of index sets

$$\mathcal{E} = (E_{\mathrm{ff}}, E_{\mathrm{lb}}, E_{\mathrm{rb}}),$$

and $\Omega^{1/2}_{\mathrm{b}}$ denotes the bundle of b half densities (which are the natural objects to consider in the absence of a metric). Densities and polyhomogeneous distributions are discussed in Appendix A.

By constructing a suitable triple space (a blown up version of X^3), these operators can be composed by the following result which follows from the general pushforward and

[1]See A for the definition of blow up.

pullback theorems A.2 and A.1 quoted in Appendix A.

Theorem 1.2. *Let* $P \in \Psi_{\mathrm{b}}^{s,\mathcal{E}}(X;V)$, $Q \in \Psi_{\mathrm{b}}^{t,\mathcal{F}}(X;V)$. *Provided* $\mathrm{Re}(E_{\mathrm{rb}}) + \mathrm{Re}(F_{\mathrm{lb}}) > 0$,

$$P \circ Q \in \Psi_{\mathrm{b}}^{s+t,\mathcal{G}}(X;V),$$

where

$$\begin{aligned} G_{\mathrm{lb}} &= (E_{\mathrm{ff}} + F_{\mathrm{lb}}) \overline{\cup} E_{\mathrm{lb}} \\ G_{\mathrm{rb}} &= (F_{\mathrm{ff}} + E_{\mathrm{rb}}) \overline{\cup} F_{\mathrm{rb}} \\ G_{\mathrm{ff}} &= (E_{\mathrm{ff}} + F_{\mathrm{ff}}) \overline{\cup} (E_{\mathrm{lb}} + F_{\mathrm{rb}}). \end{aligned}$$

A distinguished subset of $\Psi_{\mathrm{b}}^{*,\mathcal{E}}(X;V)$ is the *small calculus*, consisting of operators with $\mathcal{E} = (e_{\mathrm{ff}}, \infty_{\mathrm{lb}}, \infty_{\mathrm{rb}})$, where $e_{\mathrm{ff}} \in \mathbb{N}$ denotes a smooth index set $\{(e_{\mathrm{ff}} + k, 0)\}_{k \in \mathbb{N}}$, and ∞_* denotes the empty index set (corresponding to smooth functions which vanish to infinite order near the corresponding face). The small calculus forms a subalgebra under composition.

For the small calculus (and taking $e_{\mathrm{ff}} = 0$ for simplicity), the normal operator homomorphism is defined for $P \in \Psi_{\mathrm{b}}^m(X;V)$ to be the restriction of the operator kernel to the front face

$$N_{\mathrm{b}}(P) = (\kappa_P)_{|\mathrm{ff}} \in \Psi_{\mathrm{I}}^k({}^{\mathrm{b}}N_+ \partial X; V) \tag{1.31}$$

the I again denoting $\mathbb{R}_+$ invariance in the fiber; equivalently, $N_{\mathrm{b}}(P)$ acts by convolution in the fiber variable. This coincides with the previous definition (1.26) for differential operators, and we have the composition result

$$N_{\mathrm{b}}(P \circ Q) = N_{\mathrm{b}}(P) \circ N_{\mathrm{b}}(Q).$$

Again the *indicial family* $I(P, \lambda) \in \Psi^k(\partial X; V)$ is obtained by Mellin transform of the normal operator; its precise expression depends on choice of bdf x, but its invertibility properties (for any fixed λ) do not.

For an *elliptic*[2] element $P \in \Psi_{\mathrm{b}}^m(X;V)$ (again restricting to the small calculus where $e_{\mathrm{ff}} = 0$ for simplicity), $I(P, \lambda)$ is a family of elliptic pseudodifferential operators on ∂X, and the *b spectrum* of P is the set

$$\mathrm{spec}_{\mathrm{b}}(P) = \{\lambda \in \mathbb{C} \; ; \; I(P, \lambda) \quad \text{is not invertible on } C^\infty(\partial X)\} \tag{1.32}$$

which must necessarily be discrete by analytic Fredholm theory, and so $I(P, \lambda)$ is meromorphic as a map from $\mathbb{C} \longrightarrow \mathcal{B}(H^{s+m}(\partial X), H^s(\partial X))$.

A central result in the b-calculus is the following Fredholm result on *weighted b Sobolev spaces*

$$x^\alpha H_{\mathrm{b}}^s(X; V \otimes \Omega_{\mathrm{b}}^{1/2}) = \left\{ x^\alpha u \; ; \; \Psi_{\mathrm{b}}^s \cdot u \in L^2(X; V \otimes \Omega_{\mathrm{b}}^{1/2}) \right\}. \tag{1.33}$$

Proposition 1.3. *Let* $P \in \Psi_{\mathrm{b}}^m(X; V \otimes \Omega_{\mathrm{b}}^{1/2})$ *be elliptic, with* $m > 0$. *Then provided* $\alpha \notin \{\mathrm{Im}(\lambda) \; ; \; \lambda \in \mathrm{spec}_{\mathrm{b}}(P)\}$, *P extends as a Fredholm operator*

$$P_\alpha : x^\alpha H_{\mathrm{b}}^{s+m}(X; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow x^\alpha H_{\mathrm{b}}^s(X; V \otimes \Omega_{\mathrm{b}}^{1/2}) \tag{1.34}$$

[2]Ellipticity meaning invertibility of the principal symbol away from the zero section in ${}^{\mathrm{b}}T^*X$.

with a parametrix $Q \in \Psi_{\mathrm{b}}^{m,(0,E^+(\alpha),E^-(\alpha))}(X; V \otimes \Omega_{\mathrm{b}}^{1/2})$ *where*

$$E^{\pm}(\alpha) = \left\{ (\pm i\lambda, k) \ ; \ I(P,\lambda) \text{ has a pole of order } k, \text{ and } \operatorname{Im}(\lambda) \gtrless \alpha \right\}. \tag{1.35}$$

Furthermore, the index of P_α *is constant as* α *varies within connected components of* $\mathbb{R} \setminus \operatorname{Im}(\operatorname{spec}_{\mathrm{b}}(P))$ *and in general if* $\alpha_1 < \alpha_2$*, the index differs by*

$$\operatorname{ind}(P_{\alpha_1}) - \operatorname{ind}(P_{\alpha_2}) = \sum_{\substack{z \in \operatorname{spec}_{\mathrm{b}}(P) \\ \operatorname{Im}(z) \in (\alpha_1, \alpha_2)}} \mathit{rank}(I(P,z))$$

where $\mathit{rank}(I(P,z))$ *is a count of the "singular range" of* $I(P,z)$ *at its poles.*

Remark. The index formula above is *relative*; i.e. does not give the precise value of $\operatorname{ind}(P_\alpha)$, only the amount by which it changes as α varies. However, it is sometimes possible to find an *index zero weight*, i.e. a choice of α such that P_α is Fredholm and formally self-adjoint, whence $\operatorname{ind}(P_\alpha) = 0$.

1.6 Scattering Differential and Pseudodifferential operators

The microlocalization of scattering vector fields proceeds similarly; though in many ways the results are simpler.

The *scattering differential operators* $\operatorname{Diff}_{\mathrm{sc}}^k(X) : \dot{C}^\infty(X) \longrightarrow \dot{C}^\infty(X)$ are the k-fold compositions of scattering vector fields (with $\operatorname{Diff}_{\mathrm{sc}}^0(X) = C^\infty(X)$); in coordinates

$$\operatorname{Diff}_{\mathrm{sc}}^k(X;V) \ni D = \sum_{j+|\alpha|\leq k} a_{j,\alpha}(x,y) \left(x^2\partial_x\right)^j (x\partial_y)^\alpha, \quad a_{j,\alpha} \in C^\infty(X; \operatorname{End}(V)). \tag{1.36}$$

The *principal symbol* is a function on ${}^{\mathrm{sc}}T^*X$, which in coordinates has the form

$$C^\infty({}^{\mathrm{sc}}T^*X; \operatorname{End}(\pi^*V)) \ni \sigma_{\mathrm{int}}(D) = \sum_{j+|\alpha|=k} a_{j,\alpha}(x,y)\xi^j\eta^\alpha, \quad a_{j,\alpha} \in \operatorname{End}(V), \tag{1.37}$$

where $(\xi,\eta) = \xi\frac{dx}{x^2} + \sum_i \eta_i \frac{dy_i}{x}$.

From $[\mathcal{V}_{\mathrm{sc}}(X), \mathcal{V}_{\mathrm{sc}}(X)] \subset x\mathcal{V}_{\mathrm{sc}}(X)$ in Lemma 1.1, the restriction of $v \in \mathcal{V}_{\mathrm{sc}}(X)$ to ∂X takes values in the *abelian* Lie algebra $\mathcal{V}_{\mathrm{sc}}(X)/x\,\mathcal{V}_{\mathrm{sc}}(X)$, and this map (along with restriction of smooth functions) generates the *normal operator homomorphism*

$$N_{\mathrm{sc}} : \operatorname{Diff}_{\mathrm{sc}}^k(X;V) \longrightarrow \operatorname{Diff}_{\mathrm{I,fib}}^k({}^{\mathrm{sc}}T_{\partial X}X; V), \tag{1.38}$$

the range space denoting translation invariant differential operators along the fibers of ${}^{\mathrm{sc}}T_{\partial X}X \longrightarrow \partial X$ which are smoothly parametrized by the base. Thinking of these as fiber-wise constant coefficient differential operators and taking the fiberwise Fourier transform, we obtain the *scattering symbol*

$$\sigma_{\mathrm{sc}}(D) = \widehat{N_{\mathrm{sc}}(D)} \in C^\infty({}^{\mathrm{sc}}T^*_{\partial X}X; \operatorname{End}(V)) \tag{1.39}$$

which over $p \in \partial X$ has the form of a polynomial (not necessarily homogeneous) of order k on the vector space ${}^{\mathrm{sc}}T_p^*X$ where k is the order of D. In coordinates the scattering symbol has the form

$$D = \sum_{j+|\alpha|\leq k} a_{j,\alpha}(x,y)\,(x^2\partial_x)^j\,(x\partial_y)^\alpha \longmapsto \sigma_{\mathrm{sc}}(D) = \sum_{j+|\alpha|\leq k} a_{j,\alpha}(0,y)\,(\xi)^j\,\eta^\alpha.$$

Note that the sum is over all elements $j+|\alpha|\leq k$ and not just those with equality. The scattering and principal symbols are compatible in the sense that

$$\sigma_{\mathrm{int}}(D)(p,\xi) \sim \sigma_{\mathrm{sc}}(D)(q,\xi) \quad \text{as } p \longrightarrow q,\ |\xi| \longrightarrow \infty.$$

The scattering pseudodifferential operators are again constructed as Schwartz kernels on a space

$$X^2_{\mathrm{sc}} = [X^2_{\mathrm{b}}; \Delta \cap \mathrm{ff}].$$

which involves the additional blowup of the b double space along the set where the lifted diagonal Δ intersects the boundary $\partial X^2_{\mathrm{b}}$ at the face ff. The boundary hypersurface created by this additional blow-up is called the "scattering face" and denoted sc.

We shall only make use of a "small calculus" in this case, defined by

$$\Psi^{m,l}_{\mathrm{sc}}(X; V\otimes\Omega^{1/2}_{\mathrm{b}}) = \big\{\rho^l_{\mathrm{sc}} u\,;\ u \in I^m(X^2_{\mathrm{sc}}, \Delta; \mathrm{End}(V)\otimes\Omega^{1/2}_{\mathrm{sc}}(X^2_{\mathrm{sc}}), \\ u \equiv 0 \text{ on } \partial X^2_{\mathrm{sc}} \setminus (\mathring{\mathrm{sc}}))\big\}, \quad m\in\mathbb{R}, l\in\mathbb{N} \quad (1.40)$$

Here $u\equiv 0$ denotes vanishing to infinite order, so the kernels are required to vanish identically at all boundary faces except sc, where they vanish to order l.

By constructing a suitably blown up version of X^3, these operators can be composed, giving

$$P\in\Psi^{s,l}_{\mathrm{sc}}(X;V), \quad Q\in\Psi^{t,k}_{\mathrm{b}}(X;V) \quad \Longrightarrow \quad P\circ Q\in\Psi^{s+t,l+k}_{\mathrm{b}}(X;V).$$

For $P\in\Psi^{s,0}_{\mathrm{sc}}(X;V)$, the normal operator is defined to be the restriction of the kernel of P to the scattering face, the interior of which is diffeomorphic to the vector bundle ${}^{\mathrm{sc}}T_{\partial X}X \longrightarrow \partial X$. Under this identification, $P_{|\mathrm{sc}}$ is conormal to the 0 section and we obtain

$$N_{\mathrm{sc}}(P\circ Q) = N_{\mathrm{sc}}(P)\circ N_{\mathrm{sc}}(Q),$$

where $N_{\mathrm{sc}}(P)$ is interpreted as a convolution kernel with respect to the fibers; moreover, this is compatible with, and hence extends, the definition above for differential operators. The Fourier transform exchanges this convolution with multiplication, and we get obtain *multiplicativity* of the scattering symbols (justifying the term "symbol") with respect to composition:

$$\sigma_{\mathrm{sc}}(P\circ Q) = \sigma_{\mathrm{sc}}(P)\sigma_{\mathrm{sc}}(Q).$$

We say $P\in\Psi^{m,0}_{\mathrm{sc}}(X;V)$ is *elliptic* if, as usual, $\sigma_{\mathrm{int}}(P)$ is invertible away from the zero section. We say P is *fully elliptic* if, in addition, $\sigma_{\mathrm{sc}}(P)$ is invertible on all of ${}^{\mathrm{sc}}T^*_{\partial X}X$. Then the fundamental result concerning Fredholm extensions with respect to the weighted *scattering Sobolev spaces*

$$x^\alpha H^s_{\mathrm{sc}}(X;V) = \big\{x^\alpha u\,;\ \Psi^s_{\mathrm{sc}}\cdot u \in L^2\big\} \quad (1.41)$$

is the following.

Proposition 1.4. *Let $P\in\Psi^{m,0}_{\mathrm{sc}}(X;V)$, $m>0$, be a fully elliptic scattering pseudodiffer-*

ential operator. Then P extends to a Fredholm operator

$$P : x^\alpha H^{s+m}_{\mathrm{sc}}(X;V) \longrightarrow x^\alpha H^s_{\mathrm{sc}}(X;V)$$

for all $\alpha \in \mathbb{R}$*, with parametrix* $Q \in \Psi^{-m,0}_{\mathrm{sc}}(X;V)$ *such that* $(\mathrm{Id} - PQ), (\mathrm{Id} - QP) \in \Psi^{-\infty,\infty}_{\mathrm{sc}}(X;V)$*. In particular, the nullspace of P consists of sections vanishing to infinite order at* ∂X*:*

$$\mathrm{Null}(P) \subset \dot{C}^\infty(X;V)$$

The scattering pseudodifferential operators also have a rather explicit index theory, as we shall discuss in Chapter 2.

In Chapters 3 and 4 we shall require mixed b, scattering Sobolev spaces $H^{k,l}_{\mathrm{b,sc}}(X;\Omega^{1/2}_{\mathrm{sc}})$ which we define here for $k, l \in \mathbb{N}$. We let

$$H^{k,l}_{\mathrm{b,sc}}(X;\Omega^{1/2}_{\mathrm{sc}}) = \Big\{u \in L^2(X;\Omega^{1/2}_{\mathrm{sc}})\ ;\ \sum_{j\le k; i \le l} V_1 \cdots V_j W_1 \cdots W_i u \in L^2 \quad \text{where } V_\alpha \in \mathcal{V}_{\mathrm{b}}(X), W_\alpha \in \mathcal{V}_{\mathrm{sc}}(X)\Big\}.$$

Proposition 1.5. *The following are equivalent characterizations of* $H^{k,l}_{\mathrm{b,sc}}(X;\Omega^{1/2}_{\mathrm{sc}})$*:*

1. $H^{k,l}_{\mathrm{b,sc}}(X;\Omega^{1/2}_{\mathrm{sc}}) = \Big\{u \in L^2(X;\Omega^{1/2}_{\mathrm{sc}})\ ;\ \sum_{j\le k; i\le l} W_1 \cdots W_i V_1 \cdots V_j u \in L^2\Big\}$*, where* $V_\alpha \in \mathcal{V}_{\mathrm{b}}(X), W_\alpha \in \mathcal{V}_{\mathrm{sc}}(X)$*,*
2. $H^{k,l}_{\mathrm{b,sc}}(X;\Omega^{1/2}_{\mathrm{sc}}) = \Big\{u \in H^k_{\mathrm{b}}(X;\Omega^{1/2}_{\mathrm{sc}})\ ;\ \Psi^l_{\mathrm{sc}}(X)\cdot u \in H^k_{\mathrm{b}}(X)\Big\}$
3. $H^{k,l}_{\mathrm{b,sc}}(X;\Omega^{1/2}_{\mathrm{sc}}) = \Big\{u \in H^l_{\mathrm{sc}}(X;\Omega^{1/2}_{\mathrm{sc}})\ ;\ \Psi^k_{\mathrm{b}}(X)\cdot u \in H^l_{\mathrm{sc}}(X)\Big\}$.

Proof. (1) Follows from iteratively using the fact that $[\mathcal{V}_{\mathrm{b}}(X), \mathcal{V}_{\mathrm{sc}}(X)] \subset \mathcal{V}_{\mathrm{sc}}(X)$ from Lemma 1.1. Then (2) is obtained by noting that

$$u \in H^{k,l}_{\mathrm{b,sc}}(X) \implies \left\{\sum_{j\le k} V_1 \cdots V_j u \in L^2\right\} = H^l_{\mathrm{b}}(X),$$

and the fact that any $P \in \Psi^l_{\mathrm{sc}}(X)$ acting on L^2-based Sobolev spaces can be controlled by estimates on operators of the form $\sum_{i\le l} W_1 \cdots W_l$.

Finally, (3) follows from (1) and similar arguments. □

Chapter 2

An index theorem of Callias type for pseudodifferential operators

In [11], C. Callias obtained a formula for the Fredholm index of operators on odd-dimensional Euclidean space $\mathbb{R}^n$ having the form

$$P = \not{D} \otimes 1 + i \otimes \Phi(x) : \Gamma(V' \otimes V'') \longrightarrow \Gamma(V' \otimes V''),$$

where $\not{D}$ is a self-adjoint spin Dirac operator (associated to any connection with appropriate flatness at infinity), Φ is a Hermitian matrix-valued function which is uniformly invertible off a compact set (representing a Higgs potential in physics), and V' and V'' are trivial vector bundles. According to his formula, the index depends only on a topological invariant of Φ restricted to S^{n-1}, the sphere at infinity. In a following paper [8], R. Bott and R. Seeley interpreted Callias' result in terms of a symbol map (equivalent to the "total symbol" we define below) $\sigma(P) : S^{2n-1} = \partial\left(\overline{T^*\mathbb{R}^n}\right) \longrightarrow \mathrm{End}(V' \otimes V'')$ and point out that the resulting index formula has the form of the product of the Chern characters of $\sigma(\not{D})$ and Φ, each integrated over a copy of S^{n-1}.

The Fredholm index of Dirac operators coupled to skew-adjoint non-scalar potentials on various odd dimensional manifolds was subsequently studied by several authors, culminating in a result in [3] by N. Anghel (and obtained independently by J. Råde in [25], and U. Bunke in [10] who proved a C^* equivariant version of the theorem applicable in particular to families of Dirac operators) for operators of the above type on arbitrary odd-dimensional, complete Riemannian manifolds . Under suitable conditions on $\not{D}$, a generalized Dirac operator associated to a vector bundle $V \longrightarrow X$, and on the potential $\Phi \in \Gamma(\mathrm{End}(V))$, Anghel proved that

$$\mathrm{ind}(\not{D} + i\Phi) = \mathrm{ind}(\not{\partial}_+^+)$$

where $\not{\partial}_+^+$ is a related Dirac operator on a hypersurface $Y \subset X$, representing "infinity" in X. The proofs of these results depend on the fact that $\not{D}$ is a Dirac operator – Callias' original proof uses local trace formulas of the integral kernels, Anghel and Bunke use the relative index theorem of Gromov and Lawson [15], and Råde uses elliptic boundary conditions analogous to the Atiyah-Patodi-Singer conditions to preserve the index under various cutting and gluing procedures.

In this chapter we shall determine the index of Callias-type operators via methods in topological K-theory, in the spirit of [6] and [7], using the calculus of scattering pseudodifferential operators outlined in section 1.6 of Chapter 1. In particular, this allows us to consider a class of *pseudodifferential* Callias-type operators, which we dub Callias-Anghel operators, and to obtain a families version of the index theorem with little additional effort. Our result applies also to even-dimensional manifolds, where Callias-Anghel operators which are not of Dirac type may indeed have nontrivial index.

A Callias-Anghel operator P on an asymptotically conic manifold X has the form $P = D + i\Phi$, where $D \in \Psi^m_{\text{sc}}(X, V)$ is an elliptic scattering pseudodifferential operator with Hermitian symbols, and $\Phi \in \Gamma(\text{End}(V))$ is a compatible potential, meaning that $\Phi_{|\partial X}$ is Hermitian, invertible and commutes with the symbol of D. The main result of this chapter, proved in section 2.4, is the following.

Theorem. *A Callias-Anghel operator $P = D + i\Phi$ extends to be Fredholm, with index*

$$\text{ind}(P) = \int_{S^*_{\partial X}X} \text{ch}(V^+_+) \cdot \pi^*\text{Td}(\partial X).$$

where $V^+_+ \subset \pi^*V \longrightarrow S^*_{\partial X}X$ is the jointly positive eigenbundle of $\sigma(D)$ and $\pi^*\Phi$ on the cosphere bundle of X over ∂X.

The essence of the proof is to note that the index of $D + i\Phi$ is determined topologically by the symbolic data, which is shown to be trivial over the interior of X and completely determined by the coupling between Hermitian (from $\sigma(D)$) and skew-Hermitian (from $i\Phi$) terms on the cosphere bundle over infinity ($S^*_{\partial X}X$). Indeed, once the problem is properly formulated, the proof is a straightforward computation in K-theory.

We note that the previous results for Dirac operators hold for arbitrary complete Riemannian manifolds, whereas we restrict ourselves to asymptotically conic manifolds. This choice is justified by the following reasons.

- While this class of manifolds is geometrically more restricted than the general complete Riemannian manifolds considered by Anghel and others, the conditions for operators to be Fredholm on these spaces are much *less* restrictive and more easily verified in practice. Correspondingly, we need to assume less about D and Φ to obtain our result. We discuss a connection between our setup and the one considered by Anghel in section 2.6, and expect that using scattering models in the context of certain other noncompact index problems (essentially situations in which the Fredholm data is sufficiently local near infinity) may be possible.

- The symbolic structure of the scattering calculus has a very simple interpretation in terms of topological K-theory, permitting us to utilize a powerful families index theorem derived from [7], which is proved in section 2.1.

We begin with a brief introduction to the scattering calculus in section 2.1, culminating with the proof of the index theorem for families of scattering operators. We introduce Callias-Anghel type operators in section 2.2 and prove that they extend to Fredholm operators. Section 2.3 is the heart of our result, and consists of the reduction of the symbol to the corner $S^*_{\partial X}X$ of the total space $\partial(\overline{T^*X})$ in K-theory; it is entirely topological in nature. We present our results in section 2.4, with a particular analysis of the important case of Dirac operators. Finally, we discuss the relation to previous results in section 2.6.

2.1 The index formula in the scattering calculus

Given $D \in \Psi^m_{\text{sc}}(X; V)$, recall that we have an principal symbol or*interior symbol* map

$$\sigma_{\text{int}}(D) : {}^{\text{sc}}T^*X \longrightarrow \text{End}(\pi^*V)$$

and also a *boundary* or *scattering symbol*

$$\sigma_{\text{sc}}(D) : {}^{\text{sc}}T^*_{\partial X}X \longrightarrow \text{End}(\pi^*V).$$

These have asymptotic growth/decay of order $\leq m$ along the fibers, and they satisfy the compatibility condition that, asymptotically,

$$\sigma_{\text{int}}(D)(p, \xi) \sim \sigma_{\text{sc}}(D)(q, \xi) \qquad \text{as } p \longrightarrow q,\ |\xi| \longrightarrow \infty,$$

where $p \in X$, $q \in \partial X$, $\xi \in {}^{\text{sc}}T^*_pX$.

We will restrict ourselves to so-called "classical" operators whose symbols have asymptotic expansions in terms of $|\xi|^{m-k}$, $k \in \mathbb{N}$, as $|\xi| \longrightarrow \infty$. Then for 0th order operators, we can regard the interior symbol as a map

$$\sigma_{\text{int}}(D) : {}^{\text{sc}}S^*X \longrightarrow \text{End}(\pi^*V),$$

where ${}^{\text{sc}}S^*_pX$ is the boundary of the radially compactified fiber $\overline{{}^{\text{sc}}T^*_pX}$, and the value of $\sigma_{\text{int}}(D)$ is obtained by taking the limit of the leading term in the asymptotic expansion. Similarly, we extend $\sigma_{\text{sc}}(D)$ to a map

$$\sigma_{\text{sc}}(D) : \overline{{}^{\text{sc}}T^*_{\partial X}X} \longrightarrow \text{End}(\pi^*V),$$

again using radial compactification of the fibers. Since D is 0th order, both symbols are bounded, asymptotic compatibility is just equality of the limits, and we can combine the two symbols into a continuous *total symbol*

$$\sigma_{\text{tot}}(D) : \partial(\overline{{}^{\text{sc}}T^*X}) \longrightarrow \text{End}(\pi^*V),$$

where $\overline{{}^{\text{sc}}T^*X}$ is the total space of the compactified scattering cotangent bundle. It is a manifold with corners, with boundary $\partial(\overline{{}^{\text{sc}}T^*X})$ consisting of both ${}^{\text{sc}}S^*X$ and $\overline{{}^{\text{sc}}T^*_{\partial X}X}$, which intersect at the corner ${}^{\text{sc}}S^*_{\partial X}X$ (see Figure 2-1).

We can produce a total symbol for higher order operators as follows. For every $m \in \mathbb{R}$, we construct a trivial real line bundle $N_m \longrightarrow \overline{{}^{\text{sc}}T^*X}$ whose bounded sections consist of functions with asymptotic growth/decay of order m. In terms of an asymptotically conic metric, a trivialization of N_m over the interior is given by the section $|\xi|^m$, that is

$${}^{\text{sc}}T^*X \times \mathbb{R} \longrightarrow N_m : ((p, \xi), t) \overset{\cong}{\longmapsto} t\,|\xi|^m_p\,.$$

Symbols of mth order operators define bounded sections of N_m, which take limiting values at the boundary as above, and we define the *renormalized symbols* as

$${}_m\sigma_{\text{int}}(D) : {}^{\text{sc}}S^*X \longrightarrow N_m \otimes \text{End}(\pi^*V), \quad {}_m\sigma_{\text{sc}}(D) : \overline{{}^{\text{sc}}T^*_{\partial X}X} \longrightarrow N_m \otimes \text{End}(\pi^*V).$$

We combine these to obtain the *renormalized total symbol*

$$ {}_m\sigma_{\mathrm{tot}}(D) : \partial(\overline{{}^{\mathrm{sc}}T^*X}) \longrightarrow N_m \otimes \mathrm{End}(\pi^*V). \tag{2.1}$$

Note that full ellipticity is equivalent to invertibility of the renormalized total symbol ${}_m\sigma_{\mathrm{tot}}(D)$, since this invertibility does not depend on the chosen trivialization of N_m.

Remark. Note that in discussing the total symbols of pseudodifferential operators, we use the notation $\pi : \partial(\overline{{}^{\mathrm{sc}}T^*X}) \longrightarrow X$ to denote the generalized projection. This is not a proper fiber bundle, as the fiber over the interior is a sphere, ${}^{\mathrm{sc}}S_p^*X$, while the fiber over a boundary point is the (radially compactified) vector space $\overline{{}^{\mathrm{sc}}T_p^*X}$. Nevertheless, the notation is convenient.

Below we shall consider families of scattering pseudodifferential operators, for which we use the following notation. Suppose X has the structure of a fiber bundle $X \longrightarrow Z$, where Z is a compact manifold without boundary, and such that the fiber is a manifold Y with boundary ∂Y. We use the notation $X/Z \equiv Y$ to denote the fiber, though there is no real such quotient. Thus X has boundary ∂X which itself fibers over Z, with fiber ∂Y. X is associated to a principal Diffeo(Y)-bundle $\mathcal{P} \longrightarrow Z$, from which we derive additional associated bundles.

We suppose given a metric on X which restricts to a fixed exact scattering metric on each fiber, for instance by taking a scattering metric on the total space. A *family of scattering operators* on X, over Z, is an operator acting on sections of a vector bundle[1] $V \longrightarrow X$ which is scattering pseudodifferential in the fiber directions, and smoothly varying in the base. It is properly defined as a section of the bundle

$$\Psi^m_{\mathrm{sc}}(X/Z; V) = \mathcal{P} \times_{\mathrm{Diffeo}(Y)} \Psi^m_{\mathrm{sc}}(Y; W) \longrightarrow Z, \tag{2.2}$$

where V and W are related by $V = \mathcal{P} \times_{\mathrm{Diffeo}(Y)} W$.

In the simple case that X is a product, $X = Y \times Z$, Z is just a smooth parameter space for the operators, and we recover the case of a single operator by taking $Z = \mathrm{pt}$, $X/Z = X = Y$.

For a family $D \in \Psi^m_{\mathrm{sc}}(X/Z; V)$ of operators, the symbol maps have domain ${}^{\mathrm{sc}}T^*(X/Z)$, which is the vertical scattering cotangent bundle

$${}^{\mathrm{sc}}T^*(X/Z) = \mathcal{P} \times_{\mathrm{Diffeo}(Y)} {}^{\mathrm{sc}}T^*Y \longrightarrow Z,$$

with fibers isomorphic to the scattering cotangent bundle ${}^{\mathrm{sc}}T^*Y$ of the fiber. The renormalized total symbol is a map

$${}_m\sigma_{\mathrm{tot}}(D) : \partial\left(\overline{{}^{\mathrm{sc}}T^*(X/Z)}\right) \longrightarrow N_m \otimes \mathrm{End}(\pi^*V)$$

as before, where now everything is fibered over Z, and $\partial(\overline{{}^{\mathrm{sc}}T^*(X/Z)})$ has fibers isomorphic to $\partial(\overline{{}^{\mathrm{sc}}T^*Y})$. Note that $\pi : \partial(\overline{{}^{\mathrm{sc}}T^*(X/Z)}) \longrightarrow X$ is a family of projections modeled on $\pi : \partial(\overline{{}^{\mathrm{sc}}T^*Y}) \longrightarrow Y$, to which the remark at the end of section 2.1 applies.

As in the case of ordinary pseudodifferential operators, a family D of Fredholm opera-

[1] Any vector bundle $V \longrightarrow X$ can be exhibited as a family of vector bundles $V = \mathcal{P} \times_{\mathrm{Diffeo}(Y)} W$, where $W \longrightarrow Y$ is a fixed vector bundle with the same rank as V.

tors[2] over Z has an index $\mathrm{ind}(D) \in K^0(Z)$ given by

$$\mathrm{ind}(D) = [\mathrm{ker}D] - [\mathrm{coker}D] \in K^0(Z),$$

which is well-defined by a stabilization procedure and Kuiper's theorem [18].

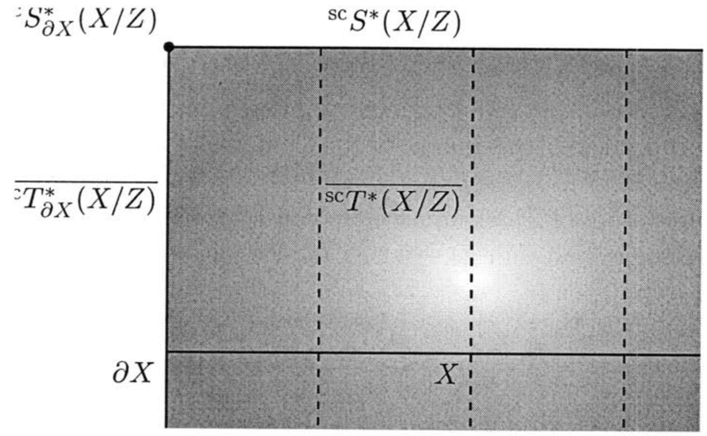

Figure 2-1: The total space $\overline{{}^{\mathrm{sc}}T^*(X/Z)}$ and its boundary

The following theorem is one of the primary reasons for using the scattering calculus in our treatment. Among calculi of pseudodifferential operators on noncompact manifolds, the scattering calculus is particularly simple since its boundary symbols are *local*[3], and hence give well-defined elements in the compactly supported topological K-theory of ${}^{\mathrm{sc}}T^*X$. In particular, this allows for the index to be computed by a reduction to the Atiyah-Singer index theorem for compact manifolds ([7], [6]). This is discussed in [24], [22] and [1] though we provide a proof here for completeness.

Remark. As our applications are to self-adjoint operators with skew-Hermitian potentials, the domain and range bundles of our operators will always be the same, which permits us to write the index formula below in terms of the *odd* Chern character of the total symbol, which in this case defines an element of the odd K-group $K^1(\partial(\overline{{}^{\mathrm{sc}}T^*X}))$. In the general case, the symbolic data of fully elliptic operators correspond to elements of $K^0_c({}^{\mathrm{sc}}T^*\mathring{X})$, and a similar formula (an extension of Fedosov's formula in [12]) can be obtained using a relative version of de Rham cohomology (see [1]).

First let us introduce the notation we use for K-theory. As usual, we write elements in even K-theory as formal differences of vector bundles up to equivalence and stabilization,

$$[V] - [W] \in K^0(M)$$

and use the notation

$$[V, W, \sigma] \in K^0(M, N)$$

for relative classes, where $\sigma : V_{|N} \xrightarrow{\cong} W_{|N}$ is an isomorphism over N. This applies in particular to K-theory with *compact support* $K^0_c(M) = K^0(M, \infty)$, which is just K-theory relative

[2] Fredholmness in the families setting is with respect to families of scattering Sobolev spaces $H^*_{\mathrm{sc}}(X/Z; V) = \mathcal{P} \times_{\mathrm{Diffeo}(Y)} H^*_{\mathrm{sc}}(Y; W)$.

[3] The trade off is that the condition of full ellipticity in the scattering calculus is stronger than the corresponding condition in calculi with less local boundary data.

to infinity with respect to any compactification (the one point compactification $M \cup \{\infty\}$ is typically used, though for vector bundles V, we use the fiberwise radial compactification $V \cup SV$).

Odd K-theory is represented by homotopy classes of maps $M \longrightarrow \lim_{n\to\infty} \mathrm{GL}(n)$, though we use the notation

$$[V, \sigma] \in K^1(M)$$

as shorthand for the element $[V \oplus V^\perp, \sigma \oplus \mathrm{Id}] \in K^1(M)$, where $V \oplus V^\perp \cong M \times \mathbb{C}^N$, so $\sigma \oplus \mathrm{Id} : M \longrightarrow \mathrm{GL}(N)$. In particular, an element $[V, V, \sigma] \in K^0(M, N)$ with identical domain and range bundles is the image of an element $[V, \sigma] \in K^1(N)$ in the long exact sequence of the pair (M, N).

Lastly, we define a topological index map for scattering pseudodifferential operators analogous to the classical one. For $D \in \Psi^m_{\mathrm{sc}}(X/Z; V, W)$ fully elliptic,

$$[\pi^* V, \pi^* W, {}_m\sigma_{\mathrm{tot}}(D)] \in K^0_c({}^{\mathrm{sc}}T^*(\mathring{X}/Z))$$

is well-defined[4], where $\mathring{X} = X \setminus \partial X$ is the interior of X (so the compact support refers both to the fiber *and* base directions). Then for the K-oriented embedding of fibrations ${}^{\mathrm{sc}}T^*(\mathring{X}/Z) \longrightarrow \mathbb{R}^{2N} \times Z$ into an even dimensional trivial Euclidean fibration which is induced by an embedding $\mathring{X} \longrightarrow \mathbb{R}^N \times Z$, we define top-ind$(D)$ to be the image of $[\pi^* V, \pi^* W, {}_m\sigma_{\mathrm{tot}}(D)]$ under the composition

$$\text{top-ind} : K^0_c({}^{\mathrm{sc}}T^*(\mathring{X}/Z)) \longrightarrow K^0_c(N({}^{\mathrm{sc}}T^*(\mathring{X}/Z))) \longrightarrow K^0_c(\mathbb{R}^{2N} \times Z) \cong K^0(Z), \qquad (2.3)$$

where the first map is the Thom isomorphism onto the normal bundle of ${}^{\mathrm{sc}}T^*(\mathring{X}/Z)$ in $\mathbb{R}^{2N} \times Z$, the second is the pushforward with respect to the open embedding

$$N({}^{\mathrm{sc}}T^*(\mathring{X}/Z)) \hookrightarrow \mathbb{R}^{2N} \times Z$$

in compactly supported K-theory[5], and the last is the Bott periodicity isomorphism (equivalently, the Thom isomorphism for a trivial bundle). That this is a well-defined map independent of choices follows exactly as in the classical case in [7].

Theorem 2.1. *Let $P \in \Psi^m_{\mathrm{sc}}(X/Z; V)$ be a family of fully elliptic scattering pseudodifferential operators. It is therefore a Fredholm family, with well-defined index* $\mathrm{ind}(P) \in K^0(Z)$, *and*

$$\mathrm{ind}(P) = \text{top-ind}(P).$$

Furthermore, the Chern character of this index is given by the cohomological formula

$$\mathrm{ch}(\mathrm{ind}(P)) = q_! \left(\mathrm{ch}(\sigma_{\mathrm{tot}}(P)) \cdot \pi^* \mathrm{Td}(X/Z)\right),$$

where $q_! : H^{\mathrm{odd}}(\partial(\overline{{}^{\mathrm{sc}}T^*(X/Z)})) \longrightarrow H^{\mathrm{even}}(Z)$ is defined below, and $\mathrm{ch}(\sigma_{\mathrm{tot}}(P))$ *is shorthand for* $\mathrm{ch}_{\mathrm{odd}}([\pi^* V, {}_m\sigma_{\mathrm{tot}}(P)])$, *with* $[\pi^* V, {}_m\sigma_{\mathrm{tot}}(P)] \in K^1(\partial(\overline{{}^{\mathrm{sc}}T^*(X/Z)}))$, *defined using any trivialization of N_m.*

---

[4]This involves choosing a trivialization of N_m, though the element of K-theory obtained is independent of this choice.

