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### *Sharing the load: Mobile robotic networks in dynamic environments*

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**M**odern technological advances make the deployment of large groups of autonomous mobile agents with onboard computing and communication capabilities increasingly feasible and attractive. In the near future, large groups of such autonomous agents will be used to perform complex tasks in dynamic environments, including transportation and distribution, logistics, surveillance, search and rescue operations, humanitarian demining, environmental monitoring, and planetary exploration [1], [2].

The potential advantages of multiple mobile agents are numerous. For instance, the availability of real-time data collected in situ would dramatically impact the modeling and study of several critical, episodic, rapidly evolving, and localized environmental phenomena such as hurricanes, oil spills, and forest fires. Similar considerations can be made for the study of ocean currents, winds, and accurate local weather forecasting. Collection of such data is currently limited by the fact that most available sensors are static (e.g., fixed monitoring stations) or remote (e.g., satellites). Moreover, the intrinsic parallelism associated with a multiagent system provides robustness to failures of single agents and, in many cases, can guarantee better time efficiency. It is also possible to reduce the total implementation and operation cost, increase reactivity and system reliability, and add flexibility and modularity to monolithic approaches. As a consequence, the interest of the systems and control, robotics, computer science, and networking communities for systems comprising multiple mobile agents has increased rapidly in the last several years. In particular, the field of multiple unmanned aerial vehicles (UAVs) has gathered a wealth of interest from the



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# Sharing the Load

## *Mobile Robotic Networks in Dynamic Environments*

BY MARCO PAVONE, KETAN SAVLA, AND EMILIO FRAZZOLI

research community, because UAVs provide an ideal platform for several of the applications mentioned earlier [3].

With such a powerful technology at hand, the major scientific challenge is to design efficient coordination policies among these agents. This article takes the viewpoint that the agents can be interpreted as resources to be allocated to customers. In surveillance and exploration missions, customers are points of interest to be visited; in transportation and logistics applications, customers are people demanding delivery of some goods or service (e.g., utility repair). Finally, consider a possible architecture for networks of autonomous agents performing distributed sensing: a set of  $n$  cheap-sensing devices (sensing nodes), distributed in the environment, provides sensor measurements, while  $m$  sophisticated agents (cluster heads) collect and process information from the sensing nodes and transmit it to the outside world. In this case, the sensing nodes represent customers, while the agents acting as cluster heads represent the resources to be allocated.

The underlying mathematical problem in many of these applications can be studied within the framework of spatial queues, with the tasks as customers and the mobile agents as servers. The solution has to typically address three key challenges: 1) task allocation among the agents, 2) service scheduling for each agent, and 3) design of loitering strategies, i.e., strategies to adopt for agents with no assigned tasks. In general, these challenges are coupled. Therefore, devising an optimal or at least provably efficient policy is a difficult problem. Considering motion constraints for the agents, as should be done for UAVs, complicates things further.

A natural way to reduce the complexity is to partition the workspace among the agents and then let each agent follow a certain set of rules in its own region. To what extent does this decoupling strategy affect optimality? The objective of this article is to illustrate specific scenarios and recently developed partitioning schemes whereby one can retain optimality, or at least some degree of optimality under this decomposition.

This article is outlined as follows: First, we describe the characteristics of a family of dynamic vehicle routing problems that indeed capture the main features of a large number of dynamic task-based motion coordination problems, appearing in a variety of foreseen application domains. Second, we are interested in identifying specific scenarios in which one can decompose, through workspace partitioning, task allocation, service scheduling, and loitering strategies, while maintaining some degree of optimality for the original problem. For these scenarios, we discuss which types of partitions should actually be used: in some cases, the optimal partitions coincide with well-known tessellation of the plane from computational geometry (e.g., Voronoi diagrams [4]); while in other cases, new types of partitions should be used. Finally, we present efficient algorithms that achieve such spatial partitions.

## Family of Dynamic Routing Problems

Consider a geographical region  $\mathcal{Q}$  in which a certain dynamic process generates spatially localized service requests (also called demands). (Henceforth, if not otherwise stated, we assume that  $\mathcal{Q}$  is a compact, convex subset of  $\mathbb{R}^2$ , with an open interior.) Service requests can represent, for example, pickup or

## *A natural strategy for load sharing is to partition the workspace among the agents and then let each agent follow a certain set of rules in its own region. To what extent does this decoupling strategy affect optimality?*

delivery points, or events of interest that require close investigation or measurement. The process generating service requests is modeled as a spatiotemporal Poisson process with temporal intensity  $\lambda > 0$  and spatial distribution  $\varphi$  over  $\mathcal{Q}$  (i.e., upon arrival, the locations of service requests are identically and independently distributed according to  $\varphi$ ) and can be known to various degrees of accuracy. (Indeed, most of the results presented in this article are also valid when the arrival process is a general renewal process.) A total of  $m$  mobile agents provide service in  $\mathcal{Q}$ ; each service request requires an independent and identically distributed amount of on-site service with finite mean duration. A service request is fulfilled when one of the agents moves to its location and completes its on-site service; we assume that agents are able to service an unlimited number of requests. Service requests can either stay active until satisfied or they might expire after a certain time. The time service requests that remain active can itself be a random variable, describing customer impatience. There could be multiple classes of service requests (i.e., heterogeneous service requests), some of them requiring higher-priority service.

Furthermore, the agents can be subject to several differential constraints because of their dynamic capabilities; in particular, most vehicles of practical interest are subject to first- or second-order nonholonomic constraints (e.g., wheeled vehicles, aircraft, and ships). The information available to the agents can also be limited in several ways. For example, agents might be able to communicate directly only with other agents (or static nodes) that lie within a certain radius or might not have any communication capability (e.g., when cheap mobile sensors are used or stealthiness is required).

Finally, it is possible that no central authority is available, which is aware of the state of all agents, and can process all the information available to the system. Thus, the design of cooperative policies is also affected by the mode of its implementation (centralized or decentralized).

It is desired to maximize the quality of service delivered by the mobile agents, for example, in terms of the average or worst-case time delay between the issuance of a service request and the time it is fulfilled. When service requests have a finite lifetime (i.e., they can expire), another parameter of interest is the fraction of service requests that is fulfilled before expiration. In general, the focus is on the quality of service as perceived by the end user, rather than, for example, fuel economies achieved by the mobile agents.

## In the near future, large groups of autonomous agents will be used to perform complex tasks in dynamic environments.

The basic version of the problem in which the agents do not have any differential constraint, service requests do not expire and do not have any priority, a central dispatcher is available, and the objective is to minimize the average waiting time of service requests is known in the literature as the dynamic traveling repairman problem (DTRP) [5].

