ELEMENTS FOR THE NUMERICAL ANALYSIS
OF WAVE MOTION IN LAYERED MEDIA

by

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ABSTRACT

A technique is developed for the numerical analysis of wave motion in layered media. Semidiscrete particular solutions satisfying inhomogeneous boundary conditions are calculated by the finite element method. These solutions are combined with semidiscrete modes of an appropriate eigenvalue problem also obtained by the finite element method. The boundary conditions corresponding to rigid and rough footings on a layered stratum are treated in detail. Applications are considered which illustrate the use of the technique developed in this work.

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CHAPTER 1

INTRODUCTION

The analysis of wave motion in layered media is of interest to both seismologists and engineers. Applications in the area of seismology focus on the study of propagation of seismic waves in layered media. On the other hand, the engineer is interested in the dynamic behavior of a structure built on a layered soil deposit. The structure may be excited by seismic waves propagating in the layered medium. Alternatively, the excitation may be due to vibrations (for example, machine vibrations) within the structure itself. In fact, this leads to a problem which has been the subject of intensive investigation in the last two decades, namely, the analysis of forced vibrations of foundations. The analytical solutions which have been calculated (for example, [5, 12, 13, 22]) are applicable to highly idealized situations. For the analysis of problems which arise in practice one must rely upon numerical methods. However, it must be noted that analytical solutions apart from being of considerable theoretical interest provide the means for checking numerical solutions.

A numerical method for the analysis of wave motion in layered media which accounts for the radiation into the far field was presented by Lysmer and Waas[16] and Waas[23]. Time-harmonic waves in plane strain or antiplane shear as well as axisymmetric waves in a layered stratum were considered. The method is based on the calculation (by the finite element method) of semidiscrete solutions (modes) satisfying homogeneous boundary conditions. The solution in the far field is written as a linear combination of such semidiscrete modes. It is combined with a fully discrete solution (obtained using the finite element method) in the part of the region where inhomogeneous boundary conditions are given. The method was extended by Kausel[6] to nonaxisymmetric waves in axisymmetric regions of a layered stratum. We note that in these developments the semidiscrete solutions were used in regions of infinite extent. An extension to wave motion in a finite region of the stratum with homogeneous boundary conditions was presented by Kausel and Roësset[9]. In this work we consider a further extension. We develop a technique for the analysis of wave motion in finite regions of a layered medium with inhomogeneous boundary conditions. We calculate semidiscrete particular solutions satisfying the inhomogeneous boundary conditions and then combine these solutions with semidiscrete modes satisfying the corresponding homogeneous boundary conditions. We show that one can find semidiscrete particular solutions for a variety of inhomogeneous boundary conditions. The resulting improvements with respect to computational effort are significant.

In Chapter 2 we review the previous work. In Chapter 3 we develop in detail some plane elements. We consider the boundary conditions corresponding to a rigid and rough strip footing. The development of other plane elements is outlined. In Chapter 4 we present some axisymmetric elements. We focus on boundary conditions corresponding to rigid and rough circular and ring footings. Other axisymmetric elements are also considered. In Chapter 5 we discuss some applications which illustrate the use of the elements developed in this work. The conclusions of this work as well as some ideas on possible extensions are summarized in Chapter 6.

CHAPTER 2

REVIEW OF PREVIOUS WORK

The elements to be developed in this study share common underlying techniques with those described in the works of Lysmer [14], Lysmer and Drake [15], Lysmer and Waas [16], Waas [23], Kausel [6], Kausel, Roësset and Waas [10], Kausel and Roësset [9]. In this chapter, we review rather briefly those earlier studies. Special attention is given to the derivation of the eigenvalue problem which is essential to the development of the elements.

2.1 SOME PRELIMINARIES

The techniques developed in the references cited above are applicable to the analysis of time-harmonic wave motion in an isotropic linearly viscoelastic layered stratum. In systems of rectangular Cartesian coordinates (x, y, z), or cylindrical coordinates (r, θ, z), the stratum is understood as the region $0 \leq z \leq h$, i.e., the region between the parallel planes $z = 0$ and $z = h$ (h denotes the depth of the stratum). The boundaries at $z = 0$ and $z = h$ will be referred to as the surface and the base of the stratum respectively. Each layer of the stratum is assumed homogeneous. Interfaces of layers are planes parallel to the surface and the base of the stratum. If there are M layers in the stratum, layer j, $1 \leq j \leq M$, is the region between the planes $z = z_j$ and $z = z_{j+1}$ with

$$0 = z_1 < z_2 < z_3 < \ldots < z_j < z_{j+1} < \ldots < z_{M+1} = h \;.$$

The depth of layer j will be denoted by h_j. It is given by

$$h_j = z_{j+1} - z_j \quad .$$

Layers are assumed to be "bonded" at interfaces, i.e., the stresses acting on the interfaces as well as all displacement components are required to be continuous there.

The mass density of layer j will be denoted by ρ_j, and the Lamé moduli by λ_j and G_j. Poisson's ratio ν_j is then given by

$$\nu_j = \frac{\lambda_j}{2(\lambda_j + G_j)} \quad .$$

The Lamé moduli λ, G are real for a linearly elastic solid. However, they must be specified as complex-valued functions of the frequency ω for a linearly viscoelastic solid. Poisson's ratio ν is real if the viscosity of the material is identical in bulk (volumetric) and shear deformations. It is well known that the differential equations which must be satisfied by a time-harmonic displacement field in a linearly visco-elastic solid are formally the same as those in a linearly elastic material. The moduli appearing in the coefficients of the equations for the linearly viscoelastic material are, however, complex numbers. The methods of analysis we are considering here may be applied to both cases with the same ease. The presence of complex coefficients results, of course, in a slightly greater number of operations using complex arithmetic. However, the dissipative behavior of a viscoelastic material excludes resonance and thus computational troubles are avoided. In this work, dissipative behavior of the hysteretic type identical in bulk and shear straining will be assumed. In this case the complex Lamé moduli are given by

$$\lambda^c = \lambda(1 + 2\beta i) \quad , \quad G^c = G(1 + 2\beta i) \quad .$$

λ and G are the moduli of the corresponding linearly elastic solid. β (real number) is the fraction of critical damping. For dissipative behavior β must be positive for $\omega > 0$ (all field quantities varying in time as $\exp(i\omega t)$). Alternatively, the above expressions for the moduli may be rewritten as

$$\lambda^c = \lambda(1 + 2\beta \operatorname{sgn}(\omega) i) \quad , \quad G^c = G(1 + 2\beta \operatorname{sgn}(\omega) i)$$

with $\beta > 0$ (sgn is the sign function). Since damping is assumed to be of the hysteretic type, β is a constant (it is independent of the frequency). Some more details are given by Waas [23] and Kausel [6]. The fraction of critical damping of layer j will be denoted by β_j. Since, as already stated, the equations we shall be dealing with are formally the same for linearly elastic and linearly viscoelastic materials, derivations need be given for, say, linearly elastic solids only. If the corresponding results are desired for a linearly viscoelastic material, then, simply, the Lamé moduli λ and G must be replaced by the complex counterparts λ^c and G^c.

2.2 A STRATUM IN PLANE STRAIN

We consider time-harmonic vibrations of a stratum in plane strain. The displacement vector, in a system of rectangular Cartesian coordinates (x, y, z) is

$$\begin{bmatrix} u(x,z) \\ 0 \\ w(x,z) \end{bmatrix} \exp(i\omega t) \quad ,$$

i.e., particle motion is in the x-z plane and independent of the y coordinate. ω is the frequency of time-harmonic vibrations. In layer j, $1 \leq j \leq M$, the governing differential equations are

$$(\lambda_j + 2G_j)\frac{\partial^2 u}{\partial x^2} + \lambda_j \frac{\partial^2 w}{\partial x \partial z} + G_j\left[\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right] + \rho_j \omega^2 u = 0 \qquad (2.1a)$$

$$(\lambda_j + 2G_j)\frac{\partial^2 w}{\partial z^2} + \lambda_j \frac{\partial^2 u}{\partial x \partial z} + G_j\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial z}\right] + \rho_j \omega^2 w = 0 \qquad (2.1b)$$

The amplitudes of the stresses are given by

$$\sigma_x = (\lambda_j + 2G_j)\frac{\partial u}{\partial x} + \lambda_j \frac{\partial w}{\partial z} \qquad (2.2a)$$

$$\tau_{xz} = G_j\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] \qquad (2.2b)$$

$$\sigma_z = (\lambda_j + 2G_j)\frac{\partial w}{\partial z} + \lambda_j \frac{\partial u}{\partial x} . \tag{2.2c}$$

The amplitudes u, w, σ_z, τ_{xz} must be continuous at the interfaces of the layers, i.e., at $z = z_j$, $2 \leq j \leq M$. The conditions on σ_z, τ_{xz} may be written as

$$(\lambda_{j-1} + 2G_{j-1})\frac{\partial w}{\partial z}\bigg|_{z=z_j^-} + \lambda_{j-1}\frac{\partial u}{\partial x}\bigg|_{z=z_j^-} = (\lambda_j + 2G_j)\frac{\partial w}{\partial z}\bigg|_{z=z_j^+} + \lambda_j \frac{\partial u}{\partial x}\bigg|_{z=z_j^+} \tag{2.3a}$$

$$G_{j-1}\left[\frac{\partial w}{\partial x}\bigg|_{z=z_j^-} + \frac{\partial u}{\partial z}\bigg|_{z=z_j^-}\right] = G_j\left[\frac{\partial w}{\partial x}\bigg|_{z=z_j^+} + \frac{\partial u}{\partial z}\bigg|_{z=z_j^+}\right] . \tag{2.3b}$$

Boundary conditions must be given on the surface and the base of the stratum, i.e., at $z = z_1 = 0$ and $z = z_{M+1} = h$. If the stratum is understood as an idealization of a soil deposit, the surface is assumed free and the base fixed. Then the boundary conditions are

$$(\lambda_1 + 2G_1)\frac{\partial w}{\partial z}\bigg|_{z=0} + \lambda_1 \frac{\partial u}{\partial x}\bigg|_{z=0} = 0 \tag{2.4a}$$

$$G_1\left[\frac{\partial w}{\partial x}\bigg|_{z=0} + \frac{\partial u}{\partial z}\bigg|_{z=0}\right] = 0 \tag{2.4b}$$

$$u(x,h) = 0 \tag{2.4c}$$

$$w(x,h) = 0 . \tag{2.4d}$$

The displacement field in a stratum with homogeneous boundary conditions such as (2.4a, b, c, d) may be written as the superposition of modes obtained by separating the variables x and z and then solving an eigenvalue problem in the interval $0 \le z \le h$. Let us look for solutions of the governing differential equations (2.1a,b) of the form

$$u(x,z) = U(z)f(x) \tag{2.5a}$$

$$w(x,z) = W(z)f(x) \ . \tag{2.5b}$$

Substituting (2.5a,b) into (2.1a,b), we obtain

$$\frac{d^2f}{dx^2} + k^2 f = 0 \quad , \tag{2.6}$$

in which k is a constant. We also find that, in layer j, U and W must satisfy the equations

$$k^2(\lambda_j + 2G_j)U + ik(\lambda_j + G_j)\frac{dW}{dz} - G_j\frac{d^2U}{dz^2} - \omega^2\rho_j U = 0 \tag{2.7a}$$

$$k^2G_jW + ik(\lambda_j + G_j)\frac{dU}{dz} - (\lambda_j + 2G_j)\frac{d^2W}{dz^2} - \omega^2\rho_j W = 0 \ . \tag{2.7b}$$

From equation (2.6) it follows that the modes are given by

$$u(x,z) = U(z) \ \exp(-ikx)$$

$$w(x,z) = W(z) \ \exp(-ikx) \quad .$$

For a mode the conditions (2.3a,b) become

$$(\lambda_{j-1} + 2G_{j-1})\frac{dW}{dz}\bigg|_{z=z_j^-} - ik\lambda_{j-1}U(z_j^-) = (\lambda_j + 2G_j)\frac{dW}{dz}\bigg|_{z=z_j^+} - ik\lambda_j U(z_j^+) \tag{2.7c}$$

$$G_{j-1}\left[-ikW(z_j^-) + \frac{dU}{dz}\bigg|_{z=z_j^-}\right] = G_j\left[-ikW(z_j^+) + \frac{dU}{dz}\bigg|_{z=z_j^+}\right] \tag{2.7d}$$

and the boundary conditions (2.4a,b,c,d)

$$(\lambda_1 + 2G_1) \left.\frac{dW}{dz}\right|_{z=0} - ik\lambda_1 U(0) = 0 \qquad (2.7e)$$

$$- ikW(0) + \left.\frac{dU}{dz}\right|_{z=0} = 0 \qquad (2.7f)$$

$$U(h) = 0 \qquad (2.7g)$$

$$W(h) = 0 \quad . \qquad (2.7h)$$

The differential equations (2.7a,b), together with the conditions (2.7c, d, e, f, g, h) define an eigenvalue problem in the interval $0 \leq z \leq h$. The values of k for which non-trivial solutions U,W (eigenfunctions) exist are the eigenvalues of the problem.

Let us write the amplitudes u, w as

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \qquad (2.8a)$$

$$w = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \quad . \qquad (2.8b)$$

ϕ and ψ are potentials satisfying, in layer j

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = - \frac{\omega^2}{[C_L^j]^2} \phi \qquad (2.9a)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = - \frac{\omega^2}{[C_T^j]^2} \psi \quad . \qquad (2.9b)$$

C_T^j denotes the velocity of transverse (shear, rotational, equivoluminal) waves in layer j and is given by

$$c_T^j = \left[\frac{G_j}{\rho_j}\right]^{1/2} .$$

c_L^j denotes the velocity of longitudinal (pressure, irrotational, dilatational) waves in layer j and is given by

$$c_L^j = \left[\frac{\lambda_j + 2G_j}{\rho_j}\right]^{1/2} .$$

We look for solutions of (2.9a,b) of the form

$$\phi(x,z) = \Phi(z) \exp(-ikx) \tag{2.10a}$$

$$\psi(x,z) = \Psi(z) \exp(-ikx) . \tag{2.10b}$$

Substituting (2.10a,b) into (2.9a,b) we find

$$\frac{d^2\Phi}{dz^2} + q_j^2\Phi = 0 \tag{2.11a}$$

$$\frac{d^2\Psi}{dz^2} + p_j^2\Psi = 0 . \tag{2.11b}$$

q_j, p_j are given by

$$q_j^2 + k^2 = \frac{\omega^2}{[c_L^j]^2} \tag{2.12a}$$

$$p_j^2 + k^2 = \frac{\omega^2}{[c_T^j]^2} . \tag{2.12b}$$

Thus we obtain

$$\Phi(z) = A_1^j \cos(q_j z) + A_2^j \sin(q_j z) \tag{2.13a}$$

$$\Psi(z) = A_3^j \cos(p_j z) + A_4^j \sin(p_j z) . \tag{2.13b}$$

From equations (2.8a,b) we find

$$U = -ik\Phi - \frac{d\Psi}{dz} \tag{2.14a}$$

$$W = \frac{d\Phi}{dz} - ik\Psi \quad . \tag{2.14b}$$

Clearly, from (2.13a,b) and (2.14a,b), the eigenfunctions U,W may be expressed in terms of elementary functions. The coefficients A_1^j, A_2^j, A_3^j, A_4^j may be obtained in terms of $U(z_j)$, $W(z_j)$, $U(z_{j+1})$, $W(z_{j+1})$. Thus the eigenfunctions U,W are completely specified by the 2M values $U(z_j)$, $W(z_j)$, $1 \leq j \leq M$, since $U(h) = W(h) = 0$ by the boundary conditions (2.7g,h). These must be such that the boundary conditions (2.7e,f) as well as the conditions (2.7c,d) at interfaces of the layers be satisfied. Thus we obtain a system of 2M homogeneous linear equations for the 2M values $U(z_j)$, $W(z_j)$, $1 \leq j \leq M$. The matrix of coefficients involves transcendental functions. Nontrivial solutions are possible for those values of k which render the matrix singular. Thus the frequency equation or dispersion relation is obtained by equating the determinant of the matrix to zero. However, k appears in the argument of transcendental functions. Finding roots of such an equation is, in general, a formidable task. Search methods are typically used. The set of eigenvalues (wave numbers) k for a given frequency ω is infinite but countable [2,17]. Values of the frequency for which k=0 is an eigenvalue are usually referred to as cut-off frequencies or natural frequencies. The corresponding modes are waves traveling up and down in the layers. Calculation of the cut-off frequencies is, in general, difficult. If the stratum is homogeneous, i.e., there is only one layer (M=1), the cut-off frequencies are easily found. They are given by

$$\text{(i)} \quad \frac{\omega h}{C_L} = (2n-1)\,\frac{\pi}{2} \quad n = 1,2,\ldots \quad , \tag{2.15a}$$

the modes being longitudinal waves with u(x,z) = 0, and

$$\text{(ii)} \quad \frac{\omega h}{C_T} = (2n-1)\,\frac{\pi}{2} \quad n = 1,2,\ldots \quad , \tag{2.15b}$$

the modes being transverse waves with w(x,z) = 0.

The displacement vector for a mode of vibration is

$$\begin{bmatrix} U(z) \\ 0 \\ W(z) \end{bmatrix} \exp(i\omega t - ikx) \quad . \tag{2.16}$$

We assume that ω is not a cut-off frequency. Then $k \neq 0$. Suppose that Im[k] = 0, i.e., the wave number k is real. The phase of the wave is propagating in the positive x-direction, if k is positive, or in the negative x-direction, if k is negative. Waas [23] shows that phase propagation and energy propagation are not always in the same direction for traveling waves in plane strain. If a region of finite extent is considered (for example, $x_1 \leq x \leq x_2$), the mode is admissible. If the region is of infinite extent (for example $x \geq 0$), the mode is admissible only if it satisfies the radiation condition. For example, energy radiation in the region $x \geq 0$ requires that energy propagation be in the positive x-direction [23]. Let us now assume that $Im[k] \neq 0$. The mode is then an evanescent wave. If $Re[k] \neq 0$, there is a propagating phase, while if $Re[k] = 0$, the mode is a standing wave. Again, the mode is admissible in a region of finite extent. However, if the region is of infinite extent, the mode is admissible only if it satisfies the boundedness condition. For example, if the region $x \geq 0$ is considered, the boundedness condition requires that waves be bounded for arbitrarily large $x > 0$, i.e., $Im[k] < 0$.

Let us consider now the derivation of an algebraic eigenvalue problem for the calculation of approximate wave numbers k and eigenfunctions U and W. Following Waas [23], we use the finite element method. Each layer of the stratum is divided into sublayers the depth of which is much smaller than the minimum wavelength of traveling waves in the layer, i.e., the wavelength of transverse waves. It is then reasonable to seek approximate eigenfunctions which are linear functions of z in each sublayer. Let us assume that the stratum is divided into N sublayers. Finite elements are the line segments $[z_j, z_{j+1}]$, $1 \leq j \leq N$, corresponding to these N sublayers. The eigenfunctions U and W are the amplitudes of the displacements at $x = 0$. Let δU and δW (functions of z) be the amplitudes of virtual displacements at $x = 0$. After multiplying the left-hand sides of equations (2.7a) and (2.7b) by δU and δW respectively and adding them, we obtain, for sublayer j:

$$k^2(\lambda_j + 2G_j) \int_{z_j}^{z_{j+1}} U\,\delta U dz + k^2 G_j \int_{z_j}^{z_{j+1}} W\,\delta W dz$$

$$+ ik\lambda_j \int_{z_j}^{z_{j+1}} \frac{dW}{dz}\,\delta U dz + ikG_j \int_{z_j}^{z_{j+1}} \frac{dU}{dz}\,\delta W dz$$

$$- G_j \int_{z_j}^{z_{j+1}} \frac{d}{dz}[\frac{dU}{dz} - ikW]\,\delta U dz - \int_{z_j}^{z_{j+1}} \frac{d}{dz}[(\lambda_j+2G_j)\frac{dW}{dz} - ik\lambda_j U]\,\delta W dz$$

$$- \omega^2\rho_j \int_{z_j}^{z_{j+1}} U\delta U dz - \omega^2\rho_j \int_{z_j}^{z_{j+1}} W\delta W dz = 0 \quad . \qquad (2.17)$$

Integrating by parts and rearranging the terms, we find:

$$k^2(\lambda_j + 2G_j) \int_{z_j}^{z_{j+1}} U\delta U dz + k^2 G_j \int_{z_j}^{z_{j+1}} W\delta W dz$$

$$+ ik\lambda_j \int_{z_j}^{z_{j+1}} \left[\frac{dW}{dz}\,\delta U - U\,\frac{d}{dz}\,[\delta W]\right] dz$$

$$+ ikG_j \int_{z_j}^{z_{j+1}} \left[\frac{dU}{dz}\,\delta W - W\,\frac{d}{dz}\,[\delta U]\right] dz$$

$$+ G_j \int_{z_j}^{z_{j+1}} \frac{dU}{dz}\frac{d}{dz}\,[\delta U]\,dz + (\lambda_j + 2G_j) \int_{z_j}^{z_{j+1}} \frac{dW}{dz}\frac{d}{dz}\,[\delta W] dz$$

$$- \omega^2\rho_j \int_{z_j}^{z_{j+1}} U\delta U dz - \omega^2\rho_j \int_{z_j}^{z_{j+1}} W\delta W dz =$$

$$= G_j \left[\frac{dU}{dz} - ikW\right]\delta U \Bigg|_{z_j}^{z_{j+1}} + \left[(\lambda_j + 2G_j)\frac{dW}{dz} - ik\lambda_j U\right]\,\delta W \Bigg|_{z_j}^{z_{j+1}} \quad . \qquad (2.18)$$

U, W, δU, δW vary linearly in the sublayer:

$$U(z) = U_j\, N_j^j(z) + U_{j+1}\, N_{j+1}^j(z) \tag{2.19a}$$

$$W(z) = W_j\, N_j^j(z) + W_{j+1}\, N_{j+1}^j(z) \tag{2.19b}$$

$$\delta U(z) = \delta U_j\, N_j^j(z) + \delta U_{j+1}\, N_{j+1}^j(z) \tag{2.20a}$$

$$\delta W(z) = \delta W_j\, N_j^j(z) + \delta W_{j+1}\, N_{j+1}^j(z) \quad , \tag{2.20b}$$

in which N_j^j, N_{j+1}^j are the shape functions corresponding to nodes j and j+1 respectively:

$$N_j^j(z) = \frac{z_{j+1} - z}{h_j} \tag{2.21a}$$

$$z_j \le z \le z_{j+1}$$

$$N_{j+1}^j(z) = \frac{z - z_j}{h_j} \tag{2.21b}$$

(the superscript indicates the sublayer number), and

$$U_\ell = U(z_\ell) \quad , \quad W_\ell = W(z_\ell) \qquad \ell = j,\ j+1$$

$$\delta U_\ell = \delta U(z_\ell) \quad , \quad \delta W_\ell = \delta W(z_\ell) \quad .$$

Substituting (2.19a,b) and (2.20a,b) into equation (2.18), which holds for arbitrary δU,δW we obtain

$$[k^2 \underset{\sim}{A}^j + ik\underset{\sim}{B}^j + \underset{\sim}{G}^j - \omega^2 \underset{\sim}{M}^j] \begin{bmatrix} U_j \\ W_j \\ U_{j+1} \\ W_{j+1} \end{bmatrix} = \begin{bmatrix} -\tau_j \\ -\sigma_j \\ \tau_{j+1} \\ \sigma_{j+1} \end{bmatrix} \tag{2.22}$$

with
$$\sigma_\ell = \sigma_z \Big|_{\substack{x=0 \\ z=z_\ell}} \qquad \ell = j,\ j+1$$
$$\tau_\ell = \tau_{xz} \Big|_{\substack{x=0 \\ z=z_\ell}} \quad .$$

$\underset{\sim}{A}^j$, $\underset{\sim}{G}^j$, $\underset{\sim}{M}^j$ are 4 x 4 symmetric matrices:

$$\underset{\sim}{A}^j = \frac{1}{6} h_j \begin{bmatrix} 2(\lambda_j + 2G_j) & 0 & \lambda_j + 2G_j & 0 \\ 0 & 2G_j & 0 & G_j \\ \lambda_j + 2G_j & 0 & 2(\lambda_j + 2G_j) & 0 \\ 0 & G_j & 0 & 2G_j \end{bmatrix} \qquad (2.23a)$$

$$\underset{\sim}{G}^j = \frac{1}{h_j} \begin{bmatrix} G_j & 0 & -G_j & 0 \\ 0 & \lambda_j + 2G_j & 0 & -(\lambda_j + 2G_j) \\ -G_j & 0 & G_j & 0 \\ 0 & -(\lambda_j + 2G_j) & 0 & \lambda_j + 2G_j \end{bmatrix} \qquad (2.23b)$$

$$\underset{\sim}{M}^j = \rho_j h_j \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{6} & 0 & \frac{1}{6} \end{bmatrix} \quad . \qquad (2.23c)$$

$\underset{\sim}{B}^j$ is a 4 x 4 antisymmetric matrix:

$$\underset{\sim}{B}^j = \frac{1}{2}\begin{bmatrix} 0 & -(\lambda_j - G_j) & 0 & \lambda_j + G_j \\ \lambda_j - G_j & 0 & \lambda_j + G_j & 0 \\ 0 & -(\lambda_j + G_j) & 0 & \lambda_j - G_j \\ -(\lambda_j + G_j) & 0 & -(\lambda_j - G_j) & 0 \end{bmatrix} \quad (2.23d)$$

Since σ_z and τ_{xz} are continuous at $z = z_j$, $2 \leq j \leq N$, we have

$$\sigma_j = \left[(\lambda_{j-1} + 2G_{j-1}) \frac{dW}{dz} - ik\lambda_{j-1}U\right]_{z=z_j^-} =$$

$$= \left[(\lambda_j + 2G_j) \frac{dW}{dz} - ik\lambda_j U\right]_{z=z_j^+} , \quad (2.24a)$$

$$\tau_j = G_{j-1} \left[\frac{dU}{dz} - ikW\right]_{z=z_j^-} = G_j \left[\frac{dU}{dz} - ikW\right]_{z=z_j^+} . \quad (2.24b)$$

Assembling the matrices for the region $0 \leq z \leq h$ and using the conditions (2.24a,b) we obtain

$$[k^2\underset{\sim}{A} + ik\underset{\sim}{B} + \underset{\sim}{G} - \omega^2\underset{\sim}{M}]\underset{\sim}{\Delta} = \underset{\sim}{F} . \quad (2.25)$$

$\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{G}$, $\underset{\sim}{M}$ are (2N+2) x (2N+2) matrices assembled from $\underset{\sim}{A}^j$, $\underset{\sim}{B}^j$, $\underset{\sim}{G}^j$, $\underset{\sim}{M}^j$ respectively. $\underset{\sim}{\Delta}$, $\underset{\sim}{F}$ are (2N+2) - vectors:

$$\left.\begin{aligned} \Delta_{2j-1} &= U_j \\ \Delta_{2j} &= W_j \end{aligned}\right\} \quad 1 \leq j \leq N+1$$

$$F_1 = -\tau_1 = -\tau_{xz}\Big|_{\substack{x=0\\z=0}} \qquad F_2 = -\sigma_1 = -\sigma_z\Big|_{\substack{x=0\\z=0}}$$

$$F_{2N+1} = \tau_{N+1} = \tau_{xz}\Big|_{\substack{x=0\\z=h}} \qquad F_{2N+2} = \sigma_{N+1} = \sigma_z\Big|_{\substack{x=0\\z=h}}$$

$$F_{2j-1} = F_{2j} = 0 \quad , \qquad 2 \leq j \leq N \quad .$$

Thus F_1 and F_{2N+1} are the amplitudes of the shear tractions, at x = 0, on the surface and the base of the stratum, respectively. Similarly, F_2 and F_{2N+2} are the amplitudes of the normal tractions, at x = 0, on the surface and the base of the stratum respectively. For the boundary conditions (2.7e, f, g, h), i.e., a free surface and a fixed base, the corresponding algebraic eigenvalue problem is obtained by deleting the last two rows and the last two columns of the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{G}$, $\tilde{M}$ as well as the last two components of the vectors $\tilde{\Delta}$, $\tilde{F}$, while F_1 and F_2 are set equal to zero:

$$[k^2\tilde{A} + ik\tilde{B} + \tilde{G} - \omega^2\tilde{M}]\,\tilde{\Delta} = \tilde{0} \quad . \qquad (2.26)$$

$\tilde{A}$, $\tilde{B}$, $\tilde{G}$, $\tilde{M}$, $\tilde{\Delta}$ differ from those appearing in equation (2.25) by the modifications stated above. $\tilde{0}$ denotes the zero 2N-vector. The eigenvalues k are roots of a polynomial of degree 4N:

$$\det\,[k^2\tilde{A} + ik\tilde{B} + \tilde{G} - \omega^2\tilde{M}] = 0 \quad . \qquad (2.27)$$

Waas [23] discusses this problem in considerable detail. For the purposes of this review, it suffices to note that if k is an eigenvalue with eigenvector $\underset{\sim}{\Delta}$, then $-k$ is another eigenvalue with eigenvector $\underset{\sim}{\bar{\Delta}}$ obtained from $\underset{\sim}{\Delta}$ by changing the sign of all even-numbered components (or, all odd-numbered components, since $-\underset{\sim}{\Delta}$ is also an eigenvector with eigenvalue k):

$$\underset{\sim}{\bar{\Delta}} = \underset{\sim}{T}\,\underset{\sim}{\Delta} \quad . \tag{2.28}$$

$\underset{\sim}{T}$ is a diagonal matrix (2N x 2N):

$$\begin{aligned} T_{2j-1,2j-1} &= 1 \\ T_{2j,2j} &= -1 \end{aligned} \qquad 1 \leq j \leq N \tag{2.29}$$

It is convenient to choose those 2N wave numbers k_j, $1 \leq j \leq 2N$, and the associated 2N linearly independent eigenvectors $\underset{\sim}{\Delta}^j$ for which the corresponding modes are such that the amplitudes of the displacements decay for large $x > 0$ (this applies to complex wave numbers) or energy propagates in the positive x-direction (this applies to real wave numbers). Thus we construct the diagonal matrix $\underset{\sim}{K}$ (2N x 2N):

$$\underset{\sim}{K} = \text{diag}\,[k_j] \quad , \tag{2.30}$$

and the modal matrix $\underset{\sim}{X}$ (2N x 2N), the columns of which are the eigenvectors $\underset{\sim}{\Delta}^j$:

$$\underset{\sim}{X} = [\underset{\sim}{\Delta}^1, \underset{\sim}{\Delta}^2, \ldots , \underset{\sim}{\Delta}^{2N}] \quad . \tag{2.31}$$

If dissipation is introduced, then all wave numbers are complex. The criterion for choosing the diagonal entries of $\underset{\sim}{K}$ is, therefore, that $\text{Im}[k_j] < 0$, $1 \leq j \leq 2N$. In the absence of dissipation, energy propagation must be considered for modes with real wave numbers. This is discussed in detail by Waas [23].

Let us now calculate consistent nodal forces acting on the region $x \geq 0$, at the section $x = 0$, for a mode with wave number k and eigenvector $\underset{\sim}{\Delta}$. We consider sublayer j. The forces are obtained by integrating the tractions, at $x = 0$, multiplied by the shape functions along $z_j \leq z \leq z_{j+1}$:

Forces in the x-direction

$$\text{(node j)} \qquad P^j_{x,j} = \int_{z_j}^{z_{j+1}} -\sigma_x\Big|_{x=0} N^j_j(z)dz \quad , \tag{2.32a}$$

$$\text{(node j+1)} \qquad P^j_{x,j+1} = \int_{z_j}^{z_{j+1}} -\sigma_x\Big|_{x=0} N^j_{j+1}(z)dz \; . \tag{2.32b}$$

Forces in the z-direction

$$\text{(node j)} \qquad P^j_{z,j} = \int_{z_j}^{z_{j+1}} - \tau_{xz}\Big|_{x=0} N^j_j(z)dz \tag{2.33a}$$

$$\text{(node j+1)} \qquad P^j_{z,j+1} = \int_{z_j}^{z_{j+1}} - \tau_{xz}\Big|_{x=0} N^j_{j+1}(z)dz \tag{2.33b}$$

(again, the superscript indicates the sublayer number). We have

$$\sigma_x\Big|_{x=0} = -ik(\lambda_j + 2G_j)U + \lambda_j \frac{dW}{dz} \tag{2.34a}$$

$$\tau_{xz}\Big|_{x=0} = G_j \; [-ikW + \frac{dU}{dz}] \quad . \tag{2.34b}$$

Substituting (2.19a,b) into (2.34a,b) and the resulting expressions into (2.32a,b), (2.33a,b), we obtain

$$[\, ik\underset{\sim}{A}^j + \underset{\sim}{D}^j] \begin{bmatrix} U_j \\ W_j \\ U_{j+1} \\ W_{j+1} \end{bmatrix} = \begin{bmatrix} P^j_{x,j} \\ P^j_{z,j} \\ P^j_{x,j+1} \\ P^j_{z,j+1} \end{bmatrix} . \tag{2.35}$$

$\underset{\sim}{A}^j$ is the same as in (2.22). $\underset{\sim}{D}^j$ is a 4 x 4 matrix:

$$\underset{\sim}{D}^j = \frac{1}{2} \begin{bmatrix} 0 & \lambda_j & 0 & -\lambda_j \\ G_j & 0 & -G_j & 0 \\ 0 & \lambda_j & 0 & -\lambda_j \\ G_j & 0 & -G_j & 0 \end{bmatrix} . \tag{2.36}$$

Assembling the matrices for the region $0 \le z \le h$, we obtain

$$\underset{\sim}{P} = [ik\underset{\sim}{A} + \underset{\sim}{D}]\, \underset{\sim}{\Delta} . \tag{2.37}$$

$\underset{\sim}{A}$ is the same as in (2.26). $\underset{\sim}{D}$ is assembled from $\underset{\sim}{D}^j$. $\underset{\sim}{P}$ is a 2N-vector with components:

$$P_1 = P^1_{x,1} \quad , \; P_2 = P^1_{z,1} \quad ,$$

$$P_{2j-1} = P^{j-1}_{x,j} + P^j_{x,j} \quad , \qquad 2 \le j \le N$$

$$P_{2j} = P^{j-1}_{z,j} + P^j_{z,j} \quad .$$

Let us now obtain the dynamic stiffness matrix for the region $x \ge 0$. Let $\underset{\sim}{U}$ be the displacement vector:

$$\underset{\sim}{U}_{2j-1} = u(0,z_j)$$

$$1 \leq j \leq N$$

$$\underset{\sim}{U}_{2j} = w(0,z_j).$$

It may be written as a linear combination of the eigenvectors:

$$\underset{\sim}{U} = \underset{\sim}{X}\ \underset{\sim}{\Gamma} \quad . \tag{2.38}$$

$\underset{\sim}{X}$ is the modal matrix given by (2.31). $\underset{\sim}{\Gamma}$ is a 2N-vector of participation factors. The force vector corresponding to a mode with wave number k_j and eigenvector $\underset{\sim}{\Delta}^j$ is according to (2.37):

$$\underset{\sim}{P}^j = [ik_j\underset{\sim}{A} + \underset{\sim}{D}]\ \underset{\sim}{\Delta}^j \quad . \tag{2.39}$$

Thus, we obtain that the force vector corresponding to $\underset{\sim}{U}$ is given by:

$$\underset{\sim}{F} = \sum_{j=1}^{2N} \Gamma_j\ \underset{\sim}{P}^j \quad ,$$

or, alternatively, by:

$$\underset{\sim}{F} = [i\ \underset{\sim}{A}\ \underset{\sim}{X}\ \underset{\sim}{K} + \underset{\sim}{D}\ \underset{\sim}{X}]\ \underset{\sim}{\Gamma}\ . \tag{2.40}$$

$\underset{\sim}{K}$ is the diagonal matrix given by (2.30). The vector of participation factors may be eliminated from (2.40) using (2.38). We find

$$\underset{\sim}{F} = \underset{\sim}{R}\ \underset{\sim}{U}\ .$$

$\underset{\sim}{R}$ is the dynamic stiffness matrix:

$$\underset{\sim}{R} = i\ \underset{\sim}{A}\ \underset{\sim}{X}\ \underset{\sim}{K}\ \underset{\sim}{X}^{-1} + \underset{\sim}{D} \quad . \tag{2.41}$$

The region $x \geq 0$ is understood as an element (see figure 2.1) with nodes at $(0,z_j)$, $1 \leq j \leq N$, nodal displacements $\underset{\sim}{U}$ and nodal forces $\underset{\sim}{F}$. The element is known as a consistent transmitting boundary. Details may be found in the work by Waas [23].

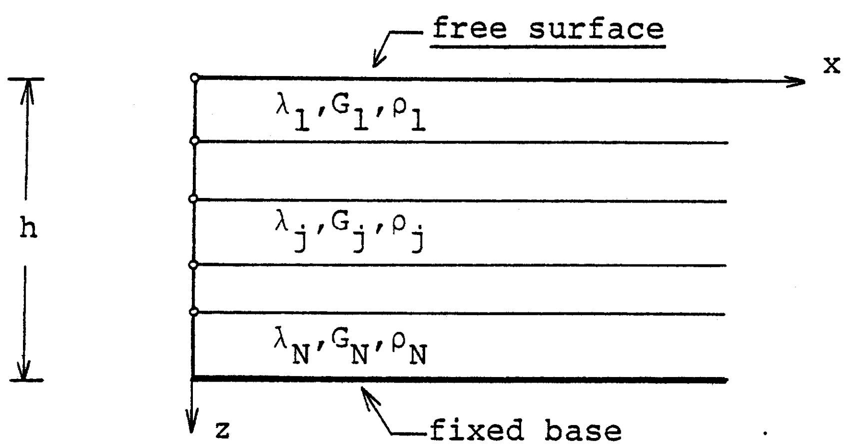


Figure 2.1 - The region $x \geq 0$, $0 \leq z \leq h$, in plane strain (free surface, fixed base) .

The procedure described above may be applied to obtain the dynamic stiffness matrix of the rectangular region $x_1 \leq x \leq x_2$. This region is, however, of finite extent and therefore all modes must be included. Let $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$ be the vectors of nodal displacements corresponding to the vertical sections at the left and the right of the region, i.e., at $x = x_1$ and $x = x_2$, respectively:

$$\begin{aligned} U^1_{2j-1} &= u(x_1, z_j) \\ U^1_{2j} &= w(x_1, z_j) \\ U^2_{2j-1} &= u(x_2, z_j) \\ U^2_{2j} &= w(x_2, z_j) \end{aligned} \qquad 1 \le j \le N \quad .$$

$\underset{\sim}{U}^1$ and $\underset{\sim}{U}^2$ may be written as linear combinations of the vectors corresponding to the 4N modes. Let us consider a mode with wave number k and eigenvector $\underset{\sim}{\Delta}$. For this mode the vector of displacements at $x = x_1$ is conveniently taken equal to $\underset{\sim}{\Delta}$. Then, the vector of displacements at $x = x_2$ is $\underset{\sim}{\Delta}\exp[-ik(x_2-x_1)]$. Alternatively, if the vector of displacements at $x = x_2$ is $\underset{\sim}{\Delta}$, then the vector of displacements at $x = x_1$ is $\underset{\sim}{\Delta}\exp[ik(x_2-x_1)]$. Thus, we may write

$$\underset{\sim}{U}^1 = \underset{\sim}{X}\ \underset{\sim}{\Gamma}^1 + \underset{\sim}{\overline{X}}\ \underset{\sim}{E}\ \underset{\sim}{\Gamma}^2 \tag{2.42a}$$

$$\underset{\sim}{U}^2 = \underset{\sim}{X}\ \underset{\sim}{E}\ \underset{\sim}{\Gamma}^1 + \underset{\sim}{\overline{X}}\ \underset{\sim}{\Gamma}^2 \ . \tag{2.42b}$$

$\underset{\sim}{\overline{X}}$ is a modal matrix obtained from $\underset{\sim}{X}$ as

$$\underset{\sim}{\overline{X}} = \underset{\sim}{T}\ \underset{\sim}{X} \quad , \tag{2.43}$$

$\underset{\sim}{T}$ being the diagonal matrix given by (2.29). Thus the columns of $\underset{\sim}{\overline{X}}$ are the eigenvectors corresponding to modes traveling in the negative x-direction or decaying for large $x < 0$. $\underset{\sim}{E}$ is a diagonal matrix (2N x 2N):

$$\underset{\sim}{E} = \text{diag}\ [\exp(-ik_j\ L)] \quad , \tag{2.44}$$

with $L = x_2 - x_1$, the length of the region. $\underset{\sim}{\Gamma}^1$, $\underset{\sim}{\Gamma}^2$ are 2N-vectors of participation factors. Let $\underset{\sim}{F}^1$, $\underset{\sim}{F}^2$ be the vectors of nodal forces acting on the left ($x = x_1$) and the right ($x = x_2$) of the region. For $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$ as given by equations (2.42a,b), using equation (2.37), we obtain

$$\underset{\sim}{F}^1 = [i\,\underset{\sim}{A}\,\underset{\sim}{X}\,\underset{\sim}{K} + \underset{\sim}{D}\,\underset{\sim}{X}]\underset{\sim}{\Gamma}^1 + [-i\,\underset{\sim}{A}\,\underset{\sim}{\bar{X}}\,\underset{\sim}{E}\,\underset{\sim}{K} + \underset{\sim}{D}\,\underset{\sim}{\bar{X}}\,\underset{\sim}{E}]\underset{\sim}{\Gamma}^2 \quad (2.45a)$$

$$\underset{\sim}{F}^2 = -[i\,\underset{\sim}{A}\,\underset{\sim}{X}\,\underset{\sim}{E}\,\underset{\sim}{K} + \underset{\sim}{D}\,\underset{\sim}{X}\,\underset{\sim}{E}]\underset{\sim}{\Gamma}^1 - [-i\,\underset{\sim}{A}\,\underset{\sim}{\bar{X}}\,\underset{\sim}{K} + \underset{\sim}{D}\,\underset{\sim}{\bar{X}}]\,\underset{\sim}{\Gamma}^2 \,. \quad (2.45b)$$

The vectors of participation factors $\underset{\sim}{\Gamma}^1$, $\underset{\sim}{\Gamma}^2$ may be eliminated from (2.45a,b) using (2.42a,b). We get

$$\begin{bmatrix} \underset{\sim}{F}^1 \\ \underset{\sim}{F}^2 \end{bmatrix} = \underset{\sim}{K} \begin{bmatrix} \underset{\sim}{U}^1 \\ \underset{\sim}{U}^2 \end{bmatrix} \quad (2.46)$$

$\underset{\sim}{K}$ is the dynamic stiffness matrix. The region $x_1 \leq x \leq x_2$ is understood as an element (see figure 2.2) with nodes at (x_1, z_j), (x_2, z_j), $1 \leq j \leq N$, nodal displacements $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$ and nodal forces $\underset{\sim}{F}^1$, $\underset{\sim}{F}^2$. The computational effort involved in obtaining the dynamic stiffness matrix is independent of the length L of the element. Details are given in [9].

2.3 A STRATUM IN ANTIPLANE SHEAR

Let us now consider time-harmonic vibrations of a stratum in antiplane shear. The displacement vector is

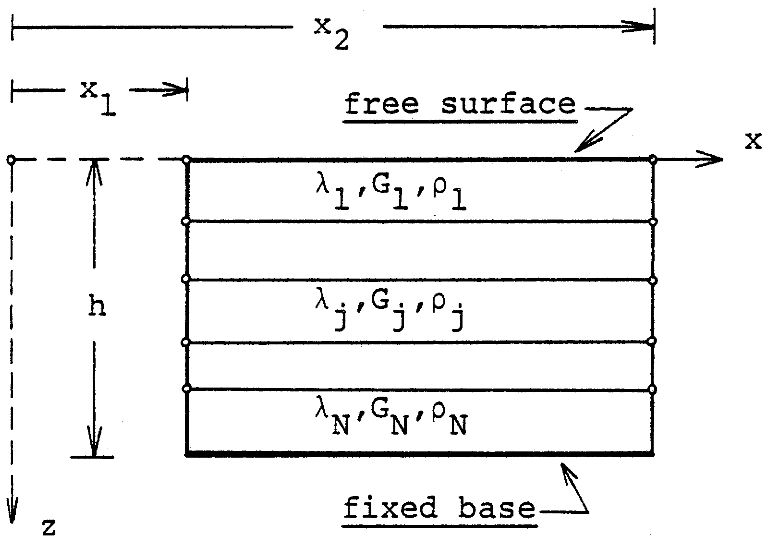


Figure 2.2 - The region $x_1 \leq x \leq x_2$, $0 \leq z \leq h$, in plane strain (free surface, fixed base) .

$$\begin{bmatrix} 0 \\ v(x,z) \\ 0 \end{bmatrix} \exp(i\omega t) \quad ,$$

i.e., particle motion is perpendicular to the x-z plane and independent of the y coordinate. We assume that there are M layers in the stratum. In layer j, the governing differential equation is

$$G_j \frac{\partial^2 v}{\partial x^2} + G_j \frac{\partial^2 v}{\partial z^2} + \rho_j \omega^2 v = 0 \quad . \tag{2.47}$$

The amplitudes of the stresses are:

$$\tau_{yz} = G_j \frac{\partial v}{\partial z} \tag{2.48a}$$

$$\tau_{yx} = G_j \frac{\partial v}{\partial x} \quad . \tag{2.48b}$$

The amplitude v must be continuous at the interfaces of the layers, i.e., at $z = z_j$, $2 \leq j \leq M$. Moreover, τ_{yz} must be continuous there:

$$G_{j-1} \left.\frac{\partial v}{\partial z}\right|_{z=z_j^-} = G_j \left.\frac{\partial v}{\partial z}\right|_{z=z_j^+} \quad . \tag{2.49}$$

If the stratum is understood as an idealization of a soil deposit, the boundary conditions are:

$$\left.\frac{\partial v}{\partial z}\right|_{z=0} = 0 \quad , \tag{2.50a}$$

$$v(x,h) = 0 \quad , \tag{2.50b}$$

i.e., the surface is free and the base is fixed.

Time-harmonic wave motion of a stratum in antiplane shear may be obtained as a superposition of modes of the form

$$v(x,z) = V(z) \exp(-ikx) \quad . \tag{2.51}$$

Substituting (2.51) into (2.47), (2.49) and (2.50a,b), we find that V must satisfy, in layer j, the equation

$$k^2 G_j V - G_j \frac{d^2V}{dz^2} - \rho_j \omega^2 V = 0 \quad , \tag{2.52a}$$

at $z = z_j$, $2 \leq j \leq M$, the condition

$$G_{j-1} \left.\frac{dV}{dz}\right|_{z=z_j^-} = G_j \left.\frac{dV}{dz}\right|_{z=z_j^+} \quad , \tag{2.52b}$$

and, finally, the boundary conditions

$$\left.\frac{dV}{dz}\right|_{z=0} = 0 \quad , \tag{2.52c}$$

$$V(h) = 0 \quad . \tag{2.52d}$$

Clearly, from equation (2.52a), we obtain that the eigenfunction V, in layer j, may be written as

$$V(z) = A_1^j \cos(p_j z) + A_2^j \sin(p_j z) \quad , \tag{2.53}$$

$$\text{with} \quad p_j^2 + k^2 = \frac{\omega^2}{[C_T^j]^2} \ .$$

Alternatively, A_1^j and A_2^j may be expressed in terms of $V(z_j)$ and $V(z_{j+1})$, using (2.53). Thus the eigenfunction V is completely specified by the M values $V(z_j)$, $1 \leq z_j \leq M$, since $V(h) = 0$ by the boundary condition (2.52d). These must be such that the M-1 conditions (2.52b) as well as the boundary condition (2.52c) be satisfied. Thus we obtain a system of M homogeneous linear equations for the M values $V(z_j)$, $1 \leq z_j \leq M$. Nontrivial solutions are possible for those values of k which render the matrix of coefficients singular. The frequency equation is, therefore, obtained by equating the determinant of the matrix to zero. However, k appears, again, in the argument of transcendental functions. If the stratum is homogeneous, i.e., there is only one layer (M=1), the frequency equation for antiplane shear, unlike the corresponding equation for plane strain, may be solved easily. For a given frequency ω, the wave numbers k are given by

$$[kh]^2 = \left[\frac{\omega h}{C_T}\right]^2 - (2n-1)^2 \frac{\pi^2}{4} \qquad (2.54)$$

$$n = 1, 2, \ldots$$

For a multilayered stratum the calculation of roots of the frequency equation is, in general, difficult. Cut-off frequencies are easily found for a homogeneous stratum. From (2.54), it is seen that they are given by

$$\frac{\omega h}{C_T} = (2n-1)\frac{\pi}{2}, \quad n = 1, 2, \ldots \qquad (2.55)$$

Again, it is, in general, difficult to calculate cut-off frequencies for a multilayered stratum.