[5]Recall that, while cohomology theories (K-theory in particular), are contravariant, there is a limited form of covariance with respect to open embeddings in any compactly supported theory. If $i : O \hookrightarrow M$ is an open embedding, we obtain a pushforward map $i_* : K^*_c(O) \longrightarrow K^*_c(M)$ via the quotient map $M^+/\infty \longrightarrow M/(M \setminus O) \cong O^+/\infty$, where M^+ denotes the one point compactification $M \cup \{\infty\}$.

Remark. The map $q_!$ is a generalized integration over the fibers map defined as follows. If we denote by p the map $p : X \longrightarrow Z$, then $q = p \circ \pi : \partial(\overline{{}^{sc}T^*(X/Z)}) \longrightarrow Z$ is not properly a fibration as per the remark in section 2.1; it really consists of two fibrations $q_1 : {}^{sc}S^*(X/Z) \longrightarrow Z$ and $q_2 : \overline{{}^{sc}T^*_{\partial X}(X/Z)} \longrightarrow Z$, with an identification of their common boundary, which is the fibration ${}^{sc}S^*_{\partial X}(X/Z) \longrightarrow Z$. We define $q_!$ as the sum

$$q_!\mu = (q_1)_!\mu + (q_2)_!\mu$$

pulling back $\mu \in H^{\mathrm{odd}}(\partial(\overline{{}^{sc}T^*(X/Z)}))$ as appropriate in each of the summands; it is well-defined on cohomology since if $\mu = d\alpha$ is exact (or more generally fiberwise exact),

$$(q_!d\alpha)(z) = \int_{\left(\overline{{}^{sc}T^*(X/Z)}\right)_z} d\alpha + \int_{({}^{sc}S^*(X/Z))_z} d\alpha = \int_{\left({}^{sc}S^*_{\partial X}(X/Z)\right)_z} \alpha - \int_{\left({}^{sc}S^*_{\partial X}(X/Z)\right)_z} \alpha = 0$$

by Stokes' Theorem, since $\overline{{}^{sc}T^*(X/Z)}$ and ${}^{sc}S^*(X/Z)$ share the common boundary ${}^{sc}S^*_{\partial X}(X/Z)$ but with opposite orientation.

Note that $q_!$ also factors as the composition of the coboundary map

$$H^{\mathrm{odd}}(\partial(\overline{{}^{sc}T^*(X/Z)})) \longrightarrow H^{\mathrm{even}}_c({}^{sc}T^*(\mathring{X}/Z))$$

with the integration map

$$p'_! : H^{\mathrm{even}}_c({}^{sc}T^*(\mathring{X}/Z)) \longrightarrow H^{\mathrm{even}}(Z),$$

a fact which will be important below.

Proof. The idea is to construct a family $\widetilde{P}$ of pseudodifferential operators on a fibration $\widetilde{X} \longrightarrow Z$ whose fibers $\widetilde{Y} = (\widetilde{X}/Z)$ are compact manifolds *without* boundary (and hence to which we can apply the Atiyah-Singer index theorem for families), and such that $\mathrm{ind}(P) = \mathrm{ind}(\widetilde{P})$. We first deform P by homotopy to be fiberwise constant (actually equal to the identity) near ∂X by pulling it up from the 0-section over ∂X and stretching it around the corner[6]. Then we can attach a mirror copy of X and extend by the identity.

By composing with an invertible operator of order $-m$, we can assume without loss of generality that $m = 0$, and therefore ${}_m\sigma_{\mathrm{tot}}(P) = \sigma_{\mathrm{tot}}(P) : \partial(\overline{{}^{sc}T^*(X/Z)}) \longrightarrow \mathrm{Aut}(\pi^*V)$. We have an $\mathbb{R}_+$ action on the fibers of ${}^{sc}T^*_{\partial X}(X/Z)$ given locally by

$$t : (p, \xi) \longmapsto \left(p, \frac{\max(|\xi| - t, 0)}{|\xi|}\xi\right),$$

where $p \in \partial X$, $\xi \in {}^{sc}T^*_p(X/Z)$, and $|\xi|$ is the norm of ξ with respect to some metric. Then $t^*\sigma_{\mathrm{tot}}(P), 0 \le t \le 1/\epsilon$ is a homotopy through invertible symbols from $\sigma_{\mathrm{tot}}(P)$ to a symbol which is fiberwise constant on $\overline{{}^{sc}T^*_{\partial X}(X/Z)}$ except in a neighborhood of the corner ${}^{sc}S^*_{\partial X}(X/Z) = \{(p, \xi)\ ;\ 1/|\xi| = 0, p \in \partial X\}$.

Actually, this homotopy can be extended "around the corner" by (again locally) first regarding $-x$ as an extension of $1/|\xi|$, where x is a boundary defining function, and rescaling so that $[-2\epsilon, \infty] \longmapsto [0, \infty]$.

This rescaling followed by the aforementioned homotopy results in a homotopy connecting $\sigma_{\mathrm{tot}}(P)$ to a section $\sigma : \partial(\overline{{}^{sc}T^*(X/Z)}) \longrightarrow \mathrm{Aut}(\pi^*V)$ such that σ is given by a vector

[6] As remarked by R. Melrose, "like pulling on a sock."

bundle isomorphism over X near ∂X. That is, for some neighborhood $U \supset \partial X$,

$$\sigma_{|\pi^{-1}(U)} = \pi^* \alpha, \qquad \alpha : U \longrightarrow \mathrm{Aut}(V),$$

since σ is fiberwise constant on $\pi^{-1}(U)$.

Smoothing the above homotopy allows us to realize it within the class of total symbols of 0th order, fully elliptic families of scattering pseudodifferential operators, and hence lift it to a homotopy

$$P \sim P' \in \Psi^0_{\mathrm{sc}}(X/Z; V)$$

such that $P'_{|U} = \alpha$. As this homotopy $P \sim P'$ produces a continuous family of Fredholm operators, $\mathrm{ind}(P) = \mathrm{ind}(P')$, and because $P'_{|U}$ is a bundle isomorphism, it is clear that

$$u \in \ker(P') \text{ or } \ker(P'^*) \implies \mathrm{supp}(u) \subset X \setminus U.$$

Let $\widetilde{X}$ be the double of X which is obtained by gluing two copies of X, one with reversed orientation, along ∂X:

$$\widetilde{X} = X \cup_{\partial X} (-X).$$

It is a fiber bundle over Z with fibers modeled on $\widetilde{Y} = Y \cup_{\partial Y} (-Y)$, a compact manifold without boundary. Let $\widetilde{\alpha}$ be a smooth extension of α across the boundary, so that $\widetilde{\alpha} : U' \supset \partial X \longrightarrow \mathrm{Aut}(V)$ is a smooth bundle isomorphism on a sufficiently small open set $U' \subset \widetilde{X}$. Let $\widetilde{V} \longrightarrow \widetilde{X}$ be a vector bundle which is constructed from two copies of $V \longrightarrow X$, glued along U' via the clutching map $\widetilde{\alpha}$.

Since $P'_{|U}$ is equal to the clutching function used to construct $\widetilde{V}$, it can be extended by the identity on $(-X)$; that is, we can produce an operator $\widetilde{P} \in \Psi^0(\widetilde{X}/Z; \widetilde{V})$ which is equal to P' on X and equal to the identity on $(-X)$. Clearly $\widetilde{P}$ is elliptic, and we have identifications

$$\ker(P') \cong \ker(\widetilde{P}), \quad \ker(P'^*) \cong \ker(\widetilde{P}^*),$$

since elements in either space have support in the region where $P' = \widetilde{P}$. These identifications are compatible with the stabilization procedure needed to identify $\mathrm{ind}(P')$ and $\mathrm{ind}(\widetilde{P})$ with well-defined elements of $K^0(Z)$, and so $\mathrm{ind}(P') = \mathrm{ind}(\widetilde{P})$.

For the first claim, we note that any embedding $\widetilde{X} \longrightarrow \mathbb{R}^N \times Z$ gives a suitable embedding of $\mathring{X} \longrightarrow \mathbb{R}^N \times Z$ for application of the topological index map, and it is straightforward to see that the image of $[\pi^*V, \pi^*V, \sigma_{\mathrm{tot}}(P')] \in K^0_c({}^{\mathrm{sc}}T^*(\mathring{X}/Z))$ under the inclusion into $K^0_c(T^*(\widetilde{X}/Z))$ is equivalent to $[\pi^*\widetilde{V}, \pi^*\widetilde{V}, \sigma(\widetilde{P})]$. Then the Atiyah-Singer index theorem [7] for families gives

$$\mathrm{ind}(P') = \mathrm{ind}(\widetilde{P}) = \text{top-ind}(\widetilde{P}) = \text{top-ind}(P').$$

For the cohomological formula, we have

$$\mathrm{ch}(\mathrm{ind}(\widetilde{P})) = p_! \left(\mathrm{ch}_{\mathrm{even}}(\sigma(\widetilde{P})) \cdot \pi^* \mathrm{Td}(\widetilde{X}/Z) \right)$$

from the Atiyah-Singer formula, where $p_! : H^{\mathrm{even}}_c({}^{\mathrm{sc}}T^*(\widetilde{X}/Z)) \longrightarrow H^{\mathrm{even}}(Z)$. Since we obtain $[\pi^*\widetilde{V}, \pi^*\widetilde{V}, \sigma(\widetilde{P})]$ in $K^0_c({}^{\mathrm{sc}}T^*(\widetilde{X}/Z))$ as a pushforward of the element $[\pi^*V, \pi^*V, \sigma_{\mathrm{tot}}(P')]$ in $K^0_c({}^{\mathrm{sc}}T^*(\mathring{X}/Z))$, it is immediate that

$$\mathrm{ch}(\mathrm{ind}(P')) = p'_! \left(\mathrm{ch}_{\mathrm{even}}(\sigma_{\mathrm{tot}}(P')) \cdot \pi^* \mathrm{Td}(X/Z) \right)$$

where $p'_! : H^{\text{even}}_c({}^{\text{sc}}T^*(\mathring{X}/Z)) \longrightarrow H^{\text{even}}(Z)$ is the integration over the fibers map mentioned in the remark above.

Since the element $[\pi^*V, \pi^*V, \sigma_{\text{tot}}(P')] = [\pi^*V, \pi^*V, \sigma_{\text{tot}}(P)] \in K^0_c({}^{\text{sc}}T^*(\mathring{X}/Z))$ is the image of $[\pi^*V, \sigma_{\text{tot}}(P)] \in K^1(\partial(\overline{{}^{\text{sc}}T^*(X/Z)}))$ under the coboundary map, which intertwines the odd and even Chern characters, we get

$$\text{ch}(\text{ind}(P)) = q_! \left(\text{ch}_{\text{odd}}(\sigma_{\text{tot}}(P)) \cdot \pi^*\text{Td}(X/Z)\right)$$

as claimed. □

Corollary 2.1. *In the special case of a single operator $P \in \Psi^m_{\text{sc}}(X;V)$, the index is an integer* $\text{ind}(P) \in \mathbb{Z}$*, and we have*

$$\text{ind}(P) = \int_{\partial(\overline{{}^{\text{sc}}T^*X})} \text{ch}(\sigma_{\text{tot}}(P)) \cdot \pi^*\text{Td}(X).$$

2.2 Callias-Anghel type operators

We shall be concerned with pseudodifferential families D whose symbols are Hermitian, coupled to skew-Hermitian potentials $i\Phi$. It is actually only necessary that $i\Phi$ be skew-Hermitian at infinity, as well as satisfy some compatibility conditions with D. We will later see why the index is only dependent on these conditions.

Let $V \longrightarrow X$ be a family of Hermitian complex vector bundles associated to the family of scattering manifolds $X \longrightarrow Z$. We will denote the inner product on V by $\langle\cdot,\cdot\rangle$. Let $D \in \Psi^m_{\text{sc}}(X/Z;V), m > 0$ be a family of elliptic (but not necessarily *fully* elliptic) scattering operators with Hermitian symbols, so

$$\sigma_{\text{int}}(D) : {}^{\text{sc}}S^*(X/Z) \longrightarrow \text{Aut}(\pi^*V) \subset \text{End}(\pi^*V)$$

is Hermitian with respect to $\langle\cdot,\cdot\rangle$ and

$$\sigma_{\text{sc}}(D) : {}^{\text{sc}}T^*_{\partial X}(X/Z) \longrightarrow \text{End}(\pi^*V)$$

is Hermitian but not necessarily invertible. Of particular interest later will be the case of a family of Dirac operators, for which $\sigma_{\text{sc}}(D)(p,\xi) = ic\ell(\xi)\cdot$ vanishes at the 0-section over $\partial(X/Z)$ and is therefore *never* fully elliptic.

Next let Φ be a section of $\text{End}(V)$. Motivated by physics, we refer to Φ as the *potential.* We will assume Φ satisfies the following conditions over the boundary ∂X, which we shall dub *compatibility with D.*

1. $\Phi_{|\partial X}$ is Hermitian with respect to $\langle\cdot,\cdot\rangle$

2. $\Phi_{|\partial X}$ is invertible

3. $\Phi_{|\partial X}$ commutes with the boundary symbol of D, that is

$$[\pi^*\Phi_{|\partial X}, \sigma_{\text{sc}}(D)] = 0 \in \text{End}(\pi^*V) \quad \text{on } \overline{{}^{\text{sc}}T^*_{\partial X}X}.$$

We refer to condition 3 as *symbolic commutativity.*

Given D and a compatible potential Φ, the *Callias-Anghel type operator*

$$P = D + i\Phi \in \Psi^m_{\mathrm{sc}}(X/Z; V)$$

is fully elliptic (and therefore Fredholm on appropriate spaces) by the following elementary lemma.

Lemma 2.1. *Let α and β be Hermitian sections of the bundle* $\mathrm{End}(V) \longrightarrow M$*, and suppose that, over a subset $\Omega \subset M$, we have $[\alpha, \beta] = \alpha\beta - \beta\alpha = 0 \in \Gamma(\Omega; \mathrm{End}(V))$. If either α or β is invertible over Ω, then the combination*

$$\alpha + i\beta \in \Gamma(\Omega; \mathrm{Aut}(V))$$

is invertible over Ω.

In particular, if both α and β are invertible over Ω, then the combination

$$t\alpha + i\, s\beta \in \Gamma(\Omega; \mathrm{End}(V)) \text{ is invertible for all } (s,t) \neq (0,0) \in \mathbb{R}^2_+.$$

Proof. It suffices to consider an arbitrary fiber V_p, $p \in \Omega$. By the assumption that α and β are Hermitian, $\alpha(p)$ has purely real eigenvalues while $i\beta(p)$ has purely imaginary ones. Since $[\alpha, \beta] = 0$, there is a basis of V_p in which $\alpha(p)$ and $\beta(p)$ are simultaneously diagonal; with respect to this basis $\alpha + i\beta$ acts diagonally with eigenvalues of the form $\lambda_j + i\mu_j$ with $\lambda_j, \mu_j \in \mathbb{R}$. If either α or β is invertible, then either $\lambda_j \neq 0$ or $\mu_j \neq 0$ for all j; therefore $\lambda_j + i\mu_j \neq 0 \in \mathbb{C}$ and $\alpha + i\beta$ must be invertible. □

Corollary 2.2. *The family of scattering operators $P = D + i\Phi$ extends to a family of Fredholm operators*[7]

$$P : x^\alpha H^{k+m}_{\mathrm{sc}}(X/Z; V) \longrightarrow x^\alpha H^k_{\mathrm{sc}}(X/Z; V)$$

for all k, α.

Proof. The interior symbol $\sigma_{\mathrm{int}}(P) = \sigma_{\mathrm{int}}(D)$ is invertible on ${}^{\mathrm{sc}}S^*(X/Z)$, since D is elliptic. The boundary symbol $\sigma_{\mathrm{sc}}(P) = \sigma_{\mathrm{sc}}(D + i\Phi) = \sigma_{\mathrm{sc}}(D) + i\pi^*\Phi$ is invertible on $\overline{{}^{\mathrm{sc}}T^*_{\partial X}(X/Z)}$ by symbolic commutativity, using Lemma 2.1 with $\alpha = \sigma_{\mathrm{sc}}(D)$, $\beta = \pi^*\Phi$ and $\Omega = \overline{{}^{\mathrm{sc}}T^*_{\partial X}(X/Z)}$. P is therefore fully elliptic, and by the theory of scattering pseudodifferential operators [21], the claim follows. □

Remark. Note how the compatibility of $\sigma_{\mathrm{int}}(P)$ and $\sigma_{\mathrm{sc}}(P)$ is satisfied. Since D is a family of operators of order $m > 0$, the leading term in the asymptotic expansion of $\sigma_{\mathrm{sc}}(P) = \sigma_{\mathrm{sc}}(D) + i\pi^*\Phi$ as $|\xi| \longrightarrow \infty$ is that of $\sigma_{\mathrm{sc}}(D)$, which grows like $|\xi|^m$, whereas $\pi^*\Phi$ is constant. In terms of the renormalized symbols and a choice of radial coordinate $|\xi|$,

$${}_m\sigma_{\mathrm{tot}}(P) = {}_m\sigma_{\mathrm{tot}}(D) + i\,|\xi|^{-m}\,\pi^*\Phi$$

and the latter term vanishes on ${}^{\mathrm{sc}}S^*(X/Z)$.

[7]See the footnote on page 2 for the definition of a family of Sobolev spaces.

2.3 Reduction to the corner

By Theorem 2.1, the index of P is determined by the element in the odd K-theory of $\partial(\overline{{}^{sc}T^*(X/Z)})$ defined by the (renormalized) total symbol ${}_m\sigma_{\text{tot}}(P)$. The remainder of the work consists in reducing this topological datum to one supported at the corner, ${}^{sc}S^*_{\partial X}(X/Z)$. To this end, we will abstract the situation somewhat, in order to simplify the notation and clarify the concepts involved. Thus we shall forget, for the time being, that our K-class is coming from the symbol of a family of pseudodifferential operators, as well as most of the structure of $\partial(\overline{{}^{sc}T^*(X/Z)})$.

Let $M = \partial(\overline{{}^{sc}T^*(X/Z)})$, and let $N = {}^{sc}S^*_{\partial X}(X/Z)$ be the corner. The important feature of N is that it is a hypersurface, separating $M \setminus N$ into disjoint components $M_1 = {}^{sc}S^*(\mathring{X}/Z)$ and $M_2 = {}^{sc}T^*_{\partial X}(X/Z)$. Actually, the fact that it is a corner is indistinguishable topologically, and we consider it just as a topological hypersurface in M.

We assume a trivialization of the line bundle N_m has been chosen, so we identify ${}_m\sigma_{\text{tot}}(P)$ and $\sigma_{\text{tot}}(P)$ and consider the index to be determined by the element $[\pi^*V, \sigma_{\text{tot}}(P)] \in K^1(M)$. Also, for notational convenience, we will write V instead of π^*V for the remainder of this section.

Proposition 2.1 clarifies the fundamental symbolic structure of P. We see that its symbol essentially consists of an invertible Hermitian term from D over M_1 and an invertible skew-Hermitian term from $i\Phi$ over M_2. These are fundamentally coupled together in a neighborhood of the corner N. Indeed, we cannot uncouple them (i.e. separate their supports via homotopy through invertible endomorphisms) at N. Also note that, were the total symbol either *entirely* Hermitian or *entirely* skew-Hermitian, we could connect it via homotopy to the identity and P would therefore have index 0. The interesting information about $\text{ind}(P)$ is therefore evidently recorded by this coupling near the corner. Proposition 2.2 confirms this, and identifies the topological object at N which encodes this coupling.

Proposition 2.1. *M is covered by two open sets $\widetilde{M_1}$ and $\widetilde{M_2}$ such that $\widetilde{M_1} \cap \widetilde{M_2} \cong N \times I$ where I is a connected, open interval. Furthermore,*

$$[V, \sigma_{\text{tot}}(P)] = [V, \chi A + i(1-\chi)B] \in K^1(M),$$

where A and B are unitary[8], Hermitian sections of $\text{End}(V)$ *such that $[A, B] = 0$ on $\widetilde{M_1} \cap \widetilde{M_2}$, and where $\chi : M \longrightarrow [0,1]$ is a cutoff function such that* $\text{supp}\chi \subset \widetilde{M_1}$ *and* $\text{supp}(1-\chi) \subset \widetilde{M_2}$. *The positive and negative eigenbundles of A and B coincide, respectively, with those of $\sigma_{\text{tot}}(D)$ and $\pi^*\Phi$.*

Proof. As remarked at the end of Section 2.2, $\sigma_{\text{tot}}(P)$ is equal to $\sigma_{\text{sc}}(D)$ on M_1 and to $\sigma_{\text{int}}(D) + \phi\pi^*\Phi$ on M_2, where $\phi \sim |\xi|^{-m}$ is a non-negative real-valued function vanishing on the closure of M_1. In particular, $\phi\pi^*\Phi$ has the same $\pm$ eigenbundles as Φ wherever $\phi \neq 0$.

Since $\sigma_{\text{tot}}(D)$ is invertible on $\overline{M_1} = M_1 \cup N$, by ellipticity, it must be invertible on a slightly larger neighborhood $\widetilde{M_1}$. We set $\widetilde{M_2} = M_2$, on which Φ is Hermitian, invertible, and commutes with $\sigma_{\text{sc}}(D)$ by the compatibility assumption. Shrinking either if necessary, we can assume that $\widetilde{M_1} \cap \widetilde{M_2} \cong N \times I$. Let χ be a cutoff function with properties as above.

Now let A and B be the (generalized) unitarizations of $\sigma_{\text{tot}}(D)$ and $\phi\pi^*\Phi$, respectively. Recall that $C \in \text{GL}(n, \mathbb{C})$ is homotopic in $GL(n, \mathbb{C})$ to its unitarization $U(C)$ via

$$C_t = C\left(t(\sqrt{C^*C})^{-1} + (1-t)\text{Id}\right) \qquad U(C) = (C_t)_{t=1}.$$

[8]at least on $\text{supp}\chi$ and $\text{supp}(1-\chi)$, respectively.

Then A is given by

$$A(p) = \begin{cases} U(\sigma_{\mathrm{tot}}(D)(p)) & \text{if } \sigma_{\mathrm{tot}}(D)(p) \text{ is invertible} \\ 0 & \text{otherwise,} \end{cases}$$

and similarly for B. Note that while A and B are not necessarily continuous sections of $\mathrm{End}(V)$ (as $\sigma_{\mathrm{tot}}(D)$ may fail to be invertible off of $\widetilde{M_1}$ and $\phi\pi^*\Phi$ vanishes away from $\widetilde{M_2}$), χA and $(1-\chi)B$ *are* continuous and have support, respectively, where $\sigma_{\mathrm{tot}}(D)$ (resp. $\phi\pi^*\Phi$) is invertible.

We claim that the homotopy $\sigma_t = (1-t)\sigma_{\mathrm{tot}}(P) + t\,(\chi A + i(1-\chi)B)$ is through invertible endomorphisms. Indeed, at a general point $p \in M$,

$$\sigma_t(p) = [(1-t)\sigma_{\mathrm{tot}}(D)(p) + t\chi(p)A(p)] + i\,[(1-t)\phi(p)\pi^*\Phi(p) + t(1-\chi)(p)B(p)]\,.$$

where the two bracketed terms commute with one another due to symbolic commutativity (where the latter is nonzero), and at least one term is invertible for any p and all t. Invertibility of $\sigma_t(p)$ is then immediate from Lemma 2.1. □

In what follows we will identify $N \times I$ with the set $\widetilde{M_1} \cap \widetilde{M_2}$, and denote its inclusion by $j : N \times I \hookrightarrow M$. Note that over $N \times I$, V splits as $V = V^+ \oplus V^-$ into ± 1 eigenbundles for A, and similarly $V = V_+ \oplus V_-$ into ± 1 eigenbundles for B. Since A and B commute over $N \times I$, these splittings are compatible, giving

$$V_{|N\times I} \cong V_+^+ \oplus V_-^- \oplus V_-^+ \oplus V_+^-.$$

The following makes use of the pushforward with respect to open embeddings in compactly supported K-theory, and also the Bott periodicity isomorphism

$$K^0(N) \xrightarrow{\cong} K^1(N \wedge S^1) = K_c^1(N \times I)$$

given by multiplication by the Bott element $\beta = [\mathbb{C}, e^{-i\theta}]$ which generates $K_c^1(S^1) = \mathbb{Z}$.

Proposition 2.2. *Let $V_{|N\times I} \cong V_+^+ \oplus V_-^- \oplus V_-^+ \oplus V_+^-$ be the splitting into joint eigenbundles of A and B as above. Identify $[V_+^+] = [V_+^+] - [0] \in K^0(N)$ with its image $[V_+^+]\cdot\beta \in K_c^1(N \times I)$ under the Bott isomorphism, and denote the image of the resulting product under the pushforward $j_* : K_c^1(N \times I) \longrightarrow K^1(M, M \setminus (N \times I)) \subset K^1(M)$ by $j_*([V_+^+])$. Then*

$$[V, \chi A + i(1-\chi)B] = j_*\left([V_+^+]\right) \in K^1(M),$$

Additionally, we have

$$j_*\left([V_+^+]\right) = j_*\left([V_-^-]\right) = -j_*\left([V_-^+]\right) = -j_*\left([V_+^-]\right),$$

where $V^\pm$ denotes the $\pm$ eigenbundle of A and $V_\pm$ the $\pm$ eigenbundle of B.

Remark. The reason for omitting the generator β from the notation is that this map $j_* : E^*(N) \longrightarrow E^{*+1}(M)$ is natural for any cohomology theory E^* with a ring structure, with the generator of the suspension isomorphism $E^*(N) \cong E_c^{*+1}(N \times I)$ playing the role of β. Any natural ring homomorphism of cohomology theories (such as the Chern character) takes such generators to one another, and thus commutes with j_*, and it is notationally convenient not to be explicit about such generating elements.

Proof. Let $\sigma = \chi A + i(1-\chi)B$. We first trivialize σ away from $N \times I$, and so note that the K-class it defines is compactly supported[9] in $N \times I$.

Let $\rho_0 : M \longrightarrow [0,1]$ be a cutoff function such that $\rho_0 \equiv 1$ on the complement of $\widetilde{M_1}$, with $\text{supp}\rho_0 \cap \text{supp}\chi = \emptyset$. Then

$$-t\rho_0 \text{Id} + (1-\chi)iB + \chi A$$

is invertible for all $0 \leq t \leq 1$, by Lemma 2.1. Next, let $\rho_1 = (1-\chi) - \rho_0$. Again, by Lemma 2.1, it follows that there is a homotopy through invertible endomorphisms

$$-\rho_0 \text{Id} + (1-\chi)iB + \chi A \sim -\rho_0 \text{Id} + \rho_1 iB + \chi A,$$

by considering $(t(1-\chi) + (1-t)\rho_1)iB$, for instance. Continuing in a similar fashion, we obtain a homotopy

$$\sigma \sim \sigma' = -\rho_0 \text{Id} + \rho_1 iB + \rho_2 A - \rho_3 i\text{Id} - \rho_4 \text{Id},$$

where $1 \equiv \sum_i \rho_i$, $\text{supp}(\rho_i) \cap \text{supp}(\rho_{i+2}) = \emptyset$, and $\text{supp}(\sigma_i) \Subset N \times I$ for $i \in \{1,2,3\}$ so that $\sigma' \equiv -\text{Id}$ on the complement of $N \times I$ (while we have trivialized σ' by $-\text{Id}$ away from $N \times I$ instead of Id, the two are equivalent up to homotopy; indeed $\sigma' \sim -\sigma'$ for any clutching function). It is now evident that $[V,\sigma] = [V,\sigma'] \in K^1(M, M \setminus (N \times I))$.

By identifying ends of the interval I, we see that σ' defines a map

$$\sigma'_{|N\times S^1} = \begin{pmatrix} \sigma_1(\theta) & & & \\ & \sigma_2(\theta) & & \\ & & \sigma_3(\theta) & \\ & & & \sigma_4(\theta) \end{pmatrix} \qquad \sigma_i : S^1 \longrightarrow \mathbb{C}\setminus\{0\}.$$

which is diagonal with respect to the splitting $V_{|N\times I} \cong V_+^+ \oplus V_-^- \oplus V_-^+ \oplus V_+^-$, with scalar entries (since A and B are unitary) independent of N, whose winding numbers are easily determined.

Indeed, by considering the effect of multiplication by $\sigma_i(t)$ as $\sigma'(t)$ passes from $-\text{Id}$, to iB, to A, to $-i\text{Id}$ and then back to $-\text{Id}$, it is easy to verify that $\text{wn}(\sigma_2) = \text{wn}(\sigma_3) = \text{wn}(\sigma_4) = 0$ and $\text{wn}(\sigma_1) = -1$. Thus there are homotopies

$$\sigma_i \sim \tilde{\sigma}_i \equiv 1, \quad i = 2,3,4, \qquad \text{and} \qquad \sigma_1(\theta) \sim \tilde{\sigma}_1(\theta) = e^{-i\theta}.$$

which, taken to be the diagonal elements of a matrix, define a homotopy $\sigma' \sim \tilde{\sigma}$. Restricting to $N \times I$, we see

$$K_c^1(N\times I) = K^1(N \wedge S^1) \ni j^*[V,\tilde{\sigma}] = [V_+^+ \oplus V_-^- \oplus V_-^+ \oplus V_+^-, e^{-i\theta} \oplus \text{Id} \oplus \text{Id} \oplus \text{Id}] = [V_+^+, e^{-i\theta}],$$

which is just the image $[V_+^+] \cdot \beta \in K_c^1(N \times I)$ of $[V_+^+] \in K^0(N)$ under Bott periodicity.

Finally, since $[V,\sigma] = [V,\tilde{\sigma}] \in K^1(M, M \setminus (N \times I))$ is in the image of the pushforward map $j_* : K_c^1(N \times I) \longrightarrow K^1(M, M \setminus (N \times I))$, we obtain

$$[V,\sigma] = j_*([V_+^+])$$

[9]Recall that an element $\alpha \in K^1(M)$ in odd K-theory has support in a set A if it is in the image of $K^1(M, M \setminus A)$, and can therefore be represented by an element $[V,\sigma]$ with $\sigma_{|M\setminus A} \equiv \text{Id}$.

as claimed.

Similar proofs, using initial trivializations to

$$\sigma \sim \sigma'' = \rho_0 \mathrm{Id} + \rho_1 iB + \rho_2 A + \rho_3 i\mathrm{Id} + \rho_4 \mathrm{Id},$$

$$\sigma \sim \sigma''' = \rho_0 \mathrm{Id} + \rho_1 iB + \rho_2 A - \rho_3 i\mathrm{Id} + \rho_4 \mathrm{Id},$$

and

$$\sigma \sim \sigma'''' = -\rho_0 \mathrm{Id} + \rho_1 iB + \rho_2 A + \rho_3 i\mathrm{Id} - \rho_4 \mathrm{Id},$$

give $[V, \sigma] = j_*([V_-^-])$, $[V, \sigma] = -j_*([V_+^-])$, and $[V, \sigma] = -j_*([V_-^+])$, respectively. □

2.4 Results

We now present our main results. To simplify notation, we drop the "sc" labels in the remainder of the paper, identifying ${}^{\mathrm{sc}}T^*(X/Z)$ with $T^*(X/Z)$ via a (non-canonical) isomorphism, which is unique up to homotopy.

Theorem 2.2. *Let $D \in \Psi^m_{\mathrm{sc}}(X/Z; V)$ be an elliptic family of scattering pseudodifferential operators with Hermitian symbols, and suppose $\Phi \in C^\infty(X; \mathrm{End}(V))$ is a compatible family of potentials as defined in section 2.2. Then the family $P = D + i\Phi$ is fully elliptic, and extends to a Fredholm operator with index satisfying*

$$\mathrm{ch}(\mathrm{ind}(P)) = p_!(\mathrm{ch}(V_+^+) \cdot \pi^* \mathrm{Td}(\partial X/Z)),$$

where $p_! : H^{\mathrm{even}}(S^*_{\partial X}(X/Z)) \longrightarrow H^{\mathrm{even}}(Z)$ is the fiber integration map, $V_+^+ \longrightarrow S^*_{\partial X}(X/Z)$ is the jointly positive eigenvector bundle with respect to $\sigma_{\mathrm{tot}}(D)_{|S^*_{\partial X}(X/Z)}$ and $\pi^*\Phi_{|\partial X}$, and $\sigma_{\mathrm{tot}}(D)$ is obtained from ${}_m\sigma_{\mathrm{tot}}(D)$ using any trivialization of N_m.

Remark. In the case of a single operator $P = D + i\Phi \in \Psi^m_{\mathrm{sc}}(X; V)$, the index formula can be written

$$\mathrm{ind}(P) = \int_{S^*_{\partial X} X} \mathrm{ch}(V_+^+) \cdot \pi^* \mathrm{Td}(\partial X).$$

Proof. By Theorem 2.1,

$$\mathrm{ch}(\mathrm{ind}(P)) = q_! \left(\mathrm{ch}(\sigma_{\mathrm{tot}}(P)) \cdot \pi^* \mathrm{Td}(X/Z)\right).$$

From Propositions 2.1 and 2.2,

$$K^1(\partial(\overline{T^*(X/Z)})) \ni [\pi^* V, \sigma_{\mathrm{tot}}(P)] = j_*[V_+^+]$$

with $[V_+^+] \in K^0(S^*_{\partial X}(X/Z)) \cong K^1_c(S^*_{\partial X}(X/Z) \times I)$, where V_+^+ is the jointly positive eigenbundle of $\sigma_{\mathrm{tot}}(D)$ and $\pi^*\Phi$.

Now, since the Chern character is a natural mapping $\mathrm{ch} : K^* \longrightarrow H^*$, we obtain

$$\mathrm{ch}(\sigma_{\mathrm{tot}}(P)) = \mathrm{ch}(j_*[V_+^+]) = j_* \mathrm{ch}(V_+^+),$$

where j_* is the composition

$$H^{\mathrm{even}}(S^*_{\partial X}(X/Z)) \xrightarrow{\cong} H^{\mathrm{odd}}_c(S^*_{\partial X}(X/Z) \times I) \longrightarrow H^{\mathrm{odd}}(\partial(\overline{T^*(X/Z)})).$$

Since $j_*\mathrm{ch}(V_+^+)$ is supported on $S^*_{\partial X}(X/Z)$, the integration over the fibers reduces to

$$\mathrm{ch}(\mathrm{ind}(P)) = p_! \left(\mathrm{ch}(V_+^+) \cdot \pi^*\mathrm{Td}(X/Z)\right),$$

where now $p : S^*_{\partial X}(X/Z) \longrightarrow Z$. Furthermore, since the Todd class is natural, $\pi^*\mathrm{Td}(X/Z)$ factors through $S^*_{\partial X}(X/Z) \longrightarrow \partial X \hookrightarrow X$ (all over Z) and we obtain $\mathrm{Td}(X/Z)_{|\partial X} = \mathrm{Td}(\partial X/Z)$; this can alternatively be seen by taking a product metric at the boundary. □

An interesting case of the above is when the family $V = E \otimes F$ is a tensor product of vector bundles, with $D \in \Psi^m_{\mathrm{sc}}(X/Z; E)$ and $\Phi \in \Gamma(X; \mathrm{End}(F))$. In this case, $\sigma_{\mathrm{sc}}(D) \otimes 1$ and $i \otimes \Phi$ commute automatically, it is sufficient that $\Phi_{|\partial X}$ be invertible and Hermitian in order to be compatible with D.

Theorem 2.3. *Given $D \in \Psi^m_{\mathrm{sc}}(X/Z; E)$ elliptic with Hermitian symbols, and a compatible potential $\Phi \in \Gamma(X; \mathrm{End}(E))$, the family $P = D \otimes 1 + i \otimes \Phi \in \Psi^m_{\mathrm{sc}}(X/Z; E \otimes F)$ is fully elliptic, with Fredholm index satisfying*

$$\mathrm{ch}(\mathrm{ind}(P)) = p_!(\mathrm{ch}(E_+) \cdot \mathrm{ch}(F_+) \cdot \pi^*\mathrm{Td}(\partial X/Z)),$$

where $p_! : H^{\mathrm{even}}(S^*_{\partial X}(X/Z)) \longrightarrow H^{\mathrm{even}}(Z)$ is the fiber integration map, and $\pi^*E = E_+ \oplus E_-$ and $\pi^*F = F_+ \oplus F_-$ are the splittings over $S^*_{\partial X}(X/Z)$ into positive and negative eigenbundles of $\sigma(D)$ and $\pi^*\Phi$, respectively.

Remark. Note that the splitting of F is actually coming from the base: $F_{\partial X} = F_+ \oplus F_-$, and $\pi^*F = \pi^*F_+ \oplus \pi^*F_-$.

Proof. The proof is as above, noting that the splitting of $\pi^*V = \pi^*E \otimes F$ over $S^*_{\partial X}(X/Z)$ into

$$\pi^*V_{|S^*_{\partial X}(X/Z)} \cong V_+^+ \oplus V_-^- \oplus V_-^+ \oplus V_+^-$$

corresponds to

$$\pi^*(E \otimes F)_{|S^*_{\partial X}(X/Z)} \cong (E_+ \otimes F_+) \oplus (E_- \otimes F_-) \oplus (E_+ \otimes F_-) \oplus (E_- \otimes F_+),$$

with $\pi^*E = E_+ \oplus E_-$ and $\pi^*F = F_+ \oplus F_-$ split into $\pm$ eigenbundles of $\sigma_{\mathrm{tot}}(D)$ and $\pi^*\Phi$, respectively. Then we note that in K-theory,

$$[E_+ \otimes F_+] - [0] = ([E_+] - [0]) \cdot ([F_+] - [0]) \in K^0(S^*_{\partial X}(X/Z)).$$

Since the pushforward $j_* : K^*_c(S^*_{\partial X}(X/Z) \times I) \longrightarrow K^*(\partial(\overline{T^*(X/Z)}))$ behaves naturally with respect to products in K-theory, and since $\mathrm{ch}([E_+] \cdot [F_+]) = \mathrm{ch}(E_+) \cdot \mathrm{ch}(F_+)$, we have

$$\mathrm{ch}(\mathrm{ind}(P)) = p_! \left(\mathrm{ch}(E_+) \cdot \mathrm{ch}(F_+) \cdot \pi^*\mathrm{Td}(\partial X/Z)\right),$$

as claimed. □

2.5 Dirac case

We further specialize to the case where $D = \not{D}$ is a family of (self-adjoint) Dirac operators, acting on sections of a family of Clifford modules V. In this case, our index formula further reduces to one over $T^*(\partial X/Z)$ rather than $S^*_{\partial X}(X/Z)$, and is given in terms of the induced boundary Dirac operators on ∂X.

