Root polytopes, triangulations, and subdivision algebras

by

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Abstract

In this thesis a geometric way to understand the relations of certain noncommutative quadratic algebras defined by Anatol N. Kirillov is developed. These algebras are closely related to the Fomin-Kirillov algebra, which was introduced in the hopes of unraveling the main outstanding problem of modern Schubert calculus, that of finding a combinatorial interpretation for the structure constants of Schubert polynomials. Using a geometric understanding of the relations of Kirillov’s algebras in terms of subdivisions of root polytopes, several conjectures of Kirillov about the reduced forms of monomials in the algebras are proved and generalized. Other than a way of understanding Kirillov’s algebras, this polytope approach also yields new results about root polytopes, such as explicit triangulations and formulas for their volumes and Ehrhart polynomials. Using the polytope technique an explicit combinatorial description of the reduced forms of monomials is also given. Inspired by Kirillov’s algebras, the relations of which can be interpreted as subdivisions of root polytopes, commutative subdivision algebras are defined, whose relations encode a variety of possible subdivisions, and which provide a systematic way of obtaining subdivisions and triangulations.

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2.2.1 This is an $S$-reduction tree with root labeled by $G^S[x_{12}x_{23}x_{34}]$, when the edge-labels are disregarded. The boldface edges indicate where the reduction is performed. We can read off the following reduced form of $x_{12}x_{23}x_{34}$ from the set of leaves: $x_{14}x_{13}x_{12} + x_{14}x_{23}x_{13} + x_{24}x_{14}x_{23} + \beta x_{14}x_{23} + x_{34}x_{14}x_{12} + x_{34}x_{24}x_{14} + \beta x_{34}x_{14} + \beta x_{14}x_{12} + \beta x_{24}x_{14} + \beta^2 x_{14}$. When the edge-labels are taken into account, this is the $B$-reduction tree corresponding to equation (2.1.3). Note that in the second child of the root we commuted edge-labels 1 and 2.

2.8.1 The edge sets of the pseudo-components in the graph depicted are

\{\{(1, 5), (5, 8}\}, \{(2, 5)\}, \{(3, 4), (4, 5)\}, \{(5, 6), (6, 7)\}\. The pseudo-component with edge set \{(1, 5), (5, 8)\} is both a left and right pseudo-component, while the pseudo-components with edge sets \{(2, 5)\}, \{(3, 4), (4, 5)\} are left pseudo-components and the pseudo-component with edge set \{(5, 6), (6, 7)\} is a right pseudo-component.

2.8.2 This figure depicts all the noncrossing alternating spanning forests of $\overline{T}$ on the vertex set $[n + 1]$ containing edge $(1, n + 1)$ and at least one edge of the form $(i_1, j)$ with $i_1 \leq i$ for each right pseudo-component of $T$ on $[i, j]$ and at least one edge of the form $(i, j_1)$ with $j \leq j_1$ for each left pseudo-component of $T$ on $[i, j]$. By the Ehrhart polynomial form of Theorem 4, see end of Section 2.8, $L_{P(T)}(t) = \binom{t+2}{2} - 6\binom{t+3}{3} + 6\binom{t+4}{4}$, since $f_{T,2} = 1, f_{T,3} = 6, f_{T,4} = 6$ and $f_{T,i} = 0$, for $i \neq 2, 3, 4$. 

\newpage
2.9.1 Trees $T_1, \ldots, T_6$ are the noncrossing alternating spanning trees of $\bar{T}$.

The root polytopes associated to them satisfy $P(T_1) <_{lex} \cdots <_{lex} P(T_6)$.

$S_T(T_1) = \emptyset, S_T(T_2) = \{(2, 4)\}, S_T(T_3) = \{3, 4\},$

$S_T(T_4) = \{(2, 5)\}, S_T(T_5) = \{(2, 5), (3, 4)\}, S_T(T_6) = \{(3, 5)\}.$

By Theorem 32, $J(P(T), x) = \frac{x^2 + 4x + 1}{(1 - x)^5}$. This is of course equivalent to $L_{P(T)}(t) = \binom{t+2}{2} - \binom{t+3}{3} + \binom{t+4}{4}$ as calculated in Figure 2.8.2. For a way to see this equivalence directly, see [BR, Lemma 3.14].

3.3.1 An $S$-reduction tree with root corresponding to the monomial $x_{12}x_{13}z_3$.

Summing the monomials corresponding to the graphs labeling the leaves of the reduction tree multiplied by suitable powers of $\beta$, we obtain a reduced form $P_n^S$ of $x_{12}x_{13}z_3$, $P_n^S = z_1x_{12}x_{13} + z_1x_{12}y_{13} + \beta z_1x_{12} + x_{12}y_{13}z_3 + \beta x_{12}y_{13}y_{13}$.

3.8.1 A $B$-reduction tree with root corresponding to the monomial $x_{13}x_{12}z_3$.

Note that in order to perform a reduction on this monomial we commute variables $x_{13}$ and $x_{12}$. In the $B$-reduction tree we only record the reductions, not the commutations. Summing the monomials corresponding to the graphs labeling the leaves of the reduction tree we obtain a reduced form $P_n^B$ of $x_{13}x_{12}z_3$, $P_n^B = z_1x_{13}x_{12} + y_{13}z_1x_{12} + y_{13}x_{12}z_3$.

3.11.1 A $B^\beta$-reduction tree with root corresponding to the monomial $x_{23}z_3y_{13}$.

Summing the monomials corresponding to the graphs labeling the leaves of the reduction tree multiplied by suitable powers of $\beta$, we obtain a reduced form $P_n^{B\beta}$ of $x_{23}z_3y_{13}$, $P_n^{B\beta} = z_2y_{12}x_{23} + z_2y_{13}y_{12} + \beta z_2y_{12} + y_{23}z_2y_{13} + z_3y_{23}y_{13} + \beta z_2y_{13} + \beta y_{23}y_{13}$.
Chapter 1

Introduction

In this thesis we study the connections of certain quadratic algebras and subdivisions of root polytopes. Three of these algebras are noncommutative quadratic algebras of types $A_n, C_n,$ and $D_n$ (the type $B_n$ and $C_n$ cases are isomorphic) which were defined by Anatol N. Kirillov [K2]. A noncommutative quotient of Kirillov's type $A_n$ quadratic algebra is the well-known Fomin-Kirillov algebra [FK] introduced in the hopes of finding a Littlewood-Richardson rule for Schubert polynomials. Such a rule is greatly anticipated; papers [BS, P2, PS, FS, RS1] contain results in special cases and further background.

Kirillov made several conjectures stating that the reduced forms of certain special monomials in his algebras are unique. We prove and generalize all of these conjectures using a geometric interpretation of the relations of Kirillov's algebras as subdivisions of certain root polytopes. Other than a way of understanding Kirillov's algebras, this polytope approach also yields new results about root polytopes, such as explicit triangulations and formulas for their volumes and Ehrhart polynomials. Moreover, using the polytope technique we not only prove that the reduced forms are unique, but also give an explicit combinatorial description of the reduced forms.

In the process, we introduce what we call subdivision algebras for the families of root polytopes we study. These are commutative algebras, whose relations describe subdivisions of root polytopes. The subdivision algebras provide a systematic way of obtaining triangulations of root polytopes.
Root polytopes were defined by Postnikov in [P1]. The full root polytope $\mathcal{P}(A_n^+)\,$ of type $A_n$ is the convex hull in $\mathbb{R}^{n+1}$ of the origin and points $e_i - e_j$ for $1 \leq i < j \leq n+1$. The polytope $\mathcal{P}(A_n^+)$ already made an appearance in the work of Gelfand, Graev and Postnikov [GGP], who gave a canonical triangulation of it in terms of noncrossing alternating trees on $[n+1] := \{1, 2, \ldots, n+1\}$. I define coned root polytopes $\mathcal{P}(T)$ (of type $A_n$) for a tree $T$ on the vertex set $[n+1]$ as the intersection of $\mathcal{P}(A_n^+)$ with the cone generated by the vectors $e_i - e_j$, where $(i,j)$ is an edge of $T$, $i < j$. Recall that a graph $G$ is noncrossing if there are no vertices $i < j < k < l$ such that $(i,k)$ and $(j,l)$ are edges in $G$. A graph $G$ is alternating if there are no vertices $i < j < k$ such that $(i,j)$ and $(j,k)$ are edges in $G$. Let

$$\overline{G} = ([n+1], \{(i,j) \mid \text{there exist edges } (i, i_1) \ldots, (i_k, j) \text{ in } G \text{ such that } i < i_1 < \ldots < i_k < j\})$$

be the (oriented) transitive closure of $G$.

The following theorem is a special case of my results on polytopes.

**Theorem 1.** If $T$ is a noncrossing tree on the vertex set $[n+1]$ and $T_1, \ldots, T_k$ are the noncrossing alternating spanning trees of $\overline{T}$, then the coned root polytopes $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)$ are $n$-dimensional simplices with disjoint interiors whose union is $\mathcal{P}(T)$. Furthermore,

$$\text{vol } \mathcal{P}(T) = f_T \frac{1}{n!},$$

where $f_T$ denotes the number of noncrossing alternating spanning trees of $\overline{T}$.

In [M1] we also calculate the Ehrhart polynomial of the coned root polytopes from Theorem 1.

There is a counterpart of Theorem 1 in terms of reduced forms of monomials in Kirillov’s type $A_n$ algebra, which is Theorem 3 below. For reference, we include the definition of Kirillov’s type $A_n$ algebra $\mathcal{B}(A_n)$ here [K2].

$\mathcal{B}(A_n)$ is the associative algebra over the polynomial ring $\mathbb{Q}[\beta]$, where $\beta$ is a variable (and a central element), generated by $\{x_{ij} \mid 1 \leq i < j \leq n+1\}$ subject to the relations
(i) $x_{ij}x_{jk} = x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}$, if $1 \leq i < j < k \leq n + 1$,

(ii) $x_{ij}x_{kl} = x_{kl}x_{ij}$ if $i, j, k, l$ are distinct.

Consider the first relation of Kirillov’s type $A_n$ algebra $B(A_n)$ as a reduction rule:

$$x_{ij}x_{jk} \rightarrow x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}. \quad (1.0.1)$$

A reduced form of the monomial $m$ in the algebra $B(A_n)$ is a polynomial obtained by successive applications of reduction (1.0.1) until no further reduction is possible, where we allow any two variables $x_{ij}$ and $x_{kl}$ where $i, j, k, l$ are distinct to commute between reductions. The following is an example of how to reduce $x_{12}x_{23}x_{34}$ in $B(A_n)$.

$$x_{12}x_{23}x_{34} \rightarrow x_{12}x_{24}x_{23} + x_{12}x_{34}x_{24} + \beta x_{12}x_{24}$$
$$\rightarrow x_{14}x_{12}x_{23} + x_{24}x_{14}x_{23} + \beta x_{14}x_{23} + x_{34}x_{12}x_{24} + \beta x_{14}x_{12}$$
$$+ \beta x_{24}x_{14} + \beta^2 x_{14}$$
$$\rightarrow x_{14}x_{13}x_{12} + x_{14}x_{23}x_{13} + \beta x_{14}x_{13} + x_{24}x_{14}x_{23} + \beta x_{14}x_{23}$$
$$+ x_{34}x_{14}x_{12} + x_{34}x_{24}x_{14} + \beta x_{34}x_{14} + \beta x_{14}x_{12} + \beta x_{24}x_{14}$$
$$+ \beta^2 x_{14} \quad (1.0.2)$$

In the example above the pair of variables on which the reductions are performed is in boldface, and the variables which we commute are underlined.

The “reason” for allowing $x_{ij}$ and $x_{kl}$ to commute only when $i, j, k, l$ are distinct might not be apparent at first, but as we prove in Section ??, it insures that, unlike in the commutative case, there are unique reduced forms for a natural set of monomials. Kirillov [K2] observed that the monomial $w = x_{12}x_{23} \cdots x_{n,n+1}$ has a unique reduced form in the quasi-classical Yang-Baxter algebra $B(A_n)$, and asked for a bijective proof. The uniqueness of the reduced form of $w$ is a special case of our results, and the desired bijection follows from our proof methods.
Kirillov [K2] conjectured that the reduced forms of certain elements in his type $A_n$, $C_n$, $D_n$ algebras are unique, independent of the order of reductions performed. All of his conjectures are special cases of more general results we prove in Chapters 2, 3 and 4 using combinatorial tools and the theory of noncommutative Gröbner bases.

**Theorem 2.** The reduced form of any monomial $m \in \mathcal{B}(A_n)$ is unique.

Not only are the reduced forms unique, they are also beautiful, as Theorem 3 depicts. Given a forest $F$ with edges $(i_1, j_1), \ldots, (i_k, j_k)$ labeled in this order, let $x^F$ be the noncommutative monomial $\prod_{t=1}^{k} x_{i_t,j_t}$. Then, the reduced form of the monomial $x^T$ for a tree can be expressed in terms of certain monomials $x^F$. For detailed definitions of the terms used in Theorem 3 see Chapter 2.

**Theorem 3.** The reduced form of $x^T$ for a “good” tree $T$ on the vertex set $[n+1]$ is $\sum_F \beta^{n-|E(F)|}x^F$, where the sum runs over all noncrossing alternating spanning forests of $\overline{T}$ with lexicographic edge-labels containing the edge $(1, n+1)$ and satisfying certain technical requirements.

This thesis is divided into four chapters, all of them based on the papers [M1, M2]. Each chapter includes an introduction of its own to outline the results contained. In Chapter 2 we study Kirillov’s type $A_n$ algebra. We develop the connection between its relations and subdivisions of type $A_n$ root polytopes. We also use the theory of noncommutative Gröbner basis to establish generalizations of the theorems stated above. In Chapter 3 we define Kirillov’s type $C_n$ algebra and establish its connection with type $C_n$ root polytopes and their subdivisions. We use our geometric interpretation to obtain results similar in flavor to the type $A_n$ results, though with a more complex nature. Finally, in Chapter 4 we define Kirillov’s type $D_n$ algebra and prove a generalized version of Kirillov’s conjecture for this algebra using noncommutative Gröbner bases.
Chapter 2

Root polytopes of type $A_n$, triangulations, and the subdivision algebra

2.1 Introduction

In this chapter we develop the connection between triangulations of type $A_n$ root polytopes and two closely related algebras: the subdivision algebra $S(A_n)$ and the algebra $B(A_n)$, which we call the quasi-classical Yang-Baxter algebra following A. N. Kirillov. The close connection of the root polytopes and the algebras $S(A_n)$ and $B(A_n)$ is displayed by the variety of results this connection yields: both in the realm of polytopes and in the realm of the algebras. Two closely related algebras with tight connections to Schubert calculus have been studied by Fomin and Kirillov in [FK] and by Kirillov in [K1]. Before stating definitions and reasons, we pause at Exercise 6.C6 of Stanley’s Catalan Addendum [S2] to learn the following.

Consider the monomial $w = x_{12}x_{23}\ldots x_{n,n+1}$ in commuting variables $x_{ij}$. Starting with $p_0 = w$, produce a sequence of polynomials $p_0, p_1, \ldots, p_m$ as follows. To obtain $p_{r+1}$ from $p_r$, choose a term of $p_r$ which is divisible by $x_{ij}x_{jk}$, for some $i, j, k$, and replace the factor $x_{ij}x_{jk}$ in this term with $x_{ik}(x_{ij} + x_{jk})$. Note that $p_{r+1}$ has one
more term than \( p_r \). Continue this process until a polynomial \( p_m \) is obtained, in which no term is divisible by \( x_{ij}x_{jk} \), for any \( i, j, k \). Such a polynomial \( p_m \) is a **reduced form** of \( w \). Exercise 6.C6 in [S2] states that, remarkably, while the reduced form is not unique, it turns out that the number of terms in a reduced form is always the **Catalan number** \( C_n = \frac{1}{n+1} \binom{2n}{n} \).

The angle from which we look at this problem gives a perspective reaching far beyond its setting in the world of polynomials. On one hand, the reductions can be interpreted in terms of root polytopes and their subdivisions, yielding a geometric, and subsequently a combinatorial, interpretation of reduced forms. On the other hand, using the combinatorial results obtained about the reduced forms, we obtain a method for calculating the volumes and Ehrhart polynomials of a family of root polytopes.

Root polytopes were defined by Postnikov in [P1]. The full root polytope \( \mathcal{P}(A_n^+) \), which is the convex hull in \( \mathbb{R}^{n+1} \) of the origin and points \( e_i - e_j \) for \( 1 \leq i < j \leq n + 1 \), already made an appearance in the work of Gelfand, Graev and Postnikov [GGP], who gave a canonical triangulation of it in terms of noncrossing alternating trees on \([n + 1]\). We obtain canonical triangulations for all acyclic root polytopes, of which \( \mathcal{P}(A_n^+) \) is a special case.

We define **acyclic root polytopes** \( \mathcal{P}(T) \) for a tree \( T \) on the vertex set \([n + 1]\) as the intersection of \( \mathcal{P}(A_n^+) \) with a cone generated by the vectors \( e_i - e_j \), where \((i, j) \in E(T), i < j \). Let

\[
\overline{G} = ([n + 1], \{(i, j) \mid \text{there exist edges } (i, i_1) \ldots (i_k, j) \text{ in } G \text{ such that } i < i_1 < \ldots < i_k < j\}),
\]

denote the (oriented) **transitive closure** of the graph \( G \). Recall that a graph \( G \) on the vertex set \([n + 1]\) is said to be **noncrossing** if there are no vertices \( i < j < k < l \) such that \((i, k)\) and \((j, l)\) are edges in \( G \). A graph \( G \) on the vertex set \([n + 1]\) is said to be **alternating** if there are no vertices \( i < j < k \) such that \((i, j)\) and \((j, k)\) are edges in \( G \). Alternating trees were introduced in [GGP]. Gelfand, Graev and Postnikov
showed that the number of noncrossing alternating trees on \([n+1]\) is counted by the Catalan number \(C_n\).

**Theorem 4.** If \(T\) is a noncrossing tree on the vertex set \([n+1]\) and \(T_1, \ldots, T_k\) are the noncrossing alternating spanning trees of \(\overline{T}\), then the root polytopes \(\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)\) are \(n\)-dimensional simplices with disjoint interiors whose union is \(\mathcal{P}(T)\). Furthermore,

\[
\text{vol } \mathcal{P}(T) = \frac{f_T}{n!},
\]

where \(f_T\) denotes the number of noncrossing alternating spanning trees of \(\overline{T}\).

Theorem 4 can be generalized in a few directions. We calculate the Ehrhart polynomial of \(\mathcal{P}(T)\); see Sections 2.5 and 2.8. We describe the intersections of the top dimensional simplices \(\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)\) in Theorem 4 in terms of noncrossing alternating spanning forests of \(\overline{T}\) in Section 2.8. Theorem 4 and its generalizations can also be proved for any forest \(F\), not necessarily noncrossing, as explained in Section 2.9. In Section 2.9 we also prove that the triangulation in Theorem 4 is shellable, and provide a second method for calculating the Ehrhart polynomial of \(\mathcal{P}(T)\).

The proof of Theorem 4 relies on relating the triangulations of a root polytope \(\mathcal{P}(T)\) to reduced forms of a monomial \(m[T]\) in variables \(x_{ij}\), which we now define. Let \(S(A_n)\) and \(B(A_n)\) be two associative algebras over the polynomial ring \(\mathbb{Q}[\beta]\), where \(\beta\) is a variable (and a central element), generated by the set of elements \(\{x_{ij} \mid 1 \leq i < j \leq n+1\}\) modulo the relation \(x_{ij}x_{jk} = x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}\). The **subdivision algebra** \(S(A_n)\) is commutative, i.e., it has additional relations \(x_{ij}x_{kl} = x_{kl}x_{ij}\) for all \(i, j, k, l\), while \(B(A_n)\), which we call the **quasi-classical Yang-Baxter algebra** following Kirillov [K2], is noncommutative and has additional relations \(x_{ij}x_{kl} = x_{kl}x_{ij}\) for \(i, j, k, l\) distinct only. The motivation for calling \(S(A_n)\) the subdivision algebra is simple; the relations of \(S(A_n)\) yield certain subdivisions of root polytopes, which we explicitly demonstrate by the Reduction Lemma (Lemma 8).

We treat the first relation as a **reduction rule**:
\[ x_{ij}x_{jk} \rightarrow x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}. \] (2.1.1)

A **reduced form** of the monomial \( m \) in the algebra \( S(A_n) \) (algebra \( B(A_n) \)) is a polynomial \( P^S_n \) (polynomial \( P^B_n \)) obtained by successive applications of reduction (2.1.1) until no further reduction is possible, where we allow commuting any two variables (commuting any two variables \( x_{ij} \) and \( x_{kl} \) where \( i, j, k, l \) are distinct) between reductions. Note that the reduced forms are not necessarily unique.

A possible sequence of reductions in algebra \( S(A_n) \) yielding a reduced form of \( x_{12}x_{23}x_{34} \) is given by

\[
\begin{align*}
    x_{12}x_{23}x_{34} & \rightarrow x_{12}x_{24}x_{23} + x_{12}x_{34}x_{24} + \beta x_{12}x_{24} \\
    & \rightarrow x_{24}x_{13}x_{12} + x_{24}x_{23}x_{13} + \beta x_{24}x_{13} + x_{34}x_{14}x_{12} + x_{34}x_{24}x_{14} \\
    & \quad + \beta x_{34}x_{14} + \beta x_{14}x_{12} + \beta x_{24}x_{14} + \beta^2 x_{14} \\
    & \rightarrow x_{13}x_{14}x_{12} + x_{13}x_{24}x_{14} + \beta x_{13}x_{14} + x_{24}x_{23}x_{13} + \beta x_{24}x_{13} \\
    & \quad + x_{34}x_{14}x_{12} + x_{34}x_{24}x_{14} + \beta x_{34}x_{14} + \beta x_{14}x_{12} + \beta x_{24}x_{14} \\
    & \quad + \beta^2 x_{14} \quad (2.1.2)
\end{align*}
\]

where the pair of variables on which the reductions are performed is in boldface. The reductions are performed on each monomial separately.

Some of the reductions performed above are not allowed in the noncommutative algebra \( B(A_n) \). The following is an example of how to reduce \( x_{12}x_{23}x_{34} \) in the noncommutative case.
In the example above the pair of variables on which the reductions are performed is in boldface, and the variables which we commute are underlined.

The "reason" for allowing $x_{ij}$ and $x_{kl}$ to commute only when $i, j, k, l$ are distinct might not be apparent at first, but as we prove in Section 2.8, it insures that, unlike in the commutative case, there are unique reduced forms for a natural set of monomials. Kirillov [K2] observed that the monomial $w = x_{12}x_{23} \cdots x_{n,n+1}$ has a unique reduced form in the quasi-classical Yang-Baxter algebra $B(A_n)$, and asked for a bijective proof. The uniqueness of the reduced form of $w$ is a special case of our results, and the desired bijection follows from our proof methods.

Before we can state a simplified version of our main result on reduced forms, we need one more piece of notation. Given a graph $G$ on the vertex set $[n+1]$ we associate to it the monomial $m^S[G] = \prod_{(i,j) \in E(G)} x_{ij}$; if $G$ is edge-labeled with labels $1, \ldots, k$, we can also associate to it the noncommutative monomial $m^B[G] = \prod_{a=1}^k x_{i_a,j_a}$, where $E(G) = \{(i_a,j_a) \mid a \in [k]\}$ and $(i,j)_a$ denotes an edge $(i,j)$ labeled $a$. In Section 2.2 we will also introduce the notations $G^S[m]$ and $G^B[m]$, which are graphs associated to a monomial.

**Theorem 5.** Let $T$ be a noncrossing tree on the vertex set $[n+1]$, and $P^S_n$ a reduced form of $m^S[T]$. Then,

$$P^S_n(x_{ij} = 1, \beta = 0) = f_T,$$

where $f_T$ denotes the number of noncrossing alternating spanning trees of $\overline{T}$. 

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If we label the edges of $T$ so that it becomes a good tree (to be defined in Section 2.6), then the reduced form $P_n^B$ of the monomial $m^B[T]$ is

$$P_n^B(x_{ij}, \beta = 0) = \sum_{T_0} x^{T_0},$$

where the sum runs over all noncrossing alternating spanning trees $T_0$ of $\overline{T}$ with reverse lexicographic edge-labels (to be defined in Section 2.7) and $x^{T_0}$ is defined to be the noncommutative monomial $\prod_{i=1}^{n} x_{i,j_i}$ if $T_0$ contains the edges $(i_1, j_1), \ldots, (i_n, j_n)$.