### Spatial Tessellations for Workload Sharing

In this section, we identify specific cases, within the previous family of dynamic vehicle routing problems, in which one can decompose, by using partitioning policies, task allocation, service scheduling, and loitering strategies, while maintaining some degree of optimality for the original problem. For each case, we discuss which types of spatial tessellations (partitions) should indeed be used. In the following, an  $m$  partition of  $\mathcal{Q}$  (where  $m \in \mathbb{N}$ ,  $m \geq 1$ ) is a collection of  $m$  closed subsets  $\{Q_i\}_{i=1}^m$ , with disjoint interiors and whose union is  $\mathcal{Q}$ .

Given a single-agent policy  $\pi$  and an  $m$  partition of  $\mathcal{Q}$ , a  $\pi$ -partitioning policy is a multiagent policy such that 1) one agent is assigned to each subregion (thus, there is a one-to-one correspondence between agents and subregions), and 2) each agent executes the single-agent policy  $\pi$  to service demands that fall within its own subregion. Note that a partitioning policy is parametrized by the single-agent policy  $\pi$  and by the  $m$  partition of  $\mathcal{Q}$ .

We start by discussing the most intuitive family of partitioning policies, namely the one that uses partitions of  $\mathcal{Q}$  with subregions of equal measure (with respect to  $\varphi$ ).

### Equitable Partitions

Arguably, the most widely applied resource allocation strategy is to equalize the total workload assigned to each resource, i.e., in our context, to each agent. Indeed, assume heavy load conditions (i.e., the fraction of time the agents are busy, i.e., they

are moving toward or providing on-site service to a service request, is close to one), that the agents are first-order holonomic vehicles, and that  $\pi^*$  is a single-agent (unbiased) optimal policy for the DTRP. Then, a  $\pi^*$ -partitioning policy that uses  $m$  partitions whose subregions have equal measure with respect to  $\varphi$  (i.e., equitable  $m$  partitions) is within a factor  $m$  of the optimal (unbiased) performance [6]. Moreover, under the same heavy load assumption, and if  $\varphi$  is uniform over  $\mathcal{Q}$ , such equitable  $\pi^*$ -partitioning policy is optimal. Two examples of possible equitable partitions are depicted in Figure 1.

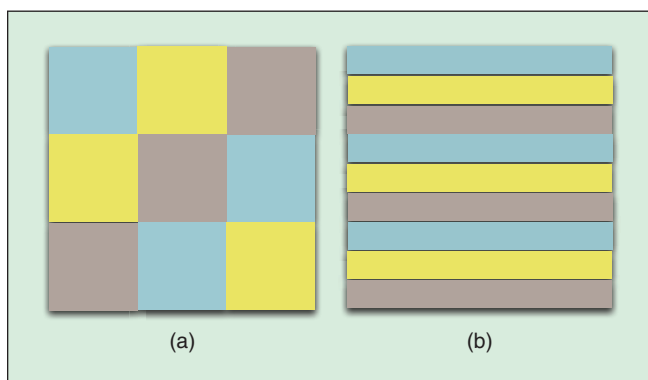
Recently, similar results have been proven for some generalizations of the DTRP problem. In particular, in [7], we studied a generalization of the DTRP problem in which service requests might expire after a certain deterministic time  $T$  (called time window). Time window constraints are indeed common in many applications, including bank deliveries, postal deliveries, grocery distribution, dial-a-ride service, bus routing, and repairmen scheduling. The objective, then, is to minimize the number of agents needed to ensure that a certain fraction  $\varepsilon$  of service requests is fulfilled before expiration. We proved that, in heavy load, when  $\varphi$  is uniform, demands do not require on-site service, and  $\varepsilon \rightarrow 1^-$  (i.e., almost all service requests have to be fulfilled before expiration), an equitable partitioning policy in which each agent services outstanding service requests inside its own subregion by repeatedly forming optimal (i.e., of minimum length) tours is within a factor 3.8 of the optimal. In particular, a number of agents sufficient to guarantee, in heavy load, that almost all service requests are serviced before expiration, is

$$m = \left\lceil \sqrt{\lambda} \beta \sqrt{\frac{2}{T}} \right\rceil,$$

where  $\beta \simeq 0.712$ .

Another generalization of the DTRP in which equitable partitioning policies perform within a constant factor of the optimal is the problem of dynamic vehicle routing with heterogeneous demands, introduced in [8]. The setup is similar to that of the DTRP problem, but there are  $n$  different classes of service requests; given coefficients  $c_1, \dots, c_n > 0$ ,  $\sum_{i=1}^n c_i = 1$ , the goal is to find the vehicle routing policy that minimizes the convex combination  $c_1 D_1 + \dots + c_n D_n$ , where  $D_i$  is the expected delay for service requests of class  $i$ . By increasing the coefficients for certain classes, a higher priority level can be given to their service requests. This problem has important applications in areas such as UAV surveillance, where targets are given different priority levels based on their urgency or potential importance. In [9], an equitable partitioning policy is proposed, in which each agent services (in its own region) demands in batches; each batch contains the outstanding  $i$  class service requests with probability  $p_i = c_i$ ,  $i = 1, \dots, n$ . This policy is shown to be within a factor  $2n^2$  of the optimal, in heavy load, and for uniform  $\varphi$ .

All previous results hold for first-order holonomic agents and in heavy load. In [10], we study a generalization of the DTRP in which agents have nonholonomic motion constraints (as it is the case when the agents are fixed-wing UAVs). In particular, we modeled the UAVs as a variant of the



**Figure 1.** Different kinds of equitable partitions for a uniform distribution  $\varphi$ . (a) Fat equitable partition. (b) Skinny equitable partition.



