The Seiberg-Witten Equations on a Surface Times a Circle

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Abstract

In this thesis I study the Seiberg-Witten equations on the product of a genus g surface Σ and a circle. I exploit S^1 invariance to reduce to the vortex equations on Σ and thus completely describe the Seiberg-Witten monopoles.

In the case where the monopoles are not Morse-Bott regular, I explicitly perturb the equations to obtain such a situation and thus find a candidate for the chain complex that calculates the Seiberg-Witten Floer homology groups.

Thesis Supervisor: Tomasz S. Mrowka
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Contents

Chapter 1

Introduction

1.1 Overview

The goal of this work is to understand the Seiberg-Witten equations on the product of a genus g surface with a circle. This has applications to the study of 4-manifolds containing an embedded genus g surface with trivial normal bundle. In particular, I intend to study the *Seiberg-Witten (monopole) Floer homology* of $S^1 \times \Sigma$. The cases $g = 0$ and $g = 1$ are worked out in [5].

It is straightforward to write down the solutions to the Seiberg-Witten equations on Y; however, we must perturb them in order to achieve transversality, and in doing so we may change the solution space. I write down an explicit perturbation of the equations for which the solution spaces are Morse-Bott nondegenerate.

In general, the behavior of the Floer groups is very different depending on whether the determinant line bundle of $\mathfrak{s}$ has torsion first Chern class. In the nontorsion case, there is really only one homology group rather than three, and solutions to the equations are irreducible. The case $c_1(\mathfrak{s}) \neq 0$ was studied by Muñoz and Wang in [10]. I calculate that in this case, the solutions are all Morse-Bott nondegenerate.

The main contribution of this work, however, comes in the case $c_1(\mathfrak{s}) = 0$; this is the unique spin^c structure that arises from a spin structure on Y. In this case both irreducible and reducible solutions arise, and there really are three groups. In this case the reducibles form a torus of dimension $2g + 1$, which can be identified as the product of the Jacobian torus of Σ with the dual circle to the S^1 factor. The irreducibles are a copy of the $(g-1)$st symmetric product of Σ.

The equations on $S^1 \times \Sigma$ are rotationally invariant and in fact reduce to equations on Σ. These are the *vortex equations* ([1]). Their solutions correspond to effective divisors on the Riemann surface Σ.

For the spin^c structure $\mathfrak{s}_k$ with $c_1(\mathfrak{s}_k)$ Poincare dual to $2k[S^1]$, the irreducible solutions correspond to $Sym^{g-1-|k|}(\Sigma)$. In the case $k=0$, all the solutions to the equations are reducible, but after perturbing to achieve transversality, we see a copy of $Sym^{g-1}(\Sigma)$ in the irreducibles here as well.

There is another 3-manifold invariant, the *Heegaard Floer homology* of Ozsváth and Szabó, that assigns three groups related by a long exact sequence to a spin^c 3-manifold. The two theories are conjectured to be isomorphic, and this work may be considered as evidence for this conjecture. More specifically, Jabuka and Mark ([3]) show that

$$HF^*_{red}(S^1 \times \Sigma_g, \mathfrak{s}_0; \mathbf{C}) \cong H^*(Sym^{g-3}\Sigma; \mathbf{C})$$

as relatively $\mathbf{Z}$-graded complex vector spaces. However, they observe that the $\mathbf{C}[U]$-module structure on this Floer cohomology group differs from that of the ordinary cohomology of this symmetric product.

The calculations here provide an explanation for this: we see the Jacobian of Σ in the reducibles and the $(g-1)$st symmetric product in the irreducibles. The Abel-Jacobi map

$$\mu : Sym^{g-1}\Sigma \to Jac,$$

which can be regarded as taking an effective divisor of degree $g-1$ to the line bundle it defines, induces a map in ordinary cohomology. The quotient

$$H^*(Sym^{g-1}\Sigma; \mathbf{C})/\mu^* H^*(Jac; C),$$

which is expected to equal $HM^*_{red}(S^1 \times \Sigma, \mathfrak{s}_0; \mathbf{C})$, is isomorphic to $H^*(Sym^{g-3}\Sigma; \mathbf{C})$

as vector spaces (after a grading shift). However, the action of U is different in these cases.

1.2 Riemann surfaces

Since, as will be seen later, the Seiberg-Witten equations on $S^1 \times \Sigma$ reduce to equations on Σ, it will be necessary to understand some facts from complex geometry relating to the Riemann surface. We will need the notions of the *Jacobian* or *Picard variety* of Σ, which classifies isomorphism classes of holomorphic line bundles; the *d-fold symmetric product* of Σ, which classifies divisors of degree d (that is, d-tuples of points); and the *Abel-Jacobi map* relating the two. Another relevant notion for us will be the *Lefschetz decomposition* of the cohomology of a compact Kahler manifold, which we will apply in the simple case of a complex torus.

Throughout this section, let Σ be a closed Riemann surface of genus g; i.e. Σ is a closed oriented 2-manifold, and we fix a complex structure, or equivalently a conformal class of metrics. This material is standard, found for instance in [2].

1.2.1 Line bundles and divisors over a Riemann surface

We consider the classifcation of complex line bundles over Σ. First, there is the topological classification. Line bundles over Σ are classified by their first Chern class in $H^2(\Sigma; \mathbf{Z}) = \mathbf{Z}$. If we take holomorphic line bundles (so the transition functions are holomorphic maps $U_\alpha \cap U_\beta \to \mathbf{C}^\times$, then we may ask about their classification up to holomorphic isomorphism. We will understand this using Čech cohomology.

Definition 1. The *Picard group* $Pic(\Sigma)$ of Σ is the group of isomorphism classes of holomorphic line bundles on Σ. It has a subgroup $Pic^0(\Sigma)$ consisting of isomorphism classes of degree zero bundles.

Proposition 2. *The group $Pic^0(\Sigma)$ is isomorphic to a $2g$-dimensional torus; in particular,*

$$Pic^0(\Sigma) \cong H^{0,1}(\Sigma)/H^1(\Sigma; \mathbf{Z}).$$

There is a short exact sequence

$$0 \to Pic^0(\Sigma) \to Pic(\Sigma) \to \mathbf{Z} \to 0$$

of abelian groups. This exact sequence splits, so $Pic(\Sigma) \cong Pic^0(\Sigma) \times \mathbf{Z}$, *but the splitting is not natural.*

Proof. Let $\{\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{C}^\times\}$ be holomorphic transition functions giving the topologically trivial line bundle L. These give rise to an element of the Čech cohomology $H^1(\Sigma; \mathcal{O}^*)$ with coefficients in the sheaf of nonvanishing holomorphic functions. The degree zero condition means that it maps to zero and thus comes from an element of $H^1(\Sigma; \mathcal{O})$. The exponential sequence

$$0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

gives rise to a long exact sequence in cohomology. Finally, the image of $H^1(\Sigma; \mathbf{Z})$ in $H^1(\Sigma; \mathcal{O})$ corresponds to the trivial line bundles. Using the Dolbeault isomorphism $H^1(\Sigma; \mathcal{O}) \cong H^{0,1}(\Sigma)$, we conclude that $Pic^0(\Sigma) \cong H^{0,1}(\Sigma)/H^1(\Sigma; \mathbf{Z})$.

Now choose any holomorphic degree 1 line bundle A. Then for a line bundle L of degree d, $L \otimes A^{-d} \in Pic^0(\Sigma)$. This gives the desired splitting. $\square$

Definition 3. The *divisor group* $Div(\Sigma)$ of Σ is the free abelian group generated by the points of Σ. Its elements are called *divisors* on Σ.

Given a divisor $D = \sum a_i x_i$, we may form the sheaf $\mathcal{M}(D)$ of meromorphic functions with zeros/poles corresponding to D. There is a holomorphic line bundle $L(D)$ that has $\mathcal{M}(D)$ as its sheaf of sections.

A divisor is *effective* if $a_i \geq 0$ for all i. Two divisors are *linearly equivalent* if their associated line bundles are isomorphic. A line bundle has a holomorphic section if and only if it is the line bundle associated to some effective divisor. We write $Div^+(\Sigma)$ for the space of effective divisors on Σ and $Div_d^+(\Sigma)$ for the space of effective divisors of degree d. Note that $Div_d^+(\Sigma)$ naturally corresponds to the dth symmetric product of Σ, the space

$$Sym^d(\Sigma) = \Sigma^d / S_d,$$

where the symmetric group S_d acts on Σ^d by permutation. The symmetric product $Sym^d(\Sigma)$ is a complex manifold of complex dimension d.

1.2.2 The Jacobian and the Abel-Jacobi map

The Jacobian $Jac(\Sigma)$ is a g-dimensional complex torus naturally associated to Σ. It comes with a map, the *Abel-Jacobi map*, defined as follows:

Let $\{\omega_i\}_{i=1}^g$ be a basis for the set of holomorphic 1-forms on Σ. Fix a basepoint $z_0 \in \Sigma$ and consider, for $z \in \Sigma$, the g-tuple of integrals

$$\left(\int_{z_0}^{z} \omega_1, \ldots, \int_{z_0}^{z} \omega_g \right).$$

This does not make sense as a map $\Sigma \to \mathbf{C}^g$ because it depends on a choice of path γ from z_0 to z. However, it is well-defined up to the lattice L generated by vectors

$$\left(\int_{\alpha} \omega_1, \ldots, \int_{\alpha} \omega_g \right)$$

for α a class in $H_1(\Sigma; \mathbf{Z})$, and so it defines a map $\mu : \Sigma \to \mathbf{C}^g/L$. The complex torus $Jac = \mathbf{C}^g/L$ is known as the *Jacobian* of Σ, and μ is the *Abel-Jacobi map.*

There is a more intrinsic description of Jac. We may write

$$Jac = H^0(\Sigma; \Omega^1)^* / H_1(\Sigma; \mathbf{Z})$$

without choosing a basis of holomorphic 1-forms. The map μ depends on the chosen basepoint, but only up to a translation.

We may extend μ to a map $Pic_d \to Jac$, still denoted μ for simplicity, for any natural number d. Simply define

$$\mu(z_1, \ldots, z_d) = \Sigma_{i=1}^d \mu(z_d).$$

There is a map $Jac \rightarrow Pic_0(\Sigma)$. This is an isomorphism, so we will frequently identify the two. There is also a map $Sym^d\Sigma \rightarrow Pic_d(\Sigma)$ that takes a degree d effective divisor to its corresponding line bundle. Making the above identification, choosing an identification of Pic_0 with Pic_d, and translating by a constant, this map is just the map μ defined above.

Chapter 2

Three-dimensional Seiberg-Witten theory

2.1 Overview

We introduce the *monopole* or *Seiberg-Witten Floer homology groups* of a closed oriented 3-manifold Y, developed in [5] and record some of their properties. This theory associates to a closed oriented manifold Y equipped with a spin^c structure $\mathfrak{s}$ three abelian groups:

$$\overline{HM}(Y,\mathfrak{s}), \widehat{HM}(Y,\mathfrak{s}), \widetilde{HM}(Y,\mathfrak{s}).$$

Roughly speaking, these groups are constructed as the Morse homology of an infinite-dimensional space $\mathcal{B}(Y,\mathfrak{s})$ together with a functional on that space. A generic perturbation of this functional will have only nondegenerate critical points, and we let these generate the chain groups. The differentials count gradient flow lines between critical points. However, because there is a group action and the stabilizers vary from point to point (either the trivial group or the circle), we need a more sophisticated picture. Instead of a single homology group, we get three groups fitting into a long exact sequence

$$\cdots \to \widetilde{HM}_k(Y,\mathfrak{s}) \to \widehat{HM}_k(Y,\mathfrak{s}) \to \overline{HM}_k(Y,\mathfrak{s}) \to \widetilde{HM}_{k-1}(Y,\mathfrak{s}) \to \cdots$$

These are also modules over the opposite ring of the ordinary cohomology of the configuration space,

$$\Lambda^*(H_1(Y;\mathbf{Z})/T)\otimes_{\mathbf{Z}}\mathbf{Z}[U].$$

The associated Floer cohomology groups are modules over the cohomology ring itself.