We generalize Theorem 5 for any $\beta$; see Sections 2.2 and 2.8. Theorem 5 can also be generalized for any forest $F$; see Sections 2.5 and 2.9. Finally, we prove using noncommutative Gröbner bases techniques that:

**Theorem 6.** The reduced form $P_n^B$ of any monomial $m$ is unique, up to commutations.

This chapter is organized as follows. In Section 2.2 we reformulate the reduction process in terms of graphs and elaborate further on Theorem 5 and its generalizations. In Section 2.3 we discuss acyclic root polytopes and relate them to reductions via the Reduction Lemma. We prove the Reduction Lemma, which translates reductions into polytope-language, in Section 2.4. In Section 2.5 we use the Reduction Lemma to prove general theorems about reduced forms of monomials, and prove formulas for the volumes and Ehrhart polynomials of $P(F)$, for any forest $F$. The lemmas of Section 2.6 indicate the significance of considering reduced forms in the noncommutative algebra $B(A_n)$. In Section 2.7 we prove Theorems 4 and 5 for a special tree $T$. Theorems 4 and 5 as well as their generalizations are proved in Section 2.8. In Section 2.9 we shell the canonical triangulation described in Theorem 4, and provide an alternative way to obtain the Ehrhart polynomial of $P(T)$ for a tree $T$. We conclude in Section 2.10 by proving that the reduced form $P_n^B$ of any monomial $m$ is unique using noncommutative Gröbner bases techniques.
2.2 Reductions in terms of graphs

We can phrase the reduction process described in Section 2.1 in terms of graphs. This view will be useful throughout the chapter. Think of a monomial $m \in A$ as a directed graph $G$ on the vertex set $[n+1]$ with an edge directed from $i$ to $j$ for each appearance of $x_{ij}$ in $m$. Let $G[m]$ denote this graph. If, however, we are in the noncommutative version of the problem, and $m = \prod_{i=1}^p x_{i,l}$, then we can think of $m$ as a directed graph $G$ on the vertex set $[n+1]$ with $p$ edges labeled $1, \ldots, p$, such that the edge labeled $l$ is directed from vertex $i_l$ to $j_l$. Let $G[m]$ denote the edge-labeled graph just described. Let $(i,j)_a$ denote an edge $(i,j)$ labeled $a$. It is straightforward to reformulate the reduction rule (2.1.1) in terms of reductions on graphs. If $m \in A$, then it reads as follows.

The reduction rule for graphs: Given a graph $G_0$ on the vertex set $[n+1]$ and $(i,j), (j,k) \in E(G_0)$ for some $i < j < k$, let $G_1, G_2, G_3$ be graphs on the vertex set $[n+1]$ with edge sets

$$
E(G_1) = E(G_0) \setminus \{(j,k)\} \cup \{(i,k)\},
$$

$$
E(G_2) = E(G_0) \setminus \{(i,j)\} \cup \{(i,k)\},
$$

$$
E(G_3) = E(G_0) \setminus \{(i,j)\} \setminus \{(j,k)\} \cup \{(i,k)\}.
$$

(2.2.1)

We say that $G_0$ reduces to $G_1, G_2, G_3$ under the reduction rules defined by equations (2.2.1).

The reduction rule for graphs $G[m]$ with $m \in B$ is explained in Section 2.6.

An $S$-reduction tree $T^S$ for a monomial $m_0$, or equivalently, for the graph $G^S[m_0]$, is constructed as follows. The root of $T^S$ is labeled by $G^S[m_0]$. Each node $G^S[m]$ in $T^S$ has three children, which depend on the choice of the edges of $G^S[m]$ on which we perform the reduction. Namely, if the reduction is performed on edges $(i,j), (j,k) \in E(G^S[m]), i < j < k$, then the three children of the node $G_0 = G^S[m]$ are labeled by the graphs $G_1, G_2, G_3$ as described by equation (2.2.1). For an example of an $S$-reduction tree; see Figure 2.2.1 (disregard the edge-labels).
Summing the monomials to which the graphs labeling the leaves of the reduction tree $T^S$ correspond multiplied by suitable powers of $\beta$, we obtain a reduced form of $m_0$.

Figure 2.2.1: This is an $S$-reduction tree with root labeled by $G^S[x_{12}x_{23}x_{34}]$, when the edge-labels are disregarded. The boldface edges indicate where the reduction is performed. We can read off the following reduced form of $x_{12}x_{23}x_{34}$ from the set of leaves: $x_{14}x_{13}x_{12} + x_{14}x_{23}x_{13} + \beta x_{14}x_{13} + x_{24}x_{14}x_{23} + \beta x_{14}x_{23} + x_{34}x_{14}x_{12} + x_{34}x_{24}x_{14} + \beta x_{34}x_{14} + \beta x_{14}x_{12} + \beta x_{24}x_{14} + \beta^2 x_{14}$. When the edge-labels are taken into account, this is the $B$-reduction tree corresponding to equation (2.1.3). Note that in the second child of the root we commuted edge-labels 1 and 2.

Let $T$ be a noncrossing tree on the vertex set $[n+1]$. In terms of reduction trees, Theorem 5 states that the number of leaves labeled by graphs with exactly $n$ edges of an $S$-reduction tree with root labeled $T$ is independent of the particular $S$-reduction tree. The generalization of Theorem 5 for any $\beta$ states that the number of leaves labeled by graphs with exactly $k$ edges of an $S$-reduction tree with root labeled $T$, is independent of the particular $S$-reduction tree for any $k$. In terms of reduced
forms we can write this as follows. If $P_n^S$ is the reduced form of a monomial $m^S[T]$ for a noncrossing tree $T$, then

$$P_n^S(x_{ij} = 1) = \sum_{m=0}^{n-1} f_{T,n-m} \beta^m,$$

where $f_{T,k}$ denotes the number of noncrossing alternating spanning forests of $T$ with $k$ edges and additional technical requirements detailed in Section 2.8. Also, if $P_n^B$ is the reduced form of a monomial $m^B[T]$ for a noncrossing good tree $T$ (defined in Section 2.6), then

$$P_n^B(x_{ij}) = \sum_F x^F,$$

where the sum runs over all noncrossing alternating spanning forests $F$ of $T$ with reverse lexicographic edge-labels (defined in Section 2.7) and additional technical requirements detailed in Section 2.8.

If we consider the reduced forms of the path monomial $w = \prod_{i=1}^{n} x_{i,i+1}$, then $T = P = ([n + 1], \{(i, i + 1) \ | \ i \in [n]\})$, and $f_{P,k}$ is simply the number of noncrossing alternating spanning forests on $[n + 1]$ with $k$ edges containing edge $(1, n + 1)$. Furthermore, $P_n^B(x_{ij}) = \sum_F x^F$, where the sum runs over all noncrossing alternating spanning forests $F$ on $[n + 1]$ with reverse lexicographic edge-labels and containing edge $(1, n + 1)$. See Section 2.7 for the treatment of this special case.

### 2.3 Acyclic root polytopes

In the terminology of [P1], a root polytope of type $A_n$ is the convex hull of the origin and some of the points $e_i - e_j$ for $1 \leq i < j \leq n+1$, where $e_i$ denotes the $i^{th}$ coordinate vector in $\mathbb{R}^{n+1}$. A very special root polytope is the full root polytope

$$\mathcal{P}(A_n^+) = \text{ConvHull}(0, e_i^- \ | \ 1 \leq i < j \leq n+1),$$

where $e_i^- = e_i - e_j$. We study a class of root polytopes including $\mathcal{P}(A_n^+)$, which we now discuss.
Let $G$ be a graph on the vertex set $[n+1]$. Define

$$ \mathcal{V}_G = \{ e_{ij} \mid (i,j) \in E(G), i < j \} $$

a set of vectors associated to $G$; $\mathcal{C}(G) = \langle \mathcal{V}_G \rangle := \{ \sum_{e_{ij} \in \mathcal{V}_G} c_{ij}e_{ij} \mid c_{ij} \geq 0 \}$, the cone associated to $G$; and

$$ \overline{V}_G = \Phi^+ \cap \mathcal{C}(G), \text{ all the positive roots of type } A_n \text{ contained in } \mathcal{C}(G), $$

where $\Phi^+ = \{ e_{ij} \mid 1 \leq i < j \leq n+1 \}$ is the set of positive roots of type $A_n$. The idea to consider the positive roots of a root system inside a cone appeared earlier in Reiner’s work [R1], [R2] on signed posets.

The root polytope $\mathcal{P}(G)$ associated to graph $G$ is

$$ \mathcal{P}(G) = \text{ConvHull}(0, e_{ij} \mid e_{ij} \in \overline{V}_G) $$

The root polytope $\mathcal{P}(G)$ associated to graph $G$ can also be defined as

$$ \mathcal{P}(G) = \mathcal{P}(A^+_n) \cap \mathcal{C}(G). $$

The equivalence of these two definitions is proved in Lemma 10 in Section 2.4.

Note that $\mathcal{P}(A^+_n) = \mathcal{P}(P)$ for the path graph $P = ([n+1], \{(i, i+1) \mid i \in [n]\})$. While the choice of $G$ such that $\mathcal{P}(A^+_n) = \mathcal{P}(G)$ is not unique, it becomes unique if we require that $G$ is minimal, that is for no edge $(i, j) \in E(G)$ can the corresponding vector $e_{ij}$ be written as a nonnegative linear combination of the vectors corresponding to the edges $E(G) \setminus \{e\}$. Graph $P$ is minimal.

We can describe the vertices in $\overline{V}_G$ in terms of paths in $G$. An increasing path of a graph $G$ is an ordered sequence of edges $(i_1, j_1), (i_2, j_2), \ldots, (i_l, j_l) \in E(G)$ such that $i_1 < j_1 = i_2 < j_2 = \ldots < j_{l-1} = i_l < j_l.$

**Lemma 7.** Let $G$ be a graph on the vertex set $[n+1]$. Any $v \in \overline{V}_G$ is $v = e_{i_1} - e_{j_l}$ for
some increasing path \((i_1, j_1), (i_2, j_2), \ldots, (i_l, j_l)\) of \(G\). If in addition \(G\) is acyclic, then the correspondence between increasing paths of \(G\) and vertices in \(\overline{V}_G\) is a bijection.

The proof of Lemma 7 is straightforward, and is left to the reader.

Define

\[ \mathcal{L}_n = \{ G = ([n+1], E(G)) \mid G \text{ is an acyclic graph} \}, \]

and

\[ \mathcal{L}(A_n^+) = \{ \mathcal{P}(G) \mid G \in \mathcal{L}_n \}, \]

the set of acyclic root polytopes.

Note that the condition that \(G\) is an acyclic graph is equivalent to \(\overline{V}_G\) being a set of linearly independent vectors.

The full root polytope \(\mathcal{P}(A_n^+) \in \mathcal{L}(A_n^+)\), since the path graph \(P\) is acyclic. We show below how to obtain central triangulations for all polytopes \(\mathcal{P} \in \mathcal{L}(A_n^+)\). A central triangulation of a \(d\)-dimensional root polytope \(\mathcal{P}\) is a collection of \(d\)-dimensional simplices with disjoint interiors whose union is \(\mathcal{P}\), the vertices of which are vertices of \(\mathcal{P}\) and the origin is a vertex of all of them. Depending on the context we at times take the intersections of these maximal simplices to be part of the triangulation.

We now state the crucial lemma which relates root polytopes and algebras \(S(A_n)\) and \(B(A_n)\) defined in Section 2.1.

**Lemma 8. (Reduction Lemma)** Given a graph \(G_0 \in \mathcal{L}_n\) with \(d\) edges let \((i, j), (j, k) \in E(G_0)\) for some \(i < j < k\) and \(G_1, G_2, G_3\) as described by equations (2.2.1). Then \(G_1, G_2, G_3 \in \mathcal{L}_n\),

\[ \mathcal{P}(G_0) = \mathcal{P}(G_1) \cup \mathcal{P}(G_2) \]

where all polytopes \(\mathcal{P}(G_0), \mathcal{P}(G_1), \mathcal{P}(G_2)\) are \(d\)-dimensional and

\[ \mathcal{P}(G_3) = \mathcal{P}(G_1) \cap \mathcal{P}(G_2) \text{ is } (d-1)\text{-dimensional}. \]

What the Reduction Lemma really says is that performing a reduction on graph \(G_0 \in \mathcal{L}_n\) is the same as "cutting" the \(d\)-dimensional polytope \(\mathcal{P}(G_0)\) into two \(d\)-dimensional polytopes \(\mathcal{P}(G_1)\) and \(\mathcal{P}(G_2)\), whose vertex sets are subsets of the vertex
set of $\mathcal{P}(G_0)$, whose interiors are disjoint, whose union is $\mathcal{P}(G_0)$, and whose intersection is a facet of both. We prove the Reduction Lemma in Section 2.4.

2.4 The proof of the Reduction Lemma

This section is devoted to proving the Reduction Lemma (Lemma 8). As we shall see in Section 2.5, the Reduction Lemma is the “secret force” that makes everything fall into its place for acyclic root polytopes. We start by providing a simple lemma which characterizes the root polytopes which are simplices, then in Lemma 10 we prove that equations (2.3.1) and (2.3.2) are equivalent definitions for the root polytope $\mathcal{P}(G)$, and finally we prove the Cone Reduction Lemma (Lemma 11), which, together with Lemma 10 implies the Reduction Lemma.

Lemma 9 is implied by the results in [P1, Lemma 13.2], but for the sake of completeness we provide a proof of it. Note that the exact definitions and notations in [P1] are different from ours. The idea for part of the proof of Lemma 10 appears in [P1, F] with different purposes.

Lemma 9. (Cf. [P1, Lemma 13.2]) For a graph $G$ on $[n+1]$ vertices and $d$ edges, the polytope $\mathcal{P}(G)$ is a simplex if and only if $G$ is alternating and acyclic. If $\mathcal{P}(G)$ is a simplex, then its $d$-dimensional normalized volume $\text{vol}_d \mathcal{P}(G) = \frac{1}{d!}$.

Proof. It follows from equation (2.3.1) that for a minimal graph $G$ the polytope $\mathcal{P}(G)$ is a simplex if and only if the vectors corresponding to the edges of $G$ are linearly independent and $\mathcal{C}(G) \cap \Phi^+ = \mathcal{V}_G$.

The vectors corresponding to the edges of $G$ are linearly independent if and only if $G$ is acyclic. By Lemma 7, $\mathcal{C}(G) \cap \Phi^+ = \mathcal{V}_G$ if and only if $G$ contains no edges $(i, j), (j, k)$ with $i < j < k$, i.e. $G$ is alternating.

That $\text{vol}_d \mathcal{P}(G) = \frac{1}{d!}$ follows from the unimodality of $\Phi^+$.

Lemma 10. For any graph $G$ on the vertex set $[n+1]$,
ConvHull(0, e_{ij}^- | e_{ij}^- ∈ V_G) = \mathcal{P}(A_n^+) ∩ C(G).

Proof. For a graph \( H \) on the vertex set \([n + 1]\), let \( \sigma(H) = \text{ConvHull}(0, e_{ij}^- | (i, j) ∈ H, i < j) \). Then, by Lemma 7, \( \sigma(G) = \text{ConvHull}(0, e_{ij}^- | e_{ij}^- ∈ V_G) \). Let \( \sigma(G) \) be a \( d \)-dimensional polytope for some \( d ≤ n \) and consider any central triangulation of it: \( \sigma(G) = \bigcup_{F ∈ \mathcal{F}} \sigma(F) \), where \( \{\sigma(F)\}_{F ∈ \mathcal{F}} \) is a set of \( d \)-dimensional simplices with disjoint interiors, \( E(F) ⊂ E(G) \), \( F ∈ \mathcal{F} \). Since \( \sigma(G) = \bigcup_{F ∈ \mathcal{F}} \sigma(F) \) is a central triangulation, it follows that \( \sigma(F) = \sigma(G) ∩ C(F) \), for \( F ∈ \mathcal{F} \), and \( C(G) = \bigcup_{F ∈ \mathcal{F}} C(F) \).

Since \( \sigma(F) \), \( F ∈ \mathcal{F} \), is a \( d \)-dimensional simplex, it follows that \( F \) is a forest with \( d \) edges. Furthermore, \( F ∈ \mathcal{F} \) is an alternating forest, as otherwise \( (i, j), (j, k) ∈ E(F) ⊂ E(G) \), for some \( i < j < k \) and while \( e_{ik}^- = e_{ij}^- + e_{jk}^- ∈ \sigma(G) ∩ C(F) \), \( e_{ik}^- ∉ \sigma(F) \), contradicting that \( \bigcup_{F ∈ \mathcal{F}} \sigma(F) \) is a central triangulation of \( \sigma(G) \). Thus, \( F = F \), and \( \sigma(F) = \sigma(F) \). It is clear that \( \sigma(F) = \text{ConvHull}(0, e_{ij}^- | e_{ij}^- ∈ V_F) ⊂ \mathcal{P}(A_n^+) ∩ C(F) \), \( F ∈ \mathcal{F} \). Since if \( x = (x_1, \ldots, x_{n+1}) \) is in the facet of \( \sigma(F) \) opposite the origin, then \( |x_1| + \cdots + |x_{n+1}| = 2 \) and for any point \( x = (x_1, \ldots, x_{n+1}) ∈ \mathcal{P}(A_n^+) \), \( |x_1| + \cdots + |x_{n+1}| ≤ 2 \) it follows that \( \mathcal{P}(A_n^+) ∩ C(F) ⊂ \sigma(F) \). Thus, \( \sigma(F) = \mathcal{P}(A_n^+) ∩ C(F) \). Finally, \( \text{ConvHull}(0, e_{ij}^- | e_{ij}^- ∈ V_G) = \sigma(G) = \bigcup_{F ∈ \mathcal{F}} \sigma(F) = \bigcup_{F ∈ \mathcal{F}} \sigma(F) = \mathcal{P}(A_n^+) ∩ C(F) \) = \( \mathcal{P}(A_n^+) ∩ (\bigcup_{F ∈ \mathcal{F}} C(F)) = \mathcal{P}(A_n^+) ∩ C(G) \) as desired.

\[ \square \]

Lemma 11. (Cone Reduction Lemma) Given a graph \( G_0 ∈ \mathcal{L}_n \) with \( d \) edges, let \( G_1, G_2, G_3 \) be the graphs described as by equations (2.2.1). Then \( G_1, G_2, G_3 ∈ \mathcal{L}_n \),

\[ \mathcal{C}(G_0) = \mathcal{C}(G_1) ∪ \mathcal{C}(G_2) \]

where all cones \( \mathcal{C}(G_0), \mathcal{C}(G_1), \mathcal{C}(G_2) \) are \( d \)-dimensional and

\[ \mathcal{C}(G_3) = \mathcal{C}(G_1) ∩ \mathcal{C}(G_2) \] is \((d - 1)\)-dimensional.

Proof. Let the edges of \( G_0 \) be \( f_1 = (i, j), f_2 = (j, k), f_3, \ldots, f_d \). Let \( v(f_1), v(f_2), v(f_3), \ldots, v(f_d) \) denote the vectors the edges of \( G_0 \) correspond to under the correspon-
\[ v : (i, j) \mapsto e_{ij}, \text{ where } i < j. \] Since \( G_0 \in \mathcal{L}_n \), the vectors \( v(f_1), v(f_2), v(f_3), \ldots, v(f_d) \) are linearly independent. By equations (2.2.1), \[ C(G_0) = \langle v(f_1), v(f_2), v(f_3), \ldots, v(f_d) \rangle, \] \[ C(G_1) = \langle v(f_1), v(f_1) + v(f_2), v(f_3), \ldots, v(f_d) \rangle, \] \[ C(G_2) = \langle v(f_1) + v(f_2), v(f_2), v(f_3), \ldots, v(f_d) \rangle, \] \[ C(G_3) = \langle v(f_1) + v(f_2), v(f_3), \ldots, v(f_d) \rangle. \] Thus, \( G_1, G_2, G_3 \in \mathcal{L}_n \), cones \( C(G_0), C(G_1) \) and \( C(G_2) \) are \( d \)-dimensional, while cone \( C(G_3) \) is \( (d - 1) \)-dimensional.

Clearly, \( C(G_1) \cup C(G_2) \subset C(G_0) \). Any vector \( v \in C(G_0) \) expressed in the basis \( v(f_1), v(f_2), v(f_3), \ldots, v(f_d) \) satisfies either \([v(f_1)]v \geq [v(f_2)]v\) or \([v(f_1)]v < [v(f_2)]v\). Thus, if \( v \in C(G_0) \), then \( v \in C(G_1) \) or \( v \in C(G_2) \). Therefore, \( C(G_0) = C(G_1) \cup C(G_2) \).

Clearly, \( C(G_3) \subset C(G_1) \cap C(G_2) \). Any \( v \in C(G_1) \) expressed in the basis \( v(f_1), v(f_2), v(f_3), \ldots, v(f_d) \) satisfies \([v(f_1)]v \geq [v(f_2)]v\), while \( v \in C(G_2) \) expressed in the basis \( v(f_1), v(f_2), v(f_3), \ldots, v(f_d) \) satisfies \([v(f_1)]v \leq [v(f_2)]v\). Thus, \( v \in C(G_1) \cap C(G_2) \) expressed in the basis \( v(f_1), v(f_2), v(f_3), \ldots, v(f_d) \) satisfies \([v(f_1)]v = [v(f_2)]v\). Therefore, \( C(G_1) \cap C(G_2) \subset C(G_3) \), leading to \( C(G_1) \cap C(G_2) = C(G_3) \).

Proof of the Reduction Lemma (Lemma 8). Straightforward corollary of Lemmas 10 and 11.

---

In Section 2.5 we use Lemmas 8 and 9 to prove general theorems about acyclic root polytopes, which can be specialized to yield proofs of parts of Theorems 4 and 5.

### 2.5 General theorems for acyclic root polytopes

In this section we prove general theorems about acyclic root polytopes and reduced forms of monomials \( m^S[F] \), for a forest \( F \).

Given a polytope \( \mathcal{P} \subset \mathbb{R}^{n+1} \), the \( t \)-th \textbf{dilate} of \( \mathcal{P} \) is

\[ t\mathcal{P} = \{(tx_1, \ldots, tx_{n+1}) | (x_1, \ldots, x_{n+1}) \in \mathcal{P}\}. \]

The \textbf{Ehrhart polynomial} of an integer polytope \( \mathcal{P} \subset \mathbb{R}^{n+1} \) is

\[ L_\mathcal{P}(t) = \#(t\mathcal{P} \cap \mathbb{Z}^{n+1}). \]
The Ehrhart polynomial of the interior $P^o$ of an integer polytope $P \subset \mathbb{R}^{n+1}$ is

$$L_{P^o}(t) = \#(tP^o \cap \mathbb{Z}^{n+1}).$$

For background on the theory of Ehrhart polynomials see [BR].

**Lemma 12.** Let $P(G)^o = \bigcup_{\sigma^o \in S} \sigma^o$, where $S$ is a collection of open simplices $\sigma^o$, such that the origin is a vertex of each simplex in $S$ and the other vertices are from $\Phi^+$. Then the number of $i$-dimensional open simplices in $S$, denoted by $f_i$, only depends on $P(G)$, not on $S$ itself.

**Proof.** Since $P(G)^o = \bigcup_{\sigma^o \in S} \sigma^o$, we have that $L_{P(G)^o}(t) = \sum_{\sigma^o \in S} L_{\sigma^o}(t)$. Since the vectors in $\Phi^+$ are unimodular, it follows that for a $d$-dimensional simplex $\sigma^o \in S$, $L_{\sigma^o}(t) = L_{\Delta^o}(t)$, where $\Delta$ is the standard $d$-simplex. By [BR, Theorem 2.2] $L_{\Delta^o}(t) = \binom{t-1}{d}$. Thus,

$$L_{P(G)^o}(t) = \sum_{i=0}^{\infty} f_i \binom{t-1}{i},$$

where $L_{P(G)^o}(t) \in \mathbb{Z}[t]$ and the set $\{\binom{t-1}{i} \mid i = 0, 1, \ldots\}$ is a basis of $\mathbb{Z}[t]$. Therefore, $f_i$ are uniquely determined for $i = 0, 1, \ldots$, by $P(G)$ and are independent of $S$. \qed

**Theorem 13.** Let $F$ be any forest on the vertex set $[n+1]$ with $l$ edges. If $T_F^S$ is an $S$-reduction tree with root labeled $F$, then the number of leaves of $T_F^S$ labeled by forests with $k$ edges, denoted by $f_{F,k}$, is a function of $F$ and $k$ only.

In other words, if $P_n^S$ is a reduced form of $m^S[F]$, then

$$P_n^S(x_{ij} = 1) = \sum_{l=0}^{l-1} f_{F,l-m} \beta^m.$$  

**Proof.** Let $T_F^S$ be a particular $S$-reduction tree with root labeled $F$. By definition, the leaves of $T_F^S$ are labeled by alternating forests with $k$ edges, where $k \in [l]$. Let the $c_k$ forests $F^k_1, \ldots, F^k_{c_k}$ label the leaves of $T_F^S$ with $k$ edges, $k \in [l]$. Repeated use
of the Reduction Lemma (Lemma 8) implies that

$$\mathcal{P}(F)^o = \bigcup_{k \in [l], i_k \in [c_k]} \mathcal{P}(F_{i_k}^k)^o,$$  \hspace{1cm} (2.5.1)$$

where the right hand side is a disjoint union of simplices by Lemma 9. By Lemma 12, the number of $k$-dimensional simplices among $\bigcup_{k \in [l], i_k \in [c_k]} \{\mathcal{P}(F_{i_k}^k)^o\}$ is independent of the particular $S$-reduction tree $T_F^S$. Thus, $f_{T,k} = c_k$ only depends on $F$ and $k$.

The formula for the reduced form of $m^S[F]$ evaluated at $x_{ij} = 1$ follows from the correspondence between the leaves of $T_F^S$ and reduced forms described in Section 2.2.

We easily obtain the Ehrhart polynomial, and thus also the volume of the polytope $\mathcal{P}(F)$ with the techniques used above.

**Theorem 14.** The Ehrhart polynomial of the polytope $\mathcal{P}(F)$, where $F$ is a forest on the vertex set $[n + 1]$ with $l$ edges, is

$$L_{\mathcal{P}(F)}(t) = (-1)^l \sum_{i=0}^{l} (-1)^i f_{F,i} \left( \binom{t+i}{i} \right),$$

where $f_{F,k}$ is the number of leaves of $T_F^S$ labeled by forests with $k$ edges.