well-studied Dubins model, where the agents are constrained to move with constant forward speed along paths whose radius of curvature is uniformly lower bounded by the parameter  $\rho$ . (Henceforth, we implicitly use this model for fixed-wing UAVs.) Moreover, for setups involving such vehicles, on-site service corresponds to the vehicle simply passing through the service location. (In the rest of the article, we maintain this notion of on-site service for all vehicles except first-order holonomic vehicles.) For the case of heavy load and uniform distribution, we proved that additively weighted equitable partitioning policies are still within a constant factor of the optimal, where the weights are related to the shape of subregions. In particular, if  $\mathcal{Q}$  is a rectangular region, the workspace should be partitioned into strips formed by placing equidistant dividers parallel to the width (i.e., the largest of the two dimensions of the rectangular region). For a general convex region, one can optimize the performance by tuning the distance between the dividers to the shape of  $\mathcal{Q}$  in the following way. Let  $W$  and  $H$  be the width and height of a minimum height rectangle enclosing  $\mathcal{Q}$ . Form strips by placing dividers parallel to the width of the enclosing rectangle in the following way. Let  $|Q_1|, \dots, |Q_m|$  be the areas of the individual strips and let  $H_1, \dots, H_m$  be the heights of the resultant strips. Note that  $\sum_{i=1}^m |Q_i| = |\mathcal{Q}|$  and  $\sum_{i=1}^m H_i = H$  and that the  $|Q_i|$ s are functions of  $H_i$ . Then, the interstrip distances  $H_i$  are selected so that

$$|Q_i| + \frac{7}{3}\pi\rho H_i = \frac{1}{m} \left( |\mathcal{Q}| + \frac{7}{3}\pi\rho H \right) \quad \forall i \in \{1, \dots, m\}. \quad (1)$$

Note that this rule for space partitioning could be regarded as a general rule for the heavy load case, in the sense that, as  $\rho \rightarrow 0^+$  (which corresponds to first-order holonomic vehicles), the corresponding strips would correspond to regular equitable partitions, which we have already stated to be optimal (note that we are assuming a uniform distribution of demands) for the first-order holonomic vehicles.

### Simultaneously Equitable Partitions

Given two functions  $\varphi_j : \mathcal{Q} \rightarrow \mathbb{R}_{>0}, j \in \{1, 2\}$ , with  $\int_{\mathcal{Q}} \varphi_j(x) dx = 1$ , an  $m$  partition is simultaneously equitable with respect to  $\varphi_1$  and  $\varphi_2$  if  $\int_{Q_i} \varphi_j(x) dx = 1/m$  for all  $i \in \{1, \dots, m\}$  and  $j \in \{1, 2\}$ . Theorem 12 in [11] shows that, given two such functions  $\varphi_j, j \in \{1, 2\}$ , there always exists an  $m$  partition that is simultaneously equitable with respect to  $\varphi_1$  and  $\varphi_2$  and whose subregions  $Q_i$  are convex.

Assume heavy load conditions, that the agents are first-order holonomic vehicles, and that  $\pi^*$  is a single-agent (unbiased) optimal policy for the DTRP; then, a  $\pi^*$ -partitioning policy that uses  $m$  partitions whose subregions have equal measure with respect to both  $\varphi$  and  $\varphi^{1/2}$  (i.e., simultaneously equitable  $m$  partitions) is optimal [6]. We will discuss in the ‘‘Algorithms for Partitioning Policies’’ section, algorithms that compute simultaneously equitable partitions; such algorithms are much more complicated than those for equitable partitions and, unfortunately, no spatially distributed implementation is known.

All the previous results hold under the heavy load assumption; when the load is only moderate, the shape of subregions

can have a significant effect. In moderate traffic and with holonomic first-order agents, a solution that turns out to be effective (although there is no certificate of optimality) is to adopt equitable partitioning policies in which the subregions are fat (i.e., with a small diameter for a given area), rather than long and thin. We next investigate which partitioning policies should be used, instead, in light load (i.e., when the fraction of time the service vehicles are busy is close to zero).

### Median Voronoi Tessellation

We first introduce the concept of Voronoi tessellations (or Voronoi diagrams). The use of Voronoi tessellations is ubiquitous in many fields of science, ranging from operations research, animal ethology (territorial behavior of animals), computer science (design of algorithms), to numerical analysis (construction of adaptive grids for partial differential equations and general quadrature rules), and algebraic geometry (moduli spaces of Abelian varieties). We now give its formal definition. A detailed exposition of Voronoi tessellations is given in [4].

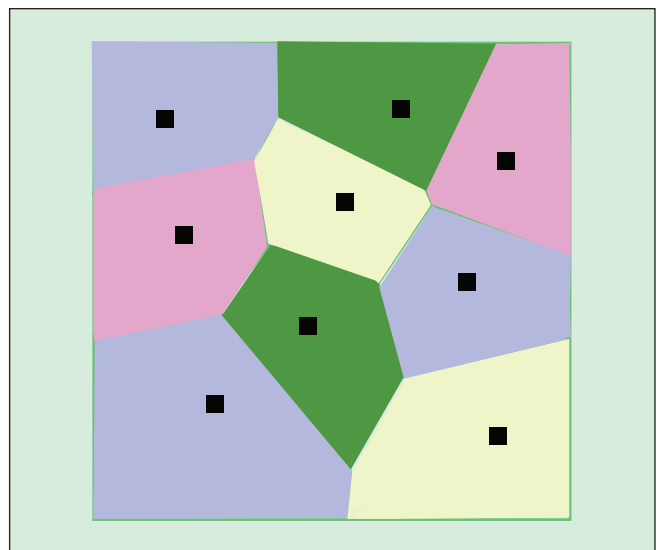
Define  $G \doteq (g_1, \dots, g_m) \in \mathcal{Q}^m$ . The Voronoi diagram  $\mathcal{V}(G) = (V_1(G), \dots, V_m(G))$  of  $\mathcal{Q}$  generated by points  $G$  is defined by

$$V_i(G) = \{q \in \mathcal{Q} \mid \|q - g_i\| \leq \|q - g_j\|, \quad \forall j \neq i, j \in I_m\},$$

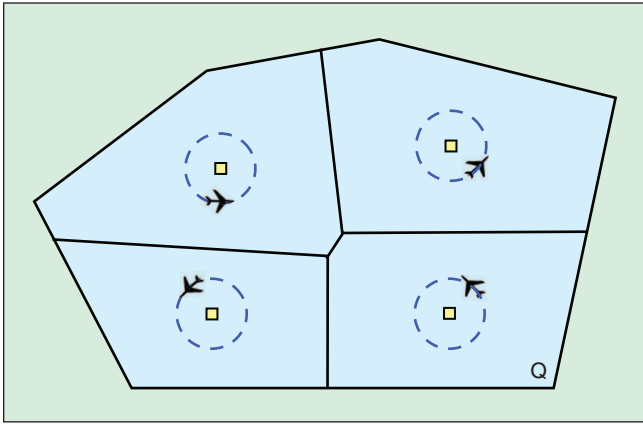
where  $I_m \doteq \{1, 2, \dots, m\}$ .  $G$  is the set of generators of  $\mathcal{V}(G)$ , and  $V_i(G)$  is the Voronoi cell of the  $i$ th generator (see Figure 2). A Voronoi diagram  $\mathcal{V}(G) = (V_1(G), \dots, V_m(G))$  of  $\mathcal{Q}$  is called a median Voronoi tessellation of  $\mathcal{Q}$  with respect to the density function  $\varphi$  if the ordered set of generators  $G$  is equal to the ordered set of generalized medians of the sets in  $\mathcal{V}(G)$  with respect to  $\varphi$ , i.e., if

$$g_i = \arg \min_{g \in \mathbb{R}^2} \int_{V_i(G)} \|g - q\| \varphi(q) dq, \quad \forall i \in \{1, \dots, m\}.$$