2.2 The Chern-Simons-Dirac functional

Our first step is to define the space $\mathcal{B}(Y,\mathfrak{s})$ and the functional on it. Given a closed, oriented, connected 3-manifold Y with spin^c structure $\mathfrak{s}=(S,\rho)$, we may define

Definition 4. The *configuration space* $\mathcal{C}(Y,\mathfrak{s})$ is the set of pairs (B,Ψ), where B is a smooth Clifford connection in E and Ψ is a smooth section of E.

The space $\mathcal{C}(Y,\mathfrak{s})$ is an affine space modeled on the vector space $\Omega^1(Y;\imath\mathbf{R})\oplus\Gamma(E)$. Suitable Sobolev completions are Banach manifolds.

Definition 5. The *Chern-Simons-Dirac functional* is the real valued function on $\mathcal{C}(Y,\mathfrak{s})$ given by

$$CSD(B,\Psi)=\frac{1}{8}\int_Y(B^t-B_0^t)\wedge(F_{B^t}+F_{B_0^t})+\frac{1}{2}\int_Y\langle\Psi,D_B\Psi\rangle\,dvol_Y.$$

Here D_B is the Dirac operator associated to the Clifford connection B.

Definition 6. The *gauge group* of Y is

$$\mathcal{G}(Y)=Map(Y,S^1).$$

If we work with sufficiently regular configurations (L^p_k with $pk>3$), then this is in fact a Banach Lie group. There is an action of $\mathcal{G}(Y)$ on $\mathcal{C}(Y,\mathfrak{s})$given by

$$u\cdot(A,\phi)=(A-u^{-1}du,u\phi).$$

Roughly speaking, we want to look at the $\mathcal{G}(Y)$-equivariant Morse theory of the function CSD on $\mathcal{C}(Y, \mathfrak{s})$. Define the quotient

$$\mathcal{B}(Y, \mathfrak{s}) = \mathcal{C}(Y, \mathfrak{s})/\mathcal{G}(Y).$$

The functional is invariant under the action of the identity component $\mathcal{G}_0(Y)$ of $\mathcal{G}(Y)$; in general, for $u \in \mathcal{G}(Y)$ and $(A, \Phi) \in \mathcal{B}(Y, s)$, we have the transformation law

$$CSD(u \cdot (B, \Psi)) = CSD(B, \Psi) - 4\pi^2 \left\langle \left[\frac{du}{u}\right] \cup c_1(\mathfrak{s}), [Y] \right\rangle.$$

(Here, $\left[\frac{du}{u}\right]$ is the de Rham cohomology class of the imaginary-valued 1-form $\frac{du}{u}$.) In particular, if $c_1(\mathfrak{s})$ is torsion, then CSD is invariant under the full gauge group and so descends to a real-valued function on $\mathcal{B}(Y, \mathfrak{s})$. In general, we have

$$CSD : \mathcal{B}(Y, \mathfrak{s}) \rightarrow \mathbf{R}/4\pi^2 \left\langle \left[\frac{du}{u}\right] \cup c_1(\mathfrak{s}), [Y] \right\rangle \mathbf{Z}.$$

Note that in any case, the gradient of CSD is fully gauge-invariant.

Definition 7. We say that a configuration $(B, \Psi) \in \mathcal{C}(Y, \mathfrak{s})$ is *reducible* if $\Psi = 0$ and *irreducible* otherwise. Denote the subset of $\mathcal{C}(Y, \mathfrak{s})$ consisting of irreducible configurations by $\mathcal{C}^*(Y, \mathfrak{s})$.

The stabilizer of the gauge group action on $\mathcal{C}(Y, \mathfrak{s})$ is trivial at the irreducible configurations; at the reducibles, there is an S^1 stabilizer consisting of the constant maps $Y \rightarrow S^1$.

2.3 The three-dimensional Seiberg-Witten equations

To find the equations that a critical point must satisfy, we calculate the gradient of CSD with respect to the L^2 inner product on $\mathcal{C}(Y, \mathfrak{s})$. It is

$$gradCSD(B, \Psi) = (*F_B - (\Psi\Psi^*)_0, D_B\Psi).$$

Here $\Psi\Psi^* \in \Gamma(EndE)$ is the endomorphism of E determined by the hermitian metric, and the 0 subscript denotes taking the traceless part. Therefore, recalling that $\rho(*\alpha) = -\rho(\alpha)$, we find that the critical points of CSD are the solutions to the *three-dimensional Seiberg-Witten equations*

$$\begin{aligned} \frac{1}{2}\rho(F_{B^t}) - (\Psi\Psi^*)_0 &= 0 \\ D_B\Psi &= 0. \end{aligned}$$

The tangent space to $\mathcal{C}(Y, \mathfrak{s})$ at any configuration is

$$T_\gamma = i\Omega^1(Y) \oplus \Gamma(S).$$

These equations define a vector field on $\mathcal{C}(Y, \mathfrak{s})$.

Note that the solution set of these equations is invariant under $\mathcal{G}(Y)$; that is, if $SW(B, \Psi) = 0$, then $SW(u \cdot (B, \Psi)) = 0$.

The linearization of the Seiberg-Witten equations at a configuration (B, Ψ) is given by

$$(b, \psi) \mapsto \left(\rho(db) - (\Psi\psi^* + \psi\Psi^*)_0, D_B\psi + \rho(b)\Psi\right).$$

2.4 Gauge fixing for irreducible configurations

The space $\mathcal{B}(Y, \mathfrak{s})$ is given as the quotient of a Hilbert manifold by the action of a Hilbert Lie group. Its points are not simply configurations but equivalence classes of configurations. It is useful to have a way to work with subspaces of $\mathcal{C}(Y, \mathfrak{s})$ rather than quotients. In other words, given a configuration $(B, \Psi) \in \mathcal{C}(Y, \mathfrak{s})$, we will find a neighborhood U of $[B, \Psi] \in \mathcal{B}(Y, \mathfrak{s})$ and a submanifold Y of $\mathcal{C}(Y, \mathfrak{s})$ containing (B, Ψ) such that the two are diffeomorphic.

Fix an irreducible configuration $\gamma = (B, \Psi)$ and let $\mathbf{d}_\gamma$ denote the linearization of the gauge group action $u \mapsto u\gamma$ at $u = 1$. Explicitly, this is given by

$$\mathbf{d}_\gamma \xi = (-d\xi, \xi\Psi).$$

Proposition 8 ([5], Corollary 9.38). *Write $\mathcal{J}_\gamma$ for the image of $\mathbf{d}_\gamma$ and $\mathcal{K}_\gamma$ for its L^2 orthogonal complement. Let $\mathcal{S}_\gamma$ be the affine subspace $\gamma + \mathcal{K}_\gamma$of $\mathcal{C}(Y, \mathfrak{s})$. Then there is a neighborhood U of γ in $\mathcal{S}$ such that the quotient map is a diffeomorphism onto a neighborhood of $[\gamma]$ in $\mathcal{B}(Y, \mathfrak{s})$.*

Note that $\mathcal{K}_\gamma$ is the kernel of the L^2 adjoint $\mathbf{d}_\gamma^*$ of $\mathbf{d}_\gamma$, which is given by

$$\mathbf{d}_\gamma^*(b, \psi) = d^*b + Im \langle \psi, \Psi \rangle .$$

Define the Hessian of CSD to be the differential of $gradCSD$at γ followed by the projection to $\mathcal{K}_\gamma$.

Proposition 9. *The extended Hessian of CSD,*

$$\begin{pmatrix} HessCSD & \mathbf{d}_\gamma \\ \mathbf{d}_\gamma^* & 0 \end{pmatrix}$$

gives a Fredholm map of index zero from $\mathcal{K} \oplus i\Omega^1(Y)$ to itself.

2.5 Gradient flows and the four-dimensional equations

We have written down the equations that a critical point of CSD must satisfy; in order to do Morse theory, we also need to understand the gradient flow lines between critical points; that is, we need to understand the equation

$$\frac{d}{dt}\gamma(t) + gradCSD(\gamma(t)) = 0$$

for a path $\gamma : \mathbf{R} \to \mathcal{C}(Y, \mathfrak{s})$. First of all, we can interpret such a path in terms of objects defined on $Z = \mathbf{R} \times Y$. Define the spin^c structure $\mathfrak{s}_Z$ on Z to have $S^+ = S^- = \pi^*S$, with the Clifford multiplication ρ_Z given by

$$\rho_Z\left(\frac{\partial}{\partial t}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\rho_Z(v) = \begin{pmatrix} 0 & -\rho(v)^* \\ \rho(v) & 0 \end{pmatrix}$$

for $v \in TY$. Then the time-dependent spinor $\Psi(t)$ can be interpreted as a section Φ of $S^+ \to Z$.

The time-dependent connection $B(t)$, on the other hand, pulls back to a Clifford connection A in $S \to Z$ with the property that

$$\nabla^A_{\frac{\partial}{\partial t}} = \frac{\partial}{\partial t}.$$

We say that such a connection is in *temporal gauge.* Conversely, given a connection on Z and a plus spinor, we can construct a path in $\mathcal{C}(Y, \mathfrak{s})$ together with a function $c : Z \to \mathbf{R}$. The Dirac operator $D_A : \Gamma(Z; \pi^*S) \to \Gamma(Z; \pi^*S)$ splits into $D_A^{\pm} : \Gamma(Z; S^{\pm}) \to \Gamma(Z; S^{\mp})$.

After reformulating the problem as an equation on the cylinder, the gradient flow equations take the form

$$\begin{aligned} \frac{1}{2}\rho_Z\left(F_A^+\right) - \left(\Phi\Phi^*\right)_0 &= 0 \\ D_A^+\Phi &= 0. \end{aligned}$$

These are the *four-dimensional Seiberg-Witten equations*, which make sense on a (not necessarily cylindrical) spin^c four-manifold X. As in the case of a three-manifold Y, denote by $\mathcal{C}(X, \mathfrak{s}_X)$ the space of configurations on which these equations are defined.

2.6 The σ and τ blowups

Note that the stabilizers of the $\mathcal{G}(Y)$-action on $\mathcal{C}(Y,\mathfrak{s})$ vary from point to point.

Proposition 10. *Let $\gamma = (B, \Psi)$ be a configuration in $\mathcal{C}(Y,\mathfrak{s})$. Then the stabilizer of the action of $\mathcal{G}(Y)$ at γ is trivial if γ is irreducible; if γ is reducible, it is the subgroup of $\mathcal{G}(Y)$ consisting of the constant maps $Y \to S^1$.*

Note that a reducible solution to the equations is simply a pair $(B, 0)$ where B is a flat connection. These exist if and only if $c_1(\mathfrak{s})$ is a torsion element of $H^2(Y;\mathbf{Z})$.

As stated earlier, the gauge group action has an S^1 stabilizer at the reducible configurations. We now introduce two "blown-up" versions of the configuration space, on which $\mathcal{G}$ acts freely. The σ model makes sense for any 3- or 4-dimensional manifold, while the τ model can be defined only in the case of a cylinder $Z = I \times Y$, where I is a (possibly infinite) interval.

Definition 11. Let M be a 3- or 4-dimensional manifold equipped with a spin^c structure $\mathfrak{s}$. The space $\mathcal{C}^\sigma(M,\mathfrak{s})$ is the set of triples (B, s, ψ) such that

- B is a Clifford connection,
- $0 \leq s < \infty$, and
- $\psi \in \Gamma(E)$ with $||\psi||_{L^2(M)} = 1$.

There is a blowdown map $\pi : \mathcal{C}^\sigma(M,\mathfrak{s}) \to \mathcal{C}(M,\mathfrak{s})$ given by $\pi(B, s, \psi) = (B, s\psi)$. This is a diffeomorphism over the space $\mathcal{C}^*(M,\mathfrak{s})$ of irreducible configurations. The gauge action on $\mathcal{C}^\sigma(M,\mathfrak{s})$ is free, and the quotient is denoted $\mathcal{B}^\sigma(M,\mathfrak{s})$.