Suppose then that $V \longrightarrow X$ is a family of scattering Clifford modules, where $\mathbb{C}\ell(X/Z) = \mathbb{C}\ell({}^{\mathrm{sc}}T(X/Z))$. Suppose the action $c\ell : \mathbb{C}\ell(X/Z) \longrightarrow \mathrm{End}(V)$ is skew-Hermitian, and we have a family of connections $\nabla \in {}^{\mathrm{b}}\mathcal{A}(X/Z;V)$ which is are lifts of b connections (see section 1.4 in Chapter 1), and which satisfy the Clifford compatibility property

$$\nabla(c\ell(\alpha)\cdot u) = c\ell\left(\nabla^{\mathrm{LC}}\alpha\right)\cdot u + c\ell(\alpha)\cdot(\nabla u), \qquad \alpha \in C^\infty(X;\mathbb{C}\ell(X/Z)), u \in C^\infty(X;V).$$

Recall that these data lead to the construction of a canonical scattering family of Dirac operators

$$\not{D} = \sum_j c\ell(e_j)\cdot\nabla_{\hat{e}_j} \in \mathrm{Diff}^1_{\mathrm{sc}}(X;V), \quad \{e_j\}_{j=1}^n \text{ an orthonormal basis for } {}^{\mathrm{sc}}T^*_p(X/Z),$$

which is essentially self-adjoint with respect to the $L^2(X;V)$ pairing $(\not{D}u, v) = (u, \not{D}v)$, and such that $\sigma_{\mathrm{sc}}(\not{D})(p,\xi) = ic\ell(\xi)\cdot$.

Recall that $V_{|\partial X}$ obtains the structure of a $\mathbb{Z}_2$-graded module over $\mathbb{C}\ell(\partial X/Z, h)$, where $h = (x^2 g)_{|\partial X}$, by

$$V_{|\partial X} = V^0 \oplus V^1, \quad V^j \text{ the } (-1)^j \text{ eigenbundle of } ic\ell(\nu),$$

for a choice of normal section $\nu : \partial X \longrightarrow {}^{\mathrm{sc}}N^*\partial X/Z$, with Clifford action

$$c\ell_0 : \mathbb{C}\ell(\partial X/Z) \longrightarrow \mathrm{End}(V), \quad c\ell_0(e_i) = c\ell(e_i\cdot\nu)$$

where we identify $\mathbb{C}\ell(\partial X/Z)$ and $\mathbb{C}\ell({}^{\mathrm{sc}}T^*\partial X, g_{|\partial X})$. This leads to the construction of the induced boundary Dirac operator

$$\not{\partial} = \sum_i c\ell_0(e_i)\nabla_{\hat{e}_i} = \begin{pmatrix} 0 & \not{\partial}^- \\ \not{\partial}^+ & 0 \end{pmatrix}.$$

Now assume $\Phi \in \Gamma(X;\mathrm{End}(V))$ is a compatible family of potentials. In particular, symbolic commutativity at ∂X implies that the positive/negative eigenbundles of Φ in $V_{|\partial X}$ are themselves Clifford modules:

$$\begin{aligned}[][\pi^*\Phi, \sigma_{\mathrm{sc}}(\not{D})] = [\pi^*\Phi, c\ell(\cdot)] = 0 \in \Gamma({}^{\mathrm{sc}}T^*(\partial X/Z); \mathrm{End}(V)) \\ \implies V_{|\partial X} = V_+ \oplus V_-, \quad \mathbb{C}\ell(X/Z)_{|\partial X} : V_\pm \longrightarrow V_\pm.\end{aligned}$$

Furthermore, it is possible to choose a Clifford connection compatible with this splitting, $\nabla : \Gamma(X;V_\pm) \longrightarrow \Gamma(X; {}^{\mathrm{sc}}T^*(X/Z)\otimes V_\pm)$, so that $\not{D}$ has the form $\not{D}_{\partial X} = \not{D}_+ \oplus \not{D}_-$ to leading order at ∂X, with $\not{D}_\pm$ families of Dirac operators on $V_\pm$.

By symbolic commutativity, the splittings $V_{|\partial X} = V_+ \oplus V_-$ and $V_{|\partial X} = V^0 \oplus V^1$ are compatible, so we have

$$V_{|\partial X} = V^0_+ \oplus V^0_- \oplus V^1_+ \oplus V^1_-,$$

with respect to which the induced boundary operator $\not{\partial}$ takes the form

$$\not{\partial} = \begin{pmatrix} 0 & \not{\partial}^-_+ & 0 & 0 \\ \not{\partial}^+_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \not{\partial}^-_- \\ 0 & 0 & \not{\partial}^+_- & 0 \end{pmatrix}$$

where the operators $\eth^{\pm}_{\pm}$ are families of Dirac operators on ∂X, a family of closed manifolds over Z, and are constructed from the Clifford action of $\mathbb{C}\ell(\partial X/Z)$ and the induced family of connections on $\partial X/Z$. Note the symbolic structure of these operators. For instance,

$$\sigma(\eth^{+}_{+}) = ic\ell_0(\cdot) : T^*(\partial X/Z) \longrightarrow \mathrm{Hom}(\pi^* V^0_+, \pi^* V^1_+).$$

It remains to show how the splitting of $\pi^* V_{|S^*_{\partial X}(X/Z)} = V^+ \oplus V^-$ into eigenbundles of $\sigma_{\mathrm{tot}}(\not{D})$ is related to the splitting $V_{|\partial X} = V^0 \oplus V^1$.

Lemma 2.2. *We have*

$$K^0(S^*_{\partial X}(X/Z)) \cong K^0_c(T^*(\partial X/Z)) \oplus K^0(\partial X),$$

with respect to which

$$[V^+_{\pm}] = [\pi^* V^0_{\pm}, \pi^* V^1_{\pm}, ic\ell_0] + [V^1_{\pm}].$$

Proof. We can identify $S^*_{\partial X}(X/Z)$ with two copies of $\overline{T^*(\partial X/Z)}$, glued along their common boundary $S^*(\partial X/Z)$. With this identification, the exact sequence

$$0 \longrightarrow K^0(S^*_{\partial X}(X/Z), \overline{T^*(\partial X/Z)}) \longrightarrow K^0(S^*_{\partial X}(X/Z)) \longrightarrow K^0(\overline{T^*(\partial X/Z)}) \longrightarrow 0$$

splits since there is an obvious retraction $S^*_{\partial X}(X/Z) \longrightarrow \overline{T^*(\partial X/Z)}$ (projecting one hemisphere of each fiber onto the other). Along with the isomorphisms

$$K^0(S^*_{\partial X}(X/Z), \overline{T^*(\partial X/Z)}) \cong K^0(\overline{T^*(\partial X/Z)}, S^*(\partial X/Z)) = K^0_c(T^*(\partial X/Z)),$$

and $K^0(T^*(\partial X/Z)) \cong K^0(\partial X)$ (by contractibility of the fibers), we obtain

$$K^0(S^*_{\partial X}(X/Z)) \cong K^0_c(T^*(\partial X/Z)) \oplus K^0(\partial X),$$

as claimed.

We will exhibit the decomposition of $[V^+_+]$ under this splitting; the case of $[V^+_-]$ is similar. We claim that, as a vector bundle,

$$V^+_+ \cong [\pi^* V^0_+, \pi^* V^1_+, ic\ell_0(\cdot)],$$

that is, V^+_+ is isomorphic to the gluing of the vector bundles $\pi^* V^0_+ \longrightarrow \overline{T^*(\partial X/Z)}$ and $\pi^* V^1_+ \longrightarrow \overline{T^*(\partial X/Z)}$ via the clutching function $ic\ell_0 : S^*(\partial X/Z) \longrightarrow \mathrm{Hom}(\pi^* V^0_+, \pi^* V^1_+)$.

To see this, observe that the two copies of $\overline{T^*(\partial X/Z)}$ in $S^*_{\partial X}(X/Z)$ retract, respectively, onto the images of ∂X under the inward and outward pointing conormal section maps $\nu : \partial X \longrightarrow N^*(\partial X/Z) \subset S^*_{\partial X}(X/Z)$ and $-\nu : \partial X \longrightarrow N^*(\partial X/Z) \subset S^*_{\partial X}(X/Z)$.

Recall that $(V^+_+)_\xi$ is the positive eigenspace of Clifford multiplication $ic\ell(\xi)$ at the point ξ, whereas $(\pi^* V^0_+)_\xi$ (resp. $(\pi^* V^1_+)_\xi$) is the positive (resp. negative) eigenspace of Clifford multiplication by the corresponding inward pointing normal, $ic\ell(\nu(\pi(\xi)))$. Thus, over a point $\nu(p) \in S^*_{\partial X}(X/Z)$, we have $(V^+_+)_{\nu(p)} = \pi^*(V^0_+)_{\nu(p)}$, while over the antipodal point $-\nu(p)$, we have $(V^+_+)_{(-\nu(p))} = \pi^*(V^1_+)_{(-\nu(p))}$, since, for $v \in (V^+_+)_{(-\nu(p))}$, $v = ic\ell(-\nu(p))v = -ic\ell(\nu(p))v$.

The bundle V^+_+ can therefore be identified with $\pi^* V^0_+$ and $\pi^* V^1_+$ over the inward and outward directed copies of $\overline{T^*(\partial X/Z)}$, respectively, and (since $c\ell_0(\xi) = c\ell(\xi)$ on $\pi^* V^0$ when $\xi \in S^*(\partial X/Z)$), it is clear that $ic\ell_0 = ic\ell : S^*(\partial X/Z) \longrightarrow \mathrm{Hom}(\pi^* V^0_+, \pi^* V^1_+)$ is the transition function gluing them together to produce V^+_+, which finishes the claim.

Consider then the element

$$[V_+^+] = [V_+^+] - [\pi^* V_+^1] + [\pi^* V_+^1] \in K^0(S^*_{\partial X}(X/Z)).$$

From the above, we see that $[V_+^+] - [\pi^* V_+^1]$ vanishes over the outward facing copy of $\overline{T^*(\partial X/Z)}$, and so maps to the element $[\pi^* V_+^0, \pi^* V_+^1, ic\ell_0] \in K_c^0(T^*(\partial X/Z))$ in the decomposition above. Clearly $K^0(\overline{T^*(\partial X/Z)}) \ni [\pi^* V_+^1] \cong [V_+^1] \in K^0(\partial X)$ under contraction along the fibers, and we therefore have

$$[V_+^+] \cong [\pi^* V_+^0, \pi^* V_+^1, ic\ell_0] + [V_+^1],$$

as claimed. □

The element $[\pi^* V_+^0, \pi^* V_+^1, ic\ell_0] \in K_c^0(T^* \partial X)$ corresponds precisely to the symbol of $\eth_+^+$, and we obtain the following:

Theorem 2.4. *Let* $\not{D} \in \mathrm{Diff}^1_{\mathrm{sc}}(X/Z; V)$ *be a family of scattering Dirac operators acting on a family of Clifford modules* $V \longrightarrow X$*, and suppose* $\Phi \in \Gamma(\mathrm{End}(V))$ *is a compatible potential. Let* $\eth_+^+ \in \mathrm{Diff}^1_{\mathrm{sc}}(\partial X/Z; V_+^0, V_+^1)$ *be the graded part of the induced boundary Dirac operator* $\eth$ *acting on the positive eigenbundle* $(V_+)_{|\partial X} = V_+^0 \oplus V_+^1$ *of* $\Phi_{|\partial X}$ *as discussed in the text preceding Lemma 2.2. Then* $P = \not{D} + i\Phi \in \mathrm{Diff}^1_{\mathrm{sc}}(X/Z; V)$ *extends to a family of Fredholm operators and*

$$\mathrm{ind}(P) = \mathrm{ind}(\eth_+^+) \in K^0(Z).$$

In particular, we have the index formula

$$\mathrm{ch}(\mathrm{ind}(P)) = p_!(\mathrm{ch}(\sigma(\eth_+^+)) \cdot \pi^* \mathrm{Td}(\partial X/Z))$$

where $p_! : H_c^{\mathrm{even}}(T^*(\partial X/Z)) \longrightarrow H^{\mathrm{even}}(Z)$ *denotes integration over the fibers and, in the case of a single operator*

$$\mathrm{ind}(P) = \mathrm{ind}(\eth_+^+) = \int_{T^*(\partial X)} \mathrm{ch}(\sigma(\eth_+^+)) \cdot \pi^* \mathrm{Td}(\partial X).$$

Proof. From Lemma 2.2, $[V_+^+] = [\pi^* V_+^0, \pi^* V_+^1, ic\ell_0] + [V_+^1] = [\pi^* V_+^0, \pi^* V_+^1, \sigma(\eth_+^+)] + [V_+^1]$. Under the composition $K^0(S^*_{\partial X}(X/Z)) \xrightarrow{j_*} K^1(S^*(X/Z)) \longrightarrow K_c^0(T^*(\mathring{X}/Z))$, $[V_+^+]$ maps to $[\pi^* V, \pi^* V, \sigma_{\mathrm{tot}}(P)]$, and we define

$$[\pi^* V, \pi^* V, \sigma_{\mathrm{tot}}(P)] = \sigma_1 + \sigma_2 \in K_c^0(T^*(\mathring{X}/Z))$$

where σ_1 and σ_2 are the images of $[\pi^* V_+^0, \pi^* V_+^1, ic\ell_0]$ and $[V_+^1]$, respectively. It suffices to show that top-ind(σ_1) = top-ind$(\eth_+^+)$ and that top-ind$(\sigma_2) = 0$. In fact, we will show that σ_2 vanishes identically.

To see the latter, note that σ_2 is equivalent to the image of $[V_+^1] \in K^0(\partial X)$ under the composition $K^0(\partial X) \xrightarrow{\pi^*} K^0(S^*_{\partial X}(X/Z)) \xrightarrow{\cdot\beta} K_c^0(S^*_{\partial X}(X/Z) \times \mathbb{R}^2) \longrightarrow K_c^0(T^*(\mathring{X}/Z))$, where the second map is the Bott periodicity isomorphism (multiplying by the generator $\beta \in K_c^0(\mathbb{R}^2)$), and the last map is the pushforward with respect to the inclusion of an open neighborhood of $S^*_{\partial X}(X/Z)$ in $T^*(\mathring{X}/Z)$. Then σ_2 is invariant with respect to the action of the rotation group $O(n)$ ($n = \dim(X/Z)$) on the fibers of $T^*(\mathring{X}/Z)$, since $O(n)$ acts fiberwise on the first factor of $S^*_{\partial X}(X/Z) \times \mathbb{R}^2$, and $\pi^*[V_+^1]$ is constant on these fibers.

Thus σ_2 is obtained by pullback of an element $\sigma_2' \in K_c^0(L)$, where L is the radial $\mathbb{R}_+$ bundle $L = T^*(X/Z)/O(n) \longrightarrow X$ and compact support means K-theory relative to the boundary hypersurfaces $S^*(X/Z)/O(n)$ and $T^*_{\partial X}(X/Z)/O(n)$. However, $K_c^0(L) \equiv 0$ for any such bundle, since it is equivalent to the (reduced) K-theory of CX, the cone on X, which is a contractible space. Therefore $\sigma_2' = 0$ and so we must have $\sigma_2 = 0$.

To see that top-ind$(\sigma_2) =$ top-ind$(\not{\partial}_+^+)$, we will make use of the following general fact. If $V \longrightarrow B$ is an oriented complex vector bundle and $i : A \hookrightarrow B$ is an open embedding of spaces (which induces an open embedding $\tilde{i} : V_{|A} \longrightarrow V$), then for any $\alpha \in K_c^*(A)$, we have $i_*(\alpha) \cdot \mathfrak{T}(V) = \tilde{i}_*\left(\alpha \cdot \mathfrak{T}(V_{|A})\right) \in K_c^*(V)$, where $\mathfrak{T}(V) \in K_c^0(V)$ is the K-orientation class (Thom class) generating $K_c^0(V)$ as a module over $K^0(B)$.

Suppose we are given a K-oriented embedding of fibrations $T^*(X/Z) \hookrightarrow \mathbb{R}^{2N} \times Z$ coming from an embedding $X \longrightarrow \mathbb{R}^N \times Z$. Let

$$g : S^*_{\partial X}(X/Z) \times \mathbb{R}^2 \hookrightarrow \mathbb{R}^{2N} \times Z$$

be the induced embedding of an open neighborhood of $S^*_{\partial X}(X/Z)$ and

$$h : S^*_{\partial X}(X/Z) \hookrightarrow \mathbb{R}^{2N} \times Z$$

be the induced embedding of $S^*_{\partial X}(X/Z)$. We denote the normal bundles of $S^*_{\partial X}(X/Z) \times \mathbb{R}^2$ and $S^*_{\partial X}(X/Z)$ in $\mathbb{R}^{2N} \times Z$ by $N_g(S^*_{\partial X}(X/Z) \times \mathbb{R}^2)$ and $N_h(S^*_{\partial X}(X/Z))$, respectively, emphasizing the corresponding embeddings. Also, let

$$f : T^*(\partial X/Z) \hookrightarrow S^*_{\partial X}(X/Z)$$

be the open embedding onto the open inward facing open disk bundle as in Lemma 2.2. Since σ_1 is supported in $S^*_{\partial X}(X/Z) \times \mathbb{R}^2$, we identify it with its corresponding element in $K_c^0(S^*_{\partial X}(X/Z) \times \mathbb{R}^2)$. Note then that $\sigma_1 = \beta \cdot f_*(\sigma(\not{\partial}_+^+))$, where $\beta \in K_c^0(\mathbb{R}^2)$ is the Bott element.

Finally, note that $h \circ f$ is a K-oriented embedding of $T^*(\partial X/Z)$ into a trivial Euclidean fibration which is homotopic to the embedding induced by the map $\partial X \longrightarrow \mathbb{R}^N \times Z$, and hence is suitable for computing top-ind$(\not{\partial}_+^+)$.

By the fact mentioned above, top-ind$(P) =$ top-ind(σ_1) is computed by the image of

$$\sigma_1 \cdot \mathfrak{T}(N_g(S^*_{\partial X}(X/Z) \times \mathbb{R}^2)) \in K_c^0(\mathbb{R}^{2N} \times Z),$$

and similarly top-ind$(\not{\partial}_+^+)$ is computed by the image of

$$\tilde{f}_*\left(\sigma(\not{\partial}_+^+) \cdot \mathfrak{T}(N_{h\circ f}(T^*(\partial X/Z)))\right) = f_*(\sigma(\not{\partial}_+^+)) \cdot \mathfrak{T}(N_h(S^*_{\partial X}(X/Z))) \in K_c^0(\mathbb{R}^{2N} \times Z).$$

Since $\sigma_1 = \beta \cdot f_*(\sigma(\not{\partial}_+^+))$, it suffices to show that

$$\mathfrak{T}(N_g(S^*_{\partial X}(X/Z) \times \mathbb{R}^2) \cdot \beta = \mathfrak{T}(N_h(S^*_{\partial X}(X/Z))),$$

but this is immediate by multiplicativity of the Thom class, since the normal bundles with respect to h and g are related by $N_h(S^*_{\partial X}(X/Z)) = \mathbb{R}^2 \times N_g(S^*_{\partial X}(X/Z) \times \mathbb{R}^2)$, and the Thom class of a trivial $\mathbb{R}^2$ bundle is just β.

Thus, we obtain that ind$(P) =$ ind$(\not{\partial}_+^+)$ since top-ind$(P) =$ top-ind$(\not{\partial}_+^+)$, and the rest follows by taking the Chern character of both sides. □

Finally, we consider the product Dirac case; that is, assume $V = E \otimes F \longrightarrow X$ where $\eth \in \mathrm{Diff}^1_{\mathrm{sc}}(X/Z; E)$ acts on E and the compatible potential $\Phi \in \Gamma(\mathrm{End}(F))$ acts on F, and $\eth \otimes 1 = \eth_F$ is obtained by equipping $E \otimes F$ with a tensor product connection. We form the Callias-Anghel type family

$$P = \eth \otimes 1 + i \otimes \Phi \in \mathrm{Diff}^1_{\mathrm{sc}}(X/Z; E \otimes F).$$

As above, the Clifford module E splits over the boundary into $E^0 \oplus E^1$, with

$$\eth = \begin{pmatrix} 0 & \eth^- \\ \eth^+ & 0 \end{pmatrix},$$

and $F_{|\partial X} = F_+ \oplus F_-$ splits into positive and negative eigenbundles of $\Phi_{|\partial X}$.

Theorem 2.5. *Let $P = \eth \otimes 1 + i \otimes \Phi \in \mathrm{Diff}^1_{\mathrm{sc}}(X/Z; E \otimes F)$ as above. Then P extends to a Fredholm family with index*

$$\mathrm{ind}(P) = \mathrm{ind}\left(\eth^+_{F_+}\right)$$

where $\eth^+_{F_+}$ is the twisted Dirac operator obtained by twisting $\eth^+ \in \mathrm{Diff}^1_{\mathrm{sc}}(\partial X/Z; E^0, E^1)$ by F_+, the positive eigenbundle of $\Phi_{|\partial X}$.

Remark. In particular, when X is an odd-dimensional spin manifold and $\eth$ is the (self-adjoint) spin Dirac operator (i.e. constructed using the fundamental representation of $\mathbb{C}\ell(X)$ on spinors), then $\eth^+ \in \mathrm{Diff}^1(\partial X; S^0, S^1)$ is the graded spin Dirac operator over the boundary, and we obtain

$$\mathrm{ind}(\eth \otimes 1 + i \otimes \Phi) = \int_{T^*\partial X} \mathrm{ch}(F_+) \cdot \hat{\mathrm{A}}(\partial X),$$

since $\mathrm{ch}(\sigma(\eth^+)) \cdot \mathrm{Td}(\partial X) = \hat{\mathrm{A}}(\partial X)$ (compare to the formula obtained by Råde in [25]).

Proof. The result follows from the previous one, after noting that V^0_+ and V^1_+ are given by $E^0 \otimes F_+$ and $E^1 \otimes F_+$, respectively, and that the clutching function

$$ic\ell_0 : S^*(\partial X/Z) \longrightarrow \mathrm{Hom}(\pi^* E^0 \otimes F_+, \pi^* E^1 \otimes F_+)$$

is given by $\sigma(\eth^+) \otimes 1 = \sigma(\eth^+_{F_+})$. □

There are a few final remarks to be made:

- First, regarding even/odd dimensionality: in the case of (families of) Dirac operators, P will only have a nonzero index when the dimension $\dim(X/Z)$ of the fiber is *odd.* Since the index of P reduces to the index of a family of *differential* operators on $\partial X \longrightarrow Z$, it must vanish when $\dim(\partial X/Z) = \dim(X/Z) - 1$ is odd for the usual reason. Because of this, previous literature on the subject was limited to the index problem on odd-dimensional manifolds, though we emphasize that, if D is allowed to be pseudodifferential, P may have nontrivial index even when $\dim(X/Z)$ is even.

- Our analysis of clutching data in the Dirac case, which related $[V^+_+] \in K^0(S^*_{\partial X}(X/Z))$ to the symbol $[\pi^* V^0_+, \pi^* V^1_+, \sigma(\eth^+_+)] \in K^0_c(T^*(\partial X/Z)$ of an operator $\eth^+_+$ on ∂X is equally valid when D is pseudodifferential. Indeed, V^+_+ can always be written as the clutching of bundles $\pi^* V^0_+$ and $\pi^* V^1_+$ coming from the base, with respect to *some* clutching function f, and then $\mathrm{ind}(P) = \mathrm{ind}(\delta)$, where $\delta \in \Psi^m(\partial X/Z; V^0, V^1)$ is any elliptic pseudodifferential operator whose symbol $\sigma(\delta) = f$. However, such a choice of δ is far from canonical without the additional structure of the Clifford bundles.

2.6 Relation to previous results

In [3], N. Anghel generalized Callias' original index theorem to the following situation[10] (adapted to our notation): Let X be a general odd-dimensional, non-compact, complete Riemannian manifold (with no particular structure assumed at infinity), with a Clifford module $V \longrightarrow X$. Let $\not{D} : \Gamma(V) \longrightarrow \Gamma(V)$ be a self-adjoint Dirac operator, and $\Phi \in \Gamma(\mathrm{End}(V))$ a potential which is assumed to be uniformly invertible away from a compact set $K \Subset X$ and such that $[\not{D}, \Phi]$ is a uniformly bounded, 0th order operator (in particular, Φ commutes with Clifford multiplication). First he proves that, for sufficiently large $\lambda > 0$,

$$P_\lambda = \not{D} + i\lambda\Phi \quad \text{is Fredholm},$$

essentially by showing that P_λ and P_λ^* satisfy what the author likes to call "*injectivity near infinity*" conditions:

$$\|P_\lambda u\|_{L^2} \geq c\,\|u\|_{L^2} \qquad \text{for all } u \in C_c^\infty(X \setminus K; V)$$

and similarly for P_λ^*. In [2] Anghel shows how such conditions are equivalent to Fredholmness for self-adjoint Dirac operators, but it is easy to see that his proof generalizes to show that any differential operator P, which is injective near infinity along with its adjoint, extends to be Fredholm.

In any case, as in Section 2.5, V splits over $X \setminus K$ into positive and negative eigenbundles of Φ: $V_{|X\setminus K} = V_+ \oplus V_-$, and choosing a compact set $L \Subset X$ such that $X \subset \mathring{L}$ with $\partial L = Y$ a separating hypersurface (compare our earlier situation in which $Y = \partial X$), we have further compatible splitting $(V_\pm)_{|Y} = V_\pm^0 \oplus V_\pm^1$ according to the decomposition $\mathbb{C}\ell(Y) \cong \mathbb{C}\ell(X)^0_{|Y}$, with $-ic\ell(\nu) \equiv (-1)^i : V_\pm^i \longrightarrow V_\pm^i$ where ν is a unit normal section. Choosing appropriate connections, we can again construct an induced Dirac operator on Y,

$$\partial_Y = \begin{pmatrix} 0 & \partial_+^- & 0 & 0 \\ \partial_+^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_-^- \\ 0 & 0 & \partial_-^+ & 0 \end{pmatrix},$$

as before, and Anghel proves that

$$\mathrm{ind}(\not{D} + i\lambda\Phi) = \mathrm{ind}(\partial_+^+).$$

His proof consists of index preserving deformations, along with the relative index theorem of Gromov and Lawson in [15] (discussed further in [2]) to reduce to a product type Dirac operator $\widetilde{\not{D}}$ on a Riemannian product $Y \times \mathbb{R}$,

$$\widetilde{\not{D}} = ic\ell(\nu)\frac{\partial}{\partial t} + \partial_+ + i\lambda\chi(t) : \Gamma(V_+) \longrightarrow \Gamma(V_+)$$

[10]See also [25] for an independently obtained proof which addresses the Dirac product case as in section 2.5.

with equivalent index. Here $\chi : \mathbb{R} \longrightarrow [-1, 1]$ is a smooth function such that $\chi \equiv -1$ near $-\infty$ and $\chi \equiv 1$ near $+\infty$. Direct computation then shows that $\operatorname{ind}(\widetilde{\not{D}}) = \operatorname{ind}(\eth_+^+)$.

We point out that the steps in his proof could just as easily reduce to a *scattering* product (i.e. $Y \times \mathbb{R}$, but with locally Euclidean ends instead of cylindrical ones), with a scattering type Dirac operator

$$\widetilde{\not{D}}' = ic\ell(\nu)\frac{\partial}{\partial t} + \frac{1}{t}\eth_+ + i\lambda\chi(t) : \Gamma(V_+) \longrightarrow \Gamma(V_+)$$

whose index is equivalent to $\operatorname{ind}(\eth_+^+)$ by our own Theorem 2.4. This is really overkill in this case, since the index of $\widetilde{\not{D}}$ is determined simply enough; however, it raises the point that scattering-type infinite ends ($\partial L \times [0, \infty)$, where $L \Subset X$ as above) may be utilized for the purpose of computing the index of Dirac operators satisfying Anghel's Fredholm conditions (injectivity near infinity for P and its adjoint). The author anticipates that cutting and gluing constructions, similar to those used in [15] to prove the relative index formula, may be able exhibit such equivalences for arbitrary Fredholm differential operators satisfying injectivity near infinity conditions.

Chapter 3

A perturbation framework for constant rank potentials

We now extend some of the results of the previous chapter to the case of a Dirac type Callias operator $P = \not{D} + i\Phi$ where we relax the invertibility condition on $\Phi_{|\partial X}$, allowing it to be of constant rank instead.

Roughly, the situation is as follows. Suppose we had a global splitting of $V = V'' \oplus V'$ such that $\Phi = 0 \oplus \Phi'$ where $\Phi'_{|\partial X}$ was invertible and satisfied the compatibility conditions with respect to $\not{D} = \not{D}'' \oplus \not{D}'$. Then $\not{D}' + i\Phi'$ would be a Callias-Anghel operator with index determined as in the previous chapter, and $\not{D}'' = \not{D}'' + i0$, though not fully elliptic, would at least be self-adjoint, and ought not to contribute to the index of P in any way. Considering $\not{D}''$ as a weighted b operator, we might well be able to find a self-adjoint Fredholm extension as an operator $x^\alpha H^k_{\mathrm{b}} \longrightarrow x^{\alpha+1} H^{k-1}_{\mathrm{b}}$ and make precise the statement that $\mathrm{ind}(P) = \mathrm{ind}(\not{D}' + i\Phi')$, as operators on suitable Sobolev spaces.

In reality, the situation is a little more complicated, but we shall proceed in the same way. We first discuss the obstruction to finding an *index zero weight* for $\not{D}$, considered as a b operator. In the absence of this obstruction, we can form a suitable parametrix for $\not{D} + i\Phi$ in the case that $\Phi_{|\partial X}$ has rank 0 (i.e. vanishes). We then show how this parametrix can be constructed as part of a family of parametrices for the family of operators $\epsilon \longmapsto \not{D} + i\,(\Phi - \epsilon \mathrm{Id})$, in such a way that the index (as a scattering operator for $\epsilon > 0$ and as a b operator for $\epsilon = 0$) is constant in ϵ. This construction uses the b-sc *transition calculus* of pseudodifferential operators, outlined in Appendix B. We then tackle the general case, showing that

$$\mathrm{ind}\,(\not{D} + i(\Phi - \epsilon \mathrm{Id})) \text{ is independent of } \epsilon,$$

where at $\epsilon = 0$ the operator is considered to act on certain weighted, hybrid Sobolev spaces $x^\alpha \mathcal{H}^{\beta,k,l}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$ which measure regularity and decay differently on sections of V over ∂X according to whether Φ acts by 0 or not.

3.1 Index zero weight

Often when presented with a scattering differential operator which fails to be fully elliptic, one may be able to view it as a (weighted) b operator instead, and find Fredholm extensions on b Sobolev spaces. There are choices involved in this process.

First a word about densities. We have been considering scattering differential operators acting on Sobolev spaces defined by the metric g. We can just as well consider $P \in \mathrm{Diff}^k_{\mathrm{sc}}(X;V)$ as acting on *scattering half densities*[1] $u \in C^\infty(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}(X))$, by composing P with the identity operator on scattering half densities, trivialized by the section $|d\mathrm{Vol}_g|^{1/2}$. Equivalently, we can think of this as equipping P with a connection on the bundle $\Omega^{1/2}_{\mathrm{sc}}(X) \longrightarrow X$ which annihilates the canonical section $|d\mathrm{Vol}_g|^{1/2}$. Either way, we will use the identification

$$g : \mathrm{Diff}^k_{\mathrm{sc}}(X;V) \xrightarrow{\cong} \mathrm{Diff}^k_{\mathrm{sc}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \tag{3.1}$$

Next, for the constructions that follow, it is more convenient to use *b half densities*, and we will use the following conjugation result to transform operators acting on scattering half densities into operators acting on b half densities and vice versa.

Lemma 3.1. *$L^2(X;\Omega^{1/2}_{\mathrm{sc}}) = x^{n/2}L^2(X;\Omega^{1/2}_{\mathrm{b}})$. Thus,*

$$P \longmapsto \widetilde{P} = x^{-n/2} P x^{n/2} \tag{3.2}$$

gives an isomorphism between bounded operators $P \in \mathcal{B}(H_1(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}), H_2(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}))$ and bounded operators $\widetilde{P} \in \mathcal{B}(H_1(X; V \otimes \Omega^{1/2}_{\mathrm{b}}), H_2(X; V \otimes \Omega^{1/2}_{\mathrm{b}}))$, where the H_i are any L^2 based Sobolev spaces.

Proof. The difference between $L^2(X;\Omega^{1/2}_{\mathrm{sc}})$ and $L^2(X;\Omega^{1/2}_{\mathrm{b}})$ consists only of integrability at ∂X, and since $\Omega_{\mathrm{sc}}(X)$ and $\Omega_{\mathrm{b}}(X)$ are trivialized near ∂X respectively by

$$\left| \frac{dx}{x^2} \frac{dy_1 \cdots dy_{n-1}}{x^{n-1}} \right| \in C^\infty(X; \Omega_{\mathrm{sc}}(X)), \quad \text{and} \quad \left| \frac{dx}{x} dy_1 \cdots dy_{n-1} \right| \in C^\infty(X; \Omega_{\mathrm{b}}(X)),$$

the claim follows. □

In order to analyze elements in $\mathrm{Diff}^*_{\mathrm{sc}}(X)$ as weighted b differential operators, we want to use the fact that that $\mathcal{V}_{\mathrm{sc}}(X) = x\,\mathcal{V}_{\mathrm{b}}(X)$. However, it is also true that

$$\mathcal{V}_{\mathrm{sc}}(X) = x^\alpha\, \mathcal{V}_{\mathrm{b}}(X) \cdot x^{1-\alpha}, \quad 0 \le \alpha \le 1$$

and the choice of α will ultimately shift the location of the indicial roots of the resulting b operator. In terms of the b-calculus, this corresponds to the following. If we consider $V \in \mathcal{V}_{\mathrm{sc}}(X)$ as a b operator, its kernel $\kappa_V \in I^1(X^2_{\mathrm{b}}, \Delta)$ vanishes to first order at the b front face, ff, and so the normal operator $N_{\mathrm{b}}(V)$ is not well-defined as a function. Rather, $N_{\mathrm{b}}(V)$ is properly defined to be the *leading order term* in the asymptotic expansion of κ_V at ff, it is well-defined as a section of a trivial line bundle $N\mathrm{ff}^{(1,0)}$, which is trivialized by a choice of boundary defining function ρ_{ff}. In particular, near $\Delta \cap \mathrm{ff}$, a canonical family of boundary defining functions consists of

$$\rho_{\mathrm{ff}} = x^\alpha x'^{1-\alpha}, \qquad \text{for } 0 \le \alpha \le 1,$$

where x and x' are the lifts from the left and right of boundary defining functions on X. Different choices of α provide different trivializations, and hence different expressions for $N_{\mathrm{b}}(V)$.

A judicious choice is $\alpha = 1/2$, which has the virtue of preserving formal self-adjointness:

[1]Defined in Appendix A.

Lemma 3.2. *If $P \in \mathrm{Diff}^k_{\mathrm{sc}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2})$ is formally self-adjoint on $L^2(X; V \otimes \Omega_{\mathrm{b}}^{1/2})$, then $x^{-k/2} P x^{-k/2} \in x^{-k}\mathrm{Diff}^k_{\mathrm{sc}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2})$ is also formally self-adjoint.*

Proof. This is immediate as multiplication by x^s is formally self-adjoint for any $s \in \mathbb{R}$, so

$$(x^s P x^s, v) = (u, x^s P^* x^s v) = (u, x^s P x^s v)$$

if P is self-adjoint. □

Of course $\mathrm{Diff}^k_{\mathrm{sc}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2})$ is filtered by order $k \in \mathbb{N}$, and different order elements are to be thought of as weighted b operators with weight equal to their order; it is not true in general that we can factor out the highest order weight from the whole operator. For instance, $\mathrm{Diff}^0_{\mathrm{sc}}(X) = C^\infty(X)$ and so additional decay must be present in the lower order terms for this process to work.

However, assuming the Clifford connection $\nabla \in {}^{\mathrm{sc}}\mathcal{A}(X; V)$ is the lift of a b connection, all goes well.

Theorem 3.1. *Let $\widetilde{\not{D}} \in \mathrm{Diff}^1_{\mathrm{sc}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2})$ be the self-adjoint Dirac operator on b half-densities, constructed from a lifted b Clifford connection $\nabla \in {}^{\mathrm{b}}\mathcal{A}(X; V)$ on a $\mathbb{C}\ell(X; g)$ bundle $V \longrightarrow X$, where $\widetilde{\not{D}} = x^{-n/2} \not{D} x^{n/2}$. Then*

$$\not{D}_{\mathrm{b}} = x^{-1/2} \widetilde{\not{D}} x^{-1/2} \in \mathrm{Diff}^1_{\mathrm{b}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2}) \tag{3.3}$$

is a self-adjoint b operator acting on b half-densities. Furthermore, $\not{D}_{\mathrm{b}}$ is a generalized Dirac operator, in that $\sigma(\not{D}_{\mathrm{b}})(\xi) = i c\ell(\xi)$, $\xi \in {}^{\mathrm{b}}T^*X$, where we identify ${}^{\mathrm{b}}T^*X$ with ${}^{\mathrm{sc}}T^*X$ by the map

$$x : {}^{\mathrm{sc}}T^*X \xrightarrow{\cong} {}^{\mathrm{b}}T^*X.$$

Remark. Note that $\not{D}_{\mathrm{b}}$ is not quite a b Dirac operator as we have defined it; while our connection $\nabla \in {}^{\mathrm{b}}\mathcal{A}(X; V)$ is a b connection, it is *not* a Clifford connection for the action of $\mathbb{C}\ell(X; g_b)$, since the Levi-Civita connection for g, $\nabla^{\mathrm{LC}(g)}$ differs from the lift of $\nabla^{\mathrm{LC}(g_{\mathrm{b}})}$ as seen in Proposition 1.1 in Chapter 1.