Let us consider a mode of vibration in antiplane shear. The displacement vector is

$$\begin{bmatrix} 0 \\ V(z) \\ 0 \end{bmatrix} \exp(i\omega t - ikx) \quad . \qquad (2.56)$$

We assume that ω is not a cut-off frequency. Then, $k \neq 0$. Suppose that $Im[k] = 0$, i.e., the wave number k is real. The mode is a wave propagating in the positive x-direction, if k is positive, or, in the negative x-direction, if k is negative. Waas [23] shows that phase propagation and energy propagation are always in the same direction for traveling waves in antiplane shear. If a region of finite extent is considered (for example, $x_1 \leq x \leq x_2$), the mode is admissible. However, if the region is of infinite extent (for example, $x \geq 0$), the mode is admissible only if it satisfies the radiation condition. In particular, radiation in the region $x \geq 0$ requires that waves be outgoing, i.e., $k > 0$. Let us now assume that $\mathrm{Im}[k] \neq 0$. The mode is an evanescent wave. If $\mathrm{Re}[k] \neq 0$, there is a propagating phase, while if $\mathrm{Re}[k] = 0$, the mode is a standing wave. Again, the mode is admissible in a region of finite extent. If the region is of infinite extent, the mode is admissible only if it satisfies the boundedness condition. For example, if the region $x \geq 0$ is considered, the boundedness condition requires that waves be bounded for arbitrarily large $x > 0$, i.e., $\mathrm{Im}[k] < 0$.

Using the finite element method, we derive an algebraic eigenvalue problem [23] for the calculation of approximate wave numbers k and eigenfunctions V. Each layer of the stratum is divided into sublayers the depth of which is much smaller than the wavelength of transverse waves in the layer. Let N be the number of sublayers into which the stratum is divided. Finite elements are the line segments $[z_j, z_{j+1}]$, $1 \leq j \leq N$, corresponding to these N sublayers. The eigenfunction V is the amplitude of the displacement at $x = 0$. Let δV (function of z) be the amplitude of a virtual displacement at $x = 0$. Multiplying the left-hand side of equation (2.52a) by δV, we obtain, for sublayer j,

$$k^2 G_j \int_{z_j}^{z_{j+1}} V\delta V dz - \omega^2 \rho_j \int_{z_j}^{z_{j+1}} V\delta V dz - G_j \int_{z_j}^{z_{j+1}} \frac{d}{dz}\left[\frac{dV}{dz}\right]\delta V dz = 0 \quad . \tag{2.57}$$

Integrating by parts, we find:

$$k^2 G_j \int_{z_j}^{z_{j+1}} V\delta V dz - \omega^2 \rho_j \int_{z_j}^{z_{j+1}} V\delta V dz$$

$$+ G_j \int_{z_j}^{z_{j+1}} \frac{dV}{dz} \frac{d}{dz} [\delta V] dz$$

$$= G_j \frac{dV}{dz} \delta V \Big|_{z_j}^{z_{j+1}} \quad . \qquad (2.58)$$

Working as in the case of plane strain, we obtain:

$$\left[k^2 \underset{\sim}{A}^j + \underset{\sim}{G}^j - \omega^2 \underset{\sim}{M}^j\right] \begin{bmatrix} V_j \\ V_{j+1} \end{bmatrix} = \begin{bmatrix} -\tau_j \\ \tau_{j+1} \end{bmatrix}, \qquad (2.59)$$

with $\quad V_\ell = V(z_\ell) \quad ,$

$$\ell = j, \; j+1$$

$$\tau_\ell = \tau_{yz}\Big|_{\substack{x=0 \\ z=z_\ell}} \quad .$$

$\underset{\sim}{A}^j$, $\underset{\sim}{G}^j$, $\underset{\sim}{M}^j$ are 2 x 2 symmetric matrices:

$$\underset{\sim}{A}^j = h_j G_j \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \qquad (2.60a)$$

$$\underset{\sim}{G}^j = \frac{G_j}{h_j} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (2.60b)$$

$$\underset{\sim}{M}^j = \rho_j h_j \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \quad . \tag{2.60c}$$

Since τ_{yz} is continuous at $z = z_j$, $2 \leq j \leq N$, we have

$$\tau_j = G_{j-1} \left.\frac{dV}{dz}\right|_{z=z_j^-} = G_j \left.\frac{dV}{dz}\right|_{z=z_j^+} \quad . \tag{2.61}$$

Assembling the matrices for the region $0 \leq z \leq h$, we obtain

$$[k^2\underset{\sim}{A} + \underset{\sim}{G} - \omega^2\underset{\sim}{M}]\underset{\sim}{\Delta} = \underset{\sim}{F} \quad . \tag{2.62}$$

$\underset{\sim}{A}$, $\underset{\sim}{G}$, $\underset{\sim}{M}$ are (N+1)x(N+1) matrices assembled from $\underset{\sim}{A}^j$, $\underset{\sim}{G}^j$, $\underset{\sim}{M}^j$ respectively. $\underset{\sim}{\Delta}$, $\underset{\sim}{F}$ are (N+1)-vectors:

$$\Delta_j = V_j \quad , \quad 1 \leq j \leq N+1 \quad ,$$

$$F_1 = -\tau_1 = -\tau_{yz}\Big|_{\substack{x=0\\z=0}} \quad , \quad F_{N+1} = \tau_{N+1} = \tau_{yz}\Big|_{\substack{x=0\\z=h}} \quad ,$$

$$F_j = 0 \quad , \quad 2 \leq j \leq N \quad .$$

Thus, F_1 and F_{N+1} are the amplitudes of the shear tractions, at x = 0, on the surface and the base of the stratum respectively. For the eigenvalue problem with boundary conditions (2.52c,d), i.e., free surface and fixed base, the corresponding algebraic eigenvalue problem is obtained by deleting the last row and the last column of the matrices $\underset{\sim}{A}$, $\underset{\sim}{G}$, $\underset{\sim}{M}$ as well as the last component of the vectors $\underset{\sim}{\Delta}$, $\underset{\sim}{F}$, while the first component of $\underset{\sim}{F}$ is set equal to zero:

$$[k^2\underset{\sim}{A} + \underset{\sim}{G} - \omega^2\underset{\sim}{M}]\underset{\sim}{\Delta} = 0 \quad . \tag{2.63}$$

$\underset{\sim}{A}$, $\underset{\sim}{G}$, $\underset{\sim}{M}$, $\underset{\sim}{\Delta}$ differ from those appearing in equation (2.62) by the changes mentioned above. The wave numbers k are roots of a polynomial of degree 2N:

$$\det[k^2\underset{\sim}{A} + \underset{\sim}{G} - \omega^2\underset{\sim}{M}] = 0 \quad . \qquad (2.64)$$

Details may be found in the work by Waas [23]. If k is an eigenvalue with eigenvector $\underset{\sim}{\Delta}$, then -k is another eigenvalue with the same eigenvector. Let us choose those N wave numbers k_j, $1 \leq j \leq N$, and the associated N linearly independent eigenvectors $\underset{\sim}{\Delta}^j$ for which the corresponding modes are waves decaying for large x > 0 or traveling in the positive x-direction, i.e., $Im[k_j] < 0$, or $Re[k_j] > 0$ and $Im[k_j] = 0$. We form the diagonal matrix $\underset{\sim}{K}$ (NxN):

$$\underset{\sim}{K} = \text{diag}\ [k_j] \quad , \qquad (2.65)$$

and the modal matrix $\underset{\sim}{X}$ (NxN), the columns of which are the eigenvectors $\underset{\sim}{\Delta}^j$:

$$\underset{\sim}{X} = [\underset{\sim}{\Delta}^1, \underset{\sim}{\Delta}^2, \ldots , \underset{\sim}{\Delta}^N] \ . \qquad (2.66)$$

Let us now obtain consistent nodal forces acting on the region $x \geq 0$, at the section x = 0, for a mode with wave number k and eigenvector $\underset{\sim}{\Delta}$. The forces on sublayer j are calculated by integrating the traction, at x = 0, multiplied by the shape functions along $z_j \leq z \leq z_{j+1}$:

$$\text{(node } j\text{)} \qquad P_j^j = \int_{z_j}^{z_{j+1}} -\tau_{yx}\Big|_{x=0} N_j^j(z)\,dz$$

$$\text{(node } j+1\text{)} \qquad P_{j+1}^j = \int_{z_j}^{z_{j+1}} -\tau_{yx}\Big|_{x=0} N_{j+1}^j(z)\,dz \quad . \tag{2.67}$$

N_j^j, N_{j+1}^j are the shape functions given by (2.21a,b). We have

$$\tau_{yx}\Big|_{x=0} = -ikG_jV \quad . \tag{2.68}$$

Since

$$V(z) = V_jN_j^j(z) + V_{j+1}\, N_{j+1}^j(z) \tag{2.69}$$

$$\text{with } V_\ell = V(z_\ell) \qquad \ell = j,\ j+1 \quad ,$$

we find

$$ik\underset{\sim}{A}^j \begin{bmatrix} V_j \\ \\ V_{j+1} \end{bmatrix} = \begin{bmatrix} P_j^j \\ \\ P_{j+1}^j \end{bmatrix} \quad . \tag{2.70}$$

$\underset{\sim}{A}^j$ is the same as in (2.59). Assembling the matrices for the region $0 \leq z \leq h$, we obtain:

$$\underset{\sim}{P} = ik\underset{\sim}{A}\,\underset{\sim}{\Delta}. \tag{2.71}$$

$\underset{\sim}{A}$ is the same as in (2.63). The components of the N-vector $\underset{\sim}{P}$ are:

$$P_1 = P_1^1$$

$$P_j = P_j^{j-1} + P_j^j \ , \ 2 \leq j \leq N \ .$$

The region $x \geq 0$ may be understood as an element (see figure 2.3) with nodes at $(0,z_j)$, $1 \leq j \leq N$. We obtain the dynamic stiffness matrix of this element. Let $\underset{\sim}{U}$ be the vector of nodal displacements:

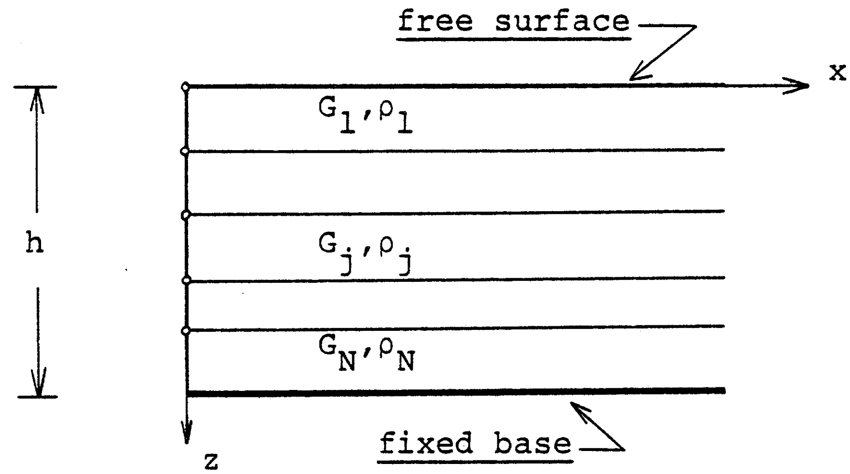


Figure 2.3 -The region $x \geq 0$, $0 \leq z \leq h$, in antiplane shear(free surface,fixed base) .

$$U_j = v(0,z_j) \quad , \quad 1 \leq j \leq N \quad .$$

We have

$$\underset{\sim}{U} = \underset{\sim}{X} \, \underset{\sim}{\Gamma} \quad , \tag{2.72}$$

$\underset{\sim}{X}$ being the modal matrix given by (2.66) and $\underset{\sim}{\Gamma}$ a vector of participation factors. The vector of nodal forces corresponding to $\underset{\sim}{U}$ is

$$\underset{\sim}{F} = i \, \underset{\sim}{A} \, \underset{\sim}{X} \, \underset{\sim}{K} \, \underset{\sim}{\Gamma} \quad . \tag{2.73}$$

$\underset{\sim}{K}$ is the diagonal matrix given by (2.65). The participation factors may be eliminated from (2.73) using (2.72). We obtain

$$\underset{\sim}{F} = \underset{\sim}{R} \, \underset{\sim}{U} \quad .$$

$\underset{\sim}{R}$ is the dynamic stiffness matrix:

$$\underset{\sim}{R} = i \, \underset{\sim}{A} \, \underset{\sim}{X} \, \underset{\sim}{K} \, \underset{\sim}{X}^{-1} \quad . \tag{2.74}$$

The element is also referred to as a consistent transmitting boundary. It is described in detail in the works by Lysmer and Waas [16] and Waas [23].

Another element is the region $x_1 \leq x \leq x_2$ with nodes at (x_1, z_j) and (x_2, z_j), $1 \leq j \leq N$. Since the region is of finite extent, all modes are admissible. Let $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$ be the vectors of nodal displacements at (x_1, z_j) and (x_2, z_j), $1 \leq j \leq N$, respectively:

$$U_j^1 = v(x_1, z_j)$$

$$U_j^2 = v(x_2, z_j) \quad .$$

We have

$$\underset{\sim}{U}^1 = \underset{\sim}{X}\,\underset{\sim}{\Gamma}^1 + \underset{\sim}{X}\,\underset{\sim}{E}\,\underset{\sim}{\Gamma}^2 \tag{2.75a}$$

$$\underset{\sim}{U}^2 = \underset{\sim}{X}\,\underset{\sim}{E}\,\underset{\sim}{\Gamma}^1 + \underset{\sim}{X}\,\underset{\sim}{\Gamma}^2 \quad . \tag{2.75b}$$

$\underset{\sim}{E}$ is a diagonal matrix (NxN):

$$\underset{\sim}{E} = \text{diag}\,[\exp(-ik_jL)] \quad , \tag{2.76}$$

with $L = x_2 - x_1$, the length of the region. $\underset{\sim}{\Gamma}^1$, $\underset{\sim}{\Gamma}^2$ are N-vectors of participation factors. Let $\underset{\sim}{F}^1$, $\underset{\sim}{F}^2$ be the vectors of nodal forces. For $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$ as given by equations (2.75a,b) we find

$$\underset{\sim}{F}^1 = i\,\underset{\sim}{A}\,\underset{\sim}{X}\,\underset{\sim}{K}\,\underset{\sim}{\Gamma}^1 - i\,\underset{\sim}{A}\,\underset{\sim}{X}\,\underset{\sim}{E}\,\underset{\sim}{K}\,\underset{\sim}{\Gamma}^2 \tag{2.77a}$$

$$\underset{\sim}{F}^2 = -\,i\,\underset{\sim}{A}\,\underset{\sim}{X}\,\underset{\sim}{E}\,\underset{\sim}{K}\,\underset{\sim}{\Gamma}^1 + i\,\underset{\sim}{A}\,\underset{\sim}{X}\,\underset{\sim}{K}\,\underset{\sim}{\Gamma}^2 \ . \tag{2.77b}$$

$\underset{\sim}{\Gamma}^1$, $\underset{\sim}{\Gamma}^2$ may be eliminated from (2.77a,b) using (2.75a,b). We obtain

$$\begin{bmatrix} \underset{\sim}{F}^1 \\ \\ \underset{\sim}{F}^2 \end{bmatrix} = \underset{\sim}{K} \begin{bmatrix} \underset{\sim}{U}^1 \\ \\ \underset{\sim}{U}^2 \end{bmatrix} . \tag{2.78}$$

$\underset{\sim}{K}$ is the dynamic stiffness matrix. It must be noted that the computational effort involved in this procedure for determining the dynamic stiffness matrix is independent of the length L of the element. Details may be found in the work by Kausel and Roësset [9].

2.4 AXISYMMETRIC ELEMENTS

In this section we consider the axisymmetric elements developed by Waas [23], Kausel [6] and Kausel and Roësset [9]. In a system of cylindrical coordinates (r,θ,z), let u, v, w denote the amplitudes of the radial, tangential and axial displacements respectively. We assume that there are M layers in the stratum. In layer j, the governing differential equations are:

$$\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{1-2\nu_j}\frac{\partial \varepsilon}{\partial r} + \frac{\omega^2}{[C_T^j]^2} u = 0 \tag{2.79a}$$

$$\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{1-2\nu_j}\frac{1}{r}\frac{\partial \varepsilon}{\partial \theta} + \frac{\omega^2}{[C_T^j]^2} v = 0 \tag{2.79b}$$

$$\nabla^2 w + \frac{1}{1-2\nu_j}\frac{\partial \varepsilon}{\partial z} + \frac{\omega^2}{[C_T^j]^2} w = 0 . \tag{2.79c}$$

∇^2 is the Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} . \tag{2.80}$$

ε is the dilatation:

$$\varepsilon = \frac{\partial u}{\partial r} + \frac{1}{r}\left(\frac{\partial v}{\partial \theta} + u\right) + \frac{\partial w}{\partial z} \ . \tag{2.81}$$

The amplitudes of the stresses are given by

$$\sigma_r = \lambda_j \varepsilon + 2G_j \frac{\partial u}{\partial r} \tag{2.82a}$$

$$\sigma_\theta = \lambda_j \varepsilon + 2G_j \left[\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}\right] \tag{2.82b}$$

$$\sigma_z = \lambda_j \varepsilon + 2G_j \frac{\partial w}{\partial z} \tag{2.82c}$$

$$\tau_{r\theta} = G_j \left[\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right] \tag{2.82d}$$

$$\tau_{\theta z} = G_j \left[\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z}\right] \tag{2.82e}$$

$$\tau_{zr} = G_j \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right] \ . \tag{2.82f}$$

As in the plane problems considered in the previous sections, the modes of wave motion are obtained by separation of variables. They are given by (see the work by Kausel [6]).

$$u(r,\theta,z) = kU(z)\ C_n'(kr) \begin{Bmatrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{Bmatrix} \tag{2.83a}$$

$$w(r,\theta,z) = -ikW(z)\ C_n(kr) \begin{Bmatrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{Bmatrix} \tag{2.83b}$$

$$v(r,\theta,z) = \frac{n}{r}\ U(z)\ C_n(kr) \begin{Bmatrix} -\sin(n\theta) \\ \\ \cos(n\theta) \end{Bmatrix} \tag{2.83c}$$

and,

$$u(r,\theta,z) = \frac{n}{r} V(z)\ C_n(kr) \begin{Bmatrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{Bmatrix} \tag{2.84a}$$

$$w(r,\theta,z) = 0 \tag{2.84b}$$

$$v(r,\theta,z) = kV(z)\ C_n^{'}(kr) \begin{Bmatrix} -\sin(n\theta) \\ \\ \cos(n\theta) \end{Bmatrix} \tag{2.84c}$$

$n = 0,\ 1,\ 2,\ \ldots$

For symmetric modes $\cos(n\theta)$ must be used for u and w, while $-\sin(n\theta)$ must be chosen for v (symmetry of the displacement field with respect to the plane $\theta=0$). For antisymmetric modes $\sin(n\theta)$, for u and w, and $\cos(n\theta)$, for v, are appropriate. $C_n(\xi)$ is any solution of Bessel's equation of order n:

$$C_n^{''} + \frac{1}{\xi} C_n^{'} + (1 - \frac{n^2}{\xi^2})\ C_n = 0\ . \tag{2.85}$$

The prime denotes differentiation with respect to the argument. If we substitute (2.83a,b,c) into the governing differential equations (2.79a,b,c), in layer j, we find that U and W must satisfy the equations

$$k^2(\lambda_j + 2G_j)U + ik(\lambda_j + G_j)\frac{dW}{dz} - G_j\ \frac{d^2U}{dz^2} - \omega^2\rho_j U = 0 \tag{2.86a}$$

$$k^2 G_j W + ik(\lambda_j + G_j)\frac{dU}{dz} - (\lambda_j + 2G_j)\ \frac{d^2W}{dz^2} - \omega^2\rho_j W = 0\ . \tag{2.86b}$$

These are identical to equations (2.7a,b) obtained for the modes of time-harmonic vibrations of the stratum in plane strain. Similarly, if we substitute (2.84a,b,c) into (2.79a, b,c), we obtain that V must satisfy the equation

$$k^2 G_j V - G_j \frac{d^2 V}{dz^2} - \rho_j \omega^2 V = 0 \quad , \tag{2.87}$$

which is identical to equation (2.52a) obtained for the modes of time-harmonic vibrations of the stratum in antiplane shear. σ_z, $\tau_{\theta z}$, τ_{zr} must be continuous at interfaces of the layers. For the modes given by (2.83a,b,c) we have, in layer j,

$$\sigma_z = -ik \left[(\lambda_j + 2G_j) \frac{dW}{dz} - ik\lambda_j U \right] C_n(kr) \left\{ \begin{matrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{matrix} \right\} \tag{2.88a}$$

$$\tau_{\theta z} = \frac{n}{r} G_j \left[\frac{dU}{dz} - ikW \right] C_n(kr) \left\{ \begin{matrix} -\sin(n\theta) \\ \\ \cos(n\theta) \end{matrix} \right\} \tag{2.88b}$$

$$\tau_{zr} = k\, G_j \left[\frac{dU}{dz} - ikW \right] C_n'(kr) \left\{ \begin{matrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{matrix} \right\} . \tag{2.88c}$$

Thus continuity of σ_z, $\tau_{\theta z}$, τ_{zr} at $z = z_j$, $2 \leq j \leq M$, requires that

$$(\lambda_{j-1} + 2G_{j-1}) \frac{dW}{dz}\bigg|_{z=z_j^-} - ik\lambda_{j-1} U(z_j^-) = (\lambda_j + 2G_j) \frac{dW}{dz}\bigg|_{z=z_j^+} - ik\lambda_j U(z_j^+) \tag{2.89a}$$

$$G_{j-1} \left[\frac{dU}{dz}\bigg|_{z=z_j^-} - ikW(z_j^-) \right] = G_j \left[\frac{dU}{dz}\bigg|_{z=z_j^+} - ikW(z_j^+) \right] . \tag{2.89b}$$

These are identical to conditions (2.7c,d) obtained for time-harmonic vibrations in plane strain. For the modes given by (2.84a,b,c) we have, in layer j,

$$\sigma_z = 0 \tag{2.90a}$$

$$\tau_{\theta z} = k\, G_j \frac{dV}{dz} C_n'(kr) \begin{Bmatrix} -\sin(n\theta) \\ \\ \cos(n\theta) \end{Bmatrix} \tag{2.90b}$$

$$\tau_{zr} = \frac{n}{r} G_j \frac{dV}{dz} C_n(kr) \begin{Bmatrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{Bmatrix} \quad . \tag{2.90c}$$

Continuity of σ_z, $\tau_{\theta z}$, τ_{zr} at $z = z_j$, $2 \le j \le M$, requires that

$$G_{j-1} \left.\frac{dV}{dz}\right|_{z=z_j^-} = G_j \left.\frac{dV}{dz}\right|_{z=z_j^+} \tag{2.91}$$

This is identical to the condition (2.52b) obtained for time-harmonic vibrations in antiplane shear. For a free surface and a fixed base, the modes given by (2.83a,b,c) must satisfy the boundary conditions

$$(\lambda_1 + 2G_1) \left.\frac{dW}{dz}\right|_{z=0} - ik\lambda_1 U(0) = 0 \tag{2.92a}$$

$$\left.\frac{dU}{dz}\right|_{z=0} - ikW(0) = 0 \tag{2.92b}$$

$$U(h) = 0 \tag{2.92c}$$

$$W(h) = 0 \quad , \tag{2.92d}$$

while the modes given by (2.84a,b,c) must satisfy

$$\left.\frac{dV}{dz}\right|_{z=0} = 0 \tag{2.93a}$$

$$V(h) = 0 \quad . \tag{2.93b}$$

Clearly, from (2.86a,b), (2.89a,b), (2.92a,b,c,d), U and W are eigenfunctions with eigenvalue k of an eigenvalue problem which is identical to that obtained for time-harmonic vibrations of the stratum (free surface, fixed base) in plane strain. From (2.87), (2.91), (2.93a,b), it follows that V is an eigenfunction with eigenvalue k of the eigenvalue problem for time-harmonic vibrations of the stratum (free surface, fixed base) in antiplane shear. We note that the eigenvalue problems are independent of the Fourier number n and the same for symmetric and antisymmetric modes.

Thus approximate eigenfunctions and eigenvalues are readily obtained by solving the algebraic eigenvalue problems derived for the plane problems which were discussed in the previous sections. Assuming that the stratum is divided into N sublayers, let $\underset{\sim}{K}$ be a diagonal matrix (3Nx3N) such that its first 2N diagonal entries are the eigenvalues of the algebraic eigenvalue problem for plane strain, chosen as in (2.30), and its last N diagonal entries are the eigenvalues of the algebraic eigenvalue problem for antiplane shear, chosen as in (2.65). It is convenient to consider the amplitudes $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{\sigma}_r$, $\bar{\sigma}_\theta$, $\bar{\sigma}_z$, $\bar{\tau}_{r\theta}$, $\bar{\tau}_{\theta z}$, $\bar{\tau}_{zr}$ (for a given Fourier number n) defined by

$$u = \bar{u} \begin{Bmatrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{Bmatrix} \qquad (2.94a)$$

$$v = \bar{v} \begin{Bmatrix} -\sin(n\theta) \\ \\ \cos(n\theta) \end{Bmatrix} \qquad (2.94b)$$

$$w = \bar{w} \begin{Bmatrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{Bmatrix} \tag{2.94c}$$

$$\sigma_r = \bar{\sigma}_r \begin{Bmatrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{Bmatrix} \tag{2.94d}$$

$$\sigma_\theta = \bar{\sigma}_\theta \begin{Bmatrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{Bmatrix} \tag{2.94e}$$

$$\sigma_z = \bar{\sigma}_z \begin{Bmatrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{Bmatrix} \tag{2.94f}$$

$$\tau_{r\theta} = \bar{\tau}_{r\theta} \begin{Bmatrix} -\sin(n\theta) \\ \\ \cos(n\theta) \end{Bmatrix} \tag{2.94g}$$

$$\tau_{\theta z} = \bar{\tau}_{\theta z} \begin{Bmatrix} -\sin(n\theta) \\ \\ \cos(n\theta) \end{Bmatrix} \tag{2.94h}$$

$$\tau_{zr} = \bar{\tau}_{zr} \begin{Bmatrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{Bmatrix} \tag{2.94i}$$

It is easily checked, from (2.82a,b,c,d,e,f), that the relations between $\bar{\sigma}_r$, $\bar{\sigma}_\theta$, $\bar{\sigma}_z$, $\bar{\tau}_{r\theta}$, $\bar{\tau}_{\theta z}$, $\bar{\tau}_{zr}$ and $\bar{u}$, $\bar{v}$, $\bar{w}$ are the same for symmetric and antisymmetric modes. Let us define some modal matrices (3N x 3N) at $r = r_o$ for a given Fourier number n. The first 2N columns correspond to the algebraic eigenvalue problem for plane strain, while the last N columns correspond to the algebraic eigenvalue problem for antiplane shear. The matrix $\underset{\sim}{\Phi}$ is defined as:

$$\left.\begin{aligned}\Phi_{3j-2,\ell} &= -U_j^{\ell}\, C_{n-1}(k_\ell r_o)\\ \Phi_{3j-1,\ell} &= -iW_j^{\ell}\, C_n(k_\ell r_o)\\ \Phi_{3j,\ell} &= 0\end{aligned}\right\}\ 1 \le \ell \le 2N,\ 1 \le j \le N \tag{2.95}$$

$$\left.\begin{aligned}\Phi_{3j-2,\ell} &= 0\\ \Phi_{3j-1,\ell} &= 0\\ \Phi_{3j,\ell} &= -V_j^{\ell-2N}\, C_{n-1}(k_\ell r_o)\end{aligned}\right\}\ 2N+1 \le \ell \le 3N,\ 1 \le j \le N$$

(the superscript indicates the particular eigenvector of the algebraic eigenvalue problem).

The matrix $\underset{\sim}{\Psi}$ is taken as

$$\left.\begin{aligned}\Psi_{3j-2,\ell} &= U_j^{\ell}\, C_n(k_\ell r_o)\\ \Psi_{3j-1,\ell} &= iW_j^{\ell}\, C_{n-1}(k_\ell r_o)\\ \Psi_{3j,\ell} &= 0\end{aligned}\right\}\ 1 \le \ell \le 2N,\ 1 \le j \le N \tag{2.96}$$

$$\left.\begin{aligned}\Psi_{3j-2,\ell} &= 0\\ \Psi_{3j-1,\ell} &= 0\\ \Psi_{3j,\ell} &= V_j^{\ell-2N}\, C_n(k_\ell r_o)\end{aligned}\right\}\ 2N+1 \le \ell \le 3N,\ 1 \le j \le N.$$

Finally, the matrix $\underset{\sim}{W}$ of modal amplitudes at $r = r_o$ for the Fourier number n is defined by

$$\left.\begin{aligned}W_{3j-2,\ell} &= k_\ell U_j^{\ell}\, C_n'(k_\ell r_o)\\ W_{3j-1,\ell} &= -ik_\ell W_j^{\ell}\, C_n(k_\ell r_o)\\ W_{3j,\ell} &= \frac{n}{r_o} U_j^{\ell}\, C_n(k_\ell r_o)\end{aligned}\right\}\ 1 \le \ell \le 2N,\ 1 \le j \le N \tag{2.97}$$

$$\left.\begin{aligned} W_{3j-2,\ell} &= \frac{n}{r_o} V_j^{\ell-2N} \, C_n \, (k_\ell r_o) \\ W_{3j-1,\ell} &= 0 \\ W_{3j,\ell} &= k_\ell V_j^{\ell-2N} \, C_n' \, (k_\ell r_o) \end{aligned}\right\} 2N+1 \le \ell \le 3N, \; 1 \le j \le N .$$

Let us consider the cylindrical surface $r = r_o$. At node j the consistent nodal forces acting on the region $r \geq r_o$ of sublayer j are (node j)

$$P^j_{r,j} = - r_o \int_{z_j}^{z_{j+1}} \bar{\sigma}_r \Big|_{r=r_o} N^j_j(z) dz \tag{2.98a}$$

$$P^j_{z,j} = - r_o \int_{z_j}^{z_{j+1}} \bar{\tau}_{zr} \Big|_{r=r_o} N^j_j(z) dz \tag{2.98b}$$

$$P^j_{\theta,j} = - r_o \int_{z_j}^{z_{j+1}} \bar{\tau}_{r\theta} \Big|_{r=r_o} N^j_j(z) dz. \tag{2.98c}$$

The consistent nodal forces at node j+1 are obtained similarly. For mode ℓ, $1 \leq \ell \leq 3N$, after integrating and assembling for the region $0 \leq z \leq h$, we find [6]

$$\underset{\sim}{P} = r_o \left[k_\ell^2 \underset{\sim}{A} \, \underset{\sim}{\Psi}^\ell + k_\ell (\underset{\sim}{D} - \underset{\sim}{E} + n\underset{\sim}{N}) \underset{\sim}{\Phi}^\ell - \left(\frac{n(n+1)}{2} \underset{\sim}{L} + n\underset{\sim}{Q}\right) \underset{\sim}{\Psi}^\ell \right] \quad . \tag{2.99}$$

$\underset{\sim}{\Psi}^\ell$ and $\underset{\sim}{\Phi}^\ell$ are the columns of matrices $\underset{\sim}{\Psi}$, $\underset{\sim}{\Phi}$ corresponding to mode ℓ. P^ℓ_{3j-2}, P^ℓ_{3j-1}, P^ℓ_{3j} are the radial, vertical and tantential forces at node j respectively. $\underset{\sim}{A}$, $\underset{\sim}{D}$, $\underset{\sim}{E}$, $\underset{\sim}{N}$, $\underset{\sim}{L}$, $\underset{\sim}{Q}$ are 3N x 3N matrices assembled from the sublayer matrices $\underset{\sim}{A}^j$, $\underset{\sim}{D}^j$, $\underset{\sim}{E}^j$, $\underset{\sim}{N}^j$, $\underset{\sim}{L}^j$, $\underset{\sim}{Q}^j$, respectively, which are given by

$$\underset{\sim}{A}^j = \frac{h_j}{6}\begin{bmatrix} 2(\lambda_j+2G_j) & 0 & 0 & \lambda_j+2G_j & 0 & 0 \\ 0 & 2G_j & 0 & 0 & G_j & 0 \\ 0 & 0 & 2G_j & 0 & 0 & G_j \\ \lambda_j+2G_j & 0 & 0 & 2(\lambda_j+2G_j) & 0 & 0 \\ 0 & G_j & 0 & 0 & 2G_j & 0 \\ 0 & 0 & G_j & 0 & 0 & 2G_j \end{bmatrix} \tag{2.100a}$$

$$\underset{\sim}{D}^j = \frac{1}{2}\begin{bmatrix} 0 & \lambda_j & 0 & 0 & -\lambda_j & 0 \\ -G_j & 0 & 0 & G_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_j & 0 & 0 & -\lambda_j & 0 \\ -G_j & 0 & 0 & G_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.100b}$$

$$\underset{\sim}{E}^j = \frac{G_j h_j}{3r_o}\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \tag{2.100c}$$

$$\underset{\sim}{N}^j = \frac{G_j h_j}{6r_o} \begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 4 & 0 & 0 \end{bmatrix} \tag{2.100d}$$

$$\underset{\sim}{L}^j = \frac{2}{3} \frac{G_j h_j}{r_o^2} \begin{bmatrix} 2 & 0 & -2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & -1 & 0 & 1 \\ 1 & 0 & -1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -2 & 0 & 2 \end{bmatrix} \tag{2.100e}$$

$$\underset{\sim}{Q}^j = \frac{G_j}{2r_o} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.100f}$$

Let us consider the region $r \geq r_o > 0$. For radiation in this region we take

$$C_n(kr) = H_n^{(2)}(kr) \quad .$$

$H_n^{(2)}$ is the Hankel function of the second kind of order n. It is bounded in the region $0 < r_o \leq r < \infty$ and, moreover, its asymptotic behavior, as $r \to \infty$, is [1]:

$$H_n^{(2)}(kr) \sim \left[\frac{2}{\pi kr}\right]^{1/2} \exp(-ikr + i\,\frac{n\pi}{2} + \frac{\pi}{4}) \quad . \qquad (2.101)$$

Thus a mode for which $\mathrm{Im}[k] < 0$ is an evanescent wave decaying for large r. This agrees with the choice of eigenvalues k for the matrix $\underset{\sim}{K}$. The region $r \geq r_o > 0$ may be understood as an element (see figure 2.4) with nodes at $r = r_o$, $z = z_j$, $1 \leq j \leq N$. We obtain the dynamic stiffness matrix of this

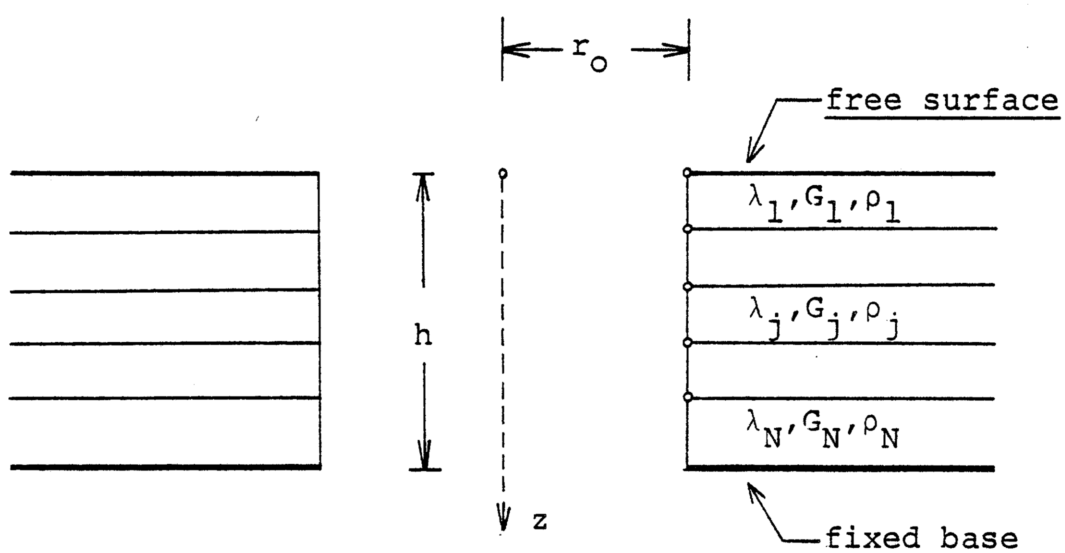


Figure 2.4 - The region $0 < r_o \leq r$, $0 \leq z \leq h$ (free surface , fixed base) .

element. Let $\underset{\sim}{U}$ be the vector of nodal displacements:

$$U_{3j-2} = \bar{u}(r_o, z_j)$$
$$U_{3j-1} = \bar{w}(r_o, z_j) \qquad 1 \leq j \leq N$$
$$U_{3j} = \bar{v}(r_o, z_j) \quad .$$

We have:

$$\underset{\sim}{U} = \underset{\sim}{W}\,\underset{\sim}{\Gamma} \ . \tag{2.102}$$

$\underset{\sim}{W}$ is the modal matrix given by (2.97). $\underset{\sim}{\Gamma}$ is a vector of modal participation factors. We use (2.99) to find the vector of nodal forces corresponding to $\underset{\sim}{U}$:

$$\underset{\sim}{F} = r_o\left[\underset{\sim}{A}\,\underset{\sim}{\Psi}\,\underset{\sim}{K}\,\underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E} + n\underset{\sim}{N})\underset{\sim}{\Phi}\underset{\sim}{K} - \left(\frac{n(n+1)}{2}\underset{\sim}{L} + n\underset{\sim}{Q}\right)\underset{\sim}{\Psi}\right]\underset{\sim}{\Gamma} . \tag{2.103}$$

We eliminate $\underset{\sim}{\Gamma}$ using (2.102). We find

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U} \quad .$$

$\underset{\sim}{R}$ is the dynamic stiffness matrix of the element:

$$\underset{\sim}{R} = r_o\left[\underset{\sim}{A}\,\underset{\sim}{\Psi}\,\underset{\sim}{K}\,\underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E} + n\underset{\sim}{N})\underset{\sim}{\Phi}\,\underset{\sim}{K} - \left(\frac{n(n+1)}{2}\underset{\sim}{L} + n\underset{\sim}{Q}\right)\underset{\sim}{\Psi}\right]\underset{\sim}{W}^{-1} . \tag{2.104}$$

The elementis also referred to as the consistent transmitting boundary (a cylindrical one). We note that the dynamic stiffness matrix is the same for symmetric and antisymmetric displacement fields. Details may be found in [6].

Let us consider the region $0 \leq r \leq r_o$. We obtain the dynamic stiffness matrix of an element (see figure 2.5) modeling the region. The nodes are at $r = r_o$, $z = z_j$, $1 \leq j \leq N$. We take

$$C_n(kr) = J_n(kr) \quad .$$

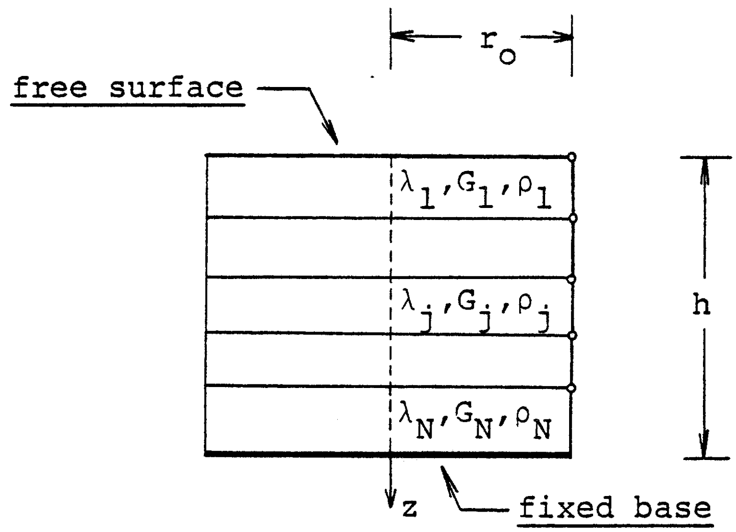


Figure 2.5-The region $0 \leq r \leq r_o$, $0 \leq z \leq h$
(free surface , fixed base) .

J_n is the Bessel function of order n. It is bounded in the region $0 \leq r \leq r_o$. Its asymptotic behavior, as $r \to 0$, is [1]:

$$J_n(kr) \sim \frac{(kr)^n}{2^n n!} \tag{2.105}$$

(solutions of Bessel's equation (2.85) which are nonsingular at $\xi = 0$ are multiples of J_n) . We note [1] that $J_n(-kr) = (-1)^n J_n(kr)$ and $J_n'(-kr) = (-1)^{n-1} J_n'(kr)$. Thus if the eigenvalue -k is used instead of k, together with the eigenfunctions U and -W (or -U and W) instead of U and W, the modes given by (2.83a,b,c) remain the same except for the factor $(-1)^n$, which affects only the modal participation factors. Similarly, if the eigenvalue -k is used instead of k, together with the eigenfunction V, the modes given by (2.84a,b,c) are left unchanged except, again, for the factor $(-1)^n$ which influences only the participation factors. Clearly, therefore, the choice of eigenvalues k need only be such that the corresponding modes are linearly independent. The choice of sign of real k or of the imaginary part of complex k is irrelevant. The vector of nodal displacements may be written as

$$\underset{\sim}{U} = \underset{\sim}{W}\,\underset{\sim}{\Gamma} \quad . \tag{2.106}$$

$\underset{\sim}{W}$ is the modal matrix given by (2.104). The nodal forces are given by

$$\underset{\sim}{F} = -r_o\left[\underset{\sim}{A}\,\underset{\sim}{\Psi}\,\underset{\sim}{K}\,\underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E} + n\underset{\sim}{N})\underset{\sim}{\Phi}\,\underset{\sim}{K} - (\frac{n(n+1)}{2}\underset{\sim}{L} + n\underset{\sim}{Q})\underset{\sim}{\Psi}\right]\underset{\sim}{\Gamma} . \tag{2.107}$$

The minus sign in front of the right-hand side is necessary since the orientation of the cylindrical surface of this element is opposite to that of the element modeling the region $r \geq r_o > 0$. Eliminating the participation factors, we obtain:

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U} \quad .$$

$\underset{\sim}{R}$ is the dynamic stiffness matrix of the element:

$$\underset{\sim}{R} = -r_o\left[\underset{\sim}{A}\,\underset{\sim}{\Psi}\,\underset{\sim}{K}\,\underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E} + n\underset{\sim}{N})\underset{\sim}{\Phi}\,\underset{\sim}{K} - (\frac{n(n+1)}{2}\underset{\sim}{L} + n\underset{\sim}{Q})\underset{\sim}{\Psi}\right]\underset{\sim}{W}^{-1} . \tag{2.108}$$

The computational effort necessary to obtain the matrix is independent of the diameter $2r_o$ of the element. Further details are given in [9].

Finally, let us consider the region $0 < r_1 \leq r \leq r_2$. The nodes of the element (see figure 2.6) modeling this region are at $r = r_1$, $z = z_j$ and at $r = r_2$, $z = z_j$, $1 \leq j \leq N$. In

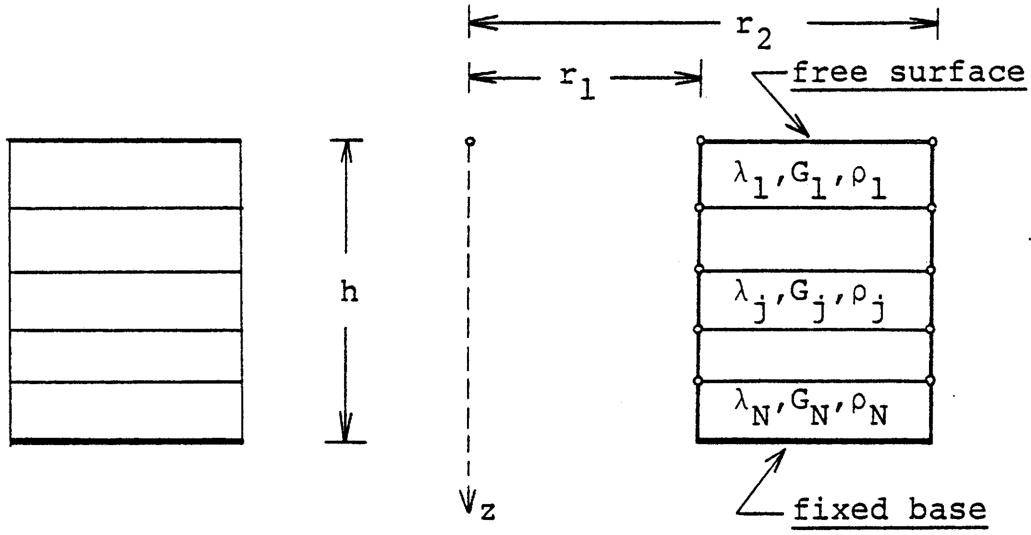


Figure 2.6 - The region $0 < r_1 \leq r \leq r_2$, $0 \leq z \leq h$ (free surface , fixed base) .

this case, all modes must be included. Apart from $H_n^{(2)}$ we also use $H_n^{(1)}$, i.e., the Hankel function of the first kind of order n. Then only the eigenvalues in matrix $\underset{\sim}{K}$ and the corresponding eigenvectors need be used. Let $\underset{\sim}{\Phi}^1$, $\underset{\sim}{\Psi}^1$, $\underset{\sim}{W}^1$ and $\underset{\sim}{\Phi}^2$, $\underset{\sim}{\Psi}^2$, $\underset{\sim}{W}^2$ be the modal matrices evaluated at $r = r_1$ and $r = r_2$ respectively using $H_n^{(2)}$. Similarly, let $\hat{\underset{\sim}{\Phi}}^1$, $\hat{\underset{\sim}{\Psi}}^1$, $\hat{\underset{\sim}{W}}^1$ and $\hat{\underset{\sim}{\Phi}}^2$, $\hat{\underset{\sim}{\Psi}}^2$, $\hat{\underset{\sim}{W}}^2$ be the modal matrices evaluated at $r = r_1$ and $r = r_2$ respectively using $H_n^{(1)}$. The displacement vectors $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$ are given by

$$\left.\begin{aligned} U^1_{3j-2} &= \bar{u}(r_1, z_j) \\ U^1_{3j-1} &= \bar{w}(r_1, z_j) \\ U^1_{3j} &= \bar{v}(r_1, z_j) \end{aligned}\right\} \qquad 1 \leq j \leq N$$

$$\left.\begin{aligned} U^2_{3j-2} &= \bar{u}(r_2,\ z_j) \\ U^2_{3j-1} &= \bar{w}(r_2,\ z_j) \\ U^2_{3j} &= \bar{v}(r_2,\ z_j) \end{aligned}\right\} \quad 1 \le j \le N \quad .$$

We have:

$$\underset{\sim}{U}^1 = \underset{\sim}{W}^1\ \underset{\sim}{\Gamma}^1 + \hat{\underset{\sim}{W}}^1\ \underset{\sim}{\Gamma}^2 \tag{2.109a}$$

$$\underset{\sim}{U}^2 = \underset{\sim}{W}^2\ \underset{\sim}{\Gamma}^1 + \hat{\underset{\sim}{W}}^2\ \underset{\sim}{\Gamma}^2 \quad . \tag{2.109b}$$

The nodal forces corresponding to $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$ are (a superscript ℓ indicates that the matrix is evaluated at $r = r_\ell$)

$$\begin{aligned} \underset{\sim}{F}^1 = {} & r_1\left[\underset{\sim}{A}\ \underset{\sim}{\Psi}^1\underset{\sim}{K}\ \underset{\sim}{K} + (\underset{\sim}{D}-\underset{\sim}{E}^1+n\underset{\sim}{N}^1)\underset{\sim}{\Phi}^1\underset{\sim}{K} - (\frac{n(n+1)}{2}\ \underset{\sim}{L}^1+n\underset{\sim}{Q}^1)\underset{\sim}{\Psi}^1\right]\underset{\sim}{\Gamma}^1 \\ & + r_1\left[\underset{\sim}{A}\ \hat{\underset{\sim}{\Psi}}^1\underset{\sim}{K}\ \underset{\sim}{K} + (\underset{\sim}{D}-\underset{\sim}{E}^1+n\underset{\sim}{N}^1)\hat{\underset{\sim}{\Phi}}^1\underset{\sim}{K} - (\frac{n(n+1)}{2}\ \underset{\sim}{L}^1+n\underset{\sim}{Q}^1)\hat{\underset{\sim}{\Psi}}^1\right]\underset{\sim}{\Gamma}^2 \end{aligned} \tag{2.110a}$$

$$\begin{aligned} \underset{\sim}{F}^2 = {} & -r_2\left[\underset{\sim}{A}\ \underset{\sim}{\Psi}^2\ \underset{\sim}{K}\ \underset{\sim}{K} + (\underset{\sim}{D}-\underset{\sim}{E}^2+n\underset{\sim}{N}^2)\underset{\sim}{\Phi}^2\underset{\sim}{K} - (\frac{n(n+1)}{2}\ \underset{\sim}{L}^2+n\underset{\sim}{Q}^2)\underset{\sim}{\Psi}^2\right]\underset{\sim}{\Gamma}^1 \\ & -r_2\left[\underset{\sim}{A}\ \hat{\underset{\sim}{\Psi}}^2\underset{\sim}{K}\ \underset{\sim}{K} + (\underset{\sim}{D}-\underset{\sim}{E}^2+n\underset{\sim}{N}^2)\hat{\underset{\sim}{\Phi}}^2\underset{\sim}{K} - (\frac{n(n+1)}{2}\ \underset{\sim}{L}^2+n\underset{\sim}{Q}^2)\hat{\underset{\sim}{\Psi}}^2\right]\underset{\sim}{\Gamma}^2 \ . \end{aligned} \tag{2.110b}$$

Using (2.109a,b) we eliminate $\underset{\sim}{\Gamma}^1$, $\underset{\sim}{\Gamma}^2$. We find

$$\begin{bmatrix} \underset{\sim}{F}^1 \\ \underset{\sim}{F}^2 \end{bmatrix} = \underset{\sim}{K} \begin{bmatrix} \underset{\sim}{U}^1 \\ \underset{\sim}{U}^2 \end{bmatrix} \tag{2.111}$$

$\underset{\sim}{K}$ is the dynamic stiffness matrix of the element. We note, again, that the computational effort required to obtain the matrix is independent of the thickness $r_2 - r_1$ of the element. Details may be found in [9].