**Proof.** It follows from the proofs of Lemma 12 and Theorem 13 that

$$L_{\mathcal{P}(F)^o}(t) = \sum_{i=0}^{l} f_{F,i} \left( \binom{t-1}{i} \right).$$

Since by the Ehrhart-Macdonald reciprocity [BR, Theorem 4.1]

$$L_{\mathcal{P}(F)}(t) = (-1)^{\dim \mathcal{P}(F)} L_{\mathcal{P}(F)^o}(-t),$$

it follows that

$$L_{\mathcal{P}(F)}(t) = (-1)^l \sum_{i=0}^{l} f_{F,i} \left( \binom{-t-1}{i} \right) = (-1)^l \sum_{i=0}^{l} (-1)^i f_{F,i} \left( \binom{t+i}{i} \right).$$

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Corollary 15. If $F$ is a forest on the vertex set $[n+1]$ with $l$ edges, then

$$\text{vol } \mathcal{P}(F) = \frac{f_{F,l}}{l!}.$$ 

Proof. By [BR, Lemma 3.19] the leading coefficient of $L_{\mathcal{P}(F)}(t)$ is equal to $\text{vol } \mathcal{P}(F)$. We also obtain $\text{vol } \mathcal{P}(F) = \frac{f_{F,l}}{l!}$ directly from the Reduction Lemma if we count the $l$-dimensional simplices in the triangulation of $\mathcal{P}(F)$.

\[\square\]

2.6 Reductions in the noncommutative case

In this section we prove two crucial lemmas about reduction (2.1.1) in the noncommutative case necessary for proving Theorem 5. While in the commutative case reductions on $G^S[m]$ could result in crossing graphs, we prove that in the noncommutative case exactly those reductions from the commutative case are allowed which result in no crossing graphs, provided that $m = m^B[T]$ for a noncrossing tree $T$ with suitable edge labels specified below. Furthermore, we also show that if there are two edges $(i,j)$ and $(j,k)$ with $i < j < k$ in a graph obtained from $G^B[m]$ by a series of reductions, then after suitably many commutations it is possible to apply reduction (2.1.1). Thus, once the reduction process terminates, the set of graphs obtained as leaves of the reduction tree are alternating forests. Now, unlike in the commutative case, they are also noncrossing. In fact, each noncrossing alternating spanning forest of $\overline{T}$ satisfying certain additional technical conditions occurs among the leaves of the reduction tree exactly once, yielding a complete combinatorial description of the reduced form of $m^B[T]$.

In terms of graphs the partial commutativity means that if $G$ contains two edges $(i,j)_a$ and $(k,l)_{a+1}$ with $i,j,k,l$ distinct, then we can replace these edges by $(i,j)_{a+1}$ and $(k,l)_a$, and vice versa. Reduction rule (2.1.1) on the other hand means that if there are two edges $(i,j)_a$ and $(j,k)_{a+1}$ in $G_0$, $i < j < k$, then we replace $G_0$ with
three graphs $G_1, G_2, G_3$ on the vertex set $[n + 1]$ and edge sets

$$
E(G_1) = E(G_0) \backslash \{(i,j)_a\} \cup \{ (j,k)_{a+1}\} \cup \{ (i,k)_a\} \cup \{(i,j)_{a+1}\}
$$

$$
E(G_2) = E(G_0) \backslash \{(i,j)_a\} \cup \{ (j,k)_{a+1}\} \cup \{ (j,k)_a\} \cup \{(i,k)_{a+1}\}
$$

$$
E(G_3) = (E(G_0) \backslash \{(i,j)_a\} \cup \{ (j,k)_{a+1}\})^a \cup \{ (i,k)_a\},
$$

where $(E(G_0) \backslash \{(i,j)_a\} \cup \{ (j,k)_{a+1}\})^a$ denotes the edges obtained from the edges $E(G_0) \backslash \{(i,j)_a\} \cup \{ (j,k)_{a+1}\}$ by reducing the label of each edge which has label greater than $a$ by 1.

A $B$-reduction tree $T^B$ is defined analogously to an $S$-reduction tree, except we use equation (2.6.1) to describe the children. See Figure 2.2.1 for an example. A graph $H$ is called a $B$-successor of $G$ if it is obtained by a series of reductions from $G$. For convenience, we refer to commutativity of $X_{ij}$ and $X_{kl}$ for distinct $i, j, k, l$ as reduction (2), by which we mean the rule $X_{ij}X_{kl} \leftrightarrow X_{kl}X_{ij}$, for $i, j, k, l$ distinct, or, in the language of graphs, exchanging edges $(i, j)_a$ and $(k, l)_{a+1}$ with $(i, j)_{a+1}$ and $(k, l)_a$ for $i, j, k, l$ distinct.

A forest $H$ on the vertex set $[n + 1]$ and $m$ edges labeled $1, \ldots, m$ is good if it satisfies the following conditions for all $1 \leq i < j < k \leq n + 1$:

(i) If edges $(i, j)_a$ and $(j, k)_b$ are in $H$, then $a < b$.

(ii) If edges $(i, j)_a$ and $(i, k)_b$ are in $H$, then $a > b$.

(iii) If edges $(i, k)_a$ and $(j, k)_b$ are in $H$, then $a > b$.

(iv) $H$ is noncrossing.

No graph $H$ with a cycle could satisfy all of (i), (ii), (iii), (iv) simultaneously, which is why we only define good forests. Note, however, that any forest $H$ has an edge-labeling that makes it a good forest.

**Lemma 16.** If the root of a $B$-reduction tree is labeled by a good forest, then all nodes of it are also labeled by good forests.

**Proof.** The root of the $B$-reduction tree is trivially labeled by a good forest. We show
that after each reduction (2.1.1) or (2) all properties (i), (ii), (iii), (iv) of good forests are preserved.

In reduction (2) we take disjoint edges \((i, j)\) and \((k, l)\) and replace them by the edges \((i, j)\) and \((k, l)\). It is easy to check that properties (i), (ii), (iii), (iv) are preserved using the fact that all edge-labels are integers and are not repeated, so the relative orders of edge-labels for edges incident to the same vertex are unchanged.

Performing reduction (2.1.1) results in three new graphs as described by equation (2.6.1). It is easy to check that properties (i), (ii), (iii) are preserved using the fact that all edge-labels are integers and are not repeated. To prove that property (iv) is also preserved, note that by (i), (ii), (iii) if edge \((i, j)\) is labeled \(a\) and edge \((j, k)\) is labeled \(a+1\), then there cannot be edges with endpoint \(j\) of the form \((i_1, j)\) with \(i_1 < i\) or \((j, k_1)\) with \(k < k_1\), or else some of the conditions (i), (ii), (iii) would be violated. That there is no edge of the form described in the previous sentence with endpoint \(j\) together with the fact that the graph \(G\) we applied reduction (2.1.1) to was noncrossing implies that edge \((i, k)\) does not cross any edges of \(G\), and therefore the resulting graph is also noncrossing.

\(\square\)

A reduction applied to a noncrossing graph \(G\) is \textbf{noncrossing} if the graphs resulting from the reduction are also noncrossing.

The following is then an immediate corollary of Lemma 16.

\textbf{Corollary 17.} If \(G\) is a good forest, then all reductions that can be applied to \(G\) and its \(B\)-successors are noncrossing.

\textbf{Lemma 18.} Let \(G\) be a good forest. Let \((i, j)\) and \((j, k)\) with \(i < j < k\) be edges of \(G\) such that no edge of \(G\) crosses \((i, k)\). Then after finitely many applications of reduction (2) we can apply reduction (2.1.1) to edges \((i, j)\) and \((j, k)\).

\textit{Proof.} By the definition of a good forest it follows that \(a < b\). If \(b = a + 1\), then we are done. Otherwise, consider all edges \((l, m)\) such that \(a < c < b\). Since \(G\) is a good forest and \((i, k)\) does not cross any edges of \(G\), we find that for any such edge \((l, m)\) is either disjoint from edges \((i, j)\) and \((j, k)\), or else \((l, m) = (i, m)\) or \((l, m) = (l, k)\).
Then reduction (2) can be applied to the edges \((l, m)\) with \(a < c < b\) until either the edges labeled \(a\) and \(a + 1\) or the edges labeled \(b - 1\) and \(b\) are disjoint, in which case we can perform reduction (2) on these edges. Once this is done, the difference between the labels of the edges \((i, j)\) and \((j, k)\) decreased, and we can repeat this process until this difference is 1, in which case reduction (2.1.1) can be applied to them. 

\[\square\]

**Corollary 19.** If \(F\) labels a leaf of a B-reduction tree whose root is labeled by a good forest, then \(F\) is a good noncrossing alternating forest.

**Proof.** By Lemma 16, \(F\) is a good forest. By definition of good, it is also noncrossing. Lemma 18 implies that \(F\) is alternating, or else reduction (2.1.1) could be applied to it, and thus it would not label a leaf of a B-reduction tree. 

\[\square\]

### 2.7 Proof of Theorems 4 and 5 in a special case

In this section we prove Theorems 4 and 5 for the special case where \(T = P = ([n+1], \{(i, i+1) \mid i \in [n]\})\). We prove the general versions of the theorems in Section 2.8.

Given a noncrossing alternating forest \(F\) on the vertex set \([n+1]\) with \(k\) edges, the reverse lexicographic order, or revlex order for short, on its edges is as follows. Edge \((i_1, j_1)\) is less than edge \((i_2, j_2)\) in the revlex order if \(j_1 > j_2\), or \(j_1 = j_2\) and \(i_1 > i_2\). The forest \(F\) is said to have revlex edge-labels if its edges are labeled with integers 1, \ldots, \(k\) such that if edge \((i_1, j_1)\) is less than edge \((i_2, j_2)\) in revlex order, then the label of \((i_1, j_1)\) is less than the label of \((i_2, j_2)\) in the usual order on the integers. Clearly, given any graph \(G\) there is a unique edge-labeling of it which is revlex. For an example of revlex edge-labels, see the graphs labeling the leaves of the B-reduction tree in Figure 2.2.1.

**Lemma 20.** If a noncrossing alternating forest \(F\) is a B-successor of a good forest, then upon some number of reductions (2) performed on \(F\), it is possible to obtain a noncrossing alternating forest \(F'\) with revlex edge-labels.

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Proof. If edges \( e_1 \) and \( e_2 \) of \( F \) share a vertex and if \( e_1 \) is less than \( e_2 \) in the revlex order, then the label of \( e_1 \) is less than the label of \( e_2 \) in the usual order on integers by Lemma 16. Since reduction (2) swaps the labels of two vertex disjoint edges labeled by consecutive integers in a graph, these swaps do not affect the relative order of the labels on edges sharing vertices. Continue these swaps until the revlex order is obtained. \( \Box \)

To avoid confusion about whether the commutative or the noncommutative version of the problem is being considered, we denote \( x_{12}x_{23}\cdots x_{n,n+1} \) by \( w_S \) in the commutative and by \( w_B \) in the noncommutative case.

**Proposition 21.** By choosing the series of reductions suitably, the set of leaves of a \( B \)-reduction tree with root labeled by \( G^B[w_B] \) can be the set of all noncrossing alternating forests \( F \) on the vertex set \([n + 1]\) containing edge \((1, n + 1)\) with revlex edge-labels.

Proof. By Corollary 19, all leaves of a \( B \)-reduction tree are noncrossing alternating forests on the vertex set \([n + 1]\). It is easily seen that they all contain edge \((1, n + 1)\). By the correspondence between the leaves of a \( B \)-reduction tree and simplices in a subdivision of \( \mathcal{P}(G^B[w_B]) \) obtained from the Reduction Lemma (Lemma 8), it follows that no forest appears more than once among the leaves. Thus, it suffices to prove that any noncrossing alternating forest \( F \) on the vertex set \([n + 1]\) containing edge \((1, n + 1)\) appears among the leaves of a \( B \)-reduction tree and that all these forests have revlex edge-labels. One can construct such a \( B \)-reduction tree by induction on \( n \). We show that starting with the path \((1, 2), \ldots, (n, n + 1)\) and performing reductions (1) and (2) we can obtain any noncrossing alternating forest \( F \) on the vertex set \([n + 1]\) containing edge \((1, n + 1)\) with revlex edge-labels.

First perform the reductions on the path \((1, 2), \ldots, (n, n + 1)\) without involving edge \((n, n + 1)\) in any of the reductions, until possible. Then we arrive to a set of trees where we have a noncrossing alternating forest \( F \) on the vertex set \([n]\) containing edge \((1, n)\) with revlex labeling and in addition edge \((n, n + 1)\). By inspection it follows that any noncrossing alternating forest \( F \) on the vertex set \([n + 1]\) containing edge \((1, n + 1)\) with revlex edge-labels can be obtained from them. \( \Box \)
Theorem 22. The set of leaves of any $B$-reduction tree with root labeled by $G^B[w_B]$ is, up to applications of reduction (2), the set of all noncrossing alternating forests with revlex edge-labels on the vertex set $[n+1]$ containing edge $(1,n+1)$.

Proof. By Proposition 21 there exists a $B$-reduction tree which satisfies the conditions above. By Theorem 13 the number of forests with a fixed number of edges among the leaves of an $S$-reduction tree is independent of the particular $S$-reduction tree, and, thus, the same is true for a $B$-reduction tree. It is clear that all forests labeling the leaves of a $B$-reduction tree with root labeled by $G^B[w_B]$ have to contain the edge $(1,n+1)$. Also, no vertex-labeled forest, with edge-labels disregarded, can appear twice among the leaves of a $B$-reduction tree. Together with Lemma 20 these imply the statement of Theorem 22. \hfill \Box

As corollaries of Theorem 22 we obtain the characterization of reduced forms of the noncommutative monomial $w_B$, as well as a way to calculate $f_{P,k}$, the number of forests with $k$ edges labeling the leaves of an $S$-reduction tree $T^S_P$ with root labeled $P = ([n+1], \{(i,i+1) | i \in [n]\})$.

Theorem 23. If the polynomial $P^B_n(x_{ij})$ is a reduced form of $w_B$, then

$$P^B_n(x_{ij}) = \sum_{F} \beta^{n-|E(F)|} x^F,$$

where the sum runs over all noncrossing alternating forests $F$ with revlex edge-labels on the vertex set $[n+1]$ containing edge $(1,n+1)$, and $x^F$ is defined to be the noncommutative monomial $\prod_{i=1}^{k} x_{i_i,j_i}$ if $F$ contains the edges $(i_1,j_1), \ldots, (i_k,j_k)$.

Proposition 24. The number of forests with $k$ edges labeling the leaves of an $S$-reduction tree $T^S_P$, $f_{P,k}$, is equal to the number of noncrossing alternating forests on the vertex set $[n+1]$ and $k+1$ edges such that edge $(1,n+1)$ is present.

Proof. Theorem 13 proves that number of leaves labeled by forests with $k$ edges in any $S$-reduction tree with root labeled $P$ is independent of the particular $S$-reduction tree. Since a $B$-reduction tree becomes an $S$-reduction tree when the edge-labels from the graphs labeling its nodes are deleted, the number of leaves labeled by forests
with \( k \) edges in any \( S \)-reduction tree with root labeled \( P \) is equal to the number of noncrossing alternating forests with revlex edge-labels on the vertex set \( [n + 1] \) with \( k \) edges containing edge \((1, n + 1)\) by Theorem 22.

The Schröder numbers \( s_n \) count the number of ways to draw any number of diagonals of a convex \((n+2)\)-gon that do not intersect in their interiors. Let \( s_{n,k} \) denote the number of ways to draw \( k \) diagonals of a convex \((n+2)\)-gon that do not intersect in their interiors. Cayley [C] in 1890 showed that \( s_{n,k} = \frac{1}{n+1} \binom{n+k+1}{n} \binom{n-1}{k} \).

**Lemma 25.** There is a bijection between the set of noncrossing alternating forests on the vertex set \([n+1]\) and \( k + 1 \) edges such that edge \((1, n + 1)\) is present and ways to draw \( k \) diagonals of a convex \((n+2)\)-gon that do not intersect in their interiors. Thus, \( f_{P,k+1} = s_{n,k} \).

**Proof.** The bijection can be described as follows. Given a forest \( F \) with edges \((i_1, j_1), \ldots, (i_k, j_k), (1, n + 1)\), correspond to it an \((n+2)\)-gon on vertices 1, \ldots, \( n+2 \) in a clockwise order, with diagonals \((i_1, j_1 + 1), \ldots, (i_k, j_k + 1)\). 

Using \( f_{P,k+1} = \frac{1}{n+1} \binom{n+k+1}{n} \binom{n-1}{k} \) we specialize Theorems 13 and 14 to Theorems 26 and 27.

**Theorem 26.** If the polynomial \( P^S_n(x_{ij}) \) is a reduced form of \( w_S \), then

\[
P^S_n(x_{ij} = 1) = \sum_{m=0}^{n-1} s_{n,n-m-1} \beta^m,
\]

where \( s_{n,k} = \frac{1}{n+1} \binom{n+k+1}{n} \binom{n-1}{k} \) is the number of noncrossing alternating forests on the vertex set \([n+1]\) with \( k + 1 \) edges, containing edge \((1, n + 1)\).

**Theorem 27.** (Cf. [S4, Exercise 6.31], [F]) The Ehrhart polynomial of the polytope \( \mathcal{P}(A^+_n) \) is

\[
L_{\mathcal{P}(A^+_n)}(t) = \frac{(-1)^n}{n+1} \sum_{i=0}^{\infty} \binom{n+i}{n} \binom{n-1}{i-1} \binom{-t-1}{i}.
\]
The generating function \( J(\mathcal{P}(A_+^1), x) = 1 + \sum_{t=1}^{\infty} L_{\mathcal{P}(A_+^1)}(t)x^t \) was previously calculated by different methods; see [S4, Exercise 6.31], [F].

2.8 Proof of Theorems 4 and 5 in the general case

In this section we find an analogue of Theorem 23 for any noncrossing good tree \( T \), and using it calculate the numbers \( f_{T,k} \). Specializing Theorems 13 and 14 to \( T \), we then conclude the proofs of Theorems 4 and 5.

Theorems 23 and 26 imply Theorem 5 for the special case \( T = P = ([n+1], \{(i, i+1) \mid i \in [n]\}) \). We generalize Theorems 22, 23 and 26 to monomials \( m^B[T] \), where \( T \) is a good tree. For this we need some technical definitions.

Consider a noncrossing tree \( T \) on \([n + 1]\). We define the pseudo-components of \( T \) inductively. The unique simple path \( P \) from 1 to \( n + 1 \) is a pseudo-component of \( T \). The graph \( T\backslash P \) is an edge-disjoint union of trees \( T_1, \ldots, T_k \), such that if \( v \) is a vertex of \( P \) and \( v \in T_l \), \( l \in [k] \), then \( v \) is either the minimal or maximal vertex of \( T_l \). Furthermore, there are no \( k - 1 \) trees whose edge-disjoint union is \( T\backslash P \) and which satisfy all the requirements stated above. The set of pseudo-components of \( T \), denoted by \( ps(T) \) is \( ps(T) = \{P\} \cup ps(T_1) \cup \cdots \cup ps(T_k) \). A pseudo-component \( P' \) is said to be on \([i, j]\), \( i < j \) if it is a path with endpoints \( i \) and \( j \). A pseudo-component \( P' \) on \([i, j]\) is said to be a left pseudo-component of \( T \) if there are no edges \((s, i) \in E(T) \) with \( s < i \) and a right pseudo-component if if there are no edges \((j, s) \in E(T) \) with \( j < s \). See Figure 2.8.1 for an example.

Figure 2.8.1: The edge sets of the pseudo-components in the graph depicted are \( \{(1, 5), (5, 8)\}, \{(2, 5)\}, \{(3, 4), (4, 5)\}, \{(5, 6), (6, 7)\} \). The pseudo-component with edge set \( \{(1, 5), (5, 8)\} \) is both a left and right pseudo-component, while the pseudo-components with edge sets \( \{(2, 5)\}, \{(3, 4), (4, 5)\} \) are left pseudo-components and the pseudo-component with edge set \( \{(5, 6), (6, 7)\} \) is a right pseudo-component.
Proposition 28. Let $T$ be a good tree. By choosing the series of reductions suitably, the set of leaves of a $B$-reduction tree with root $T$ can be the set of all noncrossing alternating spanning forests $F$ of $\overline{T}$ such that

- $F$ is on the vertex set $[n + 1]$ and contains edge $(1, n + 1)$,
- $F$ has revlex edge-labels,
- $F$ contains at least one edge of the form $(i_1, j)$ with $i_1 \leq i$ for each right pseudo-component of $T$ on $[i, j]$,
- $F$ contains at least one edge of the form $(i, j_1)$ with $j \leq j_1$ for each left pseudo-component of $T$ on $[i, j]$.

See Figure 2.8.2 for an example.

Proof. It is easily seen that all graphs labeling the leaves of a $B$-reduction tree must be noncrossing alternating spanning forests of $\overline{T}$ on the vertex set $[n + 1]$ containing edge $(1, n + 1)$ and at least one edge of the form $(i_1, j)$ with $i_1 \leq i$ for each right pseudo-component of $T$ on $[i, j]$ and at least one edge of the form $(i, j_1)$ with $j \leq j_1$ for each left pseudo-component of $T$ on $[i, j]$. The proof then follows the proof of Proposition 21. To show that any noncrossing alternating spanning forests of $\overline{T}$ on the vertex set $[n + 1]$ containing edge $(1, n + 1)$ and at least one edge of the form $(i_1, j)$ with $i_1 \leq i$ for each right pseudo-component of $T$ on $[i, j]$ and at least one edge of the form $(i, j_1)$ with $j \leq j_1$ for each left pseudo-component of $T$ on $[i, j]$ appears among the leaves of a $B$-reduction tree and that all these forests have revlex edge-labels, we use induction on the number of pseudo-components of $T$. The base case is proved in Proposition 21. Suppose now that $T$ has $p$ pseudo-components, and let $P$ be such a pseudo-component that $T\backslash P$ is a tree with $p - 1$ pseudo-components. Apply the inductive hypothesis to $T\backslash P$ and Proposition 21 to $P$ and combine the graphs obtained as outcomes in all the ways possible to obtain a set $S$ of graphs labeling the nodes of the reduction tree from which any leaf can be obtained by successive reductions. By inspection we see that any noncrossing alternating spanning forest of
Theorem 29. Let $T$ be a good tree. The set of leaves of any $\mathcal{B}$-reduction tree with root labeled $T$ is, up to applications of reduction (2), the set of all noncrossing alternating spanning forests of $\overline{T}$ with revlex edge-labels on the vertex set $[n + 1]$ containing edge $(1, n + 1)$ and at least one edge of the form $(i_1, j)$ with $i_1 \leq i$ for each right pseudo-component of $T$ on $[i, j]$ and at least one edge of the form $(i, j_1)$ with $j \leq j_1$ for each left pseudo-component of $T$ on $[i, j]$. The set of leaves can be obtained by reductions from the elements of $S$. Since no graph can be obtained twice, and no other graph can label a leaf of a $\mathcal{B}$-reduction, the proof is complete. \QED

Theorem 5. (Noncommutative part.) If the polynomial $P_n^B(x_{ij})$ is a reduced form of $m_B[T]$ for a good tree $T$, then

$$P_n^B(x_{ij}) = \sum_F \beta^{n-|E(F)|} x^F,$$

where the sum runs over all noncrossing alternating spanning forests of $\overline{T}$ with revlex edge-labels on the vertex set $[n + 1]$ containing edge $(1, n + 1)$ and at least one edge of the form $(i_1, j)$ with $i_1 \leq i$ for each right pseudo-component of $T$ on $[i, j]$ and at least one edge of the form $(i, j_1)$ with $j \leq j_1$ for each left pseudo-component of $T$ on $[i, j]$, and $x^F$ is defined to be the noncommutative monomial $\prod_{i=1}^k x_{i_1,j_1} \ldots x_{i_k,j_k}$ if $F$ contains the edges $(i_1,j_1), \ldots, (i_k,j_k)$.
Figure 2.8.2: This figure depicts all the noncrossing alternating spanning forests of $T$ on the vertex set $[n + 1]$ containing edge $(1, n + 1)$ and at least one edge of the form $(i_1, j)$ with $i_1 \leq i$ for each right pseudo-component of $T$ on $[i, j]$ and at least one edge of the form $(i, j_1)$ with $j \leq j_1$ for each left pseudo-component of $T$ on $[i, j]$.

By the Ehrhart polynomial form of Theorem 4, see end of Section 2.8, $L_{P(T)}(t) = \binom{t+2}{2} - 6\binom{t+3}{3} + 6\binom{t+4}{4}$, since $f_{T,2} = 1, f_{T,3} = 6, f_{T,4} = 6$ and $f_{T,i} = 0$, for $i \neq 2, 3, 4$. 
Proposition 30. Let $T$ be a good tree. The number of forests with $k$ edges labeling the leaves of an $S$-reduction tree $T_S^T$ with root labeled by $T$, $f_{T,k}$, is equal to the number of noncrossing alternating spanning forests $F$ of $\overline{T}$ containing edge $(1, n+1)$ and at least one edge of the form $(i_1, j)$ with $i_1 \leq i$ for each right pseudo-component of $T$ on $[i, j]$ and at least one edge of the form $(i, j_1)$ with $j \leq j_1$ for each left pseudo-component of $T$ on $[i, j]$.

Proposition 30 provides a combinatorial description of the coefficients in Theorems 13, 14 and Corollary 15, completing the proofs of Theorems 4 and 5. We state them in full generality here.