It is possible to show that a median Voronoi tessellation always exists for any domain  $\mathcal{Q}$  and density  $\varphi$ . It is interesting



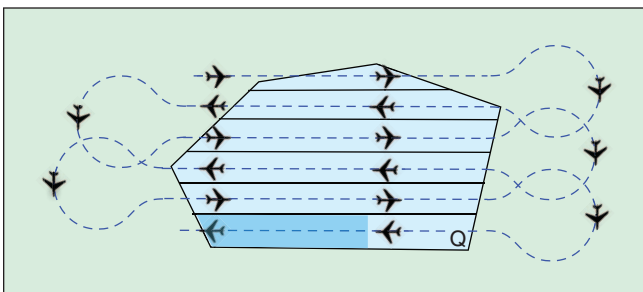
**Figure 2.** A Voronoi diagram with nine generators (generators are represented by black squares).



**Figure 3.** Fixed-wing UAVs loitering around median locations.

to note that, as the number of generators increases, the median Voronoi tessellation for a given region assumes a hexagonal honeycomb structure [12].

A median Voronoi tessellation is proven [5] to be almost optimal for the scenario involving an ensemble of first-order agents responding to service requests that are generated very rarely (i.e., in light load). The median locations in this case can be understood to be loitering locations for the agents, i.e., locations at which the agents wait when there are no outstanding service requests. The almost optimality in the light load case is to be understood as follows. If one defines the average waiting time for the service requests to be a function of the loitering locations of the agents, then the generalized median locations that give rise to the median Voronoi tessellation correspond to the local minima or the saddle point of this function. Additionally, even for networks of UAVs (Dubins vehicle), tanks (differential drive robots), or rotary wing aircraft (double integrator robots), the median Voronoi tessellation is proven [10], [13] to be almost optimal (in the same sense, as mentioned earlier) when the robotic network is sparse, i.e., the density of agents is low. For fixed-wing UAVs, the median locations in that case correspond to the center of loitering circles, as illustrated in Figure 3. The primary reason for optimality of median Voronoi tessellations for sparse robotic networks in the light load case is that, since the agents are well separated from each other, the length of the optimal path from the loitering location of an agent to the location of a service request can, on an average, be well approximated by the Euclidean distance between those locations. However, this approximation becomes weaker as the density of the robotic network increases, in which case the optimal sharing of workload



**Figure 4.** Loitering pattern for a dense network of UAVs, with the illustration of region of responsibility for one of them.

necessitates a dynamic partitioning of space, i.e., space partitions that change as a function of time.

### Dynamic Partitions

The scope of partitioning policies can be easily extended to include dynamic partitions. In [10], it was shown that for a dense group of fixed-wing UAVs, a dynamic partitioning policy performs within a constant factor of the optimal in light load. The details of the partitioning scheme are as follows. Divide  $\mathcal{Q}$  into strips of width  $w$ , where

$$w = \min \left\{ \left( \frac{4}{3\sqrt{\rho}} \frac{WH + 10.38\rho H}{m} \right)^{2/3}, 2\rho \right\}.$$

As before,  $W$  and  $H$  are the width and the height, respectively, of the minimum height rectangle enclosing  $\mathcal{Q}$ . Orient the strips along the side of length  $W$ . Construct a closed loitering path that runs along the longitudinal bisector of each strip, as shown in Figure 4. The  $m$  UAVs loiter on this path, equally spaced, in terms of path length. If  $\ell$  is the inter-UAV distance, then the subregion of  $\mathcal{Q}$  allocated to every agent is a rectangle of length  $\ell$  and width  $w$  (intersected with  $\mathcal{Q}$ ) and offset by a distance  $\sqrt{\rho w - w^2/4}$  ahead of it, as shown in Figure 4. The offset between the UAV's position and its region of responsibility is to make sure that the UAV can travel to any location in that region in a short time. Note that the dynamic partition associated with a particular agent is in fact fixed in the reference frame of that agent and that in the global frame these partitions could be regarded as a dynamic version of an equitable partition of  $\mathcal{Q}$ , modulo the boundary effects.

The same kind of dynamic partitioning policy is proven to perform within a constant factor of the optimal for a dense group of agents modeled as double integrators [13]. The reason that such dynamic partitions help to give good performance is that they help to keep the region of responsibility for an agent directly in front of it, thereby minimizing the time taken to travel to the location of a service request.

### Phase Transition Between Tessellations

We have illustrated the use of many well-known spatial tessellations for optimal or almost optimal distribution of workload for dynamic task-allocation-based scenarios. Mathematically, many of these scenarios could be considered to be variants of the same basic DTRP, in some cases representing extreme regions of the parameter space for that problem. For example, we considered the heavy load case when the fraction of time the agents are busy (i.e., they are moving toward or providing on-site service to a service request), is close to one, and the light load case, i.e., when the fraction of time the agents are busy is close to zero. We have seen that different spatial tessellations give optimal performance for these two extremes, suggesting the existence of a phase transition between tessellations driven by exogenous factors, such as  $\lambda$ , in such applications. Additionally, we have seen drastically different partitions giving optimal performance, as the network of robots with motion constraints gets denser. This suggests phase transition in tessellations also being driven

by endogenous factors like density of the robotic network. In [10], we studied such a phase transition between a median Voronoi tessellation (Figure 5) and a dynamic partition (Figure 6) with respect to the density of the robotic network in an unbounded domain. We identified a dimensionless parameter, the nonholonomic density  $\doteq \rho^2 \bar{m}$ , with  $\bar{m}$  being the number of vehicles per unit area, whose critical value ( $\approx 0.06$ ) characterizes the corresponding phase transition. Similar endogenous phase transitions for other models of robots were investigated in [13]. A formal study of the factors driving such phase transitions would help the system planner to select an appropriate tessellation for a given set of problem parameters. In addition, a fundamental understanding of such phase transitions could help to give a better insight into similar phenomena in naturally occurring systems, e.g., transition from solitary to gregarious behavior in desert locusts, as reported in [14].

### Algorithms for Partitioning Policies

In this section, we present several algorithms to implement the different partitioning policies presented in the ‘‘Spatial Tessellations for Workload Sharing’’ section.