2.5 SUMMARY

In this chapter, we reviewed the development of the following elements:

1) The element modeling the semi-infinite region $x \geq 0$ of a layered stratum in plane strain [23]. The dynamic stiffness matrix of the element is given by (2.41).

2) The element modeling the rectangular region $x_1 \leq x \leq x_2$ in plane strain [9]. The dynamic stiffness matrix is obtained as in (2.46).

3) The element modeling the semi-infinite region $x \geq 0$ of a layered stratum in antiplane shear [16,23]. The dynamic stiffness matrix of the element is given by (2.74).

4) The element modeling the rectangular region $x_1 \leq x \leq x_2$ in antiplane shear [9]. The dynamic stiffness matrix is found as in (2.78).

5) The element modeling the semi-infinite axisymmetric region $r \geq r_o > 0$ of a layered stratum (for any Fourier number n and symmetric or antisymmetric vibrations [23, 6]. The dynamic stiffness matrix is given by (2.104).

6) The element modeling the axisymmetric region $0 \leq r \leq r_o$ [9]. The dynamic stiffness matrix is given by (2. 107).

7) The element modeling the axisymmetric region $r_1 \leq r \leq r_2$ [9]. The dynamic stiffness matrix is found as in (2.111).

We note that the computational effort required to obtain the matrices is independent of the length of the element, for plane elements (2,4) or its thickness in the radial direction, for axisymmetric elements (6,7). The surface of the elements is free and the base fixed.

CHAPTER 3

PLANE ELEMENTS

The boundary conditions on the surface and the base of the plane elements described in Chapter 2 were homogeneous. The surface was assumed free and the base fixed. In this chapter we consider inhomogeneous boundary conditions. We develop in detail elements for the analysis of time-harmonic wave motion in plane strain or antiplane shear in the rectangular region $-\frac{L}{2} \leq x \leq \frac{L}{2}$, $0 \leq z \leq h$ of a layered stratum. The base of the elements is taken fixed. However, boundary conditions corresponding to a rigid and rough strip footing are prescribed on the surface of the elements. Other inhomogeneous boundary conditions are also discussed. Finally, an application is considered which shows that the method is accurate and efficient.

3.1 PLANE STRAIN

We consider time-harmonic wave motion in plane strain in the rectangular region $-\frac{L}{2} \leq x \leq \frac{L}{2}$, $0 \leq z \leq h$ of a layered stratum. Let us assume that the stratum is divided into N sublayers. We rewrite the governing differential equations, in sublayer j,

$$(\lambda_j + 2G_j)\,\frac{\partial^2 u}{\partial x^2} + \lambda_j\,\frac{\partial^2 w}{\partial x \partial z} + G_j\left[\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right] + \rho_j \omega^2 u = 0 \tag{3.1a}$$

$$(\lambda_j + 2G_j)\,\frac{\partial^2 w}{\partial z^2} + \lambda_j\,\frac{\partial^2 u}{\partial x \partial z} + G_j\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial z}\right] + \rho_j \omega^2 w = 0\ . \tag{3.1b}$$

The amplitudes u, w, σ_z, τ_{xz} must be continuous at $z = z_j$,

$2 \le j \le N$, $-\frac{L}{2} \le x \le \frac{L}{2}$. The conditions expressing continuity of σ_z, τ_{xz} are

$$(\lambda_{j-1} + 2G_{j-1})\frac{\partial w}{\partial z}\bigg|_{z=z_j^-} + \lambda_{j-1}\frac{\partial u}{\partial x}\bigg|_{z=z_j^-} = (\lambda_j + 2G_j)\frac{\partial w}{\partial z}\bigg|_{z=z_j^+} + \lambda_j\frac{\partial u}{\partial x}\bigg|_{z=z_j^+} \tag{3.2a}$$

$$G_{j-1}\left[\frac{\partial w}{\partial x}\bigg|_{z=z_j^-} + \frac{\partial u}{\partial z}\bigg|_{z=z_j^-}\right] = G_j\left[\frac{\partial w}{\partial x}\bigg|_{z=z_j^+} + \frac{\partial u}{\partial z}\bigg|_{z=z_j^+}\right] . \tag{3.2b}$$

Boundary conditions corresponding to a rigid and rough strip footing are prescribed on the surface of the region:

$$u(x,0) = \Delta_x \tag{3.3a}$$

$$w(x,0) = \Delta_z - \theta x \quad , \tag{3.3b}$$

$$-\frac{L}{2} \le x \le \frac{L}{2} .$$

Δ_x, Δ_z, θ are the amplitudes of the horizontal displacement, vertical displacement and rotation of the footing respectively. The rotation is taken positive in the counterclockwise direction. The base of the region is fixed:

$$u(x,h) = 0 \tag{3.3c}$$

$$w(x,h) = 0 \quad , \tag{3.3d}$$

$$-\frac{L}{2} \le x \le \frac{L}{2} .$$

The nodes of the element are (see figure 3.1) taken at (0,0) with degrees of freedom

$$\Delta_x, \Delta_z, \theta$$

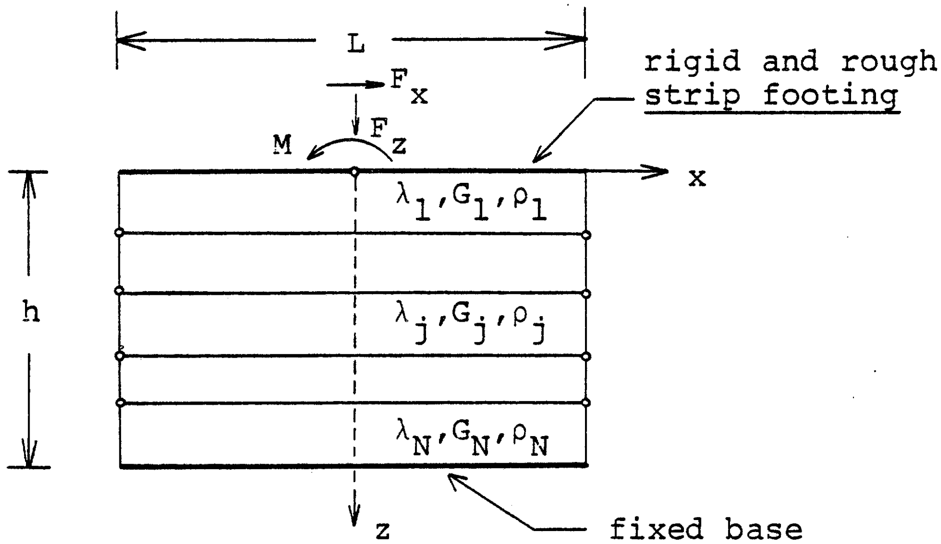


Figure 3.1 - The region $-\frac{L}{2} \leq x \leq \frac{L}{2}$, $0 \leq z \leq h$, in plane strain.

at $(-\frac{L}{2}, z_j)$, $2 \leq j \leq N$, with nodal displacements

$$u_j^1 = u(-\frac{L}{2}, z_j)$$
$$w_j^1 = w(-\frac{L}{2}, z_j) \quad ,$$

and at $(\frac{L}{2}, z_j)$, $2 \leq j \leq N$, with nodal displacements

$$u_j^2 = u(\frac{L}{2}, z_j)$$
$$w_j^2 = w(\frac{L}{2}, z_j) \quad .$$

The forces corresponding to these degrees of freedom are, at (0,0), the horizontal force F_x, the vertical force F_z and the moment M, at $(-\frac{L}{2}, z_j)$, $2 \leq j \leq N$, the horizontal force $P_{x,j}^1$ and the vertical force $P_{z,j}^1$, and at $(\frac{L}{2}, z_j)$, $2 \leq j \leq N$, the horizontal force $P_{x,j}^2$ and the vertical force $P_{z,j}^2$. In some calculations it is convenient to use the nodes at $(-\frac{L}{2}, 0)$ and $(\frac{L}{2}, 0)$ instead of the node at (0,0). The degrees of freedom at these nodes are

$$u_1^1 = u(-\frac{L}{2}, 0)$$
$$w_1^1 = w(-\frac{L}{2}, 0) \quad ,$$

and

$$u_1^2 = u(\frac{L}{2}, 0)$$
$$w_1^2 = w(\frac{L}{2}, 0) \quad ,$$

respectively. The corresponding nodal forces are $P_{x,1}^1$, $P_{z,1}^1$ and $P_{x,1}^2$, $P_{z,1}^2$. We note that the degrees of freedom at these nodes are related to Δ_x, Δ_z, θ by

$$u_1^1 = \Delta_x \qquad (3.4a)$$

$$u_1^2 = \Delta_x \qquad (3.4b)$$

$$w_1^1 = \Delta_z + \frac{L}{2}\theta \qquad (3.4c)$$

$$w_1^2 = \Delta_z - \frac{L}{2}\theta . \qquad (3.4d)$$

F_x, F_z, M are related to $P_{x,1}^1$, $P_{z,1}^1$, $P_{x,1}^2$, $P_{z,1}^2$ by

$$F_x = P_{x,1}^1 + P_{x,1}^2 \qquad (3.5a)$$

$$F_z = P_{z,1}^1 + P_{z,1}^2 \qquad (3.5b)$$

$$M = \frac{L}{2} P_{z,1}^1 - \frac{L}{2} P_{z,1}^2 \quad . \qquad (3.5c)$$

We assume that u, w are linear functions of z in each sublayer. For $z_j \leq z \leq z_{j+1}$, $1 \leq j \leq N$, we have

$$u(-\tfrac{L}{2}, z) = u_j^1 \frac{z_{j+1} - z}{h_j} + u_{j+1}^1 \frac{z - z_j}{h_j}$$

$$w(-\tfrac{L}{2}, z) = w_j^1 \frac{z_{j+1} - z}{h_j} + w_{j+1}^1 \frac{z - z_j}{h_j}$$

$$u(\tfrac{L}{2}, z) = u_j^2 \frac{z_{j+1} - z}{h_j} + u_{j+1}^2 \frac{z - z_j}{h_j}$$

$$w(\tfrac{L}{2}, z) = w_j^2 \frac{z_{j+1} - z}{h_j} + w_{j+1}^2 \frac{z - z_j}{h_j} \quad .$$

Note that $u_{N+1}^1 = w_{N+1}^1 = u_{N+1}^2 = w_{N+1}^2 = 0$ by the boundary conditions (3.3c,d). At $z = 0$, i.e., at the surface, the amplitudes u, w are constant and linear functions of x respectively by the boundary conditions (3.3a,b). We write

$$u(x,0) = u_1^1 \left(\tfrac{1}{2} - \tfrac{x}{L}\right) + u_1^2 \left(\tfrac{1}{2} + \tfrac{x}{L}\right)$$

$$w(x,0) = w_1^1 \left(\tfrac{1}{2} - \tfrac{x}{L}\right) + w_1^2 \left(\tfrac{1}{2} + \tfrac{x}{L}\right) \quad ,$$

$$-\tfrac{L}{2} \leq x \leq \tfrac{L}{2} \quad .$$

The consistent nodal forces are given by

$$P_{x,1}^1 = -\int_{z_1}^{z_2} \sigma_x\Big|_{x=-\frac{L}{2}} \left[\frac{z_2-z}{h_1}\right] dz - \int_{-\frac{L}{2}}^{\frac{L}{2}} \tau_{xz}\Big|_{z=0} \left(\tfrac{1}{2} - \tfrac{x}{L}\right) dx \qquad (3.6a)$$

$$P_{z,1}^1 = -\int_{z_1}^{z_2} \tau_{xz}\Big|_{x=-\frac{L}{2}} \left[\frac{z_2-z}{h_1}\right] dz - \int_{-\frac{L}{2}}^{\frac{L}{2}} \sigma_z\Big|_{z=0} \left(\tfrac{1}{2} - \tfrac{x}{L}\right) dx \qquad (3.6b)$$

$$P_{x,1}^2 = \int_{z_1}^{z_2} \sigma_x\Big|_{x=\frac{L}{2}} \left[\frac{z_2-z}{h_1}\right] dz - \int_{-\frac{L}{2}}^{\frac{L}{2}} \tau_{xz}\Big|_{z=0} \left(\tfrac{1}{2} + \tfrac{x}{L}\right) dx \qquad (3.6c)$$

$$P_{z,1}^2 = \int_{z_1}^{z_2} \tau_{xz}\Big|_{x=\frac{L}{2}} \left[\frac{z_2-z}{h_1}\right] dz - \int_{-\frac{L}{2}}^{\frac{L}{2}} \sigma_z\Big|_{z=0} \left(\tfrac{1}{2} + \tfrac{x}{L}\right) dx \qquad (3.6d)$$

$$P^1_{x,j} = - \int_{z_{j-1}}^{z_j} \sigma_x \Big|_{x=-\frac{L}{2}} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz - \int_{z_j}^{z_{j+1}} \sigma_x \Big|_{x=-\frac{L}{2}} \left[\frac{z_{j+1}-z}{h_j}\right] dz \tag{3.6e}$$

$$P^1_{z,j} = - \int_{z_{j-1}}^{z_j} \tau_{xz} \Big|_{x=-\frac{L}{2}} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz - \int_{z_j}^{z_{j+1}} \tau_{xz} \Big|_{x=-\frac{L}{2}} \left[\frac{z_{j+1}-z}{h_j}\right] dz \tag{3.6f}$$

$$P^2_{x,j} = \int_{z_{j-1}}^{z_j} \sigma_x \Big|_{x=\frac{L}{2}} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz + \int_{z_j}^{z_{j+1}} \sigma_x \Big|_{x=\frac{L}{2}} \left[\frac{z_{j+1}-z}{h_j}\right] dz \tag{3.6g}$$

$$P^2_{z,j} = \int_{z_{j-1}}^{z_j} \tau_{xz} \Big|_{x=\frac{L}{2}} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz + \int_{z_j}^{z_{j+1}} \tau_{xz} \Big|_{x=\frac{L}{2}} \left[\frac{z_{j+1}-z}{h_j}\right] dz \tag{3.6h}$$

$$2 \leq j \leq N.$$

Let $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$ be the vectors of nodal displacements at $(-\frac{L}{2}, z_j)$, $2 \leq j \leq N$, and $(\frac{L}{2}, z_j)$, $2 \leq j \leq N$, respectively:

$$U^\ell_{2s-1} = u^\ell_{s+1}$$

$$U^\ell_{2s} = w^\ell_{s+1}$$

$$1 \leq s \leq N-1, \ \ell = 1, 2 \quad .$$

The vectors of nodal forces at $(-\frac{L}{2}, z_j)$, $2 \leq j \leq N$, and $(\frac{L}{2}, z_j)$, $2 \leq j \leq N$, are denoted by $\underset{\sim}{F}^1$, $\underset{\sim}{F}^2$ respectively:

$$F^\ell_{2s-1} = P^\ell_{x,s+1}$$

$$F^\ell_{2s} = P^\ell_{z,s+1}$$

$$1 \leq s \leq N-1, \ \ell = 1, 2 \quad .$$

Our objective is to determine the dynamic stiffness matrix

$\underset{\sim}{K}$ of the element, i.e., the matrix relating F_x, F_z, M, $\underset{\sim}{F}^1$, $\underset{\sim}{F}^2$ to Δ_x, Δ_z, θ, $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$:

$$\left[\begin{array}{c} \underset{\sim}{F}^1 \\ \hline F_x \\ \hline F_z \\ \hline M \\ \hline \underset{\sim}{F}^2 \end{array}\right] = \left[\begin{array}{c|c|c|c|c} \underset{\sim}{K}^{11} & \underset{\sim}{K}^{1x} & \underset{\sim}{K}^{1z} & \underset{\sim}{K}^{1\theta} & \underset{\sim}{K}^{12} \\ \hline \underset{\sim}{K}^{x1} & K_{xx} & K_{xz} & K_{x\theta} & \underset{\sim}{K}^{x2} \\ \hline \underset{\sim}{K}^{z1} & K_{zx} & K_{zz} & K_{z\theta} & \underset{\sim}{K}^{z2} \\ \hline \underset{\sim}{K}^{\theta 1} & K_{\theta x} & K_{\theta z} & K_{\theta\theta} & \underset{\sim}{K}^{\theta 2} \\ \hline \underset{\sim}{K}^{21} & \underset{\sim}{K}^{2x} & \underset{\sim}{K}^{2z} & \underset{\sim}{K}^{2\theta} & \underset{\sim}{K}^{22} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{U}^1 \\ \hline \Delta_x \\ \hline \Delta_z \\ \hline \theta \\ \hline \underset{\sim}{U}^2 \end{array}\right] \quad . \tag{3.7}$$

Let us obtain the submatrices $\underset{\sim}{K}^{11}$, $\underset{\sim}{K}^{12}$, $\underset{\sim}{K}^{21}$, $\underset{\sim}{K}^{22}$. We set

$$\Delta_x = \Delta_z = 0 \ , \quad \theta = 0 .$$

The boundary conditions (3.3a,b) become

$$u(x,0) = 0 \tag{3.8a}$$

$$w(x,0) = 0 \ , \tag{3.8b}$$

$$-\frac{L}{2} \le x \le \frac{L}{2} .$$

Any displacement amplitudes u, w satisfying the differential equations (3.1a,b), the conditions (3.2a,b) at $z = z_j$, $2 \le j \le N$, and the boundary conditions (3.3c,d) and (3.8a,b) may be written, in general, as a superposition of modes of the form

$$u(x,z) = U(z)\exp(-ikx) \tag{3.9a}$$

$$w(x,z) = W(z)\exp(-ikx) \ . \tag{3.9b}$$

The eigenfunctions U, W and the wave numbers k are obtained using the procedure given in section 2.2. The difference is that the conditions

$$U(0) = 0$$
$$W(0) = 0 \quad ,$$

indicating that the surface is fixed, must be satisfied instead of (2.7e,f), which correspond to a free surface. The algebraic eigenvalue problem is

$$[k^2\underset{\sim}{A} + ik\underset{\sim}{B} + \underset{\sim}{G} - \omega^2\underset{\sim}{M}]\underset{\sim}{\Delta} = \underset{\sim}{0} \quad . \tag{3.10}$$

$\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{G}$, $\underset{\sim}{M}$ are (2N-2) x (2N-2) matrices obtained from those in (2.26) by deleting the first two rows and the first two columns. $\underset{\sim}{\Delta}$ is a (2N-2)-vector (the eigenvector corresponding to the eigenvalue k) with components

$$\left.\begin{aligned} \Delta_{2j-1} &= U(z_{j+1}) \\ \Delta_{2j} &= W(z_{j+1}) \end{aligned}\right\} \quad 1 \leq j \leq N-1 \quad .$$

We form the (2N-2) x (2N-2) diagonal matrix $\underset{\sim}{K}$ with entries the wave numbers k_j, $1 \leq j \leq 2N-2$, corresponding to modes which decay for large $x > 0$ or propagate energy in the positive x-direction:

$$\underset{\sim}{K} = \text{diag}\ [k_j] \quad . \tag{3.11}$$

The modal matrix $\underset{\sim}{X}$ is

$$\underset{\sim}{X} = [\underset{\sim}{\Delta}^1, \underset{\sim}{\Delta}^2, \ldots, \underset{\sim}{\Delta}^{2N-2}] \quad , \tag{3.12}$$

with $\underset{\sim}{\Delta}^j$ corresponding to k_j as chosen in (3.11). $\underset{\sim}{\bar{X}}$ is obtained as

$$\overline{\mathbf{X}} = \mathbf{T}\,\mathbf{X} \quad , \tag{3.13}$$

$\mathbf{T}$ being a diagonal (2N-2) x (2N-2) matrix:

$$\begin{aligned} T_{2j-1,2j-1} &= 1 \\ T_{2j,2j} &= -1 \end{aligned} \qquad 1 \leq j \leq N-1 \quad . \tag{3.14}$$

The diagonal matrix $\mathbf{E}$ is

$$\mathbf{E} = \mathrm{diag}\left[\ \exp(-ik_j L)\ \right] \quad , \tag{3.15}$$

with k_j as chosen in (3.11). The dynamic stiffness matrix of the region with fixed surface and fixed base is obtained using the procedure given in section 2.2 for the region with free surface and fixed base. We have

$$\mathbf{U}^1 = \mathbf{X}\boldsymbol{\Gamma}^1 + \overline{\mathbf{X}}\,\mathbf{E}\,\boldsymbol{\Gamma}^2 \tag{3.16a}$$

$$\mathbf{U}^2 = \mathbf{X}\,\mathbf{E}\,\boldsymbol{\Gamma}^1 + \overline{\mathbf{X}}\,\boldsymbol{\Gamma}^2 \quad . \tag{3.16b}$$

The nodal forces are given by

$$\mathbf{F}^1 = [i\,\mathbf{A}\,\mathbf{X}\,\mathbf{K} + \mathbf{D}\,\mathbf{X}]\boldsymbol{\Gamma}^1 + [-i\,\mathbf{A}\,\overline{\mathbf{X}}\,\mathbf{E}\,\mathbf{K} + \mathbf{D}\,\overline{\mathbf{X}}\,\mathbf{E}]\boldsymbol{\Gamma}^2 \tag{3.17a}$$

$$\mathbf{F}^2 = -[i\,\mathbf{A}\,\mathbf{X}\,\mathbf{E}\,\mathbf{K} + \mathbf{D}\,\mathbf{X}\,\mathbf{E}]\boldsymbol{\Gamma}^1 - [-i\,\mathbf{A}\,\overline{\mathbf{X}}\,\mathbf{K} + \mathbf{D}\,\overline{\mathbf{X}}]\boldsymbol{\Gamma}^2 . \tag{3.17b}$$

The participation factors $\boldsymbol{\Gamma}^1$, $\boldsymbol{\Gamma}^2$ are eliminated using (3.16a,b). We find

$$\begin{bmatrix} \mathbf{F}^1 \\ \hline \mathbf{F}^2 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \hline \mathbf{K}^{21} & \mathbf{K}^{22} \end{array}\right] \begin{bmatrix} \mathbf{U}^1 \\ \hline \mathbf{U}^2 \end{bmatrix} \quad . \tag{3.18}$$

$\mathbf{K}^{11}$, $\mathbf{K}^{12}$, $\mathbf{K}^{21}$, $\mathbf{K}^{22}$ are the submatrices we are looking for. After some manipulations [9] we obtain

$$\underset{\sim}{K}^{11} = (\underset{\sim}{R} + \underset{\sim}{R}')(\underset{\sim}{I} - \underset{\sim}{J}'\underset{\sim}{J})^{-1} - \underset{\sim}{R}' \qquad (3.19a)$$

$$\underset{\sim}{K}^{12} = [\underset{\sim}{K}^{21}]^T = - [\underset{\sim}{K}^{11} + \underset{\sim}{R}']\underset{\sim}{J}' \qquad (3.19b)$$

$$\underset{\sim}{K}^{22} = [\underset{\sim}{K}^{11}]' \qquad (3.19c)$$

($\underset{\sim}{I}$ is the identity matrix) .

with

$$\underset{\sim}{R} = i \, \underset{\sim}{A}\,\underset{\sim}{X}\,\underset{\sim}{K}\,\underset{\sim}{X}^{-1} + \underset{\sim}{D}$$

$$\underset{\sim}{R}' = \underset{\sim}{T}\,\underset{\sim}{R}\,\underset{\sim}{T}$$

$$\underset{\sim}{J} = \underset{\sim}{X}\,\underset{\sim}{E}\,\underset{\sim}{X}^{-1}$$

$$\underset{\sim}{J}' = \underset{\sim}{T}\,\underset{\sim}{J}\,\underset{\sim}{T}$$

$$[\underset{\sim}{K}^{11}]' = \underset{\sim}{T}\,\underset{\sim}{K}^{11}\,\underset{\sim}{T} \; .$$

It may be shown that

$$[\underset{\sim}{K}^{12}]^T = \underset{\sim}{T}\,\underset{\sim}{K}^{12}\,\underset{\sim}{T} \; . \qquad (3.20)$$

The dynamic stiffness matrix is symmetric. This is shown by the following argument. Consider a conventional finite element mesh covering the region $-\frac{L}{2} \leq x \leq \frac{L}{2}$, $0 \leq z \leq h$. The spacing of the elements in the z-direction is the same as the spacing of sublayers. The spacing in the x-direction is taken uniform. Let n be the number of columns of elements. The width of the elements is $\delta = L/n$. Thus the nodes of the mesh are at (x_ℓ, z_j), $1 \leq \ell \leq n+1$, $1 \leq j \leq N+1$, with

$$x_\ell = -\frac{L}{2} + (\ell-1)\frac{L}{n} ,$$

and the corresponding nodal degrees of freedom are

$$u_{\ell,j} = u(x_\ell, z_j)$$

$$w_{\ell,j} = w(x_\ell, z_j) \quad .$$

The elements are four-node rectangles with linear interpolation functions. Using the principle of virtual work for time-harmonic motion in plane strain, we obtain

$$\underset{\sim}{F} = \left[\underset{\sim}{K} - \omega^2 \underset{\sim}{M}\right] \underset{\sim}{U} . \qquad (3.21)$$

Both $\underset{\sim}{K}$, the stiffness matrix, and $\underset{\sim}{M}$, the mass matrix, are symmetric. The components of $\underset{\sim}{F}, \underset{\sim}{U}$ and the entries of $\underset{\sim}{K}$, $\underset{\sim}{M}$ corresponding to the nodes at the surface and the base of the region are deleted since $u_{\ell,1} = w_{\ell,1} = u_{\ell,N+1} = w_{\ell,N+1} = 0$, $1 \leq \ell \leq n+1$, by the boundary conditions. The components of $\underset{\sim}{F}$ corresponding to the interior nodes (x_ℓ, z_j), $2 \leq \ell \leq n$, $2 \leq j \leq N$, are set equal to zero since no external forces are applied there. Any solution $\underset{\sim}{U}$ of (3.21) may be written as a superposition of discrete modes of the form

$$u_{\ell,j} = U_j \exp(-ikx_\ell) \qquad (3.22a)$$

$$w_{\ell,j} = W_j \exp(-ikx_\ell) \quad . \qquad (3.22b)$$

The eigenvalue problem which yields the eigenvalues k and the discrete eigenfunctions U_j, W_j, $2 \leq j \leq N$ may be obtained by considering the equations in (3.21) corresponding to a column of interior nodes (these equations are homogeneous since the corresponding components of $\underset{\sim}{F}$ are equal to zero). It is easily seen that the frequency equation is of the form

$$f(\omega^2, \exp(-ik\delta)) = 0 \quad ,$$

in which f is a polynomial of degree 4N-4 in the term $\exp(-ik\delta)$ and of degree 2N-2 in ω^2. Thus, in general, for a given ω, there are 4N-4 discrete modes. It may be shown [14] that these modes approach the modes given by the eigenvalue problem (3.10) as $n \to \infty$, i.e., $\delta \to 0$. In fact, equation (2.18), which we used in order to obtain (2.25) and hence the eigenvalue problem (3.10) is the principle of virtual work for time-harmonic motion in plane strain (in a rectangular region, $x_1 \leq x \leq x_2$, $0 \leq z \leq h$), specialized further for x-harmonic motion. Thus our solution is the limit of the discrete solution (obtained using the finite element mesh) as the number of columns $n \to \infty$. We note that the degrees of freedom at interior nodes may be condensed out of the matrix $\underset{\sim}{K} - \omega^2 \underset{\sim}{M}$ in (3.21), since the forces at these nodes are equal to zero. The condensed matrix is symmetric and relates $\underset{\sim}{F}^1$, $\underset{\sim}{F}^2$ to $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$. Thus the dynamic stiffness matrix $\underset{\sim}{K}$ in (3.18), being the limit of this matrix as $n \to \infty$, is symmetric.

Let us now obtain a particular solution of (3.7) for which

$$\Delta_x = 1 \quad , \quad \Delta_z = 0 \quad , \theta = 0 \quad .$$

We denote the loads and displacements corresponding to this particular solution by

$$\underset{\sim}{F}^{1,1}, \ \underset{\sim}{F}^{2,1}, \ \underset{\sim}{U}^{1,1}, \ \underset{\sim}{U}^{2,1}, \ F_x^1, \ F_z^1, \ M_1 \quad .$$

Substituting

$$u(x,z) = U(z) \tag{3.23a}$$

$$w(x,z) = 0 \tag{3.23b}$$

into the differential equations (3.1a,b), the conditions (3.2a,b) and the boundary conditions (3.3a,b,c,d), we find that U must satisfy the differential equation, in sublayer j,

$$G_j \frac{d^2U}{dz^2} + \rho_j \omega^2 U = 0 \quad , \tag{3.24a}$$

the conditions at $z = z_j$, $2 \leq j \leq N$,

$$G_{j-1} \left.\frac{dU}{dz}\right|_{z=z_j^-} = G_j \left.\frac{dU}{dz}\right|_{z=z_j^+} \quad , \tag{3.24b}$$

and the boundary conditions

$$U(0) = 1 \tag{3.24c}$$

$$U(h) = 0 \quad . \tag{3.24d}$$

An exact solution of this problem is easily obtained since, from (3.24a), we have, in sublayer j,

$$U(z) = A_1^j \cos[\frac{\omega}{C_T^j} z] + A_2^j \sin[\frac{\omega}{C_T^j} z]$$

The conditions (3.24b,c,d) give a system of linear equations for A_1^j, A_2^j, $1 \leq j \leq N$. Let us obtain the corresponding discrete solution. We note that the solution (3.23a,b) is identical to (3.9a,b) with $k = 0$ and $W(z) \equiv 0$. Therefore, (2.25) applies. We obtain

$$[\underset{\sim}{G} - \omega^2 \underset{\sim}{M}]\underset{\sim}{\Delta} = \underset{\sim}{F} \quad , \tag{3.25}$$

with $\Delta_1 = 1$, $\Delta_j = U(z_j)$, $2 \leq j \leq N$,

$$F_1 = -\tau_1 = -G_1 \left.\frac{dU}{dz}\right|_{z=0}, \quad F_j = 0, \quad 2 \le j \le N .$$

$\underset{\sim}{G}$, $\underset{\sim}{M}$ are assembled from the sublayer matrices $\underset{\sim}{G}^j$, $\underset{\sim}{M}^j$:

$$\underset{\sim}{G}^j = \frac{G_j}{h_j} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.26a}$$

$$\underset{\sim}{M}^j = \rho_j \; h_j \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} . \tag{3.26b}$$

Equation (3.25) is easily solved since $\underset{\sim}{G}$, $\underset{\sim}{M}$ are symmetric tri-diagonal matrices. Let $\underset{\sim}{Y}$ be defined by

$$\left.\begin{aligned} Y_{2j-1} &= \Delta_{j+1} \\ Y_{2j} &= 0 \quad . \end{aligned}\right\} \quad 1 \le j \le N-1$$

We have

$$\underset{\sim}{U}^{1,1} = \underset{\sim}{Y} \tag{3.27a}$$

$$\underset{\sim}{U}^{2,1} = \underset{\sim}{Y} \quad . \tag{3.27b}$$

The forces $\underset{\sim}{F}^{1,1}$, $\underset{\sim}{F}^{2,1}$ are found using (2.37) with k = 0:

$$\underset{\sim}{F}^{1,1} = \underset{\sim}{D} \begin{bmatrix} 1 \\ 0 \\ \underset{\sim}{Y} \end{bmatrix} \tag{3.27c}$$

$$\underset{\sim}{F}^{2,1} = -\underset{\sim}{D} \begin{bmatrix} 1 \\ 0 \\ \underset{\sim}{Y} \end{bmatrix} . \tag{3.27d}$$

$\underset{\sim}{D}$ is obtained from that in (2.37) by deleting the first two rows. The structure of $\underset{\sim}{D}$ indicates that since in this solution the vertical displacements are equal to zero, the horizontal forces in (3.27c, d) are also equal to zero. Calculation of F_x^1, F_z^1, M_1 according to (3.5a,b) and (3.6a,b,c,d) gives

$$F_x^1 = F_1 \, L \tag{3.27e}$$

$$F_z^1 = 0 \tag{3.27f}$$

$$M_1 = \frac{L}{2} \, G_1 (1 - \Delta_2) \quad . \tag{3.27g}$$

F_1 is obtained from (3.25). It is the amplitude of the shear traction on the surface. Δ_2 is also found from (3.25).

Working similarly we calculate a particular solution of (3.7) for which

$$\Delta_x = 0, \quad \Delta_z = 1 \quad , \theta = 0 \quad .$$

The loads and displacements corresponding to this particular solution are denoted by

$$\underset{\sim}{F}^{1,2}, \ \underset{\sim}{F}^{2,2}, \ \underset{\sim}{U}^{1,2}, \ \underset{\sim}{U}^{2,2}, \ F_x^2, \ F_z^2, \ M_2 \quad .$$

Substituting

$$u(x,z) = 0 \tag{3.28a}$$

$$w(x,z) = W(z) \tag{3.28b}$$

into the differential equations (3.1a,b), the conditions (3.2a,b) and the boundary conditions (3.3a,b,c,d), we obtain that W must satisfy the differential equation, in sublayer j,

$$(\lambda_j + 2G_j) \frac{d^2W}{dz^2} + \rho_j \ \omega^2 \ W = 0 \ , \tag{3.29a}$$

the conditions at $z = z_j$, $2 \leq j \leq N$,

$$(\lambda_{j-1} + 2G_{j-1}) \left.\frac{dW}{dz}\right|_{z=z_j^-} = (\lambda_j + 2G_j) \left.\frac{dW}{dz}\right|_{z=z_j^+} , \qquad (3.29b)$$

and the boundary conditions

$$W(0) = 1 \qquad (3.29c)$$

$$W(h) = 0 \quad . \qquad (3.29d)$$

Again it is easy to solve this problem exactly. Equation (3.29a) gives

$$W(z) = A_1^j \cos\left[\frac{\omega}{C_L^j} z\right] + A_2^j \sin\left[\frac{\omega}{C_L^j} z\right] \quad .$$

The conditions (3.29b,c,d) give a system of linear equations for A_1^j, A_2^j, $1 \leq j \leq N$. Let us calculate the corresponding discrete solution. The solution (3.28a,b) is identical to (3.9a,b) with $k = 0$ and $U(z) \equiv 0$. Thus (2.25) applies. We have

$$[\underset{\sim}{G} - \omega^2 \underset{\sim}{M}]\underset{\sim}{\Delta} = \underset{\sim}{F} \quad , \qquad (3.30)$$

with $\Delta_1 = 1$, $\Delta_j = W(z_j)$, $2 \leq j \leq N$,

$$F_1 = -\sigma_1 = -(\lambda_1 + 2G_1)\left.\frac{dW}{dz}\right|_{z=0} , \; F_j = 0, \; 2 \leq j \leq N \; .$$

$\underset{\sim}{G}$, $\underset{\sim}{M}$ are assembled from the sublayer matrices $\underset{\sim}{G}^j$, $\underset{\sim}{M}^j$:

$$\underset{\sim}{G}^j = \frac{\lambda_j + 2G_j}{h_j} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (3.31a)$$

$$\underset{\sim}{M}^j = \rho_j h_j \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \quad . \qquad (3.31b)$$

Thus $\underset{\sim}{G}$, $\underset{\sim}{M}$ are symmetric tridiagonal matrices. Equation (3.30) is easily solved. Let $\underset{\sim}{Y}$ be defined by

$$\left.\begin{aligned} Y_{2j-1} &= 0 \\ Y_{2j} &= \Delta_{j+1} \end{aligned}\right\} \quad 1 \leq j \leq N-1 \; .$$

We obtain

$$\underset{\sim}{U}^{1,2} = \underset{\sim}{Y} \tag{3.32a}$$

$$\underset{\sim}{U}^{2,2} = \underset{\sim}{Y} \quad . \tag{3.32b}$$

The forces $\underset{\sim}{F}^{1,2}$, $\underset{\sim}{F}^{2,2}$ are found using (2.37) with k = 0:

$$\underset{\sim}{F}^{1,2} = \underset{\sim}{D} \begin{bmatrix} 0 \\ 1 \\ \underset{\sim}{Y} \end{bmatrix} \tag{3.32c}$$

$$\underset{\sim}{F}^{2,2} = -\underset{\sim}{D} \begin{bmatrix} 0 \\ 1 \\ \underset{\sim}{Y} \end{bmatrix} \quad . \tag{3.32d}$$

$\underset{\sim}{D}$ is obtained from that in (2.37) by deleting the first two rows. Considering the structure of $\underset{\sim}{D}$, we note that, since in this solution the horizontal displacements are equal to zero, the vertical forces in (3.32c,d) are also equal to zero. Calculation of F_x^2, F_z^2, M_2 according to (3.5a,b) and (3.6a,b,c,d) gives

$$F_x^2 = 0 \tag{3.32e}$$

$$F_z^2 = F_1 \; L \tag{3.32f}$$

$$M_2 = 0 \quad . \tag{3.32g}$$

F_1 is obtained from (3.30). It is the amplitude of the normal traction on the surface.

Let us now obtain a particular solution of equation (3.7) for which

$$\Delta_x = \Delta_z = 0 \quad , \quad \theta = 1 \quad .$$

The loads and displacements are denoted by

$$\underset{\sim}{F}^{1,3},\ \underset{\sim}{F}^{2,3},\ \underset{\sim}{U}^{1,3},\ \underset{\sim}{U}^{2,3},\ F_x^3,\ F_z^3, M_3 \quad .$$

It is interesting to calculate such a solution as a limit of solutions already available to us. We consider

$$u(x,z) = U_1(z)\ \exp(-ikx)$$

$$w(x,z) = W_1(z)\ \exp(-ikx) \quad ,$$

satisfying the differential equations (3.1a,b), the conditions (3.2a,b) at $z = z_j$, $2 \leq j \leq N$, and the boundary conditions

$$u(x,0) = 0 \tag{3.33a}$$

$$w(x,0) = \frac{1}{k} \exp\ (-ikx) \tag{3.33b}$$

$$u(x,h) = 0 \tag{3.33c}$$

$$w(x,h) = 0 \quad . \tag{3.33d}$$

The corresponding discrete solution is calculated using

$$[k^2\underset{\sim}{A} + ik\underset{\sim}{B} + \underset{\sim}{G} - \omega^2\underset{\sim}{M}]\underset{\sim}{\Delta}^1 = \frac{1}{k}\ \underset{\sim}{F}^1 \tag{3.34}$$

with $\Delta^1_{2j-1} = U_1(z_{j+1})$, $\Delta^1_{2j} = W_1(z_j)$ $1 \leq j \leq N-1$

$$F^1_1 = \frac{1}{2}\ ik(\lambda_1 + G_1), \quad F^1_2 = -\frac{1}{6}\ k^2 G_1 h_1 + \frac{\lambda_1 + 2G_1}{h_1} + \frac{1}{6}\omega^2\rho_1 h_1,$$

$$F^1_{2j-1} = F^1_{2j} = 0 \quad , \quad 2 \leq j \leq N-1 \ .$$

$\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{G}$, $\underset{\sim}{M}$ are obtained from those in (2.26) by deleting the first two rows and the first two columns since $U_1(0) = 0$ and $W_1(0) = \frac{1}{k}$ by the boundary conditions (3.33a,b). Equation (3.34) is rewritten as

$$[\underset{\sim}{R}(k)]\underset{\sim}{\Delta}^1 = \frac{1}{k}\underset{\sim}{F}^1 \quad , \tag{3.35}$$

in which $\underset{\sim}{R}(k) = k^2\underset{\sim}{A} + ik\underset{\sim}{B} + \underset{\sim}{C}$,

with $\quad \underset{\sim}{C} = \underset{\sim}{G} - \omega^2\underset{\sim}{M}$.

We assume that ω is not a cut-off frequency for the stratum with fixed surface and fixed base, i.e., $\underset{\sim}{R}(0) = \underset{\sim}{C}$ is nonsingular. Then for k sufficiently close to zero $\underset{\sim}{R}(k)$ is invertible. We calculate the inverse $\underset{\sim}{R}^{-1}(k)$ for small k. The entries of $\underset{\sim}{R}^{-1}$ are rational functions of k. They are infinitely many times differentiable at k = 0. Hence a Taylor series exists for $\underset{\sim}{R}^{-1}(k)$ around k = 0:

$$\underset{\sim}{R}^{-1}(k) = \underset{\sim}{R}^{-1}(0) + k \left.\frac{d\underset{\sim}{R}^{-1}}{dk}\right|_{k=0} + \cdots \tag{3.36}$$

We have $\quad \underset{\sim}{R}^{-1}(0) = \underset{\sim}{C}^{-1}$.

Moreover, since $\underset{\sim}{R}\underset{\sim}{R}^{-1} = \underset{\sim}{I}$ ($\underset{\sim}{I}$ being the identity matrix), we find

$$\left.\frac{d\underset{\sim}{R}}{dk}\right|_{k=0} \underset{\sim}{R}^{-1}(0) + \underset{\sim}{R}(0) \left.\frac{d\underset{\sim}{R}^{-1}}{dk}\right|_{k=0} = \underset{\sim}{0} ,$$

in which $\underset{\sim}{0}$ is the null matrix. It is easily seen that

$$\left.\frac{d\underset{\sim}{R}}{dk}\right|_{k=0} = i\underset{\sim}{B} .$$

We obtain

$$\left.\frac{d\underset{\sim}{R}^{-1}}{dk}\right|_{k=0} = -\, i\, \underset{\sim}{C}^{-1}\underset{\sim}{B}\, \underset{\sim}{C}^{-1} \quad .$$

Thus from (3.36) we find

$$\underset{\sim}{R}^{-1}(k) = \underset{\sim}{C}^{-1} - ik\, \underset{\sim}{C}^{-1}\underset{\sim}{B}\, \underset{\sim}{C}^{-1} + O(k^2) \quad , \qquad k \to 0. \qquad (3.37)$$

Equation (3.35) gives

$$\underset{\sim}{\Delta}^1 = \frac{1}{k}\, \underset{\sim}{C}^{-1}\underset{\sim}{F}^1 - i\, \underset{\sim}{C}^{-1}\underset{\sim}{B}\, \underset{\sim}{C}^{-1}\underset{\sim}{F}^1 + O(k) \quad , \qquad k \to 0. \qquad (3.38)$$

Working similarly the discrete solution corresponding to

$$u(x,z) = U_2(z)\exp(ikx)$$

$$w(x,z) = W_2(z)\exp(ikx)$$

with $U_2(0) = 0$, $W_2(0) = \frac{1}{k}$, $U_2(h) = W_2(h) = 0$, is obtained as

$$\underset{\sim}{\Delta}^2 = \frac{1}{k}\, \underset{\sim}{C}^{-1}\underset{\sim}{F}^2 + i\, \underset{\sim}{C}^{-1}\underset{\sim}{B}\, \underset{\sim}{C}^{-1}\underset{\sim}{F}^2 + O(k), \qquad k \to 0 \;, \qquad (3.39)$$

in which $F_1^2 = -F_1^1$, $F_2^2 = F_2^1$, $F_{2j-1}^2 = F_{2j}^2 = 0$, $2 \le j \le N-1$. Let us consider the superposition of the two solutions

$$\underset{\sim}{U}^x = \frac{1}{2i}\begin{bmatrix} 0 \\ \frac{1}{k} \\ \underset{\sim}{\Delta}^1 \end{bmatrix} \exp(-ikx) - \frac{1}{2i}\begin{bmatrix} 0 \\ \frac{1}{k} \\ \underset{\sim}{\Delta}^2 \end{bmatrix} \exp(ikx) \quad ,$$

with

$$\left.\begin{aligned} U_{2j-1}^x &= u(x,\, z_j) \\ U_{2j}^x &= w(x,\, z_j) \end{aligned}\right\} \quad 1 \le j \le N \quad .$$

We find that

$$\lim_{k\to 0} \underset{\sim}{U}^x = \begin{bmatrix} 0 \\ -x \\ -x\, \underset{\sim}{C}^{-1}\underset{\sim}{F} - \underset{\sim}{C}^{-1}\underset{\sim}{B}\, \underset{\sim}{C}^{-1}\underset{\sim}{F} - \underset{\sim}{C}^{-1}\underset{\sim}{f} \end{bmatrix} \tag{3.40}$$

with $F_1 = 0$, $F_2 = \frac{\lambda_1 + 2G_1}{h_1} + \frac{1}{6}\,\omega^2\rho_1 h_1$, $F_{2j-1} = F_{2j} = 0$, $2 \leq j \leq N-1$,

$$f_1 = -\tfrac{1}{2}(\lambda_1 + G_1),\ f_2 = 0,\ f_{2j-1} = f_{2j} = 0,\ 2 \leq j \leq N-1.$$

Considering the structure of the matrices $\underset{\sim}{C}$, $\underset{\sim}{B}$ and the vectors $\underset{\sim}{F}$, $\underset{\sim}{f}$, it is easily seen that the term $-x\,\underset{\sim}{C}^{-1}\underset{\sim}{F}$ corresponds to the amplitude w, i.e., the vertical displacements, while $-\underset{\sim}{C}^{-1}\underset{\sim}{B}\,\underset{\sim}{C}^{-1}\underset{\sim}{F} - \underset{\sim}{C}^{-1}\underset{\sim}{f}$ gives the amplitude of the horizontal displacements. Thus w is a linear function of x and u is independent of x. Moreover, the semidiscrete solution (3.40) satisfies at $z = 0$ the boundary conditions (3.3a,b) with $\theta = 1$, $\Delta_x = \Delta_z = 0$, i.e., unit rotation and zero translation of the footing. Obviously, the solution (3.40) is admissible only in a finite region since the amplitude w is not bounded for arbitrarily large x. Using (3.40) we obtain

$$\underset{\sim}{U}^{1,3} = \frac{L}{2}\,\underset{\sim}{C}^{-1}\underset{\sim}{F} - \underset{\sim}{C}^{-1}\underset{\sim}{B}\,\underset{\sim}{C}^{-1}\underset{\sim}{F} - \underset{\sim}{C}^{-1}\underset{\sim}{f} \tag{3.41a}$$

$$\underset{\sim}{U}^{2,3} = -\frac{L}{2}\,\underset{\sim}{C}^{-1}\underset{\sim}{F} - \underset{\sim}{C}^{-1}\underset{\sim}{B}\,\underset{\sim}{C}^{-1}\underset{\sim}{F} - \underset{\sim}{C}^{-1}\underset{\sim}{f}\ . \tag{3.41b}$$

The loads $\underset{\sim}{F}^{1,3}$, $\underset{\sim}{F}^{2,3}$, F_x^3, F_z^3, M_3 may be obtained by taking the limit, as $k \to 0$, of the loads corresponding to the superposition of the x-harmonic solutions. However, it is more convenient to substitute (3.40) directly into the expressions for the consistent nodal forces (3.6a,b,c,d,e,f,g,h). Before giving the results of this calculation, let us consider an alternative method of deriving the particular solution (3.40). We seek a priori the solution

$$u(x,z) = U(z) \tag{3.42a}$$

$$w(x,z) = -xW(z) \quad , \tag{3.42b}$$

satisfying the differential equations (3.1a,b), the conditions (3.2a,b) at $z = z_j$, $2 \leq j \leq N$, and the boundary conditions

$$u(x,0) = 0 \tag{3.43a}$$

$$w(x,0) = -x \tag{3.43b}$$

$$u(x,h) = 0 \tag{3.43c}$$

$$w(x,h) = 0 \quad . \tag{3.43d}$$

Substituting (3.42a,b) into (3.1a,b), (3.2a,b), (3.43a,b,c,d), we find that U,W must satisfy the differential equations in sublayer j

$$G_j \frac{d^2U}{dz^2} - (\lambda_j + G_j) \frac{dW}{dz} + \rho_j\omega^2 U = 0 \tag{3.44a}$$

$$(\lambda_j + 2G_j)\frac{d^2W}{dz^2} + \rho_j\omega^2 W = 0 \quad , \tag{3.44b}$$

the conditions at $z = z_j$, $2 \leq j \leq N$,

$$(\lambda_{j-1} + 2G_{j-1})\frac{dW}{dz}\bigg|_{z=z_j^-} = (\lambda_j + 2G_j)\frac{dW}{dz}\bigg|_{z=z_j^+} \tag{3.44c}$$

$$G_{j-1} \left[-W + \frac{dU}{dz}\right]_{z=z_j^-} = G_j \left[-W + \frac{dU}{dz}\right]_{z=z_j^+} , \tag{3.44d}$$

and the boundary conditions

$$U(0) = 0 \tag{3.44e}$$

$$W(0) = 1 \tag{3.44f}$$

$$U(h) = 0 \tag{3.44g}$$

$$W(h) = 0 \quad . \tag{3.44h}$$

An exact solution is easily obtained. Equation (3.44b) gives

$$W(z) = A_1^j \cos\left[\frac{\omega}{C_L^j} z\right] + A_2^j \sin\left[\frac{\omega}{C_L^j} z\right] \quad .$$

Substituting this into equation (3.44a) we find

$$U(z) = B_1^j \cos\left[\frac{\omega}{C_T^j} z\right] + B_2^j \sin\left[\frac{\omega}{C_T^j} z\right]$$

$$- \frac{C_L^j}{\omega} A_1^j \sin\left[\frac{\omega}{C_L^j} z\right] + \frac{C_L^j}{\omega} A_2^j \cos\left[\frac{\omega}{C_L^j} z\right] \quad .$$

The conditions (3.44c,d,e,f,g,h) provide a system of linear equations for A_1^j, A_2^j, B_1^j, B_2^j, $1 \leq j \leq N$. Using the finite element method, we obtain the corresponding discrete solution. Let δU and δW be virtual amplitudes. After multiplying the left-hand sides of equations (3.44a) and (3.44b) by δU and δW respectively and adding them, we obtain, for sublayer j:

$$\omega^2 \rho_j \int_{z_j}^{z_{j+1}} U\delta U dz + \omega^2 \rho_j \int_{z_j}^{z_{j+1}} W\delta W dz + G_j \int_{z_j}^{z_{j+1}} \frac{d}{dz}[-W + \frac{dU}{dz}]\,\delta U dz$$

$$-\lambda_j \int_{z_j}^{z_{j+1}} \frac{dW}{dz}\,\delta U dz + (\lambda_j + 2G_j) \int_{z_j}^{z_{j+1}} \frac{d}{dz}[\frac{dW}{dz}]\;\delta W dz = 0 \quad . \tag{3.45}$$

After integrating by parts and rearranging the terms, we find

$$-\omega^2\rho_j\int_{z_j}^{z_{j+1}} U\delta U dz - \omega^2\rho_j\int_{z_j}^{z_{j+1}} W\delta W dz - G_j\int_{z_j}^{z_{j+1}} W \frac{d}{dz}[\delta U]dz$$

$$-\lambda_j\int_{z_j}^{z_{j+1}}\frac{dW}{dz}\,\delta U dz + G_j\int_{z_j}^{z_{j+1}}\frac{dU}{dz}\,\frac{d}{dz}[\delta U]dz + (\lambda_j + 2G_j)\int_{z_j}^{z_{j+1}}\frac{dW}{dz}\,\frac{d}{dz}[\delta W]dz$$

$$= G_j\left[-W + \frac{dU}{dz}\right]\delta U\Bigg|_{z_j}^{z_{j+1}} + (\lambda_j + 2G_j)\,\frac{dW}{dz}\,\delta W\Bigg|_{z_j}^{z_{j+1}} \quad . \qquad (3.46)$$

Assuming that U, W, δU, δW are linear functions of z in each sublayer, equation (3.46),which holds for arbitrary δU, δW, gives

$$\underset{\sim}{S}^j \begin{bmatrix} U_j \\ W_j \\ U_{j+1} \\ W_{j+1} \end{bmatrix} = \begin{bmatrix} -\tau_j \\ -\sigma_j \\ \tau_{j+1} \\ \sigma_{j+1} \end{bmatrix} , \qquad (3.47)$$

with

$$\tau_\ell = G_j\left[-W + \frac{dU}{dz}\right]_{z=z_\ell}$$

$$\sigma_\ell = (\lambda_j + 2G_j)\,\frac{dW}{dz}\bigg|_{z=z_\ell} \qquad \ell = j,\ j+1$$

$$U_\ell = U(z_\ell)$$

$$W_\ell = W(z_\ell) \ .$$

$\underset{\sim}{S}^j$ is a 4x4 matrix given by

$$
\begin{bmatrix}
\frac{G_j}{h_j} - \frac{1}{3}\omega^2 \rho_j h_j & -\frac{1}{2}(\lambda_j - G_j) & -\frac{G_j}{h_j} - \frac{1}{6}\omega^2 \rho_j h_j & \frac{1}{2}(\lambda_j + G_j) \\
0 & -\frac{\lambda_j + 2G_j}{h_j} + \frac{1}{3}\omega^2 \rho_j h_j & 0 & \frac{\lambda_j + 2G_j}{h_j} + \frac{1}{6}\omega^2 \rho_j h_j \\
-\frac{G_j}{h_j} - \frac{1}{6}\omega^2 \rho_j h_j & -\frac{1}{2}(\lambda_j + G_j) & \frac{G_j}{h_j} - \frac{1}{3}\omega^2 \rho_j h_j & \frac{1}{2}(\lambda_j - G_j) \\
0 & \frac{\lambda_j + 2G_j}{h_j} + \frac{1}{6}\omega^2 \rho_j h_j & 0 & -\frac{\lambda_j + 2G_j}{h_j} + \frac{1}{3}\omega^2 \rho_j h_j
\end{bmatrix}
\qquad (3.48)
$$

We note that

$$\sigma_z\Big|_{z=z_j} = -\, x\sigma_j$$

$$\tau_{xz}\Big|_{z=z_j} = \tau_j \quad .$$

Since σ_z and τ_{xz} are continuous at $z = z_j$ $2 \leq j \leq N$, we have

$$\sigma_j = (\lambda_{j-1} + 2G_{j-1})\frac{dW}{dz}\Big|_{z=z_j^-} = (\lambda_j + 2G_j)\frac{dW}{dz}\Big|_{z=z_j^+} \tag{3.49a}$$

$$\tau_j = G_{j-1}\left[- W + \frac{dU}{dz}\right]_{z=z_j^-} = G_j\left[- W + \frac{dU}{dz}\right]_{z=z_j^+} . \tag{3.49b}$$

We obtain

$$\underset{\sim}{S}\ \underset{\sim}{\Delta} = \underset{\sim}{P} \quad . \tag{3.50}$$

$\underset{\sim}{S}$ is assembled from $\underset{\sim}{S}^j$. $\underset{\sim}{\Delta}$, $\underset{\sim}{P}$ are 2N-vectors:

$$\Delta_{2j-1} = U_j \qquad \qquad 1 \leq j \leq N$$

$$\Delta_{2j} = W_j$$

$$P_1 = -\,\tau_1 = -\,G_1\left[- W + \frac{dU}{dz}\right]_{z=0}$$

$$P_2 = -\,\sigma_1 = -\,(\lambda_1 + 2G_1)\frac{dW}{dz}\Big|_{z=0}$$

$$P_{2j-1} = P_{2j} = 0 \quad , \quad 2 \leq j \leq N \quad .$$

For the particular solution that we are considering, we have

$$\Delta_1 = 0 \quad , \quad \Delta_2 = 1 \quad ,$$

as prescribed by (3.44e,f). The semidiscrete solution $\underset{\sim}{U}^x$ corresponding to (3.42a,b) is given by

$$U^x_{2j-1} = u(x,z_j) = \Delta_{2j-1}$$
$$1 \le j \le N$$
$$U^x_{2j} = w(x,z_j) = -x\Delta_{2j} \quad .$$

It is easily checked that this solution is identical to the one given by (3.40). Actually, equation(3.40) shows that the calculations required to obtain this particular solution are simple. The matrix $\tilde{C}$ may be split into two symmetric tridiagonal matrices, one corresponding to horizontal displacements and the other to vertical displacements. Moreover, matrix $\tilde{B}$ has a simple structure.