Theorem 5. (Commutative part.) If the polynomial $P^S_n(x_{ij})$ is a reduced form of $m^S[T]$ for a good tree $T$, then

$$P^S_n(x_{ij} = 1) = \sum_{l=0}^{l-1} f_{T,l-m} \beta^m,$$

where $f_{T,k}$ is as in Proposition 30.

Theorem 4. (Ehrhart polynomial and volume.) The Ehrhart polynomial and volume of the polytope $P(T)$, for a good tree $T$ on the vertex set $[n+1]$, are, respectively,

$$L_{P(T)}(t) = (-1)^n \sum_{i=0}^{n} (-1)^i f_{T,i} \binom{t+i}{i},$$

$$\text{vol } P(T) = \frac{f_{T,n}}{n!},$$

where $f_{T,k}$ is as in Proposition 30. See Figure 2.8.2 for an example.

Theorem 4 can be generalized so that we not only describe the $n$-dimensional simplices in the triangulation of $P(T)$, but also describe their intersections in terms of noncrossing alternating spanning forests in $\overline{T}$. Using the Reduction Lemma (Lemma 8) and Theorem 29 we can deduce the following.

Theorem 4. (Canonical triangulation.) If $T$ is a noncrossing tree on the ver-
tex set \([n + 1]\) and \(T_1, \ldots, T_k\) are the noncrossing alternating spanning trees of \(\overline{T}\), then the root polytopes \(\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)\) are \(n\)-dimensional simplices forming a triangulation of \(\mathcal{P}(T)\). Furthermore, the intersections of the top dimensional simplices \(\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)\) are the simplices \(\mathcal{P}(F)\), where \(F\) run over all noncrossing alternating spanning forests of \(\overline{T}\) with revlex edge-labels on the vertex set \([n + 1]\) containing edge \((1, n + 1)\) and at least one edge of the form \((i_1, j)\) with \(i_1 \leq i\) for each right pseudo-component of \(T\) on \([i, j]\) and at least one edge of the form \((i, j_1)\) with \(j \leq j_1\) for each left pseudo-component of \(T\) on \([i, j]\).

2.9 Properties of the canonical triangulation

In this section we show that the canonical triangulation of \(\mathcal{P}(T)\) into simplices \(\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)\), and their faces, where \(T_1, \ldots, T_k\) are the noncrossing alternating spanning trees of \(\overline{T}\), as described in Theorem 4, is regular and flag. We construct a shelling and using this shelling calculate the generating function \(J(\mathcal{P}(T), x) = 1 + \sum_{i=1}^{\infty} L_{\mathcal{P}(T)}(t)x^t\), yielding another way to compute the Ehrhart polynomials. This generalizes the calculation of \(J(\mathcal{P}(A_n^+), x)\), [S4, Exercise 6.31], [F].

Recall that a triangulation of the polytope \(P\) is regular if there exists a concave piecewise linear function \(f : P \to \mathbb{R}\) such that the regions of linearity of \(f\) are the maximal simplices in the triangulation. It has been shown in [GGP, Theorem 6.3] that the noncrossing triangulation of \(\mathcal{P}(A_n^+)\) is regular. This result can be naturally extended to the canonical triangulation of any of the root polytopes \(\mathcal{P}(T)\). An attractive proof uses the following concave function constructed by Postnikov for an alternative proof of [GGP, Theorem 6.3].

Let \(f : A \to \mathbb{R}\) be a function on the set \(A\) such that polytope \(P = \text{ConvHull}(A)\). Let \(\hat{P} = \text{ConvHull}(\{a, f(a)\} \mid a \in A)\) and define then \(f(p) = \max\{x \mid \pi(a, x) = p, (a, x) \in \hat{P}\}, p \in P\). The function \(f : P \to \mathbb{R}\) is concave by definition. Consider the root polytope \(\mathcal{P}(T)\) with vertices \(0\) and \(e_i - e_j\), where \((i, j) \in I \times J\). Let \(f(0) = 0\) and \(f(e_i - e_j) = (i - j)^2\) for \((i, j) \in I \times J\). Extend this to a concave piecewise linear function as explained in the above paragraph. A check of the regions of linearity
proves the regularity of the canonical triangulation of $\mathcal{P}(T)$.

It can also be shown that the canonical triangulation of $\mathcal{P}(T)$ is flag, which we leave as an exercise to the reader. For the definition and importance of flag triangulations see [H, Section 2].

A triangulation of the $d$-polytope $\mathcal{P}(T)$ into the $d$-simplices $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)$ is \textit{shellable}, if there is a \textit{shelling}, a linear order $F_1, \ldots, F_k$ on $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)$, such that for all $2 \leq i \leq k$, $F_i$ is attached to $F_1 \cup \ldots \cup F_{i-1}$ on a union of nonzero facets of $F_i$. See [S3] for more details.

The \textbf{lexicographic ordering} on the simplices $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)$ is as follows: $\mathcal{P}(T_i) <_{\text{lex}} \mathcal{P}(T_j)$ if and only if for some $l$ the first $l$ edges of $T_i$ and $T_j$ in lexicographic ordering coincide and the $(l+1)^{\text{st}}$ edge of $T_i$ is less than the $(l+1)^{\text{st}}$ edge of $T_j$ in lexicographic ordering. In lexicographic ordering the edge $(i_1, j_1)$ is less than the edge $(i_2, j_2)$ if $i_1 < i_2$, or $i_1 = i_2$ and $j_1 < j_2$.

\textbf{Theorem 31.} Let $T$ be a noncrossing tree on the vertex set $[n+1]$. Let $T_1, \ldots, T_k$ be the noncrossing alternating spanning trees of $\overline{T}$ such that $\mathcal{P}(T_1) <_{\text{lex}} \ldots <_{\text{lex}} \mathcal{P}(T_k)$. Then $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)$ is a shelling order. See Figure 2.9.1 for an example.

\textit{Proof.} It suffices to show that for all $2 \leq m \leq k$, the intersection $\mathcal{P}(T_m) \cap (\mathcal{P}(T_1) \cup \ldots \cup \mathcal{P}(T_{m-1}))$ is a union of nonzero facets of $\mathcal{P}(T_m)$.

Let $L(T_m)$ denote the set of left vertices of $T_m$, that is, the vertices of $T_m$ which are the smaller vertex of each edge incident to them. Let

$$S(T_m) = \{(i, j) \mid i \in L(T_m) \text{ and } j \text{ is the largest vertex adjacent to } i \text{ in } T_m\}.$$ 

The set $S(T_m)$ uniquely determines $T_m$, since $T_m$ is a noncrossing alternating spanning tree.

There are exactly two noncrossing alternating trees containing $F = ([n+1], E(T_m) \backslash \{(i, j)\})$, for $(i, j) \in S(T_m) \backslash \{(1, n+1)\}$, namely, $T_m$ and $\tilde{T}_m = ([n+1], E(F) \cup \{(i', j')\})$, where $i'$ is the biggest vertex of $T_m$ smaller than $i$ such that $(i', j) \in E(T_m)$, and $j'$ is the biggest vertex of $T_m$ smaller than $j$ such that $(i, j') \in E(T_m)$, or if $(i, j)$ is the only edge incident to $i$, then $j' = i$. Let $f_{T_m} : S(T_m) \backslash \{(1, n+1)\} \to E(K_{n+1})$ be defined
by \( f_{T_m}: (i,j) \mapsto (i',j') \) according to the rule explained above. Define

\[
M_T(T_m) = \{(i, j) \in S(T_m) \mid f_{T_m}((i,j)) \not\in \overline{T}\}.
\]

The set \( S_T(T_m) = S(T_m)\setminus M_T(T_m) \) uniquely determines \( T_m \), since \( T_m \) is a noncrossing alternating spanning tree of \( \overline{T} \). Furthermore, if for some \( m' \in [k] \), \( m' \neq m \), \( S_T(T_m) \subseteq E(T_{m'}) \), then \( \mathcal{P}(T_m) \prec_{\text{lex}} \mathcal{P}(T_{m'}) \). Thus, if for a forest \( F \) on the vertex set \([n + 1], S_T(T_m) \subseteq E(F) \subseteq E(T_m) \), then \( \mathcal{P}(F) \) is not a face of \( \mathcal{P}(T_1) \cup \ldots \cup \mathcal{P}(T_{m-1}) \).

If \( F \subseteq T_m \) does not contain \( S_T(T_m) \) and \( |E(F)| = n - 1 \), then \( F \subset T_l = ([n + 1], E(T_m) \setminus \{(i,j)\} \cup \{f_{T_m}((i,j))\}) \) for \( l < m \). Thus, for all \( 2 \leq m \leq k \),

\[
\mathcal{P}(T_m) \cap (\mathcal{P}(T_1) \cup \ldots \cup \mathcal{P}(T_{m-1})) = \bigcup_{(i,j) \in S_T(T_m)} \mathcal{P}(([n + 1], E(T_m) \setminus \{(i,j)\})).
\]

See Figure 2.9.1 for an example. \( \square \)

**Theorem 32.** Let \( T \) be a good tree on the vertex set \([n + 1]\). Let \( c(n,l) \) be the number of noncrossing alternating spanning trees \( T_m \) of \( \overline{T} \) with \( |S_T(T_m)| = l \). Then,

\[
(1 - x)^{n+1} J(\mathcal{P}(T), x) = \sum_{l=1}^{n} c(n,l-1)x^{l-1}.
\]

**Proof.** It can be seen that for a forest \( F \) with \( r \) edges, \( J(\mathcal{P}(F), x) = \frac{1}{(1-x)^{r+1}} \), [BR, Theorem 2.2]. If we are adding the simplices \( \mathcal{P}(T_1), \ldots, \mathcal{P}(T_k) \) in lexicographic order one at a time, and calculating their contribution to \( J(\mathcal{P}(T), x) \), then the contribution of \( \mathcal{P}(T_m) \) such that \( \mathcal{P}(T_m) \cap (\mathcal{P}(T_1) \cup \ldots \cup \mathcal{P}(T_{m-1})) \) is a union of \((l-1)\) facets of \( \mathcal{P}(T_m) \) is

\[
\frac{1}{(1-x)^{n+1}} - (l-1)\frac{1}{(1-x)^{n}} + \cdots + (-1)^{l-1}\frac{l-l}{l+1} = \frac{x^{l-1}}{(1-x)^{n+1}}.
\]

Hence,

\[
J(\mathcal{P}(T), x) = \frac{\sum_{l=1}^{n} c(n,l-1)x^{l-1}}{(1-x)^{n+1}}.
\]

\( \square \)
Figure 2.9.1: Trees $T_1, \ldots, T_6$ are the noncrossing alternating spanning trees of $\bar{T}$. The root polytopes associated to them satisfy $\mathcal{P}(T_1) <_{lex} \cdots <_{lex} \mathcal{P}(T_6)$.

$S(T) = \{(3, 4)\}$

By Theorem 32, $J(\mathcal{P}(T), x) = \frac{x^2 + 4x^1 + 1}{(1-x)^5}$. This is of course equivalent to $L_{\mathcal{P}(T)}(t) = \binom{t+2}{2} - 6\binom{t+3}{3} + 6\binom{t+4}{4}$ as calculated in Figure 2.8.2. For a way to see this equivalence directly, see [BR, Lemma 3.14].
Remark. All the theorems proved for trees (monomials corresponding to trees) in this chapter can be formulated for forests (monomials corresponding to forests), and the proofs proceed analogously. The acyclic condition for graphs in the theorems is crucial for the proof techniques to work, but the noncrossing condition is not. Given an acyclic graph $G$ which is crossing, we can uncross it to obtain a new graph $G^u$. The graph $G^u$ is a noncrossing graph such that there is a graph isomorphism $\phi : G \rightarrow G^u$, where if $(i, j) \in E(G)$, $i < j$, then $\phi(i) < \phi(j)$. The graph $G^u$ is not uniquely determined by these conditions. All the results apply to any $G^u$, and they can be translated back for $G$ in an obvious way. E.g. the volume of $\mathcal{P}(T)$ for any tree $T$ on the vertex set $[n + 1]$ is $\text{vol} \mathcal{P}(T) = f_{Tu} \frac{1}{n!}$, where $f_{Tu}$ denotes the number of noncrossing alternating spanning trees of $Tu$, the transitive closure of the uncrossed $T$.

2.10 Unique reduced forms and Gröbner bases

The reduced form of a monomial $m \in \mathcal{B}(A_n)$ was defined in the Introduction as a polynomial $P_{n}^{\mathcal{B}}$ obtained by successive applications of the reduction rule (2.1.1) until no further reduction is possible, where we allow commuting any two variables $x_{ij}$ and $x_{kl}$ where $i, j, k, l$ are distinct, between the reductions. An alternative way of thinking of the reduced form of a monomial $m \in \mathcal{B}(A_n)$ is to view the reduction process in $\mathbb{Q}\langle \beta, x_{ij} \mid 1 \leq i < j \leq n \rangle / I_\beta$, where the generators of the (two-sided) ideal $I_\beta$ in $\mathbb{Q}\langle \beta, x_{ij} \mid 1 \leq i < j \leq n + 1 \rangle$ are the elements $x_{ij}x_{kl} - x_{kl}x_{ij}$ for $i < j, k < l$ distinct, and $\beta x_{ij} - x_{ij}\beta, i < j$. In this section we prove the following theorem.

Theorem 33. The reduced form of any monomial $m \in \mathcal{B}(A_n)$ is unique.

We use noncommutative Gröbner bases techniques, which we now briefly review. We use the terminology and notation of [G], but state the results only for our special algebra. For the more general statements, see [G]. Throughout this section we consider the noncommutative case only.
Let
\[ R = \mathbb{Q}\langle \beta, x_{ij} \mid 1 \leq i < j \leq n + 1 \rangle / I_{\beta} \]
with additive basis \( B \), the set of noncommutative monomials in variables \( \beta \) and \( x_{ij} \), where \( 1 \leq i < j \leq n \), up to equivalence under the commutativity relations described by \( I_{\beta} \).

The \textit{tip} of an element \( f \in R \) is the largest basis element appearing in its expansion, denoted by \( \text{Tip}(f) \). Let \( C\text{Tip}(f) \) denote the coefficient of \( \text{Tip}(f) \) in this expansion. A set of elements \( X \) is \textbf{tip reduced} if for distinct elements \( x, y \in X \), \( \text{Tip}(x) \) does not divide \( \text{Tip}(y) \).

A well-order \( > \) on \( B \) is \textbf{admissible} if for \( p, q, r, s \in B \):
1. if \( p < q \) then \( pr < qr \) if both \( pr \neq 0 \) and \( qr \neq 0 \);
2. if \( p < q \) then \( sp < sq \) if both \( sp \neq 0 \) and \( sq \neq 0 \);
3. if \( p = qr \), then \( p > q \) and \( p > r \).

Let \( f, g \in R \) and suppose that there are monomials \( b, c \in B \) such that
1. \( \text{Tip}(f)c = b\text{Tip}(g) \).
2. \( \text{Tip}(f) \) does not divide \( b \) and \( \text{Tip}(g) \) does not divide \( c \).

Then the \textbf{overlap relation of} \( f \) \textbf{and} \( g \) \textbf{by} \( b \) \textbf{and} \( c \) is
\[ o(f, g, b, c) = \frac{fc}{C\text{Tip}(f)} - \frac{bg}{C\text{Tip}(g)}. \]

**Proposition 34.** ([G, Theorem 2.3]) \textit{A tip reduced generating set of elements \( \mathcal{G} \) of the ideal} \( J \) \textit{of} \( R \) \textit{is a Gröbner basis, where the ordering on the monomials is admissible, if for every overlap relation}
\[ o(g_1, g_2, p, q) \Rightarrow \mathcal{G} \ 0, \]
\textit{where} \( g_1, g_2 \in \mathcal{G} \) \textit{and the above notation means that dividing} \( o(g_1, g_2, p, q) \) \textit{by} \( \mathcal{G} \) \textit{yields a remainder of} \( 0 \).

See [G, Theorem 2.3] for the more general formulation of Proposition 34 and [G, Section 2.3.2] for the formulation of the Division Algorithm.
Proposition 35. Let $J$ be the ideal generated by the elements

$$x_{ij}x_{jk} - x_{ik}x_{ij} - x_{jk}x_{ik} - \beta x_{ik}, \text{ for } 1 \leq i < j < k \leq n + 1,$$

in $R$. Then there is a monomial order in which the above generators of $J$ form a Gröbner basis $G$ of $J$ in $R$, and the tips of the generators are, $x_{ij}x_{jk}$.

Proof. Let $x_{ij} > x_{kl}$ if $(i, j)$ is less than $(k, l)$ lexicographically. The degree of a monomial is determined by setting the degrees of $x_{ij}$ to be 1 and the degrees of $\beta$ and scalars to be 0. A monomial with higher degree is bigger in the order $>$, and the lexicographically bigger monomial of the same degree is greater than the lexicographically smaller one. Since in $R$ two equal monomials can be written in two different ways due to commutations, we can pick a representative to work with, say the one which is the “largest” lexicographically among all possible ways of writing the monomial, to resolve any ambiguities. The order $>$ just defined is admissible, and in it the tip of $x_{ij}x_{jk} - x_{ik}x_{ij} - x_{jk}x_{ik} - \beta x_{ik}$, for $1 \leq i < j < k \leq n + 1$, is $x_{ij}x_{jk}$. In particular, the generators of $J$ are tip reduced. A calculation of the overlap relations shows that $o(g_1, g_2, p, q) \Rightarrow_G 0$ in $R$, where $g_1, g_2 \in G$. Proposition 34 then implies Proposition 35.

$\square$

Corollary 36. The reduced form of a noncommutative monomial $m$ in variables $\beta$ and $x_{ij}$, $1 \leq i < j \leq n + 1$, is unique in $R$.

Proof. Since the tips of elements of the Gröbner basis $G$ of $J$ are exactly the monomials which we replace in the prescribed reduction rule (2.1.1), the reduced form of a monomial $m$ is the remainder $r$ upon division by the elements of $G$ with the order $>$ described in the proof of Proposition 35. Since we proved that in $R$ the basis $G$ is a Gröbner basis of $J$, it follows by \cite{G, Proposition 2.7} that the remainder $r$ of the division of $m$ by $G$ is unique in $R$. That is, the reduced form of a good monomial $m$ is unique in $R$.

$\square$

Note that Corollary 36 is equivalent to Theorem 33.

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Chapter 3

Root polytopes of type $C_n$, triangulations, and the subdivision algebra

3.1 Introduction

In this chapter we develop the connection between triangulations of type $C_n$ root polytopes and a commutative algebra $S(C_n)$, the subdivision algebra of type $C_n$ root polytopes. A type $C_n$ root polytope is a convex hull of the origin and some of the points $e_i - e_j, e_i + e_j, 2e_k$ for $1 \leq i < j \leq n, k \in [n]$, where $e_i$ denotes the $i^{th}$ standard basis vector in $\mathbb{R}^n$. A polytope $P(m)$ corresponds to each monomial $m \in S(C_n)$, and each relation of the algebra equating a monomial with three others, $m_0 = m_1 + m_2 + m_3$, can be interpreted as cutting the polytope $P(m_0)$ into two polytopes $P(m_1)$ and $P(m_2)$ with interiors disjoint such that $P(m_1) \cap P(m_2) = P(m_3)$; thus the name subdivision algebra for $S(C_n)$.

A subdivision algebra $S(A_n)$ for type $A_n$ root polytopes studied in Chapter 2 yielded an interplay between polytopes and algebras. Using techniques for polytopes, the algebra $S(A_n)$ can be understood better, and using the properties of $S(A_n)$ results for root polytopes can be deduced. The subdivision algebra $S(C_n)$ is a type $C_n$
generalization of $S(A_n)$ and its intimate connection to type $C_n$ root polytopes is displayed by a variety of results obtained by using this connection.

Root polytopes were first defined by Postnikov in [P1], although the full root polytope of type $A_n$ already appeared in the work of Gelfand, Graev and Postnikov [GGP], where they gave a canonical triangulation of it into simplices corresponding to noncrossing alternating trees. Properties of this triangulation are studied in [S4, Exercise 6.31]. Canonical triangulations for a family of type $A_n$ root polytopes were constructed in [M1] (Chapter 2) extending the result of [GGP]. In this chapter we define type $C_n$ analogs for noncrossing and alternating graphs, and show that a family of type $C_n$ root polytopes, containing the full root polytope, has canonical triangulations into simplices corresponding to noncrossing alternating graphs. Using the canonical triangulations we compute the volumes for these root polytopes.

The subdivision algebra $S(C_n)$ is closely related to the noncommutative bracket algebra $B(C_n)$ of type $C_n$ defined by A. N. Kirillov [K2]. Kirillov conjectured the uniqueness of the reduced form of a Coxeter type element in $B(C_n)$. As the algebras $S(C_n)$ and $B(C_n)$ have over ten not-so-simple-looking relations, we postpone their definitions and the precise statement of Kirillov’s conjecture till Section 3.2. While at the first sight the relations of $B(C_n)$ might appear rather mysterious, we interpret them similarly to the relations of $S(C_n)$, as certain subdivisions of root polytopes. This connection ultimately yields a proof of Kirillov’s conjecture along with more general theorems on reduced forms, of which there are two types. In the noncommutative algebra $B(C_n)$ we show that for a family of monomials $\mathcal{M}$, including the Coxeter type element defined by Kirillov, the reduced form is unique. In the commutative algebra $S(C_n)$ and the commutative counterpart $B^c(C_n) = B(C_n)/[B(C_n), B(C_n)]$ of $B(C_n)$, the reduced forms are not unique; however, we show that the number of monomials in a reduced form of $m \in \mathcal{M}$ is independent of the order of reductions performed.

This chapter is organized as follows. In Section 3.2 we give the definition of $B(C_n)$, as well as two related commutative algebras $B^c(C_n)$ and $S(C_n)$. We also state Kirillov’s conjecture pertaining to $B(C_n)$ in Section 3.2. In Section 3.3 we introduce signed graphs, define the type $C$ analogue of alternating graphs, and show how to
reformulate the relations of the algebras $B^c(C_n), S(C_n)$ into reductions on graphs. In Section 3.4 we introduce coned root polytopes of type $C_n$ and state the Reduction Lemma which connects root polytopes and the algebras $B(C_n), B^c(C_n), S(C_n)$. In Section 3.5 we prove a characterization of the vertices of coned type $C_n$ root polytopes, while in Section 3.6 we prove the Reduction Lemma. In Section 3.7 we establish the relation between volumes of root polytopes and reduced forms of monomials in the algebras $B^c(C_n), S(C_n)$ using the Reduction Lemma. In Section 3.8 we reformulate the noncommutative relations of $B(C_n)$ in terms of edge-labeled graphs and define well-structured and well-labeled graphs, key for our further considerations. In Section 3.9 we prove a simplified version of Kirillov’s conjecture, construct a canonical triangulation for the full type $C_n$ root polytope $P(C_n^+)$ and calculate its volume. In Section 3.10 we generalize Kirillov’s conjecture to all monomials arising from well-structured and well-labeled graphs and give the triangulations and volumes of the corresponding root polytopes. Finally, in Section 3.11 we prove the general form of Kirillov’s conjecture in a weighted bracket algebra $B^\beta(C_n)$, and show a way to calculate Ehrhart polynomials of certain type $C_n$ root polytopes.

3.2 The bracket and subdivision algebras of type $C_n$

In this section the definition of the bracket algebra $B(C_n)$ is given, along with a conjecture of Kirillov pertaining to it. We introduce the subdivision algebra $S(C_n)$, which, as its name suggests, will be shown to govern subdivisions of type $C_n$ root polytopes.

Kirillov [K2] defined the algebra we are denoting $B(C_n)$ as a type $B_n$ bracket algebra $B(B_n)$, but since we can interpret its generating variables as corresponding to either the type $B_n$ and type $C_n$ roots, we refer to it as a type $C_n$ bracket algebra $B(C_n)$. The reason for our desire to designate $B(C_n)$ as a type $C_n$ algebra is its essential link to type $C_n$ root polytopes, which we develop in this chapter. Here we
define a simplified form of the bracket algebra $B(C_n)$; for a more general definition, see Section 3.11.