If $\nabla' \in {}^{\mathrm{b}}\mathcal{A}(X; V)$ is a $\mathbb{C}\ell(X; g_b)$ compatible Clifford connection, then in fact $\nabla - \nabla' = B \in C^\infty(X; {}^{\mathrm{b}}\Lambda^1 \otimes \mathrm{End}(V))$, where

$$B(c\ell(\alpha)\, u) = c\ell\,(A\alpha)\; u + c\ell(\alpha)\,(B\, u)\,,$$

where $A \in C^\infty(X; {}^{\mathrm{b}}T^*X \otimes \mathrm{End}({}^{\mathrm{b}}\Lambda^{\cdot}))$ is the connection one-form defined in (1.17):

$$A = N \otimes \frac{dx}{x}, \qquad \mathrm{End}({}^{\mathrm{b}}\Lambda^{\cdot}) \ni N = \begin{cases} (n-1-k) & \text{on } {}^{\mathrm{b}}\Lambda^k \partial X \\ k & \text{on } \frac{dx}{x} \wedge {}^{\mathrm{b}}\Lambda^k \partial X \end{cases}$$

This makes the indicial root situation somewhat complicated.

Proof. Being in $\mathrm{Diff}^1_{\mathrm{sc}}(X; V \otimes \Omega_{\mathrm{sc}}^{1/2})$, $\not{D}$ consists of elements in $\mathcal{V}_{\mathrm{sc}}(X) \otimes \Gamma(X; \mathrm{End}(V \otimes \Omega_{\mathrm{sc}}^{1/2}))$ and elements in $C^\infty(X; \mathrm{End}(V \otimes \Omega_{\mathrm{sc}}^{1/2}))$, and we need to show that the latter have $O(x)$ decay at ∂X. From the definition,

$$\not{D} = \sum c\ell(\hat{e}_i) \nabla_{e_i}$$

for an orthonormal ${}^{\mathrm{sc}}TX$ frame $\{e_i\}$. From the fact that $\nabla \in {}^{\mathrm{b}}\mathcal{A}(X; V \otimes \Omega_{\mathrm{sc}}^{1/2})$, we know that in any local trivialization near ∂X,

$$\nabla = d + A, \quad \text{for some } A \in x\, C^\infty(X; {}^{\mathrm{sc}}\Lambda^1 \otimes \mathrm{End}(V \otimes \Omega_{\mathrm{sc}}^{1/2})).$$

Locally then, taking $\{x^2\partial_x, x\partial_{y_i}\}$ as our frame (and trivializing $\Omega_{\mathrm{sc}}^{1/2}$ by our choice of coordinates), we obtain

$$\eth = \mathrm{c}\ell\left(\frac{dx}{x^2}\right)(x^2\partial_x + A_{dx/x^2}) + \sum_i \mathrm{c}\ell\left(\frac{dy_i}{x}\right)(x\partial_{y_i} + A_{dy_i/x}) \tag{3.4}$$

with $A_* \in x\, C^\infty(X; \mathrm{End}(V))$.

Conjugating $\eth$ by $x^{n/2}$ to obtain $\widetilde{\eth}$ merely alters the terms $A_.$ by other elements in $x\, C^\infty(X; \mathrm{End}(V))$, so $\widetilde{\eth}$ has the same form as in (3.4). Composing on the left and right with $x^{-1/2}$ on the other hand maps $x\, C^\infty(X; \mathrm{End}(V))$ to $C^\infty(X; \mathrm{End}(V))$, and along with the fact that

$$x^{-1/2}\left(x^2\partial_x\right) \circ x^{-1/2} = x\partial_x - 1/2, \qquad x^{-1/2}\left(x\partial_{y_i}\right) \circ x^{-1/2} = \partial_{y_i}$$

we see that $\eth_{\mathrm{b}}$ has the form (3.4) with $A_. \in C^\infty(X; \mathrm{End}(V))$. It is therefore a b differential operator, and its symbol agrees with $i c\ell : {}^{\mathrm{b}}T^*X \longrightarrow \mathrm{End}(V)$ once we use the isomorphism

$$x : \mathbb{C}\ell(X; g) \xrightarrow{\cong} \mathbb{C}\ell(X; g_b)$$

as in (1.22). □

From the theory of b pseudodifferential operators as outlined in section 1.5 of Chapter 1, we have the general result

Theorem 3.2. *If $0 \notin \mathrm{spec}_{\mathrm{b}}(\eth_{\mathrm{b}})$, then $\eth_{\mathrm{b}}$ has an index 0 Fredholm extension as an operator*

$$\eth_{\mathrm{b}} : x^\alpha H_{\mathrm{b}}^k(X; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow x^\alpha H_{\mathrm{b}}^{k-1}(X; V \otimes \Omega_{\mathrm{b}}^{1/2}), \quad \forall\, \alpha \in (\alpha_-, \alpha_+)$$

where $\alpha_+ = \inf\{\mathrm{Im}(z) > 0\,;\ z \in \mathrm{spec}_{\mathrm{b}}(\eth_{\mathrm{b}})\}$ and $\alpha_- = \sup\{\mathrm{Im}(z) < 0\,;\ z \in \mathrm{spec}_{\mathrm{b}}(\eth_{\mathrm{b}})\}$.

Hence, in this case, $\eth$ has a Fredholm extension as an operator

$$\eth : x^{\alpha-1/2} H_{\mathrm{b}}^k(X; V \otimes \Omega_{\mathrm{sc}}^{1/2}) \longrightarrow x^{\alpha+1/2} H_{\mathrm{b}}^{k-1}(X; V \otimes \Omega_{\mathrm{sc}}^{1/2}).$$

Definition. *When the conditions of the above theorem are met, i.e. that $\mathrm{spec}_{\mathrm{b}}(\eth_{\mathrm{b}}) \cap \{\mathrm{Im}(z) = 0\} = \emptyset$, we will say that $\eth_{\mathrm{b}}$ or $\eth$ "has an index zero weight."*

Proof. The first statement is a general result from the b calculus. Provided $\mathrm{spec}_{\mathrm{b}}(\eth_{\mathrm{b}}) \cap \{\mathrm{Im}(z) = 0\} = \emptyset$, we have a Fredholm extension; but as $\eth_{\mathrm{b}}$ is self-adjoint, $\mathrm{spec}_{\mathrm{b}}(\eth_{\mathrm{b}})$ lies entirely on the imaginary axis, hence the condition reduces to $0 \notin \mathrm{spec}_{\mathrm{b}}\eth_{\mathrm{b}}$. The fact that it has index 0 follows from the fact that the extension is formally (hence essentially) self-adjoint when $\alpha = 0$, and the index remains constant as α varies within (α_-, α_+).

As for the Fredholm properties of $\eth$, this follows from the definition of $\eth_{\mathrm{b}}$ and the commutative diagram

$$\begin{array}{ccc} x^{\alpha-1/2}H_{\mathrm{b}}^{k}(X;V\otimes\Omega_{\mathrm{sc}}^{1/2}) & \xrightarrow{\;\not{D}\;} & x^{\alpha+1/2}H_{\mathrm{b}}^{k-1}(X;V\otimes\Omega_{\mathrm{sc}}^{1/2}) \\ \cong\Big\downarrow x^{(1-n)/2} & & x^{(1+n)/2}\Big\uparrow\cong \\ x^{\alpha}H_{\mathrm{b}}^{k}(X;V\otimes\Omega_{\mathrm{b}}^{1/2}) & \xrightarrow{\;\not{D}_{\mathrm{b}}\;} & x^{\alpha}H_{\mathrm{b}}^{k-1}(X;V\otimes\Omega_{\mathrm{b}}^{1/2}) \end{array}$$

□

3.2 Rank zero case

We now consider the index problem for $\not{D}+i\Phi$ where $\Phi_{|\partial X}=0$. We will assume that $\Phi_{|\partial X}=O(x^2)$ for simplicity (in fact, by the arguments in section 3.3, we can always arrange for Φ to vanish identically in a neighborhood of ∂X).

Assuming $\not{D}_{\mathrm{b}}$ has an index zero weight, then $\not{D}$ has an index 0 Fredholm extension as per Theorem 3.2, and therefore $\not{D}+i\Phi$ has an index 0 extension from the following.

Lemma 3.3. *If $\Phi=O(x^2)$ near ∂X, then $\Phi: x^{\beta}H_{\mathrm{b}}^{k}(X;V\otimes\Omega_{\mathrm{sc}}^{1/2})\longrightarrow x^{\beta+1}H_{\mathrm{b}}^{k-1}(X;V\otimes\Omega_{\mathrm{sc}}^{1/2})$ is a compact operator, for any β and k.*

Proof. This is equivalent to the compactness of

$$x^{-1}\,\Phi: H_{\mathrm{b}}^{1}(X;V\otimes\Omega_{\mathrm{sc}}^{1/2})\longrightarrow H_{\mathrm{b}}^{0}(X;V\otimes\Omega_{\mathrm{sc}}^{1/2}).$$

However, if $\Phi=O(x^2)$, $x^{-1}\Phi\in xC^{\infty}(X;\mathrm{End}(V\otimes\Omega_{\mathrm{sc}}^{1/2}))\subset\Psi_{\mathrm{b}}^{0,(1,\infty,\infty)}(X;V\otimes\Omega_{\mathrm{sc}}^{1/2})$, which maps $H_{\mathrm{b}}^{1}\longrightarrow H_{\mathrm{b}}^{0}$ compactly. □

However, to use this result in the general rank case, we will need something stronger, namely that we can "connect" this weighted b operator to a fully elliptic scattering operator in some kind of continuous fashion. More specifically, we will construct a family of parametrices Q_ϵ for

$$P_\epsilon=\not{D}+i(\Phi-\epsilon\mathrm{Id}),\qquad \epsilon\in[0,\epsilon_0) \tag{3.5}$$

where Q_ϵ for $\epsilon>0$ is a scattering parametrix, and Q_0 is a weighted b parametrix, and such that

$$E_\epsilon^R=\mathrm{Id}-P_\epsilon\,Q_\epsilon,\quad E_\epsilon^L=\mathrm{Id}-Q_\epsilon\,P_\epsilon$$

are trace class operators on $H_{\mathrm{sc}}^{k}(X;V\otimes\Omega_{\mathrm{sc}}^{1/2})$ for $\epsilon>0$ and on $H_{\mathrm{b}}^{k}(X;V\otimes\Omega_{\mathrm{sc}}^{1/2})$ for $\epsilon=0$, with uniform trace in ϵ.

This construction uses the b-sc transition calculus of pseudodifferential operators, which is described in Appendix B. We briefly recall its main features. Pseudodifferential operators $\Psi_t^{m,\mathcal{E}}(X_t;V\otimes\Omega_{\mathrm{b}}^{1/2})$ are described by their Schwartz kernels on X_t^2, a blown up version of $X^2\times I$, where $I=[0,\epsilon_0)_\epsilon$, and $\mathcal{E}=(E_{\mathrm{zf}},E_{\mathrm{bf}_0},E_{\mathrm{sc}},E_{\mathrm{lb}_0},E_{\mathrm{rb}_0})$ is a tuple of index sets describing the polyhomogeneous expansions of the kernel at various faces.

Operators in the calculus naturally operate on vector bundle sections over the *single space* $X_t=[X\times I;\partial X\times\{0\}]$, though for our purposes we can consider "slices" at ϵ constant (in fact the $\epsilon=0$ "slice" is not actually the restriction to $\epsilon=0$, though it can be morally thought of as such) which represent operators in $\Psi_{\mathrm{sc}}^{*}(X;V\otimes\Omega_{\mathrm{b}}^{1/2})$ and $\Psi_{\mathrm{b}}^{*}(X;V\otimes\Omega_{\mathrm{b}}^{1/2})$

for $\epsilon > 0$ and $\epsilon = 0$, respectively. More specifically, these are pseudodifferential algebra homomorphisms

$$S_\delta : \Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_b^{1/2}) \longrightarrow \Psi_{sc}^{m,E_{sc}}(X; V \otimes \Omega_b^{1/2}), \quad \delta > 0$$
$$S_0 : \Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_b^{1/2}) \longrightarrow \Psi_b^{m,(E_{bf_0}, E_{lb}, E_{rb})}(X; V \otimes \Omega_b^{1/2})$$

and such that $S_\delta(P_\epsilon) = \not{D} + i(\Phi - \delta \mathrm{Id})$ and $S_0(P_\epsilon) = \not{D} + i\Phi$. It is in this sense that we will have a family of parametrices for P_ϵ.

As should be expected for any pseudodifferential calculus on manifolds with boundary, elliptic operators are not necessarily Fredholm; we need invert more than just the principal symbol in order to get compact errors. To this end, the transition calculus defines two normal operator/symbol homomorphisms

$$\sigma_{sc}(P) \in C^\infty({}^{sc}T^*_{\partial X}X \times I; \mathrm{End}(V)), \text{ and}$$
$$N_t(P) \in \Psi_{b,sc}^{m,(E_{zf}, E_{lb_0}, E_{rb_0}), E_{sc}}(\partial X \times {}_b[0,1]_{sc}; V \otimes \Omega_b^{1/2}).$$

The first is a simply a families version (parametrized by $\epsilon \in I$) of the boundary symbol in the scattering calculus; it is the fiberwise Fourier transform of the restriction of Schwartz kernel to the "scattering" face of the double space: $\mathrm{sc} \subset X_t^2$.

The second, which we call the *transition normal operator*, is a pseudodifferential operator on $\partial X \times [0,1]$ of b type near $\{0\}$ and of scattering type near $\{1\}$. It is the restriction of the Schwartz kernel to the face $\mathrm{bf}_0 \subset X_t^2$ which is diffeomorphic to the appropriate blow up of $\partial X^2 \times [0,1]^2$ (namely the b blow up at $\partial X^2 \times \{0\}^2$ and the scattering blow up at $\partial X^2 \times \{1\}^2$). It is to be thought of as the object which captures the essence of the transition between scattering type behavior for $\epsilon > 0$ and b type behavior at $\epsilon = 0$.

In order to construct a parametrix with compact errors, one must construct inverses to both of these objects.

The scattering symbol and transition normal operator are related to each other and to the δ-slices as follows:

$$\sigma_{sc}(P)_{|\epsilon=\delta} = \sigma_{sc}(S_\delta(P))$$
$$\sigma_{sc}(P)_{|\epsilon=0} = \sigma_{sc}(N_t(P))$$
$$N_b(N_t(P)) = N_b(S_0(P))$$

so the scattering symbol of P for $\epsilon > 0$ is just the scattering symbol of the slice at ϵ, the scattering symbol of P at $\epsilon = 0$ agrees with the scattering symbol of $N_t(P)$ at the scattering end of $\partial X \times [0,1]$, and the b normal operators of $N_t(P)$ and the slice at 0 are the same.

In the case of *differential* operators, we can obtain expressions for $\sigma_{sc}(P)$ and $N_t(P)$ by restricting P, in an appropriate sense, to certain boundary faces of the single space X_t, just as in the b and scattering calculi, where the b and scattering normal operators of differential elements are given by restriction to ∂X. In the transition calculus, we single out the following boundary hypersurfaces of the single space $X_t = [X \times I; \partial X \times \{0\}]$:

- sf, defined to be the lift of the boundary face $\partial X \times I \subset X \times I$, and
- tf, defined to be the lift of the boundary face $\partial X \times \{0\} \subset X \times I$.

See Figure B-1 in Appendix B. With these two faces defined, we have

$$P \in \mathrm{Diff}_t^k(X; V \otimes \Omega_{\mathrm{b}}^{1/2}) \implies \sigma_{\mathrm{sc}}(P) = \widehat{P_{|\mathrm{sf}}}, \text{ and } N_{\mathrm{t}}(P) = P_{|\mathrm{tf}}, \tag{3.6}$$

appropriately interpreted, and provided P does not vanish at sf and tf. We will discuss what to do when it does vanish below. For later use, we highlight the following local coordinate structure on X_t, near tf.

Lemma 3.4. *Let $(y_1, \ldots, y_{n-1})$ denote local coordinates on ∂X. Then* tf $\subset X_t$ *is covered by coordinate patches of the form*

$$(t, x, y_i) \equiv (\epsilon/x, x, y_i)$$

near the b end $\{t = 0\}$, and

$$(\tau, \epsilon, y_i) \equiv (x/\epsilon, \epsilon, y_i)$$

near the scattering end $\{\tau = 1/t = 0\}$.

tf *has local boundary defining functions x in the first case, and ϵ in the second.*

Proof. This is just the projective coordinate representation of blow up, discussed in Appendix A. As we're blowing up the set $\{x = \epsilon = 0\} \subset X \times I$, the blow up is covered by the (relatively) open sets $\{0 \leq t = \epsilon/x < \infty\}$ and $\{0 \leq \tau = x/\epsilon < \infty\}$, with coordinates and local bdfs as stated. □

Before proceeding, we note that $\Psi_t^*(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$ is constructed most naturally in terms of operators acting on b half densities $\Omega_{\mathrm{b}}^{1/2}(X_t)$. Thus in order to study P_ϵ, we will need to conjugate by $x^{n/2}$ as per Lemma 3.1, and consider $\widetilde{P}_\epsilon = x^{-n/2} P_\epsilon x^{n/2}$. For the remainder of the chapter, tildes will denote conjugation by $x^{n/2}$.

We remark that since $\widetilde{P}_\epsilon$ is a scattering differential operator, its transition calculus kernel $\kappa_{\widetilde{P}_\epsilon} \in \mathcal{A}_{\mathrm{phg}}^{(0,1_{\mathrm{bf}_0},0,\ldots)} I^1(X_t^2; \Delta)$ actually vanishes to first order at bf_0, and hence its transition normal operator $N_{\mathrm{t}}(\widetilde{P}_\epsilon)$ is not the restriction, but the *leading order term* in the asymptotic expansion at bf_0, which is valued in the trivial line bundle $N\mathrm{bf}_0{}^{(1,0)}$. This is the same phenomenon that occurred when we considered $\widetilde{P}_0$ in the b-calculus. Thus in order to compute an expression for $N_{\mathrm{t}}(\widetilde{P}_\epsilon)$, it is necessary to choose a trivialization, for instance by choosing a function which is boundary defining for bf_0 near Δ. Once again, near zf, $(x\,x')^{1/2}$ is a good choice as it preserves formal self-adjointness, where x and x' are lifts of bdfs for X from the left and right, respectively. Near sc we can take $\rho_{\mathrm{bf}_0} = \epsilon$, since this commutes with the derivatives we are interested in.

Lemma 3.5. *There exits a trivialization of $N\mathrm{bf}_0{}^{(1,0)}$ given by $(x\,x')^{1/2}$ near* zf *and ϵ near* sc *such that*

$$N_{\mathrm{t}}(\widetilde{\not{D}}) \text{ is formally self-adjoint.}$$

Proof. As $\widetilde{\not{D}}$ is a differential operator, we will evaluate $N_{\mathrm{t}}(\widetilde{\not{D}})$ by restricting to tf $\subset X_t$ after factoring out the appropriate boundary defining function. We will use the coordinates defined in Lemma 3.4 and examine the lift of $x^2\partial_x$ and $x\partial_y$.

Factoring out ϵ on the double space corresponds to factoring out ϵ on the single space, and so we compute that, near the scattering end of tf $(0 \leq \tau < \infty)$,

$$\epsilon^{-1}\left(x^2\partial_x\right) \longmapsto \epsilon^{-1}\left(\epsilon\tau^2\partial_\tau\right) = \tau^2\partial_\tau$$
$$\epsilon^{-1}\left(x\partial_y\right) \longmapsto \epsilon^{-1}\left(\epsilon\tau\partial_y\right) = \tau\partial_y$$

Thus the expression for $N_{\mathrm{t}}(\widetilde{\not{D}})$ near the scattering end is obtained by formally replacing $x^2\partial_x$ and $x\partial_y$ by $\tau^2\partial_\tau$ and $\tau\partial_y$, respectively, in the expression for $\widetilde{\not{D}}$ at ∂X.

On the other hand, factoring out $(x\,x')^{1/2}$ on the double space corresponds to examining the operator $x^{-1/2}\widetilde{\not{D}}x^{-1/2}$ on the single space. However, $x^{-1/2}\widetilde{\not{D}}x^{-1/2} = \not{D}_{\mathrm{b}}$ is our formally self-adjoint b operator, as proved in Theorem 3.1, and a simple computation shows that, near the b end of tf $(0 \leq t < \infty)$,

$$\begin{aligned} x\partial_x &\longmapsto -t\partial_t + O(x) \\ \partial_y &\longmapsto \partial_y \end{aligned}$$

so the expression for $N_{\mathrm{t}}(\widetilde{\not{D}})$ near the b end is obtained by formally replacing $x\partial_x$ and ∂_y by $-t\partial_t$ and ∂_y in the expression for $\not{D}_{\mathrm{b}}$ at ∂X.

Since formal self-adjointness is a local property, we see that $N_{\mathrm{t}}(\widetilde{\not{D}})$ is a formally self-adjoint b-sc differential operator on $\partial X \times [0,1]$ in these two regions, which overlap on $0 < \tau = 1/t < \infty$ and hence cover tf.

On the double space then, patching ϵ and $(x\,x')^{1/2}$ via a partition of unity gives a bdf for of bf_0 near Δ, yielding a trivialization of $N\mathrm{bf}_0{}^{(1,0)}$ in which $N_{\mathrm{t}}(\widetilde{\not{D}})$ is formally self-adjoint. □

Such a trivialization will be called a *self-adjoint* trivialization for $N\mathrm{bf}_0{}^{(1,0)}$. Another important fact to note is

Lemma 3.6. *Under a self-adjoint trivialization of* $N\mathrm{bf}_0{}^{(1,0)}$*, the indicial families* $I(S_0(\widetilde{\not{D}}),\lambda)$ *and* $I(N_{\mathrm{t}}(\widetilde{\not{D}}),\lambda)$ *coincide, and we have*

$$I(S_0(\widetilde{\not{D}}),\lambda) = I(N_{\mathrm{t}}(\widetilde{\not{D}}),\lambda) = I(\not{D}_{\mathrm{b}},\lambda).$$

Proof. This is immediate from the point of view of the Schwartz kernels. Indeed, for any $P \in \Psi_t^{m,\mathcal{E}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2})$ both $I(S_0(P),\lambda)$ and $I(N_{\mathrm{t}}(P),\lambda)$ are, by definition, the Mellin transform of the leading order term in the asymptotic expansion of κ_P at the boundary face $\mathrm{zf} \cap \mathrm{bf}_0$, which must therefore agree as long as we use a consistent trivialization of $N\mathrm{bf}_0{}^{(s,t)}$.

This trivialization is the equivalent to the one for $N\mathrm{ff}$ on the b double space, we considered above to get $\not{D}_{\mathrm{b}}$ in the b-calculus, proving the second claim. □

In light of the above, we conclude

Proposition 3.1. *Provided* $\not{D}_{\mathrm{b}}$ *has an index zero weight,* $N_{\mathrm{t}}(\widetilde{P}_\epsilon)$ *is invertible.*

Proof. $N_{\mathrm{t}}(\widetilde{\not{D}})$ is self-adjoint, provided we use a self-adjoint trivialization of $N\mathrm{bf}_0{}^{(1,0)}$, and with this trivialization, the 0th order term $i\epsilon\mathrm{Id}$ restricts to

$$\epsilon^{-1}(i\epsilon\mathrm{Id}) \longmapsto i\mathrm{Id}$$

near the scattering end of tf and

$$x^{-1/2}(i\epsilon\mathrm{Id})x^{-1/2} = x^{-1}(i\epsilon\mathrm{Id}) \longmapsto x^{-1}(ixt\,\mathrm{Id}) = it\,\mathrm{Id}$$

near the b end, using coordinates from Lemma 3.4.

Therefore $N_t(\widetilde{P}_\epsilon)$ has the form

$$N_t(\widetilde{P}_\epsilon) = N_t(\widetilde{\not{D}}) + \begin{cases} -it\mathrm{Id} & \text{near the b end of } I \\ -i\mathrm{Id} & \text{near the scattering end} \end{cases}$$

$N_t(\widetilde{\not{D}})$ has an index zero weight since $\not{D}_b$ does, and $I(N_t(\widetilde{\not{D}}), 0) = I(\not{D}_b, 0)$, so the b normal operator of $N_t(\widetilde{\not{D}})$ is invertible.

The scattering symbol of $N_t(\widetilde{P}_\epsilon)$ is

$$\sigma_{sc}(N_t(\widetilde{P}_\epsilon)) = \sigma_{sc}(N_t(\widetilde{\not{D}})) - i\mathrm{Id}$$

which is invertible since the first term is Hermitian and the second is skew-Hermitian.

We obtain that

$$N_t(\widetilde{P}_\epsilon) : t^\alpha \tau^\beta H^k_{b,sc}(\partial X \times I; V \otimes \Omega_b^{1/2}) \longrightarrow t^\alpha \tau^\beta H^{k-1}_{b,sc}(\partial X \times I; V \otimes \Omega_b^{1/2})$$

is Fredholm with index 0 for all k, β, and for all $\alpha \in (\alpha_-, \alpha_+)$. It remains to show that $\mathrm{Null}(N_t(\widetilde{P}_\epsilon)) = \{0\}$.

Suppose $u \in \mathrm{Null}(N_t(\widetilde{P}_\epsilon))$. Then

$$0 = (N_t(\widetilde{P}_\epsilon)u, u) = (N_t(\widetilde{\not{D}})u, u) - i(Tu, u)$$

where T is multiplication by a positive function which looks like t near the b end and 1 near the scattering end. As the first term is real and the second imaginary, they must separately vanish; in particular,

$$0 = \int_0^1 \int_{\partial X} t\,|u|^2 \frac{dt}{t}\, d\mathrm{Vol}_h \implies u \equiv 0 \text{ near } t = 0$$

and similarly,

$$0 = \int_0^1 \int_{\partial X} |u|^2 \frac{d\tau}{\tau^2} \frac{d\mathrm{Vol}_h}{\tau^{n-1}} \implies u \equiv 0 \text{ near } \tau = 0$$

so $u \equiv 0$ everywhere. Therefore $\mathrm{Null}(N_t(\widetilde{P}_\epsilon)) = \{0\}$, and $N_t(\widetilde{P}_\epsilon)$ must be invertible. □

Proposition 3.2. *The scattering symbol $\sigma_{sc}(\widetilde{P}_\epsilon)$ of $\widetilde{P}_\epsilon$ is invertible.*

Proof. We obtain the expression for $\sigma_{sc}(\widetilde{P}_\epsilon)$ by restricting to $\mathrm{sf} \subset X_t$ and taking the fiberwise Fourier transform. On the interior ($\epsilon > 0$) of sf, we have

$$\sigma_{sc}(\widetilde{P}_\epsilon) = \sigma_{sc}(\widetilde{\not{D}}) + i\epsilon\mathrm{Id}$$

which is clearly invertible since $\epsilon > 0$.

Near $\mathrm{sf} \cap \mathrm{tf}$, we can use the coordinates $(\tau, \epsilon, y) = (x/\epsilon, \epsilon, y)$. As we saw above,

$$x^2\partial_x \longmapsto \epsilon\tau^2\partial_\tau \qquad x\partial_y \longmapsto \epsilon\tau\partial_y$$

so $\sigma_{sc}(\widetilde{P}_\epsilon)$ is vanishing to first order as $\epsilon \longrightarrow 0$ on sc, where its leading term approaches

$$\epsilon^{-1}\sigma_{sc}(\widetilde{P}_\epsilon)_{|\mathrm{tf}\cap\mathrm{sf}} = \sigma_{sc}(N_t(\widetilde{P}_\epsilon))$$

as it must. As $\sigma_{\mathrm{sc}}(N_{\mathrm{t}}(\widetilde{P}_\epsilon))$ is invertible, we conclude that $\sigma_{\mathrm{sc}}(\widetilde{P}_\epsilon)$ is invertible, with inverse growing asymptotically like $O(\rho_{\mathrm{tf}}^{-1})$ near tf. □

We can finally state the main result of this section.

Proposition 3.3. *If $\not{D}$ has an index zero weight, then $P_\epsilon \in \Psi_t^{1,(0,1_{\mathrm{bf}_0},0,\dots)}(X_t; V\otimes\Omega_{\mathrm{b}}^{1/2})$ has a parametrix $Q_\epsilon \in \Psi_t^{-1,(0,-1_{\mathrm{bf}_0},0,\dots)}(X_t, V\otimes\Omega_{\mathrm{b}}^{1/2})$ such that*

$$E_\epsilon^R = \mathrm{Id} - P_\epsilon Q_\epsilon, \quad E_\epsilon^L = \mathrm{Id} - Q_\epsilon P_\epsilon \in \Psi_t^{-\infty,(0,1_{\mathrm{bf}_0},(n+1)_{\mathrm{sc}},E^+(0)_{\mathrm{lb}_0},E^-(0)_{\mathrm{rb}_0})}(X_t, V\otimes\Omega_{\mathrm{b}}^{1/2}).$$

where $E^\pm(0)$ are index sets determined by $\{z\in \mathrm{spec}_{\mathrm{b}}(\not{D}_{\mathrm{b}})\ ;\ \mathrm{Im}(z)\gtrless 0\}$ with $\mathrm{Re}(E^\pm(0)) = \alpha_\pm$.

In particular, the scattering slices $S_\delta(E_\epsilon^{R/L}) \in \Psi_{\mathrm{sc}}^{-\infty,(n+1)_{\mathrm{sc}}}(X; V\otimes\Omega_{\mathrm{b}}^{1/2})$ for $\delta > 0$ and the b slice $S_0(E_\epsilon^{R/L}) \in \Psi_{\mathrm{b}}^{-\infty,(1,E^+(0)_{\mathrm{lb}_0},E^-(0)_{\mathrm{rb}_0})}(X; V\otimes\Omega_{\mathrm{b}}^{1/2})$ are trace class, and

$$\delta \longmapsto \mathrm{Tr}\left(S_\delta(E_\epsilon^{R,L})\right) \qquad \text{is continuous for } \delta\in[0,\epsilon_0).$$

Proof. From Propositions 3.1 and 3.2, we can construct $Q_\epsilon \in \Psi_t^{-1,(0,-1_{\mathrm{bf}_0},0,\dots)}(X_t; V\otimes\Omega_{\mathrm{b}}^{1/2})$ such that $\sigma_{\mathrm{sc}}(Q_e) = \sigma_{\mathrm{sc}}(P_\epsilon)^{-1}$ and $N_{\mathrm{t}}(Q_\epsilon) = N_{\mathrm{t}}(P_\epsilon)^{-1}$. Note that Q_ϵ must have an $O(\rho_{\mathrm{bf}_0}^{-1})$ expansion at bf_0 since P_ϵ has an $O(\rho_{\mathrm{bf}_0}^{1})$ expansion there.

For the interior singularity at Δ, the usual symbolic iteration argument shows that we can produce Q_ϵ so that $P_\epsilon Q_\epsilon - \mathrm{Id}$ has symbolic order ∞; this can be chosen to be compatible with $N_{\mathrm{t}}(P_\epsilon)^{-1}$ and $\sigma_{\mathrm{sc}}(P_\epsilon)^{-1}$ since these are true inverses.

That Q_e has index sets $E^\pm(0)$ at lb and rb, respectively, follows from the b calculus, which says that $N_{\mathrm{t}}(\widetilde{P}_\epsilon)^{-1}$ must have these index sets at the left and right boundaries of $(\partial X\times[0,1])^2_{\mathrm{b,sc}} \cong \mathrm{bf}_0$, which coincide with $\mathrm{lb}\cap\mathrm{bf}_0$ and $\mathrm{rb}\cap\mathrm{bf}_0$ respectively. That the index sets are those determined by $\not{D}_{\mathrm{b}}$,

$$E^\pm(0) = \left\{(\pm i\lambda, k)\ ;\ I(\not{D}_{\mathrm{b}},\lambda) \text{ has a pole of order } k, \text{ and } \mathrm{Im}(\lambda)\gtrless 0\right\},$$

comes from the fact that $\mathrm{spec}_{\mathrm{b}}(\not{D}_{\mathrm{b}})$ and $\mathrm{spec}_{\mathrm{b}}(N_{\mathrm{t}}(\widetilde{P}_\epsilon))$ coincide.

At first pass, this produces a parametrix which inverts P_ϵ up to error terms in

$$\Psi_t^{-\infty,(0,1_{\mathrm{bf}_0},1_{\mathrm{sc}},E^+(0)_{\mathrm{lb}},E^-(0)_{\mathrm{rb}_0})}(X_t, V\otimes\Omega_{\mathrm{b}}^{1/2}).$$

The scattering slices of these error terms are not quite trace class. However, we can alter Q_ϵ by an operator with rapidly vanishing kernel at all faces except sc by using a finite Neumann series argument, obtaining a parametrix with error terms having index set $(n+1)$ at sc.

Hence we obtain that $S_\delta(E_\epsilon^{R,L})$ are trace class for $\delta\geq 0$, from the standard characterization of trace class operators in the scattering and b calculi.

In (B.2) in Appendix B, we define a trace operation

$$\mathrm{Tr}: \Psi_t^{-\infty,(1,E^+(0),E^-(0))}(X_t; V\otimes\Omega_{\mathrm{b}}^{1/2}) \longrightarrow C^0(I;\mathbb{C})$$

which, by Proposition B.6, coincides with the operator trace in that

$$\mathrm{Tr}(E_\epsilon^{R,L})(\delta) = \mathrm{Tr}(S_\delta(E_\epsilon^{R,L}))$$

and so we conclude that the latter is continuous in δ. □

3.3 The general rank case

We now describe how to deal with the general case of constant rank $\Phi_{|\partial X}$. We will see that the operator is diagonal up to order $O(x)$ near ∂X, and that for a range of particular choices of Sobolev spaces we can suitably deform the operator to be exactly diagonal in a small neighborhood of ∂X. Once this is accomplished, it is straightforward to apply our previous results, constructing a diagonal parametrix with a b-sc transition term as in section 3.2 acting on Null(Φ) and a typical scattering calculus term acting on Null$(\Phi)^{\perp}$.

Let $V_{|\partial X} = V_0 \oplus V_1 \equiv \text{Null}(\Phi) \oplus \text{Null}(\Phi)^{\perp}$ be the splitting of V at infinity into the nullspace of Φ and a Hermitian complement. We extend this splitting to a neighborhood $U \supset \partial X$ in a manner which will be determined below, so that

$$V_{|U} = V_0 \oplus V_1. \tag{3.7}$$

Definition. *Let* $\Pi_0 \in \Gamma(U; \text{End}(V_0 \oplus V_1))$ *be the projection onto the* V_0 *subbundle, and denote by* $\Pi_1 = (\text{Id} - \Pi_0)$ *the other projection. Furthermore, let* $\chi \in C^\infty(X; [0,1])$ *be a smooth cutoff supported on* U *and such that* $\chi \equiv 1$ *on an open neighborhood of* ∂X.

For $u \in L^2_{\text{loc}}(X; V \otimes \Omega_{\text{sc}}^{1/2})$, *let*

$$\begin{aligned} u_0 &= \Pi_0 \chi u \\ u_1 &= \Pi_1 \chi u \\ u_c &= (1-\chi)\, u \end{aligned}$$

so $u_0 \in L^2_{\text{loc}}(U; V_0 \otimes \Omega_{\text{sc}}^{1/2})$, $u_1 \in L^2_{\text{loc}}(U; V_1 \otimes \Omega_{\text{sc}}^{1/2})$ $u_c \in L^2_c(X; V \otimes \Omega_{\text{sc}}^{1/2})$ *and* $u = u_0 + u_1 + u_c$.

We define the hybrid Sobolev space $\mathcal{H}^{\beta,k,l}(X; V \otimes \Omega_{\text{sc}}^{1/2})$ *as follows:*

$$\mathcal{H}^{\beta,k,l}(X;V) = \left\{ u \in L^2_{\text{loc}}(X; V \otimes \Omega_{\text{sc}}^{1/2}) \; ; \; u_0 \in x^\beta H^{k+l}_{\text{b}}, u_1 \in H^{k,l}_{\text{b,sc}}, u_c \in H^{k+l}_c \right\}.$$

Remark. Thus we measure regularity using b derivatives on the V_0 component and scattering derivatives on the V_1 component of u, and allow for extra growth or decay of u_0 compared to u_1. The space is well-defined since, for u supported away from ∂X, these conditions are equivalent: $u \in x^\alpha H^k_{\text{b}} \iff u \in x^\beta H^k_{\text{sc}} \iff u \in H^k_c(\mathring{X})$, if $\text{supp}(u) \cap \partial X = \emptyset$.

Proposition 3.4. $\mathcal{H}^{\beta,k,l}$ *is a complete Hilbert space with respect to the norm*

$$\|u\|^2_{\mathcal{H}^{\beta,k,l}} = \left\| x^{-\beta} u_0 \right\|^2_{H^{k+l}_{\text{b}}} + \|u_1\|^2_{H^{k,l}_{\text{b,sc}}} + \|u_c\|^2_{H^{k+l}_c}$$

Furthermore, it is independent of the choice of χ.

Proof. That it is independent of the choice of χ follows from the fact that, for a compactly supported section, the H^{k+l}_{b} and $H^{k,l}_{\text{b,sc}}$ norms are equivalent. Indeed, with any other choice of χ, the resulting norm differs from the original by the H^{k+l}_{b}, $H^{k,l}_{\text{b,sc}}$, and H^{k+l}_c norms of a compactly supported terms, which can be estimated in terms of the original norm.

That it is a Hilbert space follows from the fact that $\|\cdot\|_{\mathcal{H}^{\beta,k,l}}$ satisfies a parallelogram law (since the three norms on the right do), hence comes from an inner product, and completeness follows from completeness of the spaces H^{k+l}_{b}, $H^{k,l}_{\text{b,sc}}$ and H^{k+l}_c. □

From the assumption $[\sigma_{\text{sc}}(\not{D}), \Phi_{|\partial X}] = 0$ of symbolic commutativity at infinity, it follows that $\not{D}$ and Φ commute to leading order at ∂X, so with respect to the splitting $V_{\partial X} = V_0 \oplus V_1$,

P has the form

$$P = \not{D} + i\Phi = \begin{pmatrix} \not{D}_0 & 0 \\ 0 & \not{D}_1 + i(\Phi) \end{pmatrix} + R$$

where the remainder term R is in $x\mathrm{Diff}^1_{\mathrm{sc}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$, in particular $R_{|\partial X} \equiv 0$.

Proposition 3.5. *P extends to a bounded operator*

$$P : x^\gamma \mathcal{H}^{\beta-1,k,1}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \longrightarrow x^\gamma \mathcal{H}^{\beta,k,0}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$$

for all $0 \leq \beta \leq 1$.

Furthermore, for the strict range $0 < \beta < 1$, the remainder term R has the form $R = x^\epsilon R'$ for some ϵ, with R' bounded.