There is a vector field SW^σ on $\mathcal{C}^\sigma$ given by

$$SW^\sigma(B,s,\psi) = \left(-\frac{1}{2} * F_{B^t} - s^2\,(\psi\psi^*)_0, -\Lambda(B,s,\psi)r, -D_B\psi + \Lambda(B,s,\psi)\psi\right).$$

At the irreducible configurations, this vector field corresponds to the SW defined above under the blowdown map. There is a description of the zeros of SW^σ in terms of the zeros of SW.

Proposition 12. *A configuration* $(B, s, \psi) \in \mathcal{C}^\sigma(Y, \mathfrak{s})$ *is a zero of* SW^σ *if and only if*

1. $s \neq 0$ and $(B, s\psi)$ is a zero of SW; or

2. $s = 0$, B is a flat connection, and ψ is an eigenvector of D_B.

There is another version of the blowup that is well suited to regarding the 4-dimensional equations as a gradient flow. It makes sense only for a cylinder $I \times Y$. The difference is that instead of normalizing the spinor to have a global L^2 norm equal to one, we require its L^2 norm *on each time slice* to be equal to one.

Definition 13. Let $Z = I \times Y$ and let $\mathfrak{s}$ be a spin^c structure on Z. Then $\mathcal{C}^\tau(Z, \mathfrak{s})$ is the set of triples (A, f, ψ) such that

- A is a Clifford connection on Z,
- f is a smooth function $\mathbf{R} \to [0, \infty]$, and
- ψ is a spinor such that $||\psi||_{L^2(\{t\}\times Y)} = 1$ for all t.

There is a blowdown map to $\mathcal{C}(Z, \mathfrak{s})$ in this case as well. In addition, for each $t \in \mathbf{R}$, there is a partially defined restriction map $\mathcal{C}^\tau(Z) \to \mathcal{C}^\sigma(Y)$. Its domain is the set

$$\{(A, f, \psi) \in \mathcal{C}^\tau(Z, \mathfrak{s}) | f(t) \neq 0\}.$$

2.7 The topology of $\mathcal{B}(Y, \mathfrak{s})$

We need to understand the algebraic topology of the space $\mathcal{B}(Y, \mathfrak{s})$. First, we want to understand $\mathcal{G}(Y) = Map(Y, S^1)$. There is a short exact sequence

$$0 \to \mathcal{G}_0(Y) \to \mathcal{G}(Y) \to H^1(Y; \mathbf{Z}) \to 0$$

where $\mathcal{G}_0$ is the identity component of $\mathcal{G}$. There is a homotopy equivalence $\mathcal{G}_0 \to S^1$ and the sequence splits. Since $\mathcal{G}$ acts freely on $\mathcal{C}^\sigma$, we conclude that $\mathcal{B}(Y, \mathfrak{s})$ has the weak homotopy type of a classifying space for $S^1 \times H^1(Y;\mathbf{Z})$. Therefore its cohomology ring is

$$H^*(\mathcal{B}(Y);\mathbf{Z}) \cong \Lambda^*(H_1(Y;\mathbf{Z})/T) \otimes_{\mathbf{Z}} \mathbf{Z}[U],$$

where T is the torsion subgroup of $H_1(Y;\mathbf{Z})$ (so $Hom(H^1(Y;\mathbf{Z}),\mathbf{Z}) \cong H_1(Y;\mathbf{Z})/T$) and $U \in H^2(\mathbf{C}P^\infty)$ is a generator.

2.8 Moduli spaces of trajectories

Given critical points α and β in $\mathcal{C}^\sigma(Y)$, we want to form suitable moduli spaces of configurations on $I \times Y$ that solve the Seiberg-Witten equations and are asymptotic to α as $t \to -\infty$ and to β as $t \to +\infty$. We will use these to define the differential in the Floer complex.

In more detail, let γ_0 be a configuration on $I \times Y$ which agrees for $|t|$ large with α and β. Define a configuration space $\mathcal{C}_k^\tau(\alpha,\beta)$ to consist of all configurations whose difference from γ_0 is in L_k^2. Then the gauge group $\mathcal{G}_{k+1} = \left\{u : \mathbf{R} \times Y \to S^1 : u - 1 \in L^2_{k+1}\right\}$ acts on this, with a quotient $\mathcal{B}_k^\tau(Y,[\alpha],[\beta])$. We want to study the moduli space

$$M(\alpha,\beta) = \{[\gamma] \in \mathcal{B}_k^\tau(Y,[\alpha],[\beta] : \mathcal{F}^\tau(\gamma) = 0\}.$$

For generic values of the perturbation, all critical points are nondegenerate and this moduli space is a finite-dimensional smooth oriented manifold. It is not necessarily connected or equidimensional. There is an $\mathbf{R}$-action by translation; denote the quotient by $\check{M}(\alpha,\beta)$.

The linearization of $\mathcal{F}^\tau$ at a configuration γ is a Fredholm operator

$$\mathcal{K}_\gamma^\tau \to V_\gamma^\tau$$

whose index is defined to be the relative grading $gr(\alpha,\beta)$ of α and β. This grading descends to the quotient space. If z is the homotopy class of γ as an element of

$\pi_1(\mathcal{B}^\sigma(Y), \alpha, \beta)$, then we may also write $gr_z([\alpha], [\beta])$ for $gr(\alpha, \beta)$. However, this index depends on z.

Lemma 14 ([5], Lemma 14.4.6). *The relative grading is defined up to* $([u] \cup c_1(\mathfrak{s}))[Y]$.

Definition 15. Denote by X the set of homotopy classes of paths from $[\alpha]$ to $[\beta]$ in $\mathcal{B}^\sigma(Y, \mathfrak{s})$.

Any $z \in X$ determines a submanifold $M_z(\alpha, \beta)$ of $M(\alpha, \beta)$. Given two paths z and w, their difference defines an element $[z - w]$ of

$$\pi_1(\mathcal{B}^\sigma(Y, \mathfrak{s})) \cong H^1(Y; \mathbf{Z}).$$

Similarly, if α and β are reducible critical points, we can form the moduli spaces $M^{red}(\alpha, \beta)$ and $\check{M}^{red}(\alpha, \beta)$ of reducible flow lines from α to β. There is a grading $\overline{gr}$ in this case as well. This reducible grading is related to the ordinary grading by

$$\overline{gr}([\alpha]) = \begin{cases} gr([\alpha]) & \lambda > 0 \\ gr([\alpha]) - 1 & \lambda < 0. \end{cases}$$

2.9 The Floer homology groups

Fix a metric g and a generic perturbation q. Let C_o be the free abelian group generated by the set of irreducible critical points of $CSD + q$; this is a finite set. Recall that reducible critical points consist of a connection B and an eigenvector of the perturbed Dirac operator $D_B + q$ with eigenvalue $\lambda \in \mathbf{R}$. Let C_u and C_s be the free abelian groups generated by the sets of unstable and stable critical points, those with $\lambda < 0$ and $\lambda > 0$ respectively. Set

$$\begin{aligned} \check{C}(Y, \mathfrak{s}, q) &= C_o \oplus C_s, \\ \hat{C}(Y, \mathfrak{s}, q) &= C_o \oplus C_u, \\ \bar{C}(Y, \mathfrak{s}, q) &= C_s \oplus C_u. \end{aligned}$$

We will define differentials on these groups, and then the homology of these chain complexes will be the three versions of Floer homology.

The boundary maps will count trajectories belonging to 0-dimensional moduli spaces $\check{M}([\alpha],[\beta])$. We define

$$\begin{aligned} \partial_o^o : C_o &\rightarrow C_o \\ \partial_s^o : C_o &\rightarrow C_s \\ \partial_s^u : C_u &\rightarrow C_s \\ \partial_o^u : C_u &\rightarrow C_o \end{aligned}$$

by

$$\begin{aligned} \partial_o^o[\alpha] &= \sum_{[\beta]\in C_o} |\check{M}_z([\alpha],[\beta])|[\beta] \\ \partial_s^o[\alpha] &= \sum_{[\beta]\in C_s} |\check{M}_z([\alpha],[\beta])|[\beta] \\ \partial_o^u[\alpha] &= \sum_{[\beta]\in C_o} |\check{M}_z([\alpha],[\beta])|[\beta] \\ \partial_s^u[\alpha] &= \sum_{[\beta]\in C_s} |\check{M}_z([\alpha],[\beta])|[\beta]. \end{aligned}$$

Here the absolute value denotes a suitably oriented count of points. There are no operators ∂_o^s or ∂_u^o because the moduli spaces that would correspond to these are empty.

There are similar maps defined using the reducible moduli spaces M^{red}. Define

$$\begin{aligned} \bar{\partial}_u^u : C_u &\rightarrow C_u \\ \bar{\partial}_s^u : C_u &\rightarrow C_s \\ \bar{\partial}_u^s : C_s &\rightarrow C_u \\ \bar{\partial}_s^s : C_s &\rightarrow C_s \end{aligned}$$

by

$$
\begin{aligned}
\bar{\partial}^u_u[\alpha] &= \sum_{[\beta]\in C_u} |\check{M}^{red}_z([\alpha],[\beta])|[\beta] \\
\bar{\partial}^u_s[\alpha] &= \sum_{[\beta]\in C_s} |\check{M}^{red}_z([\alpha],[\beta])|[\beta] \\
\bar{\partial}^s_u[\alpha] &= \sum_{[\beta]\in C_u} |\check{M}^{red}_z([\alpha],[\beta])|[\beta] \\
\bar{\partial}^s_s[\alpha] &= \sum_{[\beta]\in C_s} |\check{M}^{red}_z([\alpha],[\beta])|[\beta].
\end{aligned}
$$

These maps are the components of the boundary maps in the Floer complex. Note that both ∂^u_s and $\bar{\partial}^u_s$ are defined, but the maps are different.

Definition 16. On $\bar{C} = C_s \oplus C_u$, define

$$
\bar{\partial} = \begin{pmatrix} \bar{\partial}^s_s & \bar{\partial}^u_s \\ \bar{\partial}^s_u & \bar{\partial}^u_u \end{pmatrix}.
$$

On $\hat{C} = C_o \oplus C_u$, define

$$
\hat{\partial} = \begin{pmatrix} \partial^o_o & \partial^u_o \\ -\bar{\partial}^s_u \partial^o_s & -\bar{\partial}^u_u - \bar{\partial}^s_u \partial^u_s \end{pmatrix}.
$$

Finally, on $\check{C} = C_o \oplus C_s$, define

$$
\check{\partial} = \begin{pmatrix} \partial^o_o & \partial^u_o \bar{\partial}^s_u \\ \partial^o_s & \bar{\partial}^s_s - \partial^u_s \bar{\partial}^s_u \end{pmatrix}.
$$

It is proved in [5] that each of these operators has square zero. The homology groups of these complexes are independent of the choice of metric and perturbation; thus they are invariants of $(Y, \mathfrak{s})$. They are denoted $\overline{HM}(Y,\mathfrak{s})$, $\widecheck{HM}(Y,\mathfrak{s})$, and $\widehat{HM}(Y,\mathfrak{s})$.

2.10 Maps from cobordisms

Let W be a cobordism between closed oriented 3-manifolds Y^- and Y^+; that is, $\partial W = Y^+ \coprod(-Y^-)$ as oriented manifolds. Let $\mathfrak{s}$ be a spin^c structure on W that restricts to the spin^c structures $\mathfrak{s}^\pm$ on $Y^\pm$. Let $u \in H^*(\mathcal{B}(W, \mathfrak{s}))$ be a cohomology class. Then we will define maps

$$HM^\circ(u|W, \mathfrak{s}) : HM^\circ(Y^-, \mathfrak{s}^-) \to HM^\circ(Y^+, \mathfrak{s}^+)$$

using moduli spaces of solutions on the manifold W^* obtained by attaching cylindrical ends to W. These will satisfy a functoriality property under composition of cobordisms. Additionally, when W is a cylindrical cobordism $I \times Y$, we can identify $\mathbb{A}_Y$ with $\mathbb{A}_W$ and this will give rise to the $\mathbb{A}_Y$-module structure.

