14.126 Game Theory

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Lecture 1:

Choice under Uncertainty

Binary Relations & Preferences

X: an arbitrary set of alternatives/outcomes/prizes.

A **binary relation** $\succeq$ on X is a subset of $X \times X$. ($x \succeq y$ means $(x, y) \in \succeq$)

$\succeq$ is **complete** if for any $x, y \in X$, $x \succeq y$ or $y \succeq x$.

$\succeq$ is **transitive** if for any $x, y, z \in X$: $x \succeq y$ & $y \succeq z \Rightarrow x \succeq z$.

$\succeq$ is a **preference relation** if it is complete and transitive.

Given $\succeq$, define the binary relations:

Strict preference $\succ$ on X : $\quad x \succ y \Leftrightarrow x \succeq y$ & $y \not\succeq x$

Indifference $\sim$ on X: $\quad x \sim y \Leftrightarrow x \succeq y$ & $y \succeq x$

Descriptive Interpretation: Preference $\equiv$ Choice Behavior.

Expected utility with objective probabilities: the von Neumann-Morgenstern theory

The set of simple lotteries over X:

$$P = \left\{ p \,|\, p\colon X \to [0,1] \text{ with } |\{x \in X | p(x) > 0\}| < \infty \ \& \ \sum_{x \in X} p(x) = 1 \right\}.$$

For any $p, q \in P$ and $\alpha \in [0,1]$, define $\alpha p + (1-\alpha)q \in P$ by:

$$[\alpha p + (1-\alpha)q](x) = \alpha p(x) + (1-\alpha)q(x) \qquad x \in X.$$

δ_x: the degenerate lottery that yields x with probability 1.
We will also write x instead of δ_x.

A function $U : P \to \mathbb{R}$ is **linear** if:
$U(\alpha p + (1-\alpha)q) = \alpha U(p) + (1-\alpha)U(q)$.

Claim 1 *A function* $U : P \to \mathbb{R}$ *is* **linear** *if and only if there exists a function* $u : X \to \mathbb{R}$ *such that:*

$$U(p) = \sum_{x \in X} p(x)u(x) \qquad p \in P.$$

Given a linear U*, the* u *above is uniquely determined.*

"Under what conditions on $\succeq$ is there a utility function u (equivalently a linear U) such that the individual ranks lotteries according to their expected utility $U(p) = \sum_{x \in X} p(x)u(x)$?".

Axiom 3.1. (Preference) $\succeq$ is a preference relation on P.

Axiom 3.2. (Independence) For any $p, q, r \in P$ and $\alpha \in (0,1)$: $p \succ q \Rightarrow \alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$.

Axiom 3.3. (Continuity) For any $p, q, r \in P$, if $p \succ q \succ r$ then $\exists \alpha, \beta \in (0,1)$ s.t. $\alpha p + (1-\alpha)r \succ q \succ \beta p + (1-\beta)r$.

Theorem 1 *(von Neumann-Morgenstern) A binary relation $\succeq$ over P satisfies Axioms 3.1, 3.2, and 3.3 if and only if there is a linear function U: $P \to \mathbb{R}$ such that:*

$$\forall p, q \in P : \qquad p \succeq q \Leftrightarrow U(p) \geq U(q).$$

Moreover if U and V are two linear functions that represent $\succeq$ in the above sense, then there exist $a > 0$ and $b \in \mathbb{R}$ such that $V = aU + b$.

The meaning of the result?

-u is unobservable to the modeler. The only observable is the choice behavior $\succeq$.

-Necessity: gives a behavioral test of EU theory.

-Sufficiency: shows that this is the *maximal* test of EU theory.

-If choices are is consistent with the 3 axioms then u can be identified from $\succeq$ up to a positive affine transformation.

Proof of Sufficiency for finite X:

Suppose that $\succeq$ satisfies Axioms 3.1, 3.2., and 3.3. Then:

Step 1: $p \succ q$ & $0 \leq a < b \leq 1 \Rightarrow bp + (1-b)q \succ ap + (1-a)q$.

Step 2: (Solvability) $p \succeq q \succeq r$ & $p \succ r \Rightarrow$ there exists a *unique* a^* s.t. $q \sim a^* p + (1 - a^*)r$.