Proof. By the characterization of $H^{k,l}_{\mathrm{b,sc}}$ in Proposition 1.5 in Chapter 1, we can commute P with x^γ and k b-derivatives, so it suffices to verify the case $\gamma = 0 = k$. As boundedness is clear on the compactly supported part $u_c \in H^1_c(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$ of $u \in \mathcal{H}^{\beta-1,0,1}$, we concentrate on the terms u_0, u_1. Thus is suffices to verify boundedness of

$$\begin{pmatrix} \not{D}_0 & 0 \\ 0 & \not{D}_1 + i\Phi \end{pmatrix} + \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix} : \begin{matrix} x^{\beta-1}H^1_{\mathrm{b}} \\ \oplus \\ H^1_{\mathrm{sc}} \end{matrix} \longrightarrow \begin{matrix} x^\beta L^2 \\ \oplus \\ L^2 \end{matrix}$$

with $R_i \in x\mathrm{Diff}^1_{\mathrm{sc}}$. As $\not{D}_0$ is a homogeneous (i.e. equal to its first order part) element of $\mathrm{Diff}^1_{\mathrm{sc}}$, we can consider it equally as an element of $x\mathrm{Diff}^1_{\mathrm{b}}$, whence boundedness from $x^{\beta-1}H^1_{\mathrm{b}} \longrightarrow x^\beta H^0_{\mathrm{b}} = x^\beta L^2$ is clear. Boundedness of $\not{D}_1 + i\Phi$, R_1 and R_4 is equally clear.

Consider R_2. As an element of $x\mathrm{Diff}^1_{\mathrm{sc}}$,

$$R_2 : H^1_{\mathrm{sc}} \longrightarrow xL^2$$

is bounded, so it will be bounded from $H^1_{\mathrm{sc}} \longrightarrow x^\beta L^2$ provided $\beta \leq 1$.

Now consider $R_3 \in x\mathrm{Diff}^1_{\mathrm{sc}}$. We can decompose it into $R_3 = R_3^1 + R_3^2$, with $R_3^1 \in x^2\mathcal{V}_{\mathrm{b}}$ and $R_3^2 \in xC^\infty$. We have

$$\begin{aligned} R_3^1 &: x^{\beta-1}H^1_{\mathrm{b}} \longrightarrow x^{\beta+1}L^2, \quad \text{and} \\ R_3^2 &: x^{\beta-1}H^1_{\mathrm{b}} \longrightarrow x^\beta H^1_{\mathrm{b}}. \end{aligned}$$

We will have $x^{\beta+1}L^2 \subset L^2$ provided $\beta \geq -1$ and $x^\beta H^1_{\mathrm{b}} \subset L^2$ provided $\beta \geq 0$.

We conclude that P is bounded provided $0 \leq \beta \leq 1$. If we take the inequalities to be strict, each of the remainder terms can be factored as $x^\epsilon R_i'$, with R_i' bounded. For example, if $1 - \beta \geq \epsilon > 0$, R_2 can be written as $x^\epsilon R_2'$ with $R_2' \in x^{1-\epsilon}\mathrm{Diff}^1_{\mathrm{sc}}$, and

$$R_2' : H^1_{\mathrm{sc}} \longrightarrow x^{1-\epsilon}L^2 \subset x^\beta L^2$$

is bounded. The others follow similarly. □

Corollary 3.1. *For $0 < \beta < 1$ above, $P = P_0$ is homotopic to a differential operator P_1 which is arbitrarily close to P in norm as an operator*

$$P_t : x^\gamma \mathcal{H}^{\beta-1,k,1}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \longrightarrow x^\gamma \mathcal{H}^{\beta,k,0}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}),$$

and such that

$$(P_1)_{|U'} = \begin{pmatrix} \eth_0 & 0 \\ 0 & \eth_1 + i(\Phi) \end{pmatrix}$$

on a suitably small open neighborhood U of ∂X.

Proof. For any $\delta > 0$ we can clearly choose x_0 such that $\|R\| = x^{\epsilon}\|R'\| < \delta$ for $x < x_0$. Letting $U' = \{x < x_0\}$ and $f : X \longrightarrow [0,1]$ a cutoff function such that $f_{|X \setminus U'} \equiv 1$ and $f \equiv 0$ for $\{x < x_0/2\}$,

$$P_t = \begin{pmatrix} \eth_0 & 0 \\ 0 & \eth_1 + i(\Phi) \end{pmatrix} + tfR + (1-t)R$$

provides a homotopy such that P_1 has the required form and $\|P_t - P_0\| < \delta$. □

We can now state the main result of this chapter.

Theorem 3.3. *Let $\eth$ be the Dirac operator constructed from a $\mathbb{C}\ell(X,g)$ compatible Clifford connection $\nabla \in {}^{\mathrm{b}}\mathcal{A}(X;V)$, and Φ a compatible potential with $\Phi_{|\partial X}$ a constant rank endomorphism of V. Provided $\eth_{\mathrm{b}} = x^{-(n+1)/2}\eth x^{(n-1)/2}$ has an index zero weight, we have*

$$\operatorname{ind}(\eth + i\Phi) = \operatorname{ind}(\eth + i(\Phi - \epsilon \mathrm{Id})), \qquad \epsilon \in [0, \epsilon_0) \tag{3.8}$$

where ϵ_0 is the minimum positive eigenvalue of $\Phi_{|\partial X}$.

The left hand side of (3.8) is considered as an operator

$$\eth + i\Phi : x^{\alpha}\mathcal{H}^{-1/2,k,1}(X; V \otimes \Omega_{\mathrm{sc}}^{1/2}) \longrightarrow x^{\alpha}\mathcal{H}^{1/2,k,0}(X; V \otimes \Omega_{\mathrm{sc}}^{1/2}), \quad \alpha \in (\alpha_-, \alpha_+)$$

and the right hand side is considered as an operator

$$\eth + i(\Phi - \epsilon\mathrm{Id})) : H_{\mathrm{sc}}^{k+1}(X; V \otimes \Omega_{\mathrm{sc}}^{1/2}) \longrightarrow H_{\mathrm{sc}}^{k}(X; V \otimes \Omega_{\mathrm{sc}}^{1/2})$$

for any k.

Proof. Deforming P as in Corollary 3.1, we can assume P is diagonal with respect to $V_0 \oplus V_1$ in a neighborhood of ∂X since for $\beta = 1/2$ this is a small norm perturbation which preserves the index as the set of Fredholm operators is open.

Let $U \supset \partial X$ be the neighborhood of ∂X on which P is diagonal with respect to $V_0 \oplus V_1$, and let χ be a cutoff with support in U with $\chi \equiv 1$ in a neighborhood of ∂X.

We construct a parametrix for $P - i\epsilon$ which is of the diagonal form

$$\begin{pmatrix} Q_0 & 0 \\ 0 & Q_1 \end{pmatrix} \tag{3.9}$$

near ∂X as follows. Let $\widetilde{Q}_0$ be a parametrix for

$$\eth + i\left((1-\chi)\Phi - \epsilon\right).$$

This operator agrees with $\Pi_0(P - i\epsilon)\Pi_0$ on V_0 near ∂X. It has a rank 0 potential, so the existence of $\widetilde{Q}_0$ follows from section 3.2. Indeed, there we constructed a parametrix Q_ϵ for $\widetilde{\eth} + i\left((1-\chi)\Phi - \epsilon\mathrm{Id}\right)$, and we can then let $\widetilde{Q}_0 = x^{n/2}Q_\epsilon x^{-n/2}$ in order to act on scattering forms.

Then let $\widetilde{Q}_1$ be a parametrix for

$$\not{D} + i\left(\Phi \oplus (\chi \mathrm{Id}_{V_0}) - \epsilon\right).$$

This is a Callias-Anghel operator with invertible potential at ∂X for all ϵ, so the existence of $\widetilde{Q}_1$ follows from the results in Chapter 2. Furthermore, this operator agrees with $\Pi_1(P - i\epsilon)\Pi_1$ on V_1. Finally, the two operators above are identical off of the support of χ, so we can easily arrange for the operator kernels of $\widetilde{Q}_0$ and $\widetilde{Q}_1$ to agree on $(X \setminus U)^2 \times I \subset X^2 \times I$

Finally we set Q equal to

$$Q = \chi\left(\Pi_0\widetilde{Q}_0\Pi_0 + \Pi_1\widetilde{Q}_1\Pi_1\right)\chi + (1-\chi)\widetilde{Q}_0(1-\chi)$$

Thus Q has the form (3.9) near ∂X, where $Q_i = \Pi_i\widetilde{Q}_i\Pi_i$. By construction then, we have

$$(P - i\epsilon)Q - \mathrm{Id} = E_R(\epsilon), \qquad Q(P - i\epsilon) - \mathrm{Id} = E_L(\epsilon)$$

with the error terms which are trace-class as scattering operators for $\epsilon > 0$ and acting on the hybrid Sobolev spaces for $\epsilon = 0$, with $\epsilon \longmapsto \mathrm{Tr}(E_i(\epsilon))$ continuous.

From the trace formula for the index,

$$\mathrm{ind}(P - i\epsilon) = \mathrm{Tr}(E_R(\epsilon) - E_L(\epsilon)) \in \mathbb{Z}.$$

It follows that $\mathrm{ind}(P - i\epsilon)$ is independent of ϵ and the proof is complete. □

In fact, we can slightly weaken the hypothesis that $\not{D}$ have an index zero weight; we need only require an index zero weight for $\not{D}_0$, in a certain sense. Of course $\not{D}_0$ is not a globally defined operator on X, so what we really mean by this is that $0 \notin \mathrm{spec}_{\mathrm{b}}((\not{D}_0)_{\mathrm{b}})$, which from Theorem 3.2 is the technical condition under which we obtain an index zero weight.

Corollary 3.2. *With data as in Theorem 3.3, requiring only that* $0 \notin \mathrm{spec}_{\mathrm{b}}((\not{D}_0)_{\mathrm{b}})$*, where* $(\not{D}_0)_{\mathrm{b}} = x^{-(n+1)/2}\not{D}_0 x^{(n-1)/2} : C^\infty(U, V_0) \longrightarrow C^\infty(U, V_0)$*, we still obtain*

$$\mathrm{ind}(\not{D} + i\Phi) = \mathrm{ind}(\not{D} + i(\Phi - \epsilon\mathrm{Id})), \qquad \epsilon \in [0, \epsilon_0).$$

Proof. After the small norm perturbation, $\not{D}_{\mathrm{b}}$, and hence $I(\not{D}_{\mathrm{b}}, \lambda)$ is diagonal with respect to $V_0 \oplus V_1$ on a neighborhood of ∂X; that is,

$$I(\not{D}_{\mathrm{b}}, \lambda) = I((\not{D}_0)_{\mathrm{b}}, \lambda) \oplus I((\not{D}_1)_{\mathrm{b}}, \lambda).$$

When constructing $\widetilde{Q}_0$ in the proof above, if it happens that $\mathrm{spec}_{\mathrm{b}}(\not{D}_1) \cap \{im(z) = 0\} \neq \emptyset$, we can instead consider a perturbation of $\not{D}_1$ on U which moves all points of $\mathrm{spec}_{\mathrm{b}}(\not{D}_1)$ off of $\{\mathrm{Im}(z) = 0\}$ and so the perturbed $\not{D}_{\mathrm{b}}$ will have an index zero weight.

Such a perturbation will be *neither* compact or of small norm in general; however, since we do not alter $\not{D}_0$, such a change has no effect on $Q_0 = \Pi_0\widetilde{Q}_0\Pi_0$ since this projects off of V_1 anyway. Hence using this "cheat" still allows for the construction of a parametrix for $\not{D} + i(\Phi - \epsilon\mathrm{Id})$, and we can obtain the same result. □

Chapter 4

Magnetic Monopoles

Gauge theory in mathematics is concerned with the space of connections $\mathcal{A}$ on a principal G bundle $P \longrightarrow X$, the quotient

$$\mathcal{C} = \mathcal{A}/\mathcal{G}$$

of connections by the action of the *gauge group* $\mathcal{G} = \mathrm{Aut}(P)$, and the critical points of functionals on this space.

Yang-Mills-Higgs theory is characterized by the functional

$$\mathcal{A} \times \Gamma(X; \mathrm{ad}(P)) \ni (A, \Phi) \longmapsto \frac{1}{2} \int_M |F_A|^2 + |d_A \Phi|^2 \tag{4.1}$$

where F_A is the curvature form of a connection $A \in \mathcal{A}$ and $d_A \Phi$ is the covariant derivative of the "Higgs potential" $\Phi \in \Gamma(X; \mathrm{ad}(P))$, a section of the adjoint bundle $\mathrm{ad}(P) = P \times_{\mathrm{ad}} \mathfrak{g}$. Perhaps the most well-studied example of Yang-Mills-Higgs theory is the theory of *instantons* on a compact 4-manifold M; here $G = SU(2)$, there is no Higgs field and the local minima (among possible additional critical points) of the functional

$$A \longmapsto \frac{1}{2} \int_M |F_A|^2 = \frac{1}{2} \int_M |F_A^+|^2 + |F_A^-|^2$$

are comprised of either *self-dual* or *anti-self-dual* instantons, satisfying $F_A^{\pm} = 0$.

Parallel to this 4 dimensional theory is the theory of $SU(2)$ *monopoles* on $\mathbb{R}^3$, where solutions of the *Bogomolny equation*

$$F_A = *d_A \Phi \tag{4.2}$$

give local minima of (4.1); they are also equivalent to the dimensional reduction of instantons on $\mathbb{R}^4$ which are translation invariant in the additional variable [5].

In physics, the gauge group $SU(2)$ is that of the electroweak theory, modeling electromagnetic and weak nuclear interactions, and solutions to (4.2) represent semiclassical (i.e. non-quantum) approximations to field quanta which look approximately[1] like magnetic point charges, motivating the term *magnetic monopoles.* Moreover, these objects represent topologically nontrivial vacuum solutions which are not accessible via the traditional perturbation theory in QFT; they are therefore considered to be part of "nonperturbative field

[1] In the parlance of physicists, the gauge symmetry is broken from $SU(2)$ to $U(1)$ at infinity, hence the particles are "magnetic."

theory."

The rigorous mathematical study of monopoles was begun by C. Taubes, who showed in a series of papers [26], [27] and [28], following the book [17], that the moduli space, $\mathcal{M}$, of smooth solutions modulo the gauge action of (4.2), subject to the boundary conditions $|\Phi|_{|S^2} = 1$ (where $S^2 = \partial\overline{\mathbb{R}^3}$ is the sphere at infinity) and having finite action (4.1), consists of separate components $\mathcal{M}^k$ labeled by a *charge* parameter $k \in \mathbb{N}$, which is a topological invariant of $\Phi_{|S^2}$. $\mathcal{M}^k$ is a $4k$-dimensional manifold with a complete Riemannian metric and Kaehler structure [5] and in which widely separated monopoles of lower charge can be approximately superposed to produce higher charge monopoles – in other words, they are soliton solutions. Monopoles on hyperbolic 3-space H^3 were considered by Atiyah [4] and others.

The moduli space of monopoles has been investigated on a wider class of noncompact 3-manifolds in at least two instances. In [14] and [13], two posthumously published papers of A. Floer, he describes the quotient $\mathcal{C} = \mathcal{A}/\mathcal{G}$ and the monopole moduli spaces on 3-manifolds with Euclidean ends and asymptotically flat metrics. In [9], P. Braam extends the hyperbolic theory, examining the monopole moduli space on manifolds with an asymptotically locally hyperbolic geometry by constructing S^1 invariant instantons on a conformal compactification of $M \times S^1$. This approach neatly sidesteps the analytical difficulties of working on noncompact spaces.

In this chapter, we begin the study of magnetic monopoles on asymptotically conic manifolds, utilizing the index theory we have developed in previous chapters. In particular, our main goal will be to prove that, in the absence of nontrivial degree 1 cohomology of ∂X, i.e. whenever ∂X is homeomorphic to one or more 2-spheres, then

$$\dim(\mathcal{M}^k) = 4k \tag{4.3}$$

where $\dim(\mathcal{M}^k)$ is the (formal) dimension of the space of charge k monopoles on an asymptotically conic manifold X.

We will introduce the objects involved in the theory of monopoles in section 4.1, after which we shall examine some consequences of requiring, a priori, finite energy and polyhomogeneous regularity of our data in section 4.2. In particular, we encounter the important notions of the mass and charge of a configuration, and derive the Bogomolny equation (4.2) characterizing local minima of the Yang-Mills-Higgs action. We linearize the equations in section 4.3 and couple the result with a gauge fixing operator, designed to locally kill the action of the gauge group and produce a Fredholm operator. This has the form of a Dirac-type Callias operator with constant rank potential, whose Dirac part is a twisted version of the odd signature operator on forms. Section 4.4 is devoted to the analysis of the odd signature operator in general; we show that its induced boundary operator is the even signature operator, and that its index zero weight obstruction is half-dimensional cohomology of the boundary manifold. We finally apply the work from the previous two chapters to the monopole problem in section 4.5, arriving at the formula (4.3). The final section 4.6 is devoted to the proof that the Coulomb gauge condition does in fact fix the local gauge.

4.1 Configuration space and Yang-Mills-Higgs action

Let X be a 3 dimensional asymptotically conic manifold, equipped with a scattering metric of the form

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2}.$$

Let $P \longrightarrow X$ be a principal $SU(2)$ bundle which is trivial in a neighborhood of ∂X. In what follows, we shall take all sections of bundles to have polyhomogeneous conormal expansions at ∂X; these objects are discussed in Appendix A. Thus by $\Gamma(X;V)$ we shall mean the space $\mathcal{A}^*_{\text{phg}}(X;V)$.

Let ${}^{\text{sc}}\mathcal{A}(X;P) = \{\nabla + A \;;\; A \in \Gamma(X; {}^{\text{sc}}\Lambda^1 \otimes \text{ad}(P))\}$ be the space of (scattering) connections, which we identify by their connection one-forms relative to some fixed covariant derivative ∇, and where $\text{ad}(P) = P \times_{\text{ad}} \mathfrak{su}(2)$ is the Lie algebra bundle associated via the adjoint action $\text{ad} : SU(2) \longrightarrow \text{Aut}(\mathfrak{su}(2))$.

Let $\mathcal{G} = \Gamma(X; \text{Ad}(P)) = \text{Aut}(P)$ denote the gauge group, where $\text{Ad}(P) = P \times_{\text{Ad}} SU(2)$ is the fiber bundle associated via the adjoint action $\text{Ad} : SU(2) \longrightarrow \text{Aut}(SU(2))$.

The adjoint bundle $\text{ad}(P)$ carries a Hermitian inner product proportional to the negative of the Killing form in each fiber:

$$\text{ad}(P)_p \otimes \text{ad}(P)_p \ni \alpha(p) \otimes \beta(p) \longrightarrow \langle \alpha, \beta \rangle_p = -2\text{tr}(\alpha_p \beta_p)$$

where the trace is evaluated over the composition of α_p and β_p as endomorphisms under the standard representation on $\mathbb{C}^2$. Equivalently, we could have defined $\langle \alpha, \beta \rangle_p$ by $-\text{tr}(\text{ad}(\alpha_p)\text{ad}(\beta_p))$, using the trace on the adjoint representation $\mathfrak{su}(2) \longrightarrow \text{End}(\mathfrak{su}(2))$.

With this inner product, we can define $L^2(X; \text{ad}(P))$ as the closure of $C^\infty_c(X; \text{ad}(P))$ with respect to the Hilbert space inner product

$$\alpha \otimes \beta \longmapsto (\alpha, \beta) = \int_X \langle \alpha, *\beta \rangle = \int_X -2\text{tr}(\alpha \wedge *\beta) \tag{4.4}$$

where $*$ is the Hodge star operator. This extends to $\text{ad}(P)$-valued differential forms as well; we define $L^2(X; {}^{\text{sc}}\Lambda^k \otimes \text{ad}(P))$ to be the closure of $C^\infty_c(X; {}^{\text{sc}}\Lambda^k \otimes \text{ad}(P))$ with respect to (4.4).

Definition. *Let* $\mathcal{C} = {}^{\text{sc}}\mathcal{A}(X;P) \times \Gamma(X; \text{ad}(P))$ *be the* configuration space *consisting of connections and sections of the adjoint bundle. For* $A \in {}^{\text{sc}}\mathcal{A}(X;P)$*, let* F_A *denote its curvature, and let* d_A *denote the exterior covariant derivative associated to* A*. The* Yang-Mills-Higgs action functional $YM : \mathcal{C} \longrightarrow \mathbb{R}$ *is defined by*

$$\mathcal{C} \ni (A, \Phi) \longmapsto \frac{1}{2}\left(\|F_A\|_{L^2} + \|d_A \Phi\|_{L^2}\right) = \frac{1}{2}\int_X \langle F_A, *F_A \rangle + \langle d_A\Phi, *d_A\Phi \rangle$$

4.2 Consequences of finite action and boundary conditions

We are interested in the finite action configuration space $\{(A, \Phi) \in \mathcal{C} \;;\; YM(A, \Phi) < \infty\}$. Let us examine some consequences of requiring finite action.

First of all, a scattering form $\alpha \in \Gamma(X; {}^{\text{sc}}\Lambda^k \otimes \text{ad}(P))$ is in L^2 if and only if $|\alpha| = O(x^{3/2+\epsilon})$ for some ϵ, where $|\alpha|^2 = (* \langle \alpha, *\alpha \rangle)) = -2\,(*\text{tr}(\alpha \wedge *\alpha))$. In coordinates, relative to a basis $\{dx/x^2, dy_i/x\}$, this is equivalent to requiring leading order $O(x^{3/2+\epsilon})$ of all coefficients.

The first consequence of finite action is therefore that the one form $d_A\Phi$ satisfies $|d_A\Phi| = O(x^{3/2+\epsilon})$. By the following lemma, this implies that it descends to a polyhomogeneous b one-form with leading order $O(x^{1/2+\epsilon})$, and therefore that the restriction of $d_A\Phi$ as a true one-form on the boundary vanishes identically:

$$C^\infty(\partial X; \Lambda^1 \otimes \mathrm{ad}(P)) \ni (d_A\Phi)_{|\partial X} \equiv 0.$$

Lemma 4.1. $\alpha \in \Gamma(X; {}^{\mathrm{sc}}\Lambda^k) \iff x^k\alpha \in \Gamma(X; {}^{\mathrm{b}}\Lambda^k)$. *In particular, a scattering k-form* α *descends to a bounded b k-form only if* $|\alpha| = O(x^k)$.

Proof. This is an immediate consequence of the fact that ${}^{\mathrm{b}}T^*_pX \cong x\left({}^{\mathrm{sc}}T^*_pX\right)$ for all p in a neighborhood of ∂X, for any choice of boundary defining function x. □

Since $P_{|\partial X}$ is trivial, we can identify $d_A\Phi$ as $d\Phi + [A, \Phi]$ in a neighborhood of ∂X. Note that in the absence of A (i.e. if A vanished identically in a neighborhood of ∂X) we would obtain that $|d\Phi| = O(x^{3/2+\epsilon})$ and hence that $\Phi = O(x^{1/2+\epsilon})$, and therefore that $\Phi_{|\partial X} = 0$. More interestingly, we have the following.

Proposition 4.1. *Suppose* Φ *is bounded uniformly up to* ∂X*; then* A *is the lift of a true one-form. Equivalently,* d_A *is the lift of a true connection. In particular, as a scattering one-form,*

$$\left|A^T\right| = O(x) \qquad \left|A^N\right| = O(x^2)$$

where A^T *and* A^N *are the tangential (in the span of* $\{dy_i/x\}$*) and normal (in the span of* dx/x^2*) components of* A.

Proof. If $\Phi = O(1)$, then $d\Phi = O(1)$ (as a true one-form), and therefore $[A, \Phi]$ must have the same order in order that $(d\Phi + [A, \Phi])_{|\partial X} = 0$. But $[A, \Phi]$ has the same leading order as A, so A must be bounded as a true one-form. □

While finite action requires a priori that $|F_A| = O(x^{3/2+\epsilon})$, boundedness of Φ up to ∂X gives something stronger, in light of the above.

Corollary 4.1. *If* Φ *is bounded uniformly up to* ∂X*, then* F_A *is a true 2-form. As a scattering form, we have* $|F_A| = O(x^2)$*, and in fact* $\left|F_A^N\right|$ *has leading order* $O(x^3)$ *(where* F_A^N *consists of all terms whose basis element contains* dx/x^2*).*

Another consequence of the fact that $|d_A\Phi|_{|\partial X} \equiv 0$, i.e. that $\Phi_{|\partial X}$ is covariant constant, is that $|\Phi|_{|\partial X}$ must be constant on connected components of ∂X. For physical reasons, this quantity is called the *mass* of the configuration (A, Φ), and for purposes of normalization, we will typically assume

$$|\Phi|_{|\partial X} = m \equiv 1.$$

For the moment, let us assume ∂X is connected, but that $|\Phi|_{|\partial X} = m$ is arbitrary. Let $C = P\times_{\mathfrak{su}(2)}\mathbb{C}^2$ be the bundle associated via the standard representation $\mathfrak{su}(2) \longrightarrow \mathrm{End}(\mathbb{C}^2)$. From the assumption that $P_{|\partial X}$ is trivial, it follows that $C_{|\partial X} = \partial X \times \mathbb{C}^2$ is a trivial $\mathbb{C}^2$ bundle on which Φ acts. As a section of skew-Hermitian endomorphisms of $C_{|\partial X}$ with $|\Phi| = m$, it follows that Φ_p has eigenvalues $\pm im$ for all $p \in \partial X$.

Definition. *Let* $L_+ \oplus L_-$ *be the splitting of* $C_{|\partial X} = \partial X \times \mathbb{C}^2$ *into* $\pm im$ *eigenbundles of* $\Phi_{|\partial X}$*. We call* $L_+ \longrightarrow \partial X$ *the* charge line bundle *of* (A, Φ)*, and call the first Chern number* $k = c_1(L_+) \in \mathbb{Z}$ *the* charge *of* (A, Φ).

Remark. In the case that ∂X has multiple components, we define the mass and charge of (A, Φ) to be tuples $\overline{m} = (m_1, \ldots, m_N)$ and $\overline{k} = (c_1(L_+)_1, \ldots, c_1(L_+)_N)$, respectively. The *total charge* in this case is $k = \sum_i c_1(L_+)_i$.

From the above considerations, the charge is constant for any connected family of finite action configurations such that Φ remains bounded up to ∂X, hence components of the finite action configuration space.

Lemma 4.2. *Suppose Φ is smooth and bounded up to ∂X, so F_A is a true two-form . Then*

$$YM(A, \Phi) = \frac{1}{2} \|F_A \mp *d_A\Phi\|_{L^2}^2 \pm 8\pi\overline{k} \cdot \overline{m}$$

Proof. We first have that

$$\frac{1}{2}(F_A \mp *d_A\Phi, F_A \mp *d_A\Phi) = \frac{1}{2}\left(\|F_A\|^2 + \underbrace{\|*d_A\Phi\|^2}_{=\|d_A\Phi\|^2} \right) \mp (F_A, *d_A\Phi).$$

The last term evaluates to $(F_A, *d_A\Phi) = \langle F_A, *^2 d_A\Phi\rangle = \langle F_A, d_A\Phi\rangle$ since $*^2 \equiv 1$ in odd dimensions. Using that d_A is compatible with the inner product $\langle\cdot,\cdot\rangle$ and the Bianchi identity $d_A F_A \equiv 0$, we have

$$\langle F_A, d_A\Phi\rangle = d\langle F_A, \Phi\rangle .$$

Since F_A and Φ are smooth up to ∂X, by Stokes' theorem,

$$YM(A, \Phi) = \frac{1}{2} \|F_A \mp *d_A\Phi\|_{L^2}^2 \pm \int_{\partial X} \langle F_A, \Phi\rangle .$$

It remains to show that $\int_{\partial X} \langle F_A, \Phi\rangle = \int_{\partial X} -2\text{tr}\,(F_A\,\Phi) = 8\pi\overline{k}\cdot\overline{m}$. From the fact that $d_A(\Phi_{|\partial X}) = 0$, we have that d_A induces connections on the eigenbundles $L_\pm \longrightarrow \partial X$, with curvature forms $F_\pm$. Then since $L_+ \oplus L_- = \partial X \times \mathbb{C}^2$, we must have that $F_A = F_+ \oplus F_-$ with respect to this splitting.

We can therefore diagonalize the term $\text{tr}\,(F_A \cdot \Phi)$ as

$$\text{tr}\,(F_A \cdot \Phi) = \text{tr}\,((F_+ \oplus F_-)(im \oplus -im)) = im\,(\text{tr}(F_+) - \text{tr}(F_-))$$

and we have

$$\int_{\partial X} -2\text{tr}\,(F_A\Phi) = -2im \int_{\partial X} (\text{tr}(F_+) - \text{tr}(F_-)) = 4\pi m(c_1(L_+) - c_1(L_-)) = 8\pi m c_1(L_+)$$

since $c_1(L_+) = -c_1(L_-)$. Summing over the connected components of ∂X gives the result. □

As a consequence of Lemma 4.2, we see that minimizers of the Yang-Mills-Higgs action in each connected component satisfy the Bogomolny Equation

$$F_A = \pm * d_A\Phi \tag{4.5}$$

with action equal to $8\pi\overline{k}\cdot\overline{m}$.

Any finite action configuration with $|\Phi_{|\partial X}| = m$ is homotopic to one with $|\Phi_{|\partial X}| = 1$, so we examine solutions to (4.5) subject to the boundary condition $|\Phi_{|\partial X}| = 1$.

4.3 Linearization

We wish to examine the linearization of (4.5) acting on the space $\mathcal{C}$. ${}^{sc}\mathcal{A}(X;P)$ is an affine space modeled on $\Gamma(X;{}^{sc}\Lambda^1 \otimes \mathfrak{su}(2))$, and $\Gamma(X;\mathfrak{su}(2))$ is itself a vector space. Thus the tangent space to $\mathcal{C}$ is, formally, the space

$$T\mathcal{C} = \Gamma(X;{}^{sc}\Lambda^1 \otimes \mathfrak{su}(2)) \oplus \Gamma(X;\mathfrak{su}(2)).$$

Lemma 4.3. *Let $(A, \Phi) \in \mathcal{C}$ be a configuration, and suppose $(a, \phi) \in T\mathcal{C}$. Then*

$$\partial_{t|t=0}\left(F_{A+ta} - *d_{A+ta}(\Phi + t\phi)\right) = d_A a - *d_A\phi + *[\Phi, a] \tag{4.6}$$

Proof. First, $F_{A+ta} = F_A + t\, d_{A+ta}a$, so $\partial_{t|t=0}F_{A+ta} = d_A a$. Next, $d_{A+ta}\Phi = d_A\Phi + t[a, \Phi]$, giving the last term since $[\Phi, a] = -[a, \Phi]$. Finally, $d_{A+ta}t\phi = td_A\phi + t^2[a, \phi]$ so upon applying $\partial_{t|t=0}$ we obtain the second term above. □

The operator (4.6) fails to be an elliptic operator, owing to the action of the infinite dimensional gauge group $\mathcal{G}$. There natural choice of a *gauge fixing operator* to add to (4.6) is the Coulomb gauge operator

$$(a, \phi) \longmapsto *d_A * a + [\Phi, \phi]. \tag{4.7}$$

In section 4.6 we discuss the problem of showing that this does in fact fix the gauge locally, i.e. that there exists a unique pair (a, ϕ) in each gauge orbit of $(A + a, \Phi + \phi)$ satisfying $*d_A * a + [\Phi, \phi] = 0$, in some small neighborhood of (A, Φ).

We will write the linearized, gauge fixed operator in a slightly different form. First, $\Gamma(X;\mathrm{ad}(P)) \cong \Gamma(X;{}^{sc}\Lambda^3 \otimes \mathrm{ad}(P))$ by the Hodge star, and we set $\phi = *\psi$, for a unique $\psi \in \Gamma(X;{}^{sc}\Lambda^3 \otimes \mathrm{ad}(P))$. Then we use that $(-1)^k * d_A * = d_A^*$ acting on k forms. Thus (4.6) and (4.7) can be combined to write the linear condition

$$\begin{array}{ll} d_A a + d_A^*\psi & + * [\Phi, a] = 0 \\ -d_A^* a & + * [\Phi, \psi] = 0 \end{array}$$

Finally, since this operator maps $\Gamma(X;({}^{sc}\Lambda^1 \oplus {}^{sc}\Lambda^3) \otimes \mathrm{ad}(P))$ to $\Gamma(X;({}^{sc}\Lambda^0 \oplus {}^{sc}\Lambda^2) \otimes \mathrm{ad}(P))$, we can compose by another Hodge star operator to define

$$D_A = \begin{pmatrix} *d_A & *d_A^* \\ - * d_A^* & \end{pmatrix} : \Gamma(X;(\Lambda^1 \oplus \Lambda^3) \otimes \mathrm{ad}(P)) \longrightarrow \Gamma(X;(\Lambda^1 \oplus \Lambda^3) \otimes \mathrm{ad}(P))$$

This is the "odd signature operator" on X, twisted by the connection A:

$$D_A = (\not{D}_{\mathrm{sig}})_A \,.$$

The linearized, gauge fixed equation for monopoles is then

$$(D_A + [\Phi, \cdot])(a, \psi) = 0.$$

4.4 Odd signature operator

We take a moment to examine the odd signature operator in general. We will show that its induced boundary Dirac operator is the (even) signature operator on ∂X, and that the obstruction to the existence of an index zero weight for $\not{D}_{\text{sig}}$ is the half-dimensional cohomology of ∂X, $H^{(n-1)/2}_{\text{dR}}(\partial X; \mathbb{R})$.

On any manifold of any dimension, define the numerical operator $\tau : \Lambda^k \longrightarrow \Lambda^k$ by

$$\tau = \begin{cases} i^{k(k-1)+(n-1)/2} & \text{if } \dim(X) = n \text{ is odd, and} \\ i^{k(k-1)+m/2} & \text{if } \dim(X) = m \text{ is even} \end{cases} \tag{4.8}$$

As τ only depends on the form degree, we extend it to act on ${}^{\text{sc}}\Lambda^k$ or ${}^{\text{b}}\Lambda^k$ as well for a manifold with boundary. It is easy to check that

$$(*\tau)^2 \equiv 1$$

in any dimension. On even dimensional manifolds, the splitting $\Lambda^{\cdot} = \Lambda^+ \oplus \Lambda^-$ into ± 1 eigenbundles of $*\tau$ is the *signature splitting.*

On an odd dimensional manifold, the odd form bundle Λ^{odd} has a Clifford action

$$c\ell : \mathbb{C}\ell(X) \longrightarrow \text{End}(\Lambda^{\text{odd}}) \quad c\ell(e_i) = \frac{1}{i} * (e_i \wedge \cdot + e_i \lrcorner \cdot)\, \tau$$

which, along with the Levi-Civita connection $\nabla = \nabla^{\text{LC}(g)}$, leads to the odd signature Dirac operator

$$\not{D}_{\text{sig}} = \frac{1}{i} (*d - *d^*)\, \tau$$

On a collar neighborhood of ∂X, the odd degree form bundle ${}^{\text{sc}}\Lambda^{\text{odd}} X$ is isomorphic to the full form bundle $\Lambda^* \partial X$ on ∂X via a choice of boundary defining function; we will use the explicit identification

$${}^{\text{sc}}\Lambda^{\text{odd}} = \left\{ \begin{array}{ccc} \bigoplus_k \frac{dx}{x^2} \wedge \frac{1}{x^{2k}} \Lambda^{2k}_{\partial X} & \longmapsto & \bigoplus_k \Lambda^{2k}_{\partial X} \\ \bigoplus_k \frac{1}{x^{2k+1}} \Lambda^{2k+1}_{\partial X} & \longmapsto & \bigoplus_k \Lambda^{2k+1}_{\partial X} \end{array} \right\} = \Lambda^*_{\partial X} \tag{4.9}$$

That is, on basis elements we identify

$${}^{\text{sc}}\Lambda^{\text{odd}}_p X \ni \frac{dy_{i_1}}{x} \wedge \cdots \wedge \frac{dy_{i_k}}{x} = \frac{dy_I}{x^{|I|}} \longmapsto dy_I \in (\Lambda_{\partial X})_p\,, \quad |I| \text{ odd}$$

and

$${}^{\text{sc}}\Lambda^{\text{odd}}_p X \ni \frac{dx}{x^2} \wedge \frac{dy_{i_1}}{x} \wedge \cdots \wedge \frac{dy_{i_k}}{x} = \frac{dx}{x^2} \wedge \frac{dy_I}{x^{|I|}} \longmapsto dy_I \quad |I| \text{ even.}$$

Furthermore, we agree to orient ∂X such that

$$\frac{dx}{x^2} \wedge \frac{dy_I}{x^{|I|}} \text{ positively oriented} \Longleftrightarrow dy_I \text{ positively oriented.}$$

With this identification, we have the following two important facts.

Lemma 4.4. *The identification (4.9) intertwines the operator* $ic\ell\left(\frac{dx}{x^2}\right)\cdot$, *with the operator* $*_{\partial X}\tau$ *on boundary.*

Furthermore, it identifies the induced boundary Clifford action $c\ell_0\,(dy) = c\ell\left(\frac{dy}{x}\cdot\frac{dx}{x}^2\right)$ *with the negative of the usual Clifford action on forms:*

$$c\ell_0\,(dy) = -\,(dy\wedge\cdot - dy\lrcorner\cdot) : \Lambda^{\pm}\partial X \longrightarrow \Lambda^{\mp}\partial X$$

Proof. Set $m = n-1 = \dim\partial X$. First we note that, since $2k(2k-1) = (2k+1)(2k) \mod 4$, τ has the property that

$$\tau\left(\frac{dx}{x^2}\wedge\frac{dy_I}{x^{|I|}}\right) = \frac{dx}{x^2}\wedge\left(\tau\frac{dy_I}{x^{|I|}}\right), \quad |I| \text{ even.}$$

Thus we can compute

$$\begin{aligned} c\ell\left(\frac{dx}{x^2}\right)\cdot\left(\frac{dx}{x^2}\wedge\frac{dy_I}{x^{|I|}}\right) &= \frac{1}{i}*\frac{dx}{x^2}\lrcorner\left(\frac{dx}{x^2}\wedge\tau\frac{dy_I}{x^{|I|}}\right) = \frac{1}{i}*\left(\tau\frac{dy_I}{x^{|I|}}\right) \\ &= \frac{(-1)^{|I|}}{i}\frac{dx}{x^2}\wedge\left(\frac{(*_{\partial X}\tau)dy_I}{x^{n-1-|I|}}\right) = \frac{1}{i}\frac{dx}{x^2}\wedge\left(\frac{(*_{\partial X}\tau)dy_I}{x^{m-|I|}}\right) \end{aligned}$$

where $(-1)^{|I|} = 1$ since $|I|$ is even. Similarly,

$$c\ell\left(\frac{dx}{x^2}\right)\cdot\frac{dy_I}{x^{|I|}} = \frac{1}{i}*\frac{dx}{x^2}\wedge\left(\tau\frac{dy_I}{x^{|I|}}\right) = \frac{1}{i}\frac{(*_{\partial X}\tau)dy_I}{x^{m-|I|}}.$$

Thus under the identification (4.9), $ic\ell(dx/x^2)$ induces $*_{\partial X}\tau$ on ∂X.