Fix nondegenerate critical points $\alpha^\pm \in M(Y^\pm, \mathfrak{s}^\pm)$ and a configuration γ_0 on W^* that agrees with $\alpha^\pm$ at the ends. Define $M(W, \alpha, \beta)$ to consist of solutions γ to the four-dimensional equations on W^* such that $\gamma - \gamma_0 \in L^2$, up to gauge equivalence. Then the evaluation of u on $M(W, \alpha, \beta)$ gives the matrix coefficient of $HM^\circ(u|W, \mathfrak{s})$ from α to β.

2.11 The module structure on Floer homology

We can make the Floer homology groups into modules over the ring

$$\mathbb{A}_\dagger(Y) = H^*(\mathcal{B}(Y, \mathfrak{s}); \mathbf{Z}) = \Lambda^*(H_1(Y; \mathbf{Z})/T) \otimes_{\mathbf{Z}} \mathbf{Z}[U].$$

These maps are defined just as in the definition of the boundary maps. Instead of counting points in zero-dimensional moduli spaces, however, we now evaluate a class $u \in \mathbb{A}(Y)$ on moduli spaces of the appropriate dimension.

Chapter 3

The equations on $S^1 \times \Sigma$

3.1 The form of the equations

We want to study the Floer homology groups of $S^1 \times \Sigma$ for spin^c structures $\mathfrak{s}$. There is a general result that restricts the set of spin^c structures for which the Floer homology groups may be nonzero.

Proposition 17 ([5], Corollary 40.1.2). *Suppose $C \subset Y$ is an embedded closed surface of genus $g \geq 1$ and let $\mathfrak{s}$ be a* spin^c *structure on Y such that $c_1(\mathfrak{s})$ is not a torsion class. If*

$$|\langle c_1(\mathfrak{s}), [C] \rangle| > 2g - 2,$$

then the Floer homology groups of $(Y, \mathfrak{s})$ vanish.

Corollary 18. Let $\mathfrak{s}$ be a spin^c structure on $Y = S^1 \times \Sigma$. Then $HM^{\circ}(Y, \mathfrak{s})$ vanishes unless $c_1(\mathfrak{s})$ is Poincare dual to $2k[S^1]$ with $|k| \leq g - 1$.

Proof. This follows from taking C to be the surfaces $\{pt\} \times \Sigma$ and $S^1 \times \alpha$ for loops α in Σ.1 □

Because of this, we will confine our attention to the spin^c structures $\mathfrak{s}_k$ with $c_1(\mathfrak{s}_k) = 2kPD[S^1]$. We can write the Seiberg-Witten equations in a more explicit form suitable for computations. Fix a constant curvature metric on Σ and let $S^1 =$

$\mathbf{R}/2\pi\mathbf{Z}$. For reference, fix a smooth Clifford connection B_0 on Y that is rotationally invariant and that satisfies $\nabla^{B_0}_{\frac{\partial}{\partial\theta}}\Psi = \frac{\partial\Psi}{\partial\theta}$. Then

$$\mathcal{A}(Y) = \{B_0 + a | a \in \Omega^1(Y; i\mathbf{R})\}$$

is the space of Clifford connections. One-forms $a \in \Omega^1(Y; i\mathbf{R})$ may be decomposed as $a = a_\theta + ibd\theta$, where a_θ is a family of 1-forms on $\Sigma_g \times \{\theta\}$, and b is a real-valued function on Y. Notice that by ignoring the θ direction, a connection on Y gives rise to a family of connections on Σ parametrized by $\theta \in S^1$. However, given such a family, we do not have enough information to reconstruct the connection on Y since we do not know how to differentiate in the θ direction. For this, we need the function b as well; then covariant differentiation in the θ direction is given by

$$\nabla_{\frac{\partial}{\partial\theta}}\Psi = \frac{\partial\Psi}{\partial\theta} + ib\theta. \tag{3.1.1}$$

Since we are in a spin^c structure $\mathfrak{s}_k$, the spinor bundle S is pulled back from Σ. It decomposes into the ±1-eigenspaces for the action of $-i\rho(d\theta)$, $S^\pm$. For the spin^c structure $\mathfrak{s}_k$, $S^\pm$ has degree $k \pm (g-1)$.

Since the spinor bundle $S^+ \oplus S^-$ is pulled back from Σ, we may regard a spinor $\Phi = (\alpha, \beta)$ on Y as a family of spinors on Σ, parametrized by $\theta \in S^1$.

Each of the connections B_θ defines a $\bar\partial$-operator on the surface $\{\theta\} \times \Sigma$. With respect to these operators and their formal adjoints, the Dirac operator is given by

$$D_B = \begin{pmatrix} i\nabla_\theta & \sqrt{2}\bar\partial^*_B \\ \sqrt{2}\bar\partial_B & -i\nabla_\theta \end{pmatrix}$$

Now we turn to the curvature equation. The curvature of A is given by

$$F_B = F_{B_0} + d_2 a_\theta + (idb - \frac{\partial a_\theta}{\partial\theta}) \wedge d\theta \tag{3.1.2}$$

and therefore, since $\rho(d\theta)$ acts as multiplication by i on S^+ and $-i$ on S^-, we have

$$\rho(F_{B^t}) = \begin{pmatrix} -i\Lambda(F_{B_0^t} + 2d_2 a_\theta) & \rho(db_\theta + i\frac{\partial a_\theta}{\partial \theta}) \\ -\rho(db_\theta + i\frac{\partial a_\theta}{\partial \theta}) & i\Lambda(F_{B_0^t} + 2d_2 a_\theta) \end{pmatrix}. \tag{3.1.3}$$

Here Λ is contraction with the area form of Σ. The final term is the quadratic term $(\Phi\Phi^*)_0$. In matrix form, this term is the endomorphism

$$\begin{pmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\langle\cdot,\beta\rangle \\ \beta\langle\cdot,\alpha\rangle & \frac{1}{2}(|\beta|^2 - |\alpha|^2) \end{pmatrix} \tag{3.1.4}$$

of $S^+ \oplus S^-$.

3.2 Vanishing of a spinor component

We show that for any $\gamma \in \mathcal{B}(Y, \mathfrak{s}_k)$ solving the Seiberg-Witten equations, one of the two spinor components must vanish. In fact, we prove a slight generalization of this. The parameter λ in the lemma below is relevant for reducible solutions to the equations, and q will appear when we consider a perturbation in the case $c_1(\mathfrak{s}_k) = 0$.

Lemma 19. *Let $\lambda \in \mathbf{R}$, let $q \in \mathbf{R}$, and suppose (B, Ψ) satisfies the modified Seiberg-Witten equations*

$$\frac{1}{2}\rho(*F_{B^t}) + (\Phi\Phi^*)_0 = \rho(iqd\theta) \tag{3.2.1}$$

$$D_B\Phi = \lambda\Phi. \tag{3.2.2}$$

Then $\alpha = 0$ or $\beta = 0$.

Proof. First pull back to a 2π-periodic configuration on $\mathbf{R} \times \Sigma$. Apply the gauge transformation

$$u(\theta, z) = \exp\int_0^\theta ib(\phi, z)d\phi;$$

then the $d\theta$ component of B is zero. Our new configuration (B, α, β) is no longer 2π-periodic; however, since u is circle-valued, $|\alpha|^2$ and $|\beta|^2$ are still 2π-periodic. Now calculate:

$$\begin{aligned}
-i\frac{\partial^2\beta}{\partial\theta^2} &= -\sqrt{2}\frac{\partial}{\partial\theta}\left(\bar{\partial}_B\alpha\right) + \lambda\frac{\partial\beta}{\partial\theta} \\
&= -\sqrt{2}\left(\bar{\partial}_B\frac{\partial\alpha}{\partial\theta} + \left(\frac{\partial B}{\partial\theta}\right)^{0,1}\wedge\alpha\right) + \lambda\frac{\partial\beta}{\partial\theta} \\
&= -\sqrt{2}\left[\bar{\partial}_B\left(-i\lambda\alpha + i\sqrt{2}\bar{\partial}_B^*\beta\right) + \left(\frac{\partial B}{\partial\theta}\right)^{0,1}\wedge\alpha\right] + \lambda\frac{\partial\beta}{\partial\theta} \\
&= -2i\bar{\partial}_B\bar{\partial}_B^*\beta + i\sqrt{2}\lambda\bar{\partial}_B\alpha + \lambda\frac{\partial\beta}{\partial\theta} + \beta\left|\alpha\right|^2 \\
&= -2i\bar{\partial}_B\bar{\partial}_B^*\beta + i\lambda^2\beta + -i\sqrt{2}\beta\left|\alpha\right|^2.
\end{aligned}$$

Now take the pointwise inner product with β and integrate over Σ to conclude that

$$\int_\Sigma\left\langle\frac{\partial^2\beta}{\partial\theta^2},\theta\right\rangle = \int_\Sigma\left|\bar{\partial}_B^*\beta\right|^2 + \lambda^2\int_\Sigma|\beta|^2 + \int_\Sigma|\alpha|^2|\beta|^2$$

for every $\theta \in \mathbf{R}$. Finally, integrate this from 0 to 2π. Note that

$$\begin{aligned}
\left\langle\frac{\partial^2\beta}{\partial\theta^2},\theta\right\rangle &= \frac{\partial}{\partial\theta}\left\langle\frac{\partial\beta}{\partial\theta},\beta\right\rangle - \left|\frac{\partial\beta}{\partial\theta}\right|^2 \\
&= \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\left(|\beta|^2\right) - \left|\frac{\partial\beta}{\partial\theta}\right|^2.
\end{aligned}$$

Since $|\beta|^2$ is 2π-periodic, so are its derivatives, and so when we integrate, the boundary term drops out and we obtain

$$\lambda^2\left\|\beta\right\|^2 = \left\|\frac{\partial\beta}{\partial\theta}\right\|^2 + 2\left\|\bar{\partial}_{B_\theta}^*\beta\right\|^2 + \sqrt{2}\left\|\alpha\otimes\beta\right\|^2,$$

where the norms are all taken in $L^2([0,2\pi)\times\Sigma)$. An analogous calculation gives

$$\lambda^2\left\|\alpha\right\|^2 = \left\|\frac{\partial\alpha}{\partial\theta}\right\|^2 + 2\left\|\bar{\partial}_{B_\theta}\alpha\right\|^2 + \sqrt{2}\left\|\alpha\otimes\beta\right\|^2$$

as well. By applying the triangle inequality to each component of the Dirac equation, we also find

$$\lambda^2\left\|\alpha\right\|^2 \leq 2\left\|\bar{\partial}_{B_\theta}^*\beta\right\|^2 + \left\|\frac{\partial\alpha}{\partial\theta}\right\|^2$$

and

$$\lambda^2 \|\beta\|^2 \leq 2 \left\|\bar{\partial}_{B_\theta}\alpha\right\|^2 + \left\|\frac{\partial \beta}{\partial \theta}\right\|^2 .$$

Adding each pair of equations and comparing the results, we conclude that $\|\alpha \otimes \beta\|^2 = 0$. □

3.3 Rotational invariance and dimensional reduction

Given a solution to the modified Seiberg-Witten equations (3.2.1) on Y, pull it back to $[0, 2\pi] \times \Sigma$ and kill $(B - B_0)(\frac{d}{d\theta})$ by a gauge transformation. Then we have a path $(B_\theta, \alpha_\theta, \beta_\theta)$ of connections and spinors on Σ_g. They satisfy the following equations:

$$i\sqrt{2}\bar{\partial}^*_{B_\theta}\beta + \frac{\partial \alpha}{\partial \theta} = 0$$

$$-i\sqrt{2}\bar{\partial}_{B_\theta}\alpha + \frac{\partial \beta}{\partial \theta} = 0$$

$$i\Lambda(F_{B_\theta} + d_2 a_\theta) = \frac{1}{2}(|\beta|^2 - |\alpha|^2) - q$$

$$\alpha\beta^* = i\rho_3\left(\frac{\partial a_\theta}{\partial \theta}\right)$$

$$\beta\alpha^* = -i\rho_3\left(\frac{\partial a_\theta}{\partial \theta}\right).$$