Using (3.6e,f,g,h), we obtain

$$\tilde{F}^{1,3} = \tilde{H}^{-L/2}\,\tilde{\Delta} \tag{3.51a}$$
$$\tilde{F}^{2,3} = -\tilde{H}^{L/2}\,\tilde{\Delta}\ . \tag{3.51b}$$

The matrix $\tilde{H}^x$ is obtained by assembling the sublayer matrices $\tilde{H}^{x,j}$ (4x4) given by

$$\tilde{H}^{x,j} = \begin{bmatrix} 0 & -\frac{x}{2}\lambda_j & 0 & \frac{x}{2}\lambda_j \\ \frac{1}{2}G_j & \frac{1}{3}G_jh_j & -\frac{1}{2}G_j & \frac{1}{6}G_jh_j \\ 0 & -\frac{x}{2}\lambda_j & 0 & \frac{x}{2}\lambda_j \\ \frac{1}{2}G_j & \frac{1}{6}G_jh_j & -\frac{1}{2}G_j & \frac{1}{3}G_jh_j \end{bmatrix} \tag{3.52}$$

($\tilde{H}^{-L/2}$, $\tilde{H}^{L/2}$ in (3.51) are obtained from $\tilde{H}^x$ evaluated at $x = -L/2$ and $x = L/2$, respectively after deleting the first two rows, since the forces in (3.51) correspond to nodes below the surface). Similarly, using (3.5a,b,c) and (3.6a,b,c,d) we find

$$F_x^3 = P_1 L + \frac{1}{2}\lambda_1(1 - \Delta_4)L \qquad (3.53a)$$

$$F_z^3 = 0 \qquad (3.53b)$$

$$M_3 = \frac{1}{12} P_2 L^3 + (\frac{1}{3} G_1 h_1 - \frac{1}{2} G_1 \Delta_3 + \frac{1}{6} G_1 h_1 \Delta_4)L\ . \qquad (3.53c)$$

P_1, P_2, Δ_3,Δ_4 are obtained from (3.50).

Having found the solutions (3.9a,b) satisfying homogeneous boundary conditions, i.e., fixed surface and fixed base, and particular solutions (3.23a,b), (3.28a,b) and (3.42a,b) corresponding to unit horizontal translation, vertical translation and rotation of the footing, let us obtain the dynamic stiffness matrix $\underset{\sim}{K}$ of the element. The submatrices $\underset{\sim}{K}^{11}$, $\underset{\sim}{K}^{12}$, $\underset{\sim}{K}^{21}$, $\underset{\sim}{K}^{22}$ have already been found. They are given by (3.19a,b, c). The matrix $\underset{\sim}{K}$ in (3.7) is symmetric. This may be seen by an argument similar to the one used to show that the matrix in (3.18) is symmetric. We note that the degrees of freedom $u_{\ell,1}$, $w_{\ell,1}$, $1 \leq \ell \leq n+1$, at the surface of the region may be condensed kinematically to the three degrees of freedom, Δ_x, Δ_z, θ by the boundary conditions (3.3a,b). Moreover, the three particular semidiscrete solutions that we found are the limits of the corresponding fully discrete solutions (obtained using the finite element mesh) as the number of columns $n \to \infty$. The condensed dynamic stiffness matrix is symmetric. Therefore, $\underset{\sim}{K}$, being the limit of this matrix as $n \to \infty$, is symmetric. We have

$$\underset{\sim}{K}^{1x} = \underset{\sim}{F}^{1,1} - \underset{\sim}{K}^{11}\underset{\sim}{U}^{1,1} - \underset{\sim}{K}^{12}\underset{\sim}{U}^{2,1} \qquad (3.54a)$$

$$\underset{\sim}{K}^{2x} = \underset{\sim}{F}^{2,1} - \underset{\sim}{K}^{21}\underset{\sim}{U}^{1,1} - \underset{\sim}{K}^{22}\underset{\sim}{U}^{2,1} \qquad (3.54b)$$

$$\underset{\sim}{K}^{1z} = \underset{\sim}{F}^{1,2} - \underset{\sim}{K}^{11}\,\underset{\sim}{U}^{1,2} - \underset{\sim}{K}^{12}\,\underset{\sim}{U}^{2,2} \tag{3.54c}$$

$$\underset{\sim}{K}^{2z} = \underset{\sim}{F}^{2,2} - \underset{\sim}{K}^{21}\,\underset{\sim}{U}^{1,2} - \underset{\sim}{K}^{22}\,\underset{\sim}{U}^{2,2} \tag{3.54d}$$

$$\underset{\sim}{K}^{1\theta} = \underset{\sim}{F}^{1,3} - \underset{\sim}{K}^{11}\,\underset{\sim}{U}^{1,3} - \underset{\sim}{K}^{12}\,\underset{\sim}{U}^{2,3} \tag{3.54e}$$

$$\underset{\sim}{K}^{2\theta} = \underset{\sim}{F}^{2,3} - \underset{\sim}{K}^{21}\,\underset{\sim}{U}^{1,3} - \underset{\sim}{K}^{22}\,\underset{\sim}{U}^{2,3} \tag{3.54f}$$

Since $\underset{\sim}{K}$ is symmetric, we obtain

$$\underset{\sim}{K}^{x1} = [\underset{\sim}{K}^{1x}]^T, \; \underset{\sim}{K}^{z1} = [\underset{\sim}{K}^{1z}]^T, \;\; \underset{\sim}{K}^{\theta 1} = [\underset{\sim}{K}^{1\theta}]^T \tag{3.55a}$$

$$\underset{\sim}{K}^{x2} = [\underset{\sim}{K}^{2x}]^T, \; \underset{\sim}{K}^{z2} = [\underset{\sim}{K}^{2z}]^T, \; \underset{\sim}{K}^{\theta 2} = [\underset{\sim}{K}^{2\theta}]^T \; . \tag{3.55b}$$

Thus we find

$$K_{xx} = F_x^1 - \underset{\sim}{K}^{x1}\,\underset{\sim}{U}^{1,1} - \underset{\sim}{K}^{x2}\,\underset{\sim}{U}^{2,1} \tag{3.56a}$$

$$K_{zz} = F_z^2 - \underset{\sim}{K}^{z1}\,\underset{\sim}{U}^{1,2} - \underset{\sim}{K}^{z2}\,\underset{\sim}{U}^{2,2} \tag{3.56b}$$

$$K_{\theta\theta} = M_3 - \underset{\sim}{K}^{\theta 1}\,\underset{\sim}{U}^{1,3} - \underset{\sim}{K}^{\theta 2}\,\underset{\sim}{U}^{2,3} \tag{3.56c}$$

$$K_{xz} = F_x^2 - \underset{\sim}{K}^{x1}\,\underset{\sim}{U}^{1,2} - \underset{\sim}{K}^{x2}\,\underset{\sim}{U}^{2,2} \tag{3.56d}$$

$$K_{x\theta} = F_x^3 - \underset{\sim}{K}^{x1}\,\underset{\sim}{U}^{1,3} - \underset{\sim}{K}^{x2}\,\underset{\sim}{U}^{2,3} \tag{3.56e}$$

$$K_{z\theta} = F_z^3 - \underset{\sim}{K}^{z1}\,\underset{\sim}{U}^{1,3} - \underset{\sim}{K}^{z2}\,\underset{\sim}{U}^{2,3} \; . \tag{3.56f}$$

Using the symmetry of $\underset{\sim}{K}$ we obtain

$$K_{zx} = K_{xz}, \;\; K_{\theta x} = K_{x\theta}, \; K_{\theta z} = K_{z\theta} \; . \tag{3.57}$$

It must be pointed out that, since the horizontal translation and the rotation are uncoupled from the vertical translation,

$$K_{xz} = K_{zx} = 0 \quad , \quad K_{z\theta} = K_{\theta z} = 0 \quad . \tag{3.58}$$

The derivation of the dynamic stiffness matrix of the element is now complete. We note that the computational effort required to calculate $\underset{\sim}{K}$ is independent of the length L of the element.

3.2 ANTIPLANE SHEAR

Let us now consider time-harmonic wave motion in antiplane shear in the rectangular region $-\frac{L}{2} \leq x \leq \frac{L}{2}$, $0 \leq z \leq h$ of a layered stratum. We assume that the stratum is divided into N sublayers. The governing differential equation, in sublayer j, is

$$G_j \frac{\partial^2 v}{\partial x^2} + G_j \frac{\partial^2 v}{\partial z^2} + \rho_j \omega^2 v = 0 \quad . \tag{3.59}$$

The amplitudes v, τ_{yz} must be continuous at $z=z_j$, $2 \leq j \leq N$ $-\frac{L}{2} \leq x \leq \frac{L}{2}$. The condition expressing continuity of τ_{yz} is

$$G_{j-1} \left. \frac{\partial v}{\partial z} \right|_{z=z_j^-} = G_j \left. \frac{\partial v}{\partial z} \right|_{z=z_j^+} \quad . \tag{3.60}$$

At z=0, i.e., on the surface, the boundary condition corresponding to a rigid and rough strip footing is prescribed:

$$v(x,0) = \Delta_y \quad , \tag{3.61a}$$

$$-\frac{L}{2} \leq x \leq \frac{L}{2} \quad .$$

Δ_y is the amplitude of the antiplane horizontal displacement of the footing. At the base the element is fixed:

$$v(x,h) = 0 \quad , \tag{3.61b}$$

$$-\frac{L}{2} \leq x \leq \frac{L}{2} \quad .$$

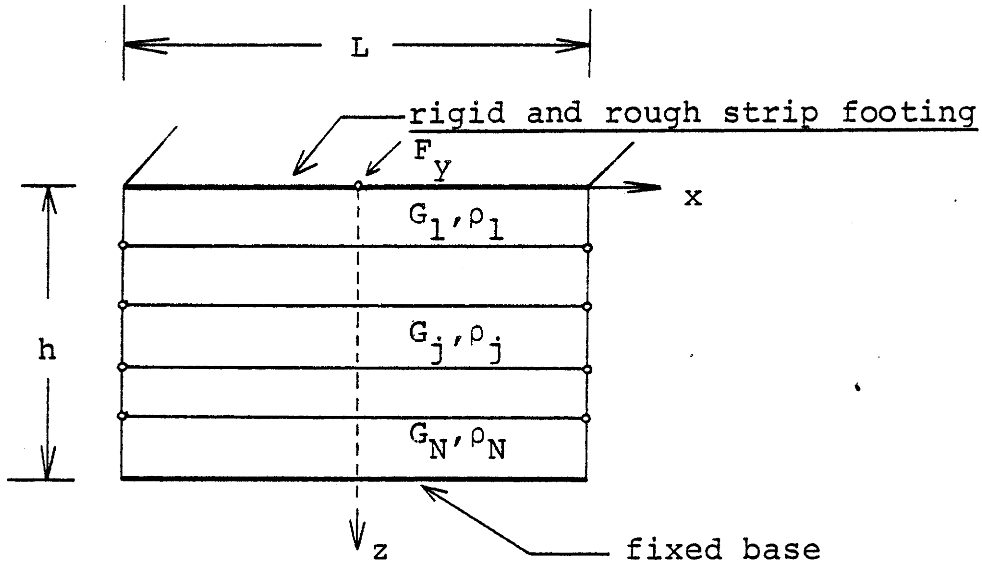


Figure 3.2 - The region $-\frac{L}{2} \le x \le \frac{L}{2}$, $0 \le z \le h$, in antiplane shear.

The nodes of the element (see figure 3.2) are at (x,0), $-\frac{L}{2} \le x \le \frac{L}{2}$, with nodal displacement Δ_y, at $(-\frac{L}{2}, z_j), 2 \le j \le N$, and at $(\frac{L}{2}, z_j)$, $2 \le j \le N$, with nodal displacements

$$v_j^1 = v(-\frac{L}{2}, z_j) \ , \ v_j^2 = v(\frac{L}{2}, z_j) \ .$$

The forces corresponding to these degrees of freedom are, at (x,0), the antiplane horizontal force F_y, and at $(-\frac{L}{2}, z_j)$, $(\frac{L}{2}, z_j)$, $2 \le j \le N$, the antiplane horizontal forces P_j^1, P_j^2 respectively. We assume that v is a linear function of z in each sublayer. For $z_j \le z \le z_{j+1}$, $1 \le j \le N$, we have

$$v(-\frac{L}{2}, z) = v_j^1 \frac{z_{j+1} - z}{h_j} + v_{j+1}^1 \frac{z - z_j}{h_j}$$

$$v(\frac{L}{2}, z) = v_j^2 \frac{z_{j+1} - z}{h_j} + v_{j+1}^2 \frac{z - z_j}{h_j} ,$$

with $v_1^1 = v_1^2 = \Delta_y$, $v_{N+1}^1 = v_{N+1}^2 = 0$. The consistent nodal forces are given by $(2 \le j \le N)$

$$F_y = -\int_{z_1}^{z_2} \tau_{xy}\Big|_{x=-\frac{L}{2}} \left[\frac{z_2-z}{h_1}\right] dz - \int_{-\frac{L}{2}}^{\frac{L}{2}} \tau_{yz}\Big|_{z=0} dx + \int_{z_1}^{z_2} \tau_{xy}\Big|_{x=\frac{L}{2}} \left[\frac{z_2-z}{h_1}\right] dz \tag{3.62a}$$

$$P_j^1 = -\int_{z_{j-1}}^{z_j} \tau_{xy}\Big|_{x=-\frac{L}{2}} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz - \int_{z_j}^{z_{j+1}} \tau_{xy}\Big|_{x=-\frac{L}{2}} \left[\frac{z_{j+1}-z}{h_j}\right] dz \tag{3.62b}$$

$$P_j^2 = \int_{z_{j-1}}^{z_j} \tau_{xy}\Big|_{x=\frac{L}{2}} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz + \int_{z_j}^{z_{j+1}} \tau_{xy}\Big|_{x=\frac{L}{2}} \left[\frac{z_{j+1}-z}{h_j}\right] dz \,. \tag{3.62c}$$

Let $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$ be the vectors of nodal displacements at $(-\frac{L}{2}, z_j)$, $2 \le j \le N$, and $(\frac{L}{2}, z_j)$, $2 \le j \le N$, respectively:

$$U_s^\ell = v_{s+1}^\ell$$

$$1 \le s \le N-1, \ \ell = 1, 2 \quad .$$

The corresponding vectors of nodal forces are denoted by $\underset{\sim}{F}^1$, $\underset{\sim}{F}^2$, respectively:

$$F_s^\ell = P_{s+1}^\ell$$

$$1 \le s \le N-1, \ \ell = 1, 2.$$

Let us calculate the dynamic stiffness matrix $\underset{\sim}{K}$ of the element:

$$\begin{bmatrix} \underset{\sim}{F}^1 \\ \hline F_y \\ \hline \underset{\sim}{F}^2 \end{bmatrix} = \left[\begin{array}{c|c|c} \underset{\sim}{K}^{11} & \underset{\sim}{K}^{1y} & \underset{\sim}{K}^{12} \\ \hline \underset{\sim}{K}^{y1} & K_{yy} & \underset{\sim}{K}^{y2} \\ \hline \underset{\sim}{K}^{21} & \underset{\sim}{K}^{2y} & \underset{\sim}{K}^{22} \end{array}\right] \begin{bmatrix} \underset{\sim}{U}^1 \\ \hline \Delta_y \\ \hline \underset{\sim}{U}^2 \end{bmatrix} \,. \tag{3.63}$$

The submatrices $\underset{\sim}{K}^{11}$, $\underset{\sim}{K}^{12}$, $\underset{\sim}{K}^{21}$, $\underset{\sim}{K}^{22}$ are obtained first. We set $\Delta_y = 0$. The boundary condition (3.61a) becomes

$$v(x,0) = 0 \qquad (3.64)$$
$$-\frac{L}{2} \leq x \leq \frac{L}{2} \quad .$$

Any displacement amplitude v satisfying the differential equation (3.59), the condition (3.60) at $z = z_j$, $2 \leq j \leq N$, and the boundary conditions (3.61b) and (3.64) may be written, in general, as a superposition of modes of the form

$$v(x,z) = V(z)\exp(-ikx) \quad . \qquad (3.65)$$

The procedure given in section 2.3 may be used to obtain the eigenfunctions V and the wave numbers k. The difference is that the surface is now fixed by the boundary condition (3.64) while (2.52c) indicates a free surface. Working as in section 2.3, we obtain the corresponding algebraic eigenvalue problem:

$$[k^2\underset{\sim}{A} + \underset{\sim}{G} - \omega^2\underset{\sim}{M}]\underset{\sim}{\Delta} = \underset{\sim}{0} \quad . \qquad (3.66)$$

$\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{M}$ are (N-1)x(N-1) matrices obtained from those in (2.63) by deleting the first row and the first column. $\underset{\sim}{\Delta}$ is given by

$$\Delta_j = V(z_{j+1}) \quad , \quad 1 \leq j \leq N-1 \quad .$$

The (N-1)x(N-1) diagonal matrix $\underset{\sim}{K}$ is formed with entries the wave numbers k_j, $1 \leq j \leq N-1$, corresponding to modes which decay for large $x > 0$, i.e., $Im(k_j) < 0$, or travel in the positive x-direction, i.e., $Re(k_j) > 0$ and $Im(k_j) = 0$:

$$\underset{\sim}{K} = \text{diag}\,[k_j] \quad . \qquad (3.67)$$

The modal matrix $\underset{\sim}{X}$ is

$$\underset{\sim}{X} = [\underset{\sim}{\Delta}^1, \underset{\sim}{\Delta}^2, \ldots, \underset{\sim}{\Delta}^{N-1}] \quad , \qquad (3.68)$$

with $\underset{\sim}{\Delta}^j$ corresponding to k_j as chosen in (3.67). Finally, we form the diagonal matrix $\underset{\sim}{E}$:

$$\underset{\sim}{E} = \text{diag}\ [\exp(-ik_j L)] \quad , \qquad (3.69)$$

with k_j as in (3.67). The dynamic stiffness matrix for the region with fixed surface and fixed base is obtained using the procedure given in section 2.3 for the region with free surface and fixed base. We write

$$\underset{\sim}{U}^1 = \underset{\sim}{X}\ \underset{\sim}{\Gamma}^1 + \underset{\sim}{X}\ \underset{\sim}{E}\ \underset{\sim}{\Gamma}^2 \qquad (2.70a)$$

$$\underset{\sim}{U}^2 = \underset{\sim}{X}\ \underset{\sim}{E}\ \underset{\sim}{\Gamma}^1 + \underset{\sim}{X}\ \underset{\sim}{\Gamma}^2 \quad . \qquad (3.70b)$$

The nodal forces are

$$\underset{\sim}{F}^1 = i\ \underset{\sim}{A}\ \underset{\sim}{X}\ \underset{\sim}{K}\ \underset{\sim}{\Gamma}^1 - i\ \underset{\sim}{A}\ \underset{\sim}{X}\ \underset{\sim}{E}\ \underset{\sim}{K}\ \underset{\sim}{\Gamma}^2 \qquad (3.71a)$$

$$\underset{\sim}{F}^2 = -\ i\ \underset{\sim}{A}\ \underset{\sim}{X}\ \underset{\sim}{E}\ \underset{\sim}{K}\ \underset{\sim}{\Gamma}^1 + i\ \underset{\sim}{A}\ \underset{\sim}{X}\ \underset{\sim}{K}\ \underset{\sim}{\Gamma}^2 \quad . \qquad (3.71b)$$

Eliminating $\underset{\sim}{\Gamma}^1$, $\underset{\sim}{\Gamma}^2$ we find

$$\begin{bmatrix} \underset{\sim}{F}^1 \\ \text{--} \\ \underset{\sim}{F}^2 \end{bmatrix} = \left[\begin{array}{c|c} \underset{\sim}{K}^{11} & \underset{\sim}{K}^{12} \\ \hline \underset{\sim}{K}^{21} & \underset{\sim}{K}^{22} \end{array}\right] \begin{bmatrix} \underset{\sim}{U}^1 \\ \text{--} \\ \underset{\sim}{U}^2 \end{bmatrix} \quad , \qquad (3.72)$$

with

$$\underset{\sim}{K}^{11} = \underset{\sim}{R}(\underset{\sim}{I} + \underset{\sim}{J}\ \underset{\sim}{J})(\underset{\sim}{I} - \underset{\sim}{J}\ \underset{\sim}{J})^{-1} \qquad (3.73a)$$

$$\underset{\sim}{K}^{12} = \underset{\sim}{K}^{21} = (\underset{\sim}{K}^{21})^T = -2\ \underset{\sim}{R}\ \underset{\sim}{J}\ (\underset{\sim}{I} - \underset{\sim}{J}\ \underset{\sim}{J})^{-1} \qquad (3.73b)$$

$$\underset{\sim}{K}^{22} = \underset{\sim}{K}^{11} \quad , \qquad (3.73c)$$

($\underset{\sim}{I}$ is the identity matrix)

in which
$$\underset{\sim}{R} = i\,\underset{\sim}{A}\,\underset{\sim}{X}\,\underset{\sim}{K}\,\underset{\sim}{X}^{-1}$$
$$\underset{\sim}{J} = \underset{\sim}{X}\,\underset{\sim}{E}\,\underset{\sim}{X}^{-1} \quad .$$

Clearly, the dynamic stiffness matrix in (3.72) is symmetric.

Let us now obtain a particular solution of (3.63) for which $\Delta_y = 1$. The forces and displacements corresponding to this solution are denoted by

$$\underset{\sim}{F}^{1,1},\ \underset{\sim}{F}^{2,1},\ \underset{\sim}{U}^{1,1},\ \underset{\sim}{U}^{2,1},\ F_y^1 \quad .$$

Substituting

$$v(x,z) = V(z) \tag{3.74}$$

into the differential equation (3.59), the condition (3.60) and the boundary conditions (3.61a,b), we find that V must satisfy the differential equation in sublayer j,

$$G_j \frac{d^2V}{dz^2} + \rho_j \omega^2 V = 0 \tag{3.75a}$$

the condition at $z=z_j$, $2 \leq j \leq N$,

$$G_{j-1} \left.\frac{dV}{dz}\right|_{z=z_j^-} = G_j \left.\frac{dV}{dz}\right|_{z=z_j^+} , \tag{3.75b}$$

and the boundary conditions

$$V(0) = 1 \tag{3.75c}$$
$$V(h) = 0 \quad . \tag{3.75d}$$

This problem is the same as the one derived in the case of plane strain for the particular solution corresponding to horizontal vibrations of the footing. For the corresponding discrete solution we have

$$[\underset{\sim}{G} - \omega^2 \underset{\sim}{M}]\,\underset{\sim}{\Delta} = \underset{\sim}{F} \quad , \tag{3.76}$$

with $$\Delta_1 = 1, \quad \Delta_j = V(z_j), \quad 2 \le j \le N \quad ,$$

$$F_1 = -\tau_1 = -G_1 \left.\frac{dV}{dz}\right|_{z=0} \quad , \; F_j = 0 \; , \; 2 \le j \le N \; .$$

$\underset{\sim}{G}$, $\underset{\sim}{M}$ are the same as those (2.63). Equation (3.76) is easily solved since $\underset{\sim}{G}$, $\underset{\sim}{M}$ are symmetric tridiagonal matrices. We obtain

$$\underset{\sim}{U}^{1,1} = \underset{\sim}{Y} \tag{3.77a}$$

$$\underset{\sim}{U}^{2,1} = \underset{\sim}{Y} \quad , \tag{3.77b}$$

with $$Y_j = \Delta_{j+1}, \qquad 1 \le j \le N-1 \; .$$

The forces $\underset{\sim}{F}^{1,1}$, $\underset{\sim}{F}^{2,1}$ are found using (2.71) with k=0:

$$\underset{\sim}{F}^{1,1} = \underset{\sim}{F}^{2,1} = 0 \quad . \tag{3.77c}$$

Using (3.62a) we find

$$F_y = F_1 L \quad . \tag{3.77d}$$

F_1 is obtained from (3.76).

The dynamic stiffness matrix $\underset{\sim}{K}$ in (3.63) is symmetric. This may be shown by an argument similar to the one used in the case of plane strain. Using the submatrices $\underset{\sim}{K}^{11}$, $\underset{\sim}{K}^{12}$, $\underset{\sim}{K}^{21}$, $\underset{\sim}{K}^{22}$ given by (3.73a,b,c) and the particular solution (3.77a,b,c,d) we find

$$\underset{\sim}{K}^{1y} = (\underset{\sim}{K}^{y1})^T = -\underset{\sim}{K}^{11}\, \underset{\sim}{U}^{1,1} - \underset{\sim}{K}^{12}\, \underset{\sim}{U}^{2,1} \tag{3.78a}$$

$$\underset{\sim}{K}^{2y} = (\underset{\sim}{K}^{y2})^T = \underset{\sim}{K}^{1y} \tag{3.78b}$$

$$K_{yy} = F_y^1 - 2\, \underset{\sim}{K}^{y1}\, \underset{\sim}{U}^{1,1} \quad . \tag{3.78c}$$

This completes the derivation of the dynamic stiffness matrix of the element. Again, the computational effort required to calculate $\underset{\sim}{K}$ is independent of the length L of the element.

3.3 OTHER ELEMENTS

The techniques discussed in the previous sections may be used to develop several other elements. We note that the method consists of two steps. The first step is to find the solutions (modes) satisfying appropriate homogeneous boundary conditions. In this step an algebraic eigenvalue problem of the form (2.26) or (2.63) is applicable for all boundary conditions which involve tractions and displacements. The second step is to obtain particular solutions satisfying the inhomogeneous boundary conditions. Let us discuss some examples. First, we assume that boundary conditions corresponding to a rigid and smooth footing are prescribed on the surface while the base is kept fixed. We consider rocking vibrations (the other cases may be handled similarly):

$$w(x,0) = -\theta x \tag{3.79a}$$

$$\tau_{xz}(x,0) = 0 \quad , \tag{3.79b}$$

$$-\frac{L}{2} \le x \le \frac{L}{2} \quad .$$

The corresponding homogeneous boundary conditions require that w and τ_{xz} vanish at $z=0$ while $u = w = 0$ at $z=h$. The algebraic eigenvalue problem is obtained from equation (2.25) by specifying $F_1 = 0$, $\Delta_2 = 0$, $\Delta_{2N+1} = \Delta_{2N+2} = 0$. The solution of this problem provides the semidiscrete modes. A particular solution satisfying (3.79a,b) is

$$w(x,z) = - xW(z)$$

$$u(x,z) = U(z) .$$

U and W must satisfy the differential equations (3.44a,b) the conditions (3.44c,d) and the boundary conditions

$$\left[-W + \frac{dU}{dz}\right]_{z=0} = 0 \tag{3.80a}$$

$$W(0) = \theta \tag{3.80b}$$

$$U(h) = 0 \tag{3.80c}$$

$$W(h) = 0 \quad . \tag{3.80d}$$

This problem is similar to the one for rocking vibrations of a rigid and rough footing. The discrete solution for U and W is found from (3.50) with $P_1 = 0$, $\Delta_2 = \theta$. The corresponding semidiscrete solution for u and w together with the semidiscrete modes already obtained suffice for the development of the element.

Let us reconsider the element developed in section 3.1. We assume that x-harmonic displacements are prescribed at the base of the element:

$$u(x,h) = u_o \exp(-ikx) \tag{3.81a}$$

$$w(x,h) = w_o \exp(-ikx) \quad . \tag{3.81b}$$

The boundary conditions on the surface are given by (3.3a,b). In this case, apart from the particular solutions that we calculated in section 3.1, we need a solution satisfying (3.81a,b). Such a solution is

$$u(x,z) = U(z) \exp(-ikx)$$

$$w(x,z) = W(z) \exp(-ikx) \quad ,$$

with U and W satisfying the differential equations (2.7a,b), the conditions (2.7c,d) and the boundary conditions

$$U(0) = 0 \tag{3.82a}$$

$$W(0) = 0 \tag{3.82b}$$

$$U(h) = u_o \tag{3.82c}$$

$$W(h) = w_o \quad . \tag{3.82d}$$

The discrete solution for U and W is found from (2.25) with $\Delta_1 = \Delta_2 = 0$, $\Delta_{2N+1} = u_o$, $\Delta_{2N+2} = w_o$. Again, the corresponding semidiscrete solution for u and w together with the solutions in section 3.1 are sufficient for the development of the element.

From the examples given above it is clear that elements may be developed for a variety of inhomogeneous boundary conditions. It must be noted that the computational effort required to obtain the semidiscrete solutions involved in the development of these elements is always independent of the length of the elements.

3.4 A NOTE ON THE MODES OF VIBRATION

When we calculated the dynamic stiffness matrix in (3.18) for the region with fixed surface and fixed base we assumed that the displacement field in the finite region may be written as a linear combination of 4N-4 linearly independent modes. It turns out, however, that in the absence of damping ($\beta = 0$), at some frequencies, the set of modes of the form

$$u(x,z) = U(z)\ \exp(-ikx) \tag{3.83a}$$

$$w(x,z) = W(z)\ \exp(-ikx) \tag{3.83b}$$

is not complete. This means that the algebraic eigenvalue problem in (3.10) does not yield 2N-2 linearly independent eigenvectors. The case arises at frequencies ω_o for which the frequency equation has a double root - wave number k_o. Let ω_o be one of the cut-off frequencies for the stratum with fixed surface and fixed base (note that the discussion which follows holds for other homogeneous boundary conditions as well) corresponding to transverse waves, i.e.,

$$\frac{\omega_o h}{C_T} = m\pi\ , \quad m = 1,2,\ldots \tag{3.84}$$

Then $k_o = 0$ is a wave number. In fact, it is a double root. The mode is of the form

$$u(x,z) = U(z) \tag{3.85a}$$

$$w(x,z) = 0\ . \tag{3.85b}$$

If ω_o is a cut-off frequency corresponding to both longitudinal and transverse waves, i.e., if in addition to (3.84), we have

$$\frac{\omega_o h}{C_L} = n\pi \ , \quad n = 1,2,... \tag{3.86}$$

then there is another mode of the form (3.83a,b), namely,

$$u(x,z) = 0 \tag{3.87a}$$

$$w(x,z) = W(z) \ . \tag{3.87b}$$

This can happen only in the exceptional cases given by

$$\frac{C_L}{C_T} = \left[\frac{2 - 2\nu}{1 - 2\nu}\right]^{\frac{1}{2}} = \frac{p}{q} \ , \tag{3.88}$$

with $p = 2,3,4,...$

$q = 1,2,...,p-1$.

If Poisson's ratio ν is <u>not</u> such that (3.88) is satisfied then there is only one mode of the form (3.83a,b) at the cut-off frequency ω_o , namely, (3.85a,b). Then the set of modes of the form (3.83a,b) is not complete. However, at such frequencies it is possible to find other modes (i.e., solutions of the differential equations satisfying the homogeneous boundary conditions) which are linearly independent of the modes of the form (3.83a,b). Let us look for such a mode in the form

$$u(x,z) = xU(z) \tag{3.89a}$$

$$w(x,z) = W(z) \ , \tag{3.89b}$$

in which U is the eigenfunction appearing in (3.85a) and W is to be determined. Substituting (3.89a,b) into the differential equations (3.1a,b) and the boundary conditions (3.3c,d), (3.8a,b) we find that

$$G \frac{d^2U}{dz^2} + \rho\omega_o^2 U = 0 \tag{3.90a}$$

$$(\lambda + 2G) \frac{d^2W}{dz^2} + \rho\omega_o^2 W = -(\lambda + G) \frac{dU}{dz} \tag{3.90b}$$

$$U(0) = 0 \tag{3.90c}$$

$$W(0) = 0 \tag{3.90d}$$

$$U(h) = 0 \tag{3.90e}$$

$$W(h) = 0 . \tag{3.90f}$$

Clearly (3.90a,c,e) are satisfied since U is the eigenfunction corresponding to $k_o = 0$ at frequency ω_o . The boundary value problem given by (3.90b,d,f), although homogeneous, has a solution. Indeed, we have

$$U(z) = A \sin[\frac{\omega_o z}{C_T}] , \tag{3.91}$$

and from (3.90b), we obtain

$$W(z) = B_1 \cos[\frac{\omega_o z}{C_L}] + B_2 \sin[\frac{\omega_o z}{C_L}] + \frac{C_T}{\omega_o} A \cos[\frac{\omega_o z}{C_T}] . \tag{3.92}$$

Imposing the conditions (3.90d,f) we find

$$B_1 = -\frac{C_T}{\omega_o} A$$

$$B_2 = \frac{C_T}{\omega_o} \frac{\cos[\frac{\omega_o h}{C_L}] - \cos[\frac{\omega_o h}{C_T}]}{\sin[\frac{\omega_o h}{C_L}]} A .$$

Thus we have found a mode of the form (3.89a,b). Clearly, this mode is linearly independent of modes of the form (3.83 a,b). In fact, the set of modes (3.83a,b) becomes complete when augmented by the mode (3.89a,b). Although the calculations in (3.91), (3.92) were done for a homogeneous stratum, the results hold for a layered stratum as well. We note that if ω_o is a cut-off frequency for longitudinal waves (i.e., (3.86) holds) but not for transverse waves (i.e., (3.88) is <u>not</u> satisfied), then apart from the mode of the form (3.87a,b) one has a mode of the form

$$u(x,z) = U(z) \tag{3.93a}$$

$$w(x,z) = xW(z) \quad , \tag{3.93b}$$

in which W is the eigenfunction in (3.87b). These additional modes which exist at cut-off frequencies provided that Poisson's ratio does not satisfy (3.88) have also been determined by Mindlin [18]. However, one has to look for modes of another form at frequencies other than the cut-off frequencies. We note that the condition which characterizes these frequencies ω_o and the corresponding wave numbers k_o is that $\frac{d\omega}{dk} = 0$ at (ω_o, k_o) along any branch of the spectrum passing through (ω_o, k_o). This condition is slightly more restrictive than the condition that the frequency equation has a double root k_o at ω_o. Indeed, let the frequency equation be written as

$$f(\omega, k) = 0 \quad , \tag{3.94}$$

f is an analytic function of ω^2, k^2. We obtain

$$\frac{\partial f}{\partial \omega}\frac{d\omega}{dk} + \frac{\partial f}{\partial k} = 0 \quad , \tag{3.95}$$

along any branch of the spectrum defined by (3.94). If $\frac{d\omega}{dk} = 0$ at (ω_o, k_o) then, from (3.95), we obtain that $\frac{\partial f}{\partial k} = 0$ at (ω_o, k_o), which means that k_o is a double root of (3.94) at ω_o. Note, however, that at cut-off frequencies $\frac{\partial f}{\partial k} = 0$ does not imply that $\frac{d\omega}{dk} = 0$, since $\frac{\partial f}{\partial \omega} = 0$ there ($\frac{\partial f}{\partial \omega}$ is an even function of k). In fact, $\frac{d\omega}{dk} = 0$ at the cut-off frequencies if and only if Poisson's ratio ν does not satisfy (3.88) (this is shown in [21]). Apart from cut-off frequencies, points (ω_o, k_o) for which $\frac{d\omega}{dk} = 0$ occur at the intersection of branches of complex wave numbers with branches of real wave numbers and branches of imaginary wave numbers. At such points the set of modes (3.83a,b) is not complete. Let U, W be the eigenfunctions corresponding to k_o at frequency ω_o. We will show that

$$u(x,z) = -ixU(z)\exp(-ik_o x) + \frac{dU}{dk}\exp(-ik_o x) \tag{3.96a}$$

$$w(x,z) = -ixW(z)\exp(-ik_o x) + \frac{dW}{dk}\exp(-ik_o x) \tag{3.96b}$$

is another mode at frequency ω_o. The derivatives with respect to wave number are taken at (ω_o, k_o) along any branch of the spectrum through that point. Substituting (3.96a,b) into the differential equations (3.1a,b) and the boundary conditions (3.3c,d), (3.8a,b) we obtain

$$-ix[-k_o^2(\lambda+2G)U - ik_o(\lambda+G)W' + GU'' + \omega_o^2\rho U]$$

$$+ [-k_o^2(\lambda+2G)\frac{dU}{dk} - ik_o(\lambda+G)\frac{dW'}{dk} + G\frac{dU''}{dk} + \omega_o^2\rho\frac{dU}{dk}]$$

$$- 2k_o(\lambda+2G)U - i(\lambda+G)W' = 0 \tag{3.97a}$$

$$- ix[-k_o^2 GW - ik_o(\lambda+G)U' + (\lambda+2G)W'' + \omega_o^2\rho W]$$

$$+ [-k_o^2 G\frac{dW}{dk} - ik_o(\lambda+G)\frac{dU'}{dk} + (\lambda+2G)\frac{dW''}{dk} + \omega_o^2\rho\frac{dW}{dk}]$$

$$- 2k_o GW - i(\lambda+G)U' = 0 \tag{3.97b}$$

$$- ixU(0) + \left.\frac{dU}{dk}\right|_{z=0} = 0 \tag{3.97c}$$

$$- ixW(0) + \left.\frac{dW}{dk}\right|_{z=0} = 0 \tag{3.97d}$$

$$- ixU(h) + \left.\frac{dU}{dk}\right|_{z=h} = 0 \tag{3.97e}$$

$$- ixW(h) + \left.\frac{dW}{dk}\right|_{z=h} = 0 \quad , \tag{3.97f}$$

in which prime indicates differentiation with respect to z. We note that since $U(0) = 0$, it follows that $\left.\frac{dU}{dk}\right|_{z=0} = 0$ and, therefore, (3.97c) is satisfied. Similarly (3.97d,e,f) are satisfied. The first term in (3.97a) is identically zero since U, W are the eigenfunctions corresponding to k_o (see equations (2.7a,b)). The second two terms in (3.97a) may be combined as

$$\frac{d}{dk}[-k^2(\lambda+2G)U - ik(\lambda+G)W' + GU'' + \omega^2\rho U]\bigg|_{k=k_o}$$

$$- 2\omega_o\left.\frac{d\omega}{dk}\right|_{k=k_o}\rho U = 0. \tag{3.98}$$

Again the first term in (3.98) is identically zero, since U, W are the eigenfunctions corresponding to k (we are using the same symbols for the eigenfunctions at k and k_o). The second term in (3.98) is zero because of our assumption that $\frac{d\omega}{dk} = 0$ at (ω_o, k_o). Thus (3.97a) is satisfied. Using a similar argument we can show that (3.97b) holds. Thus the eigenvalue problem is satisfied by modes of the form (3.96a,b). Note that these modes are derivatives of the modes (3.83a,b) with respect to the wave number along any branch of the spectrum through (ω_o, k_o). The modes (3.89a,b), (3.93a,b) which occur at cut-off frequencies are special cases of (3.96a,b).

In order to obtain the discrete solution for $\frac{dU}{dk}$, $\frac{dW}{dk}$, we differentiate with respect to k in

$$[k^2\underset{\sim}{A} + ik\underset{\sim}{B} + \underset{\sim}{G} - \omega^2\underset{\sim}{M}] = \underset{\sim}{0}$$

to obtain

$$[2k\underset{\sim}{A} + i\underset{\sim}{B} - 2\omega \frac{d\omega}{dk} M]\underset{\sim}{\Delta} + [k^2\underset{\sim}{A} + ik\underset{\sim}{B} + \underset{\sim}{G} - \omega^2\underset{\sim}{M}] \frac{d}{dk}[\underset{\sim}{\Delta}] = \underset{\sim}{0} .$$

Since $\frac{d\omega}{dk} = 0$ at (ω_o, k_o) we find

$$[k_o^2\underset{\sim}{A} + ik_o\underset{\sim}{B} + \underset{\sim}{G} - \omega_o^2 \underset{\sim}{M}] \frac{d}{dk} [\underset{\sim}{\Delta}] = - [2k_o\underset{\sim}{A} + i\underset{\sim}{B}]\underset{\sim}{\Delta} . \tag{3.99}$$

Note that the matrix

$$\underset{\sim}{R} = k_o^2\underset{\sim}{A} + ik_o\underset{\sim}{B} + \underset{\sim}{G} - \omega_o^2 \underset{\sim}{M}$$

is singular at (ω_o, k_o). Nevertheless, the problem (3.99) has a solution which may be determined by expanding $\frac{d}{dk}[\underset{\sim}{\Delta}]$ in the eigenvectors of $\underset{\sim}{R}$. The solution may contain an arbitrary multiple of $\underset{\sim}{\Delta}$. This is consistent with the fact that the eigenvalue problem is satisfied if an arbitrary multiple of the mode of the form (3.83a,b) is added to the mode of the form (3.96a,b).