We now examine the action of $c\ell_0(dy_i)$. Using what we have already shown, we have

$$\begin{aligned} c\ell\left(\frac{dy_i}{x}\frac{dx}{x^2}\right)\cdot\left(\frac{dx}{x^2}\wedge\frac{dy_I}{x^{|I|}}\right) &= \frac{1}{i^2}*\left(\frac{dy_i}{x}\wedge\cdot+\frac{dy_i}{x}\lrcorner\cdot\right)\tau\left(\frac{dx}{x^2}\wedge\frac{(*_{\partial X}\tau)dy_I}{x^{m-|I|}}\right) \\ &= *\left(\frac{dx}{x^2}\wedge\left(\frac{dy_i}{x}\wedge\cdot+\frac{dy_i}{x}\lrcorner\cdot\right)\frac{\tau(*_{\partial X}\tau)dy_I}{x^{m-|I|}}\right) \\ &= \frac{1}{x^l}*_{\partial X}(dy_i\wedge\cdot+dy_i\lrcorner\cdot)\tau(*_{\partial X}\tau)dy_I \end{aligned}$$

where $l = |I|\pm 1$. Also,

$$\begin{aligned} c\ell\left(\frac{dy_i}{x}\frac{dx}{x^2}\right)\cdot\left(\frac{dy_I}{x^{|I|}}\right) &= \frac{1}{i^2}*\left(\frac{dy_i}{x}\wedge\cdot+\frac{dy_i}{x}\lrcorner\cdot\right)\frac{\tau(*_{\partial X}\tau)dy_I}{x^{m-|I|}} \\ &= -\frac{dx}{x^2}\wedge\frac{1}{x^l}*_{\partial X}(dy_i\wedge\cdot+dy_i\lrcorner\cdot)\tau(*_{\partial X}\tau)dy_I \end{aligned}$$

We conclude that

$$c\ell_0(dy_i) = (-1)^k *_{\partial X}(dy_i\wedge\cdot+dy_i\lrcorner\cdot)\tau(*_{\partial X}\tau), \qquad \text{on } \Lambda^k$$

However, inserting $(*_{\partial X})^2 = (-1)^{m-k} = (-1)^k$ before the leftmost τ, and using $(*_{\partial X}\tau)^2 = 1$, we have

$$c\ell_0(dy_i) = *_{\partial X}(dy_i\wedge\cdot+dy_i\lrcorner\cdot)*_{\partial X} = dy_i\lrcorner\cdot - dy_i\wedge\cdot$$

□

Corollary 4.2. *The induced boundary Dirac operator of the odd signature operator* $\not{D}_{\text{sig}}$ *is* *minus the (even) signature operator*

$$-\eth_{\text{sig}} = \begin{pmatrix} 0 & -(d+d^*) \\ -(d+d^*) & 0 \end{pmatrix} \in \text{Diff}^1(\partial X; \Lambda^{\pm}, \Lambda^{\mp}),$$

where $\Lambda^{\cdot} = \Lambda^+ \oplus \Lambda^- \longrightarrow \partial X$ *is the signature splitting.*

We now investigate the problem of finding an index zero weight for the associated b operator. Recall from Chapter 1 that, near $\partial X, \not{D}_{\text{sig}}$ can be written

$$\not{D}_{\text{sig}} = c\ell\left(\frac{dx}{x^2}\right)\left(\underbrace{\nabla_{x^2\partial_x} - x\,\eth_{\text{sig}}}_{=D}\right).$$

We will examine the term D in parentheses for convenience, since $\not{D}_{\text{sig}}$ and D differ only by the invertible bundle endomorphism $c\ell(dx/x^2)$.

Setting $m = n-1$, and using the identification (4.9), we recall from Proposition 1.1 that $\nabla_{x^2\partial_x} = \nabla^{\text{LC}(g)}_{x^2\partial_x}$ acts by

$$\nabla_{x^2\partial_x} = \begin{cases} x^2\partial_x - x(2k) & \text{on } \Lambda^2_{\partial X}k, \text{ and} \\ x^2\partial_x - x(m-(2k+1)) & \text{on } \Lambda^{2k+1}_{\partial X}. \end{cases}$$

Recall how we obtain D_{b}. We first conjugate by $x^{n/2}$ to get $\widetilde{D} = x^{-n/2}Dx^{n/2}$ acting on b half densities rather than scattering half densities, and then set

$$D_{\text{b}} = x^{-1/2}\widetilde{D}x^{-1/2} = x^{-(n+1)/2}Dx^{(n-1)/2}$$

to get a b operator. From section 3.1 in Chapter 3, D (and hence $\not{D}_{\text{sig}}$) has an index zero weight provided the indicial family $I(D_{\text{b}}, \lambda)$ is invertible for all $\{\lambda\ ;\ \text{Im}(\lambda) = 0\}$.

Proposition 4.2. *Provided ∂X is a Witt manifold, i.e. its half-dimensional real cohomology group is trivial, $H^{m/2}_{\text{dR}}(\partial X; \mathbb{C}) = 0$, then $\not{D}_{\text{sig}}$ has an index zero weight.*

Proof. Using

$$x^{-(n+1)/2}\left(x^2\partial_x\right)x^{(n-1)/2} = x\partial_x + (n-1)/2 = x\partial_x + m/2,$$

and $x^{-(n+1)/2}\left(x\,\eth_{\text{sig}}\right)x^{(n-1)/2} = \eth_{\text{sig}}$ since the latter commutes with x, we have

$$D_{\text{b}} = x\partial_x + M - \eth_{\text{sig}},$$

where $M : C^\infty(\partial X; \Lambda^{\cdot}) \longrightarrow C^\infty(\partial X; \Lambda^{\cdot})$ is a number operator acting on forms by

$$M = (-1)^k\left(\frac{m}{2} - k\right) \tag{4.10}$$

Then we have

$$I(D_{\text{b}}, \lambda) = i\lambda + M - \eth_{\text{sig}}.$$

Since the latter two terms are self-adjoint, (and hence have real spectrum), it suffices to check invertibility at $\lambda = 0$. Furthermore, since $I(D_{\text{b}}, 0) = M - \eth_{\text{sig}}$ is self-adjoint, we need only examine its nullspace.

We claim that $\mathrm{Null}(I(D_{\mathrm{b}},0)) = \mathcal{H}^{m/2}$, the space of harmonic $m/2$ forms. First off, the nullspace must be smooth since $M - \eth_{\mathrm{sig}}$ is elliptic. Then, using the Hodge decomposition

$$C^\infty(\partial X; \Lambda^\cdot) = \mathcal{H}^\cdot \oplus d\,(C^\infty(\partial X; \Lambda^\cdot)) \oplus d^*\,(C^\infty(\partial X; \Lambda^\cdot))$$

we see that $I(D_{\mathrm{b}},0)$ acts on harmonic forms $\mathcal{H}^\cdot$ by

$$I(D_{\mathrm{b}},0)_{|\mathcal{H}^\cdot} = M, \qquad \Longrightarrow \qquad \mathrm{Null}(I(D_{\mathrm{b}},0)) \cap \mathcal{H}^\cdot = \mathcal{H}^{m/2}$$

using (4.10).

On the complement of the harmonic forms, $\mathcal{H}^\perp = d\,(C^\infty(\partial X; \Lambda^\cdot)) \oplus d^*\,(C^\infty(\partial X; \Lambda^\cdot))$, $M - (d + d^*)$ acts by a matrix whose determinant can be explicitly computed to be nonzero. Thus,

$$\mathrm{Null}(I(D_{\mathrm{b}},0)) = \mathcal{H}^{m/2} = H^{m/2}_{\mathrm{dR}}(X;\mathbb{C}).$$

□

4.5 Monopole moduli dimension

Recall that the linearized, gauge fixed equation for monopoles is $(D_A + [\Phi,\cdot])(a,\psi) = 0$, where

$$D_A = (\not{D}_{\mathrm{sig}})_A$$

is the odd signature operator twisted by $A \in \mathcal{A}(X;P)$.

Lemma 4.5. *The induced boundary Dirac operator of D_A is*

$$-(\eth_{\mathrm{sig}})_A = \begin{pmatrix} 0 & -(d_A + d_A^*) \\ -(d_A + d_A^*) & 0 \end{pmatrix}$$

acting on $C^\infty(\partial X; (\Lambda^+ \otimes \mathrm{ad}(P)) \oplus (\Lambda^- \otimes \mathrm{ad}(P)))$, where d_A denotes the induced connection on ∂X.

Proof. The proof is the same as that leading up to Corollary 4.2 above; twisting the Levi-Civita connection by the connection A. □

Similarly, using that the normal component A^N of A vanishes to second order at ∂X and therefore doesn't contribute to the related b operator, i.e.

$$x^{-(n+1)/2}\left[(\nabla_A)_{x^2\partial_x}\right] x^{(n-1)/2} = \nabla_{x\partial_x} + M + O(x),$$

we obtain that D_A has an index zero weight provided

$$M - (d_A + d_A^*) : C^\infty(\partial X; \Lambda^\cdot \otimes \mathrm{ad}(P)) \longrightarrow C^\infty(\partial X; \Lambda^\cdot \otimes \mathrm{ad}(P))$$

is invertible, where

$$M = \begin{cases} -1 & \text{on } \Lambda^0, \\ 0 & \text{on } \Lambda^1, \text{ and }. \\ 1 & \text{on } \Lambda^2. \end{cases}$$

Of course, since $(F_A)_{|\partial X} \neq 0$, d_A is not a flat connection on $\mathrm{ad}(P) \longrightarrow \partial X$, and so there is no Hodge theory available. However, from Corollary 3.2, we recall that we need only obtain an index zero weight for the Dirac operator acting on the nullspace bundle of the potential. That is, we need only invert $M - (d_A + d_A^*)$ on $\mathrm{Null}([\Phi,\cdot]_{|\partial X}) = \mathbb{C}\Phi \longrightarrow \partial X$.

$\mathbb{C}\Phi \longrightarrow \partial X$ is a trivial line bundle, and $(d_A)_{|\mathbb{C}\Phi}$ is a flat connection since $(d_A\Phi)_{|\partial X} = 0$ by finite energy. Thus we have a Hodge theory for $d_A + d_A^*$ on $\mathbb{C}\Phi \longrightarrow \partial X$, and the d_A-harmonic forms are isomorphic to the ordinary harmonic forms since $\mathbb{C}\Phi \longrightarrow \partial X$ is trivial:

$$\mathcal{H}^k(\partial X; \mathbb{C}\Phi) = \mathcal{H}^k(\partial X; \mathbb{C}) = H^k_{\mathrm{dR}}(\partial X; \mathbb{C})$$

Thus we have proved

Proposition 4.3. *If ∂X is homeomorphic to a disjoint union of 2-spheres, i.e.*

$$H^1_{\mathrm{dR}}(\partial X) = 0,$$

then D_A has an index zero weight on $\mathbb{C}\Phi$.

We finally come to the main result of this chapter, and the main application for the index theorems we have developed in Chapters 2 and 3.

Theorem 4.1. *Let X be an asymptotically conic 3 manifold with $\partial X \cong \bigsqcup_{i=1}^N S^2$. The (formal) dimension of the space of magnetic monopoles of total charge k is $4k$.*

Specifically, if (A, Φ) is a solution to 4.5 with charge $\overline{k} = (c_1(L_+)_1, \ldots, c_1(L_+)_N)$, then

$$\mathrm{ind}(D_A + [\Phi, \cdot]) = 4k$$

where $k = \sum_{i=1}^N \overline{k}_i$.

Proof. By the previous Proposition, the condition on ∂X guarantees that D_A has an index zero weight, whence Corollary 3.2 from Chapter 3 applies to show that

$$\mathrm{ind}(D_A + [\Phi, \cdot]) = \mathrm{ind}(D_A + [\Phi, \cdot] - i\epsilon) \tag{4.11}$$

for suitably small ϵ. This has the effect of pushing the nullspace bundle of $\mathrm{ad}(\Phi)$ into the $-i$-eigenbundle of $\mathrm{ad}(\Phi)$.

The operator has the form

$$\not{D}_{\mathrm{sig}} \otimes 1 + 1 \otimes \mathrm{ad}(\Phi)$$

in the notation of Chapter 2, since $V = {}^{\mathrm{sc}}\Lambda^{\cdot} \otimes \mathrm{ad}(P)$ is a tensor product and the connection has the form $\nabla^{\mathrm{LC}(g)} \otimes \nabla_A$. Thus, using (4.11) and Theorem 2.5 from Chapter 2, we conclude

$$\mathrm{ind}(D_A + [\Phi, \cdot]) = \mathrm{ind}(\eth^+_{\mathrm{ad}(P)_+})$$

where $\eth^+ = (d + d^*)^+$ is the $\Lambda^+ \longrightarrow \Lambda^-$ signature operator on ∂X, twisted by $\mathrm{ad}(P)^+$, the $+i$-eigenbundle of $\mathrm{ad}(\Phi)$ in $\mathrm{ad}(P)$. We'll assume for the moment that ∂X is connected.

From [18], the twisted signature operator on a compact manifold has index

$$\mathrm{ind}(\eth^+_{\mathrm{ad}(P)_+}) = \int_{\partial X} \mathrm{ch}_2(\mathrm{ad}(P)_+) \cdot \mathbf{L}(\partial X) = 2c_1(\mathrm{ad}(P)_+)$$

where $\mathrm{ch}_2(E) = \sum_k 2^k \mathrm{ch}^k(E)$ and $\mathrm{ch}^k(E) \in H^2k(\partial X)$ is the kth component of the usual Chern character. The proof is therefore concluded once we show that

$$c_1(\mathrm{ad}(P)_+) = 2c_1(L_+).$$

Indeed, we claim that $\mathrm{ad}(P)_+ \cong L_+^{\otimes 2}$ by some elementary representation theory. The irreducible complex representations of $\mathfrak{su}(2) \cong \mathfrak{sl}(2, \mathbb{C})$ are determined by their dimension, so letting π_0, π_1 and π_2 denote the trivial, standard and adjoint representations of $\mathfrak{su}(2)$, with complex dimension 1, 2, and 3 respectively, we must have

$$\pi_1 \otimes \pi_1 \cong \pi_2 \oplus \pi_0$$

and a highest weight vector $w \otimes w \in \pi_1 \otimes \pi_1$ on the left must be identified with a highest weight vector $v = v \oplus 0 \in \pi_2 \oplus \pi_0$ on the right (also note that highest weight vector in $\pi_2 \oplus \pi_0$ must lie in π_2). Since $\mathfrak{su}(2)$ is simple, any element can be taken to be the generator of the 1-dimensional Cartan subalgebra.

Transferring this to $P \longrightarrow X$, we have that $\mathrm{ad}(P) \oplus \mathbb{C} \cong C \otimes C$, where $C = P \times_{\mathfrak{su}(2)} \mathbb{C}^2$ is the standard representation associated bundle. Moreover, in $\mathrm{ad}(P) \longrightarrow \partial X$ we can let $\Phi_p \in \mathfrak{su}(2)$ be the Cartan generator at each p; then the $+i$ eigenspace of Φ_p in any representation can be taken as (span of the) highest weight vector, and we conclude that

$$(\mathrm{ad}(P)_+)_p \cong (L_+ \otimes L_+)_p, \qquad \text{for all } p \in \partial X.$$

We conclude that

$$\mathrm{ind}(D_A + [\Phi, \cdot]) = \mathrm{ind}(D_A + [\Phi, \cdot] - i\epsilon) = \mathrm{ind}(\bar{\partial}^+_{\mathrm{ad}(P)_+}) = 2c_1(\mathrm{ad}(P)_+) = 4c_1(L_+)$$

which finishes the result.

In the case that ∂X has multiple components, we simply modify the above by

$$\mathrm{ind}(D_A + [\Phi, \cdot]) = \sum_i \mathrm{ind}(\bar{\partial}^+_{\mathrm{ad}(P)_+}) = 4 \sum_i c_1(L_+)_i = 4k$$

□

4.6 Gauge Fixing

In introducing the Coulomb gauge operator (4.7), we sought to kill the action of the gauge group locally. In other words, the claim is that

Theorem. *For all $(a', \phi') \in TC$ sufficiently small, there exists a unique $\gamma \in \mathcal{G}$ such that*

$$\begin{aligned} &\gamma \cdot (A + a', \Phi + \phi') = (A + a, \Phi + \phi), \quad \textit{with} \\ &-d_A^* a + [\Phi, \phi] = 0 \end{aligned} \tag{4.12}$$

In the above, "sufficiently small" means with respect to a Sobolev space completion of TC and $\mathcal{G}$. We will not prove this theorem completely. As the focus of this thesis is on linear analysis, we will go through the the details of the linear analysis only; in fact this is quite similar to what we did in Chapter 3.

The condition (4.12) for γ translates to the following nonlinear equation for γ:

$$\mathrm{GF}(a', \phi', \gamma) = -d_A^* \left(a' - (d_{A+a'}\gamma)\gamma^{-1}\right) + [\Phi, \gamma(\Phi - \phi')\gamma^{-1}] = 0 \tag{4.13}$$

This follows easily from the following.

Lemma 4.6. *The action of $\mathcal{G}$ on $\mathcal{A}(P)$ can be written in the intrinsic manner (i.e. without reference to a specific trivialization)*

$$\gamma \cdot A = A - (d_A\gamma)\,\gamma^{-1}.$$

In particular,

$$\gamma \cdot (A + a') = a' - (d_{A+a'}\gamma)\,\gamma^{-1}.$$

Proof. Recall that we are identifying $\Gamma(X; \Lambda^1 \otimes \mathrm{ad}(P))$ with $\mathcal{A}(P)$ by choosing some reference connection ∇. Therefore in a local trivialization,

$$\nabla = d + [A', \cdot] \implies d_A = \nabla + [A, \cdot] = d + [A + A', \cdot].$$

From the local trivialization formula for a change of gauge,

$$\begin{aligned}\gamma \cdot (A+A') &= \gamma\,(A + A')\,\gamma^{-1} - (d\gamma)\,\gamma^{-1} = \gamma\,(A + A')\,\gamma^{-1} - (d\gamma)\,\gamma^{-1} + (A+A') - (A+A') \\ &= (A + A') - \big(d\gamma + [A + A', \gamma]\big)\,\gamma^{-1} = (A + A') - (d_A\gamma)\,\gamma^{-1}.\end{aligned}$$

Thus the effect of γ on the covariant derivative $d_A = \nabla + [A, \cdot]$ is to transform A to $A - (d_A\gamma)\,\gamma^{-1}$ as stated, which is an expression which is well-defined independent of coordinates. □

By inspection, it is clear that

$$\mathrm{GF}(0, 0, \mathrm{Id}) = 0,$$

and the idea is to use the implicit function theorem to see that $\mathrm{GF}(a', \phi', \gamma(a', \phi')) = 0$ defines a unique function $(a', \phi') \longmapsto \gamma(a', \phi')$ in a neighborhood of $(a', \phi') = (0, 0)$.

Formally evaluating $\partial_\gamma \mathrm{GF}(0, 0, \mathrm{Id})$, we obtain

$$L\eta \equiv \partial_\gamma \mathrm{GF}(0, 0, \mathrm{Id}) \cdot \eta = d_A^* d_A \eta - [\Phi, [\Phi, \eta]], \quad \eta \in \Gamma(X; \mathrm{ad}(P)) = T\mathcal{G} \tag{4.14}$$

and we will show that this is invertible for a suitable Sobolev completion of $\Gamma(X; \mathrm{ad}(P))$.

This is an elliptic, though not fully elliptic, scattering operator, as we see from the following.

Lemma 4.7. $(\mathrm{ad}(\Phi))^2 = [\Phi, [\Phi, \cdot]] \in C^\infty(X; \mathrm{End}(\mathrm{ad}(P)))$ *is non-positive definite and Hermitian. Where $\Phi \neq 0$ (in particular near ∂X), it has one-dimensional nullspace bundle* $\mathrm{span}_{\mathbb{C}}(\Phi) \subset \mathrm{ad}(P)$.

Proof. From the fact that $\mathrm{ad}(\Phi) = [\Phi, \cdot]$ is skew Hermitian, it follows that $\mathrm{ad}(\Phi)^2$ is Hermitian. It is non-positive since

$$\big\langle \mathrm{ad}(\Phi)^2 u, u \big\rangle = -\langle \mathrm{ad}(\Phi)u, \mathrm{ad}(\Phi)u \rangle = -\left|\mathrm{ad}(\Phi)u\right|^2 \leq 0$$

That $\mathrm{ad}(\Phi)^2$ has nullspace equal to $\mathrm{span}_{\mathbb{C}}(\Phi)$ is equivalent to the fact that, for any $B \in \mathfrak{su}(2)$, the centralizer $Z_{\mathfrak{su}(2)}(B) = \mathrm{span}_{\mathbb{C}} B$. This can be seen by using the basis elements

$$\mathbf{i} = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

which satisfy

$$[\mathbf{i},\mathbf{j}]=\mathbf{k},\quad [\mathbf{j},\mathbf{k}]=\mathbf{i},\quad [\mathbf{k},\mathbf{i}]=\mathbf{j}.$$

□

Our approach to inverting L will mirror the analysis of Callias-type Dirac operators with constant rank potentials. Namely, we will first find a self-adjoint weight for the twisted Laplacian $\Delta_A = d_A^* d_A$ as a weighted b operator, then construct a parametrix for L on hybrid Sobolev spaces. We will consider L as an operator acting on scattering half-densities $\Omega_{\mathrm{sc}}^{1/2}(X)$.

Lemma 4.8. *Near ∂X, $\Delta_A = d_A^* d_A$ has the form*

$$-(x^2\partial_x)^2 + (n-1)x(x^2\partial_x) + x^2\Delta_{A,\partial X} + O(x^3)$$

where $\Delta_{A,\partial X}$ is the twisted Laplacian induced by $(d_A)_{|\partial X}$, and where the remainder terms are in $x^3\mathrm{Diff}^2_{\mathrm{sc}}(X;\mathrm{ad}(P)\otimes\Omega_{\mathrm{sc}}^{1/2})$.

Proof. We trivialize P near the boundary. Then from the fact that $d_A = d + [A,\cdot]$ is the lift of a true connection as we proved in Proposition 4.1, the normal component of A (as a scattering form) must have expansion $O(x^2)$. Thus the normal part of Δ_A (the one involving $x^2\partial_x$ derivatives), must agree with the standard Laplacian d^*d on functions, up to terms of order $O(x^3)$ at worst.

From the fact that g is asymptotically conic, the standard Laplacian d^*d has the form

$$-(x^2\partial_x)^2 + (n-1)x(x^2\partial_x) + x^2\Delta_{\partial X},$$

where $\Delta_{\partial X}$ is the standard Laplacian induced on ∂X by $d_{|\partial X} = d$. We therefore need only modify this expression by introducing the connection terms induced by A^T on ∂X. □

We introduce the operator $\widetilde{\Delta}_A = x^{-n/2}\Delta_A x^{n/2}$ acting on b half-densities, and then examine the related b operator

$$(\Delta_A)_{\mathrm{b}} = x^{-1}\widetilde{\Delta}_A x^{-1} = x^{-n/2-1}\Delta_A x^{n/2-1}$$

Corollary 4.3. *$(\Delta_A)_{\mathrm{b}}$ has an index zero weight at $\alpha = 0$.*

Proof. Using $x\partial x \circ x^\alpha = x^\alpha(x\partial_x + \alpha)$, we obtain that $(\Delta_A)_{\mathrm{b}}$ has the form

$$-(x\partial_x)^2 + \left(\frac{n-2}{2}\right)^2 + \Delta_{A,\partial X} + O(x)$$

and therefore $I((\Delta_A)_{\mathrm{b}},\lambda)$ has the form

$$I((\Delta_A)_{\mathrm{b}},\lambda) = \lambda^2 + \left(\frac{n-2}{2}\right)^2 + \Delta_{A,\partial X}.$$

Evidently, $\mathrm{spec}_{\mathrm{b}}((\Delta_A)_{\mathrm{b}})$ consists of the set

$$\mathrm{spec}_{\mathrm{b}}((\Delta_A)_{\mathrm{b}}) = \left\{\pm i\sqrt{\nu + \left(\frac{n-2}{2}\right)^2}\ ;\ \nu\in\mathrm{spec}(\Delta_{A,\partial X})\right\}$$

which does not intersect the line $\alpha = \mathrm{Im}(\lambda) = 0$, since $n = 3$ and $\Delta_{A,\partial X}$ is a positive self-adjoint operator on ∂X with discrete spectrum $0 = \nu_0 < \nu_1 < \cdots$. □

From the theory of b pseudodifferential operators, we conclude that

Proposition 4.4. $(\Delta_A)_{\mathrm{b}}$ *has an index zero Fredholm extension as an operator*

$$(\Delta_A)_{\mathrm{b}} : x^\alpha H^k_{\mathrm{b}}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{b}}) \longrightarrow x^\alpha H^{k-2}_{\mathrm{b}}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{b}}), \quad \alpha \in (-1/2, 1/2)$$

and therefore, Δ_A *has an index zero Fredholm extension as an operator*

$$\Delta_A : x^{\alpha-1} H^k_{\mathrm{b}}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{sc}}) \longrightarrow x^{\alpha+1} H^{k-2}_{\mathrm{b}}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{sc}})$$

Proof. This follows directly from the theory of b pseudodifferential operators, and the fact that $(\Delta_A)_{\mathrm{b}}$ is formally self-adjoint for $\alpha = 0$. The upper and lower limits for α come from the expression for $\mathrm{spec}_{\mathrm{b}}((\Delta_A)_{\mathrm{b}})$ in the proof above, with $\pm i\frac{(n-2)}{2} = \pm\frac{i}{2}$ the closest elements of $\mathrm{spec}_{\mathrm{b}}((\Delta_A)_{\mathrm{b}})$ to 0.

The extension for Δ_A then follows from the commutative diagram

$$\begin{array}{ccc}
x^{\alpha-1} H^k_{\mathrm{b}}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{sc}}) & \xrightarrow{\Delta_A} & x^{\alpha+1} H^{k-1}_{\mathrm{b}}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{sc}}) \\
\cong \big\downarrow x^{1-n/2} & & x^{1+n/2} \big\uparrow \cong \\
x^{\alpha} H^k_{\mathrm{b}}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{b}}) & \xrightarrow{(\Delta_A)_{\mathrm{b}}} & x^{\alpha} H^{k-1}_{\mathrm{b}}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{b}})
\end{array}$$

□

Now let U be a collar neighborhood of ∂X such that $\Phi_{|U} \neq 0$. We split $\mathrm{ad}(P)_{|U}$ as the nullspace of $\mathrm{ad}(\Phi)$ and an orthogonal complement,

$$\mathrm{ad}(P)_{|U} = \mathrm{ad}(P)_0 \oplus \mathrm{ad}(P)_1 = \mathbb{C}\Phi \oplus \Phi^\perp \tag{4.15}$$

and define the following *hybrid Sobolev spaces* as in section 3.3 of Chapter 3.

Definition. *Let* $\Pi_0 \in \Gamma(U; \mathrm{End}(\mathrm{ad}(P)_0 \oplus \mathrm{ad}(P)_1))$ *be the projection onto the* $\mathrm{ad}(P)_0$ *subbundle, and denote by* $\Pi_1 = (\mathrm{Id} - \Pi_0)$ *the other projection. Furthermore, let* $\chi \in C^\infty(X; [0,1])$ *be a smooth cutoff supported on* U *and such that* $\chi \equiv 1$ *on an open neighborhood of* ∂X.

For $u \in L^2_{\mathrm{loc}}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{sc}})$, *let*

$$\begin{aligned}
u_0 &= \Pi_0 \chi u \\
u_1 &= \Pi_1 \chi u \\
u_c &= (1-\chi)\, u
\end{aligned}$$

Define the hybrid Sobolev spaces $\mathcal{H}^{\beta,k,l}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{sc}})$ *by*

$$\mathcal{H}^{\beta,k,l}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{sc}}) = \left\{ u \in L^2_{\mathrm{loc}}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{sc}}) \; ; \; u_0 \in x^\beta H^{k+l}_{\mathrm{b}}, u_1 \in H^{k,l}_{\mathrm{b,sc}}, u_c \in H^{k+l}_c \right\}.$$

We proved in Proposition 3.4 in Chapter 3 that this is a complete Hilbert space which is independent of the choice of χ.

Lemma 4.9. *With respect to the splitting (4.15), L has the form*

$$L = L_0 + R = \begin{pmatrix} \Delta_A & 0 \\ 0 & \Delta_A - [\Phi, [\Phi, \cdot]] \end{pmatrix} + R$$

with the remainder term $R \in x^2 \mathrm{Diff}^1_{\mathrm{b}}(X; \mathrm{ad}(P))$.

Proof. The term $[\Phi, [\Phi, \cdot]]$ commutes with Π_i, with $[\Phi, [\Phi, \cdot]] \circ \Pi_0 = 0$ and $[\Phi, [\Phi, \cdot]] \circ \Pi_1 = [\Phi, [\Phi, \cdot]]$. Hence the error term comes from the commutator $[\Delta_A, \Pi_0]$.

From the above, $\Delta_A \in x^2 \mathrm{Diff}^2_b(X; \mathrm{ad}(P))$, with scalar principal symbol

$$\sigma(x^{-2}\Delta_A)(p, \xi) = \|\xi\|^2 \, \mathrm{Id} \in \mathrm{End}(\mathrm{ad}(P)).$$

Letting $\widetilde{\Delta}_A = x^{-2}\Delta_A \in \mathrm{Diff}^2_b(X; \mathrm{ad}(P))$, we have

$$[\Delta_A, \Pi_0] = [x^2 \widetilde{\Delta}_A, \Pi_0] = x^2 [\widetilde{\Delta}_A, \Pi_0]$$

since Π_0 commutes with x^2. The term in brackets must be of order 1 since $\widetilde{\Delta}_A$ has scalar principal symbol, and hence

$$\sigma\left([\widetilde{\Delta}_A, \Pi_0]\right) = [\sigma(\widetilde{\Delta}_A), \sigma(\Pi_0)] = 0$$

We conclude that $R \in x^2 \mathrm{Diff}^1_{\mathrm{b}}(X; \mathrm{ad}(P))$. □

Just as in section 3.3, we have the following result.

Proposition 4.5. *L extends to a bounded operator*

$$L : x^\gamma \mathcal{H}^{\beta-2,k,2}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \longrightarrow x^\gamma \mathcal{H}^{\beta,k,0}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$$

for all $0 \leq \beta \leq 1$.

Furthermore, for the strict range $0 < \beta < 1$, *the remainder term R has the form* $R = x^\epsilon R'$ *for some* ϵ, *with* R' *bounded.*

Proof. By commuting P with x^γ and k b-derivatives, it suffices to verify the case $\gamma = 0 = k$. It suffices to verify boundedness of

$$\begin{pmatrix} \Delta_A & 0 \\ 0 & \Delta_A - [\Phi, [\Phi, \cdot]] \end{pmatrix} + \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix} : \begin{matrix} x^{\beta-2} H^2_{\mathrm{b}} \\ \oplus \\ H^2_{\mathrm{sc}} \end{matrix} \longrightarrow \begin{matrix} x^\beta L^2 \\ \oplus \\ L^2 \end{matrix}$$

with $R_i \in x^2 \mathrm{Diff}^1_{\mathrm{b}}$.

As before, the key terms to verify are R_2 and R_3. We can view R_2 as an element of $x\mathrm{Diff}^1_{\mathrm{sc}}(X; \mathrm{ad}(P))$, since $x^2 \mathrm{Diff}^1_{\mathrm{b}} \subset x \mathrm{Diff}^1_{\mathrm{sc}}$; we have

$$R_2 : H^2_{\mathrm{sc}} \longrightarrow xL^2 \subset x^\beta L^2, \text{ if } \beta \leq 1.$$

We split $R_3 \in x^2 \mathrm{Diff}^1_{\mathrm{b}}$ into terms $R_3 = R^1_3 + R^2_3$, with $R^1_3 \in x^2(\mathcal{V}_{\mathrm{b}})$, and $R^2_3 \in x^2 C^\infty$. Then

$$R^1_3 : x^{\beta-2} H^2_{\mathrm{b}} \longrightarrow x^\beta H^1_{\mathrm{b}}, \quad \text{and}$$
$$R^2_3 : x^{\beta-2} H^2_{\mathrm{b}} \longrightarrow x^\beta H^2_{\mathrm{b}}.$$

Each term requires that $x^\beta H^i_{\mathrm{b}} \subset L^2$, which requires $0 \leq \beta$.

We conclude that L is bounded provided $0 \leq \beta \leq 1$, and the result for strict inequality is the same as before. $\square$

Corollary 4.4. *Taking the strict range* $0 < \beta < 1$, *L is homotopic by an arbitrarily small norm perturbation to an operator which is diagonal with respect to* $\mathrm{ad}(P) = \mathrm{ad}(P)_0 \oplus \mathrm{ad}(P)_1$ *on a neighborhood of* ∂X.

Proof. Choosing β in the strict range, and taking $U = \partial X \times [0,1)_x$, we can choose x_0 sufficiently small so that

$$L = \begin{pmatrix} \Delta_A & 0 \\ 0 & \Delta_A - [\Phi, [\Phi, \cdot]] \end{pmatrix} + R$$

with $\|R_{|\partial X \times [0,x_0)}\|$ as small as we like. Then for a smooth cutoff function $f : X \longrightarrow [0,1]$ with $f \equiv 1$ on $X \setminus \partial X \times [0, x_0)$,

$$L_t = \begin{pmatrix} \Delta_A & 0 \\ 0 & \Delta_A - [\Phi, [\Phi, \cdot]] \end{pmatrix} + tfR + (1-t)R$$

provides a homotopy with the required properties. $\square$

Proposition 4.6. *For k sufficiently large, L is invertible as an operator*

$$L : x^\gamma \mathcal{H}^{\beta-2,k,2}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{sc}}) \longrightarrow x^\gamma \mathcal{H}^{\beta,k,0}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{sc}}),$$

where $\beta \in (0,1), \gamma \in (1/2 - \beta, 3/2 - \beta)$.

Proof. By taking $0 < \beta < 1$, we ensure that we are in the strict range mentioned in Proposition 4.5, so by deforming L slightly, we may assume that it is diagonal with respect to the splitting $\mathrm{ad}(P)_0 \oplus \mathrm{ad}(P)_1$ in a neighborhood of ∂X, preserving Fredholmness and the index if the deformation is suitably small.

As in the proof of Theorem 3.3 in Chapter 3, we can construct a parametrix which has the diagonal form

$$Q = \begin{pmatrix} Q_0 & 0 \\ 0 & Q_1 \end{pmatrix} \tag{4.16}$$

on a neighborhood of ∂X. Q_0 is a b parametrix for an operator of the form Δ_A, and Q_1 is a scattering parametrix for an operator of the form $\Delta_A - U$, where U is a negative-definite, Hermitian potential, by Lemma 4.7.

Then for $\gamma \in (1/2 - \beta, 3/2 - \beta)$ (which corresponds to taking $\alpha \in (-1/2, 1/2)$, Δ_A has index 0 as a b operator, and $\Delta_A - U$ has index 0 as a scattering operator. Indeed, the latter can be seen by taking $\gamma = 0$, for some $\beta > 1/2$, for which Δ_A is a positive, self-adjoint operator (hence with purely real, non-negative spectrum) and therefore $\Delta_A - U$ is invertible by spectral considerations.

Thus in the range $\gamma \in (1/2 - \beta, 3/2 - \beta)$, Q furnishes an index 0 parametrix for L. We claim that that $\mathrm{Null}(L)$ must be empty. For $u \in \mathrm{Null}(L)$,

$$0 = (Lu, u) = (d_A u, d_A u) + ([\Phi, u]) \geq \|d_A u\|^2$$

so u must be covariant constant. But $u \in x^\gamma \mathcal{H}^{\beta-2,k,2}(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{sc}})$ requires in particular that $u_{|\partial X} = 0$ by integrability (for $\gamma \geq -3/2$, $u \in x^\gamma L^2(X; \mathrm{ad}(P) \otimes \Omega^{1/2}_{\mathrm{sc}}) \cap C^0$ implies that

$u_{|\partial X} = 0$) and continuity ($u \in C^l_{\mathrm{loc}}(\mathring{X}; \mathrm{ad}(P) \otimes \Omega_{\mathrm{sc}}^{1/2})$ if $k+2 > 3/2+l$, and can be extended continuously to ∂X). Finally, we see that u must vanish identically by parallel transport from ∂X into the interior. □

Appendix A

Manifolds with corners and polyhomogeneous distributions

To avoid distracting from the main exposition, we collect in this appendix some facts about manifolds with corners, polyhomogeneous conormal distributions and densities which we use at various points throughout the text. The primary reference for all of this material is [19].

A.1 Manifolds with corners

A manifold with corners X is a Hausdorff space modeled smoothly on

$$\mathbb{R}^n_k = \mathbb{R}^{n-k} \times [0, \infty)^k.$$

That is, every point $p \in X$ has a neighborhood homeomorphic to the above, for some k, with smooth transition maps between coordinate charts. We shall only consider compact manifolds with corners. Points are in the boundary of X if, in some (hence any) coordinate chart, $x_i(p) = 0$ for at least one i, and in fact, the *codimension*

$$\operatorname{codim}(p) = \# \{i \ ; \ x_i(p) = 0\} \in \mathbb{N}_0$$

is well-defined for all p independent of coordinates; with respect to the codimension, X is a stratified space:

$$X = \bigsqcup_{l \geq 0} X_l$$

where $X_l = \{p \ ; \ \operatorname{codim}(p) = l\}$. In particular, note that $X_0 = \mathring{X}$.

A *boundary face of codimension* k is the closure of a connected component of X_k. We let $M_k(X)$ denote the set of boundary faces of codimension k. Note that $H \in M_k(X), J \in M_l(X) \implies H \cap J \in M_{k+l}(X)$ if it is nonempty. As part of the definition of a manifold with corners, we typically require that all boundary hypersurfaces $H \in M_1(X)$, are embedded. Among other things, this prevents boundary faces from having higher codimension intersection with themselves.