Let the **bracket algebra** $B(C_n)$ of type $C_n$ be an associative algebra over $\mathbb{Q}$ with a set of generators $\{x_{ij}, y_{ij}, z_i \mid 1 \leq i \neq j \leq n\}$ subject to the following relations:

1. $x_{ij} + x_{ji} = 0, y_{ij} = y_{ji},$ for $i \neq j,$
2. $z_i z_j = z_j z_i$
3. $x_{ij} x_{kl} = x_{kl} x_{ij},$ $y_{ij} x_{kl} = x_{kl} y_{ij},$ $y_{ij} y_{kl} = y_{kl} y_{ij},$ for $i < j, k < l$ distinct.
4. $z_i x_{kl} = x_{kl} z_i,$ $z_i y_{kl} = y_{kl} z_i,$ for all $i \neq k, l$
5. $x_{ij} x_{jk} = x_{ik} x_{ij} + x_{jk} x_{ik},$ for $1 \leq i < j < k \leq n,$
5'. $x_{jk} x_{ij} = x_{ij} x_{ik} + x_{ik} x_{jk},$ for $1 \leq i < j < k \leq n,$
6. $x_{ij} y_{jk} = y_{ik} x_{ij} + y_{jk} y_{ik},$ for $1 \leq i < j < k \leq n,$
6'. $y_{jk} x_{ij} = x_{ij} y_{ik} + y_{ik} y_{jk},$ for $1 \leq i < j < k \leq n,$
7. $x_{ik} y_{jk} = y_{jk} y_{ij} + y_{ij} x_{ik},$ for $1 \leq i < j < k \leq n,$
7'. $y_{jk} x_{ik} = y_{ij} y_{jk} + x_{ik} y_{ij},$ for $1 \leq i < j < k \leq n,$
8. $y_{ik} x_{jk} = x_{jk} y_{ij} + y_{ij} y_{ik},$ for $1 \leq i < j < k \leq n,$
8'. $x_{jk} y_{ik} = y_{ij} x_{jk} + y_{ik} y_{ij},$ for $1 \leq i < j < k \leq n,$
9. $x_{ij} z_j = z_i x_{ij} + y_{ij} z_i + z_j y_{ij},$ for $i < j$
9'. $z_j x_{ij} = x_{ij} z_i + z_i y_{ij} + y_{ij} z_j,$ for $i < j$

Let $w_{C_n} = \prod_{i=1}^{n-1} x_{i,i+1} z_n$ be a Coxeter type element in $B(C_n)$ and let $P_n^B$ be the polynomial in variables $x_{ij}, y_{ij}, z_i, 1 \leq i \neq j \leq n$ obtained from $w_{C_n}$ by successively applying the defining relations (1) - (9') in any order until unable to do so. We call $P_n^B$ a **reduced form** of $w_{C_n}$ and consider the process of successively applying the defining relations (5) - (9') as a reduction process, with possible commutations (2)-(4) between reductions, as we show in the following example.
\[ x_{12}x_{23}z_3 \rightarrow x_{13}x_{12}z_3 + x_{23}x_{13}z_3 \]
\[ \rightarrow x_{13}z_3x_{12} + x_{23}z_1x_{13} + x_{23}y_{13}z_1 + x_{23}z_3y_{13} \]
\[ \rightarrow z_1x_{13}x_{12} + y_{13}z_1x_{12} + z_3y_{13}x_{12} + x_{23}z_1x_{13} + y_{12}x_{23}z_1 + y_{13}y_{12}z_1 \]
\[ + z_2x_{23}y_{13} + y_{23}z_2y_{13} + z_3y_{23}y_{13} \]
\[ \rightarrow z_1x_{13}x_{12} + y_{13}z_1x_{12} + z_3y_{13}x_{12} + x_{23}z_1x_{13} + y_{12}x_{23}z_1 \]
\[ + y_{13}y_{12}z_1 + z_2y_{12}x_{23} + z_2y_{13}y_{12} + y_{23}z_2y_{13} + z_3y_{23}y_{13} \]

In the example above the pair of variables on which one of reductions (5) – (9') is performed is in boldface, and the variables which we commute according to one of (2)-(4) are underlined.

**Conjecture 1. (Kirillov [K2])** Apart from applying the relations (1)-(4), the reduced form \( P_n^S \) of \( w_{cn} \) does not depend on the order in which the reductions are performed.

Note that the above statement does not hold true for any monomial. We show one simple example of how it fails.

\[ x_{12}x_{23}y_{13} \rightarrow x_{13}x_{12}y_{13} + x_{23}x_{13}y_{13} \quad (3.2.1) \]

\[ x_{12}x_{23}y_{13} \rightarrow x_{12}y_{12}x_{23} + x_{12}y_{13}y_{12} \quad (3.2.2) \]

Note that we reduced the monomial \( x_{12}x_{23}y_{23} \) in two different ways yielding two different polynomials. The reader can also check another example of this phenomenon by reducing the monomial \( y_{14}x_{24}y_{34} \) in two different ways to obtain two different reduced forms.

We prove Conjecture 1 in Section 3.9, as well as its generalizations in Sections
3.10 and 3.11. We first define and study a commutative algebra $S(C_n)$ closely related to $B(C_n)$, though more complicated than its commutative counterpart, $B^c(C_n) = B(C_n)/[B(C_n), B(C_n)]$, which is simply the commutative associative algebra over $\mathbb{Q}$ with a set of generators $\{x_{ij}, y_{ij}, z_i, 1 \leq i \neq j \leq n\}$ subject to relations (1) and (5) from above. Our motivation for defining $S(C_n)$ is a natural correspondence between the relations of $S(C_n)$ and ways to subdivide type $C_n$ root polytopes, which correspondence is made precise in the Reduction Lemma (Lemma 39). In order to emphasize this connection, we call $S(C_n)$ the subdivision algebra of type $C_n$. The subalgebra $S(A_{n-1})$ of $S(C_n)$ generated by $\{x_{ij} | 1 \leq i \neq j \leq n\}$ has been studied in [M1] (Chapter 2), and an analogous correspondence between the relations of $S(A_{n-1})$ and ways to subdivide type $A_{n-1}$ root polytopes has been established. Moreover, results in the spirit of Conjecture 1 for type $A_{n-1}$ can also be found in [M1] (Chapter 2).

Let the subdivision algebra $S(C_n)$ be the commutative algebra over $\mathbb{Q}[\beta]$, where $\beta$ is a variable (and a central element), with a set of generators $\{x_{ij}, y_{ij}, z_i, 1 \leq i \neq j \leq n\}$ subject to the following relations:

1. $x_{ij} + x_{ji} = 0, y_{ij} = y_{ji}$, for $i \neq j$,
2. $x_{ij}x_{jk} = x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}$, for $1 \leq i < j < k \leq n$,
3. $x_{ij}y_{jk} = y_{ik}x_{ij} + y_{jk}y_{ik} + \beta y_{ik}$, for $1 \leq i < j < k \leq n$,
4. $x_{ik}y_{jk} = y_{jk}y_{ij} + y_{ij}x_{ik} + \beta y_{ij}$, for $1 \leq i < j < k \leq n$,
5. $y_{ik}x_{jk} = x_{jk}y_{ij} + y_{ij}y_{ik} + \beta y_{ij}$, for $1 \leq i < j < k \leq n$,
6. $y_{ij}x_{ij} = z_i x_{ij} + y_{ij}z_i + \beta z_i$, for $i < j$,
7. $x_{ij}z_j = y_{ij}x_{ij} + z_j y_{ij} + \beta y_{ij}$, for $i < j$.

Notice that when we set $\beta = 0$ relations (2)-(5) of $S(C_n)$ become relations (5)-(8) of $B(C_n)$, and if we combine relations (6) and (7) of $S(C_n)$ we obtain relation (9) of $B(C_n)$. In some cases we will in fact simply work with the commutative counterpart of $B(C_n)$, $B^c(C_n)$.

We treat relations (2)-(7) of $S(C_n)$ as reduction rules:

$$x_{ij}x_{jk} \rightarrow x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}, \quad (3.2.3)$$
\[ x_{ij}y_{jk} \rightarrow y_{ik}x_{ij} + y_{jk}y_{ik} + \beta y_{ik}, \quad (3.2.4) \]
\[ x_{ik}y_{jk} \rightarrow y_{jk}y_{ij} + y_{ij}x_{ik} + \beta y_{ij}, \quad (3.2.5) \]
\[ y_{ik}x_{jk} \rightarrow x_{jk}y_{ij} + y_{ij}y_{ik} + \beta y_{ij}, \quad (3.2.6) \]
\[ y_{ij}x_{ij} \rightarrow z_{i}x_{ij} + y_{ij}z_{i} + \beta z_{i} \quad (3.2.7) \]
\[ x_{ij}z_{j} \rightarrow y_{ij}x_{ij} + z_{j}y_{ij} + \beta y_{ij} \quad (3.2.8) \]

A reduced form of the monomial \( m \) in variables \( x_{ij}, y_{ij}, z_{k}, 1 \leq i < j \leq n, k \in [n] \), in the algebra \( S(C_n) \) is a polynomial \( P_n^S \) obtained by successive applications of reductions (3.2.3)-(3.2.8) until no further reduction is possible, where we allow commuting any two variables. Requiring that \( m \) is in variables \( x_{ij}, y_{ij}, z_{k}, 1 \leq i < j \leq n, k \in [n] \), is without loss of generality, since otherwise we can simply replace \( x_{ij} \) with \( -x_{ji} \) and \( y_{ij} \) with \( y_{ji} \). Note that the reduced forms are not necessarily unique. However we show in Section 3.7 that the number of monomials in a reduced form of a suitable monomial \( m \) is independent of the order of the reductions performed.

### 3.3 Commutative reductions in terms of graphs

In this section we rephrase the reduction process described in Section 3.2 in terms of graphs. This view will be useful throughout the chapter. We use the language of signed graphs. Signed graphs have appeared in the literature before, for example in Zaslavsky's and Reiner's work [Z1, Z2, R1, R2]. Their notation is not the same, and we use a notation closer to Reiner's. In particular, positive and negative edges in our notation mean something different than in Zaslavsky's language. We request the reader to read the definitions with full attention for this reason.

A signed graph \( G \) on the vertex set \([n]\) is a multigraph with each edge labeled by \(+\) or \(-\). All graphs in this chapter are signed and in each of them the loops are labeled positive. We denote an edge with endpoints \( i, j \) and sign \( \epsilon \in \{+, -\} \) by \((i, j, \epsilon)\). Note that \((i, j, \epsilon) = (j, i, \epsilon)\). As a result, we drop the signs from the loops
in figures. A positive edge, that is an edge labeled by +, is said to be **positively incident**, or, **incident with a positive sign**, to both of its endpoints. A negative edge is positively incident to its smaller vertex and **negatively incident** to its greater endpoint. We say that a graph is **alternating** if for any vertex \( v \in V(G) \) the edges of \( G \) incident to \( v \) are incident to \( v \) with the same sign.

Think of a monomial \( m \in \mathcal{S}(C_n) \) in variables \( x_{ij}, y_{ij}, z_k, 1 \leq i < j \leq n, k \in [n] \), as a signed graph \( G \) on the vertex set \([n]\) with a negative edge \((i, j, -)\) for each appearance of \( x_{ij} \) in \( m \) and with a positive edge \((i, j, +)\) for each appearance of \( y_{ij} \) in \( m \) and with a loop \((i, i, +)\) for each appearance of \( z_i \) in \( m \). Let \( G^S[m] \) denote this graph. It is straightforward to reformulate the reduction rules (3.2.3)-(3.2.8) in terms of reductions on graphs. If \( m \in \mathcal{S}(C_n) \), then we replace each monomial \( m \) in the reductions by corresponding graphs \( G^S[m] \).

**Reduction rules for graphs:**

Given a graph \( G_0 \) on the vertex set \([n]\) and \((i, j, -), (j, k, -) \in E(G_0)\) for some \( i < j < k \), let \( G_1, G_2, G_3 \) be graphs on the vertex set \([n]\) with edge sets

\[
E(G_1) = E(G_0) \setminus \{(j, k, -)\} \cup \{(i, k, -)\},
E(G_2) = E(G_0) \setminus \{(i, j, -)\} \cup \{(i, k, -)\},
E(G_3) = E(G_0) \setminus \{(i, j, -)\} \cup \{(j, k, -)\} \cup \{(i, k, -)\}.
\]  

(3.3.1)

Given a graph \( G_0 \) on the vertex set \([n]\) and \((i, j, -), (j, k, +) \in E(G_0)\) for some \( i < j < k \), let \( G_1, G_2, G_3 \) be graphs on the vertex set \([n]\) with edge sets

\[
E(G_1) = E(G_0) \setminus \{(j, k, +)\} \cup \{(i, k, +)\},
E(G_2) = E(G_0) \setminus \{(i, j, -)\} \cup \{(i, k, +)\},
E(G_3) = E(G_0) \setminus \{(i, j, -)\} \cup \{(j, k, +)\} \cup \{(i, k, +)\}.
\]  

(3.3.2)

Given a graph \( G_0 \) on the vertex set \([n]\) and \((i, k, -), (j, k, +) \in E(G_0)\) for some \( i < j < k \), let \( G_1, G_2, G_3 \) be graphs on the vertex set \([n]\) with edge sets

\[
E(G_1) = E(G_0) \setminus \{(j, k, +)\} \cup \{(i, k, +)\},
E(G_2) = E(G_0) \setminus \{(i, j, -)\} \cup \{(i, k, +)\},
E(G_3) = E(G_0) \setminus \{(i, j, -)\} \cup \{(j, k, +)\} \cup \{(i, k, +)\}.
\]  

(3.3.3)
Given a graph $G_0$ on the vertex set $[n]$ and $(i, k, +), (j, k, -) \in E(G_0)$ for some $i < j < k$, let $G_1, G_2, G_3$ be graphs on the vertex set $[n]$ with edge sets

$$
E(G_1) = E(G_0) \setminus \{(j, k, -)\} \cup \{(i, j, +)\},
$$
$$
E(G_2) = E(G_0) \setminus \{(i, k, +)\} \cup \{(i, j, +)\},
$$
$$
E(G_3) = E(G_0) \setminus \{(i, k, +)\} \setminus \{(j, k, -)\} \cup \{(i, j, +)\}.
$$

(3.3.4)

Given a graph $G_0$ on the vertex set $[n]$ and $(i, j, +), (i, j, -) \in E(G_0)$ for some $i < j$, let $G_1, G_2, G_3$ be graphs on the vertex set $[n]$ with edge sets

$$
E(G_1) = E(G_0) \setminus \{(i, j, -)\} \cup \{(i, i, +)\},
$$
$$
E(G_2) = E(G_0) \setminus \{(i, j, +)\} \cup \{(i, i, +)\},
$$
$$
E(G_3) = E(G_0) \setminus \{(i, j, +)\} \setminus \{(i, i, +)\} \cup \{(i, i, +)\}.
$$

(3.3.5)

Given a graph $G_0$ on the vertex set $[n]$ and $(i, j, -), (j, j, +) \in E(G_0)$ for some $i < j$, let $G_1, G_2, G_3$ be graphs on the vertex set $[n + 1]$ with edge sets

$$
E(G_1) = E(G_0) \setminus \{(j, j, +)\} \cup \{(i, j, +)\},
$$
$$
E(G_2) = E(G_0) \setminus \{(i, j, -)\} \cup \{(i, j, +)\},
$$
$$
E(G_3) = E(G_0) \setminus \{(i, j, +)\} \setminus \{(i, j, -)\} \cup \{(i, j, +)\}.
$$

(3.3.6)

We say that $G_0$ reduces to $G_1, G_2, G_3$ under the reduction rules defined by equations (3.3.1)-(3.3.6).

An $S$-reduction tree $T^S$ for a monomial $m_0$, or equivalently, the graph $G^S[m_0]$,
is constructed as follows. The root of $T^S$ is labeled by $G^S[m_0]$. Each node $G^S[m]$ in $T^S$ has three children, which depend on the choice of the edges of $G^S[m]$ on which we perform the reduction. E.g., if the reduction is performed on edges $(i, j, -), (j, k, -) \in E(G^S[m]), i < j < k$, then the three children of the node $G_0 = G^S[m]$ are labeled by the graphs $G_1, G_2, G_3$ as described by equation (3.3.1). For an example of an $S$-reduction tree, see Figure 3.3.1.

Figure 3.3.1: An $S$-reduction tree with root corresponding to the monomial $x_{12}x_{13}z_3$. Summing the monomials corresponding to the graphs labeling the leaves of the reduction tree multiplied by suitable powers of $\beta$, we obtain a reduced form $P^S_n$ of $x_{12}x_{13}z_3$, $P^S_n = z_1x_{12}y_{13} + z_1x_{12}y_{13} + \beta z_1x_{12} + x_{12}y_{13}z_3 + \beta x_{12}y_{13}$.

Of course, given a graph we can also easily recover the corresponding monomial. Namely, given a graph $G$ on the vertex set $[n]$ we associate to it the monomial $m^S[G] = m^S[G] = \prod_{(i, j, e) \in E(G)} w(i, j, e)$, where $w(i, j, -) = x_{ij}$ for $i < j$, $w(i, j, -) = x_{ji}$ for $i > j$, $w(i, j, +) = y_{ij}$ and $w(i, i, +) = z_i$. Summing the monomials corresponding to the graphs labeling the leaves of the reduction tree $T^S$ multiplied by suitable powers of $\beta$, we obtain a reduced form of $m_0$. 

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3.4 Coned type $C$ root polytopes

Generalizing the terminology of [P1, Definition 12.1], a root polytope of type $C_n$ is the convex hull of the origin and some of the points $e_i - e_j$, $e_i + e_j$ and $2e_k$ for $1 \leq i < j \leq n$, $k \in [n]$, where $e_i$ denotes the $i^{th}$ coordinate vector in $\mathbb{R}^n$. A very special root polytope is the full type $C_n$ root polytope

$$\mathcal{P}(C_n^+) = \text{ConvHull}(0, e^-_{ij}, e^+_{ij}, 2e_k \mid 1 \leq i < j \leq n, k \in [n])$$

$$= \text{ConvHull}(0, e^-_{ij}, 2e_k \mid 1 \leq i < j \leq n, k \in [n]),$$

where $e^-_{ij} = e_i - e_j$ and $e^+_{ij} = e_i + e_j$. We study a class of root polytopes including $\mathcal{P}(C_n^+)$, which we now discuss.

Let $G$ be a graph on the vertex set $[n]$. Let

$$v(i, j, \epsilon) = \begin{cases} 
  e^\epsilon_{ij} & \text{if } i \leq j \\
  e^\epsilon_{ji} & \text{if } i > j,
\end{cases}$$

Define

$$\mathcal{V}_G = \{v(i, j, \epsilon) \mid (i, j, \epsilon) \in E(G)\},$$
a set of vectors associated to $G$;

$$\mathcal{C}(G) = \langle \mathcal{V}_G \rangle := \{ \sum_{v(i, j, \epsilon) \in \mathcal{V}_G} c_{ij}v(i, j, \epsilon) \mid c_{ij} \geq 0\},$$

the cone associated to $G$; and

$$\overline{\mathcal{V}}_G = \Phi^+ \cap \mathcal{C}(G),$$

all the positive roots of type $C_n$ contained in $\mathcal{C}(G)$,

where $\Phi^+ = \{e^-_{ij}, e^+_{ij}, 2e_k \mid 1 \leq i < j \leq n, k \in [n]\}$ is the set of positive roots of type $C_n$. The idea to consider the positive roots of a root system inside a cone appeared earlier in Reiner’s work [R1], [R2] on signed posets. Coned type $A_n$ root polytopes were studied in [M1].
Define the **transitive closure** of a graph $G$ as

$$
\overline{G} = \{(i, j, \epsilon) \mid v(i, j, \epsilon) \in \overrightarrow{V}_G\}
$$

The **root polytope** $\mathcal{P}(G)$ associated to graph $G$ is

$$
\mathcal{P}(G) = \text{ConvHull}(0, v(i, j, \epsilon) \mid (i, j, \epsilon) \in \overline{G})
$$

(3.4.1)

The root polytope $\mathcal{P}(G)$ associated to graph $G$ can also be defined as

$$
\mathcal{P}(G) = \mathcal{P}(C^+_n) \cap \mathcal{C}(G).
$$

(3.4.2)

The equivalence of these two definitions is proved in Lemma 44 in Section 3.6.

Note that $\mathcal{P}(C^+_n) = \mathcal{P}(P^l)$ for the graph $P^l = ([n], \{(n, n, +), (i, i + 1, -) \mid i \in [n - 1]\})$. While the choice of $G$ such that $\mathcal{P}(C^+_n) = \mathcal{P}(G)$ is not unique, it becomes unique if we require that $G$ is **minimal**, that is for no edge $(i, j, \epsilon) \in E(G)$ can the corresponding vector $v(i, j, \epsilon)$ be written as a nonnegative linear combination of the vectors corresponding to the edges $E(G) \setminus \{(i, j, \epsilon)\}$. Graph $P^l$ is minimal.

We can describe the vertices in $\overrightarrow{V}_G$ in terms of paths in $G$. A **playable route** $P$ of a graph $G$ is an ordered sequence of edges $(i_1, j_1, \epsilon_1), \ldots, (i_l, j_l, \epsilon_l) \in E(G)$, $j_k = i_{k+1}$ for $k \in [l - 1]$, such that $(i_k, j_k, \epsilon_k)$ and $(i_{k+1}, j_{k+1}, \epsilon_{k+1})$, $k \in [l - 1]$, are incident to $j_k = i_{k+1}$ with opposite signs. For a playable route of $G$, $v(i_1, j_1, \epsilon_1) + \cdots + v(i_l, j_l, \epsilon_l) \in \Phi^+$.

A **playable pair** $(P_1, P_2)$ in a graph $G$ is a pair of playable routes $(i_1, j_1, \epsilon_1), \ldots, (i_l, j_l, \epsilon_l)$ and $(i'_1, j'_1, \epsilon'_1), \ldots, (i'_{l'}, j'_{l'}, \epsilon'_{l'})$ such that $i_1 = j_l$ and $i'_1 = j'_{l'}$. It follows that $\frac{1}{2}(v(i_1, j_1, \epsilon_1) + \cdots + v(i_l, j_l, \epsilon_l)) + \frac{1}{2}(v(i'_1, j'_1, \epsilon'_1) + \cdots + v(i'_{l'}, j'_{l'}, \epsilon'_{l'})) \in \Phi^+$.

Define a map $\phi$ from the playable routes and playable pairs to $\Phi^+$ as follows.
\[ \phi(P) = v(i_1, j_1, \epsilon_1) + \cdots + v(i_t, j_t, \epsilon_t), \] where \( P \) is the playable route above,
\[
\phi(P_1, P_2) = \frac{1}{2} (v(i_1, j_1, \epsilon_1) + \cdots + v(i_t, j_t, \epsilon_t)) + \frac{1}{2} (v(i'_1, j'_1, \epsilon'_1) + \cdots + v(i'_t, j'_t, \epsilon'_t)), \]
where \((P_1, P_2)\) is the playable pair above. (3.4.3)

**Proposition 37.** Let \( G \) be a graph on the vertex set \([n]\). Any \( v \in \overline{V}_G \) is \( v = \phi(P) \) or \( v = \phi(P_1, P_2) \) for some playable route \( P \) or playable pair \((P_1, P_2)\) of \( G \). If the set of vectors \( V_G \) is linearly independent, then the correspondence between playable routes \( G \) and vertices in \( \overline{V}_G \) is a bijection.

The proof of Proposition 37 appears in Section 3.5.

Define
\[
\mathcal{L}_n = \{ G = ([n], E(G)) \mid V_G \text{ is a linearly independent set} \},
\]
and
\[
\mathcal{L}(C_n^+) = \{ P(G) \mid G \in \mathcal{L}_n \}, \] the set of type \( C_n \) coned root polytopes

with linearly independent generators. Since all polytopes in this chapter are coned root polytopes with linearly independent generators, we simply refer to them as coned root polytopes.

The next lemma characterizes graphs \( G \) which belong to \( \mathcal{L}_n \); a version of it appears in [F, p. 42].

**Lemma 38.** ([F, p. 42]) A graph \( G \) on the vertex set \([n]\) belongs to \( \mathcal{L}_n \) if and only if each connected component of \( G \) is a tree or a graph whose unique simple cycle has an odd number of positively labeled edges.

The full root polytope \( P(C_n^+) \notin \mathcal{L}(C_n^+) \), since the graph \( P^t \in \mathcal{L}_n \) by Lemma 38.
We show below how to obtain central triangulations for all polytopes \( \mathcal{P} \in \mathcal{L}(C_n^+) \). A **central triangulation** of a \( d \)-dimensional root polytope \( \mathcal{P} \) is a collection of \( d \)-dimensional simplices with disjoint interiors whose union is \( \mathcal{P} \), the vertices of which are vertices of \( \mathcal{P} \) and the origin is a vertex of all of them. Depending on the context we at times take the intersections of these maximal simplices to be part of the triangulation.

We now state the crucial lemma which relates root polytopes and the algebras \( \mathcal{B}(C_n), \mathcal{B}^o(C_n) \) and \( \mathcal{S}(C_n) \) defined in Section 3.2.

**Lemma 39. (Reduction Lemma)** Given a graph \( G_0 \in \mathcal{L}_n \) with \( d \) edges let \( G_1, G_2, G_3 \) be as described by any one of the equations (3.3.1)-(3.3.6). Then \( G_1, G_2, G_3 \in \mathcal{L}_n \),

\[
\mathcal{P}(G_0) = \mathcal{P}(G_1) \cup \mathcal{P}(G_2)
\]

where all polytopes \( \mathcal{P}(G_0), \mathcal{P}(G_1), \mathcal{P}(G_2) \) are \( d \)-dimensional and

\[
\mathcal{P}(G_3) = \mathcal{P}(G_1) \cap \mathcal{P}(G_2) \text{ is } (d - 1)\text{-dimensional}.
\]

What the Reduction Lemma really says is that performing a reduction on graph \( G_0 \in \mathcal{L}_n \) is the same as “cutting” the \( d \)-dimensional polytope \( \mathcal{P}(G_0) \) into two \( d \)-dimensional polytopes \( \mathcal{P}(G_1) \) and \( \mathcal{P}(G_2) \), whose vertex sets are subsets of the vertex set of \( \mathcal{P}(G_0) \), whose interiors are disjoint, whose union is \( \mathcal{P}(G_0) \), and whose intersection is a facet of both. We prove the Reduction Lemma in Section 3.6.

### 3.5 Characterizing the vertices of coned root polytopes

In this section we prove Proposition 37, which characterizes the vertices of any root polytope \( \mathcal{P}(G) \). We start by proving the statement for connected \( G \in \mathcal{L}_n \).

**Proposition 40.** Let \( G \in \mathcal{L}_n \) be a connected graph. The correspondence between
playable routes of $G$ and vertices in $\overline{V}_G$ given by

$$\phi : P = \{(i_1, j_1, e_1), (i_2, j_2, e_2), \ldots, (i_t, j_t, e_t)\} \mapsto v(i_1, j_1, e_1) + \cdots + v(i_t, j_t, e_t),$$

is a bijection.