3.5 AN APPLICATION

In order to verify the developments presented in the previous sections we consider a simple application, namely, time-harmonic vibrations of a rigid and rough strip footing on the surface of a stratum in plane strain. Let 2b be the width of the footing. The boundary conditions on the surface of the stratum are

$$\left.\begin{aligned} u(x,0) &= \Delta_x \\ w(x,0) &= \Delta_z - \theta x \end{aligned}\right\} \qquad |x| \le b \quad ,$$

$$\left.\begin{aligned} \sigma_z\Big|_{z=0} &= 0 \\ \tau_{xz}\Big|_{z=0} &= 0 \end{aligned}\right\} \qquad |x| > b \quad .$$

The base of the stratum is fixed. Let F_x, F_z, M be the amplitudes of the horizontal force, vertical force and moment applied on the footing. We have

$$\begin{bmatrix} \Delta_z \\ \Delta_x \\ \theta \end{bmatrix} = \underset{\sim}{F} \begin{bmatrix} F_z \\ F_x \\ M \end{bmatrix} \quad .$$

$\underset{\sim}{F}$ is the dynamic compliance matrix (symmetric):

$$\underset{\sim}{F} = \begin{bmatrix} F_{zz} & 0 & 0 \\ 0 & F_{xx} & F_{x\theta} \\ 0 & F_{\theta x} & F_{\theta\theta} \end{bmatrix} \quad .$$

We assume that the stratum is homogeneous. Then the nondimensional compliances

$$GF_{zz},\ GF_{xx},\ Gb^2F_{\theta\theta},\ GbF_{x\theta}$$

are functions of the nondimensional quantities $\frac{\omega h}{C_T}$ (nondimensional frequency), $\frac{h}{b}$, ν, β. For given values of these quantities the compliances may be calculated by combining the element (modeling the region $-b \leq x \leq b$, $0 \leq z \leq h$) developed in section 3.1 with the transmitting boundaries (modeling the regions $x < -b$ and $x > b$, $0 \leq z \leq h$) developed by Waas [23] and described in section 2.2 (see figure 3.3). Results obtained using different schemes have been reported by Chang-Liang [3] and Gazetas [4]. Actually, Chang-Liang's scheme

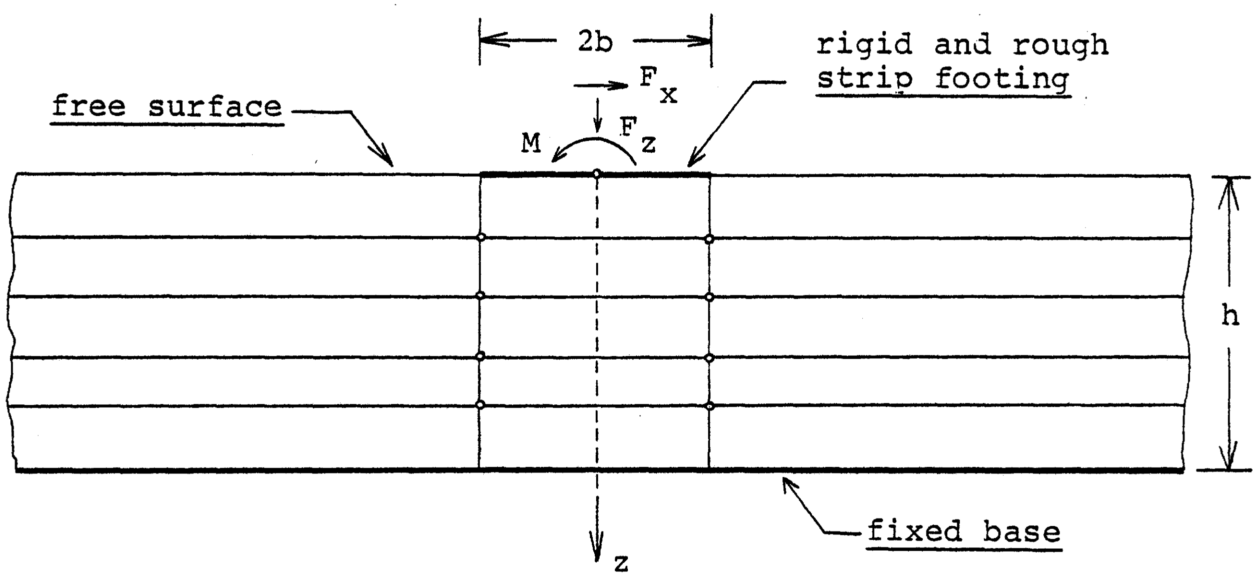
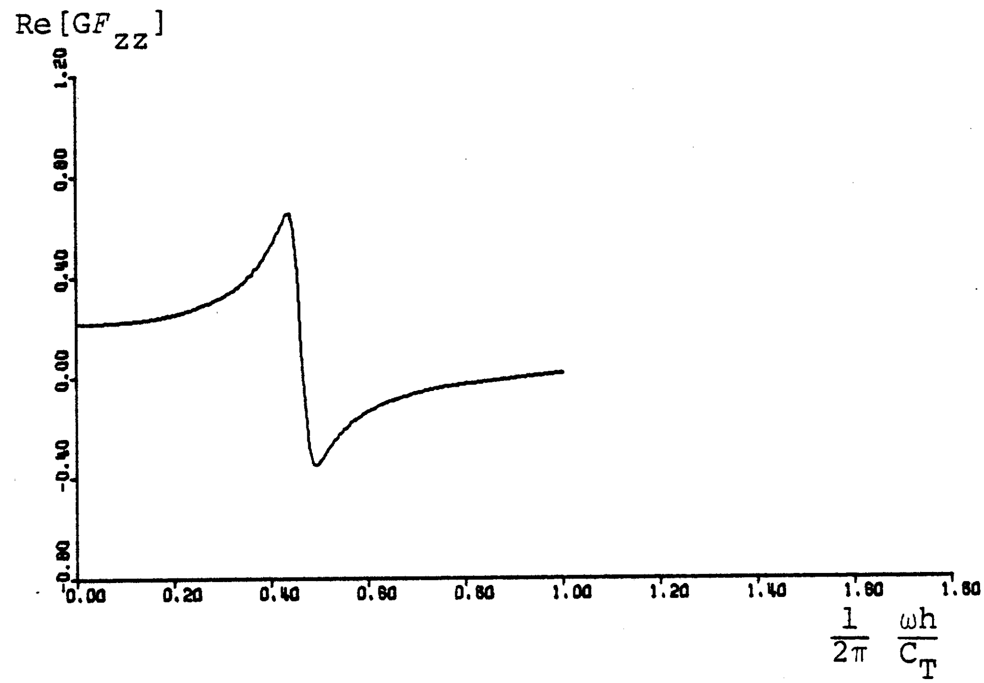
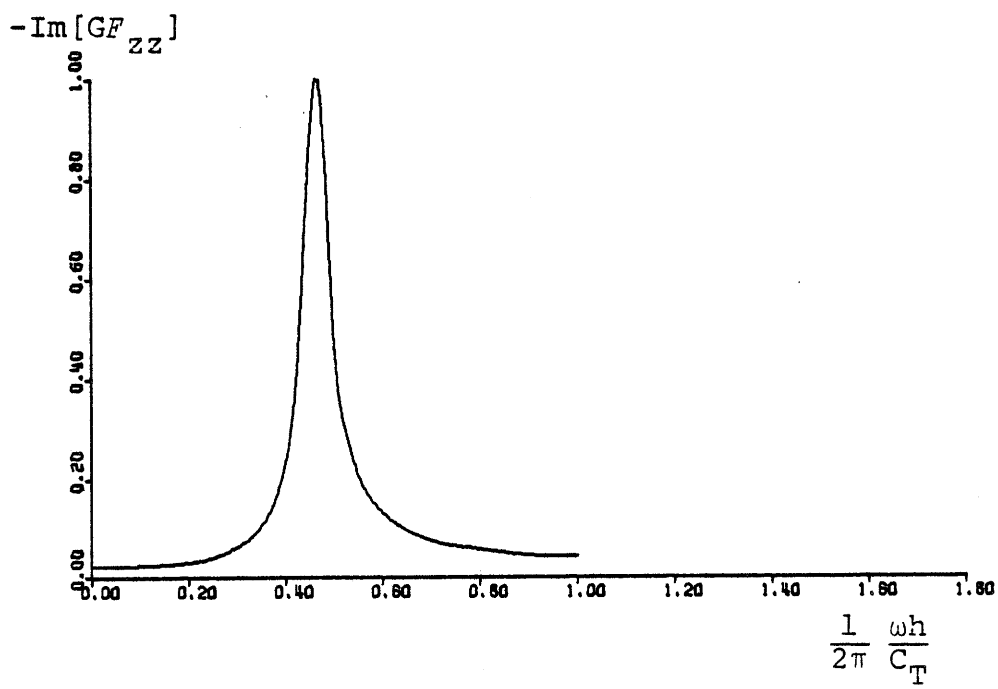


Figure 3.3 - Scheme for the calculation of the stiffness of a strip footing on the surface of a layered stratum.

differs in that the transmitting boundaries are combined with a conventional finite element mesh modeling the rectangular region below the footing. As noted in section 3.1 the element that we are using may be understood as a mesh with an infinite number of columns of finite elements. Figures 3.4 , 3.5 , 3.6 show plots of the nondimensional compliances GF_{zz}, GF_{xx}. $Gb^2F_{\theta\theta}$ versus the nondimensional frequency $\frac{1}{2\pi}\frac{\omega h}{C_T}$, for $\frac{h}{b} = 2$, $\nu = 0.30$, $\beta = 0.05$. The stratum was divided into ten sublayers of equal depth. For each frequency the computations take approximately 5.0 seconds on IBM 370/165. It must be pointed out that only fast memory is necessary. Indeed, this is a great advantage of using the elements developed in this work, since the storage requirements are very low compared with those of a conventional finite element mesh having several columns of elements for accurate results (the spacing in the z-direction being the same as the spacing of sublayers in the element). This is because the number of nodal degrees of freedom in the finite element mesh is much larger. Moreover, we note, again, that the computational effort associated with the elements considered in this work is independent of their length. Clearly, this is not the case with a finite element mesh (fine enough for accurate results). The agreement of the results shown in figures 3.4 , 3.5 , 3.6 with those reported in [3 , 4] is excellent; in fact, the difference between the results cannot be resolved within the scale of the drawings.



<u>Figure 3.4</u> - The nondimesional vertical compliance (strip footing , ν = 0.30 , β = 0.05 , $\frac{h}{b}$ = 2).



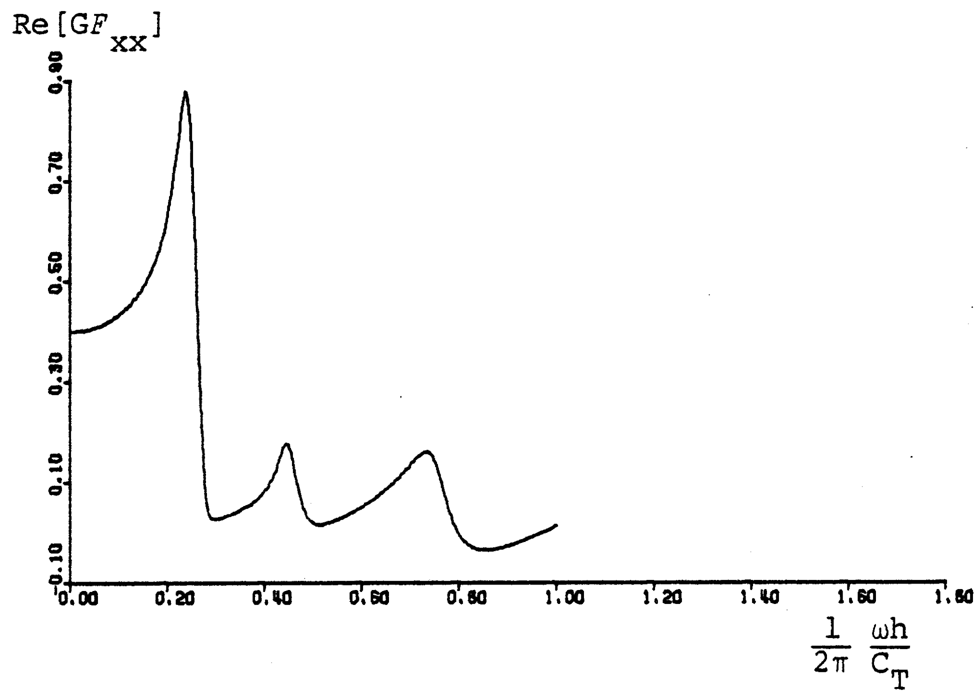
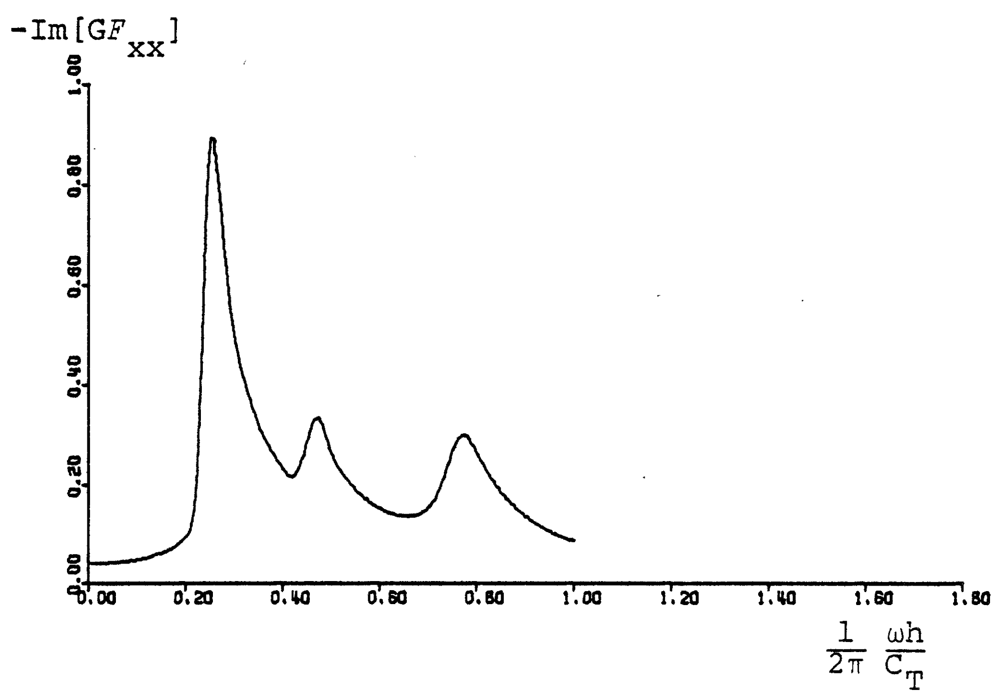


Figure 3.5-The nondimensional horizontal compliance (strip footing , $\nu = 0.30$, $\beta = 0.05$, $\frac{h}{b} = 2$).



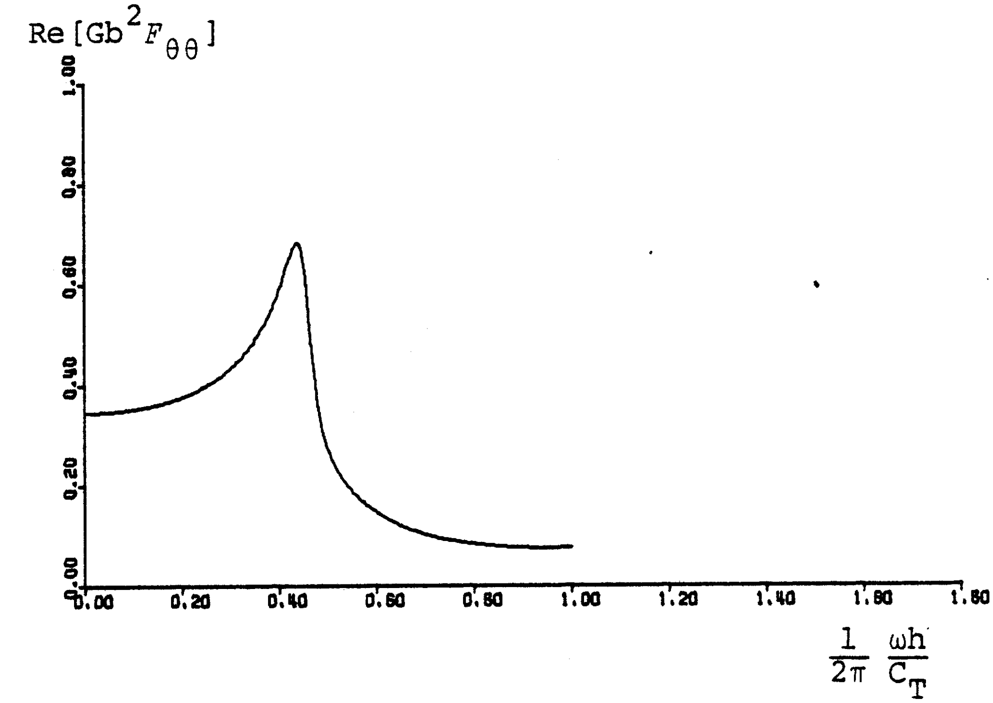
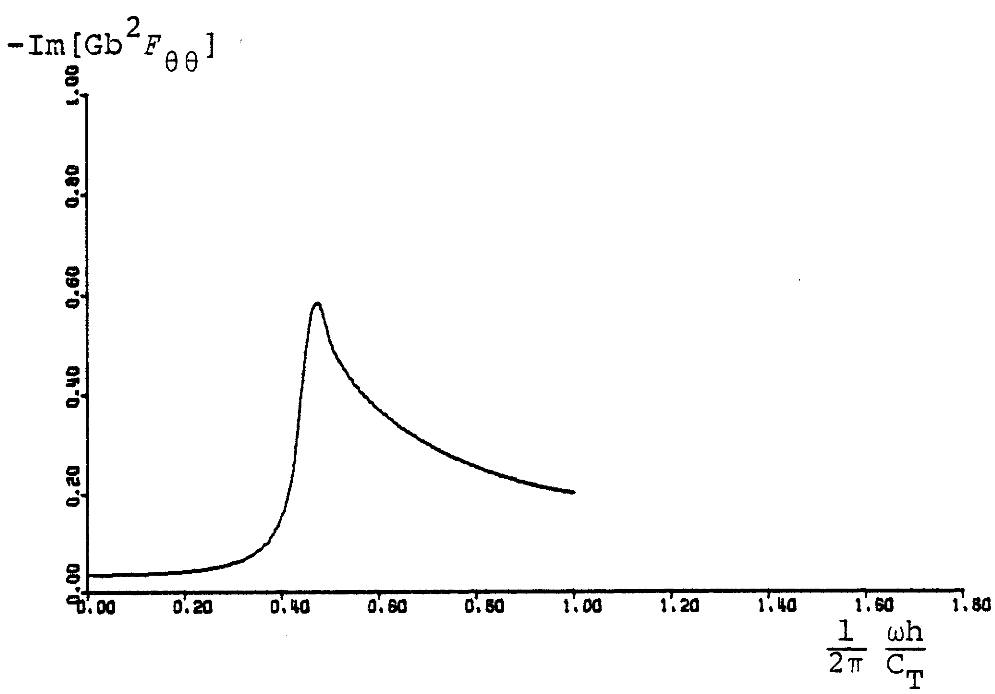


Figure 3.6 - The nondimensional rocking compliance (strip footing , ν = 0.30 , β = 0.05 , $\frac{h}{b}$ = 2).



CHAPTER 4

AXISYMMETRIC ELEMENTS

The surface of the axisymmetric elements described in Chapter 2 was assumed free while the base was taken fixed, i.e., the boundary conditions were homogeneous. In this chapter we turn our attention to inhomogeneous boundary conditions. We develop in detail elements for the analysis of time-harmonic wave motion in the axisymmetric regions $0 \leq r \leq r_o$, $0 \leq z \leq h$, and $r_1 \leq r \leq r_2$, $0 \leq z \leq h$ of a layered stratum. Boundary conditions corresponding to a rigid and rough footing are prescribed on the surface of the elements while the base is assumed fixed. Other inhomogeneous boundary conditions are also considered. An application demonstrates the accuracy and efficiency of the method.

4.1 TORSIONAL VIBRATIONS

We consider time-harmonic antisymmetric vibrations of axisymmetric regions of a layered stratum for the Fourier number $n = 0$. Particle motion is perpendicular to vertical planes through the axis. Moreover, it is independent of the θ coordinate. Thus the amplitudes u and w vanish while v is a function of r and z but not of θ (we use the same notation as in section 2.4). Let us assume that the stratum is divided into N sublayers. The governing differential equation for the amplitude v, in sublayer j, is obtained from equations (2.79a,b,c) as

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2} + \frac{\omega^2}{[C_T^j]^2} v = 0 \quad . \qquad (4.1)$$

The amplitudes of the stresses are given by

$$\sigma_r = 0 \qquad (4.2a)$$

$$\sigma_\theta = 0 \qquad (4.2b)$$

$$\sigma_z = 0 \qquad (4.2c)$$

$$\tau_{r\theta} = G_j \left[\frac{\partial v}{\partial r} - \frac{v}{r}\right] \qquad (4.2d)$$

$$\tau_{\theta z} = G_j \frac{\partial v}{\partial z} \qquad (4.2e)$$

$$\tau_{zr} = 0 \quad . \qquad (4.2f)$$

Continuity of the amplitudes v, σ_z, $\tau_{\theta z}$, τ_{zr} is required at $z=z_j$, $2 \leq j \leq N$. Thus we obtain the condition

$$G_{j-1} \left.\frac{\partial v}{\partial z}\right|_{z=z_j^-} = G_j \left.\frac{\partial v}{\partial z}\right|_{z=z_j^+} \quad . \qquad (4.3)$$

Let us first consider the axisymmetric region $0 \leq r \leq r_o$, $0 \leq z \leq h$. The boundary condition corresponding to a rigid and rough circular footing is prescribed at z=0, i.e., on the surface of the region:

$$v(r,0) = r\phi \;, \quad 0 \leq r \leq r_o \quad . \qquad (4.4a)$$

ϕ is the amplitude of the rotation of the footing. It is taken positive in the counterclockwise direction. The base of the region is assumed fixed:

$$v(r,h) = 0, \qquad 0 \leq r \leq r_o. \qquad (4.4b)$$

As degrees of freedom of the element (see figure 4.1), we take ϕ, associated with the footing and, at $(r_o, z_j), 2 \leq j \leq N$,

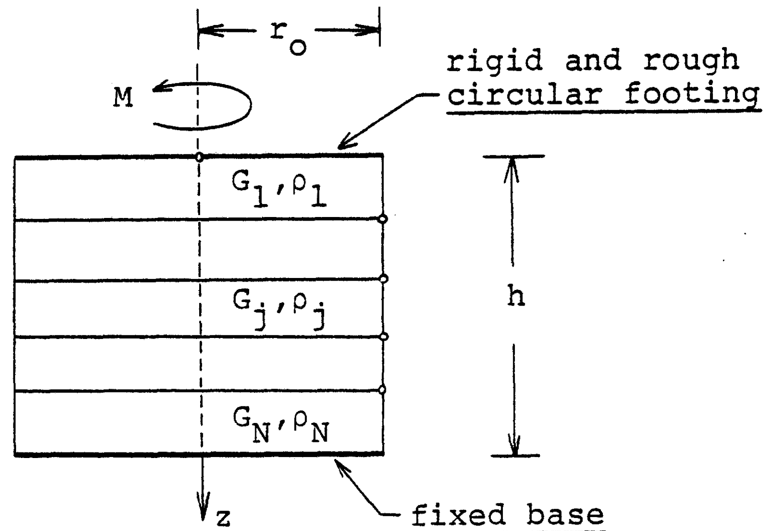


Figure 4.1 -The region $0 \leq r \leq r_o$, $0 \leq z \leq h$ (torsional vibrations).

the nodal displacements

$$v_j = v(r_o, z_j) \quad .$$

The loads corresponding to these degrees of freedom are the moment M and, at (r_o, z_j), $2 \leq j \leq N$, the tangential force P_j. We assume that v is a linear function of z in each sublayer. For $z_j \leq z \leq z_{j+1}$, $1 \leq j \leq N$, we have

$$v(r_o, z) = v_j \frac{z_{j+1} - z}{h_j} + v_{j+1} \frac{z - z_j}{h_j} \quad .$$

We note that $v_1 = v(r_o, z_1) = r_o\phi$ and $v_{N+1} = 0$ by the boundary conditions (4.4a,b). The consistent nodal forces are given by

$$M = -2\pi\int_0^{r_o} \tau_{\theta z}\Big|_{z=0} r^2 dr + 2\pi r_o^2 \int_{z_1}^{z_2} \tau_{r\theta}\Big|_{r=r_o} \left[\frac{z_2 - z}{h_1}\right] dz \qquad (4.5a)$$

$$P_j = 2\pi r_o \int_{z_{j-1}}^{z_j} \tau_{r\theta}\Big|_{r=r_o} \left[\frac{z - z_{j-1}}{h_{j-1}}\right] dz$$

$$+ 2\pi r_o \int_{z_j}^{z_{j+1}} \tau_{r\theta}\Big|_{r=r_o} \left[\frac{z_{j+1} - z}{h_j}\right] dz \, , \qquad (4.5b)$$

$$2 \leq j \leq N.$$

We denote the vector of nodal displacements at (r_o, z_j), $2 \leq j \leq N$, by $\underset{\sim}{U}^o$:

$$U_s^o = v_{s+1} \quad , \qquad 1 \leq s \leq N-1 \quad .$$

The vector of nodal forces at (r_o, z_j), $2 \leq j \leq N$, is denoted by $\underset{\sim}{F}^o$:

$$F_s^o = P_{s+1} \quad , \qquad 1 \leq s \leq N-1 \quad .$$

Let us calculate the dynamic stiffness matrix $\underset{\sim}{K}$ of the element, i.e., the matrix relating M, $\underset{\sim}{F}^o$ to ϕ, $\underset{\sim}{U}^o$:

$$\begin{bmatrix} M \\ \hline \underset{\sim}{F}^o \end{bmatrix} = \left[\begin{array}{c|c} K_{\phi\phi} & \underset{\sim}{K}^{\phi o} \\ \hline \underset{\sim}{K}^{o\phi} & \underset{\sim}{K}^{oo} \end{array}\right] \begin{bmatrix} \phi \\ \hline \underset{\sim}{U}^o \end{bmatrix} . \qquad (4.6)$$

First, we determine the submatrix $\underset{\sim}{K}^{oo}$. We set

$$\phi = 0.$$

The boundary condition (4.4a) becomes

$$v(r,0) = 0 \ , \ 0 \leq r \leq r_o \quad . \tag{4.7}$$

Any displacement amplitude v satisfying the differential equation (4.1), the condition (4.3) at $z=z_j$, $2 \leq j \leq N$, and the boundary conditions (4.4b), (4.7) may be written as a superposition of modes which, according to (2.84a,b,c) are given by:

$$v(r,z) = kV(z)J_o'(kr) \quad . \tag{4.8}$$

The Bessel function J_o has been used in (4.8) since it is bounded for $0 \leq r \leq r_o$ (it is nonsingular at r=0). As shown in section 2.4, V is an eigenfunction with eigenvalue k of an eigenvalue problem which is identical to that obtained for time-harmonic vibrations of the stratum in antiplane shear. The difference is that the condition

$$V(0) = 0 \quad ,$$

indicating that the surface is fixed, must be satisfied instead of (2.93a), which corresponds to a free surface. This eigenvalue problem has already been considered in connection with the plane element developed in section 3.2. We rewrite the corresponding algebraic eigenvalue problem

$$[k^2\underset{\sim}{A} + \underset{\sim}{G} - \omega^2\underset{\sim}{M}]\underset{\sim}{\Delta} = \underset{\sim}{0} \quad . \tag{4.9}$$

$\underset{\sim}{A}$, $\underset{\sim}{G}$, $\underset{\sim}{M}$, $\underset{\sim}{\Delta}$ are the same as in (3.66). Again, the (N-1) x (N-1) diagonal matrix $\underset{\sim}{K}$ is conveniently formed with entries the wave numbers k_j, $1 \leq j \leq N-1$, chosen so that either $Im[k_j] < 0$, or

$Re[k_j] > 0$ and $Im[k_j] = 0$:

$$\underset{\sim}{K} = \text{diag}\ [k_j] \quad . \tag{4.10}$$

In fact, for the development of the element modeling the region $0 \leq r \leq r_o$, $0 \leq z \leq h$, the wave numbers chosen in (4.10), as noted in section 2.4, need only correspond to linearly independent modes. The dynamic stiffness matrix for the region with fixed surface and fixed base is obtained using the procedure given in section 2.4 for the region with free surface and fixed base. The modal matrix $\underset{\sim}{\Phi}$ is now given by

$$\Phi_{j,\ell} = V_j^{\ell}\ J_1(k_\ell r_o) \quad , \tag{4.11}$$

$$1 \leq \ell \leq N-1,\ 1 \leq j \leq N-1 \ .$$

(the superscript indicates the particular eigenvector of the algebraic eigenvalue problem). The matrix $\underset{\sim}{\Psi}$ is taken as

$$\Psi_{j,\ell} = V_j^{\ell}\ J_o(k_\ell r_o) \quad , \tag{4.12}$$

$$1 \leq \ell \leq N-1,\ 1 \leq j \leq N-1 \ .$$

Finally, the matrix $\underset{\sim}{W}$ is given by

$$W_{j,\ell} = k_\ell V_j^{\ell}\ J_o'(k_\ell r_o) \tag{4.13}$$

$$1 \leq \ell \leq N-1,\ 1 \leq j \leq N-1 \ .$$

We have

$$\underset{\sim}{U}^o = \underset{\sim}{W}\,\underset{\sim}{\Gamma} \ . \tag{4.14}$$

Using (2.107) and integrating with respect to θ, the nodal forces are obtained as

$$\underset{\sim}{F}^{o} = - 2\pi r_o \, [\underset{\sim}{A}\,\underset{\sim}{\Psi}\,\underset{\sim}{K}\,\underset{\sim}{K} - \underset{\sim}{E}\,\underset{\sim}{\Phi}\,\underset{\sim}{K}]\underset{\sim}{\Gamma} \quad . \tag{4.15}$$

$\underset{\sim}{A}$ (the same as in (4.9)), $\underset{\sim}{E}$ are assembled from the sublayer matrices $\underset{\sim}{A}^j$, $\underset{\sim}{E}^j$, respectively, which are given by

$$\underset{\sim}{A}^j = \frac{G_j h_j}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{4.16a}$$

$$\underset{\sim}{E}^j = \frac{G_j h_j}{3r_o}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad . \tag{4.16b}$$

Note that $\underset{\sim}{A}^j$, $\underset{\sim}{E}^j$ are obtained from those in (2.100a,c) by deleting the rows and columns corresponding to radial and axial displacements. Eliminating the participation factors $\underset{\sim}{\Gamma}$ we obtain:

$$\left[\underset{\sim}{F}^{o}\right] = \left[\underset{\sim}{K}^{oo}\right] \left[\underset{\sim}{U}^{o}\right] \quad . \tag{4.17}$$

$\underset{\sim}{K}^{oo}$ is the matrix we are looking for. It is given by

$$\underset{\sim}{K}^{oo} = -2\pi r_o [\underset{\sim}{A}\,\underset{\sim}{\Psi}\,\underset{\sim}{K}\,\underset{\sim}{K} - \underset{\sim}{E}\,\underset{\sim}{\Phi}\,\underset{\sim}{K}]\,\underset{\sim}{W}^{-1} \quad . \tag{4.18}$$

This expression may be simplified . Let $\underset{\sim}{X}$ be the modal matrix the columns of which are the eigenvectors $\underset{\sim}{\Delta}^j$ corresponding to k_j, $1 \leq j \leq N-1$, as chosen in (4.10):

$$\underset{\sim}{X} = [\underset{\sim}{\Delta}^1, \underset{\sim}{\Delta}^2, \ldots , \underset{\sim}{\Delta}^{N-1}] \quad . \tag{4.19}$$

The eigenvectors of the algebraic eigenvalue problem (4.9) are orthogonal with respect to the matrix $\underset{\sim}{A}$. Let us assume that they are normalized so that

$$\underset{\sim}{X}^T \underset{\sim}{A} \underset{\sim}{X} = \underset{\sim}{I} \quad ,$$

$\underset{\sim}{I}$ being the identity matrix. Then (4.18) becomes

$$\underset{\sim}{K}^{oo} = - 2\pi \underset{\sim}{A} \underset{\sim}{X} \underset{\sim}{\Lambda} \underset{\sim}{X}^T \underset{\sim}{A} \quad , \tag{4.20}$$

$\underset{\sim}{\Lambda}$ being a diagonal matrix given by

$$\underset{\sim}{\Lambda} = \text{diag} \left[-k_j r_o \frac{J_o(k_j r_o)}{J_1(k_j r_o)} + 2 \right] .$$

It is clearly seen, from (4.20), that $\underset{\sim}{K}^{oo}$ is symmetric.

Let us now obtain a particular solution of (4.6) for which ϕ=1. We denote the loads and displacements corresponding to this particular solution by

$$\underset{\sim}{F}^{o,1}, \underset{\sim}{U}^{o,1}, M_1 \quad .$$

Substituting

$$v(r,z) = rV(z) \tag{4.21}$$

into the differential equation (4.1), the condition (4.3) and the boundary conditions (4.4a,b),we find that V must satisfy the differential equation, in sublayer j,

$$G_j \frac{d^2V}{dz^2} + \omega^2 \rho_j V = 0 \tag{4.22a}$$

the conditions at $z=z_j$, $2 \leq j \leq N$,

$$G_{j-1} \left.\frac{dV}{dz}\right|_{z=z_j^-} = G_j \left.\frac{dV}{dz}\right|_{z=z_j^+} \tag{4.22b}$$

and the boundary conditions

$$V(0) = 1 \tag{4.22c}$$

$$V(h) = 0 \quad . \tag{4.22d}$$

This problem is the same as the one obtained in section 3.2. To find the corresponding discrete solution, we solve

$$[\underset{\sim}{G} - \omega^2 \underset{\sim}{M}]\, \underset{\sim}{\Delta} = \underset{\sim}{F} \quad , \tag{4.23}$$

with $\quad \Delta_1 = 1, \quad \Delta_j = V(z_j), \quad 2 \leq j \leq N \quad ,$

$$F_1 = -\tau_1 = -G_1 \left.\frac{dV}{dz}\right|_{z=0}, F_j = 0 \;, \quad 2 \leq j \leq N \;.$$

$\underset{\sim}{G}$, $\underset{\sim}{M}$ are the same as those in (2.63). The solution of (4.23) is easily calculated since $\underset{\sim}{G}$, $\underset{\sim}{M}$ are symmetric tridiagonal matrices. Thus we obtain

$$\underset{\sim}{U}^{o,1} = \underset{\sim}{Y} \quad , \tag{4.24a}$$

with $\quad Y_j = r_o \, \Delta_{j+1} \quad , \quad 1 \leq j \leq N-1 \quad .$

From (4.2d) we find that $\tau_{r\theta} = 0$ for this particular solution. Therefore,

$$\underset{\sim}{F}^{o,1} = \underset{\sim}{0} \quad . \tag{4.24b}$$

Finally, using (4.5a) we find

$$M_1 = \frac{\pi}{2} F_1 r_o^4 \quad . \tag{4.24c}$$

F_1 is obtained from (4.23).

The dynamic stiffness matrix $\underset{\sim}{K}$ in (4.6) is symmetric. Using the submatrix $\underset{\sim}{K}^{oo}$ given by (4.18) and the particular solution (4.24a,b,c), we find

$$\underset{\sim}{K}^{o\phi} = [\underset{\sim}{K}^{\phi o}]^T = - \underset{\sim}{K}^{oo} \underset{\sim}{U}^{o,1} \tag{4.25a}$$

$$K_{\phi\phi} = M_1 - \underset{\sim}{K}^{\phi o} \underset{\sim}{U}^{o,1} \quad . \tag{4.25b}$$

This completes the derivation of the dynamic stiffness matrix of the element. The computational effort required to obtain $\underset{\sim}{K}$ is independent of the diameter $2r_o$ of the element.

Let us now consider the axisymmetric region $0 < r_1 \leq r \leq r_2$, $0 \leq z \leq h$. The boundary condition corresponding to a rigid and rough ring footing is prescribed on the surface of the region:

$$v(r,0) = r\phi , \qquad r_1 \leq r \leq r_2 \quad . \tag{4.26a}$$

The base is again taken fixed:

$$v(r,h) = 0 \quad , \qquad r_1 \leq r \leq r_2 \quad . \tag{4.26b}$$

As degrees of freedom of the element (see figure 4.2) we take ϕ, the amplitude of the rotation of the footing and, at (r_1,z_j) and (r_2,z_j), $2 \leq j \leq N$, the nodal displacements

$$v_j^{\ell} = v(r_{\ell},z_j) \quad , \ell = 1, 2 \quad .$$

The loads corresponding to these degrees of freedom are the moment M and, at (r_{ℓ},z_j), $2 \leq j \leq N$, the tangential forces

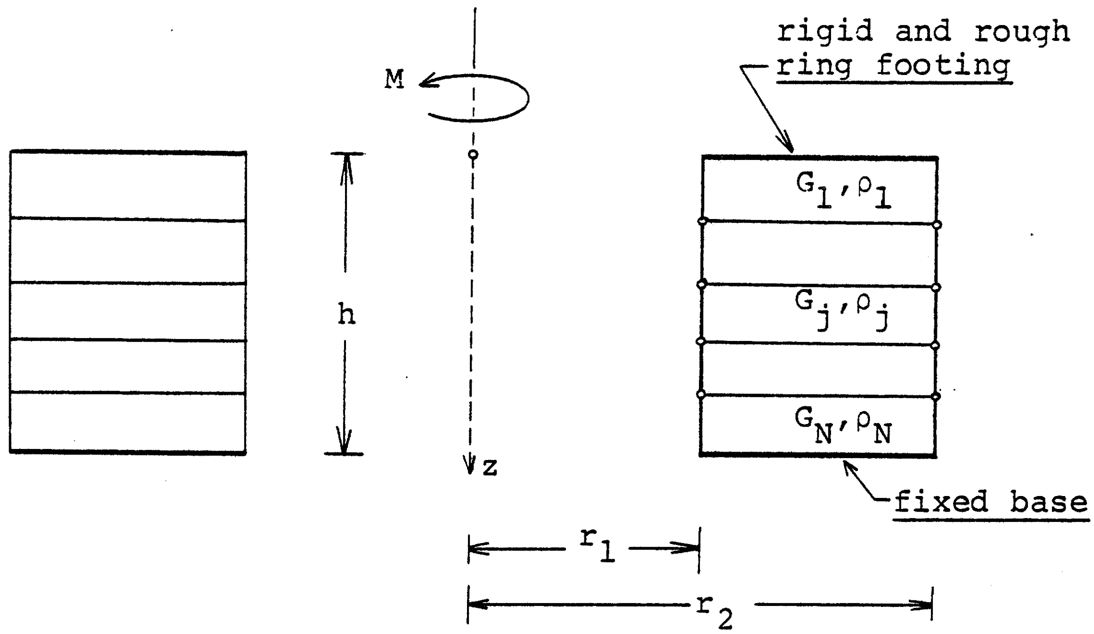


Figure 4.2 -The region $r_1 \le r \le r_2$, $0 \le z \le h$ (torsional vibrations).

P_j^ℓ $(\ell=1,2)$. The amplitude v is assumed to be a linear function of z in each sublayer. For $z_j \le z \le z_{j+1}$, $1 \le j \le N$, we have

$$v(r_\ell, z) = v_j^\ell \frac{z_{j+1} - z}{h_j} + v_{j+1}^\ell \frac{z - z_j}{h_j} , \qquad \ell = 1,\ 2 .$$

Note that $v_1^\ell = r_\ell \phi$, $v_{N+1}^\ell = 0$ $(\ell = 1,2)$ by the boundary conditions (4.26a,b). The consistent nodal forces are given by

$$M = -\,2\pi \int_{r_1}^{r_2} \tau_{\theta z}\Big|_{z=0} r^2 dr - 2\pi r_1^2 \int_{z_1}^{z_2} \tau_{r\theta}\Big|_{r=r_1} \left[\frac{z_2 - z}{h_1}\right] dz$$

$$+ 2\pi r_2^2 \int_{z_1}^{z_2} \tau_{r\theta}\Big|_{r=r_2} \left[\frac{z_2 - z}{h_1}\right] dz \qquad (4.27a)$$

$$P_j^1 = -2\pi r_1 \int_{z_{j-1}}^{z_j} \tau_{r\theta}\Big|_{r=r_1} \left[\frac{z - z_{j-1}}{h_{j-1}}\right] dz - 2\pi r_1 \int_{z_j}^{z_{j+1}} \tau_{r\theta}\Big|_{r=r_1} \left[\frac{z_{j+1} - z}{h_j}\right] dz$$

(4.27b)

$$P_j^2 = 2\pi r_2 \int_{z_{j-1}}^{z_j} \tau_{r\theta}\Big|_{r=r_2} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz + 2\pi r_2 \int_{z_j}^{z_{j+1}} \tau_{r\theta}\Big|_{r=r_2} \left[\frac{z_{j+1}-z}{h_j}\right] dz \ , \tag{4.27c}$$

$$2 \leq j \leq N \ .$$

Let $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$ denote the vectors of nodal displacements at (r_1, z_j) and (r_2, z_j), $2 \leq j \leq N$:

$$U_s^\ell = v_{s+1}^\ell \ , \quad 1 \leq s \leq N-1, \quad \ell = 1, 2.$$

The vectors of nodal forces are denoted by $\underset{\sim}{F}^1$, $\underset{\sim}{F}^2$:

$$F_s^\ell = P_{s+1}^\ell \ , \quad 1 \leq s \leq N-1, \quad \ell = 1, 2.$$

Let us determine the dynamic stiffness matrix $\underset{\sim}{K}$ of the element

$$\begin{bmatrix} \underset{\sim}{F}^1 \\ \hline M \\ \hline \underset{\sim}{F}^2 \end{bmatrix} = \left[\begin{array}{c|c|c} \underset{\sim}{K}^{11} & \underset{\sim}{K}^{1\phi} & \underset{\sim}{K}^{12} \\ \hline \underset{\sim}{K}^{\phi 1} & K_{\phi\phi} & \underset{\sim}{K}^{\phi 2} \\ \hline \underset{\sim}{K}^{21} & \underset{\sim}{K}^{2\phi} & \underset{\sim}{K}^{22} \end{array}\right] \begin{bmatrix} \underset{\sim}{U}^1 \\ \hline \phi \\ \hline \underset{\sim}{U}^2 \end{bmatrix} \ . \tag{4.28}$$

First, we calculate the submatrices $\underset{\sim}{K}^{11}$, $\underset{\sim}{K}^{12}$, $\underset{\sim}{K}^{21}$, $\underset{\sim}{K}^{22}$. We set $\phi = 0$. The modes (fixed surface, fixed base) are now of the form

$$v(r,z) = kV(z)H_o^{'(1)}(kr) \ , \tag{4.29a}$$

and

$$v(r,z) = kV(z)H_o^{'(2)}(kr) \ . \tag{4.29b}$$

Since we are using both Hankel functions, i.e., of the first and the second kind, only the eigenvalues chosen in (4.10) need be used. Working as for the element modeling the region $0 \leq r \leq r_o$, $0 \leq z \leq h$, we obtain

$$\underset{\sim}{U}^1 = \underset{\sim}{W}^1 \underset{\sim}{\Gamma}^1 + \hat{\underset{\sim}{W}}^1 \underset{\sim}{\Gamma}^2 \tag{4.30a}$$

$$\underset{\sim}{U}^2 = \underset{\sim}{W}^2 \underset{\sim}{\Gamma}^2 + \hat{\underset{\sim}{W}}^2 \underset{\sim}{\Gamma}^2 \quad . \tag{4.30b}$$

$\underset{\sim}{W}^1, \underset{\sim}{W}^2$ are obtained as in (4.13) but using $H_o^{'(2)}$ instead of $J_o^{'}$ and evaluating at $r = r_1$ and $r = r_2$ respectively. $\hat{\underset{\sim}{W}}^1, \hat{\underset{\sim}{W}}^2$ are calculated using $H_o^{'(1)}$. The forces are given by

$$\underset{\sim}{F}^1 = 2\pi r_1 \; [\underset{\sim}{A}\underset{\sim}{\Psi}^1\underset{\sim}{K}\underset{\sim}{K} - \underset{\sim}{E}^1\underset{\sim}{\Phi}^1\underset{\sim}{K}]\underset{\sim}{\Gamma}^1 + 2\pi r_1 [\underset{\sim}{A}\hat{\underset{\sim}{\Psi}}^1\underset{\sim}{K}\underset{\sim}{K} - \underset{\sim}{E}^1\hat{\underset{\sim}{\Phi}}^1\underset{\sim}{K}]\underset{\sim}{\Gamma}^2 \tag{4.31a}$$

$$\underset{\sim}{F}^2 = -2\pi r_2 \; [\underset{\sim}{A}\underset{\sim}{\Psi}^2\underset{\sim}{K}\underset{\sim}{K} - \underset{\sim}{E}^2\underset{\sim}{\Phi}^2\underset{\sim}{K}]\underset{\sim}{\Gamma}^1 - 2\pi r_2 \; [\underset{\sim}{A}\hat{\underset{\sim}{\Psi}}^2\underset{\sim}{K}\underset{\sim}{K} - \underset{\sim}{E}^2\hat{\underset{\sim}{\Phi}}^2\underset{\sim}{K}]\underset{\sim}{\Gamma}^2 \quad . \tag{4.31b}$$

$\underset{\sim}{\Psi}^1$, $\underset{\sim}{\Phi}^1$, $\underset{\sim}{\Psi}^2$, $\underset{\sim}{\Phi}^2$ are obtained from $\underset{\sim}{\Psi}$, $\underset{\sim}{\Phi}$, given by (4.12), (4.11), using Hankel functions of the second kind instead of Bessel functions and evaluating at $r=r_1$ and $r=r_2$ respectively. $\hat{\underset{\sim}{\Psi}}^1$, $\hat{\underset{\sim}{\Phi}}^1$, $\hat{\underset{\sim}{\Psi}}^2$, $\hat{\underset{\sim}{\Phi}}^2$ are calculated using Hankel functions of the first kind. $\underset{\sim}{E}^1$, $\underset{\sim}{E}^2$ are obtained from $\underset{\sim}{E}$ by evaluating at $r=r_1$ and $r=r_2$ respectively. The participation factors may be eliminated from (4.31a,b) using (4.30a,b). We find

$$\begin{bmatrix} \underset{\sim}{F}^1 \\ \hline \underset{\sim}{F}^2 \end{bmatrix} = \left[\begin{array}{c|c} \underset{\sim}{K}^{11} & \underset{\sim}{K}^{12} \\ \hline \underset{\sim}{K}^{21} & \underset{\sim}{K}^{22} \end{array}\right] \begin{bmatrix} \underset{\sim}{U}^1 \\ \hline \underset{\sim}{U}^2 \end{bmatrix} \quad . \tag{4.32}$$

A particular solution of (4.28) for which $\phi=1$ is obtained as for the element modeling the region $0 \leq r \leq r_o$, $0 \leq z \leq h$. We have

$$\underset{\sim}{U}^{\ell,1} = \underset{\sim}{Y}^{\ell} \quad , \tag{4.33a}$$

with $Y_j^{\ell} = r_{\ell}\Delta_{j+1}$, $1 \leq j \leq N-1$, $\ell = 1, 2$ ($\underset{\sim}{\Delta}$ is the same as in (4.23)). Similarly, for this particular solution

$$\underset{\sim}{F}^{\ell,1} = 0 \qquad \ell = 1, 2 \quad . \tag{4.33b}$$

Using (4.27a) we find

$$M_1 = \frac{\pi}{2} F_1(r_2^4 - r_1^4) \quad . \tag{4.33c}$$

The dynamic stiffness matrix in (4.28) is symmetric. Using the submatrices $\underset{\sim}{K}^{11}$, $\underset{\sim}{K}^{12}$, $\underset{\sim}{K}^{21}$, $\underset{\sim}{K}^{22}$ in (4.32) and the particular solution (4.33a,b,c) we obtain

$$\underset{\sim}{K}^{1\phi} = [\underset{\sim}{K}^{\phi 1}]^T = - \underset{\sim}{K}^{11} \underset{\sim}{U}^{1,1} - \underset{\sim}{K}^{12} \underset{\sim}{U}^{2,1} \tag{4.34a}$$

$$\underset{\sim}{K}^{2\phi} = [\underset{\sim}{K}^{\phi 2}]^T = - \underset{\sim}{K}^{21} \underset{\sim}{U}^{1,1} - \underset{\sim}{K}^{22} \underset{\sim}{U}^{2,1} \tag{4.34b}$$

$$K_{\phi\phi} = M_1 - \underset{\sim}{K}^{\phi 1} \underset{\sim}{U}^{1,1} - \underset{\sim}{K}^{\phi 2}\underset{\sim}{U}^{2,1} \quad . \tag{4.34c}$$

The derivation of the matrix is now complete. Note, again, that the computational effort required is independent of the thickness $r_2 - r_1$ of the element.

4.2 VERTICAL VIBRATIONS

Let us now consider time-harmonic symmetric vibrations of axisymmetric regions of a layered stratum for the Fourier number n=0. In this case, particle motion is in vertical planes through the axis. Again, it is independent of the θ coordinate. The amplitude v vanishes while u and w are functions of r and z.

From equations (2.79a,b,c) we obtain the governing differential equations for the amplitudes u and w, in sublayer j,

$$(\lambda_j + 2G_j)\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right] + G_j \frac{\partial^2 u}{\partial z^2} + (\lambda_j + G_j)\frac{\partial^2 w}{\partial r \partial z}$$

$$+ \omega^2 \rho_j u = 0. \qquad (4.35a)$$

$$(\lambda_j + 2G_j)\frac{\partial^2 w}{\partial z^2} + G_j\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right] + (\lambda_j + G_j)\left[\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial z}\right]$$

$$+ \omega^2 \rho_j w = 0 . \qquad (4.35b)$$

The amplitudes of the stresses are given by

$$\sigma_r = (\lambda_j + 2G_j)\frac{\partial u}{\partial r} + \lambda_j \left[\frac{u}{r} + \frac{\partial w}{\partial z}\right] \qquad (4.36a)$$

$$\sigma_\theta = (\lambda_j + 2G_j)\frac{u}{r} + \lambda_j \left[\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right] \qquad (4.36b)$$

$$\sigma_z = (\lambda_j + 2G_j)\frac{\partial w}{\partial z} + \lambda_j \left[\frac{u}{r} + \frac{\partial u}{\partial r}\right] \qquad (4.36c)$$

$$\tau_{r\theta} = 0 \qquad (4.36d)$$

$$\tau_{\theta z} = 0 \qquad (4.36e)$$

$$\tau_{zr} = G_j \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right] . \qquad (4.36f)$$

The amplitudes u, w, σ_z, $\tau_{\theta z}$, τ_{zr} must be continuous at $z=z_j$, $2 \leq j \leq N$. The conditions expressing continuity of σ_z, $\tau_{\theta z}$, τ_{zr} are

$$\left[(\lambda_{j-1} + 2G_{j-1})\frac{\partial w}{\partial z} + \lambda_{j-1}\left(\frac{u}{r} + \frac{\partial u}{\partial r}\right)\right]_{z=z_j^-} =$$

$$= \left[(\lambda_j + 2G_j)\frac{\partial w}{\partial z} + \lambda_j\left(\frac{u}{r} + \frac{\partial u}{\partial r}\right)\right]_{z=z_j^+} \qquad (4.37a)$$

$$G_{j-1}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right]_{z=z_j^-} = G_j\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right]_{z=z_j^+} \quad . \qquad (4.37b)$$

First, we consider the axisymmetric region $0 \leq r \leq r_o$, $0 \leq z \leq h$. The boundary conditions corresponding to a rigid and rough circular footing are prescribed on the surface of the region:

$$u(r,0) = 0 \qquad (4.38a)$$

$$w(r,0) = \Delta_z \ , \qquad (4.38b)$$

$$0 \leq r \leq r_o \ .$$

Δ_z is the amplitude of the vertical translation of the footing. The base of the region is assumed fixed:

$$u(r,h) = 0 \qquad (4.38c)$$

$$w(r,h) = 0 \ , \qquad (4.38d)$$

$$0 \leq r \leq r_o \ .$$

As degrees of freedom of the element (see figure 4.3) we take Δ_z , associated with the footing, and, at (r_o,z_j), $2 \leq j \leq N$, the nodal displacements

$$u_j = u(r_o,z_j)$$

$$w_j = w(r_o,z_j) \quad .$$

The loads corresponding to these degrees of freedom are the vertical force F_z and at (r_o,z_j), $2 \leq j \leq N$, the radial force $P_{r,j}$ and the vertical force $P_{z,j}$. Again, we assume that u and w are linear functions of z in each sublayer. Thus, for $z_j \leq z \leq z_{j+1}$, $1 \leq j \leq N$, we have

$$u(r_o,z) = u_j \frac{z_{j+1}-z}{h_j} + u_{j+1}\frac{z-z_j}{h_j}$$

$$w(r_o,z) = w_j \frac{z_{j+1}-z}{h_j} + w_{j+1}\frac{z-z_j}{h_j} \quad .$$

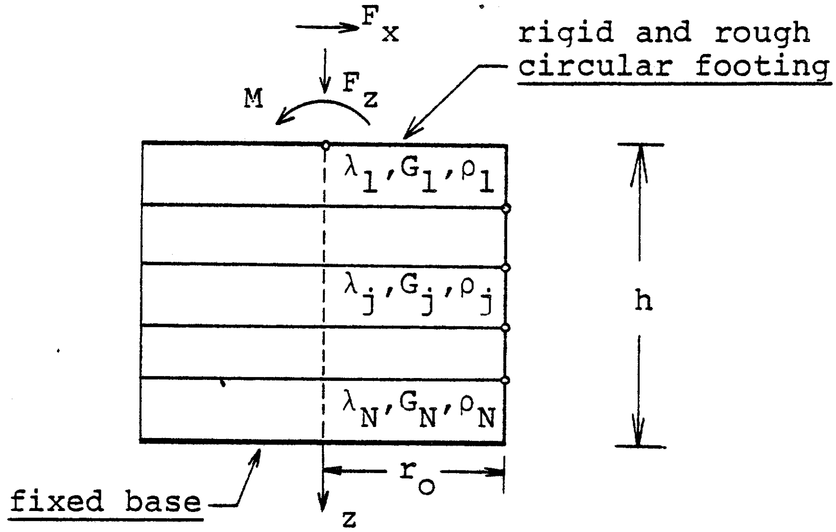


Figure 4.3 The region $0 \leq r \leq r_o$, $0 \leq z \leq h$ (vertical, horizontal vibrations, rocking).