Step 3: $p \sim q \Rightarrow ap + (1 - a)r \sim aq + (1 - a)r$.

Step 4: $p \succeq q \Rightarrow ap + (1 - a)r \succeq aq + (1 - a)r$.

Step 5: There are $x^*, x_* \in X$ s.t. $x^* \succeq p \succeq x_*$ for all $p \in P$.

Then define $U(p)$ by:

$$p \sim U(p)x^* + (1 - U(p))x_*.$$

Verify that U is linear. $\square$

EU with subjective probabilities: the Savage theory

"To say that a decision has to be made is to say that one of two or more acts has to be chosen or decided upon. In deciding on an act, account must be taken of the possible states of the world, and also of the consequences implicit in each act for each possible state of the world."
(Savage, 1954)

	STATE	
ACT	Good	Rotten
break into bowl	6 egg omelet	no omelet and 5 good eggs destroyed
break into saucer	6 egg omelet and a saucer to wash	5 egg omelet and a saucer to wash
throw away	5 egg omelet, 1 good egg destroyed	5 egg omelet

The Savage Model

S: A set of states of the world.

Simple Savage acts: $F = \{f \mid f : S \to X \text{ and } |f(S)| < \infty\}$.

$\mathcal{A}$: the set of all subsets of S.

Generically: $f, g, h, f', g', h', \ldots \in F$
$A, B, C, A', B', C', \ldots \in \mathcal{A}$
$x, y, z, x', y'z', \ldots \in X$.

x also denotes the constant act that gives x at every state.

Savage's question:

Let $\succeq$ be a binary relation $\succeq$ over the Savage acts F that represents the individual's choices over acts in F.

"Under what conditions on $\succeq$ is there a subjective probability measure μ on the states of the world and a vNM utility function u on the set of prizes such that the individual evaluates acts according to the expected utility with respect to μ and u, i.e.:

$$f \succeq g \;\Leftrightarrow\; \mathbb{E}_\mu[u \circ f] \geq \mathbb{E}_\mu[u \circ g] \qquad f, g \in F\,\text{?''}$$

Qualitative Probability

Extracting perceived relative likelihoods of events from choices:
$$\begin{pmatrix} \$100 & A \\ \$0 & A^c \end{pmatrix} \text{ or } \begin{pmatrix} \$100 & B \\ \$0 & B^c \end{pmatrix} \quad ?$$

$\mu : \mathcal{A} \to [0,1]$ is a **probability measure** if $\mu(S) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ when $A \cap B = \emptyset$.

A binary relation $\succeq^*$ on $\mathcal{A}$ is a **qualitative probability** if:

(a) $\succeq^*$ is complete and transitive,

(b) $A \succeq^* \emptyset$ for any A,

(c) $S \succ^* \emptyset$, and

(d) If $A \cap C = B \cap C = \emptyset$, then: $A \succeq^* B \Leftrightarrow A \cup C \succeq^* B \cup C$.

Properties of Qualitative Probability

Proposition 1 *Let $\succeq^*$ be a qualitative probability, then the following are satisfied:*

(i) *If $A \supset B$ then $S \succeq^* A \succeq^* B \succeq^* \emptyset$.*

(ii) *If $A \sim^* B$ and $A \cap C = \emptyset$ then $A \cup C \succeq^* B \cup C$.*

(ii') *If $A \succ^* B$ and $A \cap C = \emptyset$ then $A \cup C \succ^* B \cup C$.*

(ii'') *If $A \succeq^* B$ and $A \cap C = \emptyset$ then $A \cup C \succeq^* B \cup C$.*

(iii) *If $A \sim^* B$, $C \sim^* D$, and $A \cap C = \emptyset$ then $A \cup C \succeq^* B \cup D$.*

(iii') *If $A \succ^* B$, $C \succeq^* D$, and $A \cap C = \emptyset$ then $A \cup C \succ^* B \cup D$.*

(iv) *If $A \sim^* B$, $C \sim^* D$, and $A \cap C = B \cap D = \emptyset$ then $A \cup C \sim^* B \cup D$.*