The last two equations are to be interpreted as equations for sections of $Hom(S^+, S^-)$ and $Hom(S^-, S^+)$ respectively. Differentiate the first equation with respect to θ to get

$$\begin{aligned}
\frac{1}{\sqrt{2}}\frac{\partial^2\alpha}{\partial\theta^2} &= i\frac{\partial}{\partial\theta}\left(i\bar{\partial}^*_{B_0}\beta + ia^{0,1}_\theta \llcorner \beta\right) \\
&= i\bar{\partial}^*_{B_\theta}\frac{\partial\beta}{\partial\theta} + i\frac{\partial a^{0,1}_\theta}{\partial\theta}\llcorner\beta \\
&= i\bar{\partial}^*_{B_\theta}\left(\sqrt{2}i\bar{\partial}_{B_\theta}\alpha\right) + i\frac{\partial a^{0,1}_\theta}{\partial\theta}\llcorner\beta \\
&= -\bar{\partial}^*_{B_\theta}\left(\sqrt{2}\bar{\partial}_{A_\theta}\alpha\right) + \frac{-1}{\sqrt{2}}\alpha|\beta|^2.
\end{aligned}$$

Now take the inner product with α and integrate along Σ to get

$$-\frac{1}{\sqrt{2}}\int_\Sigma \left\langle \frac{\partial^2 \alpha}{\partial \theta^2}, \alpha \right\rangle - \sqrt{2}|\bar{\partial}_{B_\theta}\alpha|^2 - \frac{1}{\sqrt{2}}\int_\Sigma |\alpha|^2|\beta|^2 = 0$$

for all $\theta \in [0,1]$. This makes sense in S^1, so

$$\frac{1}{\sqrt{2}}\int_Y \left|\frac{\partial \alpha}{\partial \theta}\right|^2 + \sqrt{2}\int_Y |\bar{\partial}_{B_\theta}\alpha|^2 + \sqrt{2}\int_Y |\alpha|^2|\beta|^2 = 0.$$

In particular, this implies that α is independent of θ. A similar argument shows that β is independent of θ as well.

To summarize, we have the following result.

Proposition 20. *If (B, Ψ) satisfies (3.2.1), then there is a gauge transformation $u : Y \to S^1$ such that the configuration $u \cdot (B, \Psi)$ is invariant under rotation in the S^1 factor.*

Proposition 21. *Suppose that (B, Ψ) satisfies the equations (3.2.1) and $q = 0$ if $k \neq 0$. Then at least one component of the spinor Ψ must vanish: if $k \leq 0$, then $\beta = 0$ and if $k \geq 0$, then $\alpha = 0$. In particular, in the case $k = 0$, all solutions to the equations with $q = 0$ are reducible.*

Proof. This follows immediately from the Chern-Weil formula

$$2k = \frac{i}{2}\int_\Sigma F_B = \frac{1}{4\pi}\int_\Sigma (|\beta|^2 - |\alpha|^2 - q)dvol_\Sigma, \tag{3.3.1}$$

together with the fact that $\alpha\beta = 0$. □

Rotational invariance allows us to reduce the equations (3.2.1) to a system of equations on Σ. First, we consider irreducible solutions. We may suppose that either $k < 0$ or $k = 0$ and $q < 0$. Then $\beta = 0$, and the curvature equation gives

$$\frac{1}{2}|\alpha|^2 \quad = \quad -i\Lambda F_{B_\theta} - q = -i\Lambda(2F_{B_+} + F_{T^*\Sigma}) + |q|$$

or

$$i\Lambda F_{B_+} + \frac{1}{2}|\alpha|^2 - ((2g-2)\pi + |q|) = 0$$

since we gave $T\Sigma$ a standard connection of constant curvature. Here B_+ denotes the connection in S^+ obtained by restricting B. The Dirac equation becomes simply

$$\bar{\partial}_B \alpha = 0.$$

These are the *vortex equations* on Σ.

In the case of reducible solutions, the equation is simply

$$F_{B^t} = 0.$$

Thinking of harmonic forms, for instance, we see that the gauge equivalence classes of connections on S with flat determinant are given by the $(2g+1)$-dimensional torus

$$T_Y = iH^1(Y;\mathbf{R})/2\pi i H^1(Y;\mathbf{Z}).$$

3.4 The vortex equations

Let M be a Kähler manifold and $E \to M$ a complex vector bundle with a hermitian metric H.

The vortex equations are

$$\begin{aligned} F_A^{0,2} &= 0 & (3.4.1)\\ \bar{\partial}_A \phi &= 0 & (3.4.2)\\ i\Lambda F_A + \frac{1}{2}\phi\otimes\phi^* &= \frac{\tau}{2} I & (3.4.3) \end{aligned}$$

for a unitary connection A and a section ϕ. The group of bundle automorphisms of E acts on the space of pairs (A,ϕ), and we want to understand the space of solutions up to gauge equivalence. We restate the problem using the following fact, proved for instance in [2].

Fact 22. *Let $E \longrightarrow M$ be a holomorphic vector bundle over a complex manifold, and let H be a hermitian metric on E. Then there is a unique connection $D_{E,H}$ on E that is compatible with the metric and satisfies $(D_{E,H})^{0,1} = \bar{\partial}_E$.*

This fact allows us to restate the problem, following Bradlow ([1]). We will fix a holomorphic structure ($\bar{\partial}$-operator) on E together with a section ϕ of E holomorphic with respect to that structure. Then we will look for a hermitian metric K such that $D_{E,K}$ satisfies the vortex equations.

In our case, E is a line bundle, so once we fix a background hermitian metric H, any metric K has the form

$$K = e^{2u}H$$

for some real-valued function $u : X \longrightarrow \mathbf{R}$. We need to rewrite the equations in terms of u. If ∇_H and ∇_K are the connections associated to $\bar{\partial}$ and H, K respectively, then

$$iF_K = iF_H + \Delta u.$$

Proposition 23. *([1], Lemma 4.1) Let $K = e^{2u}H$. Then the vortex equation for u becomes*

$$i\Lambda F_H + \Delta u + \frac{1}{2}|\phi|_H^2 e^{2u} - \frac{\tau}{2} = 0. \tag{3.4.4}$$

As a first step toward solving this, observe that the Laplacian

$$\Delta : L_k^2(X) \longrightarrow L_{k-2}^2(X)$$

has one-dimensional kernel and cokernel. The kernel consists of the constant functions, and the image consists of those functions that have average value zero. As a consequence of this, we may find a function $v \in C^\infty(X; \mathbf{R})$ such that

$$-\Delta v = \left(i\Lambda F_H - \frac{\tau}{2}\right) - \int_X \left(i\Lambda F_H - \frac{\tau}{2}\right),$$

and v is unique up to a constant. Let

$$w = 2(u - v);$$

then u solves 3.4.4 if and only if w solves

$$-\Delta w - \left(|\phi|_H^2 e^{2v}\right) e^w - \left(\int i\Lambda F_H - \frac{\tau}{2}\right) = 0.$$

This is an equation studied by Kazdan and Warner in [4]. Their result is the following.

Theorem 24. *([4]) Let M be a closed Riemannian manifold and consider the equation*

$$-\Delta u + he^u - c = 0, \tag{3.4.5}$$

where c is a real constant and $h \in C^\infty(M, \mathbf{R})$ is not identically zero. Suppose $h \leq 0$ everywhere.

- *If $c = 0$, then there are no solutions unless h changes sign.*
- *If $c < 0$, then there exists a unique $u \in C^\infty(M, \mathbf{R})$ solving Equation 3.4.5.*

In our case, we have

$$c = \left(\int i\Lambda F_H - (2g-2)\pi\right) = 2\pi\left(c_1 - (g-1)\right) = -2\pi k.$$

In the case $k \neq 0$, we can conclude that given an effective divisor, there is a unique solution to the vortex equations. In addition, this gives another proof that there are no irreducible solutions in the case $k = 0$.

3.5 The case $c_1 \neq 0$

When the spin^c structure has nonzero Chern class, we have seen that the equations reduce to the vortex equations on Σ. Since vortices correspond to effective divisors of the appropriate degree, we conclude that the moduli space is a copy of $Sym^{g-1-|k|}\Sigma$.

Next we will show that the solutions are Morse-Bott nondegenerate. This will follow from nondegeneracy for the vortex equations.

Lemma 25. *At any solution (A,Φ) to the vortex equations, the linearized vortex operator is a surjection*

$$\Gamma(S^+\oplus iT^*\Sigma)\to\Gamma(S^-\oplus i\mathbf{R}).$$

Proof. The operator is given by

$$(a,\phi)\longmapsto\left(2\Lambda da+2Re\left\langle\Phi,\phi\right\rangle,\bar{\partial}_A\phi+a^{0,1}\Phi\right). \tag{3.5.1}$$

We will show that the image is dense. So suppose we have some pair (f,Ψ) orthogonal to the image. Here f is a function and Ψ a section of $S^-=\Lambda^{0,1}\otimes S^+$. By assumption, for any pair (a,ϕ),

$$\begin{aligned}0=&\langle 2*da+2iReh(\Phi,\phi),f\rangle+\left\langle\bar{\partial}_A\phi+a^{0,1}\Phi,\Psi\right\rangle\\ =&2\left\langle a,d^*(f\omega)\right\rangle+2\left\langle\phi,if\Phi\right\rangle+\left\langle\phi,\bar{\partial}_A^*\Psi\right\rangle+\left\langle a^{0,1},\Psi\otimes\Phi^*\right\rangle.\end{aligned}$$

Taking $\phi=0$, we find that

$$-2d^*(f\omega)+\Psi\otimes\Phi^*=0,$$

which implies immediately (by comparing types) that $d(*f\omega)=df$ has type (1,0), and so that f is a holomorphic function and therefore constant. Since f is constant, the equation becomes $\Psi\otimes\Phi^*=0$, which implies that Ψ vanishes wherever Φ does not. But the zero set of Φ is a finite set, so $\Psi=0$. Now taking $a=0$, we find that $f\Phi=0$ and so $f=0$. Therefore the linearized operator has dense image.

The linearized operator is the second map in an elliptic complex, in which the first map is the map $iC^\infty(\Sigma) \to \Gamma(S^+ \oplus iT^*\Sigma)$ given by linearizing the gauge action:

$$\xi \longmapsto (-d\xi, \xi\Phi)\,. \tag{3.5.2}$$

Ellipticity of the complex implies that the linearized vortex operator has closed image.

□

We can form an "extended Hessian" for the vortex equations just as we did for the Seiberg-Witten equations. As in the case of the Seiberg-Witten equations, we use a gauge-fixing condition to replace the elliptic complex with a single elliptic operator. The gauge-fixing condition for the vortex operator is

$$d^*a + Im\,\langle\Phi, \phi\rangle = 0. \tag{3.5.3}$$

Combining this gauge-fixing map with the linearized vortex map, we obtain a map

$$\Gamma(S^+ \oplus iT^*\Sigma) \to \Gamma(S^- \oplus i\mathbf{R} \oplus i\mathbf{R})$$

given by

$$\begin{pmatrix} 2\Lambda d & 2Re\,\langle\Phi, \cdot\rangle \\ \pi^{0,1}\Phi & \bar{\partial}_A \\ d^* & Im\,\langle\Phi, \cdot\rangle \end{pmatrix}.$$

Since we are on a closed manifold, the zeroth-order terms are compact operators, and so this operator has the same index as the direct sum of

$$d + d^* : \Omega^1 \to \Omega^2 \oplus \Omega^0$$

and

$$\bar{\partial}_A : \Gamma(S^+) \to \Gamma(S^-).$$

The index of the first operator is minus the Euler characteristic of Σ, or $2 - 2g$. By the Riemann-Roch theorem, the second has complex index

$$deg(S^+) - (g-1) = k.$$

We showed above that the operator is surjective. So our operator has a kernel of real dimension $2(g-1+|k|)$, which is exactly the dimension of $Sym^{g-1+|k|}\Sigma$ as expected.