Denote by $[e_i]w$ the coefficient of $e_i$ when $w \in \mathbb{R}^n$ is expressed in terms of the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$.

**Proof of Proposition 40.** Given a playable route $P$ of $G$, $\phi(P) \in \overline{V}_G$ by definition. It remains to show that for each vertex $v \in \overline{V}_G$ there exists a playable route $P$ in $G$ such that $v = \phi(P)$. The uniqueness of such a route follows from the linear independence of the set of vectors $\mathcal{V}_G$ for $G \in \mathcal{L}_n$.

Consider $v \in \overline{V}_G$. Then $v = e_i \pm e_j$, for some $1 \leq i < j \leq n$, or $v = 2e_k = e_k + e_k$, for $k \in [n]$, and

$$v = \sum_{e \in E(G)} c_e v(e), \text{ for some real } c_e \geq 0. \quad (3.5.1)$$

Let $H = ([n], \{e \in E(G) \mid c_e \neq 0\})$. Observe that $H$ has at most one connected component containing edges. This follows since a connected $G \in \mathcal{L}_n$ contains at most one simple cycle, and if there were two connected components of $H$, one would be a tree contributing at least two nonzero coordinates to the right hand side of (3.5.1) and each connected component containing edges contributes at least one nonzero coordinate to the right hand side of (3.5.1). But, the left hand side of (3.5.1) has one or two nonzero coordinates.

If $k$ is a leaf of $H$ then $[e_k]v \neq 0$. Therefore, $H$ can have at most two leaves. We consider three cases depending on the number of leaves $H$ has: 0, 1, 2. In all cases we show that there exists a playable route $P$ of $G$ with all its edges among the edges of $H$, such that $\phi(P) = v$, yielding the desired conclusion.

**Case 1.** $H$ has 0 leaves. Since $H \subset G \in \mathcal{L}_n$, it follows that $H$ is a simple cycle. Relabel the vertices of the cycle so that $H$ is now a graph on $[m]$. Then $i = 1$ since 1 only has edges positively incident to it. Regardless of which vertex of $H$ is $j > 1$, there is a playable route $P$ starting at vertex $i$ and ending at $j$ such that $\phi(P) = v$. 

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Case 2. \( H \) has 1 leaf. Then \( H \) is a union of a simple cycle \( C \) and a simple path \( Q \). Relabel the vertices of \( H \) so that it is a graph on the vertex set \([m]\). Let \( l \) be the leftmost vertex of the cycle \( C \) of \( H \) and let \( p \) be the vertex in common to \( C \) and \( Q \). Let \( k \) be the unique leaf.

If \( l \neq p \), then \( \{i, j\} = \{l, k\} \). Thus, at least one of the edges of \( C \) incident to \( p \) are incident with an opposite sign to \( p \) than the edge of \( Q \) incident to \( p \). Therefore, the edges on the path from \( l \) to \( p \) through the edge that is incident to \( p \) in \( C \) with the opposite sign to that of the edge of \( Q \), and then the edges of path \( Q \) form a playable route \( P \) such that \( \phi(P) = v \).

If \( l = p \) then we consider two possibilities, depending on whether \( l \notin \{i, j\} \) or \( l \in \{i, j\} \). If \( l \notin \{i, j\} \) then \( i = k = 1 \) and \( l \neq j \). If \( j \in C \), then the edges of \( Q \) (from 1 to \( l \)) and the edges on the path from \( l \) to \( j \) through the edge that is incident to \( j \) in \( C \) with the sign of \( e_j \) in \( v \) make up a playable route \( P \) with \( \phi(P) = v \). If \( j \in Q \) however, then, either the edges on the path from \( i \) to \( j \) along \( Q \) make up a playable route \( P \) with \( \phi(P) = v \), or the the edges of \( Q \) (from 1 to \( l \)) and the edges of \( C \) and then the edges on the path from \( l \) to \( j \) make up a playable route \( P \) with \( \phi(P) = v \).

If \( l = p \) and \( l \in \{i, j\} \) then either \( i = l \) or \( j = l \). If \( i = l \) then the edges on the path \( Q \) from \( l = 1 \) to \( j = k \) make up a playable route \( P \) with \( \phi(P) = v \). On the other hand if \( j = l \) then \( i = 1 \) and if the edge of \( Q \) is incident to \( l \) with the same sign as that of the sign of \( e_j \) in \( v \), than the edges of \( Q \) make up a playable route \( P \) with \( \phi(P) = v \). If, however, that sign is different, then it must be that \( [e_j]v = 1 \) in which case all edges of \( H \) (suitably ordered) make up a playable route \( P \) with \( \phi(P) = v \).

Case 3. \( H \) has 2 leaves. Then \( H \) could be a path, or a union of a simple cycle \( C \) and two disjoint paths \( Q_1, Q_2 \) attached to \( C \) at vertices \( p_1 \neq p_2 \), or a union of a cycle \( C \) and a tree \( T \) with two leaves attached to \( C \) at \( t \). As in cases 1 and 2, in each case we can identify a playable route by inspection. We omit the details here. \( \square \)

Proposition 40 yields a characterization of the vertices of \( \mathcal{P}(G) \) for a connected \( G \in \mathcal{L}_n \).

Proposition 41. Let \( G \in \mathcal{L}_n \). The map \( \phi \) defined by (3.4.3) is a one-to-one correspondence between playable routes and playable pairs of \( G \) and the vertices in \( \overline{V_G} \).
Proof. The proof is almost identical to that of Proposition 40. The only difference is that the graph $H$ defined in the proof of Proposition 40 could have two connected components containing edges. The case of $H$ with one connected component containing edges is the same as in the proof of Proposition 40.

Let the two connected components of $H$ containing edges be $H_1$ and $H_2$. Then, $H_1$ and $H_2$ each contributes exactly one coordinate with a nonzero coefficient, and thus each of them is a union of a simple cycle (since $G \in \mathcal{L}_n$) and a possibly empty simple path. The edges of $H_1$ and $H_2$, in a suitable order, constitute playable pairs. □

Proposition 42. For any graph $G$ the set of vertices $V(G)$ is the image of playable routes and pairs of $G$ under the map $\phi$ defined by (3.4.3).

Proof. Let $P(G) = \text{ConvHull}(0, v(i, j, \epsilon) \mid v(i, j, \epsilon) \in V(G))$, and let $\Delta$ be a central triangulation of $P(G)$. For each $\sigma \in \Delta$ we define $C(\sigma) = C(G')$, where the vertex set of $\sigma$ is $\{0, v(i, j, \epsilon) \mid (i, j, \epsilon) \in G'\}$, $G' \subset G$ and $G' \in \mathcal{L}_n$. Then,

$$V(G) \subset C(G) = \bigcup_{\sigma \in \Delta} C(\sigma).$$

Thus, any $v \in V(G)$ belongs to some $C(G')$. Therefore, $v \in V(G')$, for $G' \in \mathcal{L}_n$, $G' \subset G$. By Proposition 41, there is a playable route $P$ or pair $(P_1, P_2)$ in $G'$, such that $v = \phi(P)$ or $v = \phi(P_1, P_2)$. But all playable routes and pairs of $G'$ are also playable routes and pairs of $G$. □

Propositions 40, 41 and 42 imply Proposition 37.

3.6 The proof of the Reduction Lemma

This section is devoted to proving the Reduction Lemma (Lemma 39). As we shall see in Section 3.7, the Reduction Lemma is the “secret force” that makes everything fall into its place for coned root polytopes. We start by characterizing the root polytopes which are simplices, then in Lemma 44 we prove that equations (3.4.1) and (3.4.2) are equivalent definitions for the root polytope $P(G)$, and finally we prove the
Lemma 43. For a graph $G$ on the vertex set $[n]$ with $d$ edges, the polytope $P(G)$ as defined by (3.4.1) is a simplex if and only if $G$ is alternating and $G \in \mathcal{L}_n$.

Proof. It follows from equation (3.4.1) that for a minimal graph $G$ the polytope $P(G)$ is a simplex if and only if the vectors corresponding to the edges of $G$ are linearly independent and $C(G) \cap \Phi^+ = \mathcal{V}_G$.

The vectors corresponding to the edges of $G$ are linearly independent if and only if $G \in \mathcal{L}_n$. By Proposition 37, $C(G) \cap \Phi^+ = \mathcal{V}_G$ if and only if $G$ contains no edges incident to a vertex $v \in V(G)$ with opposite signs, i.e. $G$ is alternating. \hfill \square

Lemma 44. For any graph $G$ on the vertex set $[n]$,

$$
\text{ConvHull}(0, v(i, j, \epsilon) \mid (i, j, \epsilon) \in \overline{G}) = P(C^+_n) \cap C(G).
$$

Proof. For a graph $H$ on the vertex set $[n]$, let $\sigma(H) = \text{ConvHull}(0, v(i, j, \epsilon) \mid (i, j, \epsilon) \in H)$. Then, $\sigma(\overline{G}) = \text{ConvHull}(0, v(i, j, \epsilon) \mid (i, j, \epsilon) \in \overline{G})$. Let $\sigma(\overline{G})$ be a $d$-dimensional polytope for some $d \leq n$ and consider any central triangulation of it: $\sigma(\overline{G}) = \bigcup_{F \in \mathcal{F}} \sigma(F)$, where $\{\sigma(F)\}_{F \in \mathcal{F}}$ is a set of $d$-dimensional simplices with disjoint interiors, $E(F) \subset E(\overline{G})$, $F \in \mathcal{F}$. Since $\sigma(\overline{G}) = \bigcup_{F \in \mathcal{F}} \sigma(F)$ is a central triangulation, it follows that $\sigma(F) = \sigma(\overline{G}) \cap C(F)$, for $F \in \mathcal{F}$, and $C(G) = \bigcup_{F \in \mathcal{F}} C(F)$.

Since $\sigma(F)$, $F \in \mathcal{F}$, is a $d$-dimensional simplex, it follows that $F \in \mathcal{L}_n$ and has $d$ edges. Furthermore, $F \in \mathcal{F}$ is alternating, as otherwise there are edges $(i, j, \epsilon_1), (j, k, \epsilon_2) \in E(F) \subset E(\overline{G})$ incident to $j$ with opposite signs, and while $v(i, j, \epsilon_1) + v(j, k, \epsilon_2) \in \sigma(\overline{G}) \cap C(F)$, $v(i, j, \epsilon_1) + v(j, k, \epsilon_2) \not\in \sigma(F)$, contradicting that $\bigcup_{F \in \mathcal{F}} \sigma(F)$ is a central triangulation of $\sigma(\overline{G})$. Thus, $\overline{F} = F$, and $\sigma(F) = \sigma(\overline{F})$.

It is clear that $\sigma(\overline{F}) = \text{ConvHull}(0, v(i, j, \epsilon) \mid (i, j, \epsilon) \in \overline{F}) \subset P(C^+_n) \cap C(F)$, $F \in \mathcal{F}$. Since if $x = (x_1, \ldots, x_{n+1})$ is in the facet of $\sigma(\overline{F})$ opposite the origin, then $|x_1| + \cdots + |x_{n+1}| = 2$ and for any point $x = (x_1, \ldots, x_{n+1}) \in P(C^+_n)$, $|x_1| + \cdots + |x_{n+1}| \leq 2$ it follows that $P(C^+_n) \cap C(F) \subset \sigma(\overline{F})$. Thus, $\sigma(\overline{F}) = P(C^+_n) \cap C(F)$. 

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Finally, ConvHull($0, v(i, j, \epsilon) \mid (i, j, \epsilon) \in \overline{G}$) = $\sigma(\overline{G}) = \bigcup_{F \in \mathcal{F}} \sigma(F) = \bigcup_{F \in \mathcal{F}} \sigma(\overline{F}) = \\
\bigcup_{F \in \mathcal{F}} (P(C_n^+) \cap C(F)) = P(C_n^+) \cap (\bigcup_{F \in \mathcal{F}} C(F)) = P(C_n^+) \cap C(G)$ as desired.

\[ \square \]

**Lemma 45. (Cone Reduction Lemma)** Given a graph $G_0 \in \mathcal{L}_n$ with $d$ edges, let $G_1, G_2, G_3$ be the graphs described by any one of the equations (3.3.1)-(3.3.6). Then $G_1, G_2, G_3 \in \mathcal{L}_n,$

$$C(G_0) = C(G_1) \cup C(G_2)$$

where all cones $C(G_0), C(G_1), C(G_2)$ are $d$-dimensional and

$$C(G_3) = C(G_1) \cap C(G_2)$$

is $(d - 1)$-dimensional.

The proof of Lemma 45 is the same as that of the Cone Reduction Lemma in the type $A_n$ case; see [M1, Lemma 7].

**Proof of the Reduction Lemma (Lemma 39).** Straightforward corollary of Lemmas 44 and 45. \[ \square \]

### 3.7 Volumes of root polytopes and the number of monomials in reduced forms

In this section we use the Reduction Lemma to establish the link between the volumes of root polytopes and the number of monomials in reduced forms. In fact we shall see that if we know either of these quantities, we also know the other.

**Proposition 46.** Let $G_0 \in \mathcal{L}_n$ be a connected graph on the vertex set $[n]$ with $n$ edges, and let $T^S$ be an $S$-reduction tree with root labeled $G_0$. Then,

$$\text{vol}_n(\mathcal{P}(G_0)) = \frac{2f(G_0)}{n!},$$

where $f(G_0)$ denotes the number of leaves of $T^S$ labeled by graphs with $n$ edges.
Proof. By the Reduction Lemma (Lemma 39) \( \vol_n(\mathcal{P}(G_0)) = \sum_G \vol_n(\mathcal{P}(G)) \), where \( G \) runs over the leaves of \( T^S \) labeled by graphs with \( n \) edges. We now prove that for each \( G \) with \( n \) edges labeling a leaf of \( T^S \) with root labeled \( G_0 \), \( \vol_n(\mathcal{P}(G)) = \frac{2}{n!} \). Since \( G_0 \in \mathcal{L}_n \) is a connected graph on the vertex set \([n]\) with \( n \) edges, so are all its successors with \( n \) edges. If \( G \) labels a leaf of \( T^S \), then \( G \) satisfies the conditions of Lemma 43. Thus, \( \mathcal{P}(G) \) is a simplex.

The volume of \( \mathcal{P}(G) \) can be calculated by calculating the determinant \( \det(M) \) of the matrix \( M \) whose rows are the vectors \( v(e), e \in E(G) \), written in the standard basis. If \( v \in [n] \) is a vertex of degree 1 in \( G \), the \( v^{th} \) column contains a single 1 or \(-1\) in the row corresponding to the edge incident to \( v \). Let this row be the \( v_{r}^{th} \). Delete the \( v^{th} \) column and \( v_{r}^{th} \) row from \( M \) and delete the edge incident to \( v \) in \( G \) obtaining a new graph. Successively identify the leaves in the new graphs and delete the corresponding columns and rows from their matrices until we obtain a graph \( C \) that is a simple cycle and the corresponding matrix \( M' \). The rows of \( M' \) are the vectors \( v(e), e \in E(C) \). By Laplace expansion, \( \det(M) = \det(M') \). Since \( G \in \mathcal{L}_n \), so is \( C \in \mathcal{L}_n \). Thus, \( \det(M') \neq 0 \). Expand \( M' \) by any of its rows obtaining matrices \( M_1 \) and \( M_2 \). Then we get \( \det(M') = \det(M_1) + \det(M_2) = 2 \), since both \( M_1 \) and \( M_2 \) are such that their entries are all 0, 1 or \(-1\), each row (column) except one has exactly two nonzero entries, and the remaining one exactly one nonzero entry. Thus, \( \vol_n(\mathcal{P}(G)) = \det(M)/n! = 2/n! \).

\( \square \)

A general version of Proposition 46 can be proved for any connected \( G_0 \in \mathcal{L}_n \) using the following lemma.

**Lemma 47.** Let \( G \in \mathcal{L}_n \) be an alternating graph on the vertex set \([n]\) with \( d \) edges, with \( c \) connected components of which \( k \leq c \) contain simple cycles. Then,

\[
\vol_d(\mathcal{P}(G)) = \frac{2^k}{d!}.
\]

**Proof.** Let \( M^a \) be the matrix whose rows are the vectors \( v(i, j, \epsilon), (i, j, \epsilon) \in E(G) \), written in the standard basis. Matrix \( M^a \) is a \( d \times n \) matrix. The rows and columns of
$M^a$ can be rearranged so that it has a block form in which the blocks $B_1, \ldots, B_c$ on the diagonal correspond to the connected components of $G$, while all other blocks are 0. Since $G \in \mathcal{L}_n$ satisfies the conditions of Lemma 43, $\mathcal{P}(G)$ is a simplex, $\text{vol}_d(\mathcal{P}(G)) \neq 0$ and $\text{vol}_d(\mathcal{P}(G))$ can be calculated by dropping some $n - d$ columns of $M^a$ such that the resulting matrix $M$ has nonzero determinant. Then, $\text{vol}_d(\mathcal{P}(G)) = |\det(M)|/d!$. Drop a column $b_i$ from the block matrix $B_i$ if the block $B_i$ corresponds to a tree on $m$ vertices, obtaining matrix $B'_i$ with nonzero determinant. Then, $|\det(B'_i)| = 1$. If $B_i$ corresponds to a connected component of $G_0$ with $m$ vertices and $m$ edges, then $B'_i = B_i$ and $|\det(B_i)| = 2$. Since there are $n - d$ connected components which are trees, if we drop the columns $b_i$ from $M^a$ for all blocks $B_i$ corresponding to a tree obtaining a matrix $M$, then $\text{vol}_d(\mathcal{P}(G)) = \frac{|\det(M)|}{d!}$. Since $M$ has a special block form with blocks $B'_i$ along diagonal and zeros otherwise, we have that $|\det(M)| = |\prod_{i=1}^c \det(B'_i)| = 2^k$.

**Proposition 48.** Let $G_0 \in \mathcal{L}_n$ be a graph on the vertex set $[n]$ with $d$ edges, with $c$ connected components of which $k \leq c$ contain cycles. Let $T^S$ be an $S$-reduction tree with root labeled $G_0$. Then,

$$\text{vol}_d(\mathcal{P}(G_0)) = \frac{2^k f(G_0)}{d!},$$

where $f(G_0)$ denotes the number of leaves of $T^S$ labeled by graphs with $d$ edges.

The proof of Proposition 48 proceeds analogously to Proposition 46, in view of Lemma 47.

**Corollary 49.** Let $G_0 \in \mathcal{L}_n$ and let $m^S[G_0]$ be the monomial corresponding to it. Then for any reduced form $P_n^S$ of $m^S[G_0]$, the value of $P_n^S(x_{ij} = y_{ij} = z_i = 1, \beta = 0)$ is independent of the order of reductions performed.

**Proof.** Note that $P_n^S(x_{ij} = y_{ij} = 1, \beta = 0) = f(G_0)$, as defined in Proposition 48. Since $\text{vol}_d(\mathcal{P}(G_0))$ is only dependent on $G_0$, the value of $P_n^S(x_{ij} = y_{ij} = z_i = 1, \beta = 0)$ is independent of the particular reductions performed. \qed
With analogous methods the following proposition about reduced forms in $\mathcal{B}^c(C_n)$ can also be proved.

**Proposition 50.** Let $G_0 \in \mathcal{L}_n$ and let $m^S[G_0] = m^{\mathcal{B}c}[G_0]$ be the monomial corresponding to it. Then for any reduced form $P_n^{\mathcal{B}c}$ of $m^S[G_0]$ in $\mathcal{B}^c(C_n)$, the value of $P_n^{\mathcal{B}c}(x_{ij} = y_{ij} = z_i = 1)$ is independent of the order of reductions performed.

### 3.8 Reductions in the noncommutative case

In this section we turn our attention to the noncommutative algebra $\mathcal{B}(C_n)$. We consider reduced forms of monomials in $\mathcal{B}(C_n)$ and the reduction rules correspond to the relations (5) – (9') of $\mathcal{B}(C_n)$:

1. $x_{ij}x_{jk} \rightarrow x_{ik}x_{ij} + x_{jk}x_{ik}$, for $1 \leq i < j < k \leq n$,
2. $x_{jk}x_{ij} \rightarrow x_{ij}x_{ik} + x_{ik}x_{jk}$, for $1 \leq i < j < k \leq n$,
3. $x_{ij}y_{jk} \rightarrow y_{ik}x_{ij} + y_{jk}y_{ik}$, for $1 \leq i < j < k \leq n$,
4. $y_{jk}x_{ij} \rightarrow x_{ij}y_{ik} + y_{ik}y_{jk}$, for $1 \leq i < j < k \leq n$,
5. $x_{ik}y_{jk} \rightarrow y_{jk}y_{ik} + y_{ij}x_{ik}$, for $1 \leq i < j < k \leq n$,
6. $y_{jk}x_{ik} \rightarrow x_{ik}y_{jk} + x_{ik}y_{ij}$, for $1 \leq i < j < k \leq n$,
7. $y_{ik}x_{jk} \rightarrow x_{jk}y_{ik} + y_{ij}y_{ik}$, for $1 \leq i < j < k \leq n$,
8. $x_{jk}y_{ik} \rightarrow y_{ij}x_{jk} + y_{ik}y_{ij}$, for $1 \leq i < j < k \leq n$,
9. $x_{ij}z_j \rightarrow z_i x_{ij} + y_{ij}z_i + z_j y_{ij}$, for $i < j$,
10. $z_j x_{ij} \rightarrow x_{ij}z_i + z_i y_{ij} + y_{ij}z_j$, for $i < j$.

As observed in Proposition 50, in the commutative counterpart of $\mathcal{B}(C_n)$, $\mathcal{B}^c(C_n)$, the number of monomials in a reduced form of $w_{C_n}$ is the same, regardless of the order of the reductions performed. In this section we develop the tools necessary for proving the uniqueness of the reduced form in $\mathcal{B}(C_n)$ for $w_{C_n}$ and other monomials. The key concept is that of a “good” graph, which property is preserved under the reductions.
As in the commutative case before, we can phrase the reduction process in terms of graphs. Let \( m = \prod_{i=1}^{p} w(i, j, \epsilon_i) \) be a monomial in variables \( x_{ij}, y_{ij}, z_k, 1 \leq i < j \leq n, k \in [n] \), where \( w(i, j, -) = x_{ij} \) for \( i < j \), \( w(i, j, +) = x_{ji} \) for \( i > j \), \( w(i, j, +) = y_{ij} \) and \( w(i, i, +) = z_i \). We can think of \( m \) as a graph \( G \) on the vertex set \([n]\) with \( p \) edges labeled \( 1, \ldots, p \), such that the edge labeled \( l \) is \((i_l, j_l, \epsilon_l)\). Let \( G^S[m] \) denote the edge-labeled graph just described. Let \((i, j, \epsilon)_a\) denote an edge \((i, j, \epsilon)\) labeled \( a \). Recall that in our edge notation \((i, j, \epsilon) = (j, i, \epsilon)\), i.e., vertex-label \( i \) might be smaller or greater than \( j \). We can reverse the process and obtain a monomial from an edge labeled graph \( G \). Namely, if \( G \) is edge-labeled with labels \( 1, \ldots, p \), we can also associate to it the noncommutative monomial \( m^E[G] = \prod_{a=1}^{p} w(i_a, j_a, \epsilon_a) \), where \( E(G) = \{(i_a, j_a, \epsilon_a)_a | a \in [p]\} \).

In terms of graphs the partial commutativity of \( B(C_n) \), as described by relations (2)-(4), means that if \( G \) contains two edges \((i, j, \epsilon_1)_a\) and \((k, l, \epsilon_2)_{a+1}\) with \( i, j, k, l \) distinct, then we can replace these edges by \((i, j, \epsilon_1)_{a+1}\) and \((k, l, \epsilon_2)_a\), and vice versa. For illustrative purposes we write out the graph reduction for relation (5) of \( B(C_n) \). If there are two edges \((i, j, -)_a\) and \((j, k, -)_{a+1}\) in \( G_0 \), \( i < j < k \), then we replace \( G_0 \) with two graphs \( G_1, G_2 \) on the vertex set \([n]\) and edge sets

\[
E(G_1) = E(G_0) \setminus \{(i, j, -)_a\} \cup \{(i, k, -)_{a+1}\} \cup \{(i, j, -)_{a+1}\}
\]

\[
E(G_2) = E(G_0) \setminus \{(i, j, -)_a\} \cup \{(j, k, -)_{a+1}\} \cup \{(i, k, -)_{a+1}\}
\]

Relations \((5') - (9')\) of \( B(C_n) \) can be translated into graph language analogously. We say that \( G_0 \) reduces to \( G_1 \) and \( G_2 \) under reductions \((5) - (9')\).

While in the commutative case reductions on \( G^S[m] \) could result in crossing graphs, we prove that in \( B(C_n) \) all reductions preserve the noncrossing nature of graphs, provided that we started with a suitable noncrossing graph \( G \). A graph \( G \) is noncrossing if there are no vertices \( i < j < k < l \) such that \((i, k, \epsilon_1)\) and \((j, l, \epsilon_2)\) are edges of \( G \). We also show that under reasonable circumstances, if in \( B^n(C_n) \) a reduction could be applied to edges \( \epsilon_1 \) and \( \epsilon_2 \), then after suitably many allowed

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commutations in $B(C_n)$ it is possible to perform a reduction on $e_1$ and $e_2$ in $B(C_n)$.