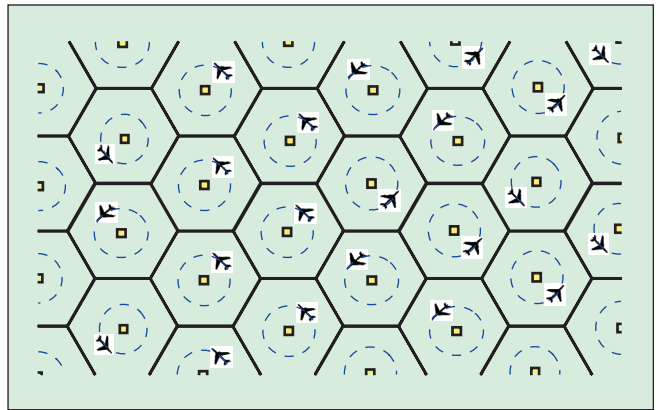
### Algorithms for Equitable Partitions

If we model the workload for subregion  $S \subseteq \mathcal{Q}$  as  $\lambda_S \doteq \lambda \int_S \varphi(q) dq$ , then the workload for agent  $i$  is  $\lambda_{Q_i}$ . Then, equitable partitioning entails equalizing the workload  $\lambda_{Q_i}$  in the  $m$  subregions.

We first assume that a central dispatcher is available. In such case, many simple algorithms provide equitable partitions. For example, one could sweep  $\mathcal{Q}$  from a point in the interior of  $\mathcal{Q}$  using an arbitrary starting ray until  $\lambda \int_{Q_i} \varphi(q) dq = \lambda/m$ , continuing the sweep until  $\lambda \int_{Q_2} \varphi(q) dq = \lambda/m$ , etc. Sweeping techniques can also be devised for dividing  $\mathcal{Q}$  into strips of equal area or width or according to the rule described in (1).

Indeed, to the best of our knowledge, all equitable partitioning policies inherently assume a central dispatcher that computes a tessellation of the workspace. This fact is in sharp contrast with the desire of a fully distributed architecture for a multiagent system. The lack of a fully distributed architecture limits the applicability of equitable partitioning policies to limited-size, multiagent systems operating in a known static environment. If a central dispatcher is not available, then a possible solution is to run a distributed leader election algorithm, let the leader execute one of the centralized algorithms discussed earlier, and finally let the leader broadcast subregion’s assignments to all other agents. Such conceptually simple solution, however, can be impractical in scenarios where the density  $\varphi$  changes over time or agents can fail, since at every parameter’s change a new time-consuming leader election is needed. In [15], we introduced a radically different, spatially distributed algorithm for equitable partitioning, which does not require any leader election. Such algorithm is described next.

- 1) *Power diagrams and virtual generators:* In the solution proposed in [15], power diagrams are the key geometrical concept. Define  $G_W \doteq ((g_1, w_1), \dots, (g_m, w_m)) \in (\mathcal{Q} \times \mathbb{R})^m$ . We refer to the pair  $(g_i, w_i)$  as a power point. The power diagram  $\mathcal{V}(G_W) = (V_1(G_W), \dots,$



**Figure 5.** The median Voronoi tessellation for an unbounded domain with UAVs loitering about median locations.

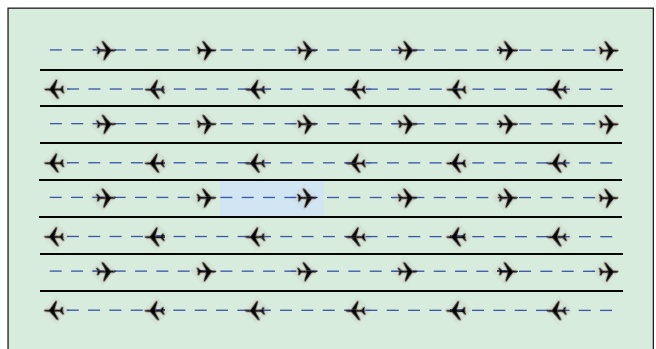
$V_m(G_W))$  of  $\mathcal{Q}$  generated by power points  $G_W$  is defined by

$$V_i(G_W) = \{q \in \mathcal{Q} \mid \|q - g_i\|^2 - w_i \leq \|q - g_j\|^2 - w_j, \forall j \neq i, j \in I_m\}.$$

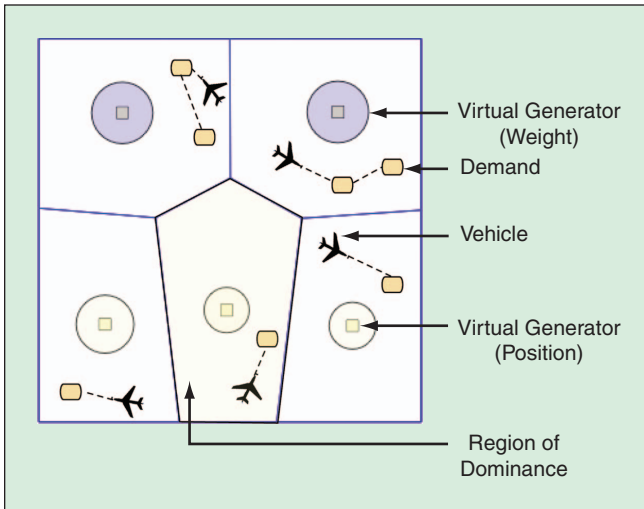
We refer to  $G_W$  as the set of power generators of  $\mathcal{V}(G_W)$  and to  $V_i(G_W)$  as the power cell of the  $i$ th power generator. Notice that, when all weights are the same, the power diagram coincides with the Voronoi diagram; indeed, power diagrams are a generalization of Voronoi diagrams. A useful property of power diagrams is that each cell  $V_i(G_W)$  is a convex set. In the following, we simply refer to  $V_i(G_W)$  as  $V_i$ .

The key advantage of power diagrams is that an equitable power diagram always exists for any density  $\varphi$  [15]. More precisely, given  $m \geq 1$  distinct points  $(g_1, \dots, g_m)$  in  $\mathcal{Q}$ , there exist weights  $w_i, i \in \{1, \dots, m\}$ , such that the power points  $((g_1, w_1), \dots, (g_m, w_m))$  generate a power diagram that is equitable with respect to  $\varphi$  (i.e.,  $\int_{V_i} \varphi(q) dq = \int_{\mathcal{Q}} \varphi(q) dq / m$ ).