Morse-Bott nondegeneracy for the Seiberg-Witten equations is a consequence of this; we consider the extended Seiberg-Witten Hessian at a solution $(A, \alpha, 0)$ coming from the vortex equations.

3.6 The case $c_1(\mathfrak{s}) = 0$: perturbing the equations

We now consider the case of a spin structure, i.e. $c_1 = 0$. As stated above, critical points of the Chern-Simons-Dirac functional downstairs on $\mathcal{B}(Y, \mathfrak{s})$ are all reducible, and they are given by the gauge equivalence classes of connections that induce flat connections in the determinant line bundle. These are classified by $iH^1(Y; \mathbf{R})/2\pi i H^1(Y; \mathbf{Z})$.

On the blowup $\mathcal{B}^\sigma(Y, \mathfrak{s})$, solutions to the equations are triples (B, s, Ψ), where B is a flat connection, $s = 0$, and Ψ is an eigenvector for D_B of unit L^2 norm. The equations on Σ become

$$\begin{aligned} F_B &= 0 \\ \sqrt{2}\bar{\partial}_B^*\beta + i\left(\frac{\partial}{\partial\theta} + ib\right)\alpha &= \lambda\alpha \\ \sqrt{2}\bar{\partial}_B\alpha - i\left(\frac{\partial}{\partial\theta} - ib\right)\beta &= \lambda\beta. \end{aligned}$$

The bundles S^+ and S^- are complex line bundles of degrees $g-1$ and $1-g$ respectively.

We need to know which connections B give rise to Dirac operators D_B that have a nontrivial kernel.

Proposition 26. *If $D_B\Psi = 0$ and $\Psi \neq 0$, then B is gauge equivalent to a connection with $b = 0$.*

Proof. Decompose the sections α and β as $\alpha = \sum_{n\in\mathbf{Z}} \alpha_n e^{in\theta}$ and $\beta = \sum_{n\in\mathbf{Z}} \beta_n e^{in\theta}$; then for each integer n, we get the equations

$$\begin{aligned} -i\,(n+b)\,\alpha_n + \bar{\partial}_A^* \beta_n &= 0 \\ i\,(n+b)\,\alpha_n + \bar{\partial}_A \alpha_n &= 0. \end{aligned}$$

Apply $\bar{\partial}_B^*$ to the second equation, substitute into the first, and integrate against α_n to find

$$\left\|\bar{\partial}_B \alpha_n\right\|^2_{L^2(\Sigma)} = -\,(n+b)^2 \left\|\alpha_n\right\|^2_{L^2(\Sigma)}.$$

Thus there is just one n for which $\alpha_n \neq 0$, and we must have $n = -b$ and $\alpha_n \in \ker \bar{\partial}_B$. It follows immediately that $\beta_n \in \ker \bar{\partial}_B^*$ as well. $\square$

So D_B has kernel precisely when the holomorphic line bundle defined by B lies in the theta divisor and the $d\theta$ component b of B is an integer (so it can be made to vanish by a gauge transformation). In other words, it lies in

$$\Theta \times \{0\} \subset Jac \times S^1.$$

Such critical points fail to be Morse-Bott nondegenerate. However, we can write down a perturbation of the equations that removes these singularities but will introduce irreducibles. For any spin^c 3-manifold $(Y, \mathfrak{s})$, there is a map

$$\pi : \mathcal{B}^\sigma(Y, \mathfrak{s}) \longrightarrow T_Y$$

defined as follows: first fix a base connnection B_0. Then given a configuration $(B, s, \phi) \in \mathcal{C}^\sigma(Y, \mathfrak{s})$, project $B - B_0$ to the space $i\mathcal{H}^1(Y)$ of imaginary-valued harmonic 1-forms. This projection is invariant under the action of the identity component of $\mathcal{G}(Y)$. In our case, there is even a projection

$$\pi' : \mathcal{B}^\sigma(S^1 \times \Sigma, \mathfrak{s}_0) \longrightarrow S^1$$

given by composing π with projection to $iH^1(S^1;\mathbf{R})/2\pi iH^1(S^1;\mathbf{Z})$.

We will perturb the Chern-Simons-Dirac functional by adding a term $f\circ\pi'$ for some smooth $f: S^1 \to \mathbf{R}$; by Theorem 11.1.2 of [5], such a perturbation is allowable for calculating Floer homology. The critical points of $CSD+f\circ\pi'$ downstairs satisfy the Dirac equation and the following curvature equation:

$$\frac{1}{2}\rho(*F_{B^t}) + (\Psi\Psi^*)_0 = \rho(\eta(B_h)) \tag{3.6.1}$$

Here η is an imaginary-valued harmonic 1-form defined using the inner product on

$$T_{B-B_0}T_Y = i\mathcal{H}^1(Y)$$

as the L^2 gradient of f. If we choose f to be a function only of the $d\theta$ component of B_h, then η will have the form

$$\eta = iq\left(\left(B-B_0\right)_h\right)d\theta \tag{3.6.2}$$

for a function $q: T_Y \to \mathbf{R}$.

Proposition 27. *Suppose the function $f: T_Y \to \mathbf{R}$ is such that $f(B)$ depends only on the $d\theta$ component b of $(B-B_0)_h$. Then a critical point of $CSD+f\circ\pi$ is rotationally invariant, and at least one of its spinor components vanishes.*

Proof. An irreducible solution satisfies $D_B\Psi = 0$ and so by (26), it has $b=0$ after a gauge transformation. Critical points of $CSD+f\circ\pi$ satisfy (3.2.1) with the constant q given by (3.6.2) and are thus rotationally invariant by (20).

By definition, a reducible solution has both spinor components equal to zero. An irreducible, by Proposition 26, has $b=0$. Then by Lemma19, one component vanishes. By Proposition 21, the sign of q tells us which component must vanish. $\square$

We observe that the irreducible solutions form a copy of the symmetric product. In the $q<0$ case, for instance, $\beta=0$, and reducing the equations to Σ, we get the vortex equations for the line bundle $S^+ \to \Sigma$ with the parameter τ equal to

$4(g-1)\pi + 2|q|$. This is greater than the critical value $4(g-1)\pi$, so by Theorem 24, the set of solutions up to gauge is identified with $Sym^{g-1}\Sigma$.

Proposition 28. *A reducible solution (B, Ψ) satisfies $q = 0$; that is, $h(B)$ is a critical point of $f : T(Y) \to \mathbf{R}$.*

Proof. This follows immediately from the Chern-Weil formula (3.3.1). □

It follows that above each critical point of f there is a $2g$-torus of reducible solutions.

To summarize, we have proved the following:

Theorem 29. *Let $f : S^1 \to \mathbf{R}$ be a smooth function with $f'(0) = 0$ and suppose f has exactly two critical points, a maximum at $q_+ \in S^1$ and a minimum at $q_- \in S^1$. Let M be the moduli space of gauge equivalence classes of critical points of $CSD + f \circ \pi'$. Then M intersects exactly three fibers of the map π'. The fiber over 0 consists of irreducible solutions and is a Morse-Bott nondegenerate copy of $Sym^{g-1}\Sigma$. The fibers over q_+ and q_- are $2g$-tori consisting of reducible solutions.*

The statement about the reducibles follows from the next lemma.

Lemma 30. *A configuration $(B, 0)$ is a reducible critical point of $CSD + \pi \circ f$ in $\mathcal{B}(S^1 \times \Sigma, \mathfrak{s}_0)$ if and only if*

- *B is a flat connection and*
- *$\pi'(B)$ is a critical point of f.*

Proof. By integrating (3.6.1) with $\Psi = 0$, we see that for a reducible solution $(B_\theta + ibd\theta, 0)$, $ibd\theta$ must be a critical point of the function f. □

Before we perturbed the equations, the solutions were a copy of T^{2g+1} corresponding to the flat connections on Y. In other words, over each point in the dual circle there was a $2g$-torus. The perturbation splits this torus into two $2g$-tori, located above the points q_- and q_+ of the dual circle. At the same time, it introduces irreducibles that did not exist before.

3.6.1 Morse-Bott nondegeneracy

Using the fact that the irreducibles are Morse-Bott nondegenerate, we will write down a further perturbation in the next chapter in order to show that they contribute precisely $H^*(Sym^{g-1}\Sigma)$ to the chain complex. The reducibles are not yet nondegenerate, but we should be able to make them so.

Chapter 4

Perturbing the equations

4.1 Overview

The definition of the Seiberg-Witten Floer homology groups involves a perturbation of the Chern-Simons-Dirac functional to a Morse function. The aim of this chapter is to extend the definition to the Morse-Bott situation, in which the critical points are no longer fully nondegenerate but instead form nondegenerate critical manifolds.

4.2 Sobolev completions

We will need to consider suitable Sobolev completions of the spaces we are using. For this, fix a smooth connection B_0 on Y .

Definition 31. The L^2_k configuration space $\mathcal{C}_k(Y,\mathfrak{s})$ is the space

$$\mathcal{C}_k(Y,\mathfrak{s}) = \left\{(B_0+b,\Psi)|b \in L^2_k(Y;iT^*Y), \Psi \in L^2_k(Y;S)\right\}.$$

There is also a Sobolev completion of the gauge group.

Definition 32. Let $k \geq \frac{3}{2}$. The L^2_k gauge group $\mathcal{G}_k(Y,\mathfrak{s})$ is

$$\mathcal{G}_k(Y,\mathfrak{s}) = L^2_k(Y,S^1).$$

If $k > \frac{3}{2}$, then this is in fact a Hilbert Lie group and there is a smooth action

$$\mathcal{G}_k \times \mathcal{C}_{k-1} \to \mathcal{C}_{k-1}.$$

The quotient of this action is denoted $\mathcal{B}_k(Y, \mathfrak{s})$.

Given a configuration $\gamma \in \mathcal{C}_k(Y, \mathfrak{s})$, the tangent space to the Hilbert manifold $\mathcal{C}_k(Y, \mathfrak{s})$ at γ is

$$T_{k,\gamma}(Y, \mathfrak{s}) = \left\{(b, \Psi) | b \in L^2_k(Y; iT^*Y), \Psi \in L^2_k(Y; S)\right\}.$$

In addition, for $j \leq k$, we can also define the completion of $T_{k,\gamma}$ in the L^2_j norm:

$$T_{j,\gamma}(Y, \mathfrak{s}) = \left\{(b, \Psi) | b \in L^2_j(Y; iT^*Y), \Psi \in L^2_j(Y; S)\right\}.$$

Then we have bundles $T_j \to \mathcal{C}_k$ for $j \leq k$. At an irreducible configuration $\gamma \in \mathcal{C}_k(Y, \mathfrak{s})$, define $\mathbf{d}_\gamma$ toe be the linearization at 1 of the gauge group multiplication by γ; this is given by

$$\mathbf{d}_\gamma \xi = (-d\xi, \xi\Psi).$$

The map $\mathbf{d}_\gamma$ extends to a map $L^2_{j+1}(Y; i\mathbf{R}) \to T_{j,\gamma}$ for each $j \leq k$; let $\mathcal{J}_{j,\gamma}$ be the image of this map and $\mathcal{K}_{k,\gamma}$ the L^2 orthogonal complement of $\mathcal{J}_{j,\gamma}$ in $T_{j,\gamma}$.

4.3 Global slices

Let $\gamma_0 = (B_0, 0)$ be a reducible configuration in $\mathcal{C}_k(Y, \mathfrak{s})$. Following section 9.6 of [5], we will exhibit $\mathcal{B}_k(Y, \mathfrak{s})$ as the quotient of an affine Hilbert manifold by the group $\mathcal{G}^h$.