We now define two central notions of the noncommutative case, that of a well-structured graph and that of a well-labeled graph.

A graph $H$ on the vertex set $[n]$ is **well-structured** if it satisfies the following conditions:

(i) $H$ is noncrossing.

(ii) For any two edges $(i, j, +), (k, l, +) \in H$, $i < j, k < l$, it must be that $i < l$ and $k < j$.

(iii) For any two edges $(i, i, +), (k, l, +) \in H$, $k < l$, it must be that $k \leq i \leq l$.

(iv) There are no edges $(i, j, +), (k, j, -) \in H$ with $k < i < j$.

(v) There are no edges $(i, j, +), (k, l, -) \in H$ with $k < i < j < l$.

(vi) Graph $H$ is connected, contains exactly one loop, and contains no nonloop cycles.

Condition (vi) implies that any well-structured graph on the vertex set $[n]$ contains $n$ edges.

A graph $H$ on the vertex set $[n]$ and $p$ edges labeled $1, \ldots, p$ is **well-labeled** if it satisfies the following conditions:

(i) If edges $(i, j, \epsilon_1)_a$ and $(j, k, \epsilon_2)_b$ are in $H$, $i < j < k$, $\epsilon_1, \epsilon_2 \in \{-, +\}$, then $a < b$.

(ii) If edges $(i, j, \epsilon_1)_a$ and $(i, k, \epsilon_2)_b$ in $H$ are such that $i < j < k$, $\epsilon_1, \epsilon_2 \in \{-, +\}$, then $a > b$.

(iii) If edges $(i, j, \epsilon_1)_a$ and $(k, j, \epsilon_2)_b$ in $H$ are such that $i < k < j$, $\epsilon_1, \epsilon_2 \in \{-, +\}$, then $a > b$.

(iv) If edges $(i, i, +)_a$ and $(i, j, -)_b$ in $H$ are such that $i < j$, then $a < b$.

(v) If edges $(j, j, +)_a$ and $(i, j, -)_b$ in $H$ are such that $i < j$, then $a > b$.

(vi) If edges $(i, i, +)_a$ and $(i, j, +)_b$ in $H$ are such that $i < j$, then $a > b$.

(vii) If edges $(j, j, +)_a$ and $(i, j, +)_b$ in $H$ are such that $i < j$, then $a < b$.

Note that no graph $H$ with a nonloop cycle can be well-labeled. However, every well-structured graph can be well-labeled. We call graphs that are both well-structured and well-labeled **good** graphs.
A B-reduction tree $T^B$ is defined analogously to an S-reduction tree, except we use the noncommutative reductions to describe the children. See Figure 3.8.1 for an example. A graph $H$ is called a B-successor of $G$ if it is obtained by a series of reductions from $G$.

Figure 3.8.1: A B-reduction tree with root corresponding to the monomial $x_{13}x_{12}z_3$. Note that in order to perform a reduction on this monomial we commute variables $x_{13}$ and $x_{12}$. In the B-reduction tree we only record the reductions, not the commutations. Summing the monomials corresponding to the graphs labeling the leaves of the reduction tree we obtain a reduced form $P^B_n$ of $x_{13}x_{12}z_3$, $P^B_n = z_1 x_{13}x_{12} + y_{13} z_1 x_{12} + z_3 y_{13} x_{12}$.

**Lemma 51.** If the root of a B-reduction tree is labeled by a good graph, then all nodes of it are also labeled by good graphs.

The proof of Lemma 51 is an analysis of the local changes that happen during the noncommutative reduction process. An analogous lemma for type $A_n$ is proved in [M1, Lemma 12].

A reduction applied to a noncrossing graph $G$ is noncrossing if the graphs resulting from the reduction are also noncrossing.

The following is then an immediate corollary of Lemma 51.

**Corollary 52.** If $G$ is a good graph, then all reductions that can be applied to $G$ and its B-successors are noncrossing.
Let \( e_1 = (i_1, j_1, \epsilon_1)_{a_1}, e_2 = (i_2, j_2, \epsilon_2)_{a_2}, e_3 = (i_3, j_3, \epsilon_3)_{a_3} \) be edges of the graph \( H \) such that in the commutative algebra \( \mathcal{B}^c(C_n) \) a reduction could be performed on \( e_1 \) and \( e_2 \) as well as on \( e_1 \) and \( e_3 \). Suppose that \( a_1 < a_2 < a_3 \). Then we say, in the noncommutative case \( \mathcal{B}(C_n) \), that performing reduction on edges \( e_1 \) and \( e_2 \) is a priority over performing reduction on edges \( e_1 \) and \( e_3 \). We give a few concrete examples of this priority below.

**Example.** Performing reduction (6) on edges \( (i, j, -), (j, k, +) \in H, i < j < k \), is a priority over performing reduction (9) on edges \( (i, j, -), (j, j, +) \in H \). Performing reduction (9) on edges \( (i, j, -), (j, j, +) \in H \), is a priority over performing reduction (5) on edges \( (i, j, -), (j, k, -) \in H, i < j < k \). Performing reduction (9) on edges \( (i, j, -), (j, j, +) \in H \), is a priority over performing reduction (9) on edges \( (k, j, -), (j, j, +) \in H, i < k < j \). Performing reduction (9) on edges \( (i, j, -), (k, j, +) \in H \), is a priority over performing reduction (8) on edges \( (i, j, -), (k, j, +) \in H, k < i < j \).

**Lemma 53.** Let \( G \) be a good graph. Let \( e_1 \) and \( e_2 \) be edges of \( G \) such that one of the reductions (5) – (9') could be applied to them in the commutative case, and such that the reduction would be noncrossing. Then after finitely many applications of allowed commutations in \( \mathcal{B}(C_n) \) we can perform a reduction on edges \( e_1 \) and \( e_2 \), provided there is no edge \( e_3 \) in the graph such that reducing \( e_1 \) and \( e_3 \) or \( e_2 \) and \( e_3 \) is a priority over reducing \( e_1 \) and \( e_2 \).

The proof of Lemma 53 proceeds by inspection. An analogous lemma for type \( A_n \) is proved in [M1, Lemma 14].

### 3.9 The Proof of Kirillov’s Conjecture

In this section we prove Conjecture 1, construct a triangulation of \( \mathcal{P}(C_n^+) \) and compute its volume. In order to do this we study alternating well-structured graphs. Recall that an alternating well-structured graph \( T^d \) is the union of a noncrossing alternating tree \( T \) on the vertex set \([n]\) and a loop, that is, \( T^d = ([n], E(T) \cup \{(k, k, +)\}) \), for
some \( k \in [n] \) for which \( T' \) is alternating. A well-labeling that will play a special role in this section is the lexicographic labeling, defined below.

The **lexicographic order** on the edges of a graph \( G \) with \( m \) edges is as follows. Edge \( (i_1, j_1, \epsilon) \) is less than edge \( (i_2, j_2, \epsilon) \), \( \epsilon \in \{+, -\} \), in the lexicographic order if \( j_1 > j_2 \), or \( j_1 = j_2 \) and \( i_1 > i_2 \). Furthermore, any positive edges is less than any negative edges in the lexicographic ordering. Graph \( G \) is said to have **lexicographic edge-labels** if its edges are labeled by integers \( 1, \ldots, m \) such that if edge \( (i_1, j_1, \epsilon_1) \) is less than edge \( (i_2, j_2, \epsilon_2) \) in lexicographic order, then the label of \( (i_1, j_1, \epsilon_1) \) is less than the label of \( (i_2, j_2, \epsilon_2) \) in the usual order on the integers. Given any graph \( G \) there is a unique edge-labeling of it which is lexicographic. Note that our definition of lexicographic is closely related to the conventional definition, but it is not the same. For an example of lexicographic edge-labels, see the graphs labeling the leaves of the \( B \)-reduction tree in Figure 3.8.1.

**Lemma 54.** If \( T' \) is an alternating good graph, then upon some number of commutations performed on \( T' \), it is possible to obtain \( T_1 \) with lexicographic edge-labels.

**Proof.** If edges \( e_1 \) and \( e_2 \) of \( T' \) share a vertex and if \( e_1 \) is less than \( e_2 \) in the lexicographic order, then the label of \( e_1 \) is less than the label of \( e_2 \) in the usual order on integers by the definition of well-labeling on alternating well-structured graphs. Since commutation swaps the labels of two vertex disjoint edges labeled by consecutive integers in a graph, these swaps do not affect the relative order of the labels on edges sharing vertices. Continue these swaps until the lexicographic order is obtained. \( \square \)

**Proposition 55.** By choosing the series of reductions suitably, the set of leaves of a \( B \)-reduction tree with root labeled by \( G^B[w_{Cn}] \) can be all alternating well-structured graphs \( T' \) on the vertex set \([n]\) with lexicographic edge-labels. The number of such graphs is \( \binom{2n-1}{n} \).

**Proof.** By the correspondence between the leaves of a \( B \)-reduction tree and simplices in a subdivision of \( \mathcal{P}(G^B[w_{Cn}]) \) obtained from the Reduction Lemma (Lemma 39), it follows that no graph with edge labels disregarded appears more than once among
the leaves of a \( B \)-reduction tree. Thus, it suffices to prove that any alternating well-structured graph \( T' \) on the vertex set \([n]\) appears among the leaves of a \( B \)-reduction tree and that all these graphs have lexicographic edge-labels.

First perform all possible reductions on the graph and its successors not involving the loop \((n, n, +)\). According to [M1, Theorem 18] the outcome is all noncrossing alternating spanning trees with lexicographic ordering on the vertex set \([n]\) and edge \((1, n, -)\) present. Let \( T_1, \ldots, T_w \) be the trees just described and \( T_{i} = ([n], E(T_i) \cup \{(n, n, +)\}), i \in [w] \). It is clear from the definition of reductions that the only edges involved in further reducing \( T_i, i \in [w] \) are the ones incident to vertex \( n \). Thus, in order to understand what the leaves of a reduction tree with root labeled \( T_i, i \in [w] \), are, it suffices to understand the leaves of a reduction tree with root labeled \( G = ([k + 1], \{(k + 1, k + 1, +), (i, k + 1, -) | i \in [k]\}), k \in \{1, 2, \ldots, n - 1\} \). It follows by inspection that the leaves of a reduction tree with root labeled \( G \) are of the form \(([k + 1], E(G_1) \cup E(G_2))\), where \( G_1 \) is a connected well-structured graph with only positive edges (having exactly one loop) on \([l]\), \( l \in [k + 1] \), of which there are \( 2^{l-1} \) and \( G_2 = ([k + 1], \{(i, k + 1) | i \in \{l, l + 1, \ldots, k\}\}) \). It follows that all alternating well-structured graphs \( T_i \) are among the leaves of the particular \( B \)-reduction tree described. Since all these graphs are well-labeled, having started with a good graph, by Lemma 54 we can assume they have lexicographic edge-labels.

From the description of the reductions above it is clear that the number of leaves of this particular reduction tree is

\[
\sum_{k=1}^{n-1} T(n, k) \cdot (2^{k+1} - 1),
\]

where

\[
T(n, k) = \binom{2n - k - 3}{n - k - 1} \frac{k}{n - 1}
\]

is the number of noncrossing alternating trees on the vertex set \([n]\) with exactly \( k \) edges incident to \( n \), and \( 2^{k+1} - 1 \) is the number of leaves of the reduction tree with root labeled \( G([k + 1], \{(k + 1, k + 1, +), (i, k + 1, -) | i \in [k]\}) \) as above. The formula
for $T(n, k)$ follows by a simple bijection between noncrossing alternating trees on the vertex set $[n]$ with exactly $k$ edges incident to $n$ and ordered trees on the vertex set $[n]$ with the root having degree $k$. By equations (6.21), (6.22), (6.28) and the bijection presented in Appendix E.1. in [D], ordered trees on the vertex set $[n]$ with the root having degree $k$ are enumerated by $T(n, k)$. Since $\sum_{k=1}^{n-1} T(n, k) \cdot (2^{k+1} - 1) = \binom{2n-1}{n}$, the proof is complete.

\[ \square \]

**Theorem 56.** The set of leaves of a $B$-reduction tree with root labeled by $G^B[w_{C_n}]$ is, up to commutations, the set of all alternating well-structured graphs on the vertex set $[n]$ with lexicographic edge-labels.

**Proof.** By Proposition 55 there exists a $B$-reduction tree which satisfies the conditions above. By Proposition 48 the number of graphs with $n$ of edges among the leaves of an $S$-reduction tree is independent of the particular $S$-reduction tree, and, thus, the same is true for a $B$-reduction tree. Since all graphs labeling the leaves of a $B$-reduction tree with root labeled by $G^B[w_{C_n}]$ have to be good by Lemma 51, and no graph, with edge-labels disregarded, can appear twice among the leaves of a $B$-reduction tree, imply, together with Lemma 54, the statement of Theorem 56. \[ \square \]

As corollaries of Theorem 56 we obtain the characterization of reduced forms of the noncommutative monomial $w_{C_n}$, a triangulation of $\mathcal{P}(C_n^+)$ and a way to compute its volume.

**Theorem 57.** If the polynomial $P^B_n(x_{ij}, y_{ij}, z_i)$ is a reduced form of $w_{C_n}$, then up to commutations

$$P^B_n(x_{ij}, y_{ij}, z_i) = \sum_{T^l} m^B[T^l],$$

where the sum runs over all alternating well-structured graphs $T^l$ on the vertex set $[n]$ with lexicographic edge-labels.

**Theorem 58.** If the polynomial $P^{B^e}_n(x_{ij}, y_{ij}, z_i)$ is a reduced form of $w_{C_n}$ in $B^e(C_n)$, then
Proof. Proposition 50 and Theorem 57 imply $P_n(x_{ij} = y_{ij} = z_i = 1) = \binom{2n-1}{n}$. □

Theorem 59. Let $T^l_1, \ldots, T^l_m$ be all alternating well-structured graphs on the vertex set $[n]$. Then $\mathcal{P}(T^l_1), \ldots, \mathcal{P}(T^l_m)$ are $n$-dimensional simplices forming a triangulation of $\mathcal{P}(C^+_n)$. Furthermore,

$$\text{vol}_n(\mathcal{P}(C^+_n)) = \binom{2n-1}{n} \frac{2}{n!}.$$ 

Proof. The Reduction Lemma implies the first claim, and Proposition 46 implies $\text{vol}_n(\mathcal{P}(C^+_n)) = \binom{2n-1}{n} \frac{2}{n!}$. □

The value of the volume of $\mathcal{P}(C^+_n)$ has previously been observed by Fong [F, p. 55].

3.10 The general case

In this section we find analogues of Theorems 56, 57, 58 and 59 for any well-structured graph $T^l$ on the vertex set $[n]$.

Proposition 60. Let $T^l$ be a well-structured graph on the vertex set $[n]$. By choosing the series of reductions suitably, the set of leaves of a $B$-reduction tree with root labeled by $T^l$ can be all alternating well-structured spanning graphs $G$ of $\overline{T^l}$ on the vertex set $[n]$ with lexicographic edge-labels.

Proof. All graphs labeling the leaves of a $B$-reduction tree must be alternating well-structured spanning graphs $G$ of $\overline{T^l}$. Also, it is possible to obtain any well-structured graph $T^l$ on the vertex set $[n]$ as a $B$-successor of $P^l$. Furthermore, if $T^l$ and $T^l_1$ are two $B$-successor of $P^l$ in the same $B$-reduction tree, and neither is the $B$-successor of the other, then the intersection of $\overline{T^l}$ and $\overline{T^l_1}$ does not contain a well-structured graph $G$, as the existence of such a graph would imply that $\mathcal{P}(T^l)$ and $\mathcal{P}(T^l_1)$ have a common interior point, contrary to the Reduction Lemma. Since the set of leaves
of a $B$-reduction tree with root labeled by $P^l$ is, up to commutations, the set of all alternating well-structured graphs on the vertex set $[n]$ with lexicographic edge-labels according to Theorem 56, Proposition 60 follows.

**Theorem 61.** Let $T^l$ be a well-structured graph on the vertex set $[n]$. The set of leaves of a $B$-reduction tree with root labeled $T^l$ is, up commutations, the set of all alternating well-structured spanning graphs $G$ of $T^l$ on the vertex set $[n]$ with lexicographic edge-labels.

**Proof.** The proof is analogous to that of Theorem 56 using Proposition 60 instead of Proposition 55.

As corollaries of Theorem 61 we obtain the characterization of reduced forms of the noncommutative monomial $m^B[T^l]$, a triangulation of $\mathcal{P}(T^l)$ and a way to compute its volume, for a well-structured graph $T^l$ on the vertex set $[n]$.

**Theorem 62.** (Noncommutative part.) If the polynomial $P^B_n(x_{ij}, y_{ij}, z_i)$ is a reduced form of $m^B[T^l]$ for a well-structured graph $T^l$ on the vertex set $[n]$, then up to commutations

$$P^B_n(x_{ij}, y_{ij}, z_i) = \sum_G m^B[G],$$

where the sum runs over all alternating well-structured spanning graphs $G$ of $T^l$ on the vertex set $[n]$ with lexicographic edge-labels.

**Theorem 63.** (Commutative part.) If the polynomial $P^B_n(x_{ij}, y_{ij}, z_i)$ is a reduced form of $m^B[T^l]$ for a well-structured graph $T^l$ on the vertex set $[n]$, then

$$P^B_n(x_{ij} = y_{ij} = z_i = 1) = f_{T^l},$$

where $f_{T^l}$ is the number of alternating well-structured spanning graphs $G$ of $T^l$. 

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Theorem 64. (Triangulation and volume.) Let $T_1, \ldots, T_m$ be all alternating well-structured spanning graphs of $\overline{T}$ for a well-structured graph $T$ on the vertex set $[n]$. Then $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_m)$ are $n$-dimensional simplices forming a triangulation of $\mathcal{P}(T)$. Furthermore, 
\[ \text{vol}_n(\mathcal{P}(T)) = \frac{2}{n!} f_{\overline{T}}, \]
where $f_{\overline{T}}$ is the number of alternating well-structured spanning graphs $G$ of $\overline{T}$.

3.11 A more general noncommutative algebra $B^\beta(C_n)$

In this section we define the noncommutative algebra $B^\beta(C_n)$, which specializes to $B(C_n)$ when we set $\beta = 0$. We prove analogs of the results presented so far for this more general algebra. We also provide a way for calculating Ehrhart polynomials for certain type $C_n$ root polytopes.

Let the $\beta$-bracket algebra $B^\beta(C_n)$ of type $C_n$ be an associative algebra over $\mathbb{Q}[\beta]$, where $\beta$ is a variable (and a central element), with a set of generators \{x_{ij}, y_{ij}, z_i \mid 1 \leq i \neq j \leq n\} subject to the following relations:

(1) $x_{ij} + x_{ji} = 0$, $y_{ij} = y_{ji}$, for $i \neq j$,
(2) $z_i z_j = z_j z_i$,
(3) $x_{ij} x_{kl} = x_{ki} x_{ij} + y_{ij} x_{kl} = x_{kl} y_{ij}$, $y_{ij} y_{kl} = y_{kl} y_{ij}$, for $i < j$, $k < l$ distinct.
(4) $z_i x_{kl} = x_{kl} z_i$, $z_i y_{kl} = y_{kl} z_i$, for all $i \neq k, l$,
(5) $x_{ij} x_{jk} = x_{ik} x_{ij} + x_{jk} x_{ik} + \beta x_{ik}$, for $1 \leq i < j < k \leq n$,
(5') $x_{jk} x_{ij} = x_{ij} x_{ik} + x_{ik} x_{jk} + \beta x_{ik}$, for $1 \leq i < j < k \leq n$,
(6) $x_{ij} y_{jk} = y_{ik} x_{ij} + y_{jk} y_{ik} + \beta y_{ik}$, for $1 \leq i < j < k \leq n$,
(6') $y_{jk} x_{ij} = x_{ij} y_{ik} + y_{jk} y_{ik} + \beta y_{ik}$, for $1 \leq i < j < k \leq n$,
(7) $x_{ik} y_{jk} = y_{jk} x_{ij} + y_{ij} x_{ik} + \beta y_{ij}$, for $1 \leq i < j < k \leq n$,
(7') $y_{jk} x_{ik} = y_{ij} y_{jk} + x_{ik} y_{ij} + \beta y_{ij}$, for $1 \leq i < j < k \leq n$,
(8) $y_{ik} x_{jk} = x_{jk} y_{ij} + y_{ij} y_{ik} + \beta y_{ij}$, for $1 \leq i < j < k \leq n$,
(8') $x_{jk} y_{ik} = y_{ij} x_{jk} + y_{ij} y_{ik} + \beta y_{ij}$, for $1 \leq i < j < k \leq n$,
(9) $x_{ij} z_j = z_i x_{ij} + y_{ij} z_i + z_j y_{ij} + \beta z_i + \beta y_{ij}$, for $1 \leq i < j \leq n$,
(9') $z_j x_{ij} = x_{ij} z_i + z_i y_{ij} + y_{ij} z_j + \beta z_i + \beta y_{ij}$, for $1 \leq i < j \leq n$. 

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Kirillov [K2] made Conjecture 1 not just for $B(C_n)$, but for a more general $\beta$-bracket algebra of type $C_n$, which is almost identical to $B^\beta(C_n)$; it differs in a term in relations (9) and (9'). We prove the analogue of Conjecture 1 for $B^\beta(C_n)$.

Notice that the commutativization of $B^\beta(C_n)$ yields the relations of $S(C_n)$, except for relations (9) and (9') of $B^\beta(C_n)$, which can be obtained by combining relations (6) and (7) of $S(C_n)$. Since the Reduction Lemma (Lemma 39) hold for $S(C_n)$, so does it for $B^\beta(C_n)$, keeping in mind that relations (9) and (9') of $B^\beta(C_n)$ are obtained by combining relations (6) and (7) of $S(C_n)$. As a result, we can think of relations (5) – (9') of $B^\beta(C_n)$ as operations subdividing root polytopes into smaller polytopes and keeping track of their lower dimensional intersections.

A $B^\beta$-reduction tree is analogous to an $S$-reduction tree, just that the children of the nodes are obtained by the relations (5) – (9') of $B^\beta(C_n)$, and now some nodes have five, and some nodes have three children. See Figure 3.11.1 for an example. If $T^{B^\beta}$ is a $B^\beta$-reduction tree with root labeled $G$ and leaves labeled by graphs $G_1, \ldots, G_q$, then

$$\mathcal{P}^\circ(G) = \mathcal{P}^\circ(G_1) \cup \cdots \cup \mathcal{P}^\circ(G_q),$$

by an analogue of the Reduction Lemma.

In order to prove an analogue of Proposition 55 for the algebra $B^\beta(C_n)$, we need a definition more general than well-structured. Thus we now define weakly-well-structured graphs.

A graph $H$ on the vertex set $[n]$ and $p \leq n$ edges is weakly-well-structured if it satisfies the following conditions:

(i) $H$ is noncrossing.

(ii) For any two edges $(i, j, +), (k, l, +) \in H$, $i < j, k < l$, it must be that $i < l$ and $k < j$.

(iii) For any two edges $(i, i, +), (k, l, +) \in H$, $k < l$, it must be that $k \leq i \leq l$.

(iv) There are no edges $(i, i, +), (k, j, -) \in H$ with $k < i < j$.

(v) There are no edges $(i, j, +), (k, l, -) \in H$ with $k \leq i < j \leq l$.

(vi) Graph $H$ contains at most one loop, and $H$ contains no nonloop cycles.
Figure 3.11.1: A $B^\beta$-reduction tree with root corresponding to the monomial $x_{23}z_3y_{13}$. Summing the monomials corresponding to the graphs labeling the leaves of the reduction tree multiplied by suitable powers of $\beta$, we obtain a reduced form $P_{\beta}^{B_{\alpha}}$ of $x_{23}z_3y_{13}$, $P_{\beta}^{B_{\alpha}} = z_2y_{12}x_{23} + z_2y_{13}y_{12} + \beta z_2y_{12} + y_{23}z_2y_{13} + z_3y_{23}y_{13} + \beta z_2y_{13} + \beta y_{23}y_{13}$. 
(vii) Graph $H$ contains a positive edge incident to vertex 1.

Note that well-structured graphs are also weakly-well-structured.

**Proposition 65.** By choosing the set of reductions suitably, the set of leaves of a $B^3$-reduction tree $T^{B^3}$ with root labeled by $P^t = ([n], \{(n, n, +), (i, i + 1, -) \mid i \in [n - 1]\})$ can be the set of all alternating weakly-well-structured subgraphs $G$ of $\overline{P^t}$ with lexicographic edge-labels.

**Proof.** The proof of Proposition 65 proceeds analogously as that of Proposition 55, using equation (3.11.1), instead of the original statement of the Reduction Lemma, and using the full statement of [M1, Theorem 18] which says that the leaves of a reduction tree with root labeled by $([n], \{(i, i + 1, -) \mid i \in [n - 1]\})$ are all noncrossing alternating forests with negative edges on the vertex set $[n]$ containing edge $(1, n, -)$ with lexicographic edge-labels. □

**Theorem 66.** The set of leaves of a $B^3$-reduction tree $T^{B^3}$ with root labeled $P^t$ is, up commutations, the set of all alternating weakly-well-structured subgraphs $G$ of $\overline{P^t}$ with lexicographic edge-labels.