Any boundary face H of a manifold with corners X is again a manifold with corners, and the boundary faces of H correspond (though not necessarily bijectively) with $G \in M_*(X)$ such that $G \cap H \neq \emptyset$.

For a boundary hypersurface $H \in M_1(X)$, a *boundary defining function* consists of a function

$$\rho_H \in C^\infty(X;[0,\infty)), \qquad \text{such that } \rho_H^{-1}(0) \equiv H, \quad (d\rho_H)_{|H} \neq 0.$$

Near a point p with $\operatorname{codim}(p) = k$, coordinates $(x_1,\dots,x_k,y_1,\dots,y_{n-k}) \in \mathbb{R}^n_k$ can always be chosen so that the x_i are (locally) boundary defining functions for the hypersurfaces whose intersection contains p.

Of major importance in the theory of manifolds with corners are *p-submanifolds*, which are those submanifolds which appear as a product in every coordinate neighborhood:

$$Y = \{(x_1,\dots,x_k,y_1,\dots,y_{n-k}) \ ; \ x_1 = \cdots x_l = y_1 = \cdots y_m = 0\} \quad \text{locally,}$$

where $m+l$ is the codimension of Y. In particular, boundary faces are always p-submanifolds.

A.2 Morphisms

For the purposes of analysis on manifolds with corners, a good class of morphisms consists of the b-maps. If $\{\rho_H \ ; \ H \in M_1(X)\}$ and $\{\rho'_G \ ; \ G \in M_1(Y)\}$ are complete sets of boundary defining functions for X and Y, then a smooth map $f : X \longrightarrow Y$ is said to be a *b-map* provided

$$f^*\rho'_G = a_G \prod_{H \in M_1(X)} \rho_H^{e(H,G)}, \qquad a_G > 0 \in C^\infty(X), \quad e(H,G) \in \mathbb{N}_0.$$

In particular, $f^{-1}(\partial Y) \subset \partial X$ or $f^{-1}(\partial Y) = X$. The non-negative matrix elements $e(\cdot,\cdot)$ are called the *boundary exponents* of f. Equipped with b-maps as morphisms, manifolds with corners form a category.

As on manifolds with boundary, the *b vector fields* $\mathcal{V}_b(X)$, consisting of all vector fields tangent to the boundary, form a subalgebra. Correspondingly, we have the b tangent bundle

$$^{\mathrm{b}}TX \longrightarrow X \qquad \text{such that } \mathcal{V}_b(X) = C^\infty(X; {^{\mathrm{b}}TX})$$

with the local form

$$^{\mathrm{b}}T_pX = \operatorname{span}_{\mathbb{R}} \{x_1\partial_{x_1}, \cdots x_k\partial_{x_k}, \partial_{y_1}, \cdots, \partial_{y_{n-k}}\}$$

at a point p with $\operatorname{codim}(p) = k$.

At $p \in \partial X$, contained in a boundary face $H \in M_{\operatorname{codim}(p)}(X)$, the b-normal bundle is defined by the exact sequence

$$0 \longrightarrow {^{\mathrm{b}}N_pH} \longrightarrow {^{\mathrm{b}}T_pX} \longrightarrow T_pX \longrightarrow 0$$

where the map $^{\mathrm{b}}TX \longrightarrow TX$ is induced by the inclusion $\mathcal{V}_b(X) \subset \mathcal{V}(X)$. When $H \in M_1(X)$ is a hypersurface, $^{\mathrm{b}}NH \longrightarrow \mathring{H}$ is canonically trivial over the interior, as $(\rho\partial_\rho)_p$ is independent of the defining function ρ for H at $p \in \mathring{H}$.

For any interior b-map (meaning a b-map such that $f(X) \not\subset \partial Y$), the differential map

$f_* : T_pX \longrightarrow T_{f(p)}Y$ extends by continuity to produce

$$f_* : {}^{\mathrm{b}}T_pX \longrightarrow {}^{\mathrm{b}}T_{f(p)}Y \tag{A.1}$$

$$f_* : {}^{\mathrm{b}}N_pH \longrightarrow {}^{\mathrm{b}}N_{f(p)}G \tag{A.2}$$

where $f(p) \in G \in M_{\mathrm{codim}(f(p))}(Y)$. Provided f_* is surjective as a map (A.1), f is said to be a *b-submersion*, and if f_* is surjective as a map (A.2), f is called *b-normal.* An equivalent characterization of b-normal maps is that they do not increase codimension of points; i.e. we must have $\mathrm{codim}(p) \geq \mathrm{codim}(f(p))$ for all p.

An important class of maps are the *b-fibrations* consisting of b-normal b-submersions. An equivalent characterization is

$$f \text{ is a b-fibration} \iff \text{for each } H \in M_1(X), e(H,G) \neq 0 \text{ for at most one } G \in M_1(Y)$$

where $e(H, G)$ are the boundary exponents of f. A b-fibration is a fibration over the interior, and for $p \in \partial Y$, $f^{-1}(p)$ is a union of boundary hypersurfaces, the restrictions of f to which are b-fibrations.

A.3 Blow up

Given a p-submanifold H of X, the *blow up* of X at H is the space

$$[X; H] = X \setminus H \cup S^+NH$$

where S^+NH is the inward pointing spherical normal bundle. There is a natural *blow down map*

$$\beta : [X; H] \longrightarrow X$$

equal to the identity on $X \setminus H$ and the bundle projection $\pi : S^+NH \longrightarrow H$ on S^+NH. $[X; H]$ is equipped with the minimal C^∞ structure containing $\beta^*(C^\infty(X))$ along with quotients ρ_1/ρ_2 of functions ρ_i vanishing simply at H (which have a continuous limit in S^+NH by L'Hopital's rule). With this structure, $[X; H]$ is a manifold with corners and β is a b-map[1]. The boundary hypersurface consisting of S^+NH is called the *front face* of $[X; H] \longrightarrow X$, and denoted

$$\mathrm{ff} = S^+NH \subset [X; H].$$

If, on a coordinate patch $(U, x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$, H is given by the vanishing of local coordinates

$$H = \{x_1 = \cdots = x_l = y_1 = \cdots = y_m = 0\},$$

then $[X; H] \cap \beta^{-1}(U)$ is covered by coordinate patches U_i and V_j, $1 \leq i \leq l$, $1 \leq j \leq m$, with coordinates

$$\left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, x_i, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_l}{x_i}, x_{l+1}, \ldots, x_k, \frac{y_1}{x_i}, \ldots, \frac{y_m}{x_i}, y_{m+1}, \ldots, y_{n-k}\right)$$

[1]Note that it is *not* b-normal.

on U_i and

$$\left(\frac{x_1}{y_i}, \dots, \frac{x_l}{y_i}, x_{l+1}, \dots, x_k, \frac{y_1}{y_i}, \dots, \frac{y_{i-1}}{y_i}, y_i, \frac{y_{i+1}}{y_i}, \dots, \frac{y_m}{y_i}, y_{m+1}, \dots, y_{n-k}\right)$$

on V_i. These are called *projective coordinates* on $[X; H]$. Boundary defining functions for ff are given by x_i on U_i and y_i on V_i.

Let G be another p-submanifold of X. The *lift of* G in $[X; H]$ is a p-submanifold given by

1. $\beta^{-1}(G)$ if $G \subset H$, or
2. $\overline{\beta^{-1}(G \setminus H)}$ otherwise.

It is often a convenient abuse of notation to also use the letter G to refer to the lift of G in $[X; H]$.

Given a sequence $H_1, \dots, H_n$ of p-submanifolds of X, the *iterated blow up*

$$[X; H_1, H_2, \dots, H_n] = [\cdots [[X; H_1]; H_2]; \cdots ; H_n]$$

is understood to mean the manifold obtained first by blowing up $H_1 \subset X$, and then blowing up the lift of H_2 in $[X; H_1]$, and so on. By composing the individual blow down maps, we have a total blow down map

$$\beta : [X; H_1, \dots, H_n] \longrightarrow X.$$

Note that $[X; H_1, \dots, H_n]$ will in general depend on the order in which $H_1, \dots, H_n$ are blown up. However, there are important cases in which the order may be interchanged. In particular,

$$[X; H_1, H_2] = [X; H_2, H_1]$$

provided

- H_1 and H_2 are *separated:* $H_1 \cap H_2 = \emptyset$ in X, or
- H_2 is *contained in* H_1: $H_2 \subset H_1$ (or vice versa), or
- H_1 intersects H_2 *transversally:* $H_1 \pitchfork H_2$.

A.4 Polyhomogeneous conormal distributions

The polyhomogeneous conormal distributions on X play essentially the same role as the smooth functions on manifolds without boundary; though they are allowed to have non-smooth asymptotic expansions at ∂X.

First, we say $E \subset \mathbb{C} \times \mathbb{N}_0$ is a (smooth) *index set* provided

- E is discrete,
- if $E = \{(s_j, p_j)\}_{j=0}^{\infty}$, and $|(s_j, p_j)| \longrightarrow \infty$ only if $\mathrm{Re}(s_j) \longrightarrow \infty$; and
- $(s, p) \in E$ implies $(s + k, p - l) \in E$ for all $k, l \in \mathbb{N}_0$ with $p \geq l$.

We define $\mathrm{Re}(E) = \inf\{\mathrm{Re}(s)\ ;\ (s,p) \in E\}$ and similarly $\mathrm{Im}(E) = \inf\{\mathrm{Im}(s)\ ;\ (s,p) \in E\}$.

If $\partial X = \emptyset$, we set $\mathcal{A}_{\mathrm{phg}}(X) = C^\infty(X)$. Then suppose X has boundary hypersurfaces $H_1, \ldots, H_N \in M_1(X)$. Let $\mathcal{E} = (E_1, \ldots, E_N) \subset (\mathbb{C} \times \mathbb{N}_0)^N$ be a vector of index sets corresponding to the boundary hypersurfaces, and for each k, let $\mathcal{E}(k)$ be the vector of index sets derived from $\mathcal{E}$ corresponding to boundary hypersurfaces of H_k (i.e. if H_k has a boundary hypersurface F_j such that F_j is a component of $H_k \cap H_l$ for some $l \neq k$, then $\mathcal{E}(k)_j = E_l$). We define the polyhomogeneous distributions with index sets $\mathcal{E}$ inductively, by setting

$$\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X) = \left\{ u \in C^{-\infty}(X)\ ;\ u \sim \sum_{(s,p)\in E_k} \rho_k^s (\log \rho_k)^p u_{s,p},\ u_{s,p} \in \mathcal{A}^{\mathcal{E}(k)}_{\mathrm{phg}}(H_k),\ \text{near } H_k \right\}.$$

Note the asymptotic notation, which implies that the expansion of u is only well-defined up to terms in $\dot{C}^\infty(X)$. Note also that the expansion depends on the choice of ρ_i. Provided $\mathrm{Re}(E)$ is finite, the *leading order* terms

$$\sum_{(\mathrm{Re}(E)\leq s<\mathrm{Re}(E)+1,p)} \rho_k^s (\log \rho_k)^p u_{s,p}$$

are well-defined over H_k, as a section of a line bundle $NH_k^{(s,p)}$ whose bounded sections are multiples of $\rho_k^s (\log \rho_k)^p$.

A.5 Densities

For any vector space V of dimension n, the s-densities on V consist of

$$\Omega^s(V) = \{u : \textstyle\bigwedge^n V^* \setminus 0 \longrightarrow \mathbb{R}\ ;\ u(tv) = |t|^s\ u(v)\}.$$

$\Omega^s(V)$ is a one dimensional vector space; if $\{e_i\}$ is a basis of V, then $|e_1 \wedge \cdot \wedge e_n|^s$ trivializes $\Omega^s(V)$. There are natural isomorphisms

$$\Omega^s(V) \otimes \Omega^t(V) \cong \Omega^{s+t}(V), \qquad \Omega^0(V) \cong \mathbb{R}.$$

With respect to a direct sum $V \oplus W$ we have

$$\Omega^s(V \oplus W) \cong \Omega^s(V) \otimes \Omega^s(W)$$

coming from the identification $\bigwedge^{n+l}(V^* \oplus W^*) = \bigwedge^n V^* \otimes \bigwedge^l W^*$ if $\dim(V) = n$, $\dim(W) = l$.

This construction is functorial, hence it can be transferred to the category of vector bundles. For a manifold X, the density bundle

$$\Omega^s(X) \longrightarrow X, \qquad \Omega^s(X)_p = \Omega^s(T_p^* X)$$

is a trivial line bundle for any s; in particular, for $s = 1$, integration is well-defined (provided X is compact):

$$\int_X : C^\infty(X; \Omega^1(X)) \longrightarrow \mathbb{R}. \tag{A.3}$$

In a similar vein, we define the b and scattering s-density bundles respectively by

$$\Omega^s_{\mathrm{b}}(X)_p = \Omega^s({}^{\mathrm{b}}T^*_pX), \qquad \text{and} \qquad \Omega^s_{\mathrm{sc}}(X)_p = \Omega^s({}^{\mathrm{sc}}T^*_pX).$$

When $s = 1$, integration as in (A.3) is defined (say for sections vanishing with all derivatives at ∂X).

For b densities, there is a natural restriction map to hypersurfaces

$$_{|H} : C^\infty(X; \Omega^s_{\mathrm{b}}(X)) \longrightarrow C^\infty(H; \Omega^s_{\mathrm{b}}(H))$$

coming from the canonical triviality of ${}^{\mathrm{b}}NH$ over the interior. In coordinates, this corresponds to canceling off a factor of $\left|\frac{dx}{x}\right|^s$ where x is a boundary defining coordinate for H.

Half densities play a special role. On any vector bundle $V \longrightarrow X$ with an inner product, we have a pairing

$$\dot{C}^\infty(X; V \otimes \Omega^{1/2}_{\mathrm{b}}(X)) \times \dot{C}^\infty(X; V \otimes \Omega^{1/2}_{\mathrm{b}}(X)) \longrightarrow \dot{C}^\infty(X; \Omega_{\mathrm{b}}(X))$$

which, composed with the integration map as in (A.3), gives a bilinear pairing

$$\dot{C}^\infty(X; V \otimes \Omega^{1/2}_{\mathrm{b}}(X)) \times \dot{C}^\infty(X; V \otimes \Omega^{1/2}_{\mathrm{b}}(X)) \longrightarrow \mathbb{R}, \quad (u, v) \longmapsto \int_X \langle u, v\rangle$$

whose completion defines the L^2 space $L^2(X; V \otimes \Omega^{1/2}_{\mathrm{b}})$. The case of scattering densities and true densities is analogous.

With respect to a fibration $f : X \longrightarrow Y$, with fiber F, we have the natural density identification

$$\Omega^s(X) \cong \Omega^s(Y) \otimes \Omega^s(F)$$

since, though the splitting $T_pX \cong T_pF \oplus T_pY$ is not canonical, the resulting isomorphism $\bigwedge^n(T_pX) \cong \bigwedge^l(T_pF) \otimes \bigwedge^m(T_pY)$ is independent of the choice of splitting. This allows us to construct the pushforward, or fiber integration, map

$$f_* : C^\infty(X; \Omega(X)) \longrightarrow C^\infty(Y; \Omega(Y)).$$

In the case of a b-fibration $f : X \longrightarrow Y$ between manifolds with corners, similar considerations give rise to the pushforward map in the b category,

$$f_* : \dot{C}^\infty(X; \Omega_{\mathrm{b}}(X)) \longrightarrow \dot{C}^\infty(Y; \Omega_{\mathrm{b}}(Y)),$$

whose completion to the space of polyhomogeneous distributions we shall discuss below.

A.6 Pullback and pushforward

Fundamental to the use of polyhomogeneous distributions on manifolds with corners are two results dictating their behavior with respect to pullback and pushforward operations. These are proved in [20].

We first define some operations on index sets. Given index sets E and F, their *extended* *union* is the index set

$$E\overline{\cup}F = \{(s,p)\ ;\ (s,p) \in E \text{ or } (s,p) \in F\} \cup \{(s,p+q+1)\ ;\ (s,p) \in E \text{ and } (s,q) \in F\}$$

Suppose X and Y have boundary hypersurfaces $H_1, \dots, H_N \in M_1(X)$ and $G_1, \dots G_M \in M_1(Y)$ respectively. Let $f : X \longrightarrow Y$ be a b-map, with boundary exponent matrix $e(i,j) = e(H_i, G_j)$, $1 \le i \le N$, $1 \le j \le M$. For a collection $\mathcal{F} = (F_1, \dots, F_M)$ of index sets on Y, we define

$$f^{\#}\mathcal{F} = (E_1, \dots, E_N), \quad \text{where } E_j = \left\{ \sum\nolimits_{e(i,j)\neq 0} (e(i,j)s_i, p_i)\ ;\ (s_i, p_i) \in F_i \right\}.$$

Suppose f is a b-fibration. We set

$$f_{\#}\mathcal{E} = (F_1, \dots, F_N), \quad \text{where } F_i = \overline{\bigcup}_{H_j \subset f^{-1}(G_i)} \left\{ \left(\frac{s}{e(i,j)}, p \right)\ ;\ (s,p) \in E_j \right\}.$$

For applications to pseudodifferential operators, we will combine the pullback and push-forward theorems for polyhomogeneous distributions with those for conormal (in the sense of Hörmander) distributions with respect to submanifolds. Recall that if $W \subset X$ is a submanifold, with $\dim(X) = n$, $\operatorname{codim}(W) = k$, the space $I^m(X, W)$ of conormal distributions of order m consists of distributions which have the local form

$$u(y) = \int e^{i\langle y', \xi' \rangle} a(y'', \xi')\, d\xi', \qquad a \in S^{m+(n-2k)/4}(\mathbb{R}^{n-k} \times \mathbb{R}^k; \mathbb{C})$$

where $(y) = (y', y'')$ such that $W = \{y'_i = 0\}_{i=1}^k$, and a is a symbol of order $m + (n-2k)/4$. This definition extends to the case where W is a p-submanifold of a manifold with corners. We let

$$\mathcal{A}^{\mathcal{E}}_{\text{phg}} I^m(X, W; V)$$

denote denote a space of distributional sections of $V \longrightarrow X$ with polyhomogeneous expansions at $\partial X \setminus W$, which have Hörmander-type conormal singularity of order m at W. This is well-defined, for instance by allowing $a(y'', \xi')$ to have an appropriate polyhomogeneous expansion as $y''_i \longrightarrow 0$ if $W \cap \partial X \neq \emptyset$.

Theorem A.1. *Let $f : X \longrightarrow Y$ be a b-map. Then the pullback $f^* : \dot{C}^\infty(Y) \longrightarrow \dot{C}^\infty(X)$ extends to a map*

$$f^* : \mathcal{A}^{\mathcal{F}}_{\text{phg}}(Y) \longrightarrow \mathcal{A}^{f^{\#}\mathcal{F}}_{\text{phg}}(X).$$

Let $Z \subset Y$ be a p-submanifold, with f transversal to Z. Then pullback extends to a map

$$f^* : \mathcal{A}^{\mathcal{F}}_{\text{phg}} I^m(Y, Z) \longrightarrow \mathcal{A}^{f^{\#}\mathcal{F}}_{\text{phg}} I^{m+(\dim(X)-\dim(Y))/4}(X, f^{-1}(Z)).$$

The following is a major motivation for the use of both polyhomogeneous distributions as well as b densities on manifolds with corners.

Theorem A.2. *Let $f : X \longrightarrow Y$ be a b-fibration. If $\operatorname{Re}(E_j) > 0$ whenever $f(E_j) \cap \mathring{Y} \neq \emptyset$, then the pushforward map $f_* : \dot{C}^\infty(X; \Omega_b(X)) \longrightarrow \dot{C}^\infty(Y; \Omega_b(Y))$ extends to a map*

$$f_* : \mathcal{A}^{\mathcal{E}}_{\text{phg}}(X; \Omega_b(X)) \longrightarrow \mathcal{A}^{f_{\#}\mathcal{E}}_{\text{phg}}(Y; \Omega_b(Y)).$$

Let $W \subset X$ be a p-submanifold. Then under the same conditions on $\mathcal{E}$,

$$f_* : \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}} I^m(X, W; \Omega_{\mathrm{b}}(X)) \longrightarrow \mathcal{A}^{f_\# \mathcal{E}}_{\mathrm{phg}} I^m(Y, f(W); \Omega_{\mathrm{b}}(X)).$$

Furthermore, if f is transversal to W, then

$$f_* : \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}} I^m(X, W; \Omega_{\mathrm{b}}(X)) \longrightarrow \mathcal{A}^{f_\# \mathcal{E}}_{\mathrm{phg}} (Y; \Omega_{\mathrm{b}}(X)).$$

Appendix B

b-sc transition calculus

We include here a self-contained summary of the b-sc transition calculus of pseudodifferential operators. This idea is due to Melrose and Sa Barretto, and was used by Guillarmou and Hassell in [16], which serves as a reference for this Appendix. The proofs here are brief and do not go into full detail; a complete construction of the transition calculus will be much longer.

Let X be a compact manifold with boundary, and let $\mathcal{V}_{\mathrm{b}}(X)$, $\mathcal{V}_{\mathrm{sc}}(X)$ denote the Lie algebras of b vector fields and scattering vector fields, respectively. Let $I = [0, 1)$ be a half open interval. The calculus is meant to microlocalize families of differential operators, parametrized by $\epsilon \in I$, which fail to be fully elliptic in the scattering sense precisely as $\epsilon \longrightarrow 0$, where we treat them as weighted b type operators, though the calculus allows us to analyze this behavior in something of a uniform way.

B.1 Single, Double, and Triple Spaces

Our operators are constructed as Schwartz kernels on $X^2 \times I$, acting on functions on $X \times I$, with composition occurring on $X^3 \times I$ all via pullback, multiplication and pushforward. In order to have a nice description of the singularities of these kernels, in terms of polyhomogeneous expansions, we consider blow ups of these spaces.

The *single space* is defined to be

$$X_t = [X \times I; \partial X \times \{0\}].$$

We name the boundary faces of X_t by

$$\begin{aligned} \mathrm{sf} &= \text{lift of } \partial X \times I \\ \mathrm{tf} &= \text{lift of } \partial X \times \{0\} \\ \mathrm{zf} &= \text{lift of } X \times \{0\} \end{aligned}$$

See Figure B-1.

The double space is defined in two steps. Let C_n denote the union of boundary faces of codimension n of $X^2 \times I$. Thus $C_3 = \partial X \times \partial X \times \{0\}$, while C_2 is a union of the faces $\partial X \times \partial X \times I$, $\partial X \times X \times \{0\}$ and $X \times \partial X \times \{0\}$. The *b blowup* or *total boundary blowup* is well-defined for any manifold with corners to be the blow up of all boundary faces in order

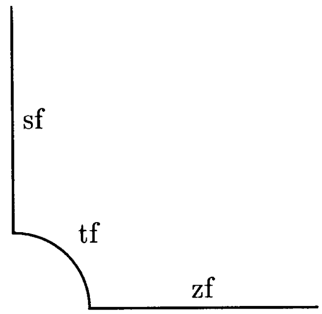

Figure B-1: The single space X_t and its boundary faces

of decreasing codimension. In our case we get

$$(X^2 \times I)_{\mathrm{b}} = [X^2 \times I; C_3, C_2]$$

Denote by C_V and Δ the lifts of $\partial X \times \partial X \times I$ and the fiber diagonal $\Delta \times I$, respectively. These intersect transversally in $(X^2 \times I)_{\mathrm{b}}$ and we define the *double space* to be

$$X_t^2 = [(X^2 \times I)_{\mathrm{b}}; C_V \cap \Delta]$$

We denote by $\beta_2 : X_t^2 \longrightarrow X^2 \times I$ the composite blow down map, and we name the boundary faces of X_t^2 by

$$\begin{aligned} \mathrm{sc} &= \text{lift of } C_V \cap \Delta & \mathrm{bf} &= \text{lift of } \partial X \times \partial X \times I \\ \mathrm{bf}_0 &= \text{lift of } C_3 & \mathrm{lb}_0 &= \text{lift of } X \times \partial X \times \{0\} \\ \mathrm{rb}_0 &= \text{lift of } \partial X \times X \times \{0\} & \mathrm{lb} &= \text{lift of } X \times \partial X \times I \\ \mathrm{rb} &= \text{lift of } \partial X \times X \times I & \mathrm{zf} &= \text{lift of } X \times X \times \partial I \end{aligned}$$

See Figure B-2.

The triple space is similarly defined in two steps. We start with the b blowup of $X^3 \times I$; again letting C_n denote the union of boundary faces of codimension n, we have

$$(X^3 \times I)_{\mathrm{b}} = [M^3; C_4, C_3, C_2].$$

Note that it is well-defined since, first of all, $C_4 = \partial X^3 \times \{0\}$ is a single face, and all the boundary faces of codimension $n-1$ become separated after blowing up C_n. Now consider the b-fibrations $\pi_{L/R/C} : (X^3 \times I)_{\mathrm{b}} \longrightarrow (X^2 \times I)_b$. The double space X_t^2 is obtained from $(X^2 \times I)_{\mathrm{b}}$ by blowing up the intersection $C_V \cap \Delta$, which has preimage under $\pi_{L/R/C}$ in two boundary faces. Let G_L denote the preimage of $C_V \cap \Delta$ with respect to π_L intersecting the preimage of $(\partial X \times X^2 \times I)$ in $(X^3 \times I)_{\mathrm{b}}$ with respect to the blowup β, and J_L denote the preimage of $C_V \cap \Delta$ intersecting the preimage of $(\mathring{X} \times X^2 \times I)$. Define $G_{R/C}$ and $J_{R/C}$ similarly. Some thought reveals that $G_L \cap G_R \cap G_C = K$ is a nonempty submanifold, while the J_* only intersect the corresponding G_*.

Finally, we set

$$X_t^3 = [(X^3 \times I)_{\mathrm{b}}; K, G_L, G_R, G_C, J_L, J_R, J_C]$$

which is well-defined since the G_* are separated after blowing up K.

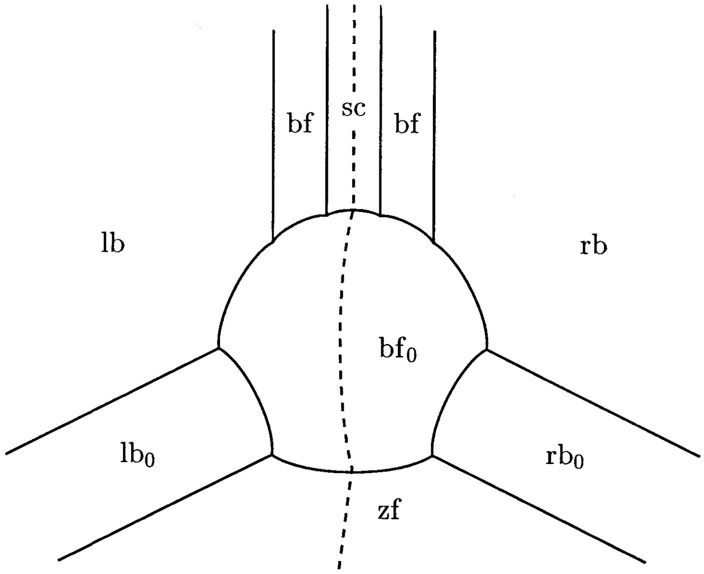

Figure B-2: The double space X_t^2 and its boundary faces

The important features of these spaces is that they lift the obvious projections to b-fibrations. A proof of the following theorem can be found in [16].

Theorem B.1. *There are b-fibrations π_* making the following diagram commute*

$$\begin{array}{ccccccc} X_t^3 & \overset{\pi_{L,R,C}}{\Rrightarrow} & X_t^2 & \overset{\pi_{L,R}}{\rightrightarrows} & X_t & \overset{\pi_X}{\longrightarrow} & X \\ \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_1} & \overset{\pi_I}{\times} & \\ X^3 \times I & \overset{\pi_{L,R,C}}{\Rrightarrow} & X^2 \times I & \overset{\pi_{L,R}}{\rightrightarrows} & X \times I & \longrightarrow & I \end{array} \tag{B.1}$$

Remark. We shall "overload" the notation for projections, so that for instance, $\pi_I : X_t^3 \longrightarrow I$ should mean the unique b-fibration lifting $X^3 \times I \longrightarrow I$, which by the above theorem is given by any appropriate composition.

We also wish to single out some submanifolds of X_t^2, namely, we have the obvious isomorphism

$$\mathrm{zf} \cong X_\mathrm{b}^2$$

between the "zero face" and the b double space. Also, for $\epsilon \in (0,1) \subset I$, the submanifold $\pi_I^{-1}(\epsilon)$ is well-defined, and we have the isomorphism

$$\pi_I^{-1}(\epsilon) \cong X_\mathrm{sc}^2, \quad \epsilon > 0$$

between this submanifold and the scattering double space X_sc^2. Finally, we have the isomorphism

$$\Delta \cong X_t$$

between the lifted diagonal Δ, and the single space X_t.

Similarly, on the triple space, we have the identification

$$X_t^3 \supset \pi_I^{-1}(\epsilon) \cong X_{\mathrm{sc}}^3, \quad \epsilon > 0.$$

B.2 Densities

We shall use primarily b half densities for our kernels, in order to facilitate the invocation of the pushforward theorem for polyhomogeneous conormal distributions. We have the obvious identifications

$$\begin{aligned}
&\Omega_{\mathrm{b}}^{1/2}(X \times I) \cong \pi_X^*(\Omega_{\mathrm{b}}^{1/2}(X)) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{1/2}(I)) \\
&\Omega_{\mathrm{b}}^{1/2}(X^2 \times I) \cong \pi_{X,L}^*(\Omega_{\mathrm{b}}^{1/2}(X)) \otimes \pi_{X,R}^*(\Omega_{\mathrm{b}}^{1/2}(X)) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{1/2}(I)) \\
&\Omega_{\mathrm{b}}^{1/2}(X^3 \times I) \cong \pi_{X,L}^*(\Omega_{\mathrm{b}}^{1/2}(X)) \otimes \pi_{X,C}^*(\Omega_{\mathrm{b}}^{1/2}(X)) \otimes \pi_{X,R}^*(\Omega_{\mathrm{b}}^{1/2}(X)) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{1/2}(I))
\end{aligned}$$

where $\pi_{X,R}$, etc. are shorthand for $\pi_X \circ \pi_R$ and so on.

Lemma B.1. *On X_t^2, we have a natural identification*

$$\pi_R^*(\Omega_{\mathrm{b}}^{1/2}(X_t)) \otimes \pi_L^*(\Omega_{\mathrm{b}}^{1/2}(X_t)) \cong \rho_{\mathrm{sc}}^{n/2}\Omega_{\mathrm{b}}^{1/2}(X_t^2) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{1/2}(I)).$$

Similarly, on X_t^3 we can identify

$$\pi_R^*(\Omega_{\mathrm{b}}^{1/2}(X_t^2)) \otimes \pi_C^*(\Omega_{\mathrm{b}}^{1/2}(X_t^2)) \otimes \pi_L^*(\Omega_{\mathrm{b}}^{1/2}(X_t^2)) \cong (\rho_{\mathrm{sc},R}\rho_{\mathrm{sc},L}\rho_{\mathrm{sc},C})^{n/2}\Omega_{\mathrm{b}}(X_t^3) \otimes \pi_I^*(\Omega_{\mathrm{b}}^2(I))$$

where $\rho_{\mathrm{sc},R/L/C}$ denote boundary defining functions for the faces obtained by blowing up $G_{R/L/C}$ and $J_{R/L/C}$ in the process of obtaining X_t^3.

Proof. First we will show that

$$\beta^*(\Omega_{\mathrm{b}}^{1/2}(X^2 \times I)) \cong \rho_{\mathrm{sc}}^{n/2}\Omega_{\mathrm{b}}^{1/2}(X_t^2),$$

where $\beta : X_t^2 \longrightarrow X^2 \times I$ is the blow down.

For the blowup of any p-submanifold $Y \subset X$, a local coordinate computation shows that

$$\beta^*(\Omega^{1/2}(X)) = \rho_{\mathrm{ff}}^{(d-1)/2}\Omega^{1/2}([X;Y])$$

where ρ_{ff} is a bdf for the front face (the lift of Y), and $d = \dim(X) - \dim(Y)$ is the codimension. In particular, under boundary blowup, b half densities lift to b half densities:

$$\beta^*(\Omega_{\mathrm{b}}^{1/2}(X)) = \Omega_{\mathrm{b}}^{1/2}([X;H]), \quad \text{if } H \subset X \text{ is a boundary face.}$$

With this we conclude that

$$\beta^*(\Omega_{\mathrm{b}}^{1/2}(X^2 \times I)) \cong \Omega_{\mathrm{b}}^{1/2}((X^2 \times I)_b)$$

so it suffices to examine the lift under the final blow up. $C_V \cap \Delta$ has codimension $n+1$ and sits inside a boundary face of codimension 1. Using that $\Omega_{\mathrm{b}}^{1/2}((X^2 \times I)_{\mathrm{b}}) \cong \rho^{-1/2}\Omega((X^2 \times I)_{\mathrm{b}})^{1/2}$ where ρ is a total boundary defining function, we see that

$$\beta^*\Omega_{\mathrm{b}}^{1/2}((X^2 \times I)_{\mathrm{b}}) \cong \rho^{-1/2}\rho_{\mathrm{sc}}^{n/2}\Omega(X_t^2)^{1/2} = \rho_{\mathrm{sc}}^{n/2}\Omega_{\mathrm{b}}^{1/2}(X_t^2).$$

To prove the Lemma, it suffices to verify the simpler statement that

$$\Omega_{\mathrm{b}}^{1/2}(X^2 \times I) \cong \pi_R^* \left(\Omega_{\mathrm{b}}^{1/2}(X \times I)\right) \otimes \pi_L^* \left(\Omega_{\mathrm{b}}^{1/2}(X \times I)\right) \otimes \pi_I^* \left(\Omega_{\mathrm{b}}^{-1/2}(I)\right)$$

by the commutativity of the diagram in Theorem B.1. As these spaces are products, at a point $(p, q, \epsilon) \in X^2 \times I$, the right hand side is isomorphic to

$$\begin{aligned}\Omega_{\mathrm{b}}^{1/2}(X)_p \otimes \Omega_{\mathrm{b}}^{1/2}(X)_q \otimes \left(\Omega_{\mathrm{b}}^{1/2}(I)_\epsilon\right)^2 \otimes \Omega_{\mathrm{b}}^{-1/2}(I)_\epsilon &= \Omega_{\mathrm{b}}^{1/2}(X)_p \otimes \Omega_{\mathrm{b}}^{1/2}(X)_q \otimes \Omega_{\mathrm{b}}^{1/2}(I)_\epsilon \\ &\cong \Omega_{\mathrm{b}}^{1/2}(X^2 \times I)_{(p,q,\epsilon)},\end{aligned}$$

verifying the first claim. The second claim is similarly proved by considering the un-blown up space $X^3 \times I$ and maps to $X^2 \times I$. □

Let the *kernel density bundle* $\omega_{\mathrm{kd}} \longrightarrow X_t^2$ be defined by

$$\omega_{\mathrm{kd}} = \rho_{\mathrm{sc}}^{-n/2} \Omega_{\mathrm{b}}^{1/2}(X_t^2).$$

This convention normalizes the densities in such a way that the kernel of the identity operator on b half densities has smooth asymptotic expansion of order 0 at all boundary faces meeting the lifted diagonal.

Lemma B.2. *The restriction of the kernel density bundle gives the following identifications*

$$\begin{aligned}(\omega_{\mathrm{kd}})_{|\mathrm{zf}} &\cong \Omega_{\mathrm{b}}^{1/2}(X_b^2) \\ (\omega_{\mathrm{kd}})_{|\pi_I^{-1}(e)} &\cong \rho_{\mathrm{sc}}^{-n/2} \Omega_{\mathrm{b}}^{1/2}(X_{\mathrm{sc}}^2) \cong \Omega_{\mathrm{sc}}^{1/2}(X_{\mathrm{sc}}^2), \quad \epsilon > 0 \\ (\omega_{\mathrm{kd}})_{|\Delta} &\cong \rho_{\mathrm{sf}}^{-n} \Omega_{\mathrm{b}}((X_t)_{\mathrm{fib}}) \otimes \Omega_{\mathrm{b}}^{1/2}(I)\end{aligned}$$

where $(X_t)_{\mathrm{fib}}$ denotes the (generalized) fiber of the b-fibration $\pi_I : X_t \longrightarrow I$.

Remark. The last claim says that, restricted to the diagonal, the kernel density bundle looks like an ordinary density in the "X" directions, and a half density in the "I" directions. The factor ρ_{sf}^{-n} accounts for the fact that the operators act like scattering operators for $\epsilon > 0$, where $\rho_{\mathrm{sf}}^{-n} \Omega_{\mathrm{b}}(X) \cong \Omega_{\mathrm{sc}}(X)$.

Proof. Recall from Appendix A that the restriction of b half densities to boundary faces is well-defined, corresponding locally to the cancellation of a boundary defining factor. Thus, on a neighborhood U of zf, where $(\omega_{\mathrm{kd}})_{|U} \cong \Omega_{\mathrm{b}}^{1/2}(U)$, we have $(\omega_{\mathrm{kd}})_{|\mathrm{zf}} \cong \Omega_{\mathrm{b}}^{1/2}(\mathrm{zf}) \cong \Omega_{\mathrm{b}}^{1/2}(X_b^2)$, using the diffeomorphism $\mathrm{zf} \cong X_b^2$.

Along the slice $\pi_I^{-1}(\epsilon), \epsilon > 0$, we have

$$\Omega_{\mathrm{b}}^{1/2}(X_t^2)_{|\pi_I^{-1}(\epsilon)} \cong \Omega_{\mathrm{b}}^{1/2}(\pi_I^{-1}(\epsilon)) \otimes \Omega_{\mathrm{b}}^{1/2}(I) \cong \Omega_{\mathrm{b}}^{1/2}(\pi_I^{-1}(\epsilon))$$

using the trivialization of $\Omega_{\mathrm{b}}^{1/2}(I)$ provided by ϵ, namely $\left|\frac{d\epsilon}{\epsilon}\right|^{1/2} : I \longrightarrow \Omega_{\mathrm{b}}^{1/2}(I)$. The rest follows from the diffeomorphism $\pi_I^{-1}(\epsilon) \cong X_{\mathrm{sc}}^2$.