The gauge group $\mathcal{G}_{k+1}$ contains a subgroup $\mathcal{G}^\perp_{k+1}$ defined by

$$\mathcal{G}^\perp_{k+1} = \left\{e^\xi | \xi \in L^2_{k+1}(Y; i\mathbf{R}), \int_Y \xi = 0\right\}.$$

Denote the space of harmonic gauge transformations by $\mathcal{G}^h$ (this is independent of

k). Let

$$\mathcal{G}^{h,0} = \{u \in \mathcal{G}^h : u(x_0) = 1\}$$

for some fixed basepoint $x_0 \in Y$, and let $\mathcal{G}^0_{k+1} = \mathcal{G}^{h,0} \times \mathcal{G}^{\perp}_{k+1}$. Define $\mathcal{B}^0_k$ to be the quotient of $\mathcal{C}_k$ by the action of $\mathcal{G}^0_{k+1}$. There is a product decomposition $\mathcal{G}_{k+1} = \mathcal{G}^{\perp}_{k+1} \times \mathcal{G}^h$. Define a map $\mathcal{G}^{\perp}_{k+1} \times \mathcal{K}_{k,\gamma_0} \to \mathcal{C}_k(Y, \mathfrak{s})$ by

$$\left(e^{\xi}, (a, \phi)\right) \mapsto (B_0 + a - d\xi, \phi).$$

We claim that this is a diffeomorphism. Given a pair $(B, \psi) \in \mathcal{C}_k(Y, \mathfrak{s})$, we want to find ξ and (a, ϕ) such that $d^*a = 0$, $b = a - d\xi$, and $e^{\xi}\phi = \psi$. Applying the first equation to the second gives

$$d^*b = -\Delta\xi.$$

The Laplacian on Y is an isomorphism $\Delta : L^2_{k+1,0}(Y; i\mathbf{R}) \to L^2_{k-1,0}(Y; i\mathbf{R})$, where the 0 subscript denotes the space of functions with integral zero. So we can define the inverse map by

$$(B, \Psi) \mapsto \left(e^{-\Delta^{-1}d^*b}, \left(a - \Delta^{-1}d^*b, e^{\Delta^{-1}d^*b}\psi\right)\right).$$

Note that it is essential here that γ_0 is reducible; otherwise the equation would have a zeroth order term in addition to the Laplacian, and we would not have such a simple description of the solutions.

4.4 Finite-dimensional approximation

According to Proposition 11.2.1 of [5], we can approximate $\mathcal{B}^0(Y, \mathfrak{s})$ by a finite-dimensional manifold E in such a way that we get an embedding of the moduli space M^0 into E. More precisely,

Proposition 33. *Suppose $M^0 \subset \mathcal{B}^0_k(Y, \mathfrak{s})$ is an S^1-invariant finite-dimensional com-*

pact C^1 manifold. We may define an S^1-invariant map

$$\Psi : \mathcal{B}_k^0(Y, \mathfrak{s}) \to \mathbf{R}^n \times T^{b_1} \times \mathbf{C}^m$$

and a neighborhood U of M^0 in $\mathcal{B}_0^k(Y, \mathfrak{s})$ such that Ψ embeds U in $\mathbf{R}^n \times T^{b_1} \times \mathbf{C}^m$.

We briefly describe the construction of Ψ. Let $\{c_i\}_{i=1}^{n+t}$ be coclosed one-forms, with the first n coexact and the last $t = b_1(Y)$ a basis for the space of harmonic one-forms. Then we can define maps $r_i : \mathcal{C}(Y, \mathfrak{s}) \to \mathbf{R}$ by

$$r_i(B_0 + b, \Psi) = \int_Y b \wedge *\bar{c}_i.$$

Then define Ψ to be

$$\Psi(B, \psi) = \begin{pmatrix} r_1(B, \Psi), \ldots, r_n(B, \Psi), \\ r_{n+1}(B, \Psi), \ldots, r_{n+t}(B, \Psi), \\ q_1(B, \psi), \ldots, q_m(B, \psi) \end{pmatrix}.$$

Then $\Psi$is a map $\mathcal{C}(Y, \mathfrak{s}) \to \mathbf{R}^n \times \mathbf{R}^t \times \mathbf{C}^m$ and descends to an S^1-invariant map $\mathcal{B}^0(Y, \mathfrak{s}) \to \mathbf{R}^n \times T^t \times \mathbf{C}^m$. The proposition states that by taking enough forms and spinors, we can arrange that $\Psi|_U$ is an embedding.

This finite-dimensional approximation gives a way of constructing perturbations.

Definition 34. Suppose $g : \mathbf{R}^n \times T^{b_1} \times C^m \to \mathbf{R}$ is a smooth S^1-invariant function with compact support. Then we say the composition $g \circ \Psi$ is a *cylinder function.*

Kronheimer and Mrowka definte a class of *tame perturbations*, which are sections q of $T_0 \to \mathcal{C}$ having the following properties (among others):

- For every $k \geq 2$, q extends to a smooth section of $T_k \to \mathcal{C}_k$;
- The linearization dq extends to a smooth section of $End(T_k) \to \mathcal{C}_k$.

If q is a tame perturbation such that the critical points of $CSD + q$ are nondegenerate and the moduli spaces of trajectories are regular, thatn the perturbation q can be used to calculate the Floer groups. Kronheimer and Mrowka ([5], Theorem 11.6.1) construct a Banach space of tame perturbations that is large enough to ensure the needed transversality.

Proposition 35 (([5], Theorem 11.1.2)). *If f is a cylinder function, then its gradient $gradf$ is a tame perturbation.*

4.5 Defining the perturbations

Definition 36. Let γ be a critical point of CSD. If the image M of $CritCSD$ in $\mathcal{B}(Y, \mathfrak{s})$ is a manifold and the kernel of $Hess_{\gamma}CSD$ is the tangent space to M, then we say γ is *Morse-Bott regular.*

Suppose the critical set M of CSD is Morse-Bott regular. Our goal is to find a perturbation h of CSD such that $CSD + h$ has only nondegenerated critical points. We would like h to be such that $h|_M$ is a Morse function, and then we want to show that the critical points of $CSD + h$ correspond naturally to those of $h|_M$.

Choose Ψ so that it restricts to an embedding of M^0 into $\mathbf{R}^n \times T^t \times C^m$. Let f^0 be a smooth S^1-invariant function on M^0 that descends to a Morse function f on M. Then regarding f^0 as a function on $\Psi[M^0]$, extend f^0 to a tubular neighborhood of $\Psi[M^0]$ in $\mathbf{R}^n \times T^t \times \mathbf{C}^m$ by choosing a normal bundle structure and making f^0 constant on fibers; finally, extend to all of $\mathbf{R}^n \times T^t \times \mathbf{C}^m$ by a cutoff function. Then take our perturbing function h to be $h = \Psi \circ f^0$.

4.6 Nondegeneracy for the perturbed equations

Now that we have constructed the perturbing function f, we need to show that it actually brings us into a nondegenerate situation.

Theorem 37. *There exists $\epsilon > 0$ such that for $0 < t < \epsilon$, each critical point x of $f|_{CritCSD}$, x is a nondegenerate critical point of $CSD + tf$.*

Proof. Write $H = Hess_\gamma CSD$ and $F = Hess_\gamma f$. Since $gradf$ is a section of $T_k \to C_k$, $d[gradf]_\gamma$ has image contained in L^2_k. This implies that

$$H : \mathcal{K}_{k,\gamma} \to \mathcal{K}_{k-1,\gamma}$$

is a compact operator, and so for any t, $H + tf$ is a Fredholm operator of index zero. Therefore $H + tf$ is surjective if and only if it is injective.

Now we will show that we can choose t small enough that $H + tf$ has no kernel. Let ν_k be a complement of $T_\gamma M$ in $\mathcal{K}_{k,\gamma}$. Then the operator H has the form

$$H = \begin{pmatrix} L_0 & 0 \\ 0 & 0 \end{pmatrix}$$

where L_0 is an isomorphism $\nu_k \to \nu_k$. In this same decomposition, f decomposes as

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where D is an isomorphism $T_\gamma M \to T_\gamma M$. Now suppose we have a pair (x, y) in the kernel of $H + tf$. This means that

$$\begin{aligned} (L_0 + tA)x + tBy &= 0 \\ tCx + tDy &= 0. \end{aligned}$$

Since L_0 is an isomorphism, $L_0 + tA$ is also an isomorphism if we choose t such that $0 < t \|A\| < \frac{1}{2} \|L_0^{-1}\|^{-1}$, and then we have

$$\|(L_0 + tA)^{-1}\| \leq \frac{\|L_0^{-1}\|}{1 - t\|L_0^{-1}A\|} < 2\|L_0^{-1}\|.$$

Now solve the first equation for x and substitute into the second to get

$$t^2 C(L_0 + tA)^{-1}By = tDy.$$

Now take t even smaller so that

$$0 < 2t \|C\| \cdot \left\|L_0^{-1}\right\| \cdot \|B\| \leq \frac{1}{2} \left\|D^{-1}\right\|^{-1};$$

then $tC(L_0 + tA)^{-1}B - D$ is an isomorphism, and therefore $y = 0$. But since $L_0 + tA$ is an isomorphism, this implies $x = 0$ as well. Therefore $H + tA$ is surjective. $\square$

Chapter 5

The conjectured Seiberg-Witten groups

5.1 Overview

Based on the analysis of the critical points in the previous chapter, we write down conjectured groups of Seiberg-Witten cochains.

Definition 38. Let

$$C_o^* = H^*(Sym^{g-1}\Sigma),$$

$$C_s^* = (H^*(Jac;\mathbf{Z}) \oplus H^*(Jac;\mathbf{Z})[-1]) \otimes_{\mathbf{Z}} \mathbf{Z}[U],$$

and

$$C_u^* = (H^*(Jac;\mathbf{Z}) \oplus H^*(Jac;\mathbf{Z})[-1]) \otimes_{\mathbf{Z}} \left(\mathbf{Z}[U,U^{-1}]/\mathbf{Z}[U]\right).$$

For now, these are simply graded abelian groups. We will make each into a cochain complex and the cohomology of each complex will be a module over $\mathbb{A}(Y)$.

Define

$$\bar{C} = C_s \oplus C_u,$$

$$\hat{C} = C_o \oplus C_u,$$

and

$$\check{C} = C_s \oplus C_o.$$

To write down the boundary maps of these chain complexes, we will need some facts about Riemann surfaces. First of all, we relate $\overline{HM}(Y)$ to the cohomology ring of Y, as is done in [5].

5.2 The Lefschetz decomposition and $\overline{HM}(Y)$

Theorem 39. *Let $\mathfrak{s}$ be a* spin^c *structure on Y with $c_1(\mathfrak{s})$ torsion. Then there is a spectral sequence with E_2 term*

$$E_2 = \Lambda^*(H_1(Y;\mathbf{Z})/T) \otimes_{\mathbf{Z}} \mathbf{Z}[U, U^{-1}]$$

that converges to $\overline{HM}(Y)$. The only nonzero differential is d_3, given by $d_3(x \otimes U^k) = (\xi \cdot x) \otimes U^{k-1}$. Here $\xi \in \Lambda^3(H_1(Y;\mathbf{Z})/T)$ arises from the triple cup product map $\Lambda^3 H^1(Y;\mathbf{Z}) \to \mathbf{Z}$.

Now we can define the boundary map $\bar{\partial}$ to be

$$\bar{\partial} = \begin{pmatrix} 0 & 0 \\ d_3 & 0 \end{pmatrix}.$$

Here juxtaposition represents the exterior product in $\Lambda^*(H_1(Y;\mathbf{Z})/T)$, which is not to be confused with the cup product in $H^*(Y;\mathbf{Z})$.