**Proof.** Proposition 65 proves the existence of one such $B^3$-reduction tree. An analogue of Lemma 51 states that if the root of a $B^3$-reduction tree is a weakly-well-structured well-labeled graph, then so are all its nodes. Together with equation (3.11.1) these imply Theorem 66. □

As corollaries of Theorem 66 we obtain the characterization of reduced forms of the noncommutative monomial $w_{C_n}$ in $B^3(C_n)$ as well as a canonical triangulation of $\mathcal{P}(P^t)$ and an expression for its Ehrhart polynomial.

**Theorem 67.** If the polynomial $P_n^{B^3}(x_{ij}, y_{ij}, z_i)$ is a reduced form of $w_{C_n}$ in $B^3(C_n)$, then

\[
P_n^{B^3}(x_{ij}, y_{ij}, z_i) = \sum_G \beta^{n - |E(G)|}m^{B^3}[G],
\]

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where the sum runs over all alternating weakly-well-structured graphs \( G \) on the vertex set \([n]\) with lexicographic edge-labels.

**Theorem 68. (Canonical triangulation.)** Let \( G_1, \ldots, G_k \) be all the alternating well-structured graphs on the vertex set \([n]\). Then the root polytopes \( \mathcal{P}(G_1), \ldots, \mathcal{P}(G_k) \) are \( n \)-dimensional simplices forming a triangulation of \( \mathcal{P}(P^i) \). Furthermore, the intersections of the top dimensional simplices \( \mathcal{P}(G_1), \ldots, \mathcal{P}(G_k) \) are simplices \( \mathcal{P}(H) \), where \( H \) runs over all alternating weakly-well-structured graphs on the vertex set \([n]\).

Given a polytope \( \mathcal{P} \subset \mathbb{R}^n \), the \( t^{th} \) dilate of \( \mathcal{P} \) is

\[
t \mathcal{P} = \{(tx_1, \ldots, tx_n)|(x_1, \ldots, x_n) \in \mathcal{P}\}.
\]

The **Ehrhart polynomial** of an integer polytope \( \mathcal{P} \subset \mathbb{R}^n \) is

\[
L_{\mathcal{P}}(t) = \#(t\mathcal{P} \cap \mathbb{Z}^n).
\]

For background on the theory of Ehrhart polynomials see [BR].

**Theorem 69. (Ehrhart polynomial.)**

\[
L_{\mathcal{P}(P^i)}(t) = (-1)^n \left( \sum_{d=1}^{n} f^1(d)(-1)^d \left( \binom{d+t}{d} + \binom{d+t-1}{d} \right) + \sum_{d=1}^{n-1} f(d)(-1)^d \binom{d+t}{d} \right),
\]

where \( f^1(d) \) is the number of alternating weakly-well-structured graphs on the vertex set \([n]\) with \( d \) edges one of which is a loop and \( f(d) \) is the number of alternating weakly-well-structured graphs on the vertex set \([n]\) with \( d \) edges and no loops.

**Proof.** By Theorem 68, \( \mathcal{P}(P^i)^{\circ} = \bigcup_{F \in W} \mathcal{P}(F)^{\circ} \bigcup \bigcup_{F^i \in W^i} \mathcal{P}(F^i)^{\circ} \), where \( W \) is the set of all alternating weakly-well-structured graphs on the vertex set \([n]\) with no loops and \( W^i \) is the set of all alternating weakly-well-structured graphs on the vertex set \([n]\) with a loop. Then

\[
L_{\mathcal{P}(P^i)^{\circ}}(t) = \sum_{F \in W} L_{\mathcal{P}(F)^{\circ}}(t) + \sum_{F^i \in W^i} L_{\mathcal{P}(F^i)^{\circ}}(t).
\]
By [S1, Theorem 1.3] the Ehrhart series of $P(F)$, $F \in W$, $\#E(F) = d$, and $P(F')$, $F' \in W'$, $\#E(F') = d$, respectively, are $J(P(F), x) = 1 + \sum_{t=1}^\infty L_P(F)(t)x^t = \frac{1}{(1-x)^{d+1}}$ and $J(P(F'), x) = \frac{1+x}{(1-x)^{d+1}}$. Equivalently, $L_{P(F)}(t) = \binom{t-1}{d}$, $L_{P(F')}^{(d)}(t) = \binom{t}{d}$. Thus,

$$L_{P(F)}^{(d)}(t) = \sum_{d=1}^{n} f^i(d) \left( \binom{t-1}{d} + \binom{t}{d} \right) + \sum_{d=1}^{n-1} f(d) \binom{t-1}{d},$$

where $f^i(d) = \#\{F^i \in W | \#E(F^i) = d\}$, $f(d) = \#\{F \in W | \#E(F) = d\}$. Using the Ehrhart-Macdonald reciprocity [BR, Theorem 4.1]

$$L_{P(F)}^{(d)}(t) = (-1)^n L_{P(F')}^{(d)}(-t) =$$

$$= (-1)^n \left( \sum_{d=1}^{n} f^i(d)(-1)^d \left( \binom{d+t}{d} + \binom{d+t-1}{d} \right) + \sum_{d=1}^{n-1} f(d)(-1)^d \binom{d+t}{d} \right).$$

Theorems 66, 67, 68 and 69 can be generalized to any well-structured graph $G$ by adding further technical requirements on the weakly-well-structured graphs that can appear among the leaves of a $B^\beta$-reduction tree with root labeled by $G$. Due to the technical nature of these results, we omit them here.
Chapter 4

Reduced forms in Kirillov’s type $D_n$ bracket algebra

4.1 Introduction

In this chapter we study the noncommutative bracket algebra $\mathcal{B}(D_n)$ of type $D_n$ defined by A. N. Kirillov [K2]. Using noncommutative Gröbner bases techniques we prove that a family of monomials has unique reduced forms in it. A special case of our results proves a conjecture of A. N. Kirillov about the uniqueness of the reduced form of a Coxeter type element in the bracket algebra of type $D_n$.

In Section 4.2 the definition of $\mathcal{B}(D_n)$ is given along with Kirillov’s conjecture pertaining to it. In Section 4.3 combinatorial results regarding a family of monomials are proved. Finally, in Section 4.4 we prove a general result on the reduced forms of monomials implying Kirillov’s type $D_n$ conjecture.

4.2 The type $D_n$ bracket algebra

In the rest of the chapter we study the reduced forms of elements in the type $D_n$ bracket algebra with combinatorial methods fused with noncommutative Gröbner basis theory. There is a connection with subdivisions of type $C_n$ root polytopes is present in this case in a manner analogous to that shown in Chapter ??; we do not
study this aspect further.

Let the \textbf{\textit{\textbeta-bracket algebra}} \(B^{\textbeta}(D_n)\) of type \(D_n\) be an associative algebra over \(\mathbb{Q}[\beta]\), where \(\beta\) is a variable (and a central element), with a set of generators \(\{x_{ij}, y_{ij} \mid 1 \leq i \neq j \leq n\}\) subject to the following relations:

(1) \(x_{ij} + x_{ji} = 0, y_{ij} = y_{ji}, \text{ for } i \neq j,\)

(2) \(z_i z_j = z_j z_i,\)

(3) \(x_{ij} x_{kl} = x_{kl} x_{ij}, y_{ij} x_{kl} = y_{kl} y_{ij}, y_{ij} y_{kl} = y_{kl} y_{ij}, \text{ for } i < j, k < l \text{ distinct.}\)

(4) \(z_i x_{kl} = x_{kl} z_i, z_i y_{kl} = y_{kl} z_i, \text{ for all } i \neq k, l\)

(5) \(x_{ij} x_{jk} = x_{ik} x_{ij} + x_{jk} x_{ik} + \beta x_{ik}, \text{ for } 1 \leq i < j < k \leq n,\)

(5') \(x_{jk} x_{ij} = x_{ij} x_{ik} + x_{ik} x_{jk} + \beta x_{ik}, \text{ for } 1 \leq i < j < k \leq n,\)

(6) \(x_{ij} y_{jk} = y_{ik} x_{ij} + y_{jk} y_{ik} + \beta y_{ik}, \text{ for } 1 \leq i < j < k \leq n,\)

(6') \(y_{jk} x_{ij} = x_{ij} y_{ik} + y_{ik} y_{jk} + \beta y_{ik}, \text{ for } 1 \leq i < j < k \leq n,\)

(7) \(x_{ik} y_{jk} = y_{jk} x_{ik} + y_{ij} x_{ik} + \beta y_{ij}, \text{ for } 1 \leq i < j < k \leq n,\)

(7') \(y_{jk} x_{ik} = y_{ij} x_{jk} + y_{ik} y_{ij} + \beta y_{ij}, \text{ for } 1 \leq i < j < k \leq n,\)

(8) \(x_{ik} y_{jk} = y_{jk} x_{ik} + y_{ij} x_{ik} + \beta y_{ij}, \text{ for } 1 \leq i < j < k \leq n,\)

(8') \(y_{jk} x_{ik} = x_{ij} y_{jk} + y_{ik} y_{jk} + \beta y_{ik}, \text{ for } 1 \leq i < j < k \leq n,\)

Note that \(B^{\textbeta}(C_n)\) is the quotient of \(B^{\textbeta}(D_n)\), since \(B^{\textbeta}(C_n)\) has all the above relations and in addition relations (9), (9'); see Section ??.

Let \(w_{D_n} = \prod_{i=1}^{n-1} x_{i,i+1} y_{n-1,n}\) be a Coxeter type element in \(B^{\textbeta}(D_n)\) and let \(P^n_{\textbeta}\) be the polynomial in variables \(x_{ij}, y_{ij}, 1 \leq i \neq j \leq n\) obtained from \(w_{D_n}\) by successively applying the defining relations (5) – (8') in any order until unable to do so, in the algebra \(\mathbb{Q}[\beta][x_{ij}, y_{ij} \mid 1 \leq i < j \leq n]/I,\) where \(I\) is the (two-sided) ideal generated by the relations (1) – (4). We call \(P^n_{\textbeta}\) a \textbf{\textit{\textbeta-reduced form}} of \(w_{D_n}\) and consider the process of successively applying the defining relations (5) – (8') as a reduction process in \(\mathbb{Q}[\beta][x_{ij}, y_{ij} \mid 1 \leq i < j \leq n]/I,\) with the \textbf{\textit{\textbeta-reduction rules}}:

(5) \(x_{ij} x_{jk} \to x_{ik} x_{ij} + x_{jk} x_{ik} + \beta x_{ik}, \text{ for } 1 \leq i < j < k \leq n,\)

(5') \(x_{jk} x_{ij} \to x_{ij} x_{ik} + x_{ik} x_{jk} + \beta x_{ik}, \text{ for } 1 \leq i < j < k \leq n,\)

(6) \(x_{ij} y_{jk} \to y_{ik} x_{ij} + y_{jk} y_{ik} + \beta y_{ik}, \text{ for } 1 \leq i < j < k \leq n,\)

(6') \(y_{jk} x_{ij} \to x_{ij} y_{ik} + y_{ik} y_{jk} + \beta y_{ik}, \text{ for } 1 \leq i < j < k \leq n,\)
The reduced form of any other element of $B^3(D_n)$ is defined analogously. As in the type $C_n$ case, the relations of $B^3(D_n)$ can be interpreted as subdividing type $C_n$ root polytopes and the reduced form of an element as a subdivision, though not a triangulation, of a type $C_n$ polytope. We pursue a different approach to studying reduced forms here.

We can think of the reduction process in $\mathbb{Q}(\beta, x_{ij}, y_{ij} | 1 \leq i < j \leq n)/I_\beta$, where the generators of the (two-sided) ideal $I_\beta$ are those of $I$ and in addition the commutators of $\beta$ with all the other variables $x_{ij}, y_{ij}, 1 \leq i < j \leq n$.

**Conjecture 2. (Kirillov [K2])** Apart from applying the relations (1)-(4), the reduced form $P^{B^3}_n$ of $w_{D_n}$ does not depend on the order in which the reductions are performed.

Note that the above statement does not hold true for any monomial; some examples illustrating this were already explained in the comments after Conjecture 1 in Section 3.2.

### 4.3 Graphs for type $D_n$

It is straightforward to reformulate the reduction rules (5)-(8') in terms of reductions on graphs. If $m \in B^3(D_n)$, then we replace each monomial $m$ in the reductions by corresponding graphs $G^B[m]$. The analogous procedure for type $C_n$ is explained in detail in Section 3.8.

We now define a central notion for those signed graphs whose corresponding monomials turn out to have a unique reduced form in $B^3(D_n)$. We reuse the expression "good graph" from the type $C_n$ case, though the meaning in type $D_n$ is different.
Previously we used good in the type $C_n$ sense; in the following we use good in the type $D_n$ sense.

A graph $H$ on the vertex set $[n]$ and $k$ edges labeled $1, \ldots, k$ is good if it satisfies the following conditions:

(i) The negative edges of $H$ form a noncrossing graph.

(ii) If edges $(i, j, -)_a$ and $(j, k, \epsilon_2)_b$ are in $H$, $i < j < k$, $\epsilon_2 \in \{-, +\}$, then $a < b$.

(iii) If edges $(i, j, -)_a$ and $(i, k, \epsilon_2)_b$ are in $H$, $i < j < k$, $\epsilon_2 \in \{-, +\}$, then $a > b$.

(iv) If edges $(j, k, -)_a$ and $(i, k, \epsilon_2)_b$ are in $H$, $i < j < k$, $\epsilon_2 \in \{-, +\}$, then $a < b$.

(v) If edges $(j, k, +)_a$ and $(i, k, -)_b$ are in $H$, $i < j < k$, then $a > b$.

(vi) If edges $(i, k, -)_a$ and $(j, l, +)_b$ are in $H$, $i < j < k < l$, then $a > b$.

**Lemma 70.** If $H$ is a good graph, then reduction rules $(5')$, $(6')$, $(7')$, $(8)$ cannot be performed on it. If we perform any of the reduction rules $(5)$, $(6)$, $(7)$, $(8')$ on $H$, then we obtain a graph $H'$, which is also a good graph.

**Proof.** Note that there is no way of commuting the labels of good graphs as to obtain an order on the edges which would allow rules $(5')$, $(6')$, $(7')$, $(8)$ to be performed.

That the following properties carry over from $H$ to $H'$ follows from [M1, Lemma 12], noting that only reduction rule $(5)$ creates new negative edges:

- The negative edges of $H$ form a noncrossing graph.
- If edges $(i, j, -)_a$ and $(j, k, -)_b$ are in $H$, $i < j < k$, then $a < b$.
- If edges $(i, j, -)_a$ and $(i, k, -)_b$ are in $H$, $i < j < k$, then $a > b$.
- If edges $(j, k, -)_a$ and $(i, k, -)_b$ are in $H$, $i < j < k$, then $a < b$.
- If edges $(i, j, -)_a$ and $(j, k, +)_b$ are in $H$, $i < j < k$, then $a < b$.
- If edges $(j, k, -)_a$ and $(i, k, +)_b$ are in $H$, $i < j < k$, then $a < b$.
- If edges $(j, k, +)_a$ and $(i, k, -)_b$ are in $H$, $i < j < k$, then $a > b$. 

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• If edges \((i, k, -)\) and \((j, l, +)\) are in \(H\), \(i < j < k < l\), then \(a > b\). 

Finally, given that all the above properties carry over from \(H\) to \(H'\), it follows that the property

• If edges \((i, j, -)\) and \((i, k, +)\) are in \(H\), \(i < j < k\), then \(a > b\).

also carries over.

Why are good graphs so good? Well, if the relations (5), (6), (7), (8') were a non-commutative Gröbner basis for the ideal they generate in \(\mathbb{Q}(\beta, x_{ij}, y_{ij} \mid 1 \leq i < j \leq n)/I_\beta\), with tips \(x_{ij}x_{jk}\), \(x_{ij}y_{jk}\), \(x_{ik}y_{jk}\), \(x_{jk}y_{ik}\), respectively, then it would follow immediately that the reduced form of monomials corresponding to good graphs are unique by results in noncommutative Gröbner bases theory. As it turns out the previous is not the case, however, we can still use Gröbner bases to prove the uniqueness of the reduced forms of the monomials corresponding to good graphs, which we call **good monomials**, with a little bit more work. We show how to do this in the next section.

### 4.4 Gröbner bases

In this section we briefly review some facts about noncommutative Gröbner bases and use them to show that the reduced forms of good monomials are unique.

We use the terminology and notation of [G], but state the results only for our special algebra. For the more general statements, see [G]. Throughout this section we consider the noncommutative case only.

Let

\[
R = \mathbb{Q}(\beta, x_{ij}, y_{ij} \mid 1 \leq i < j \leq n)/I_\beta
\]

with multiplicative basis \(B\), the set of noncommutative monomials up to equivalence under the commutativity relations described by \(I_\beta\).

The **tip** of an element \(f \in R\) is the largest basis element appearing in its expansion, denoted by \(\text{Tip}(f)\). Let \(C\text{Tip}(f)\) denote the coefficient of \(\text{Tip}(f)\) in this expansion.
A set of elements $X$ is **tip reduced** if for distinct elements $x, y \in X$, $\text{Tip}(x)$ does not divide $\text{Tip}(y)$.

A well-order $>$ on $B$ is **admissible** if for $p, q, r, s \in B$:
1. if $p < q$ then $pr < qr$ if both $pr \neq 0$ and $qr \neq 0$;
2. if $p < q$ then $sp < sq$ if both $sp \neq 0$ and $sq \neq 0$;
3. if $p = qr$, then $p > q$ and $p > r$.

Let $f, g \in R$ and suppose that there are monomials $b, c \in B$ such that
1. $\text{Tip}(f)c = b\text{Tip}(g)$.
2. $\text{Tip}(f)$ does not divide $b$ and $\text{Tip}(g)$ does not divide $c$.

Then the overlap relation of $f$ and $g$ by $b$ and $c$ is

$$o(f, g, b, c) = \frac{fc}{\text{CTip}(f)} - \frac{bg}{\text{CTip}(g)}.$$

**Proposition 71.** ([G, Theorem 2.3]) A tip reduced generating set of elements $G$ of the ideal $J$ of $R$ is a Gröbner basis, where the ordering on the monomials is admissible, if for every overlap relation

$$o(g_1, g_2, p, q) \Rightarrow 0,$$

where $g_1, g_2 \in G$ and the above notation means that dividing $o(g_1, g_2, p, q)$ by $G$ yields a remainder of 0.

See [G, Theorem 2.3] for the more general formulation of Proposition 71 and [G, Section 2.3.2] for the formulation of the Division Algorithm.

**Proposition 72.** Let $J$ be the ideal generated by the elements

- $x_{ij}x_{jk} - x_{ik}x_{ij} - x_{jk}x_{ik} - \beta x_{ik}$, for $1 \leq i < j < k \leq n$,
- $x_{ij}y_{jk} - y_{ik}x_{ij} - y_{jk}y_{ik} - \beta y_{ik}$, for $1 \leq i < j < k \leq n$,
- $x_{ik}y_{jk} - y_{jk}y_{ij} - y_{ij}x_{ik} - \beta y_{ij}$, for $1 \leq i < j < k \leq n$,
- $x_{jk}y_{ik} - y_{ij}x_{jk} - y_{ik}y_{ij} - \beta y_{ij}$, for $1 \leq i < j < k \leq n$,
in $R/Y$, where $Y$ is the ideal in $R$ generated by the elements

\[ x_{ik}x_{ij}y_{ik} + x_{jk}x_{ik}y_{ik} + \beta x_{ik}y_{ik} - x_{ij}y_{ij}x_{jk} - x_{ij}y_{ij}y_{ij} - \beta x_{ij}y_{ij}, \text{ for } 1 \leq i < j < k \leq n. \]

Then there is a monomial order in which the above generators of $J$ form a Gröbner basis $G$ of $J$ in $R/Y$, and the tips of the generators are, respectively,

- $x_{ij}x_{jk}$,
- $x_{ij}y_{jk}$,
- $x_{ik}y_{jk}$,
- $x_{jk}y_{ik}$.

**Proof.** Let $x_{ij} > y_{kl}$ for any $i < j$, $k < l$, and let $x_{ij} > x_{kl}$ and $y_{ij} > y_{kl}$ if $(i, j)$ is less than $(k, l)$ lexicographically. The degree of a monomial is determined by setting the degrees of $x_{ij}, y_{ij}$ to be 1 and the degrees of $\beta$ and scalars to be 0. A monomial with higher degree is bigger in the order $>$, and the lexicographically bigger monomial of the same degree is greater than the lexicographically smaller one. Since in $R$ two equal monomials can be written in two different ways due to commutations, we can pick a representative to work with, say the one which is the “largest” lexicographically among all possible ways of writing the monomial, to resolve any ambiguities. The order $>$ just defined is admissible, in it the tips of

- $x_{ij}x_{jk} - x_{ik}x_{ij} - x_{jk}x_{ik} - \beta x_{ik}$, for $1 \leq i < j < k \leq n$,
- $x_{ij}y_{jk} - y_{ik}x_{ij} - y_{jk}y_{ik} - \beta y_{ik}$, for $1 \leq i < j < k \leq n$,
- $x_{ik}y_{jk} - y_{jk}y_{ij} - y_{ij}x_{ik} - \beta y_{ij}$, for $1 \leq i < j < k \leq n$,
- $x_{jk}y_{ik} - y_{ij}x_{jk} - y_{ik}y_{ij} - \beta y_{ij}$, for $1 \leq i < j < k \leq n$,

are

- $x_{ij}x_{jk}$,
In particular the generators of $J$ are tip reduced. A calculation of the overlap relations shows that $o(g_1, g_2, p, q) \Rightarrow \mu 0$ in $R/Y$, where $g_1, g_2 \in \mathcal{G}$. Proposition 71 then implies Proposition 72.

Corollary 73. The reduced form of a good monomial $m$ is unique in $R/Y$.

Proof. Since the tips of elements of the Gröbner basis $\mathcal{G}$ of $J$ are exactly the monomials which we replace in the prescribed reduction rules (5), (6), (7), (8'), the reduced form of a good monomial $m$ is the remainder $r$ upon division by the elements of $\mathcal{G}$ with the order $>$ described in the proof of Proposition 72. Since we proved that in $R/Y$ the basis $\mathcal{G}$ is a Gröbner basis of $J$, it follows by [G, Proposition 2.7] that the remainder $r$ of the division of $m$ by $\mathcal{G}$ is unique in $R/Y$. That is, the reduced form of a good monomial $m$ is unique in $R/Y$. □

We would, however, like to prove uniqueness of the reduced form of a good monomial $m$ in $R$. This is what the next series of statements accomplish.

Lemma 74. There is a monomial order in which the elements

- $x_{ij}y_{jk}$,
- $x_{ik}y_{jk}$,
- $x_{jk}y_{ik}$.

are a Gröbner basis of $Y$ in $R$, and the tip of $x_{ik}x_{ij}y_{ik} + x_{jk}x_{ik}y_{ik} + \beta x_{ik}y_{ik} - x_{ij}y_{ij}x_{jk} - x_{ij}y_{ij}y_{ij} - \beta x_{ij}y_{ij} = x_{ij}y_{ik}y_{ij}$.

Proof. Let $x_{ij} < y_{kl}$ for any $i < j$, $k < l$, and let $x_{ij} > x_{kl}$ and $y_{ij} > y_{kl}$ if $(i, j)$ is less than $(k, l)$ lexicographically. The degree of a monomial is determined by setting the degrees of $x_{ij}, y_{ij}$ to be 1 and the degrees of $\beta$ and scalars to be 0. A monomial with higher degree is bigger in the order $>$, and the lexicographically bigger monomial of the same degree, the variables being read from left to right, is greater than the
lexicographically smaller one. Since in \( R \) two equal monomials can be written in two different ways due to commutations, we can pick a representative to work with, say the one which is the “largest” lexicographically among all possible ways of writing the monomial, to resolve any ambiguities. The order \( > \) just defined is admissible, the tip of \( x_{ik}x_{ij}y_{ik} + x_{jk}x_{ik}y_{ik} + \beta x_{ik}y_{ik} - x_{ij}y_{ij}x_{jk} - x_{ij}y_{ij}y_{ij} - \beta x_{ij}y_{ij} \) is \( x_{ij}y_{ik}y_{ij} \), and thus the generators of \( Y \) are tip reduced. Since there are no overlap relations at all, by Proposition 71 Lemma 74 follows.

\[ \Box \]

**Corollary 75.** If \( f \in Y \) then there is a term of \( f \) which can be written as \( m_1 \cdot x_{ij}y_{ik}y_{ij} \cdot m_2 \) for some \( 1 \leq i < j < k \leq n \), where \( m_1, m_2 \) are some monomials in \( R \).

**Proof.** Lemma 74 implies that

\[ \langle x_{ij}y_{ik}y_{ij} \mid 1 \leq i < j < k \leq n \rangle = \langle \text{Tip}(Y) \rangle. \]

From here the statement follows.

\[ \Box \]

**Theorem 76.** The reduced form of a good monomial \( m \) is unique in \( R \).

**Proof.** By Corollary 73 the reduced form of a good monomial \( m \) is unique in \( Q(\beta, x_{ij}, y_{ij} \mid 1 \leq i < j \leq n)/I_\beta/Y \). Since by Corollary 75 every \( f \in Y \) contains a term divisible by \( x_{ij}y_{ik}y_{ij} \) for some \( 1 \leq i < j < k \leq n \), it follows that the reduced form of a good monomial \( m \) is unique in \( Q(\beta, x_{ij}, y_{ij} \mid 1 \leq i < j \leq n)/I_\beta \), since a good monomial cannot contain any term divisible by \( x_{ij}y_{ik}y_{ij} \) because of property (iii), with \( \epsilon_2 = + \).

A special case of Theorem 76 is the statement of Conjecture 2, since \( w_{D_n} \) is a good monomial.
Bibliography