Note that $u_1 = 0$, $w_1 = \Delta_z$, $u_{N+1} = w_{N+1} = 0$ by the boundary conditions (4.38a,b,c,d). The consistent forces are given by

$$F_z = -2\pi \int_0^{r_o} \sigma_z \Big|_{z=0} r\,dr + 2\pi r_o \int_{z_1}^{z_2} \tau_{rz} \Big|_{r=r_o} \left[\frac{z_2 - z}{h_1}\right] dz \tag{4.39a}$$

$$P_{r,j} = 2\pi r_o \int_{z_{j-1}}^{z_j} \sigma_r \Big|_{r=r_o} \left[\frac{z - z_{j-1}}{h_{j-1}}\right] dz + 2\pi r_o \int_{z_j}^{z_{j+1}} \sigma_r \Big|_{r=r_o} \left[\frac{z_{j+1} - z}{h_j}\right] dz \ , \tag{4.39b}$$

$$P_{z,j} = 2\pi r_o \int_{z_{j-1}}^{z_j} \tau_{rz} \Big|_{r=r_o} \left[\frac{z - z_{j-1}}{h_{j-1}}\right] dz + 2\pi r_o \int_{z_j}^{z_{j+1}} \tau_{rz} \Big|_{r=r_o} \left[\frac{z_{j+1} - z}{h_j}\right] dz \ , \ 2 \leq j \leq N. \tag{4.39c}$$

The vector of displacements at (r_o, z_j), $2 \leq j \leq N$, is denoted by $\underset{\sim}{U}^o$:

$$U^o_{2s-1} = u_{s+1} \qquad\qquad 1 \leq s \leq N-1$$

$$U^o_{2s} = w_{s+1} \quad .$$

We denote the vector of forces at (r_o, z_j), $2 \leq j \leq N$, by $\underset{\sim}{F}^o$:

$$F^o_{2s-1} = P_{r,s+1} \qquad\qquad 1 \leq s \leq N-1$$

$$F^o_{2s} = P_{z,s+1}.$$

Let us now determine the dynamic stiffness matrix $\underset{\sim}{K}$ of the element, i.e., the matrix relating F_z, $\underset{\sim}{F}^o$, to Δ_z, $\underset{\sim}{U}^o$:

$$\begin{bmatrix} F_z \\ \hline \underset{\sim}{F}^o \end{bmatrix} = \left[\begin{array}{c|c} K_{zz} & \underset{\sim}{K}^{zo} \\ \hline \underset{\sim}{K}^{oz} & \underset{\sim}{K}^{oo} \end{array}\right] \begin{bmatrix} \Delta_z \\ \hline \underset{\sim}{U}^o \end{bmatrix} \quad . \tag{4.40}$$

First, we calculate the submatrix $\underset{\sim}{K}^{oo}$. We consider solutions for which $\Delta_z = 0$. The boundary condition (4.38b) becomes

$$w(r,0) = 0 \quad , \qquad 0 \leq r \leq r_o \quad . \tag{4.41}$$

Any amplitudes u, w satisfying the differential equations (4.35a,b), the conditions (4.37a,b) at $z = z_j, 2 \leq j \leq N$, and the boundary conditions (4.38a,c,d),(4.41) may be obtained as the superposition of modes which, according to (2.83a,b,c) are given by

$$u(r,z) = kU(z)J_o'(kr) \tag{4.42a}$$

$$w(r,z) = -ikW(z)J_o(kr). \tag{4.42b}$$

Again, the Bessel function has been used since it is nonsingular at r=0. It was shown in section 2.4 that U and W are eigenfunctions with eigenvalue k of an eigenvalue problem which is identical to that obtained for time-harmonic vibrations of the stratum in plane strain. In this case, the conditions

$$U(0) = W(0) = 0 \quad ,$$

indicating that the surface is fixed must be satisfied instead of (2.92a,b), which correspond to a free surface. The eigenvalue problem has been considered in connection with the plane element developed in section 3.1. The corresponding algebraic eigenvalue problem is given by

$$[k^2\underset{\sim}{A} + ik\underset{\sim}{B} + \underset{\sim}{G} - \omega^2\underset{\sim}{M}]\underset{\sim}{\Delta} = 0 \quad . \tag{4.43}$$

$\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{G}$, $\underset{\sim}{M}$, $\underset{\sim}{\Delta}$ are the same as in (3.10). We form the diagonal matrix $\underset{\sim}{K}$ with entries the wave numbers k_j, $1 \leq j \leq 2N-2$, chosen as in (3.11):

$$\underset{\sim}{K} = \text{diag}\,[k_j] \quad . \tag{4.44}$$

Again, it must be noted that for the element under consideration the wave numbers chosen in (4.44) need only correspond to linearly independent modes. The modal matrix $\underset{\sim}{\Phi}$ is given by

$$\Phi_{2j-1,\ell} = U_j^\ell \, J_1(k_\ell r_o) \tag{4.45a}$$

$$\Phi_{2j,\ell} = -iW_j^\ell \, J_o(k_\ell r_o) \quad , \tag{4.45b}$$

$$1 \leq \ell \leq 2N-2, \quad 1 \leq j \leq N-1 \quad .$$

(The superscript indicates the particular eigenvector of the algebraic eigenvalue problem). The matrix $\underset{\sim}{\Psi}$ is taken as

$$\Psi_{2j-1,\ell} = U_j^\ell \, J_o(k_\ell r_o) \tag{4.46a}$$

$$\Psi_{2j,\ell} = -iW_j^\ell \, J_1(k_\ell r_o), \tag{4.46b}$$

$$1 \leq \ell \leq 2N-2, \quad 1 \leq j \leq N-1 \quad .$$

Finally, the matrix $\underset{\sim}{W}$ is given by

$$W_{2j-1,\ell} = k_\ell U_j^\ell J_o'(k_\ell r_o) \tag{4.47a}$$

$$W_{2j,\ell} = -\, ik_\ell W_j^\ell J_o(k_\ell r_o) \quad , \tag{4.47b}$$

$$1 \leq \ell \leq 2N-2, \quad 1 \leq j \leq N-1 \quad .$$

We write

$$\underset{\sim}{U}^o = \underset{\sim}{W} \, \underset{\sim}{\Gamma} \quad . \tag{4.48}$$

The nodal forces are given by

$$\underset{\sim}{F}^o = -2\pi r_o [\underset{\sim}{A} \; \underset{\sim}{\Psi} \; \underset{\sim}{K} \; \underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E}) \underset{\sim}{\Phi} \; \underset{\sim}{K}] \underset{\sim}{\Gamma} \quad . \tag{4.49}$$

$\underset{\sim}{A}$, $\underset{\sim}{D}$, $\underset{\sim}{E}$ are obtained from those in (2.107) by deleting the rows and columns corresponding to tangential displacements. Using (4.48) we eliminate the participation factors $\underset{\sim}{\Gamma}$:

$$\left[\underset{\sim}{F}^o\right] = \left[\underset{\sim}{K}^{oo}\right] \left[\underset{\sim}{U}^o\right] \quad . \tag{4.50}$$

$\underset{\sim}{K}^{oo}$ is given by

$$\underset{\sim}{K}^{oo} = -2\pi r_o[\underset{\sim}{A}\ \underset{\sim}{\Psi}\ \underset{\sim}{K}\ \underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E})\underset{\sim}{\Phi}\ \underset{\sim}{K}]\underset{\sim}{W}^{-1} \quad . \tag{4.51}$$

Let us now obtain a particular solution of (4.40) for which $\Delta_z = 1$. The loads and displacements corresponding to this particular solution are denoted by

$$\underset{\sim}{F}^{o,1},\ \underset{\sim}{U}^{o,1}\ F_z^1 \quad .$$

Substituting

$$u(r,z) = 0 \tag{4.52a}$$

$$w(r,z) = W(z) \tag{4.52b}$$

into the differential equations (4.35a,b), the conditions (4.37a,b) and the boundary conditions (4.38a,b,c,d), we find that W must satisfy the differential equation, in sublayer j,

$$(\lambda_j + 2G_j)\frac{d^2W}{dz^2} + \omega^2\rho_j W = 0 \quad . \tag{4.53a}$$

the conditions at $z=z_j$, $2 \leq j \leq N$,

$$(\lambda_{j-1} + 2G_{j-1})\frac{dW}{dz}\bigg|_{z=z_j^-} = (\lambda_j + 2G_j)\frac{dW}{dz}\bigg|_{z=z_j^+} \tag{4.53b}$$

and the boundary conditions

$$W(0) = 1 \tag{4.53c}$$

$$W(h) = 0 \quad . \tag{4.53d}$$

This problem is the same as the one obtained in section 3.1 for the particular solution corresponding to vertical vibrations of the rigid and rough strip footing. To find the corresponding discrete solution we solve

$$[\underset{\sim}{G} - \omega^2 \underset{\sim}{M}] \underset{\sim}{\Delta} = \underset{\sim}{F} \quad , \tag{4.54}$$

with $\Delta_1 = 1,\ \Delta_j = W(z_j), \qquad 2 \leq j \leq N \quad ,$

$$F_1 = -\sigma_1 = -(\lambda_1 + 2G_1)\frac{dW}{dz}\bigg|_{z=0} \;, \quad F_j = 0, \quad 2 \leq j \leq N.$$

$\underset{\sim}{G}$, $\underset{\sim}{M}$ are the same as those in (3.30). Again, equation (4.54) may be solved easily since $\underset{\sim}{G}$, $\underset{\sim}{M}$ are symmetric tridiagonal matrices. We obtain

$$\underset{\sim}{U}^{o,1} = \underset{\sim}{Y} \tag{4.55a}$$

with $Y_{2j-1} = 0,\ Y_{2j} = \Delta_{j+1}, \quad 1 \leq j \leq N-1$.

Using (4.39b,c) we find

$$\underset{\sim}{F}^{o,1} = -2\pi r_o \underset{\sim}{D} \begin{bmatrix} 0 \\ 1 \\ \underset{\sim}{Y} \end{bmatrix} \quad . \tag{4.55b}$$

The matrix $\underset{\sim}{D}$ is the same as in (3.32c,d). Considering the structure of $\underset{\sim}{D}$ or directly from (4.36f) which gives $\tau_{zr} = 0$, it is seen that vertical forces are equal to zero in this particular solution. Finally, using (4.39a) we obtain

$$F_z^1 = \pi F_1 r_o^2 \quad . \tag{4.55c}$$

F_1 is obtained from (4.54).

The dynamic stiffness matrix $\underset{\sim}{K}$ in (4.40) is symmetric. Using the submatrix $\underset{\sim}{K}^{oo}$ given by (4.51) and the particular solution calculated above (4.55a,b,c) we find

$$\underset{\sim}{K}^{oz} = [\underset{\sim}{K}^{zo}]^T = \underset{\sim}{F}^{o,1} - \underset{\sim}{K}^{oo}\,\underset{\sim}{U}^{o,1} \qquad (4.56a)$$

$$K_{zz} = F_z^1 - \underset{\sim}{K}^{zo}\,\underset{\sim}{U}^{o,1} \quad . \qquad (4.56b)$$

This completes the calculation of the dynamic stiffness matrix of the element. Note, again, that the computational effort is independent of the diameter $2r_o$ of the element.

Let us now consider the axisymmetric region $0 < r_1 \leq r \leq r_2$, $0 \leq z \leq h$. On the surface of the region we prescribe the boundary condition corresponding to a rigid and rough ring footing:

$$u(r,0) = 0 \qquad (4.57a)$$

$$w(r,0) = \Delta_z \quad , \quad r_1 \leq r \leq r_2 \qquad (4.57b)$$

As degrees of freedom of the element (see figure 4.4), we take Δ_z, the amplitude of the vertical translation of the

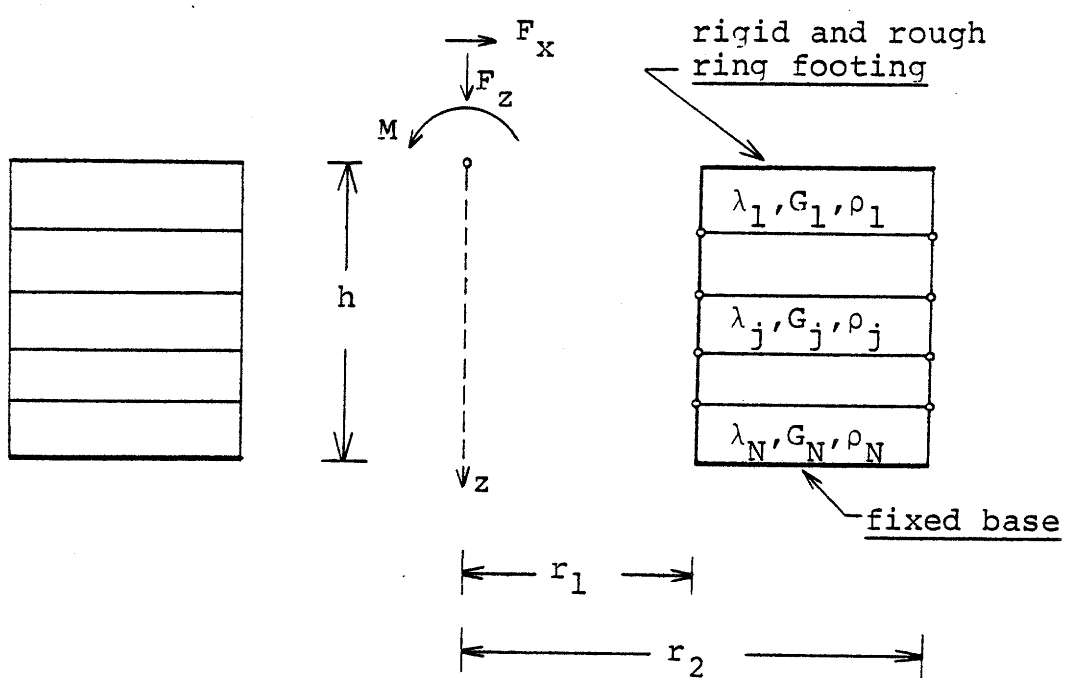


Figure 4.4 The region $r_1 \leq r \leq r_2$, $0 \leq z \leq h$ (vertical, horizontal vibrations, rocking).

footing and, at (r_1, z_j) and (r_2, z_j), $2 \leq j \leq N$, the nodal displacements

$$u_j^\ell = u(r_\ell, z_j) \qquad \ell = 1,\ 2$$
$$w_j^\ell = w(r_\ell, z_j) \quad .$$

The loads corresponding to these degrees of freedom are the vertical force F_z and, at (r_ℓ, z_j), $2 \leq j \leq N$, the radial forces $P^\ell_{r,j}$ and the vertical forces $P^\ell_{z,j}$ $(\ell = 1,2)$. The amplitudes u,w are assumed to be linear functions of z in each sublayer. For $z_j \leq z \leq z_{j+1}$, $1 \leq j \leq N$, we have

$$u(r_\ell, z_j) = u_j^\ell \frac{z_{j+1} - z}{h_j} + u_{j+1}^\ell \frac{z - z_j}{h_j} \qquad \ell = 1,2$$
$$w(r_\ell, z_j) = w_j^\ell \frac{z_{j+1} - z}{h_j} + w_{j+1}^\ell \frac{z - z_j}{h_j}$$

with $u_1^\ell = 0$, $w_1^\ell = \Delta_z$, $u_{N+1}^\ell = w_{N+1}^\ell = 0$ $(\ell = 1,2)$ by the boundary conditions. The consistent forces are given by

$$F_z = -2\pi \int_{r_1}^{r_2} \sigma_z \Big|_{z=0} r dr$$
$$-2\pi r_1 \int_{z_1}^{z_2} \tau_{rz} \Big|_{r=r_1} \left[\frac{z_2 - z}{h_1}\right] dz + 2\pi r_2 \int_{z_1}^{z_2} \tau_{rz} \Big|_{r=r_2} \left[\frac{z_2 - z}{h_1}\right] dz \tag{4.58a}$$

$$P^1_{r,j} = -2\pi r_1 \int_{z_{j-1}}^{z_j} \sigma_r \Big|_{r=r_1} \left[\frac{z - z_{j-1}}{h_{j-1}}\right] dz - 2\pi r_1 \int_{z_j}^{z_{j+1}} \sigma_r \Big|_{r=r_1} \left[\frac{z_{j+1} - z}{h_j}\right] dz \tag{4.58b}$$

$$P^1_{z,j} = -2\pi r_1 \int_{z_{j-1}}^{z_j} \tau_{rz}\Big|_{r=r_1} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz - 2\pi r_1 \int_{z_j}^{z_{j+1}} \tau_{rz}\Big|_{r=r_1} \left[\frac{z_{j+1}-z}{h_j}\right] dz \tag{4.58c}$$

$$P^2_{r,j} = 2\pi r_2 \int_{z_{j-1}}^{z_j} \sigma_r\Big|_{r=r_2} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz + 2\pi r_2 \int_{z_j}^{z_{j+1}} \sigma_r\Big|_{r=r_2} \left[\frac{z_{j+1}-z}{h_j}\right] dz \tag{4.58d}$$

$$P^2_{z,j} = 2\pi r_2 \int_{z_{j-1}}^{z_j} \tau_{rz}\Big|_{r=r_2} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz + 2\pi r_2 \int_{z_j}^{z_{j+1}} \tau_{rz}\Big|_{r=r_2} \left[\frac{z_{j+1}-z}{h_j}\right] dz$$

$$2 \leq j \leq N \; . \tag{4.58e}$$

We denote the vectors of nodal displacements at (r_1, z_j) and (r_2, z_j), $2 \leq j \leq N$, by $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$ respectively.

$$\left.\begin{aligned} U^\ell_{2s-1} &= u^\ell_{s+1} \\ U^\ell_{2s} &= w^\ell_{s+1} \end{aligned}\right. \qquad 1 \leq s \leq N-1, \; \ell = 1,2 \; .$$

The vectors of nodal forces are denoted by $\underset{\sim}{F}^1$, $\underset{\sim}{F}^2$:

$$\left.\begin{aligned} F^\ell_{2s-1} &= P^\ell_{r,s+1} \\ F^\ell_{2s} &= P^\ell_{z,s+1} \end{aligned}\right. \qquad 1 \leq s \leq N-1, \; \ell = 1, 2 \; .$$

Let us calculate the dynamic stiffness matrix of the element

$$\begin{bmatrix} \underset{\sim}{F}^1 \\ \hline F_z \\ \hline \underset{\sim}{F}^2 \end{bmatrix} = \left[\begin{array}{c|c|c} \underset{\sim}{K}^{11} & \underset{\sim}{K}^{1z} & \underset{\sim}{K}^{12} \\ \hline \underset{\sim}{K}^{z1} & K_{zz} & \underset{\sim}{K}^{z2} \\ \hline \underset{\sim}{K}^{21} & \underset{\sim}{K}^{2z} & \underset{\sim}{K}^{22} \end{array}\right] \begin{bmatrix} \underset{\sim}{U}^1 \\ \hline \Delta_z \\ \hline \underset{\sim}{U}^2 \end{bmatrix} . \tag{4.59}$$

First, we obtain the submatrices $\underset{\sim}{K}^{11}$, $\underset{\sim}{K}^{12}$, $\underset{\sim}{K}^{21}$, $\underset{\sim}{K}^{22}$. We consider solutions for which $\Delta_z = 0$. The modes (fixed surface, fixed base) are now of the form

$$u(r,z) = kU(z)H_o^{'(1)}(kr) \tag{4.60a}$$

$$w(r,z) = -ikW(z)H_o^{(1)}(kr) \quad . \tag{4.60b}$$

and

$$u(r,z) = kU(z)H_o^{'(2)}(kr) \tag{4.60c}$$

$$w(r,z) = -ikW(z)H_o^{(2)}(kr) \quad . \tag{4.60d}$$

Note that since we are using both Hankel functions only the eigenvalues chosen in (4.44) need be used. We write

$$\underset{\sim}{U}^1 = \underset{\sim}{W}^1 \underset{\sim}{\Gamma}^1 + \hat{\underset{\sim}{W}}^1 \underset{\sim}{\Gamma}^2 \tag{4.61a}$$

$$\underset{\sim}{U}^2 = \underset{\sim}{W}^2 \underset{\sim}{\Gamma}^1 + \hat{\underset{\sim}{W}}^2 \underset{\sim}{\Gamma}^2$$

$\underset{\sim}{W}^1$, $\underset{\sim}{W}^2$ are obtained as in (4.47a,b) but using Hankel functions of the second kind (instead of Bessel functions) and evaluating at $r = r_1$ and $r = r_2$ respectively. $\hat{\underset{\sim}{W}}^1$, $\hat{\underset{\sim}{W}}^2$ are calculated using Hankel functions of the first kind. The forces are given by

$$\underset{\sim}{F}^1 = 2\pi r_1 [\underset{\sim}{A} \underset{\sim}{\Psi}^1 \underset{\sim}{K} \underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E}^1)\underset{\sim}{\Phi}^1 \underset{\sim}{K}]\underset{\sim}{\Gamma}^1$$
$$+ 2\pi r_1 [\underset{\sim}{A} \hat{\underset{\sim}{\Psi}}^1 \underset{\sim}{K} \underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E}^1)\hat{\underset{\sim}{\Phi}}^1 \underset{\sim}{K}]\underset{\sim}{\Gamma}^2 \tag{4.62a}$$

$$\underset{\sim}{F}^2 = -2\pi r_2 [\underset{\sim}{A} \underset{\sim}{\Psi}^2 \underset{\sim}{K} \underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E}^2)\underset{\sim}{\Phi}^2 \underset{\sim}{K}]\underset{\sim}{\Gamma}^1$$
$$-2\pi r_2 [\underset{\sim}{A} \hat{\underset{\sim}{\Psi}}^2 \underset{\sim}{K} \underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E}^2)\hat{\underset{\sim}{\Phi}}^2 \underset{\sim}{K}]\underset{\sim}{\Gamma}^2 \quad . \tag{4.62b}$$

(a superscript ℓ indicates that the matrix is evaluated at $r=r_\ell$). $\underset{\sim}{\Psi}^1$, $\underset{\sim}{\Phi}^1$, $\underset{\sim}{\Psi}^2$, $\underset{\sim}{\Phi}^2$ are calculated using Hankel functions of the second kind while $\hat{\underset{\sim}{\Psi}}^1$, $\hat{\underset{\sim}{\Phi}}^1$, $\hat{\underset{\sim}{\Psi}}^2$, $\hat{\underset{\sim}{\Phi}}^2$ are determined using Hankel functions of the first kind. Eliminating the participation factors, we find .

$$\begin{bmatrix} \underset{\sim}{F}^1 \\ \hline \underset{\sim}{F}^2 \end{bmatrix} = \left[\begin{array}{c|c} \underset{\sim}{K}^{11} & \underset{\sim}{K}^{12} \\ \hline \underset{\sim}{K}^{21} & \underset{\sim}{K}^{22} \end{array}\right] \begin{bmatrix} \underset{\sim}{U}^1 \\ \hline \underset{\sim}{U}^2 \end{bmatrix} . \tag{4.62}$$

$\underset{\sim}{K}^{11}$, $\underset{\sim}{K}^{12}$, $\underset{\sim}{K}^{21}$, $\underset{\sim}{K}^{22}$ are the submatrices we are looking for.

A particular solution of (4.59) for which $\Delta_z = 1$ is obtained as for the element modeling the region $0 \leq r \leq r_o$, $0 \leq z \leq h$. We have

$$\underset{\sim}{U}^{\ell,1} = \underset{\sim}{Y} \quad , \qquad \ell = 1, 2 \ , \tag{4.63a}$$

$\underset{\sim}{Y}$ being the same as in (4.55a). The forces are given by

$$\underset{\sim}{F}^{1,1} = 2\pi r_1 \underset{\sim}{D} \begin{bmatrix} 0 \\ 1 \\ \underset{\sim}{Y} \end{bmatrix} \tag{4.63b}$$

$$\underset{\sim}{F}^{2,1} = -2\pi r_2 \underset{\sim}{D} \begin{bmatrix} 0 \\ 1 \\ \underset{\sim}{Y} \end{bmatrix} , \tag{4.63c}$$

$\underset{\sim}{D}$ being the same as in (4.55b). Finally, we find

$$F_z^1 = \pi F_1 \ (r_2^2 - r_1^2) \ . \tag{4.63d}$$

The dynamic stiffness matrix in (4.59) is symmetric. Using the submatrices $\underset{\sim}{K}^{11}$, $\underset{\sim}{K}^{12}$, $\underset{\sim}{K}^{21}$, $\underset{\sim}{K}^{22}$ in (4.62) and the particular solution above, we obtain

$$\underset{\sim}{K}^{1z} = [\underset{\sim}{K}^{z1}]^T = \underset{\sim}{F}^{1,1} - [\underset{\sim}{K}^{11} + \underset{\sim}{K}^{12}] \underset{\sim}{U}^{1,1} \tag{4.64a}$$

$$\underset{\sim}{K}^{2z} = [\underset{\sim}{K}^{z2}]^T = \underset{\sim}{F}^{2,1} - [\underset{\sim}{K}^{21} + \underset{\sim}{K}^{22}] \underset{\sim}{U}^{1,1} \tag{4.64b}$$

$$K_{zz} = F_z^1 - [\underset{\sim}{K}^{z1} + \underset{\sim}{K}^{z2}]\ \underset{\sim}{U}^{1,1} \ . \tag{4.64c}$$

The derivation of the dynamic stiffness matrix of the element is now complete. Again, the number of operations necessary to calculate the matrix is independent of the thickness $r_2 - r_1$ of the element.

4.3 HORIZONTAL VIBRATIONS AND ROCKING

We consider time-harmonic symmetric vibrations of axisymmetric regions of a layered stratum for the Fourier number n=1. This case involves all three displacements. We have

$$u(r,\theta,z) = \bar{u}(r,z)\ \cos\theta \tag{4.65a}$$

$$w(r,\theta,z) = \bar{w}(r,z)\ \cos\theta \tag{4.65b}$$

$$v(r,\theta,z) = -\ \bar{v}(r,z)\ \sin\theta \quad . \tag{4.65c}$$

The governing differential equations for the amplitudes $\bar{u}$, $\bar{w}$, $\bar{v}$, in sublayer j, are obtained from equations (2.79a,b,c) as

$$\frac{\partial^2\bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{u}}{\partial r} - \frac{2\bar{u}}{r^2} + \frac{\partial^2\bar{u}}{\partial z^2} + \frac{2\bar{v}}{r^2} + \frac{1}{1-2\nu_j}\frac{\partial\bar{\varepsilon}}{\partial r} + \frac{\omega^2}{[C_T^j]^2}\bar{u} = 0 \tag{4.66a}$$

$$\frac{\partial^2\bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{v}}{\partial r} - \frac{2\bar{v}}{r^2} + \frac{\partial^2\bar{v}}{\partial z^2} + \frac{2\bar{u}}{r^2} + \frac{1}{1-2\nu_j}\frac{\bar{\varepsilon}}{r} + \frac{\omega^2}{[C_T^j]^2}\bar{v} = 0 \tag{4.66b}$$

$$\frac{\partial^2\bar{w}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{w}}{\partial r} - \frac{\bar{w}}{r^2} + \frac{\partial^2\bar{w}}{\partial z^2} + \frac{1}{1-2\nu_j}\frac{\partial\bar{\varepsilon}}{\partial z} + \frac{\omega^2}{[C_T^j]^2}\bar{w} = 0 \ . \tag{4.66c}$$

$\bar{\varepsilon}$ is the amplitude of the dilatation:

$$\bar{\varepsilon} = \frac{\partial\bar{u}}{\partial r} + \frac{\bar{u}}{r} - \frac{\bar{v}}{r} + \frac{\partial\bar{w}}{\partial z} \quad . \tag{4.67}$$

The amplitudes of the stresses are given by

$$\sigma_r = \bar{\sigma}_r \cos\theta = [\lambda_j \bar{\varepsilon} + 2G_j \frac{\partial \bar{u}}{\partial r}] \cos\theta \tag{4.68a}$$

$$\sigma_\theta = \bar{\sigma}_\theta \cos\theta = [\lambda_j \bar{\varepsilon} + 2G_j \frac{\bar{u}}{r} - 2G_j \frac{\bar{v}}{r}] \cos\theta \tag{4.68b}$$

$$\sigma_z = \bar{\sigma}_z \cos\theta = [\lambda_j \bar{\varepsilon} + 2G_j \frac{\partial \bar{w}}{\partial z}] \cos\theta \tag{4.68c}$$

$$\tau_{r\theta} = - \bar{\tau}_{r\theta} \sin\theta = - G_j [\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r} + \frac{\bar{u}}{r}] \sin\theta \tag{4.68d}$$

$$\tau_{\theta z} = - \bar{\tau}_{\theta z} \sin\theta = -G_j [\frac{\bar{w}}{r} + \frac{\partial \bar{v}}{\partial z}] \sin\theta \tag{4.68e}$$

$$\tau_{zr} = \bar{\tau}_{zr} \cos\theta = G_j [\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial r}] \cos\theta \quad . \tag{4.68f}$$

The amplitudes u, w, v, σ_z, $\tau_{\theta z}$, τ_{zr} must be continuous at $z=z_j$, $2 \leq j \leq N$. The conditions expressing continuity of σ_z, $\tau_{\theta z}$, τ_{zr} are:

$$\left[\lambda_{j-1} \bar{\varepsilon} + 2G_{j-1} \frac{\partial \bar{w}}{\partial z}\right]_{z=z_j^-} = \left[\lambda_j \bar{\varepsilon} + 2G_j \frac{\partial \bar{w}}{\partial z}\right]_{z=z_j^+} \tag{4.69a}$$

$$G_{j-1} \left[\frac{\bar{w}}{r} + \frac{\partial \bar{v}}{\partial z}\right]_{z=z_j^-} = G_j \left[\frac{\bar{w}}{r} + \frac{\partial \bar{v}}{\partial z}\right]_{z=z_j^+} \tag{4.69b}$$

$$G_{j-1}\left[\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial r}\right]_{z=z_j^-} = G_j \left[\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial r}\right]_{z=z_j^+} . \tag{4.69c}$$

Let us first consider the axisymmetric region $0 \leq r \leq r_o$, $0 \leq z \leq h$. The boundary conditions corresponding to a rigid and rough circular footing are prescribed on the surface of the region:

$$\bar{u}(r,0) = \Delta_x \tag{4.70a}$$

$$\bar{w}(r,0) = -r\phi \tag{4.70b}$$

$$\bar{v}(r,0) = \Delta_x, \quad 0 \leq r \leq r_o \quad . \tag{4.70c}$$

Δ_x and ϕ are the amplitudes of the horizontal translation (in the direction of the x-axis) and the rotation (about the y-axis) respectively (the system of coordinates (x,y,z) is understood as in the plane regions considered in the previous chapters: the x-z plane is the plane $\theta=0$). The rotation is taken positive in the counterclockwise direction. The base of the region is assumed fixed:

$$\bar{u}(r,h) = 0 \tag{4.70d}$$

$$\bar{w}(r,h) = 0 \tag{4.70e}$$

$$\bar{v}(r,h) = 0 \quad , \quad 0 \le r \le r_o \ . \tag{4.70f}$$

Δ_x, ϕ and the nodal displacements

$$\left.\begin{aligned} \bar{u}_j &= \bar{u}(r_o, z_j) \\ \bar{w}_j &= \bar{w}(r_o, z_j) \\ \bar{v}_j &= \bar{v}(r_o, z_j) \end{aligned}\right\} \quad 2 \le j \le N$$

are taken as the degrees of freedom of the element (see figure 4.3) modeling the region under consideration. The loads corresponding to these degrees of freedom are the horizontal force F_x, the rocking moment M and, at (r_o, z_j), $2 \le j \le N$, the radial force $P_{r,j}$, the vertical force $P_{z,j}$ and the tangential force $P_{\theta,j}$. We assume that $\bar{u}$, $\bar{w}$, $\bar{v}$ are linear functions of z in each sublayer. For $z_j \le z \le z_{j+1}$, $1 \le j \le N$, we have

$$\bar{u}(r_o, z) = \bar{u}_j \frac{z_{j+1}-z}{h_j} + \bar{u}_{j+1} \frac{z-z_j}{h_j} \tag{4.71a}$$

$$\bar{w}(r_o, z) = \bar{w}_j \frac{z_{j+1}-z}{h_j} + \bar{w}_{j+1} \frac{z-z_j}{h_j} \tag{4.71b}$$

$$\bar{v}(r_o, z) = \bar{v}_j \frac{z_{j+1}-z}{h_j} + \bar{v}_{j+1} \frac{z-z_j}{h_j} \quad , \tag{4.71c}$$

with $\bar{u}_1 = \bar{v}_1 = \Delta_x$, $\bar{w}_1 = -r_o\phi$, $\bar{u}_{N+1} = \bar{w}_{N+1} = \bar{v}_{N+1} = 0$ by the boundary conditions (4.70a,b,c,d,e,f). The consistent forces are given by

$$F_x = -\pi \int_0^{r_o} \bar{\tau}_{zr}\Big|_{z=0} r\,dr - \pi \int_0^{r_o} \bar{\tau}_{\theta z}\Big|_{z=0} r dr$$

$$+\pi r_o \int_{z_1}^{z_2} \bar{\sigma}_r\Big|_{r=r_o} \left[\frac{z_2-z}{h_1}\right] dz + \pi r_o \int_{z_1}^{z_2} \bar{\tau}_{r\theta}\Big|_{r=r_o} \left[\frac{z_2-z}{h_1}\right] dz \qquad (4.72a)$$

$$M = \pi \int_0^{r_o} \bar{\sigma}_z\Big|_{z=0} r^2 dr - \pi r_o^2 \int_{z_1}^{z_2} \bar{\tau}_{zr}\Big|_{r=r_o} \left[\frac{z_2-z}{h_1}\right] dz \qquad (4.72b)$$

$$P_{r,j} = \pi r_o \int_{z_{j-1}}^{z_j} \bar{\sigma}_r\Big|_{r=r_o} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz + \pi r_o \int_{z_j}^{z_{j+1}} \bar{\sigma}_r\Big|_{r=r_o} \left[\frac{z_{j+1}-z}{h_j}\right] dz \qquad (4.72c)$$

$$P_{z,j} = \pi r_o \int_{z_{j-1}}^{z_j} \bar{\tau}_{zr}\Big|_{r=r_o} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz + \pi r_o \int_{z_j}^{z_{j+1}} \bar{\tau}_{zr}\Big|_{r=r_o} \left[\frac{z_{j+1}-z}{h_j}\right] dz \qquad (4.72d)$$

$$P_{\theta,j} = \pi r_o \int_{z_{j-1}}^{z_j} \bar{\tau}_{r\theta}\Big|_{r=r_o} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz + \pi r_o \int_{z_j}^{z_{j+1}} \bar{\tau}_{r\theta}\Big|_{r=r_o} \left[\frac{z_{j+1}-z}{h_j}\right] dz, \qquad (4.72e)$$

$$2 \le j \le N.$$

The vector of displacements at (r_o, z_j), $2 \le j \le N$, is denoted by :

$$U^o_{3s-2} = \bar{u}_{s+1}$$

$$U^o_{3s-1} = \bar{w}_{s+1} \qquad 1 \le s \le N-1$$

$$U^o_{3s} = \bar{v}_{s+1} \quad .$$

We denote the vector of forces by $\underset{\sim}{F}^{o}$:

$$F^{o}_{3s-2} = P_{r,s+1}$$
$$F^{o}_{3s-1} = P_{z,s+1} \qquad 1 \le s \le N-1$$
$$F^{o}_{3s} = P_{\theta,s+1} \ .$$

Let us calculate the dynamic stiffness matrix $\underset{\sim}{K}$ of the element, i.e., the matrix relating F_x, M, $\underset{\sim}{F}^{o}$ to Δ_x, ϕ, $\underset{\sim}{U}^{o}$:

$$\begin{bmatrix} F_x \\ \hline M \\ \hline \underset{\sim}{F}^{o} \end{bmatrix} = \left[\begin{array}{c|c|c} K_{xx} & K_{x\phi} & \underset{\sim}{K}^{xo} \\ \hline K_{\phi x} & K_{\phi\phi} & \underset{\sim}{K}^{\phi o} \\ \hline \underset{\sim}{K}^{ox} & \underset{\sim}{K}^{o\phi} & \underset{\sim}{K}^{oo} \end{array}\right] \begin{bmatrix} \Delta_x \\ \hline \phi \\ \hline \underset{\sim}{U}^{o} \end{bmatrix} . \qquad (4.73)$$

First, we determine the submatrix $\underset{\sim}{K}^{oo}$. We consider solutions for which $\Delta_x = 0$, $\phi = 0$. The boundary conditions (4.70a,b,c) become

$$\bar{u}(r,0) = 0 \qquad (4.74a)$$
$$\bar{w}(r,0) = 0 \qquad 0 \le r \le r_o \qquad (4.74b)$$
$$\bar{v}(r,0) = 0 \ . \qquad (4.74c)$$

Any displacement amplitudes $\bar{u}$, $\bar{w}$, $\bar{v}$ satisfying the differential equations (4.66a,b,c), the conditions (4.69a,b,c) at $z=z_j$, $2 \le j \le N$, and the boundary conditions (4.70d,e,f), (4.74a,b,c) may be written as the superposition of modes which, according to (2.83a,b,c) and (2.84a,b,c) are given by

$$\bar{u}(r,z) = kU(z)J_1'(kr) \tag{4.75a}$$

$$\bar{w}(r,z) = -ikW(z)J_1(kr) \tag{4.75b}$$

$$\bar{v}(r,z) = \frac{1}{r}U(z)J_1(kr) \tag{4.75c}$$

and

$$\bar{u}(r,z) = \frac{1}{r}V(z)J_1(kr) \tag{4.76a}$$

$$\bar{w}(r,z) = 0 \tag{4.76b}$$

$$\bar{v}(r,z) = kV(z)J_1'(kr) \quad . \tag{4.76c}$$

We have used the Bessel function since we are considering the region $0 \leq r \leq r_o$. It was shown in section 2.4 that U and W are eigenfunctions with eigenvalue k of an eigenvalue problem which is identical to that obtained for time-harmonic vibrations in plane strain. Similarly, V is an eigenfunction with eigenvalue k of the eigenvalue problem obtained for antiplane shear. The difference is that the boundary conditions

$$U(0) = W(0) = 0$$

$$V(0) = 0 \quad ,$$

which indicate that the surface is fixed must be satisfied instead of (2.92a,b) and (2.93a), which correspond to a free surface. The eigenvalue problems have been considered in connection with the plane elements developed in Chapter 3. The algebraic eigenvalue problem corresponding to modes given by (4.75a,b,c) is

$$[k^2\underset{\sim}{A} + ik\underset{\sim}{B} + \underset{\sim}{G} - \omega^2\underset{\sim}{M}]\underset{\sim}{\Delta} = 0 \quad . \tag{4.77}$$

$\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{G}$, $\underset{\sim}{M}$, $\underset{\sim}{\Delta}$ are the same as in (3.10). The algebraic eigenvalue problem corresponding to modes given by (4.76a,b,c) is

$$[k^2\underset{\sim}{A} + \underset{\sim}{G} - \omega^2\underset{\sim}{M}]\underset{\sim}{\Delta} = 0 \quad . \tag{4.78}$$

$\underset{\sim}{A}$, $\underset{\sim}{G}$, $\underset{\sim}{M}$, $\underset{\sim}{\Delta}$ are those in (3.66). We form the (3N-3)x(3N-3) diagonal matrix $\underset{\sim}{K}$ in which the first 2N-2 entries are the wave numbers k_j, $1 \leq j \leq 2N-2$, chosen as in (3.11), and the last N-1 entries are the wave numbers k_j, $1 \leq j \leq N-1$, chosen as in (3.67):

$$\underset{\sim}{K} = \text{diag } [k_j] \quad . \tag{4.79}$$

Again, it must be noted that for the element modeling the region $0 \leq r \leq r_o$ the wave numbers chosen in (4.79) need only correspond to linearly independent modes. The matrix $\underset{\sim}{\Phi}$ is now given by

$$\left.\begin{aligned} \Phi_{3j-2,\ell} &= -\, U_j^\ell \, J_o(k_\ell r_o) \\ \Phi_{3j-1,\ell} &= -\, i\, W_j^\ell \, J_1(k_\ell r_o) \\ \Phi_{3j,\ell} &= 0 \end{aligned}\right\} \quad 1 \leq \ell \leq 2N-2, \; 1 \leq j \leq N-1$$

$$\left.\begin{aligned} \Phi_{3j-2,\ell} &= 0 \\ \Phi_{3j-1,\ell} &= 0 \\ \Phi_{3j,\ell} &= -V_j^{\ell-2N+2} J_o(k_\ell r_o) \end{aligned}\right\} \quad 2N-1 \leq \ell \leq 3N-3, \; 1 \leq j \leq N-1 \tag{4.80}$$

(the superscript indicates the particular eigenvector of the algebraic eigenvalue problem). The matrix $\underset{\sim}{\Psi}$ is taken as

$$\left.\begin{aligned} \Psi_{3j-2,\ell} &= U_j^\ell \, J_1(k_\ell r_o) \\ \Psi_{3j-1,\ell} &= iW_j^\ell \, J_o(k_\ell r_o) \\ \Psi_{3j,\ell} &= 0 \end{aligned}\right\} \quad 1 \leq \ell \leq 2N-2, \; 1 \leq j \leq N-1 \tag{4.81}$$

$$\left.\begin{aligned} \Psi_{3j-2,\ell} &= 0 \\ \Psi_{3j-1,\ell} &= 0 \\ \Psi_{3j,\ell} &= V_j^{\ell-2N+2}\, J_1(k_\ell r_o) \end{aligned}\right\} 2N-1 \le \ell \le 3N-3,\; 1 \le j \le N-1 \;.$$

The matrix $\underset{\sim}{W}$ of modal amplitudes at $r = r_o$ is given by

$$\left.\begin{aligned} W_{3j-2,\ell} &= k_\ell\, U_j^\ell\, J_1'(k_\ell r_o) \\ W_{3j-1,\ell} &= -ik_\ell\, W_j^\ell\, J_1(k_\ell r_o) \\ W_{3j,\ell} &= \frac{1}{r_o} U_j^\ell\, J_1(k_\ell r_o) \end{aligned}\right\} \; 1 \le \ell \le 2N-2,\; 1 \le j \le N-1$$

$$\left.\begin{aligned} W_{3j-2,\ell} &= \frac{1}{r_o} V_j^{\ell-2N+2}\, J_1(k_\ell r_o) \\ W_{3j-1,\ell} &= 0 \\ W_{3j,\ell} &= k_\ell V_j^{\ell-2N+2}\, J_1'(k_\ell r_o) \end{aligned}\right\} \; 2N-1 \le \ell \le 3N-3,\; 1 \le j \le N-1 \;. \tag{4.82}$$

We have

$$\underset{\sim}{U}^o = \underset{\sim}{W}\, \underset{\sim}{\Gamma} \quad . \tag{4.83}$$

Using (2.107) with n=1 and integrating with respect to θ we find the nodal forces

$$\underset{\sim}{F}^o = -\pi r_o [\underset{\sim}{A}\, \underset{\sim}{\Psi} \underset{\sim}{K}\, \underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E} + \underset{\sim}{N})\, \underset{\sim}{\Phi}\, \underset{\sim}{K} - (\underset{\sim}{L} + \underset{\sim}{Q}) \underset{\sim}{\Psi}]\, \underset{\sim}{\Gamma} \;. \tag{4.84}$$

$\underset{\sim}{A}$, $\underset{\sim}{D}$, $\underset{\sim}{E}$, $\underset{\sim}{N}$, $\underset{\sim}{L}$, $\underset{\sim}{Q}$ are obtained from those in (2.107) by deleting the first three rows and columns. Using (4.83), we eliminate the participation factors. We find

$$\left[\underset{\sim}{F}^o\right] = \left[\underset{\sim}{K}^{oo}\right] \left[\underset{\sim}{U}^o\right] \quad . \tag{4.85}$$

$\underset{\sim}{K}^{oo}$ is given by

$$\underset{\sim}{K}^{oo} = -\pi r_o[\underset{\sim}{A}\ \underset{\sim}{\Psi}\ \underset{\sim}{K}\ \underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E} + \underset{\sim}{N})\underset{\sim}{\Phi}\ \underset{\sim}{K} - (\underset{\sim}{L} + \underset{\sim}{Q})\underset{\sim}{\Psi}]\underset{\sim}{W}^{-1} \quad . \quad (4.86)$$

Let us now obtain a particular solution of (4.73) for which $\Delta_x = 1$, $\phi = 0$. The loads and displacements corresponding to this particular solution are denoted by

$$\underset{\sim}{F}^{o,1},\ \underset{\sim}{U}^{o,1},\ F_x^1,\ M_1 \quad .$$

Substituting

$$\bar{u}(r,z) = U(z) \quad (4.86a)$$

$$\bar{w}(r,z) = 0 \quad (4.86b)$$

$$\bar{v}(r,z) = U(z) \quad (4.86c)$$

into differential equations (4.66a,b,c), the conditions (4.69a, b,c) and the boundary conditions (4.70a,b,c,d,e,f), we find that U must satisfy the differential equation, in sublayer j,

$$G_j \frac{d^2U}{dz^2} + \omega^2\rho_j U = 0 \quad , \quad (4.87a)$$

the condition at $z=z_j$, $2 \le j \le N$,

$$G_{j-1} \left.\frac{dU}{dz}\right|_{z=z_j^-} = G_j \left.\frac{dU}{dz}\right|_{z=z_j^+} \quad (4.87b)$$

and the boundary conditions

$$U(0) = 1 \quad (4.87c)$$

$$U(h) = 0 \quad . \quad (4.87d)$$

This problem is the same as the one obtained in section 3.1 for the particular solution corresponding to horizontal vibrations of the rigid and rough strip footing. To find the corresponding discrete solution we solve

$$[\underset{\sim}{G} - \omega^2 \underset{\sim}{M}] \underset{\sim}{\Delta} = \underset{\sim}{F} \quad , \qquad (4.88)$$

with $\Delta_1 = 1, \quad \Delta_j = U(z_j), \quad 2 \leq j \leq N \quad ,$

$$F_1 = -\tau_1 = -G_1 \left. \frac{dU}{dz} \right|_{z=0} , \quad F_j = 0, \quad 2 \leq j \leq N \quad .$$

$\underset{\sim}{G}$, $\underset{\sim}{M}$ are the same as those in (3.25). We obtain

$$\underset{\sim}{U}^{o,1} = \underset{\sim}{Y} \quad . \qquad (4.89a)$$

with $Y_{3j-2} = \Delta_{j+1}, \quad Y_{3j-1} = 0, \quad Y_{3j} = \Delta_{j+1}, \quad 1 \leq j \leq N-1.$

Using (4.72c,d,e) we find

$$\underset{\sim}{F}^{o,1} = -\pi r_o \underset{\sim}{D} \begin{bmatrix} 1 \\ 0 \\ 1 \\ \underset{\sim}{Y} \end{bmatrix} \quad . \qquad (4.89b)$$

The matrix $\underset{\sim}{D}$ is obtained from that in (2.107) by deleting the first three rows and multiplying columns $3j-2$, $1 \leq j \leq N$, by -1. It is easily seen that radial and tangential forces in (4.89b) are equal to zero. Finally, using (4.72a,b) we obtain

$$F_x^1 = \pi r_o^2 F_1 \qquad (4.89c)$$

$$M_1 = \frac{\pi}{2} r_o^2 G_1 (1 - \Delta_2) \quad . \qquad (4.89d)$$

F_1, Δ_2 are found from (4.88).