The basic idea is to associate to each agent  $i$  a virtual power generator (virtual generator for short)  $(g_i, w_i)$ ; then, the power cell  $V_i$  becomes the region of dominance for agent  $i$  (see Figure 7). A virtual generator  $(g_i, w_i)$  is simply an artificial (or logical) variable, whose value is locally controlled by the  $i$ th agent; in



**Figure 6.** UAVs loitering in an unbounded domain. The dynamic partition associated with a UAV is also illustrated.

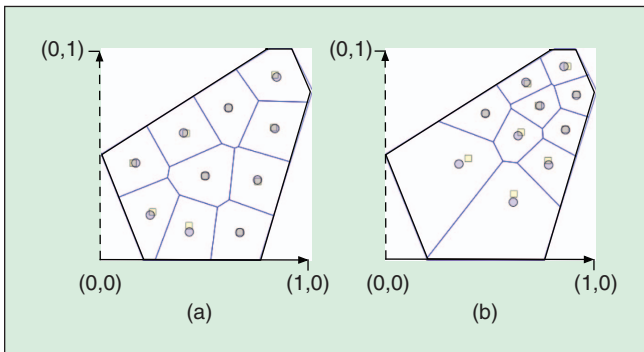


**Figure 7.** Vehicles, virtual generators, demands, and regions of dominance. A positive weight  $w$  is represented by a yellow circle with radius  $\sqrt{w}$ ; a negative weight  $w$  is represented by a blue circle with radius  $\sqrt{|w|}$ . Note that the position of a vehicle and the position of its virtual generator are, in general, distinct.

particular,  $g_i$  is a virtual point and  $w_i$  is its weight. In general, the position of an agent and the position of its virtual generator are distinct, i.e., the position of an agent inside its own region of dominance  $V_i$  is independent from the position of its virtual generator.

- 2) *Locational optimization:* In light of the previous result on the existence of equitable power diagrams, we enable the weights of the virtual generators to follow a spatially distributed gradient descent law (while maintaining the positions of the virtual generators fixed) such that an equitable partition is reached. Assume, henceforth, that the positions of the virtual generators are distinct, i.e.,  $g_i \neq g_j$  for  $i \neq j$ , and belong to  $\mathcal{Q}$ . Let  $W \doteq (w_1, \dots, w_m) \in \mathbb{R}^m$  and define the set

$$S \doteq \{W \mid \lambda_{V_i} > 0 \quad \forall i \in I_m\}.$$



**Figure 8.** Typical equitable partitions achieved by using the control laws in [15]. The yellow squares represent the position of the generators, while the blue circles represent the agents. (a) Typical equitable partition of  $\mathcal{Q}$  for  $\varphi(q) = 1$ . (b) Typical equitable partition of  $\mathcal{Q}$  for  $\varphi(q) = e^{-5\|q-c\|^2}$ ,  $c = (0.8, 0.8)$ . (We are omitting the normalization constant for  $\varphi$ .)

Set  $S$  contains all possible vectors of weights such that no region of dominance has zero measure. We introduce the locational optimization function  $H_V : S \rightarrow \mathbb{R}_{>0}$ :

$$H_V(W) \doteq \sum_{i=1}^m \left( \lambda \int_{V_i(W)} \varphi(q) dq \right)^{-1} = \sum_{i=1}^m \lambda_{V_i(W)}^{-1}.$$

Assume that the virtual generators' weights obey a first-order dynamical behavior described by  $\dot{w}_i = u_i$ . Then, we set up the following control law defined over the set  $S$

$$u_i = -\frac{\partial H_V}{\partial w_i}(W), \quad (2)$$

where we assume that the power diagram  $\mathcal{V}(W) = (V_1, \dots, V_m)$  is continuously updated. One can prove that virtual generators' weights starting at  $t = 0$  at  $W(0) \in S$  (this is the case, for example, if all weights are initialized to zero), and evolving under (2) remain in  $S$  and converge asymptotically to a vector of weights that yields an equitable power diagram.

It is possible to show (see [15]) that the computation of the partial derivative in (2) only requires information from the agents with neighboring power cells. Therefore, the gradient descent law (2) is indeed spatially distributed over the dual graph of the power diagram; thus, the overall algorithm is spatially distributed. We mention that, in a power diagram, each power generator has an average number of neighbors less than or equal to six; therefore, the computation of gradient (2) is scalable with the number of agents.

- 3) *Optimizing secondary objectives:* The previous gradient descent law, although effective in providing an equitable partition (with convex subregions), can yield long and skinny subregions. Notice that, to obtain an equitable power diagram, changing the virtual generators' weights, while maintaining their positions fixed, is sufficient. Then, we can use the degrees of freedom given by the positions of the virtual generators to optimize secondary objectives, e.g., to obtain power diagrams similar to Voronoi diagrams or to obtain cells whose shapes show circular symmetry (which are especially useful for moderate loads, as discussed earlier). These extensions are discussed in [15]; typical equitable partitions achieved by using the control laws in [15] are shown in Figure 8.

### Algorithms for Simultaneously Equitable Partitions

An approximate algorithm to compute an  $m$  partition that is simultaneously equitable with respect to two functions  $\varphi_j$ ,  $j \in \{1, 2\}$ , can be found in [11] (specifically, see discussion on p. 621); in the particular case when both functions  $\varphi_j$ ,  $j \in \{1, 2\}$ , are uniform, the problem reduces to the previous case. To the best of our knowledge, there are no spatially distributed algorithms to compute simultaneously equitable partitions.

### Algorithms for Dynamic Partitions

Dynamic partitions (that are almost equitable) that have been shown to be useful for dense networks of fixed-wing UAVs in



light load scenarios can be obtained using a central dispatcher. As noted earlier, the dynamic partition associated with an agent is fixed in its own reference frame. Therefore, it can be completely specified by its dimensions and its offset with respect to the agent's position. The central dispatcher can compute these quantities along with the closed loitering path as shown in Figure 4. The specification of the initial location and orientation of the vehicles along this path then completes the specification of the dynamic partitioning policy.

### Algorithms for Median Voronoi Tessellations

For the case of equitable partitions, we presented a spatially distributed equitable partitioning policy. The strategy is scalable in the sense that every agent needs to talk only to a few others to determine its action. A similar algorithm can be given for the problem of achieving median Voronoi tessellations. Indeed, assume that the agents update generators' locations (in this case, the position of the generators should coincide with the position of the agents) and weights according to the following law:

$$\begin{cases} \dot{w}_i = 0, & w_i(0) = 0, \\ \dot{g}_i = - \int_{V_i(G_W)} \frac{g_i - q}{\|g_i - q\|} \varphi(q) dq, & g_i(0) \in \mathcal{Q}, \end{cases}$$

where the partition  $\mathcal{V}(G_W)$  is updated continuously (note that, under this control law, the weights do not have any dynamics; since all weights are the same, the partition  $\mathcal{V}(G_W)$  is always a Voronoi diagram). One can prove that, under this control law, the generator locations converge asymptotically to a set that contains the generalized median locations, i.e., the locations that generate a median Voronoi tessellation.

