14.126 Game Theory

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Lecture 3:

Choice under Uncertainty (Wrap up)

Simultaneous Action Games

The Allais Paradox

Problem 1:

$$p = 1 \times \$300 \quad \text{versus} \quad q = 0.8 \times \$500 + 0.2 \times \$0$$

Problem 2:

$$p' = 0.5 \times \$300 + 0.5 \times \$0 \quad \text{versus} \quad q' = 0.4 \times \$500 + 0.6 \times \$0$$

Typical choices $p \succ q$ and $q' \succ p'$ are inconsistent with independence:

$$p \succ q \quad \Leftrightarrow \quad p' = 0.5 \times p + 0.5 \times \$0 \succ 0.5 \times q + 0.5 \times \$0 = q'.$$

The Ellsberg Paradox (Single Urn)

An urn contains three balls. One of the balls is RED. The other two are either GREEN or WHITE.

Problem 1:

$$f = \begin{pmatrix} \$100 & G \\ \$0 & W \cup R \end{pmatrix} \quad \text{versus} \quad g = \begin{pmatrix} \$100 & R \\ \$0 & G \cup W \end{pmatrix}$$

Problem 2:

$$f' = \begin{pmatrix} \$100 & G \cup W \\ \$0 & R \end{pmatrix} \quad \text{versus} \quad g' = \begin{pmatrix} \$100 & R \cup W \\ \$0 & G \end{pmatrix}$$

Typical choices $g \succ f$ and $f' \succ g'$ are inconsistent with any subjective probability assessment on $\{G, W, R\}$.

The Ambiguity Aversion interpretation.

Machina and Schmeidler (1992)

Same model as Savage.

A function $V : P \to \mathbb{R}$ satisfies **stochastic dominance** if for any $x, y \in X$, $p \in P$ and $\alpha \in (0, 1)$:

$$V(\alpha\delta_x + (1-\alpha)p) \geq V(\alpha\delta_y + (1-\alpha)p) \;\; \Leftrightarrow \;\; V(\delta_x) \geq V(\delta_y).$$

A function $V : P \to \mathbb{R}$ is **mixture continuous** if for any $p, q, r \in P$ the sets

$$\{\alpha \in [0,1] : V(\alpha p + (1-\alpha)r) \geq V(q))\}$$

$$\{\alpha \in [0,1] : V(\alpha p + (1-\alpha)r) \leq V(q))\}$$

are closed.

Probabilistic Sophistication

Definition 1 $\succeq$ **probabilistically sophisticated** if there exist a probability μ on S and a mixture continuous and stochastic dominance satisfying $V : P \to \mathbb{R}$ s.t.:

$$f \succeq g \Leftrightarrow V(p_f^\mu) \geq V(p_g^\mu).$$

Axiom 5.2.1. (Strong Comparative Probability) For any two disjoint events A and B, $h, h' \in F$ and $x, y, x', y' \in X$ such that $x \succ y$ and $x' \succ y'$:

$$\begin{pmatrix} x & A \\ y & B \\ h & (A \cup B)^c \end{pmatrix} \succeq \begin{pmatrix} x & B \\ y & A \\ h & (A \cup B)^c \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x' & A \\ y' & B \\ h' & (A \cup B)^c \end{pmatrix} \succeq \begin{pmatrix} x' & B \\ y' & A \\ h' & (A \cup B)^c \end{pmatrix}.$$

Theorem 4 (M&S, 1992)

$\succeq$ satisfies 4.2.1–4.2.4 and 5.2.1 iff there exist a non-atomic probability measure μ on S and a non-constant $V : P \to \mathbb{R}$ s.t. $\succeq$ is probabilistically sophisticated w.r.t. μ and V. Moreover, the probability measure μ is unique.

Probabilistic sophistication is consistent with Allais, it is inconsistent with Ellsberg.

Schmeidler (1989)

$\nu\colon \mathcal{A} \to [0,1]$ is a **capacity** (non-additive measure) if $\nu(\emptyset) = 0$, $\nu(S) = 1$, and $\nu(A) \geq \nu(B)$ whenever $B \subset A$.

Choquet Integral:

Let $\varphi\colon S \to \mathbb{R}$ be a simple function

$$\int_S \varphi \, d\nu = \int_{-\infty}^{0} [\nu(\{s : \varphi(s) \geq \alpha\}) - 1] \, d\alpha + \int_0^{+\infty} \nu(\{s : \varphi(s) \geq \alpha\}) \, d\alpha.$$

Simple **Anscombe-Aumann acts**:

$$H = \{h \,|\, h : S \to P \text{ and } |h(S)| < \infty\}.$$

Mixtures of Anscombe-Aumann acts:

$$[\alpha h + (1-\alpha)h'](s) = \alpha h(s) + (1-\alpha)h'(s) \qquad s \in S.$$

Two acts $f, g \in H$ are **comonotonic** if it is never the case that $f(s) \succ f(t)$ and $g(s) \prec g(t)$ for some $s, t \in S$.