We can apply the *Lefschetz decomposition* to determine the group $\overline{HM}(Y, \mathfrak{s}_0; \mathbf{C})$. For any vector space V, we may define a bilinear contraction

$$\cdot \llcorner \cdot : V^* \times \Lambda^k V \to \Lambda^{k-1} V$$

by

$$\alpha \llcorner v_1 \wedge \cdots \wedge v_k = \Sigma_i (-1)^{i-1} \alpha(v_i) v_1 \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_k.$$

This extends naturally to a bilinear map $\Lambda^l V^* \times \Lambda^k V \to \Lambda^{k-l} V$ for $l \leq k$. Now suppose V is a complex vector space of dimension $2g$ equipped with a symplectic form $\omega \in \Lambda^2 V^*$. Then contraction with ω defines a map $\Lambda^k V \to \Lambda^{k-2} V$ for any k. There is also an isomorphism $V \to V^*$ defined by $v \mapsto \omega(\cdot, v)$; using this, we may regard ω as an element of $\Lambda^2 V$ and can thus take the wedge product with ω.

Define operators L,Λ, and H on $\Lambda^* V$ by $L\alpha = \omega \wedge \alpha$,$\Lambda\alpha = -\omega \llcorner \alpha$, and $H\alpha = (p-g)\alpha$ for a p-form α. These operators give $\Lambda^* V$ the structure of a representation of $\mathfrak{sl}_2(\mathbf{C})$. Therefore we obtain a decomposition

$$\Lambda^k V = P^k \oplus \omega \wedge P^{k-2} \oplus \omega^2 \wedge P^{k-4} \wedge \cdots ,$$

where $P^i = ker \Lambda \cap \Lambda^i V$ is the *primitive part* of $\Lambda^i V$.

The above is a special case of the *Lefschetz decomposition* of the cohomology of a compact Kähler manifold (X, ω). If we represent cohomology by harmonic forms, then the above operators still act on $H^*(X)$, making it into an $\mathfrak{sl}_2(\mathbf{C})$ representation. The case above corresponds to the case when our manifold X is a complex torus, and we will apply it in the case of the Jacobian of Σ. Geometrically, we will see the cap product with ω on $H_*(\mathfrak{J})$, which corresponds to the cup product with ω on $H^*(\mathfrak{J})$.

However, with integral coefficients, this representation theory no longer applies. The kernel and cokernel of wedge product with ω were calculated by Lee and Packer in [6]. However, because of the extension problems arising from the spectral sequence, we cannot identify this with the groups $\overline{HM}(Y)$.

5.3 The Abel-Jacobi map and the zero spin^c structure

Now we turn to $\widetilde{HM}(Y)$, the definition of which which involves the irreducible solutions. We will construct the boundary map d_s^o in our complex using the Abel-Jacobi map for Σ. We noted above that for the given perturbation, the irreducibles form a copy of $Sym^{g-1}\Sigma_g$.

There is a map

$$\uparrow : H^*(\Sigma) \to H^*(Sym^d\Sigma)$$

arising as follows (the discussion and notation follow [[11]]):

There is a universal divisor

$$\Delta = \{(x, D) \in \Sigma \times Sym^d\Sigma | x \in D\}$$

inside the product $\Sigma \times Sym^g\Sigma$. We can use this to define the map $\uparrow$ by

$$\uparrow \alpha = (pr_2)_!(\Delta \cup pr_1^*(\alpha)),$$

where the pr_i are the ith projections and the ! denotes "integration along the fibers." Note that using the notation of Macdonald, $\uparrow 1 = \eta$, $\uparrow \alpha_i = \xi_i$, and $\uparrow \alpha_i^{'} = \xi_i^{'}$. The map $\uparrow$ gives $H^*(Sym^d\Sigma)$ the structure of a module over $\Lambda^* H^1(\Sigma) \otimes Sym^* H^0(\Sigma)$.

Now we return to our situation, in which $d = g - 1$. The degree $g - 1$ Abel-Jacobi map induces a map

$$\mu^* : H^*(Jac) \to H^*(Sym^{g-1}\Sigma).$$

. Define the boundary map d_s^o to be $(\mu^*, 0)$. Then the contribution of C_s^* to the cohomology of our complex is the quotient

$$H_s^* = H^*(Sym^{g-1}\Sigma_g)/\mu^* H^*(Jac).$$

As stated above, μ^* is an isomorphism on first cohomology.

5.4 The cohomology of symmetric products

Let $Sym^d\Sigma$ be the dth symmetric product of a genus g Riemann surface Σ. Let $\alpha_1, \cdots, \alpha_g, \alpha_1^{'}, \cdots, \alpha_g^{'}$ be a standard basis for the first cohomology of C and let $\beta = \alpha_i \wedge \alpha_i^{'}$ be a generator of H^2. In the d-fold product C^d, denote the corresponding elements for the jth factor by $\alpha_{i,j}$,$\alpha_{i,j}^{'}$, and β_j.

Consider the jth projection map $pr_j : C^d \to C$. Define elements of $H^*(C^d)$ by $\alpha_{i,j} = pr_j^* \alpha_i$, $\alpha'_{i,j} = pr_j^* \alpha'_i$, and $\beta_j = pr^* \beta$. Following Macdonald's notation, write $\eta = \beta_1 + \cdots + \beta_d$, $\xi_i = \alpha_{i,1} + \cdots + \alpha_{i,d}$, and $\xi'_i = \alpha'_{i,1} + \cdots + \alpha'_{i,d}$;these elements of $H^*(C^d)$ are invariant under the S^d action. The following is a theorem of Macdonald ([7]).

Theorem 40. *The cohomology of $Sym^d C$ is isomorphic to the subring of $H^*(C^d)$ invariant under the S_d action. $H^*(C_d) \cong H^*(C^d)^{S^d}$. The elements η and ξ_i are generators of the ring. As an abelian group, $H^k(Sym^d \Sigma)$ is freely generated by elements*

$$\xi_I \xi'_{I'} \eta^q$$

with $|I| + |I'| + 2q = k$. In particular, the Betti numbers are

$$b_i(Sym^d \Sigma) = \begin{cases} \sum_j \binom{2g}{i-2j} & d \leq g \\ \sum_j \binom{2g}{i-2j} & d \geq g. \end{cases}$$

In addition, Macdonald gives a description of the ring structure of $Sym_d \Sigma$.

Theorem 41. *The cohomology ring of $Sym^d \Sigma$ is generated by the ξ_i and η subject to the relations*

$$\xi_{i_1} \cdots \xi_{j_1} \cdots \xi_{j_j} (\xi_{k_1} \xi'_{k_1} - \eta) \cdots (\xi_{k_k} \xi'_{k_k} - \eta) \eta^q = 0$$

for $i + j + 2k + q = d + 1$.

We perform the following calculation using ths relation:

$$0 = \prod_{i \in I} (\sigma_i - \eta) = \sum_{J \subseteq I} \sigma_J (-\eta)^{k-|J|}$$

and therefore

$$\begin{aligned}
0 &= k! \sum_{|I|=k} \sum_{J \subseteq I} \sigma_J (-\eta)^{k-|J|} \\
&= k! \sum_{|I|=k} \left(\sigma_I + \sum_{J \subsetneq I} \sigma_J (-\eta)^{k-|J|} \right) \\
&= \omega^k + k! \sum_{|I|=k} \sum_{J \subsetneq I} \sigma_J (-\eta)^{k-|J|} \\
&= \omega^k + k! \sum_{j<k} \binom{g-j}{k-j} \sum_{|J|=j} \sigma_J (-\eta)^{k-j} \\
&= \omega^k + k! \sum_{j<k} \binom{g-j}{k-j} \frac{1}{j!} \omega^j (-\eta)^{k-j}.
\end{aligned}$$

Therefore we can write ω^k as a sum of terms $\omega^j \eta^{k-j}$ with nonzero integer coefficients.

Proposition 42. *The map $\mu^* : H^1(T^{2g}) \to H^1(Sym_d \Sigma)$ is an isomorphism of abelian groups taking α_i to ξ_i and α_i' to ξ_i'.*

Chapter 6

The 4-dimensional equations

6.1 Overview

In order to understand the differentials in the Seiberg-Witten chain complex, we need to understand the equations on the cylinder $Z = R \times S^1 \times \Sigma$. First we write down the equations on a Kähler manifold, following, for instance, [8] or [9]:

$$\begin{aligned} \Lambda F_A^+ &= i\left(|\alpha|^2 - |\beta|^2\right) \\ F_A^{0,2} &= \alpha^* \otimes \beta \\ \bar{\partial}_A \alpha + \bar{\partial}_A^* \beta &= 0. \end{aligned}$$

Let $\tau = t + i\theta$ be a complex coordinate on $\mathbf{R} \times S^1$. Then we can write a plus spinor on Z as a pair $(\alpha, \beta d\bar{\tau})$ where α is a section of E and β is an E-valued one-form. A connection on Z has the form $A_0 + \xi + ihd\theta$ where ξ is a section of $\pi^* T^* \Sigma$ and q is a complex-valued function.

With these conventions, the curvature equation, perturbed as in the previous chapter, becomes

$$\rho(\frac{1}{2} F_{A^t} + g'(b) vol(\sigma)) = (\Phi\Phi^*)_0$$

The idea for solving the equation is to reduce it to a vortex equation on the 4-manifold Z. As a first step, we consider the unperturbed equations on Z. Given two critical points γ_o and γ_1, necessarily reducible by the arguments of Chapter 3, we want to study the 4-dimensional solutions with these limits at $t = \pm\infty$.

6.2 Flow lines and vortices

We will study the relation between solutions on Z and vortices on the closed complex surface $R = S^2 \times \Sigma$.

Proposition 43. *A flow line on Z that converges exponentially to critical points of the unperturbed equations as $t \longrightarrow \pm\infty$ must satisfy*

$$F_A^{0,2} = 0 \tag{6.2.1}$$

$$\alpha = 0 \quad or \quad \beta = 0 \tag{6.2.2}$$

$$\bar{\partial}_A \alpha = 0 \quad and \quad \bar{\partial}_A^* \beta = 0. \tag{6.2.3}$$

Proof. This follows from the same argument as in [9], Proposition 6.0.10. Applying $\bar{\partial}_A$ to 6.2.3, we see that

$$\begin{aligned} 0 &= \bar{\partial}_A \bar{\partial}_A \alpha + \bar{\partial}_A \bar{\partial}_A^* \beta \\ &= F_A^{0,2} \alpha + \bar{\partial}_A \bar{\partial}_A^* \beta \\ &= |\alpha|^2 \beta + \bar{\partial}_A \bar{\partial}_A^* \beta \end{aligned}$$

and so,

$$|\alpha|^2 |\beta|^2 + \langle \bar{\partial}_A \bar{\partial}_A^* \beta, \beta \rangle = 0$$

pointwise on $\mathbf{R} \times S^1 \times \Sigma$. Now integrate along $[-T, T] \times S^1 \times \Sigma$, let $T \longrightarrow \infty$, and use the exponential decay. □

This tells us that we can have exponentially decaying flow lines only between critical points that have one of the spinor components vanishing. Next, we want to show that these flow lines can be extended over the closed manifold. We map

$\mathbf{R} \times S^1 \times \Sigma$ into R by sending

$$(t, \theta, z) \mapsto (e^{t+i\theta}, z).$$

This is a holomorphic map onto R minus the two copies of Σ at the poles. Write $(\tilde{A}, \tilde{\alpha})$ for the image of (A, α) under this map.

As $t \longrightarrow -\infty$, the exponential decay on the cylinder says that

$$\begin{aligned} |A - A_0| &< Ce^{\delta t} \\ |\alpha| &< Ce^{\delta t}. \end{aligned}$$

The second condition tells us that $\tilde{\alpha}$ extends over the disk; however, the norm of one-forms changes, so we do not know that $\tilde{A}$ extends.

Conversely, suppose we are given an effective divisor in R. It has an associated holomorphic line bundle E and a holomorphic section α, and we can find a connection A giving rise to the holomorphic structure. Pulling back to $\mathbf{R} \times S^1 \times \Sigma$, we get a pair (A, α) with exponential convergence at the ends. As above, we would like to find a complex gauge transformation u that takes us to a solution to the vortex equations.

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