The success of the previous strategy depends on the fidelity of communication channels between agents. In fact, a common theme in cooperative control is the investigation of the effects of different communication and information sharing protocols on the system performance. Clearly, the ability to access more information at each single agent cannot decrease the performance level; hence, it is commonly believed that providing better communication among agents will improve the system's performance. In this section, we show that median Voronoi tessellations can be attained without any explicit communication between agents; in other words, the no-communication constraint in such case is not binding and does not limit the steady-state performance.

In [16], a median Voronoi partitioning policy that does not rely on interagent communication is proposed. Moreover, this policy does not assume any prior knowledge of the spatial density function  $\varphi$ . This policy is closely related to an algorithm due to MacQueen [17], originally designed as a simple online adaptive algorithm to solve clustering problems and later used as the method of choice in several vector quantization problems, where little information about the underlying geometric structure is available. MacQueen's algorithm has the advantage to be a learning adaptive algorithm, not requiring a priori knowledge of the distribution of the objects, but rather allowing the online generation of samples.

The behavior of the policy in the light load, i.e., for small values of  $\lambda$ , can be summarized as follows:

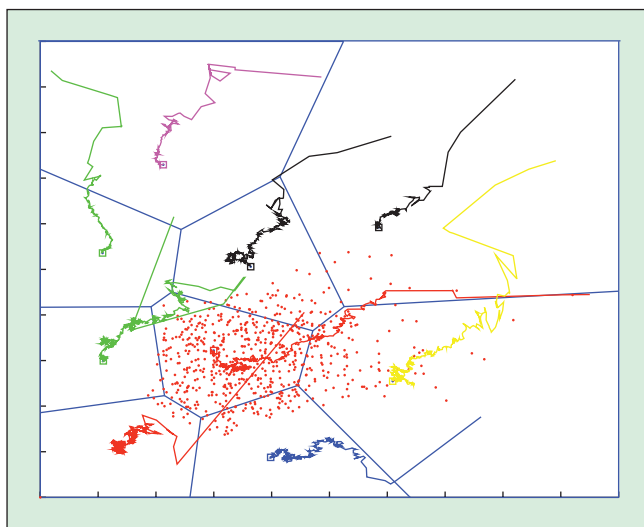
**The most widely applied resource allocation strategy is to equalize the total work-load assigned to each resource, i.e., in our context, to each robot.**

- 1) At the initial time,  $m$  agents are assumed to be deployed at general positions in  $\mathcal{Q}$ , and there are no outstanding service requests.
- 2) The agents do not move until the first service request appears. On the appearance of a service request, every agent moves toward the service request location.
- 3) As soon as one agent reaches the service request, all agents start moving toward their current reference point, which is the point minimizing the average distance to demands serviced in the past by each agent (if there is no unique minimizer then move to the nearest one) and the process continues.

It was proven in [16] that, in the light load, almost surely, the set of reference points,  $\{p_1^*(t_j), \dots, p_m^*(t_j)\}$ , converges to the set of generalized  $m$  median locations of  $\mathcal{Q}$  as  $j \rightarrow \infty$ .

Figure 9 shows the results of numerical experiments with a nonuniform distribution, namely an isotropic normal distribution centered at  $(0.25, 0.25)$ , with a standard deviation equal to 0.25.

It is interesting to compare the performance of this no-communication policy with a sensor-based policy that allows agents to have access to each other's positions at all times. The outcome of this additional capability in the light load case is that an agent will move toward a service request location only



**Figure 9.** Numerical simulation in the light load case, for a normal spatial distribution. Paths followed by the reference points up to  $t = 10^4$  (corresponding to approximately 5,000 demands), using the no-communication policy. The locations of all demands visited by one of the agents are also shown.

## The scope of partitioning policies can be easily extended to include dynamic partitions.

if it is the closest among all the agents. The sensor-based policy is more efficient than the no-communication policy in terms of the length of the path traveled by each agent, since there is no duplication of effort, as several agents pursue the same demand (unless a demand is on the boundary of two or more Voronoi regions). However, in terms of quality of service, it is shown that there is no difference between the two policies, for low-demand generation rates. Numerical results show that the sensor-based policy is more efficient in a broader range of demand generation rates and, in fact, provides almost optimal performance both in light- and heavy load conditions. However, as  $\lambda$  is increased from low values, the performance of the no-communication policy degrades significantly, almost approaching the performance of a single-vehicle system over an intermediate range of values of  $\lambda$ . The intuition in this phenomenon is the following. As agents do not return to their own reference points between visiting successive demands, their efficiency decreases since they are no longer able to effectively separate regions of responsibility. In practice, unless they communicate and concentrate on their Voronoi region, agents are likely to duplicate efforts as they pursue the same demand and effectively behave as a single-vehicle system. Interestingly, this efficiency loss seems to decrease for large  $\lambda$ , and the numerical results suggest that the no-communication policy recovers a similar performance as the sensor-based policy in the heavy load limit. Unfortunately, at this time, there exists no rigorous analysis of the proposed policies for general values of the demand-generation rate.

Another attractive feature of the no-communication-based policy is that the distribution  $\phi$  need not be constant: Indeed, the algorithm will provide a good approximation to a local optimum for the cost function as long as the characteristic time it takes for the demand-generation process to vary significantly is much greater than the relaxation time of the algorithm. In summary, the no-communication policy can be seen as a learning mechanism in which the agents learn the demand-generation process, the ensuing demand spatial distribution, and adapt their own behavior to it.

## Game-Theoretic Perspective for the No-Communication Policy

Interestingly, the no-communication-based policy can be regarded as a learning algorithm in the context of the following game [16]. The service requests are considered as resources and the agents as selfish entities. The resources offer rewards in a continuous fashion, and the agents can collect these rewards by traveling to the resource locations. Every resource offers reward at a rate, which depends on the number of agents present at its location: the reward rate is unity when there is one agent, and it is zero when there are more than one agent. Moreover, the life of the resource ends when more than one agent is present at its location. This setup can be understood to be an extreme form of congestion game, where the resource cannot be shared between agents and is cutoff at the first attempt to share it. The total reward for agent  $i$  from a particular resource is the time difference between its arrival and the arrival of the next vehicle, if  $i$  is the first vehicle to reach the location of the resource, and zero otherwise. The utility function of agent  $i$  is then defined to be the expected value of reward, where the expectation is taken over the location of the next resource. Hence, the goal of every vehicle is to select their reference location to maximize the expected value of the reward from the next resource. In [16], we prove that the median locations, as a choice for reference positions, are an efficient pure Nash equilibrium for this game. Moreover, we prove that, by maximizing their own utility function, the agents also maximize the common global utility function, which is the negative of the average wait time for service requests.