Axiom 5.3.1. (Preference) $\succeq$ is a preference over H.

Axiom 5.3.2. (Non-degeneracy) There exist some $h^*, h_* \in H$ with $h^* \succ h_*$.

Axiom 5.3.3. (Comonotonic Independence) For any pairwise comonotonic acts $f, g, h \in H$ and $\alpha \in (0, 1)$:

$$f \succ g \Rightarrow \alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h.$$

Axiom 5.3.4. (vNM-Continuity) For any $f, g, h \in H$, if $f \succ g \succ h$ then there exist $\alpha, \beta \in (0, 1)$ such that:

$$\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h.$$

Axiom 5.3.5. (Monotonicity) For any $f, g \in H$, if $f(s) \succeq g(s)$ for all $s \in S$ then $f \succeq g$.

Theorem 5 *(Schmeidler, 1989)* $\succeq$ *satisfies 5.3.1–5.3.5 iff there is a capacity* $\nu\colon \mathcal{A} \to [0,1]$ *and a non-constant linear function* $U\colon P \to \mathbb{R}$ *s.t.:*

$$f \succeq g \quad \Leftrightarrow \quad \int_S U \circ f \; d\nu \geq \int_S U \circ g \; d\nu \qquad f, g \in H$$

Moreover ν *is unique and* U *is unique up to a positive affine transformation.*

Example: (Choquet-EU & Ellsberg) $U(\$100) = 1$, $U(\$0) = 0$, $\nu(\emptyset) = \nu(G) = \nu(W) = 0$, $\nu(R) = \nu(R \cup G) = \nu(R \cup W) = 1/3$, $\nu(G \cup W) = 2/3$, and $\nu(S) = 1$.

$$\int_S U \circ f \; d\nu = 0, \; \int_S U \circ g \; d\nu = 1/3, \; \int_S U \circ f' \; d\nu = 2/3, \; \int_S U \circ g' \; d\nu = 1/3.$$

Uncertainty Aversion

$\succeq$ exhibits **uncertainty aversion** if:

$$f \succeq g \;\Rightarrow\; \alpha f + (1-\alpha) g \succeq g.$$

Example:

$$f = \begin{pmatrix} \$100 & G \\ \$0 & W \cup R \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} \$100 & W \\ \$0 & G \cup R \end{pmatrix}$$

The 1/2-1/2 mixture of these acts yield:

$$\frac{1}{2}f + \frac{1}{2}h = \begin{pmatrix} \frac{1}{2}\$100 + \frac{1}{2}\$0 & G \cup W \\ \$0 & R \end{pmatrix} \succ f \sim h.$$

The **core** of ν:

$$core(\nu) = \{\mu | \mu \text{ is a probability measure and } \mu \geq \nu\}.$$

ν is **convex** if $\nu(A) + \nu(B) \leq v(A \cup B) + \nu(A \cap B)$.

Theorem 6*(Schmeidler, 1989) Let $\succeq$ be, ν and U be as in Theorem 5. Then the following are equivalent:*

(i) $\succeq$ exhibits uncertainty aversion,

(ii) ν is convex,

(iii) For any simple function $\varphi \colon S \to \mathbb{R}$:

$$\int_S \varphi \, d\nu = \min_{\mu \in core(\nu)} \int_S \varphi \, d\mu.$$

The Maxmin Model

5.3.1-5.3.5 and uncertainty aversion imply:

$$f \succeq g \quad \Leftrightarrow \quad \min_{\mu \in core(\nu)} \int_S U \circ f \; d\mu \geq \min_{\mu \in core(\nu)} \int_S U \circ g \; d\mu$$

Example: ν is convex and

$$core(\nu) = \{\mu \,|\, \mu(G) + \mu(W) = 2/3, \,\&\, \mu(R) = 1/3\}.$$

Rank-dependent Model: (Quiggin, 1982) Intersection of the Choquet-EU model and probabilistic sophistication.

$$\nu = \gamma \circ \mu$$

It is consistent with Allais, inconsistent with Ellsberg.

Simultaneous Action Games:

1. Normal Form Games

(no payoff uncertainty)

2. Bayesian Games
(with payoff uncertainty)

Preliminaries

$\Delta(X)$: the set of **probability distributions** over X.
(*Technical:* If X is infinite, we will assume that X has a topology and set $\Delta(X)$ to be the set of all Borel probability measures)

If $X = \prod_{i \in N} X_i$, then for any $x \in X$ and $i \in N$:

$$X_{-i} = \prod_{j \in N \setminus \{i\}} X_j \quad \& \quad x_{-i} = (x_j)_{j \in N \setminus \{i\}}.$$

An event E is **Mutual Knowledge** (**MK**) if everybody knows E.