For the last claim, we note that, near $\mathrm{zf} \cap \Delta$,

$$\begin{aligned}\Omega_{\mathrm{b}}^{1/2}(X_t^2) &\cong \Omega_{\mathrm{b}}^{1/2}(X) \otimes \Omega_{\mathrm{b}}^{1/2}(N\Delta) \otimes \Omega_{\mathrm{b}}^{1/2}(I) \\ &\cong \Omega_{\mathrm{b}}^{1/2}(X) \otimes \Omega_{\mathrm{b}}^{1/2}({}^{\mathrm{b}}TX) \otimes \Omega_{\mathrm{b}}^{1/2}(I) \\ &\cong \Omega_{\mathrm{b}}(X) \otimes \Omega_{\mathrm{b}}^{1/2}(I).\end{aligned}$$

Similarly, near $\mathrm{sc} \cap \Delta$, we have

$$\begin{aligned}\omega_{\mathrm{kd}} &= \rho_{\mathrm{sc}}^{-n/2}\Omega_{\mathrm{b}}^{1/2}(X) \otimes \Omega_{\mathrm{b}}^{1/2}(N\Delta) \otimes \Omega_{\mathrm{b}}^{1/2}(I) \cong \Omega_{\mathrm{sc}}^{1/2}(X) \otimes \Omega_{\mathrm{sc}}^{1/2}(N\Delta) \otimes \Omega_{\mathrm{b}}^{1/2}(I) \\ &\cong \Omega_{\mathrm{sc}}^{1/2}(X) \otimes \Omega_{\mathrm{sc}}^{1/2}({}^{\mathrm{sc}}TX) \otimes \Omega_{\mathrm{b}}^{1/2}(I) \cong \Omega_{\mathrm{sc}}(X) \otimes \Omega_{\mathrm{b}}^{1/2}(I) \\ &= \rho_{\partial X}^{-n}\Omega_{\mathrm{b}}(X) \otimes \Omega_{\mathrm{b}}^{1/2}(I),\end{aligned}$$

from which the proposition follows. □

B.3 The Calculus

Fix a vector bundle $V \longrightarrow X$, and denote also by $V \longrightarrow X_t^i$ $i \in \{1,2,3\}$ the pullback of V to the single, double and triple spaces. Then the *b-sc transition pseudodifferential operators* are defined to be

$$\begin{aligned}\Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) &= \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}} I^m(X_t^2, \Delta; \mathrm{End}(V) \otimes \omega_{\mathrm{kd}}), \\ \mathcal{E} &= (E_{\mathrm{zf}}, E_{\mathrm{bf_0}}, E_{\mathrm{sc}}, E_{\mathrm{lb_0}}, E_{\mathrm{rb_0}}, \infty_{\mathrm{lb}}, \infty_{\mathrm{rb}}, \infty_{\mathrm{bf}})\end{aligned}$$

where $I^m(X_t^2, \Delta)$ denotes the space of distributions conormal to Δ in the sense of Hörmander, with symbol order m, and the $\mathcal{A}_{\mathrm{phg}}^{\mathcal{E}}$ means the distributions are meant to be classical conormal (see Appendix A) with respect to the boundary faces of X_t^2 with polyhomogeneous expansions given by the index sets $\mathcal{E}$. In particular, these distributions always vanish to infinite order at lb, rb, and bf, which is what is meant by the special index set ∞.

Also, the vector bundle $\mathrm{End}(V) \longrightarrow X_t^2$ is notational shorthand for the vector bundle $\mathrm{End}(\pi_R^* V, \pi_L^* V) \longrightarrow X_t^2$.

A distinguished subclass of these operators form the *small calculus*

$$\begin{aligned}\Psi_t^{m,(e_{\mathrm{zf}}, e_{\mathrm{bf_0}}, e_{\mathrm{sc}})}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) &= \Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}), \\ &\text{with } \mathcal{E} = (e_{\mathrm{zf}}, e_{\mathrm{bf_0}}, e_{\mathrm{sc}}, \infty_{\mathrm{lb_0}}, \infty_{\mathrm{rb_0}}, \infty_{\mathrm{lb}}, \infty_{\mathrm{rb}}, \infty_{\mathrm{bf}})\end{aligned}$$

where $e_i \in \mathbb{Z}$ are identified with the "smooth" index sets $\{(e_i + k, 0)\ ;\ k \in \mathbb{N}\}$.

Let $\nu = \left|\frac{d\epsilon}{\epsilon}\right| \in C^\infty(I, \Omega_{\mathrm{b}}(I))$ be the canonical trivializing section of $\Omega_{\mathrm{b}}(I) \longrightarrow I$; in particular ν^s trivializes $\Omega_{\mathrm{b}}^s(I) \longrightarrow I$ for any $s \in \mathbb{R}$.

We define the action of $P \in \Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$ on $u \in \dot{C}^\infty(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$ by

$$Pu = (\pi_L)_* \left(\kappa_P \cdot \pi_R^*(u)\, \pi_I^*(\nu^{-1/2})\right),$$

where κ_P is the Schwartz kernel of P.

Proposition B.1. *The above composition is well-defined, and extends to an operation*

$$\Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \cdot \mathcal{A}_{\mathrm{phg}}^{\mathcal{F}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \subset \mathcal{A}_{\mathrm{phg}}^{\mathcal{G}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$$

where

$$\begin{aligned} G_{\mathrm{sf}} &= F_{\mathrm{sf}} + E_{\mathrm{sc}} \\ G_{\mathrm{tf}} &= (F_{\mathrm{tf}} + E_{\mathrm{bf}_0}) \,\overline{\cup}\, (F_{\mathrm{zf}} + E_{\mathrm{rb}_0}) \\ G_{\mathrm{zf}} &= (F_{\mathrm{zf}} + E_{\mathrm{zf}}) \,\overline{\cup}\, (F_{\mathrm{tf}} + E_{\mathrm{lb}_0}) \end{aligned}$$

Remark. In particular, the small calculus maps $\mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}$ to $\mathcal{A}^{\mathcal{G}}_{\mathrm{phg}}$ with

$$\begin{aligned} G_{\mathrm{sf}} &= F_{\mathrm{sf}} + E_{\mathrm{sc}} \\ G_{\mathrm{tf}} &= F_{\mathrm{tf}} + E_{\mathrm{bf}_0} \\ G_{\mathrm{zf}} &= F_{\mathrm{zf}} + E_{\mathrm{zf}} \end{aligned}$$

Proof. This is a direct consequence of the pullback and pushforward Theorems A.1 and A.2 for polyhomogeneous distributions. Indeed, Taking $u \in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$, we have

$$\kappa_P \cdot \pi_R^*(u)\, \pi_I^*(\nu^{-1/2}) \in \mathcal{A}^{\mathcal{H}}_{\mathrm{phg}} I^m(X_t^2, \Delta; V \otimes \rho_{\mathrm{sc}}^{-n/2} \Omega_{\mathrm{b}}^{1/2}(X_t^2) \otimes \pi_R^* \Omega_{\mathrm{b}}^{1/2}(X_t) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{-1/2}(I)))$$

where

$$\begin{aligned} H_{\mathrm{sc}} &= E_{\mathrm{sc}} + F_{\mathrm{sf}} & H_{\mathrm{bf}} &= E_{\mathrm{bf}} + F_{\mathrm{sf}} = \infty + F_{\mathrm{sf}} = \infty \\ H_{\mathrm{lb}} &= E_{\mathrm{lb}} + F_{\mathrm{sf}} = \infty + F_{\mathrm{sf}} = \infty & H_{\mathrm{bf}_0} &= E_{\mathrm{bf}_0} + F_{\mathrm{tf}} \\ H_{\mathrm{lb}_0} &= E_{\mathrm{lb}_0} + F_{\mathrm{tf}} & H_{\mathrm{rb}_0} &= E_{\mathrm{rb}_0} + F_{\mathrm{zf}} \\ H_{\mathrm{zf}} &= E_{\mathrm{zf}} + F_{\mathrm{zf}}. \end{aligned}$$

We obtain the index sets G_i from the pushforward theorem, since

$$\begin{aligned} \pi_L^{-1}(\mathrm{sf}) &= \mathrm{bf} \cup \mathrm{sc} \cup \mathrm{rb} \\ \pi_L^{-1}(\mathrm{tf}) &= \mathrm{bf}_0 \cup \mathrm{rb}_0 \\ \pi_L^{-1}(\mathrm{zf}) &= \mathrm{lb}_0 \cup \mathrm{zf}, \end{aligned}$$

and the interior conormal singularity is killed since π_L is transversal to Δ.

It remains to verify that the density bundles behave as expected. The claim is that the pushforward under π_L of $\kappa_P\, \pi_R^*(u)\pi_I^*(\nu^{-1/2})$ can be identified with a section of $\Omega_{\mathrm{b}}^{1/2}(X_t) \otimes V$; an equivalent claim is that it pairs with $\Omega_{\mathrm{b}}^{1/2}(X_t)$ to produce an element of $\Omega_{\mathrm{b}}(X_t) \otimes V$. Thus let $\gamma \in C^\infty(X_t; \Omega_{\mathrm{b}}^{1/2}(X_t))$ and consider

$$\gamma\, (\pi_L)_* \left(\kappa_P \cdot \pi_R^*(u)\pi_I^*(\nu^{-1/2})\right) = (\pi_L)_* \left(\pi_L^*(\gamma)\, \kappa_P \cdot \pi_R^*(u)\pi_I^*(\nu^{-1/2})\right).$$

The element in parentheses on the right hand side is a section of

$$\pi_L^*(\Omega_{\mathrm{b}}^{1/2}(X_t)) \otimes \pi_R^*(\Omega_{\mathrm{b}}^{1/2}(X_t)) \otimes \rho_{\mathrm{sc}}^{-n/2} \Omega_{\mathrm{b}}^{1/2}(X_t^2) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{-1/2}(I)) \otimes V,$$

By Lemma B.1, we can identify this with the bundle

$$\rho_{\mathrm{sc}}^{n/2} \Omega_{\mathrm{b}}^{1/2}(X_t^2) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{1/2}(I)) \otimes \rho_{\mathrm{sc}}^{-n/2} \Omega_{\mathrm{b}}^{1/2}(X_t^2) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{-1/2}(I)) \otimes V \cong \Omega_{\mathrm{b}}(X_t^2) \otimes V,$$

and $(\pi_L)_*$ maps this into $\Omega_{\mathrm{b}}(X_t) \otimes V$ as claimed. We were a bit cavalier with the bundle

V for notational simplicity, but this also works out, since $\pi_R^*(u)$ gives a section of $\pi_R^* V$ and κ_P is a section of $\mathrm{End}(V) = \mathrm{End}(\pi_R^* V, \pi_L^* V)$, so they evaluate to a section of $\pi_L^* V$. $\square$

We define the composition of operators by pulling back to the triple space, multiplying and pushing forward:

$$\kappa_{P\circ Q} = (\pi_C)_* \left(\pi_L^*(\kappa_P) \cdot \pi_R^*(\kappa_Q)\, \pi_I^*(\nu^{-2})\right)$$

Proposition B.2. *The composition of operators is well-defined, with*

$$\Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \circ \Psi_t^{m',\mathcal{F}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \subset \Psi_t^{m+m',\mathcal{G}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$$

where

$$\begin{aligned}
G_{\mathrm{sc}} &= E_{\mathrm{sc}} + F_{\mathrm{sc}} \\
G_{\mathrm{zf}} &= (E_{\mathrm{zf}} + F_{\mathrm{zf}}) \overline{\cup} (E_{\mathrm{rb}_0} + F_{\mathrm{lb}_0}) \\
G_{\mathrm{bf}_0} &= (E_{\mathrm{bf}_0} + F_{\mathrm{bf}_0}) \overline{\cup} (E_{\mathrm{lb}_0} + F_{\mathrm{rb}_0}) \\
G_{\mathrm{lb}_0} &= (E_{\mathrm{lb}_0} + F_{\mathrm{zf}}) \overline{\cup} (E_{\mathrm{bf}_0} + F_{\mathrm{lb}_0}) \\
G_{\mathrm{rb}_0} &= (E_{\mathrm{zf}} + F_{\mathrm{rb}_0}) \overline{\cup} (E_{\mathrm{rb}_0} + F_{\mathrm{bf}_0})
\end{aligned}$$

Remark. In particular, the small calculus composes as

$$\Psi_t^{m,(e_{\mathrm{zf}}, e_{\mathrm{bf}_0}, e_{\mathrm{sc}})} \circ \Psi_t^{m',(f_{\mathrm{zf}}, f_{\mathrm{bf}_0}, f_{\mathrm{sc}})} \subset \Psi_t^{m+m',(e_{\mathrm{zf}}+f_{\mathrm{zf}}, e_{\mathrm{bf}_0}+f_{\mathrm{bf}_0}, e_{\mathrm{sc}}+f_{\mathrm{sc}})}.$$

Proof. First, we note that the pushforward under π_C of $\pi_L^*(\kappa_P) \cdot \pi_R^*(\kappa_Q)\pi_I^*(\nu^{-2})$ can be identified with an element of the kernel density bundle on X_t^2 by pulling back a b half density $\gamma \in C^\infty(X_t^2, \Omega_{\mathrm{b}}^{1/2})$ by π_C and examining

$$(\pi_C)_* \left(\pi_C^*(\gamma)\pi_L^*(\kappa_P) \cdot \pi_R^*(\kappa_Q)\pi_I^*(\nu^{-2})\right).$$

By (B.1) in Lemma B.1, this is naturally an element of the bundle $\rho_{\mathrm{sc},C}^{-n/2}\Omega_{\mathrm{b}}(X_t^3)$, since the term $\pi_R^*(\rho_{\mathrm{sc}})^{-n/2} \cdot \pi_L^*(\rho_{\mathrm{sc}})^{-n/2}$ from the kernel density bundles of P and Q contributes terms $\rho_{\mathrm{sc},R}^{-n/2}\rho_{\mathrm{sc},L}^{-n/2}\rho_{\mathrm{sc},C}^{-n}$ on X_t^2.

As in the classical case, the interior conormal singularities of κ_P and κ_Q compose transversally, and the only contribution to survive the pushforward comes from $\pi_C^{-1}(\Delta)$, since the conormal singularity everywhere else is transversal to π_C.

Finally, the index sets are determined by the pushforward theorem, once we identify the relationships between the inverse images under $\pi_{L/R/C}$ of the boundary hypersurfaces of X_t^2, in X_t^3. In fact, as all boundary faces in question are the lifts of boundary faces of the product spaces $X^2 \times I$ and $X^3 \times I$ under blowups, it suffices to consider the maps $\pi_{L/R/C} : X^3 \times I \longrightarrow X^2 \times I$, using commutativity of (B.1).

For instance, $\mathrm{bf}_0 \subset X_t^2$ is the lift under β of the face $\partial X \times \partial X \times \{0\} \subset X^2 \times I$. So consider $\pi_C^{-1}(\partial X \times \partial X \times \{0\}) \subset X^3 \times I$. This consists of two boundary faces[1], namely

$$\partial X \times \partial X \times \partial X \times \{0\} \quad \text{and} \quad \partial X \times X \times \partial X \times \{0\}.$$

[1]Though one face is included in the other in $X^3 \times I$, this inclusion relationship is not preserved under blowup to X_t^3, so we consider them separately.

The first face projects down to $\partial X \times \partial X \times \{0\}$ under both π_L and π_R, corresponding to the face $\mathrm{bf}_0 \subset X_t^2$, while the second face projects to $\partial X \times X \times \{0\}$ under π_L (corresponding to $\mathrm{lb}_0 \subset X_t^2$) and to $X \times \partial X \times \{0\}$ under π_R (corresponding to $\mathrm{rb}_0 \subset X_t^2$). From this we conclude that

$$G_{\mathrm{bf}_0} = (E_{\mathrm{bf}_0} + F_{\mathrm{bf}_0}) \overline{\cup} (E_{\mathrm{lb}_0} + F_{\mathrm{rb}_0}) .$$

The index sets for the other faces are obtained similarly. □

B.4 Normal operators and symbols

As with any pseudodifferential operator theory on manifolds with corners, given $P \in \Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_b^{1/2})$ with kernel κ_P, the leading order term in the asymptotic expansion of κ_P at the boundary faces intersecting the lifted diagonal play a special role.

Recall that for a boundary face $F \subset M$ of a manifold with corners and a polyhomogeneous section $u \in \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}}(M; V)$ of a vector bundle, the "restriction" of u to F is well defined, as a section

$$u_{|F} \in \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}_F}(F; V \otimes NF^{\mathrm{top}(E_F)})$$

where NF is the normal bundle to F and $NF^{(s,p)}, s \in \mathbb{C}, p \in \mathbb{N}$ is the (trivial) line bundle with sections $\rho^s(\log(\rho))^p$ where ρ is a positive section of NF. The index set E_F corresponding to the face F, $\mathrm{top}(E_F) = \{(s,p) \in E_F \ ; \ \mathrm{Re}(s) = \inf \mathrm{Re}(E_F), p = \max(q) \text{ s.t. } (s,q) \in E_F\}$. Finally, $\mathcal{E}_F$ refers to the vector of index sets corresponding to faces intersecting F.

Thus unless $\mathrm{top}(E_F) = (0,0)$, the restriction map is a section of a bundle which, though trivial, is not canonically so. Nonetheless, we have the identifications $NF^{(s,p)} \otimes NF^{(t,q)} = NF^{(s+t,p+q)}$, and of course any choice of boundary defining function ρ_F provides a natural trivialization. We shall suppress this bundle from the notation which follows, though it is important to bear in mind that it is there.

Definition. *Given $P \in \Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_b^{1/2})$, let*

- *$N_t(P)$ be the restriction of κ_P to* bf_0,
- *$N_{\mathrm{sc}}(P)$ be the restriction of κ_P to* sc,
- *$\sigma_{\mathrm{sc}}(P) = \widehat{N_{\mathrm{sc}}(P)}$ be the fiberwise Fourier transform of $N_{\mathrm{sc}}(P)$ with respect to the fibration* $\mathrm{sc} \longrightarrow \partial X \times I$,
- *$S_\delta(P)$ be the restriction of κ_P to $\pi_I^{-1}(\delta), \delta > 0$, and*
- *$S_0(P)$ be the restriction of κ_P to* zf.

The $N_*(P)$ and $S_*(P)$ are distributions conormal to the intersection of the lifted diagonal, Δ, with the submanifold in question; the restrictions are well-defined as these submanifolds are transversal to Δ. In the case of restriction to the boundary faces, we mean to take the leading order term in the asymptotic expansion, as described above.

The composition theorem allows us to interpret these terms as various model operators.

Proposition B.3. *With* $P \in \Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_{\rm b}^{1/2}), Q \in \Psi_t^{m',\mathcal{F}}(X_t; V \otimes \Omega_{\rm b}^{1/2})$, *we have identifications*

$$\begin{aligned}
&\sigma_{\rm sc}(P) \in C^\infty({}^{\rm sc}T^*_{\partial X}X \times I; \operatorname{End}(V) \otimes \Omega_{\rm sc}^{1/2})) \\
&N_{\rm t}(P) \in \Psi_{\rm b,sc}^{m,(E_{\rm zf}, E_{\rm lb_0}, E_{\rm rb_0}), E_{\rm sc}}(\partial X \times {}_{\rm b}[0,1]_{\rm sc}; V \otimes \Omega_{\rm b}^{1/2}) \\
&S_0(P) \in \Psi_{\rm b}^{m,(E_{\rm bf_0}, E_{\rm lb_0}, E_{\rm rb_0})}(X; V \otimes \Omega_{\rm b}^{1/2}) \\
&S_\delta(P) \in \Psi_{\rm sc}^{m,E_{\rm sc}}(X; V \otimes \Omega_{\rm b}^{1/2}) \quad \delta > 0
\end{aligned}$$

With respect to composition, we have

$$\begin{aligned}
\sigma_{\rm sc}(P \circ Q) &= \sigma_{\rm sc}(P)\sigma_{\rm sc}(Q) \\
N_{\rm t}(P \circ Q) &= N_{\rm t}(P) \circ N_{\rm t}(Q) \\
S_0(P \circ Q) &= S_0(P) \circ S_0(Q) \\
S_\delta(P \circ Q) &= S_\delta(P) \circ S_\delta(Q)
\end{aligned}$$

provided $\operatorname{Re}(E_{\rm rb_0} + F_{\rm lb_0}) > \operatorname{Re}(E_{\rm zf} + F_{\rm zf})$ *in the case of* $S_0(P \circ Q)$, *and provided* $\operatorname{Re}(E_{\rm lb_0} + F_{\rm rb_0}) > \operatorname{Re}(E_{\rm bf_0} + F_{\rm bf_0})$ *in the case of* $N_{\rm t}(P \circ Q)$.

Remark. $N_{\rm bf_0}(P)$ is an operator on $\partial X \times {}_{\rm b}[0,1]_{\rm sc}$ which is of b type near 0 and scattering type near 1. It has index sets $(E_{\rm zf}, E_{\rm lb_0}, E_{\rm rb_0})$ as a b operator, and the index set $E_{\rm sc}$ as a scattering operator (in particular this index set will usually be smooth, indicating an integral rate of decay as $[0,1] \ni s \longrightarrow 1$).

$\sigma_{\rm sc}(P)$ is just a family of scattering symbols, parametrized smoothly by $\epsilon \in (0,1)$.

Proof. We noted above that, in the triple space, $\pi_I^{-1}(\epsilon) \cong X^3_{\rm sc}$ for $\epsilon > 0$; so in fact for fixed, positive ϵ, composition is exactly the composition of scattering operators. From this, the characterizations of $\sigma_{\rm sc}$ and $S_\delta, \delta > 0$ follow immediately.

Consider then the slice $S_0(P)$. From the identification $\mathrm{zf} \cong X^2_{\rm b}$, we can view this as a Schwartz kernel of a b operator on X, and we need to show that $S_0(P \circ Q) = S_0(P) \circ S_0(Q)$ under appropriate conditions. As $\mathrm{zf} \subset X^2_t$ is the lift of $X \times X \times \{0\} \subset X^2 \times I$, we consider the inverse image of the latter boundary face under π_C in $X^3 \times I$. Clearly, $\pi_C^{-1}(X \times X \times \{0\})$ consists of boundary faces

$$X \times X \times X \times \{0\} \quad \text{and} \quad X \times \partial X \times X \times \{0\}.$$

The first face projects to $X^2 \times \{0\} \subset X^2 \times I$ under both π_L and π_R, while the second projects to $X \times \partial X \times \{0\}$ under π_L and to $\partial X \times X \times \{0\}$ under π_R. This is the reason for the index set behavior

$$G_{\rm zf} = (E_{\rm zf} + F_{\rm zf}) \overline{\cup} (E_{\rm rb_0} + F_{\rm lb_0})$$

with respect to composition. Provided $\operatorname{Re}(E_{\rm rb_0} + F_{\rm lb_0}) > \operatorname{Re}(E_{\rm zf} + F_{\rm zf})$, the contribution of the former will vanish upon restriction to zf, and so under this condition, the value $(\kappa_{P \circ Q})_{|\rm zf}$ will be determined by the composition on the lift of $X \times X \times X \times \{0\}$ in X^3_t.

By examining local coordinates (or otherwise), it can be checked that the lift of this face is isomorphic to the b triple space $X^3_{\rm b}$, with $\pi_{L/R/C}$ inducing the usual b fibrations in that context. Thus we obtain that

$$S_0(P \circ Q) = S_0(P) \circ S_0(Q), \quad \text{provided } \operatorname{Re}(E_{\rm rb_0} + F_{\rm lb_0}) > \operatorname{Re}(E_{\rm zf} + F_{\rm zf}).$$

Similar considerations apply to $N_t(P)$. Indeed local coordinate examination shows that bf_0 is isomorphic to the b, scattering blow up of $\partial X \times [0,1]$. From the composition formula, provided $\mathrm{Re}(E_{\mathrm{lb}_0} + F_{\mathrm{rb}_0}) > \mathrm{Re}(E_{\mathrm{bf}_0} + F_{\mathrm{bf}_0})$, the contribution to $(\kappa_{P\circ Q})_{|\mathrm{bf}_0}$ will consist only of the composition of P and Q on the boundary hypersurface in X_t^3 which is the lift of $\partial X \times \partial X \times \partial X \times \{0\}$ in $X^3 \times I$. Local coordinate considerations again show that this restricted composition is precisely that of b, scattering operators on $\partial X \times [0,1]$.

□

By the definition of the scattering symbol and the transition normal operator, we have the following *symbol sequences*, for operators in the small calculus.

$$0 \longrightarrow \rho_{\mathrm{sc}} \Psi_t^{l,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow \Psi_t^{l,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \xrightarrow{\sigma_{\mathrm{sc}}} C^\infty({}^{\mathrm{sc}}T^*_{\partial X} X \times I; \mathrm{End}(V \otimes \Omega_{\mathrm{sc}}^{1/2})) \longrightarrow 0$$

and

$$0 \longrightarrow \rho_{\mathrm{bf}_0} \Psi_t^{l,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow \Psi_t^{l,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \xrightarrow{N_{\mathrm{bf}_0}} \Psi_{\mathrm{b,sc}}^{m,(E_{\mathrm{zf}}, E_{\mathrm{lb}_0}, E_{\mathrm{rb}_0}), E_{\mathrm{sc}}}(\partial X \times {}_{\mathrm{b}}[0,1]_{\mathrm{sc}}; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow 0$$

Proposition B.4. *The scattering symbol, normal operator and the slices S_δ and S_0 are related by the following*

$$\sigma_{\mathrm{sc}}(P)_{|\epsilon=\delta} = \sigma_{\mathrm{sc}}(S_\delta(P))$$
$$\sigma_{\mathrm{sc}}(P)_{|\epsilon=0} = \sigma_{\mathrm{sc}}(N_t(P))$$
$$N_{\mathrm{b}}\left(N_t(P)\right) = N_{\mathrm{b}}\left(S_0(P)\right)$$

Proof. This is a straightforward consequence of the geometry. For instance, the b normal operator $N_{\mathrm{b}}(S_0(P))$ consists of the restriction of $S_0(P)$ to the front face of $X_{\mathrm{b}}^2 \cong \mathrm{zf}$; this is just the restriction of κ_P to $\mathrm{zf} \cap \mathrm{bf}_0 \subset X_t^2$. On the other hand, $N_{\mathrm{b}}(N_t(P))$ is the restriction of $N_t(P)$ to the (b) front face of $(\partial X \times [0,1])^2_{\mathrm{b,sc}}$, but this also coincides with the boundary face $\mathrm{zf} \cap \mathrm{bf}_0 \subset X_t^2$; therefore they must agree.

The others follow similarly. □

B.5 Symbols of Differential operators

We give a detailed examination of the Lie algebra giving rise to the transition calculus, and an alternate definition of the normal operator and scattering symbol for differential operators.

Consider the b fibration $\pi_I : X_t \longrightarrow I$. Let $\mathcal{V}_{\mathrm{b,f}}(X_t) \subset \mathcal{V}_{\mathrm{b}}(X_t)$ denote the subset of the b vector fields on X_t (so those tangent to the boundary hypersurfaces), which are tangent to the fibers of π_I. Tangency is preserved under Lie bracket, so $\mathcal{V}_{\mathrm{b,f}}(X_t)$ is a Lie subalgebra. Then set

$$\mathcal{V}_t(X_t) = \rho_{\mathrm{sf}} \mathcal{V}_{\mathrm{b,f}}(X_t)$$

These are the *b-sc transition vector fields* which the calculus is meant to microlocalize.

Lemma B.3. $[\mathcal{V}_t(X_t), \mathcal{V}_t(X_t)] \subset \rho_{\mathrm{sf}} \mathcal{V}_t(X_t)$, *so* $\mathcal{V}_t(X_t)$ *is a subalgebra, and evaluation at* sc *takes values in the trivial Lie algebra*

$$\mathcal{V}_t(X_t)/\rho_{\rm sf}\mathcal{V}_t(X_t)$$

Also, evaluation at tf *takes values in* $\mathcal{V}_t(X_t)/\rho_{\rm tf}\mathcal{V}_t(X_t) \cong \mathcal{V}_{\rm b,sc}(\partial X \times {}_{\rm b}[0,1]_{\rm sc})$, *and evaluation at* zf *takes values in* $\mathcal{V}_t(X_t)/\rho_{\rm zf}\mathcal{V}_t(X_t) \cong \mathcal{V}_{\rm b}(X)$.

Proof. That $[\mathcal{V}_t(X_t), \mathcal{V}_t(X_t)] \subset \rho_{\rm sf}\mathcal{V}_t(X_t)$ follows exactly as in the proof of the identity $[\mathcal{V}_{\rm sc}(X), \mathcal{V}_{\rm sc}(X)] \subset x\mathcal{V}_{\rm sc}(X)$, using that $\mathcal{V}_{\rm b,f}(X_t) \cdot \rho_{\rm sf}C^\infty(X_t) \subset \rho_{\rm sf}C^\infty(X_t)$.

That $\mathcal{V}_t(X_t)/\rho_{\rm tf}\mathcal{V}_t(X_t) \cong \mathcal{V}_{\rm b,sc}(\partial X \times [0,1])$ follows from the fact that tf $\cong \partial X \times [0,1]$, and that tf itself is a generalized fiber of $\pi_I : X_t \longrightarrow I$. Similarly for zf. □

The transition differential operators are the enveloping algebra of $\mathcal{V}_t(X_t)$; equivalently, they are defined by iterated composition of $\mathcal{V}_t(X_t)$ as operators on $C^\infty(X_t; V)$.

$$\mathrm{Diff}_t^k(X_t; V) = \left\{ \sum_{j \le k} a_j\, V_1 \cdots V_j \; ; \; V_i \in \mathcal{V}_t(X_t), a_j \in C^\infty(X_t; \mathrm{End}(V)) \right\}$$

The evaluation maps above then extend to normal operator homomorphisms

$$\begin{aligned} N_{\rm sc} &: \mathrm{Diff}_t^k(X_t; V) \longrightarrow \mathrm{Diff}^k_{\rm I,fib}(({}^{\rm sc}T_{\partial X}X) \times I; V) \\ N_{\rm t} &: \mathrm{Diff}_t^k(X_t; V) \longrightarrow \mathrm{Diff}^k_{\rm b,sc}(\partial X \times I; V) \\ S_\delta &: \mathrm{Diff}_t^k(X_t; V) \longrightarrow \mathrm{Diff}^k_{\rm sc}(X; V) \\ S_0 &: \mathrm{Diff}_t^k(X_t; V) \longrightarrow \mathrm{Diff}^k_{\rm b}(X; V), \end{aligned}$$

where $\mathrm{Diff}^k_{\rm I,fib}(({}^{\rm sc}T_{\partial X}X) \times I; V)$ denotes translation invariant differential operators along the fibers of ${}^{\rm sc}T_{\partial X}X \times I \longrightarrow \partial X \times I$ which are smoothly parametrized by the base. Fiberwise Fourier transform of the first of these gives the scattering symbol

$$\sigma_{\rm sc} : \mathrm{Diff}_t^k(X_t; V) \longrightarrow C^\infty({}^{\rm sc}T^*_{\partial X}X \times I; \mathrm{End}(V)).$$

Proposition B.5. *For* $P \in \mathrm{Diff}_t^k(X; V \otimes \Omega_{\rm b}^{1/2})$, *these definitions agree with those defined for Schwartz kernels on* X_t^2.

Proof. The Schwartz kernels of differential operators can be obtained by lifting the differential operators under π_L to X_t^2 and acting on the kernel of the identity operator. As the latter is completely supported on the diagonal $\Delta \subset X_t^2$, it suffices to verify that the map

$$\mathcal{V}_t(X_t) \xrightarrow{\pi_L^*} \mathcal{V}(X_t^2) \xrightarrow{|\Delta} \mathcal{V}(\Delta) \xrightarrow{\Delta \cong X_t} \mathcal{V}_t(X_t)$$

is the identity. □

B.6 The trace

From the theory of b and scattering pseudodifferential operators, if $P \in \Psi_t^{-\infty,\mathcal{E}}$, $S_\delta(P)$ and $S_0(P)$ will extend to trace class operators provided $\mathrm{Re}(E_{\rm sc}) > n$ in the case of $S_\delta(P)$, and $\mathrm{Re}(E_{\rm bf_0}), \mathrm{Re}(E_{\rm lb_0}), \mathrm{Re}(E_{\rm rb_0}) > 0$ in the case of $S_0(P)$.

Let $P \in \Psi_t^{-\infty,\mathcal{E}}$, with $\mathrm{Re}\,(E_{\mathrm{sc}}) > n$, $\mathrm{Re}\,(E_i) > 0$ for $i \in \{\mathrm{bf}_0, \mathrm{lb}_0, \mathrm{rb}_0\}$. Since the conormal singularity at the diagonal is of order $-\infty$, $\kappa_P \in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X_t^2; \mathrm{End}(V) \otimes \Omega_{\mathrm{b}}^{1/2})$, and its restriction to the diagonal, $(\kappa_P)_{|\Delta}$, is well-defined, as an element

$$(\kappa_P)_{|\Delta} \in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X_t; \mathrm{End}(V) \otimes \rho_{\mathrm{sf}}^{-n}\Omega_{\mathrm{b}}((X_t)_{\mathrm{fib}}) \otimes \Omega_{\mathrm{b}}^{1/2}(I)), \quad \mathcal{F} = (F_{\mathrm{sf}}, F_{\mathrm{tf}}, F_{\mathrm{zf}}) = (E_{\mathrm{sc}}, E_{\mathrm{bf}_0}, E_{\mathrm{zf}})$$

using the density bundle isomorphism in Lemma B.2.

Then consider the composition

$$\tau : X_t \cong \Delta \xrightarrow{i_\Delta} X_t^2 \xrightarrow{\pi_I} I$$

where i_Δ is the natural inclusion and π_I is the composition of the blow down $X_t^2 \longrightarrow X^2 \times I$ with the projection $X^2 \times I \longrightarrow I$.

Lemma B.4. *First, $\tau : X_t \longrightarrow I$ is a b-fibration equivalent to $\pi_I : X_t \longrightarrow I$. Suppose $u \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X_t; \rho_{\mathrm{sf}}^{-n}\Omega_{\mathrm{b}}((X_t)_{\mathrm{fib}}) \otimes \Omega_{\mathrm{b}}^{1/2}(I))$ with $E_{\mathrm{zf}} = 0$, $\mathrm{Re}(E_{\mathrm{tf}}) > 0$, $\mathrm{Re}(E_{\mathrm{sf}}) > n$, then $\tau_*(u) \in \mathcal{A}^{(E_{\mathrm{zf}} \overline{\cup} E_{\mathrm{tf}})}_{\mathrm{phg}}(I, \Omega_{\mathrm{b}}^{1/2})$.*

In particular, the leading order of the index index set of $\tau_*(u)$ at $\{0\} \subset I$ is $0 = \mathrm{Re}(E_{\mathrm{zf}} \overline{\cup} E_{\mathrm{tf}})$ since $E_{\mathrm{zf}} = 0$.

Thus,

$$\tau_*(u) \in C^0(I; \Omega_{\mathrm{b}}^{1/2}(I))$$

and using the trivialization $\left|\frac{d\epsilon}{e}\right|^{1/2} : I \longrightarrow \Omega_{\mathrm{b}}^{1/2}(I)$, we can identify

$$\tau_*(u) \in C^0(I; \mathbb{C})$$

Proof. That τ is equivalent to $\pi_I : X_t \longrightarrow I$ is easy to see, for instance by examining local coordinates. As π_I is a b-fibration, the pushforward of any polyhomogeneous section of the fiber density bundle is defined, provided integrability conditions are met.

Note that the requirement $\mathrm{Re}(E_{\mathrm{sf}}) > n$ for a section of $\rho_{\mathrm{sf}}^{-n}\Omega_{\mathrm{b}}((X_t)_{\mathrm{fib}})$ is equivalent to requiring $\mathrm{Re}(E_{\mathrm{sf}}) > 0$ for a section of $\Omega_{\mathrm{b}}((X_t)_{\mathrm{fib}})$. Then the pushforward Theorem A.2 for polyhomogeneous sections shows that, provided the integrability condition $\mathrm{Re}(E_{\mathrm{sf}}) > 0$ is satisfied,

$$\tau_*(u) \in \mathcal{A}^{F}_{\mathrm{phg}}(I, \Omega_{\mathrm{b}}^{1/2}(I)), \quad F = E_{\mathrm{zf}} \overline{\cup} E_{\mathrm{tf}}.$$

□

Letting $\mathrm{tr} : \mathrm{End}(V) \longrightarrow \mathbb{C}$ denote the fiberwise trace on $\mathrm{End}(V)$, we define a provisional operator trace on $P \in \Psi_t^{-\infty,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$ where $\mathcal{E} = (0_{\mathrm{zf}}, E_{\mathrm{bf}_0}, E_{\mathrm{lb}_0}, E_{\mathrm{rb}_0}, E_{\mathrm{sc}})$, $\mathrm{Re}(E_i) > 0$, $\mathrm{Re}(E_{\mathrm{sc}}) > n$, by

$$\mathrm{Tr}(P) = \tau_*(\mathrm{tr}(\kappa_P)_{|\Delta}) \in C^0(I), \tag{B.2}$$

using the identifications in the above Lemma.

Proposition B.6. *This operator trace coincides with the operator trace of $S_\delta(P)$, $\delta \geq 0$. That is,*

$$\mathrm{Tr}(P)(\delta) = \mathrm{Tr}(S_\delta(P)), \delta \geq 0$$

Proof. The conditions on P imply that $S_\delta(P), \delta \geq 0$ extend to trace class elements of $\Psi_{\mathrm{sc}}^{-\infty,E_{\mathrm{sc}}}(X; V \otimes \Omega_{\mathrm{sc}}^{1/2})$ or $\Psi_{\mathrm{b}}^{-\infty,\mathcal{E}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2})$, as appropriate. In either of these calculi, the trace can realized as an integration along the diagonal, which we can reinterpret the pushforward under the canonical b-fibration

$$\Delta \subset X^2_{\mathrm{b/sc}} \longrightarrow \{\mathrm{pt}\}\,.$$

For $\delta > 0$, the preimage $\tau^{-1}(\delta)$ consists precisely of the intersection of Δ with the slice at δ, so $\mathrm{Tr}(S_\delta(P))$ is explicitly realized by $\mathrm{Tr}(P)(\delta)$.

For $\delta = 0$, the preimage $\tau^{-1}(\delta)$ consists not only of $\Delta \cap \mathrm{zf}$ but $\Delta \cap \mathrm{bf}_0 \cong \mathrm{tf} \subset X_t$ as well. However, the integrability condition $\mathrm{Re}(E_{\mathrm{tf}}) > 0$ implies that the contribution from this face vanishes relative to the contribution from E_{zf}, since $\mathrm{Re}(E_{\mathrm{zf}}) = 0$. Thus for $\delta = 0$ we can also conclude that $\mathrm{Tr}(P)(0) = \mathrm{Tr}(S_0(P))$. □

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