Working similarly, we obtain a particular solution of (4.73) for which $\Delta_x = 0$, $\phi = 1$. The loads and displacements corresponding to this particular solution are denoted by

$$\underset{\sim}{F}^{o,2},\ \underset{\sim}{U}^{o,2},\ F_x^2,\ M_2 \quad .$$

Substituting

$$\bar{u}(r,z) = U(z) \tag{4.90a}$$

$$\bar{w}(r,z) = -rW(z) \tag{4.90b}$$

$$\bar{v}(r,z) = U(z) \tag{4.90c}$$

into the differential equations (4.66a,b,c), the conditions (4.69a,b,c) and the boundary conditions (4.70a,b,c,d,e,f), we find that U and W must satisfy the differential equations, in sublayer j,

$$G_j \frac{d^2U}{dz^2} - (\lambda_j + G_j)\frac{dW}{dz} + \omega^2\rho_j U = 0 \tag{4.91a}$$

$$(\lambda_j + 2G_j)\frac{d^2W}{dz^2} + \omega^2\rho_j W = 0 \tag{4.91b}$$

the conditions at $z=z_j$, $2 \le j \le N$

$$(\lambda_{j-1} + 2G_{j-1})\frac{dW}{dz}\bigg|_{z=z_j^-} = (\lambda_j + 2G_j)\frac{dW}{dz}\bigg|_{z=z_j^+} \tag{4.91c}$$

$$G_{j-1}\left[-W + \frac{dU}{dz}\right]_{z=z_j^-} = G_j\left[-W + \frac{dU}{dz}\right]_{z=z_j^+} \tag{4.91d}$$

and the boundary conditions

$$U(0) = 0 \tag{4.91e}$$

$$W(0) = 1 \tag{4.91f}$$

$$U(h) = 0 \tag{4.91g}$$

$$W(h) = 0 \quad . \tag{4.91h}$$

This problem is the same as the one obtained in section 3.1 for the particular solution corresponding to rocking of the rigid and rough strip footing. To find the corresponding discrete solution we solve

$$\underset{\sim}{S}\ \underset{\sim}{\Delta} = \underset{\sim}{P} \quad , \tag{4.92}$$

with $\Delta_1 = 0$, $\Delta_2 = 1$, $\Delta_{2j-1} = U(z_j)$, $\Delta_{2j} = W(z_j)$, $2 \leq j \leq N$,

$$P_1 = -\tau_1 = -G_1 \left[-W + \frac{dU}{dz}\right]_{z=0}, \quad P_2 = -\sigma_1 = -(\lambda_1+2G_1)\frac{dW}{dz}\bigg|_{z=0}$$

$$P_{2j-1} = P_{2j} = 0 \ , \quad 2 \leq j \leq N.$$

$\underset{\sim}{S}$ is the same as in (3.50). Thus we obtain

$$\underset{\sim}{U}^{o,2} = \underset{\sim}{Y} \quad , \tag{4.93a}$$

with $Y_{3j-2}=\Delta_{2j+1}$, $Y_{3j-1}=-r_o\Delta_{2j+2}$, $Y_{3j}=\Delta_{2j+1}$, $1 \leq j \leq N-1$.

Using (4.72 c,d,e) we find

$$\underset{\sim}{F}^{o,2} = -\pi r_o\ \underset{\sim}{H}^{r_o}\ \underset{\sim}{\Delta}' \ , \tag{4.93b}$$

with $\Delta'_{3j-2} = \Delta_{2j-1}$, $\Delta'_{3j-1} = \Delta_{2j}$, $\Delta'_{3j} = 0$, $1 \leq j \leq N$.

The matrix $\underset{\sim}{H}^r$ is obtained by assembling the sublayer matrices $\underset{\sim}{H}^{r,j}$ given by

$$\underset{\sim}{H}^{r,j} = \begin{bmatrix} 0 & -\frac{r}{2}\lambda_j & 0 & 0 & \frac{r}{2}\lambda_j & 0 \\ \frac{1}{2}G_j & \frac{1}{3}G_jh_j & 0 & -\frac{1}{2}G_j & \frac{1}{6}G_jh_j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{r}{2}\lambda_j & 0 & 0 & \frac{r}{2}\lambda_j & 0 \\ \frac{1}{2}G_j & \frac{1}{6}G_jh_j & 0 & -\frac{1}{2}G_j & \frac{1}{3}G_jh_j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.94}$$

($\underset{\sim}{H}^{r_o}$ in (4.93) is obtained from $\underset{\sim}{H}^r$ evaluated at $r = r_o$ after deleting the first three rows, since the forces in (4.93) correspond to nodes below the surface). Finally, using (4.72a,b) we obtain

$$F_x^2 = \pi r_o^2 P_1 + \frac{\pi}{2} r_o^2 \lambda_1(1-\Delta_4) \tag{4.95a}$$

$$M_2 = \frac{\pi}{4} r_o^4 P_2 + (\frac{1}{3} G_1 h_1 - \frac{1}{2} G_1 \Delta_3 + \frac{1}{6} G_1 h_1 \Delta_4) \pi r_o^2 \quad . \tag{4.95b}$$

The dynamic stiffness matrix $\underset{\sim}{K}$ in (4.73) is symmetric. Using the submatrix $\underset{\sim}{K}^{oo}$ given by (4.86) and the particular solutions (4.89a,b,c,d) and (4.93a,b), (4.95a,b) we obtain

$$\underset{\sim}{K}^{ox} = [\underset{\sim}{K}^{xo}]^T = \underset{\sim}{F}^{o,1} - \underset{\sim}{K}^{oo} \underset{\sim}{U}^{o,1} \tag{4.69a}$$

$$\underset{\sim}{K}^{o\phi} = [\underset{\sim}{K}^{\phi o}]^T = \underset{\sim}{F}^{o,2} - \underset{\sim}{K}^{oo} \underset{\sim}{U}^{o,2} \tag{4.96b}$$

$$K_{xx} = F_x^1 - \underset{\sim}{K}^{xo} \underset{\sim}{U}^{o,1} \tag{4.96c}$$

$$K_{\phi\phi} = M_2 - \underset{\sim}{K}^{\phi o} \underset{\sim}{U}^{o,2} \tag{4.96d}$$

$$K_{x\phi} = K_{\phi x} = M_1 - \underset{\sim}{K}^{\phi o} \underset{\sim}{U}^{o,1} \tag{4.96e}$$

The calculation of the dynamic stiffness matrix of the element is now complete. We note that the computational effort required to obtain the matrix is independent of the diameter $2r_o$ of the element.

Let us now consider the axisymmetric region $0 \leq r_1 \leq r \leq r_2$, $0 \leq z \leq h$. The boundary conditions corresponding to a rigid and rough ring footing are prescribed on the surface of the region:

$$\bar{u}(r,0) = \Delta_x \tag{4.97a}$$

$$\bar{w}(r,0) = -r\phi \qquad r_1 \le r \le r_2 \tag{4.97b}$$

$$\bar{v}(r,0) = \Delta_x \ . \tag{4.97c}$$

Δ_x, ϕ and the nodal displacements

$$\left.\begin{aligned} \bar{u}_j^\ell &= \bar{u}(r_\ell, z_j) \\ \bar{w}_j^\ell &= \bar{w}(r_\ell, z_j) \\ \bar{v}_j^\ell &= \bar{v}(r_\ell, z_j) \end{aligned}\right\} \quad 2 \le j \le N, \ \ell = 1, 2$$

are taken as the degrees of freedom of the element (see figure 4.4). The corresponding loads are the horizontal force F_x, the rocking moment M and, at (r_ℓ, z_j), $2 \le j \le N$, the radial forces $P_{r,j}^\ell$, the vertical forces $P_{z,j}$ and the tangential forces $P_{\theta,j}^\ell$ $(\ell = 1, 2)$. The amplitudes $\bar{u}$, $\bar{w}$, $\bar{v}$ are assumed to be linear functions of z in each sublayer. For $z_j \le z \le z_{j+1}$, $1 \le j \le N$, we have

$$\bar{u}(r_\ell, z_j) = \bar{u}_j^\ell \frac{z_{j+1} - z}{h_j} + \bar{u}_{j+1}^\ell \frac{z - z_j}{h_j}$$

$$\bar{w}(r_\ell, z_j) = \bar{w}_j^\ell \frac{z_{j+1} - z}{h_j} + \bar{w}_{j+1}^\ell \frac{z - z_j}{h_j} \qquad \ell = 1, 2$$

$$\bar{v}(r_\ell, z_j) = \bar{v}_j^\ell \frac{z_{j+1} - z}{h_j} + \bar{v}_{j+1}^\ell \frac{z - z_j}{h_j} \ ,$$

with $\bar{u}_1^\ell = v_1^\ell = \Delta_x$, $\bar{w}_1^\ell = -r_\ell\phi$, $\bar{u}_{N+1}^\ell = \bar{w}_{N+1}^\ell = \bar{v}_{N+1}^\ell = 0$ $(\ell=1,2)$ by the boundary conditions. The consistent forces are given by

$$F_x = -\pi \int_{r_1}^{r_2} \bar{\tau}_{zr}\Big|_{z=0} rdr - \pi \int_{r_1}^{r_2} \bar{\tau}_{\theta z}\Big|_{z=0} rdr$$

$$-\pi r_1 \int_{z_1}^{z_2} \bar{\sigma}_r\Big|_{r=r_1}\left[\frac{z_2-z}{h_1}\right]dz - \pi r_1 \int_{z_1}^{z_2} \bar{\tau}_{r\theta}\Big|_{r=r_1}\left[\frac{z_2-z}{h_1}\right]dz$$

$$+ \pi r_2 \int_{z_1}^{z_2} \bar{\sigma}_r\Big|_{r=r_2}\left[\frac{z_2-z}{h_1}\right]dz + \pi r_2 \int_{z_1}^{z_2} \bar{\tau}_{r\theta}\Big|_{r=r_2}\left[\frac{z_2-z}{h_1}\right]dz \quad (4.98a)$$

$$M = \pi \int_{r_1}^{r_2} \bar{\sigma}_z\Big|_{z=0} r^2 dr$$

$$+ \pi r_1^2 \int_{z_1}^{z_2} \bar{\tau}_{zr}\Big|_{r=r_1}\left[\frac{z_2-z}{h_1}\right]dz - \pi r_2^2 \int_{z_1}^{z_2} \bar{\tau}_{zr}\Big|_{r=r_2}\left[\frac{z_2-z}{h_1}\right]dz \quad (4.98b)$$

$$P^1_{r,j} = -\pi r_1 \int_{z_{j-1}}^{z_j} \bar{\sigma}_r\Big|_{r=r_1}\left[\frac{z-z_{j-1}}{h_{j-1}}\right]dz - \pi r_1 \int_{z_j}^{z_{j+1}} \bar{\sigma}_r\Big|_{r=r_1}\left[\frac{z_{j+1}-z}{h_j}\right]dz \quad (4.98c)$$

$$P^1_{z,j} = -\pi r_1 \int_{z_{j-1}}^{z_j} \bar{\tau}_{zr}\Big|_{r=r_1}\left[\frac{z-z_{j-1}}{h_{j-1}}\right]dz - \pi r_1 \int_{z_j}^{z_{j+1}} \bar{\tau}_{zr}\Big|_{r=r_1}\left[\frac{z_{j+1}-z}{h_j}\right]dz \quad (4.98d)$$

$$P^1_{\theta,j} = -\pi r_1 \int_{z_{j-1}}^{z_j} \bar{\tau}_{r\theta}\Big|_{r=r_1}\left[\frac{z-z_{j-1}}{h_{j-1}}\right]dz - \pi r_1 \int_{z_j}^{z_{j+1}} \bar{\tau}_{r\theta}\Big|_{r=r_1}\left[\frac{z_{j+1}-z}{h_j}\right]dz \quad (4.98e)$$

$$P^2_{r,j} = \pi r_2 \int_{z_{j-1}}^{z_j} \bar{\sigma}_r\Big|_{r=r_2}\left[\frac{z-z_{j-1}}{h_{j-1}}\right]dz + \pi r_2 \int_{z_j}^{z_{j+1}} \bar{\sigma}_r\Big|_{r=r_2}\left[\frac{z_{j+1}-z}{h_j}\right]dz \quad (4.98f)$$

$$P^2_{z,j} = \pi r_2 \int_{z_{j-1}}^{z_j} \bar{\tau}_{zr}\Big|_{r=r_2}\left[\frac{z-z_{j-1}}{h_{j-1}}\right]dz + \pi r_2 \int_{z_j}^{z_{j+1}} \bar{\tau}_{zr}\Big|_{r=r_2}\left[\frac{z_{j+1}-z}{h_j}\right]dz \quad (4.98g)$$

$$P^2_{\theta,j} = \pi r_2 \int_{z_{j-1}}^{z_j} \bar{\tau}_{r\theta}\Big|_{r=r_2} \left[\frac{z-z_{j-1}}{h_{j-1}}\right] dz + \pi r_2 \int_{z_j}^{z_{j+1}} \bar{\tau}_{r\theta}\Big|_{r=r_2} \left[\frac{z_{j+1}-z}{h_j}\right] dz \,, \tag{4.98h}$$

$$2 \leq j \leq N.$$

The vectors of displacements at (r_1,z_j), (r_2,z_j), $2 \leq j \leq N$, are denoted by $\underset{\sim}{U}^1$, $\underset{\sim}{U}^2$ respectively:

$$\begin{aligned} U^\ell_{3s-2} &= \bar{u}^\ell_{s+1} \\ U^\ell_{3s-1} &= \bar{w}^\ell_{s+1} \qquad\qquad 1 \leq s \leq N-1, \quad \ell = 1,\ 2 \\ U^\ell_{3s} &= \bar{v}^\ell_{s+1} \quad . \end{aligned}$$

The vectors of nodal forces are denoted by $\underset{\sim}{F}^1$, $\underset{\sim}{F}^2$:

$$\begin{aligned} F^\ell_{3s-2} &= P^\ell_{r,s+1} \\ F^\ell_{3s-1} &= P^\ell_{z,s+1} \qquad\qquad 1 \leq s \leq N-1, \quad \ell = 1,\ 2 \\ F^\ell_{3s} &= P^\ell_{\theta,s+1} \quad . \end{aligned}$$

Let us determine the dynamic stiffness matrix $\underset{\sim}{K}$ of the elements:

$$\begin{bmatrix} \underset{\sim}{F}^1 \\ \hline F_x \\ \hline M \\ \hline \underset{\sim}{F}^2 \end{bmatrix} = \left[\begin{array}{c|c|c|c} \underset{\sim}{K}^{11} & \underset{\sim}{K}^{1x} & \underset{\sim}{K}^{1\phi} & \underset{\sim}{K}^{12} \\ \hline \underset{\sim}{K}^{x1} & K_{xx} & K_{x\phi} & \underset{\sim}{K}^{x2} \\ \hline \underset{\sim}{K}^{\phi 1} & K_{\phi x} & K_{\phi\phi} & \underset{\sim}{K}^{\phi 2} \\ \hline \underset{\sim}{K}^{21} & \underset{\sim}{K}^{2x} & \underset{\sim}{K}^{2\phi} & \underset{\sim}{K}^{22} \end{array}\right] \begin{bmatrix} \underset{\sim}{U}^1 \\ \hline \Delta_x \\ \hline \phi \\ \hline \underset{\sim}{U}^2 \end{bmatrix} . \tag{4.99}$$

First, we calculate the submatrices $\underset{\sim}{K}^{11}$, $\underset{\sim}{K}^{12}$, $\underset{\sim}{K}^{21}$, $\underset{\sim}{K}^{22}$. We consider solutions for which $\Delta_x = 0$, $\phi = 0$. The modes (fixed surface, fixed base) are obtained from (4.75a,b,c) and (4.76a, b,c) using Hankel functions instead of Bessel functions. If both Hankel functions (i.e. of the first and the second kind) are used, then only the eigenvalues chosen in (4.79) need be used. We write

$$\underset{\sim}{U}^1 = \underset{\sim}{W}^1\underset{\sim}{\Gamma}^1 + \underset{\sim}{\hat{W}}^1\underset{\sim}{\Gamma}^2 \tag{4.100a}$$

$$\underset{\sim}{U}^2 = \underset{\sim}{W}^2\underset{\sim}{\Gamma}^1 + \underset{\sim}{\hat{W}}^2\underset{\sim}{\Gamma}^2 \quad . \tag{4.100b}$$

$\underset{\sim}{W}^1$, $\underset{\sim}{W}^2$ are obtained as in (4.82) but using Hankel functions of the second kind and evaluating at $r=r_1$ and $r=r_2$ respectively. $\underset{\sim}{\hat{W}}^1$, $\underset{\sim}{\hat{W}}^2$ are calculated using Hankel functions of the first kind. The nodal forces are given by

$$\underset{\sim}{F}^1 = \pi r_1[\underset{\sim}{A}\,\underset{\sim}{\Psi}^1\underset{\sim}{K}\,\underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E}^1 + \underset{\sim}{N}^1)\underset{\sim}{\Phi}^1\underset{\sim}{K} - (\underset{\sim}{L}^1 + \underset{\sim}{Q}^1)\underset{\sim}{\Psi}^1]\underset{\sim}{\Gamma}^1$$
$$+ \pi r_1[\underset{\sim}{A}\,\underset{\sim}{\hat{\Psi}}^1\underset{\sim}{K}\,\underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E}^1 + \underset{\sim}{N}^1)\underset{\sim}{\hat{\Phi}}^1\underset{\sim}{K} - (\underset{\sim}{L}^1 + \underset{\sim}{Q}^1)\underset{\sim}{\hat{\Psi}}^1]\underset{\sim}{\Gamma}^2 \tag{4.101a}$$

$$\underset{\sim}{F}^2 = -\pi r_2[\underset{\sim}{A}\,\underset{\sim}{\Psi}^2\underset{\sim}{K}\,\underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E}^2 + \underset{\sim}{N}^2)\underset{\sim}{\Phi}^2\underset{\sim}{K} - (\underset{\sim}{L}^2 + \underset{\sim}{Q}^2)\underset{\sim}{\Psi}^2]\underset{\sim}{\Gamma}^1$$
$$- \pi r_2[\underset{\sim}{A}\,\underset{\sim}{\hat{\Psi}}^2\,\underset{\sim}{K}\,\underset{\sim}{K} + (\underset{\sim}{D} - \underset{\sim}{E}^2 + \underset{\sim}{N}^2)\underset{\sim}{\hat{\Phi}}^2\underset{\sim}{K} - (\underset{\sim}{L}^2 + \underset{\sim}{Q}^2)\underset{\sim}{\hat{\Psi}}^2]\Gamma^2 \tag{4.101b}$$

(a superscript ℓ indicates that the matrix is evaluated at $r=r_\ell$). $\underset{\sim}{\Psi}^1$, $\underset{\sim}{\Phi}^1$, $\underset{\sim}{\Psi}^2$, $\underset{\sim}{\Phi}^2$ are calculated using Hankel functions of the second kind, while $\underset{\sim}{\hat{\Psi}}^1$, $\underset{\sim}{\hat{\Phi}}^1$, $\underset{\sim}{\hat{\Psi}}^2$, $\underset{\sim}{\hat{\Phi}}^2$ are determined using Hankel functions of the first kind. Eliminating the participation factors, we obtain

$$\begin{bmatrix} \underset{\sim}{F}^1 \\ \hdashline \underset{\sim}{F}^2 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{K}^{11} & \underset{\sim}{K}^{12} \\ \hdashline \underset{\sim}{K}^{21} & \underset{\sim}{K}^{22} \end{bmatrix} \begin{bmatrix} \underset{\sim}{U}^1 \\ \hdashline \underset{\sim}{U}^2 \end{bmatrix} . \quad (4.102)$$

A particular solution of (4.99) for which $\Delta_x = 1$, $\phi = 0$ is obtained as for the element modeling the region $0 \leq r \leq r_o$, $0 \leq z \leq h$. We have

$$\underset{\sim}{U}^{\ell,1} = \underset{\sim}{Y} \quad , \qquad \ell = 1, 2 \quad . \quad (4.103a)$$

$\underset{\sim}{Y}$ being the same as in (4.89a). The nodal forces are given by

$$\underset{\sim}{F}^{1,1} = \pi r_1 \underset{\sim}{D} \begin{bmatrix} 1 \\ 0 \\ 1 \\ \underset{\sim}{Y} \end{bmatrix} \quad (4.103b)$$

$$\underset{\sim}{F}^{2,1} = -\pi r_2 \underset{\sim}{D} \begin{bmatrix} 1 \\ 0 \\ 1 \\ \underset{\sim}{Y} \end{bmatrix} \quad , \quad (4.103c)$$

$\underset{\sim}{D}$ being the same as in (4.89b). Finally, we obtain

$$F_x^1 = \pi(r_2^2 - r_1^2)\, F_1 \quad (4.103d)$$

$$M_1 = \frac{\pi}{2}\,(r_2^2 - r_1^2) G_1 (1 - \Delta_2) \quad . \quad (4.103e)$$

F_1, Δ_2 are obtained from (4.88).

Working similarly, we obtain a particular solution for which $\Delta_x = 0$, $\phi = 1$. We have

$$\underset{\sim}{U}^{\ell,2} = \underset{\sim}{Y}^{\ell} \quad , \quad (4.104a)$$

with $Y^{\ell}_{3j-2} = \Delta_{2j+1}$, $Y^{\ell}_{3j-1} = - r_{\ell}\Delta_{2j+2}$, $Y^{\ell}_{3j} = \Delta_{2j+1}$, $1 \leq j \leq N-1$.

$\underset{\sim}{\Delta}$ is the same as in (4.92). The nodal forces are given by

$$\underset{\sim}{F}^{1,2} = \pi r_1 \underset{\sim}{H}^{r_1} \underset{\sim}{\Delta}' \tag{4.104b}$$

$$\underset{\sim}{F}^{2,2} = -\pi r_2 \underset{\sim}{H}^{r_2} \underset{\sim}{\Delta}' \quad . \tag{3.104c}$$

$\underset{\sim}{H}^{r_1}$, $\underset{\sim}{H}^{r_2}$ are obtained as $\underset{\sim}{H}^{r_o}$ in (4.93b). $\underset{\sim}{\Delta}'$ is the same as in (4.93b). Finally, we find

$$F^2_x = \pi(r^2_2 - r^2_1)P_1 + \frac{\pi}{2}(r^2_2 - r^2_1)\lambda_1(1 - \Delta_4) \tag{4.104d}$$

$$M_2 = \frac{\pi}{4}(r^4_2 - r^4_1)P_2 + (\frac{1}{3}G_1h_1 - \frac{1}{2}G_1\Delta_3 + \frac{1}{6}G_1h_1\Delta_4)\pi(r^2_2 - r^2_1) \, . \tag{4.104c}$$

The dynamic stiffness matrix $\underset{\sim}{K}$ in (4.99) is symmetric. Using the submatrices $\underset{\sim}{K}^{11}$, $\underset{\sim}{K}^{12}$, $\underset{\sim}{K}^{21}$, $\underset{\sim}{K}^{22}$ in (4.102) and the particular solutions (4.103a,b,c,d,e) and (4.104a,b,c,d,e), we obtain

$$\underset{\sim}{K}^{1x} = [\underset{\sim}{K}^{x1}]^T = \underset{\sim}{F}^{1,1} - \underset{\sim}{K}^{11} \underset{\sim}{U}^{1,1} - \underset{\sim}{K}^{12} \underset{\sim}{U}^{2,1} \tag{4.105a}$$

$$\underset{\sim}{K}^{2x} = [\underset{\sim}{K}^{x2}]^T = \underset{\sim}{F}^{2,1} - \underset{\sim}{K}^{21} \underset{\sim}{U}^{1,1} - \underset{\sim}{K}^{22} \underset{\sim}{U}^{2,1} \tag{4.105b}$$

$$\underset{\sim}{K}^{1\phi} = [\underset{\sim}{K}^{\phi 1}]^T = \underset{\sim}{F}^{1,2} - \underset{\sim}{K}^{11} \underset{\sim}{U}^{1,2} - \underset{\sim}{K}^{12} \underset{\sim}{U}^{2,2} \tag{4.105c}$$

$$\underset{\sim}{K}^{2\phi} = [\underset{\sim}{K}^{\phi 2}]^T = \underset{\sim}{F}^{2,2} - \underset{\sim}{K}^{21} \underset{\sim}{U}^{1,2} - \underset{\sim}{K}^{22} \underset{\sim}{U}^{2,2} \tag{4.105d}$$

$$K_{xx} = F^1_x - \underset{\sim}{K}^{x1} \underset{\sim}{U}^{1,1} - \underset{\sim}{K}^{x2} \underset{\sim}{U}^{2,1} \tag{4.105e}$$

$$K_{\phi\phi} = M_2 - \underset{\sim}{K}^{\phi 1} \underset{\sim}{U}^{1,2} - \underset{\sim}{K}^{\phi 2} \underset{\sim}{U}^{2,2} \tag{4.105f}$$

$$K_{x\phi} = K_{\phi x} = M_1 - \underset{\sim}{K}^{\phi 1} \underset{\sim}{U}^{1,1} - \underset{\sim}{K}^{\phi 2} \underset{\sim}{U}^{2,1} \quad . \tag{4.105g}$$

This completes the derivation of the dynamic stiffness matrix of the element. The number of operations necessary to obtain the matrix is independent of the thickness $r_2 - r_1$ of the element.

4.4 OTHER ELEMENTS

Several other elements may be developed using the procedure described in the previous sections. As in the case of the plane elements discussed in Chapter 3, the first step is to find solutions (modes) satisfying appropriate homogeneous boundary conditions. This step requires the solution of algebraic eigenvalue problems of the form (2.26) or (2.63). The second step is to calculate particular solutions satisfying the inhomogeneous boundary conditions. Let us consider some examples. First, we assume that boundary conditions corresponding to a rigid and smooth footing are prescribed on the surface of the region $0 \leq r \leq r_o$, $0 \leq z \leq h$, while the base is kept fixed. We restrict our attention to rocking (the other cases are similar):

$$\bar{\tau}_{zr}\Big|_{z=0} = 0 \tag{4.106a}$$

$$\bar{\tau}_{\theta z}\Big|_{z=0} = 0 \tag{4.106b}$$

$$\bar{w}(r,0) = -\, r\phi \,, \tag{4.106c}$$

$$0 \leq r \leq r_o \,.$$

Note that the Fourier number for rocking is n = 1. The corresponding homogeneous boundary conditions are that $\bar{w}$, $\bar{\tau}_{zr}$, $\bar{\tau}_{\theta z}$

vanish at $z = 0$, while $\bar{u} = \bar{w} = \bar{v} = 0$ at $z = h$. The algebraic eigenvalue problems are obtained from (2.25) by specifying $F_1 = 0$, $\Delta_2 = 0$, $\Delta_{2N+1} = \Delta_{2N+2} = 0$ and from (2.62) by specifying $F_1 = 0$, $\Delta_{N+1} = 0$. Solving these problems, we obtain the semidiscrete modes. Next, we look for a particular solution satisfying (4.106a,b,c). Such a solution is given by

$$\bar{u}(r,z) = U(z)$$

$$\bar{w}(r,z) = -rW(z)$$

$$\bar{v}(r,z) = U(z) \quad .$$

U and W must satisfy the differential equations (4.91a,b), the conditions (4.91c,d) and the boundary conditions

$$\left[-W + \frac{dU}{dz}\right]_{z=0} = 0 \qquad (4.107a)$$

$$W(0) = \phi \qquad (4.107b)$$

$$U(h) = 0 \qquad (4.107c)$$

$$W(h) = 0 \quad . \qquad (4.107d)$$

This problem is similar to the one obtained for rocking of a rigid and rough footing. The discrete solution for U and V is found from (4.92) with $P_1 = 0$, $\Delta_2 = \phi$. The corresponding semidiscrete solution for $\bar{u}$, $\bar{w}$, $\bar{v}$, together with the semidiscrete modes, are sufficient to develop an element modeling the region.

Let us now assume that x-harmonic displacements are prescribed at the base of the region $0 \leq r \leq r_o$, $0 \leq z \leq h$. Thus the amplitudes, at $z = h$, of the displacements in the x-direction, y-direction, z-direction are taken as $u_o \exp(-ikx)$, $v_o \exp(-ikx)$, $w_o \exp(-ikx)$, respectively. We consider $k \neq 0$. In cylindrical coordinates we have

$$u(r,\theta,h) = u_o \cos\theta \exp(-ikr\cos\theta) - v_o \sin\theta \exp(-ikr\cos\theta) \tag{4.108a}$$

$$w(r,\theta,h) = w_o \exp(-ikr\cos\theta) \tag{4.108b}$$

$$v(r,\theta,h) = -u_o \sin\theta \exp(-ikr\cos\theta) - v_o \cos\theta \exp(-ikr\cos\theta). \tag{4.108c}$$

The boundary conditions corresponding to a rigid and rough circular footing are prescribed on the surface of the region. In this case, apart from the particular solutions that we calculated in the previous sections, we need a solution satisfying (4.108a,b,c). Such a solution is

$$u(r,\theta,z) = U(z)\cos\theta \exp(-ikr\cos\theta) - V(z)\sin\theta \exp(-ikr\cos\theta) \tag{4.109a}$$

$$w(r,\theta,z) = W(z)\exp(-ikr\cos\theta) \tag{4.109b}$$

$$v(r,\theta,z) = -U(z)\sin\theta \exp(-ikr\cos\theta) - V(z)\cos\theta \exp(-ikr\cos\theta). \tag{4.109c}$$

For this particular solution the surface is taken fixed. U and W must satisfy the differential equations (2.7a,b), the conditions (2.7c,d) at $z = z_j$, $2 \leq j \leq N$, and the boundary conditions

$$U(0) = 0 \tag{4.110a}$$

$$W(0) = 0 \tag{4.110b}$$

$$U(h) = u_o \tag{4.110c}$$

$$W(h) = w_o\,. \tag{4.110d}$$

The discrete solution for U and W is found from (2.25) with $\Delta_1 = \Delta_2 = 0$, $\Delta_{2N+1} = u_o$, $\Delta_{2N+2} = w_o$. Similarly, V must satisfy the differential equation (2.52a), the condition (2.52b) at $z = z_j, 2 \le j \le N$, and the boundary conditions

$$V(0) = 0 \qquad (4.111a)$$

$$V(h) = v_o \; . \qquad (4.111b)$$

The discrete solution for V is obtained from (2.62) with $\Delta_1 = 0$, $\Delta_{N+1} = v_o$. Let us calculate the Fourier series expansion of the particular solution (4.109a,b,c). We have

$$u(r,\theta,z) = \bar{u}_o^s(r,z) + \sum_{n=1}^{\infty}\left[\bar{u}_n^s(r,z)\cos(n\theta) + \bar{u}_n^a(r,z)\sin(n\theta)\right] \qquad (4.112a)$$

$$w(r,\theta,z) = \bar{w}_o^s(r,z) + \sum_{n=1}^{\infty}\left[\bar{w}_n^s(r,z)\cos(n\theta) + \bar{w}_n^a(r,z)\sin(n\theta)\right] \qquad (4.112b)$$

$$v(r,\theta,z) = \bar{v}_o^a(r,z) + \sum_{n=1}^{\infty}\left[-\bar{v}_n^s(r,z)\sin(n\theta) + \bar{v}_n^a(r,z)\cos(n\theta)\right] \qquad (4.112c)$$

(the superscripts s, a indicate symmetric and antisymmetric components respectively.). Using properties of Bessel functions [1], we find

$$\bar{u}_o^s(r,z) = i\,U(z)J_o'(kr) \qquad (4.113a)$$

$$\bar{w}_o^s(r,z) = W(z)J_o(kr) \qquad (4.113b)$$

$$\bar{v}_o^a(r,z) = i\,V(z)J_o'(kr) \qquad (4.113c)$$

$$\bar{u}_n^s(r,z) = 2i^{1-n}\,U(z)J_n'(kr) \qquad (4.114a)$$

$$\bar{u}_n^a(r,z) = 2i^{1-n}\,\frac{1}{k}\,\frac{n}{r}\,V(z)J_n(kr) \qquad (4.114b)$$

$$\bar{w}_n^s(r,z) = 2i^{-n}\, W(z) J_n(kr) \tag{4.115a}$$

$$\bar{w}_n^a(r,z) = 0 \tag{4.115b}$$

$$\bar{v}_n^s(r,z) = 2i^{1-n} \frac{1}{k} \frac{n}{r} U(z) J_n(kr) \tag{4.116a}$$

$$\bar{v}_n^a(r,z) = 2i^{1-n}\, V(z) J_n'(kr), \qquad n \geq 1. \tag{4.116b}$$

We note that the amplitudes above multiplied by $c_n i^{n-1} k$ ($c_o=1$, $c_n = \frac{1}{2}$ for $n \geq 1$) are the same as those in (2.83a,b,c) (symmetric components) or (2.84a,b,c) (antisymmetric components). Thus the forces at (r_o, z_j), $2 \leq j \leq N$, for a given component may be obtained using (2.99). The components which contribute to the loads acting on the footing are

$$\bar{v}_o^a(r,z) \tag{4.117a}$$

$$\bar{u}_o^s(r,z), \qquad \bar{w}_o^s(r,z) \tag{4.117b}$$

$$\bar{u}_1^s(r,z)\cos\theta, \quad \bar{w}_1^s(r,z)\cos\theta, \quad - \bar{v}_1^s(r,z)\sin\theta. \tag{4.117c}$$

The calculation of the moments and forces for these components is straightforward. For example, let us consider the component

$$u(r,\theta,z) = 0$$

$$w(r,\theta,z) = 0$$

$$v(r,\theta,z) = \bar{v}_o^a(r,z) = iV(z) J_o'(kr) \; .$$

The torsional moment is given by (4.5a). It is trivial to calculate the second term in that expression since $\tau_{r\theta}\Big|_{r=r_o}$ is a linear function of z. We rewrite the first term

$$-2\pi \int_0^{r_o} \tau_{\theta z}\Big|_{z=0} r^2 dr \quad .$$

We have

$$\tau_{\theta z}\Big|_{z=0} = -iF_1 J_o'(kr) \quad ,$$

with $F_1 = -G_1 \frac{dV}{dz}\Big|_{z=0}$ (F_1 is found from (2.62) which gives the discrete solution for V). Thus the integral becomes

$$2\pi i F_1 \int_0^{r_o} r^2 J_o'(kr)\,dr.$$

This integral may be calculated using properties of Bessel functions. We obtain

$$-2\pi \int_0^{r_o} \tau_{\theta z}\Big|_{z=0} r^2 dr = 2\pi i \; F_1 \left[\frac{r_o^2}{k} J_o(kr_o) - \frac{2r_o}{k^2} J_1(kr_o)\right] \quad .$$

The displacements and loads corresponding to the particular solution (4.109a,b,c), together with the results of the previous sections, are sufficient for the development of the element.

Clearly, the variety of inhomogeneous boundary conditions for which elements may be developed cannot be exhausted here. It is important to note, once more, that the computational effort required to obtain the semidiscrete solutions which are necessary for the development of the elements is independent of the thickness of the elements in the radial direction.

4.5 AN APPLICATION

As a verification of the developments presented in the previous sections, let us consider time-harmonic vibrations of a rigid and rough circular footing on the surface of a stratum. Let R be the radius of the footing. The boundary conditions on the surface of the stratum are

$$\left.\begin{aligned} u(r,\theta,0) &= \Delta_x \cos\theta \\ w(r,\theta,0) &= \Delta_z - r\phi_r \cos\theta \\ v(r,\theta,0) &= r\phi_t - \Delta_x \sin\theta \end{aligned}\right\} \quad 0 \le r \le R \quad ,$$

$$\left.\begin{aligned} \sigma_z\Big|_{z=0} &= 0 \\ \tau_{\theta z}\Big|_{z=0} &= 0 \\ \tau_{zr}\Big|_{z=0} &= 0 \end{aligned}\right\} \quad r > R$$

(ϕ_r, ϕ_t are the amplitudes of the rocking and torsional rotation of the footing respectively). The base of the stratum is fixed. Let F_x, F_z, M_r, M_t be the amplitudes of the horizontal force, vertical force, rocking moment and torsional moment respectively. We write

$$\begin{bmatrix} M_t \\ F_z \\ F_x \\ M_r \end{bmatrix} = \underset{\sim}{K} \begin{bmatrix} \phi_t \\ \Delta_z \\ \Delta_x \\ \phi_r \end{bmatrix} \quad ,$$

$\underset{\sim}{K}$ being the dynamic stiffness matrix (symmetric):

$$\underset{\sim}{K} = \begin{bmatrix} K_{tt} & 0 & 0 & 0 \\ 0 & K_{zz} & 0 & 0 \\ 0 & 0 & K_{xx} & K_{xr} \\ 0 & 0 & K_{rx} & K_{rr} \end{bmatrix}$$

Let us assume that the stratum is homogeneous. The nondimensional stiffness

$$\frac{K_{tt}}{GR^3}$$

is a function of the nondimensional quantities $\frac{\omega R}{C_T}$ (nondimensional frequency), $\frac{h}{R}$, β. The nondimensional stiffnesses

$$\frac{K_{zz}}{GR}, \quad \frac{K_{xx}}{GR}, \quad \frac{K_{rr}}{GR^3}, \quad \frac{K_{xr}}{GR^2},$$

are functions of $\frac{\omega R}{C_T}$, $\frac{h}{R}$, ν, β. In order to calculate the stiffnesses we combine (see figure 4.5) the elements (modeling the region $0 \leq r \leq R$, $0 \leq z \leq h$) developed in the previous sections with the transmitting boundaries (modeling the region $r > R$, $0 \leq z \leq h$) developed by Waas [23] and Kausel [6] (see section 2.4). Waas [23] and Kausel [6] have obtained results by combining the transmitting boundaries with a conventional finite element mesh modeling the cylindrical region below the footing. In fact, the elements that we are using may be understood as meshes with an infinite number of rings of finite elements. It is convenient to write the nondimensional stiffnesses as

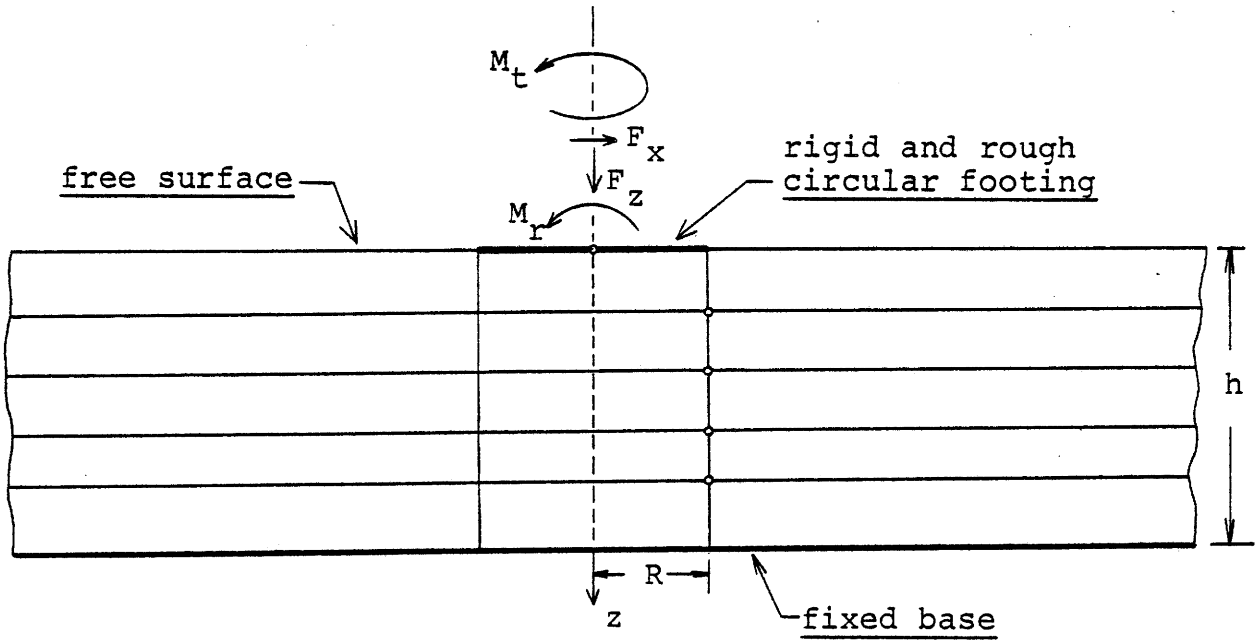


Figure 4.5 Scheme for the calculation of the stiffness of a circular footing on the surface of a stratum.

$$\frac{K_{tt}}{GR^3} = \frac{K^o_{tt}}{GR^3}\left(k_{tt} + i\,\frac{\omega R}{C_T}\,c_{tt}\right)(1+2i\beta)$$

$$\frac{K_{zz}}{GR} = \frac{K^o_{zz}}{GR}\left(k_{zz} + i\,\frac{\omega R}{C_T}\,c_{zz}\right)(1+2i\beta)$$

$$\frac{K_{xx}}{GR} = \frac{K^o_{xx}}{GR}\left(k_{xx} + i\,\frac{\omega R}{C_T}\,c_{xx}\right)(1+2i\beta)$$

$$\frac{K_{rr}}{GR^3} = \frac{K^o_{rr}}{GR^3}\left(k_{rr} + i\,\frac{\omega R}{C_T}\,c_{rr}\right)(1+2i\beta).$$

The superscript o indicates the static stiffnesses. k_{tt}, k_{zz}, k_{xx}, k_{rr}, c_{tt}, c_{zz}, c_{xx}, c_{rr} are referred to as the stiffness coefficients. We note that the normalization of the stiffnesses may be more effective if

$$C_T^c = \left[\frac{G(1+2i\beta)}{\rho}\right]^{1/2} , \qquad Re[C_T^c] > 0 ,$$

is used instead of C_T in the above expressions [7] (then the stiffness coefficients are rather independent of damping). The stiffness coefficients as defined above are, however, those used in [6,11] (results reported in these references are used here for the purposes of comparison). Figures 4.6, 4.7, 4.8, 4.9 show plots of the stiffness coefficients versus the nondimensional frequency $\frac{1}{2\pi}\frac{\omega R}{C_T}$, for $\frac{h}{R} = 2$, $\nu = \frac{1}{3}$, $\beta =$ 0.05. The static stiffnesses are

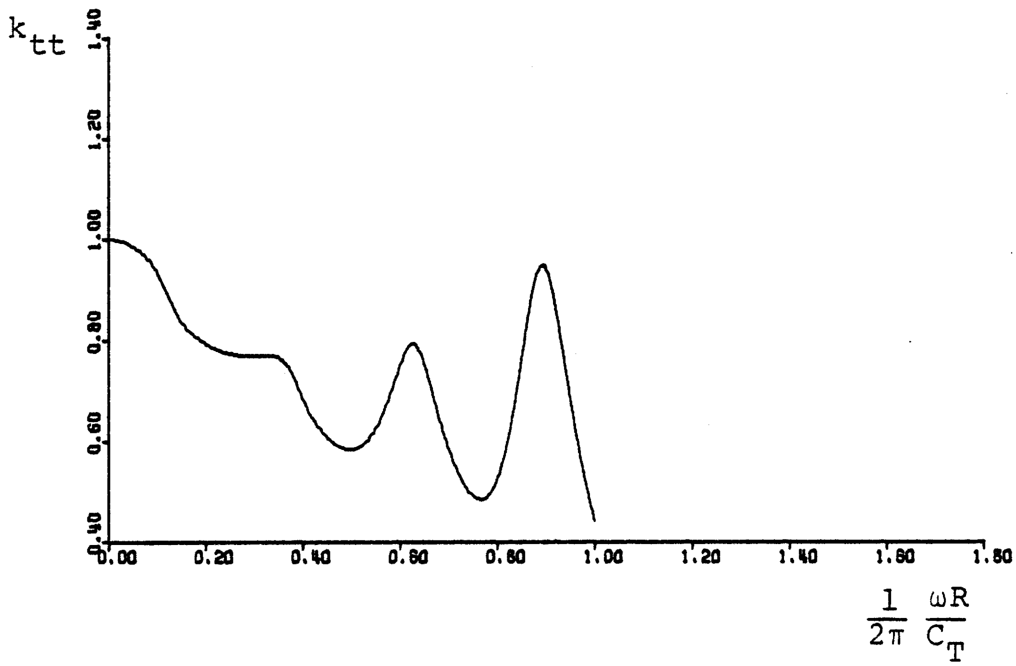
$$\frac{K_{tt}^o}{GR^3} = 5.79$$

$$\frac{K_{zz}^o}{GR} = 10.37$$

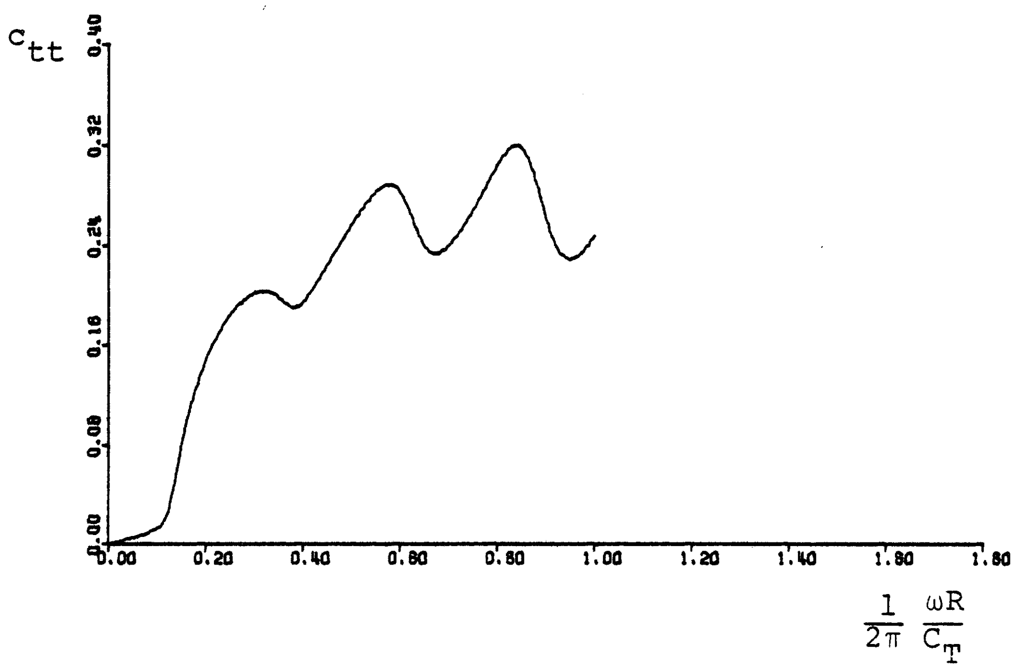
$$\frac{K_{xx}^o}{GR} = 6.36$$

$$\frac{K_{rr}}{GR^3} = 4.63 \quad .$$

The stratum was divided into twelve sublayers of equal depth. For each frequency the computation of the torsional stiffness takes approximately 0.8 second on IBM 370/165, the computation of the vertical stiffness 6.0 seconds and the computation of the swaying and rocking stiffnesses 9.5 seconds. Again, a great advantage of using the elements developed in this work is that the storage requirements are very low compared with those of a conventional finite element mesh fine enough for accurate results. For the application considered here only fast memory is necessary. Moreover, as already pointed out, the computational effort associated with the elements considered in this work is independent of their thickness in the radial direction. This is not the case with a conventional finite element mesh. The agreement of the results shown in figures 4.6, 4.7, 4.8, 4.9 with those reported in [23, 6 , 11] is excellent. It is not possible to resolve the difference between the results within the scale of the drawings.



<u>Figure 4.6</u> - Torsional stiffness coefficients (circular footing, $\beta = 0.05$, $\frac{h}{R} = 2$).