E is **Common Knowledge** (**CK**) if everybody knows E, everybody knows that everybody knows E, everybody knows that everybody knows that everybody knows E,...

Normal Form Games

Normal Form Games & Strategies

A **normal form game** is a triplet $(N, A = \prod_{i \in N} A_i, u = (u_i)_{i \in N})$:

- $N = \{1, \ldots, n\}$ is a finite set of players.
- A_i is the set of actions (pure strategies) of player i.
- $u_i \colon A \to \mathbb{R}$ is player i's vNM utility function over action profiles.

$\Delta(A_i)$: **mixed strategies** of player i. (deliberate randomization by i, j's belief about i's play, steady state population proportions, pure strategies in a perturbed game)

A mixed strategy profile can be **independent** ($\sigma = (\sigma_1 \times \ldots \times \sigma_n)$) or **correlated** ($\sigma \in \Delta(A)$.)

Payoffs are extended to mixed strategies by $u_i(\sigma) = \mathbb{E}_\sigma u_i$.

A (normal form) game is **finite** if A is finite.

Best Reply

The game is common knowledge among players.

Player i is **rational** if he maximizes his expected payoff subject to a belief about others' play.

Let $\sigma_{-i} \in \Delta\left(A_{-i}\right)$. a_i^* is a **pure best reply** to σ_{-i} if:

$$\forall a_i \in A_i : \quad u_i(a_i^*, \sigma_{-i}) \geq u_i(a_i, \sigma_{-i}).$$

σ_i^* is a **mixed best reply** of i to σ_{-i} if:

$$\forall \sigma_i \in \Delta(A_i) : \quad u_i(\sigma_i^*, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i}).$$

$B_i^p(\sigma_{-i})$: i's pure best replies to σ_{-i}.

$B_i(\sigma_{-i})$: i's mixed best replies to σ_{-i}.

Note: $B_i(\sigma_{-i}) = \Delta\left(B_i^p(\sigma_{-i})\right)$.

Domination

σ_i' **strictly dominates** σ_i if:

$$\forall \sigma_{-i} \in \Delta(A_{-i}) : \quad u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}).$$

σ_i' **weakly dominates** σ_i if:

$$\forall \sigma_{-i} \in \Delta(A_{-i}) : \quad u_i(\sigma_i', \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i}) \text{ and}$$

$$\exists \sigma_{-i} \in \Delta(A_{-i}) : \quad u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}).$$

Note: Alternative definitions where quantifiers are changed to independently mixed strategy profiles σ_{-i}, or to action profiles a_{-i} are the same.

Theorem: In a finite normal form game, an action a_i^* is never a best reply to any (possibly correlated) conjecture σ_{-i} of i iff a_i^* is strictly dominated to a mixed strategy σ_i.

A strategy may be strictly dominated to a mixed strategy but not to a pure strategy

Consider the row player's payoffs in a 2 person game:

	L	R
U	3	0
M	0	3
D	1	1

Allowing Correlated Conjectures is Crucial

Consider the row player's payoffs in a 3 person game:

L

	l	r
U	1	1
M	1	-1
D	0	0

R

	l	r
U	-1	1
M	1	1
D	0	0

Separation: Suppose C and D are nonempty, convex, disjoint sets in $\mathbb{R}^m$, and C is closed. Then, $\exists r \in \mathbb{R}^m \setminus \{0\}$:

$$\forall x \in C, y \in cl(D): \quad r \cdot x \geq r \cdot y.$$

Proof of Thm: Suppose that a_i^* is not strictly dominated.

Let $A_{-i} = \{a_{-i}^k \,|\, k = 1, ..., m\}$, $u_i(\sigma_i, \cdot) = \left(u_i(\sigma_i, a_{-i}^k)\right)_{k=1}^m$,

$$C = \{u_i(a_i^*, \cdot) - u_i(\sigma_i, \cdot) \mid \sigma_i \in \Delta(A_i)\}.$$

Assumptions above are satisfied for C and $D = (-\infty, 0)^m$. So there is $r \in \mathbb{R}^m \setminus \{0\}$ as in above.

Verify $r \geq 0$. Let $\sigma_{-i}(a_{-i}^k) = r_k / \sum_{l=1}^m r_l$. For any σ_i:

$$u_i(a_i^*, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i}) = \left(\sum_{l=1}^m r_l\right)^{-1} r \cdot [u_i(a_i^*, \cdot) - u_i(\sigma_i, \cdot)] \geq 0.$$