## Conclusions

In this article, we discussed the use of various spatial tessellations to determine, in the framework of partitioning policies, optimal workload share in a mobile robotic network. We provide a succinct summary in Table 1, where the various superscripts have the following meaning: 1) no subscript implies that the corresponding partitioning policy is within a constant factor of the optimal for a uniform distribution 2) superscript \* implies that the partition gives optimal performance for a uniform distribution, 3) \*, $\dagger$  implies that the partition gives optimal performance for a general nonuniform distribution  $\phi$ .

We also proposed efficient and spatially distributed algorithms for achieving some of these tessellations with minimum or no communication between the agents. Because of space limitations, we have not reported results of numerical experiments in this article but provided bibliographic references to publications

containing such results and further details. It is interesting to note that these tessellations appear while considering different variations of the same basic problem (DTRP). It is then natural to investigate the existence of a single objective function, whose optima correspond to the various tessellations under these different variations. The game theory approach seems to be a promising one. We have already

**Table 1. Spatial partitions for load sharing by mobile robotic networks in dynamic environments.**

	Light Load	Heavy Load
First-order holo-nomic vehicle	Median Voronoi tessellations <sup>*,<math>\dagger</math></sup>	Simultaneously equitable partitions <sup>*,<math>\dagger</math></sup>
Dubins vehicle	Median Voronoi tessellations* (sparse robotic network) Dynamic partition (dense robotic network)	Additively weighted equitable partitions

shown the equilibrium status of the median locations in the context of an appropriate game.

Another interesting field of study in the context of spatial tessellations is the emergent group behavior, i.e., territorial versus gregarious behavior arising out of foraging and hunting strategies of animals. The utility function approach again is a promising tool, since it provides a natural framework to study the behavior of animals as selfish agents. This line of research would be in stark contrast to the discipline of biorobotics, where instead of taking inspiration from nature, we would try to identify possible motives explaining the observed behavior of animal populations.

## Keywords

Dynamic vehicle routing, cooperative control, robotic networks, distributed algorithms, spatial partitions.

## References

- [1] M. Schwager, J. McLurkin, and D. Rus, "Distributed coverage control with sensory feedback for networked robots," in *Proc. Robotics: Science and Systems*, Philadelphia, USA, Aug. 2006.
- [2] A. Ahmadzadeh, J. Keller, G. Pappas, A. Jadbabaie, and V. Kumar, "An optimization-based approach to time critical cooperative surveillance and coverage with unmanned aerial vehicles," in *Proc. Int. Symp. Experimental Robotics*, 2006.
- [3] A. Ryan, M. Zennaro, A. Howell, R. Sengupta, and J. K. Hedrick, "An overview of emerging results in cooperative UAV control," in *Proc. IEEE Conf. Decision and Control*, 2004, pp. 602–607.
- [4] A. Okabe, B. Boots, K. Sugihara, and S. Chiu, *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, 2nd ed. (Wiley Series in Probability and Statistics). Chichester, U.K.: Wiley, 2000.
- [5] D. J. Bertsimas and G. J. van Ryzin, "Stochastic and dynamic vehicle routing in the Euclidean plane with multiple capacitated vehicles," *Oper. Res.*, vol. 41, no. 1, pp. 60–76, 1993.
- [6] M. Pavone, E. Frazzoli, and F. Bullo. (2009, Mar.). Distributed and adaptive algorithms for vehicle routing in a stochastic and dynamic environment [Online]. Available: <http://arXiv.org/abs/0903.3624>
- [7] M. Pavone, N. Bisnik, E. Frazzoli, and V. Isler, "A stochastic and dynamic vehicle routing problem with time windows and customer impatience," *ACM/Springer Mobile Netw. Applicat. J.*, vol. 14, no. 3, pp. 350–364, 2009.
- [8] S. L. Smith, M. Pavone, F. Bullo, and E. Frazzoli, "Dynamic vehicle routing with heterogeneous demands," in *Proc. IEEE Conf. Decision and Control*, Cancun, Mexico, Dec. 2008, pp. 1206–1211.
- [9] M. Pavone, S. Smith, F. Bullo, and E. Frazzoli, "Dynamic multi-vehicle routing with multiple classes of demands," in *Proc. Amer. Control Conf.*, St. Louis, Missouri, June 2009.
- [10] J. J. Enright, K. Savla, E. Frazzoli, and F. Bullo, "Stochastic and dynamic routing problems for multiple UAVs," *AIAA J. Guid. Control Dyn.*, to be published.
- [11] S. Bespamyatnikh, D. Kirkpatrick, and J. Snoeyink, "Generalizing ham sandwich cuts to equitable subdivisions," *Discrete Comput. Geom.*, vol. 24, no. 4, pp. 605–622, 2000.
- [12] E. Zemel, "Probabilistic analysis of geometric location problems," *Ann. Oper. Res.*, vol. 1, no. 3, pp. 215–238, Oct. 1984.
- [13] K. Savla and E. Frazzoli, "On endogenous reconfiguration for mobile robotic networks," in *Proc. Workshop Algorithmic Foundations of Robotics (WAFR)*, Guanajuato, Mexico, Dec. 2008.
- [14] J. Buhl, D. J. Sumpter, I. D. Couzin, J. Hale, E. Despland, E. Miller, and S. J. Simpson, "From disorder to order in marching locusts," *Science*, vol. 312, no. 5778, pp. 1402–1406, 2006.
- [15] M. Pavone, A. Arsie, E. Frazzoli, and F. Bullo. (2009, Mar.). Equitable partitioning policies for mobile robotic networks. [Online]. Available: <http://arxiv.org/abs/0903.5267>

- [16] A. Arsie, K. Savla, and E. Frazzoli, "Efficient routing of multiple vehicles with no communications," *IEEE Trans. Automat. Contr.*, to be published.
- [17] J. MacQueen, "Some methods for the classification and analysis of multivariate observations," in *Proc. Fifth Berkeley Symp. Mathematics Statistics and Probability*, L. M. LeCam and J. Neyman, Eds. Berkeley, CA: Univ. of California Press, 1967, pp. 281–297.

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