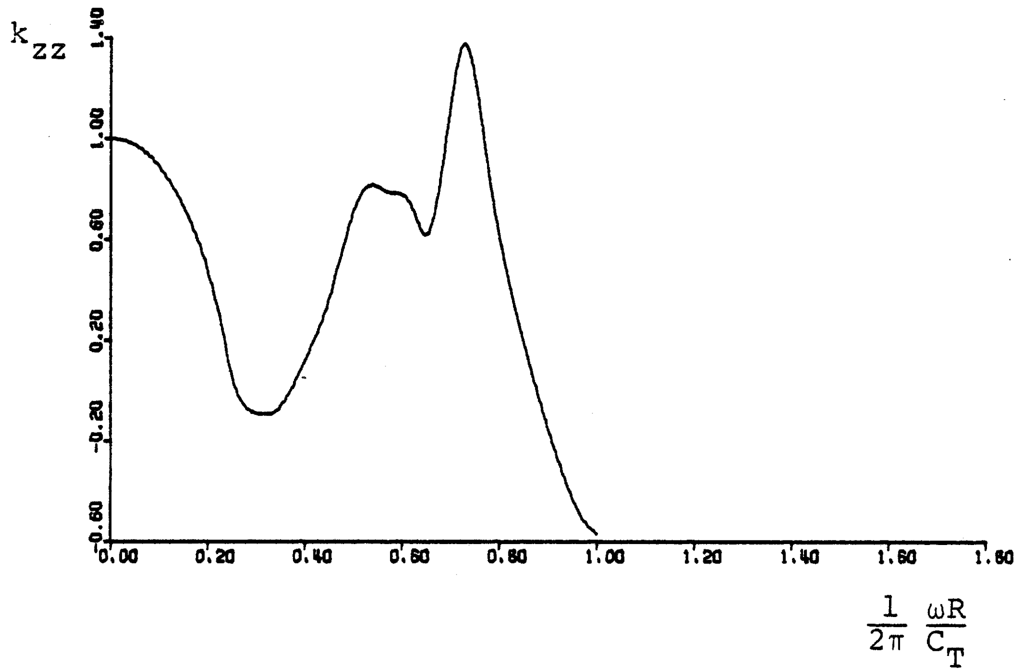
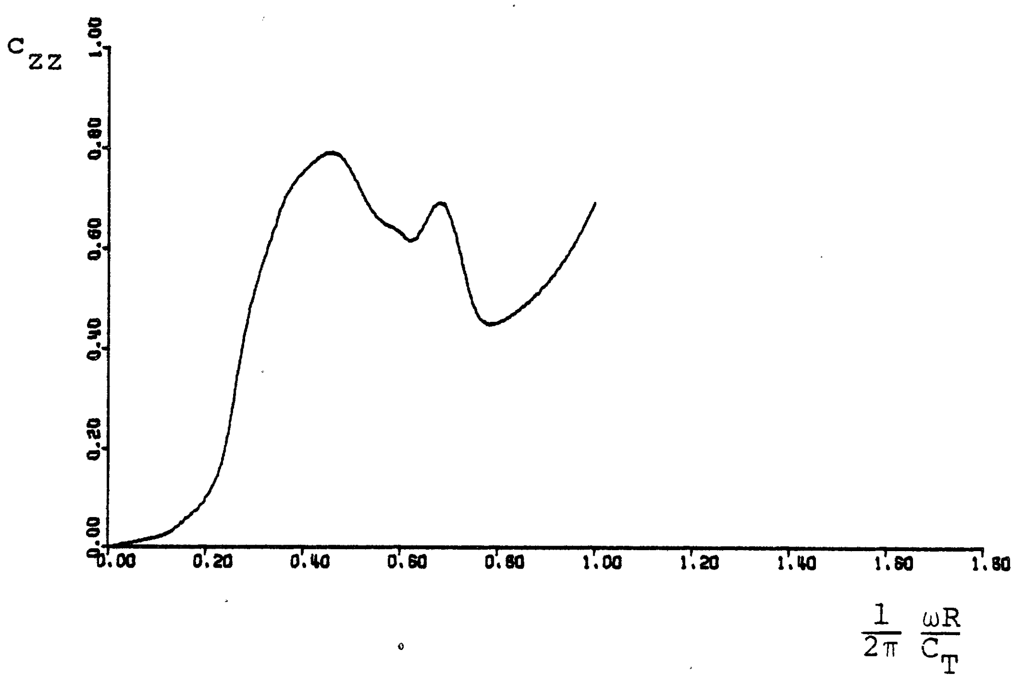


Figure 4.7 - Vertical stiffness cofficients (circular footing, $\nu = 1/3$, $\beta = 0.05$, $\frac{h}{R} = 2$).



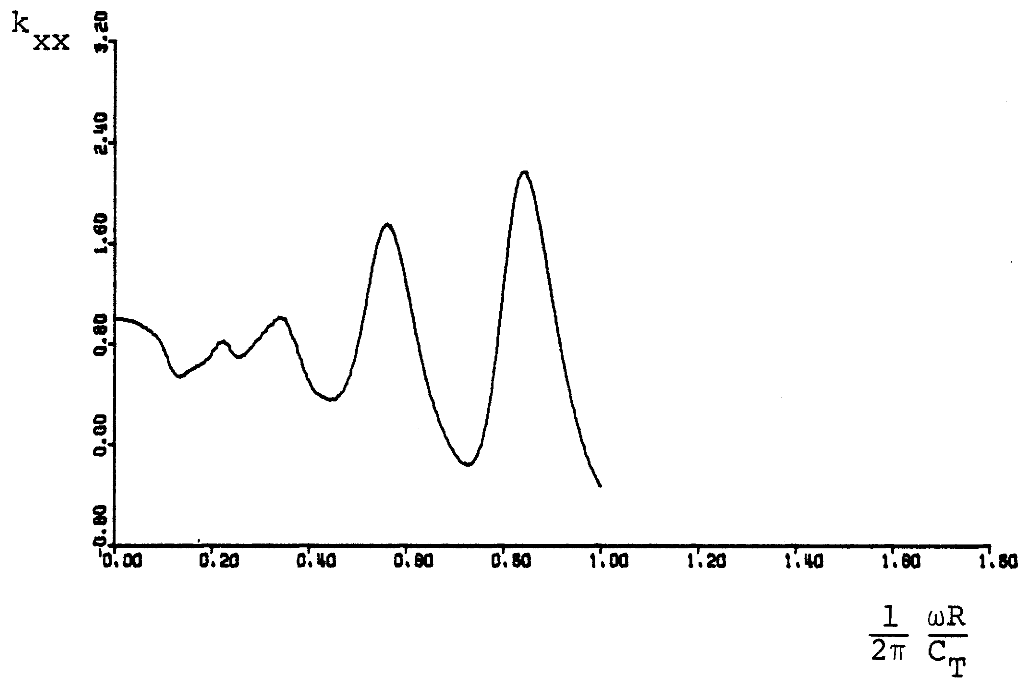
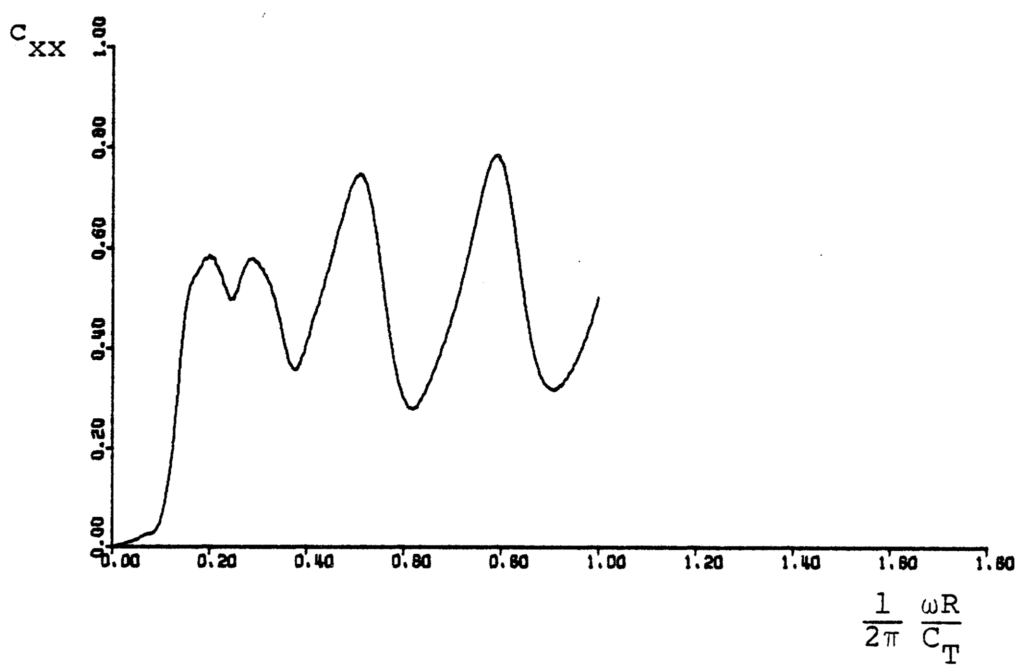


Figure 4.8 - Horizontal stiffness coefficients (circular footing, $\nu = 1/3, \beta = 0.05, \frac{h}{R} = 2$).



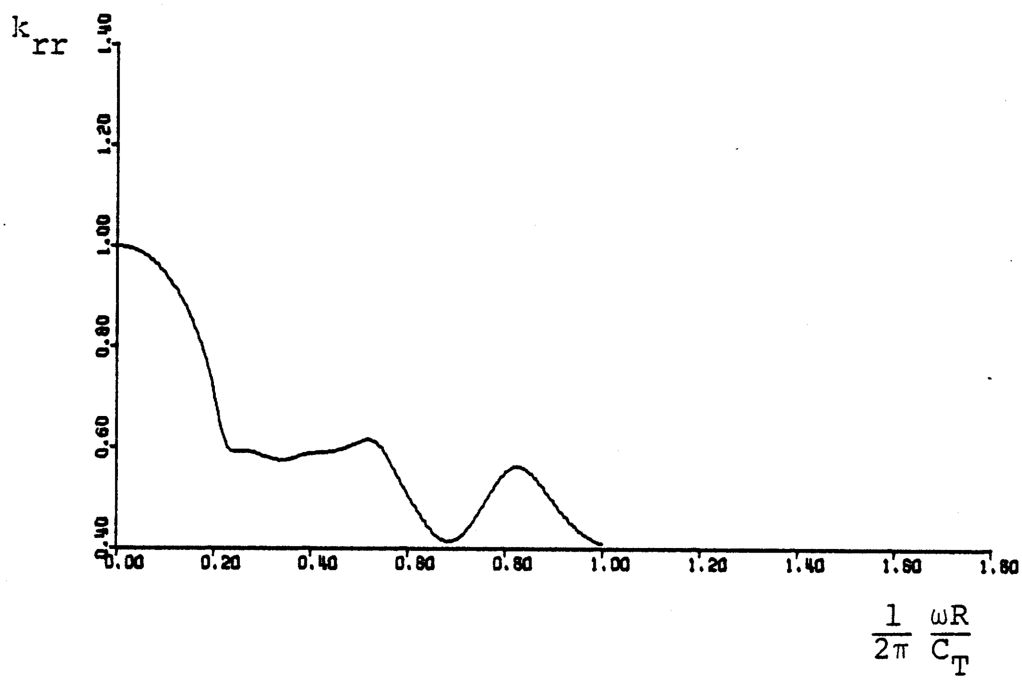
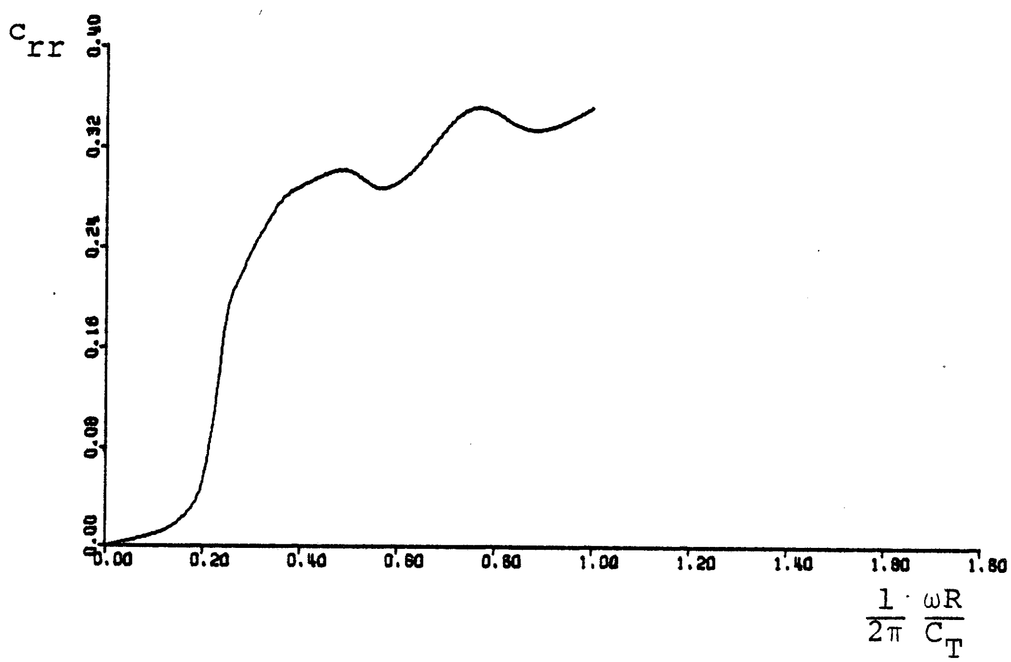


Figure 4.9 - Rocking stiffness coefficients (circular footing, $\nu = 1/3$, $\beta = 0.05$, $\frac{h}{R} = 2$).



CHAPTER 5

SOME APPLICATIONS

In this chapter we illustrate the use of the elements developed in the present work. We consider the effect of the rigidity of the side-wall on the dynamic stiffness of circular footings embedded in a layered stratum. In another application we investigate the behavior of rigid and rough ring footings on a layered stratum.

5.1 THE EFFECT OF THE RIGID SIDE-WALL ON THE DYNAMIC STIFFNESS OF EMBEDDED CIRCULAR FOOTINGS

Let us consider a circular footing of radius R embedded at depth E in a stratum of depth h (see figure 5.1). The footing

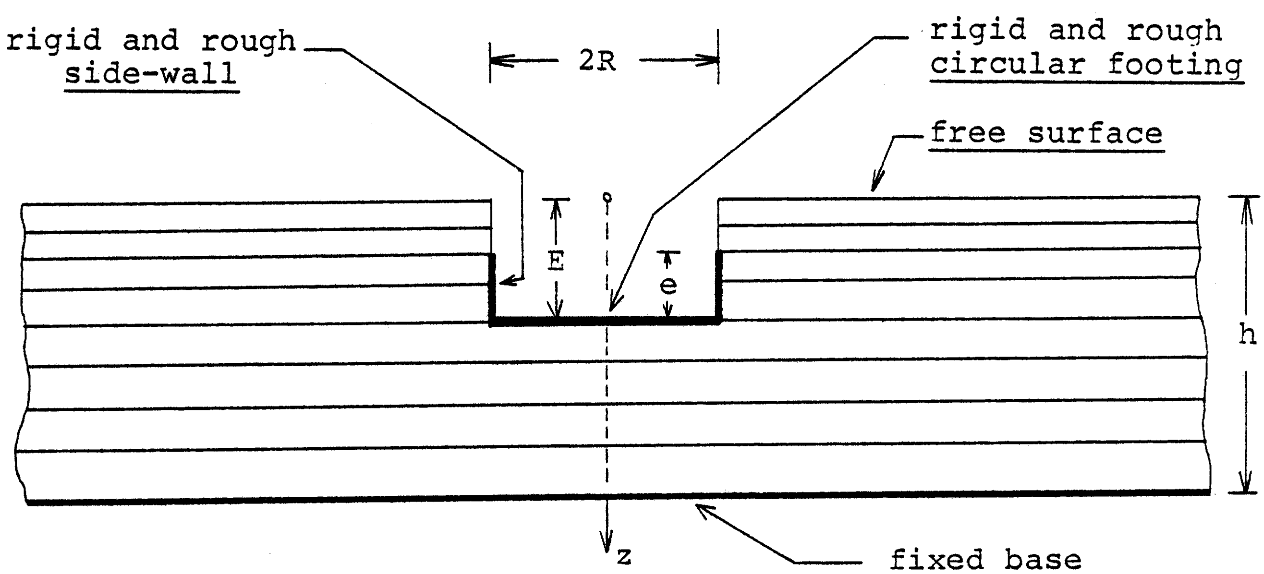


Figure 5.1 - A circular footing, combined with side-wall, embedded in a layered stratum.

is taken rigid and rough. Moreover, let us assume that a rigid and rough (cylindrical) side-wall of height e (radius R) is combined with the footing. The depth e ranges between 0 (no side-wall) and E (side-wall extending throughout the entire depth to which the circular footing is embedded). We consider a homogeneous stratum. The boundary conditions are

i) under the footing:

$$\left.\begin{aligned} u(r,\theta,E) &= \Delta_x \cos\theta \\ w(r,\theta,E) &= \Delta_z - r\phi_r \cos\theta \\ v(r,\theta,E) &= r\phi_t - \Delta_x \sin\theta \end{aligned}\right\} \qquad 0 \le r \le R$$

ii) on the wall:

$$\left.\begin{aligned} u(R,\theta,z) &= \Delta_x \cos\theta - (E-z)\phi_r \cos\theta \\ w(R,\theta,z) &= \Delta_z - R\phi_r \cos\theta \\ v(R,\theta,z) &= R\phi_t - \Delta_x \sin\theta + (E-z)\phi_r \sin\theta \end{aligned}\right\} \qquad E-e < z \le E$$

$$\left.\begin{aligned} \sigma_r\Big|_{r=R} &= 0 \\ \tau_{rz}\Big|_{r=R} &= 0 \\ \tau_{r\theta}\Big|_{r=R} &= 0 \end{aligned}\right\} \qquad 0 \le z \le E-e$$

iii) on the surface

$$\left.\begin{aligned} \sigma_z\Big|_{z=0} &= 0 \\ \tau_{rz}\Big|_{z=0} &= 0 \\ \tau_{\theta z}\Big|_{z=0} &= 0 \end{aligned}\right\} \qquad r > R \quad .$$

The base of the stratum is fixed. As in Chapter 4, Δ_x, Δ_z, ϕ_t, ϕ_r are the amplitudes of horizontal vibrations, vertical vibrations, torsional vibrations and rocking respectively (rocking is understood to be about a horizontal axis through the center of the footing, i.e. at depth z=E). The nondimensional torsional stiffness

$$\frac{K_{tt}}{GR^3}$$

is a function of the nondimensional quantities $\frac{\omega R}{C_T}$ (nondimensional frequency), $\frac{h}{R}$, $\frac{E}{R}$, $\frac{e}{E}$, β. The nondimensional stiffnesses

$$\frac{K_{zz}}{GR}, \qquad \frac{K_{xx}}{GR}, \qquad \frac{K_{rr}}{GR^3}, \qquad \frac{K_{xr}}{GR^2}$$

further depend on Poisson's ratio ν (we use the notation of Chapter 4). The static stiffnesses and the dynamic stiffness coefficients are defined as in Chapter 4. In this application the dynamic stiffnesses were calculated for two different depths: $\frac{h}{R} = 2$, $\frac{h}{R} = 3$. The embedment of the footing was taken the same for both depths: $\frac{E}{R} = 1$. In both cases $\nu = 1/3$, $\beta = 0.05$, and seven different side-wall heights were considered:

$$\frac{e}{E} = 0,\ 1/6,\ 1/3,\ 1/2,\ 2/3,\ 5/6,\ 1 \quad .$$

For the calculations, the elements (modeling the region under the footing) developed in Chapter 4 were combined (see figure 5.2) with the transmitting boundaries developed by Waas [23] and Kausel [6] and described in Section 2.4. The stratum was divided into twelve and eighteen sublayers of equal

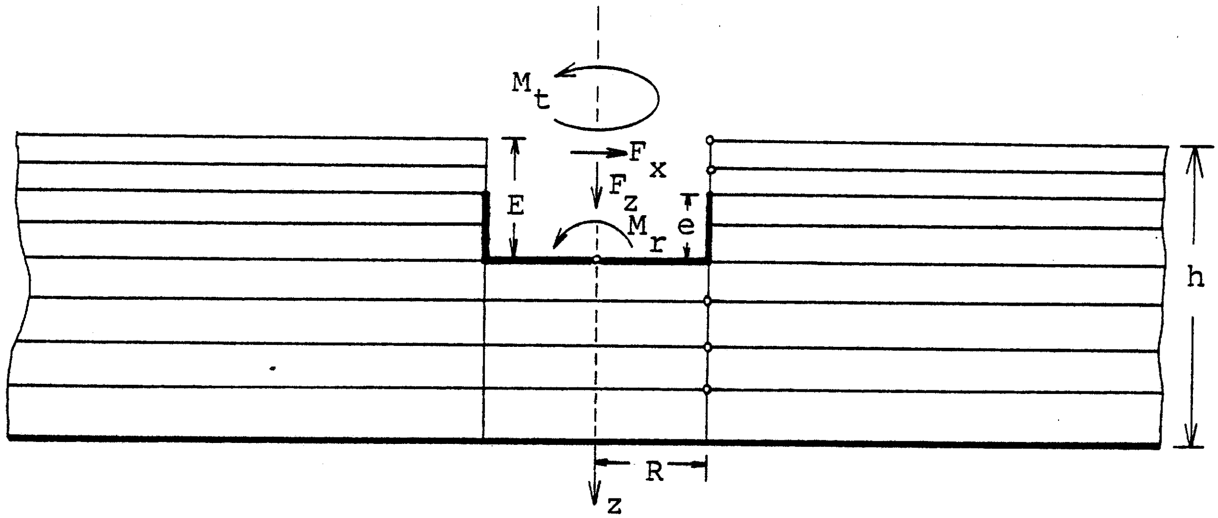


Figure 5.2 - Scheme for the calculation of the stiffness of a circular footing, combined with side-wall, embedded in a layered stratum.

depth for $\frac{h}{R} = 2$ and $\frac{h}{R} = 3$ respectively. Let us first consider the static stiffnesses. The results are given in Tables 5.1, 5.2, 5.3, 5.4 and plotted in figures 5.3, 5.4, 5.5, 5.6. As one would expect, for increasing height of the rigid side-wall (increasing $\frac{e}{E}$) the static stiffnesses increase. The increase is most significant in the case of rocking. The rocking stiffness for $\frac{e}{E} = 1$ (side-wall extending throughout the entire depth E) is approximately 2.7 times the stiffness of the circular footing with no side-wall. The increase is quite significant for the torsional and horizontal stiffnesses as well. However, the increase of vertical stiffness for e increasing from 0 to E is relatively small. This may be understood by the fact that the additional vertical stiffness provided by the side-wall is only through shear stresses ($\tau_{zr}\big|_{r=R}$) and

Table 5.1 - The nondimensional torsional static stiffness ($E/R = 1$).

$\frac{e}{E}$	$\frac{K^0_{tt}}{GR^3}$	
	$h/R = 2$	$h/R = 3$
0	9.14	8.82
1/6	12.6	12.1
1/3	15.3	14.7
1/2	17.5	16.8
2/3	19.4	18.7
5/6	20.8	19.9
1	21.2	20.4

Table 5.2 - The nondimensional vertical static stiffness ($E/R = 1$, $\nu = 1/3$).

$\frac{e}{E}$	$\frac{K^0_{zz}}{GR}$	
	$h/R = 2$	$h/R = 3$
0	18.8	12.5
1/6	20.6	13.6
1/3	21.7	14.3
1/2	22.4	14.8
2/3	23.0	15.3
5/6	23.7	15.7
1	24.4	16.2

Table 5.3 - The nondimensional horizontal static stiffness ($E/R = 1$, $\nu = 1/3$).

$\frac{e}{E}$	$\frac{K^o_{xx}}{GR}$	
	h/R = 2	h/R = 3
0	10.5	8.84
1/6	12.8	10.6
1/3	14.3	11.8
1/2	15.4	12.6
2/3	16.2	13.3
5/6	16.8	13.7
1	17.0	13.8

Table 5.4 - The nondimensional rocking static stiffness ($E/R = 1$, $\nu = 1/3$).

$\frac{e}{E}$	$\frac{K^o_{rr}}{GR^3}$	
	h/R = 2	h/R = 3
0	7.0	5.94
1/6	8.6	7.23
1/3	10.2	8.66
1/2	12.1	10.4
2/3	14.4	12.5
5/6	16.9	14.8
1	18.8	16.5

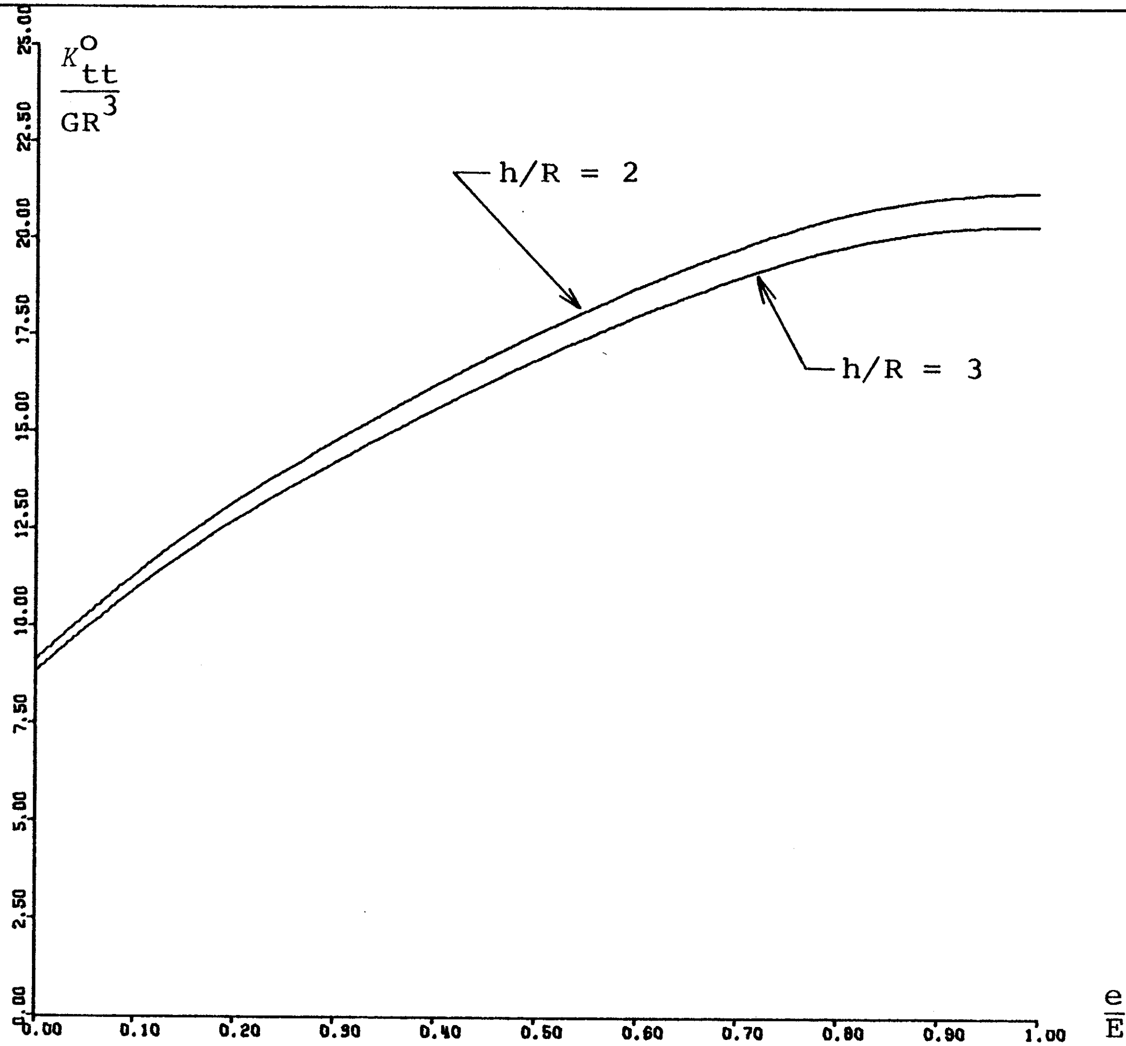


Figure 5.3 - The nondimensional torsional static stiffness (E/R = 1).

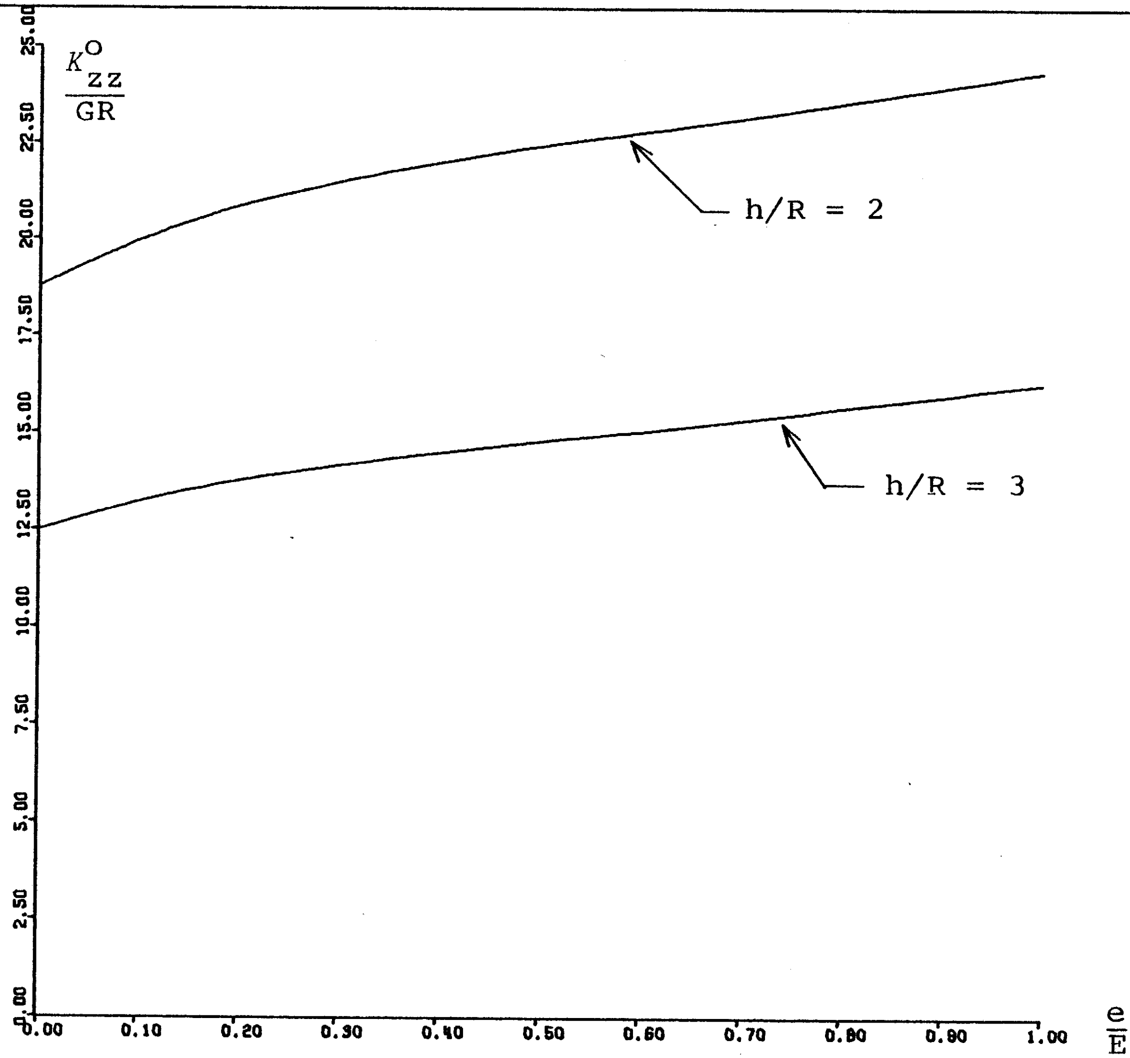


Figure 5.4 - The nondimensional vertical static stiffness ($E/R = 1$, $\nu = 1/3$).

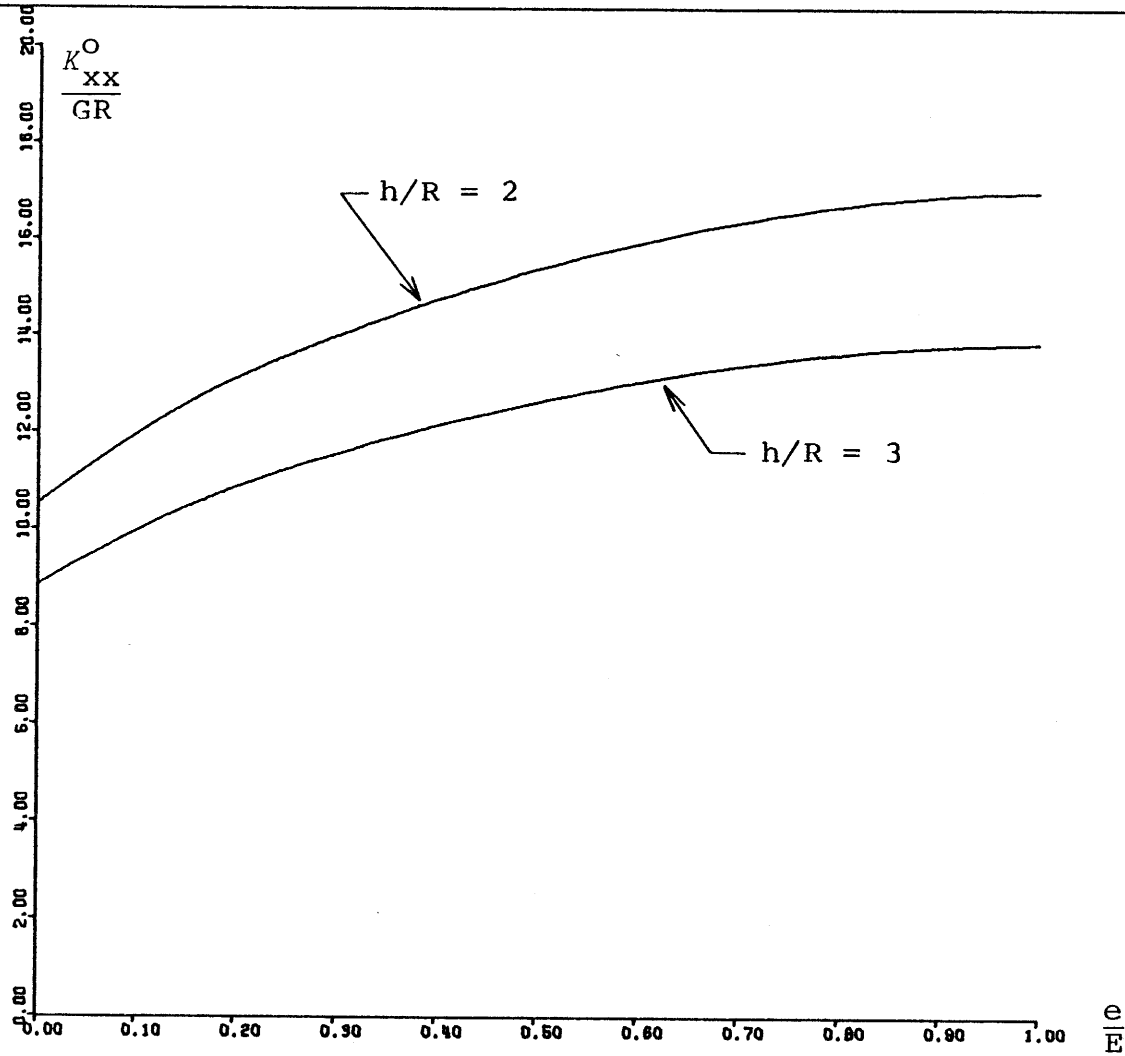


Figure 5.5 - The nondimensional horizontal static stiffness ($E/R = 1$, $\nu = 1/3$).

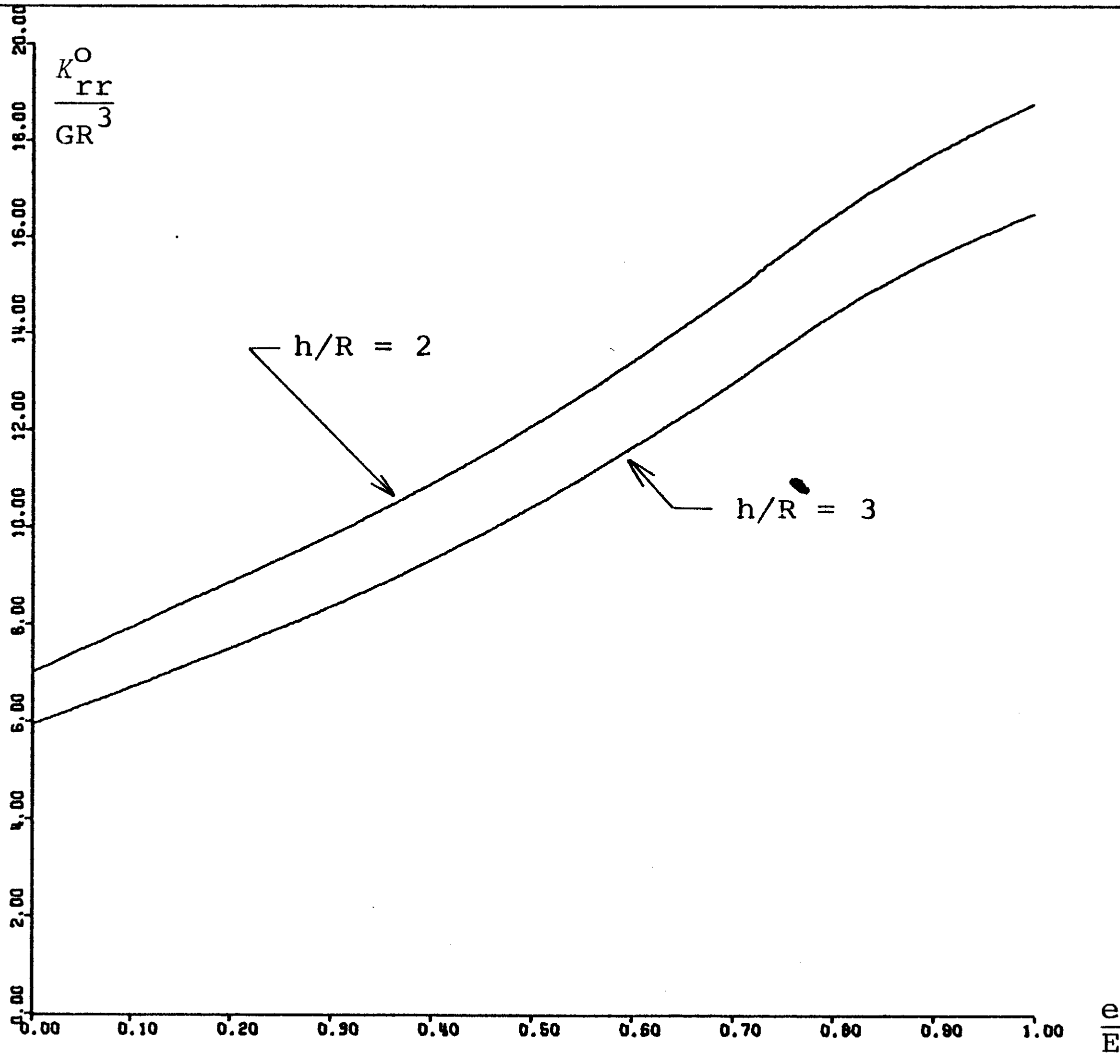


Figure 5.6 - The nondimensional rocking static stiffness ($E/R = 1$, $\nu = 1/3$).

therefore it is not "direct." The rocking stiffness increases at an increasing rate at least up to $\frac{e}{E} = 0.75$. This is a clear indication that the rigid side-wall is most essentially contributing to the rocking stiffness. Figures 5.7, 5.8, 5.9, 5.10 show the static stiffnesses normalized with respect to the stiffness of the footing with no side-wall versus the non-dimensional height of the side-wall. It is clearly seen in these figures that the normalized static stiffnesses are rather independent of the depth to radius $\frac{h}{R}$ ratio. Finally, let us consider the height of the center of stiffness, δ, of the footing combined with the side-wall (horizontal vibrations and rocking about a horizontal axis through the center of stiffness are uncoupled). We have

$$\delta = - \frac{\overset{o}{K}_{xr}}{\overset{o}{K}_{xx}} .$$

Of interest here is the relative increase of the height of the center of stiffness for increasing height of the side-wall. Figure 5.11 shows the nondimensional increase of the height of the center of stiffness

$$\frac{\delta - \delta(e = 0)}{e}$$

versus the nondimensional height of the side-wall. Clearly, it varies between rather narrow limits: approximately between 0.5 and 0.4 both for $\frac{h}{R} = 2$ and for $\frac{h}{R} = 3$.

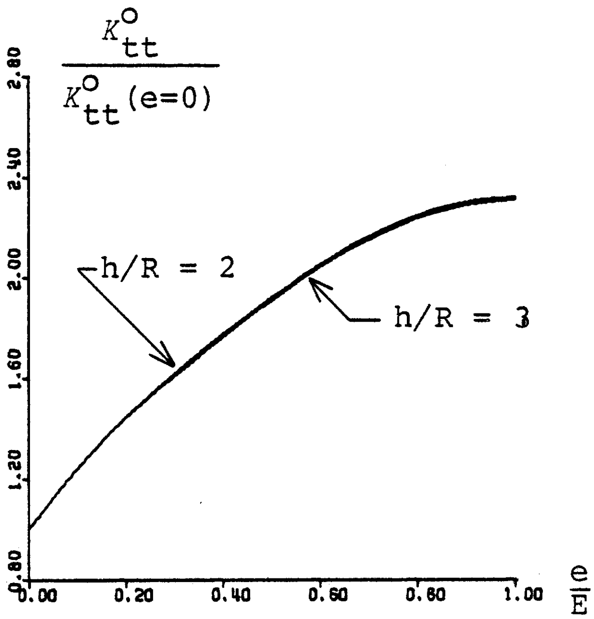


Figure 5.7 - Normalized torsional static stiffness (E/R = 1).

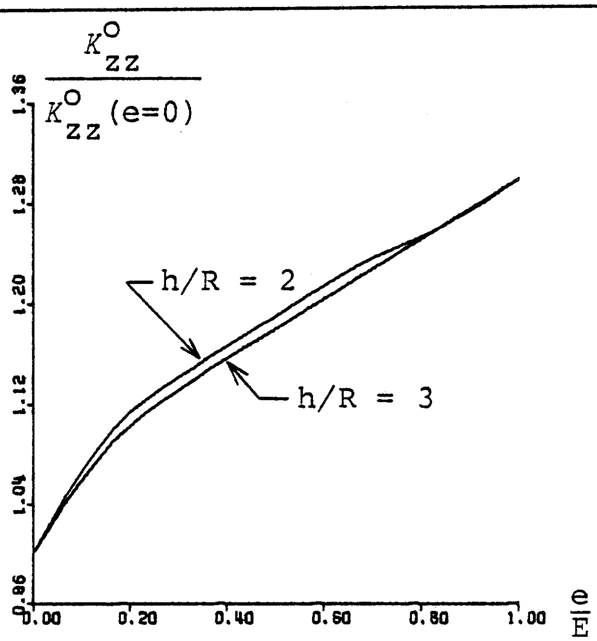


Figure 5.8 - Normalized vertical static stiffness (E/R = 1 , $\nu = 1/3$).

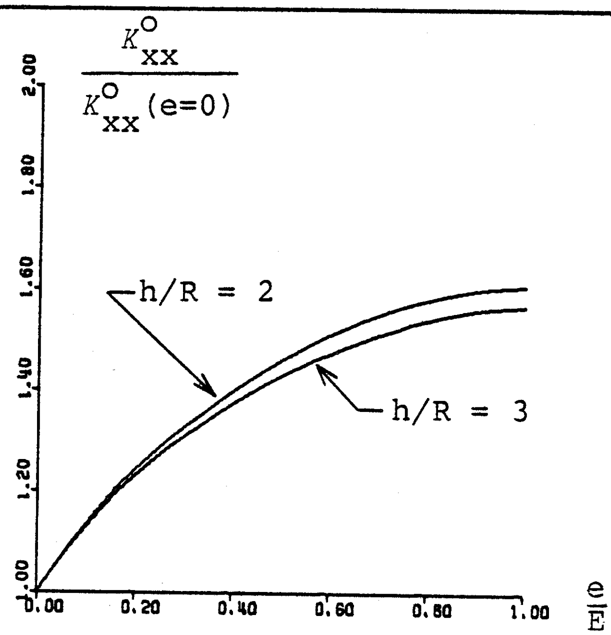


Figure 5.9 - Normalized horizontal static stiffness (E/R = 1 , ν = 1/3).

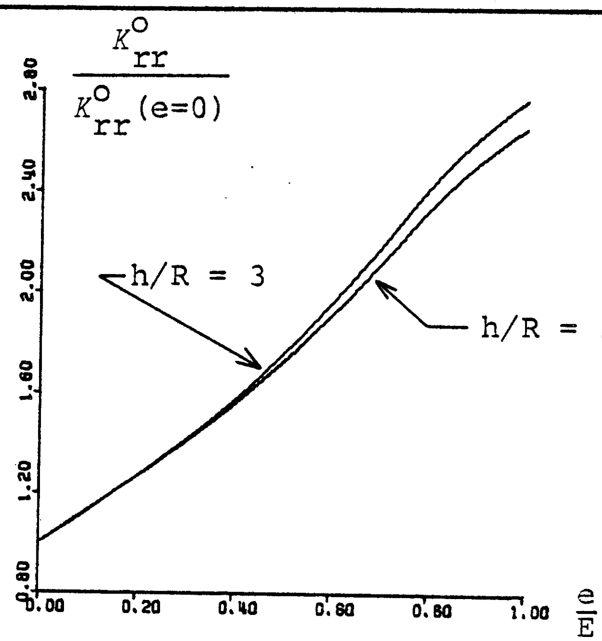


Figure 5.10 - Normalized rocking static stiffness (E/R = 1 , ν = 1/3).

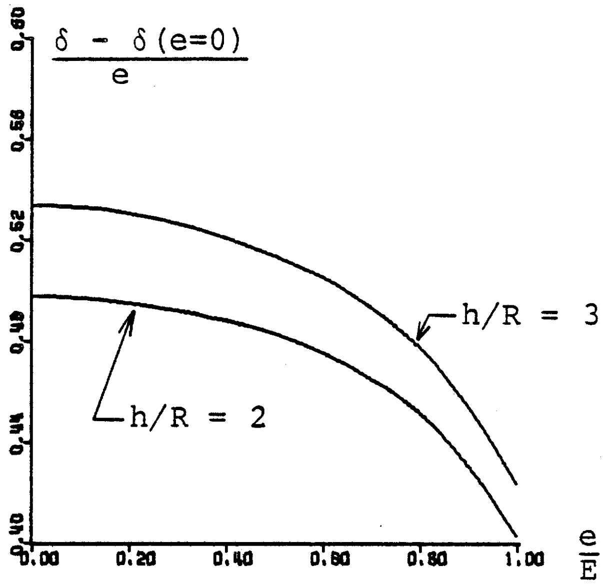


Figure 5.11 - Nondimensional increase of the height of the center of stiffness ($E/R = 1$, $\nu = 1/3$).

The stiffness coefficients are plotted versus the nondimensional frequency $\frac{1}{2\pi}\frac{\omega R}{C_T}$ in figures 5.12, 5.13, 5.14, 5.15, 5.16, 5.17, 5.18, 5.19. The curves are identified by the index $i = 1, 2, \ldots, 7$ which indicates the nondimensional height of the side-wall (for the curve with index i, $\frac{e}{E} = \frac{i-1}{6}$). From these figures it appears that the height of the side-wall has little effect on the stiffness coefficients k_{tt}, k_{zz}, k_{xx}, k_{rr} (corresponding to the real part of the dynamic stiffness). It is noted, however, that the peaks are less sharp for increasing height of the side-wall. This indicates that resonance or near-resonance becomes less sharp for a footing combined with the rigid side-wall. The effect of the ratio $\frac{e}{E}$ on the stiffness coefficients c_{tt}, c_{zz}, c_{xx}, c_{rr} (corresponding to the imaginary part of the dynamic stiffness) is much more interesting. These coefficients provide a measure of the

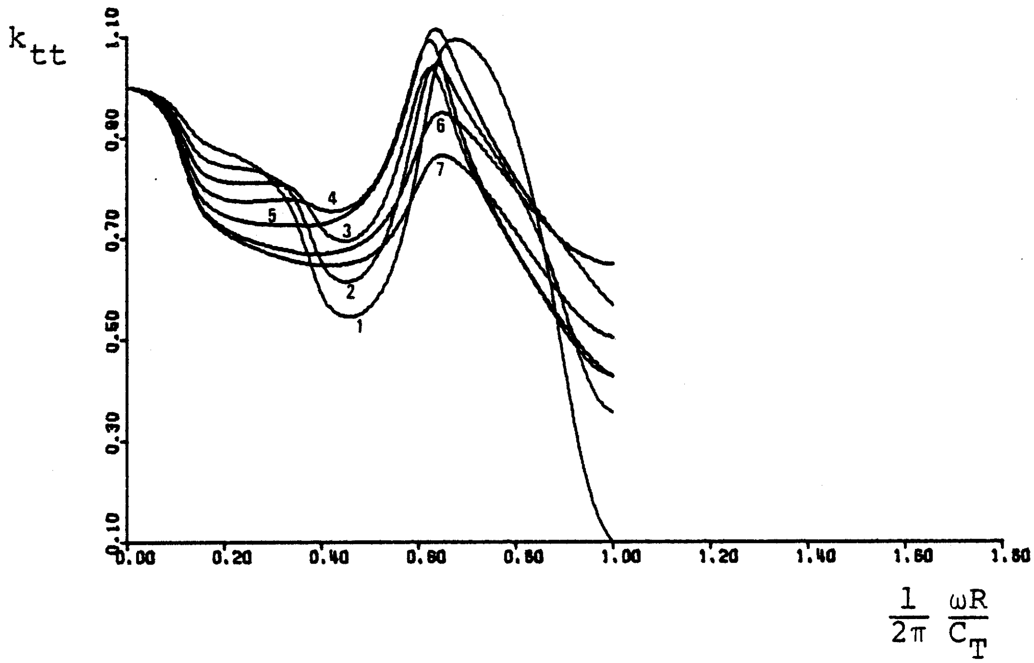
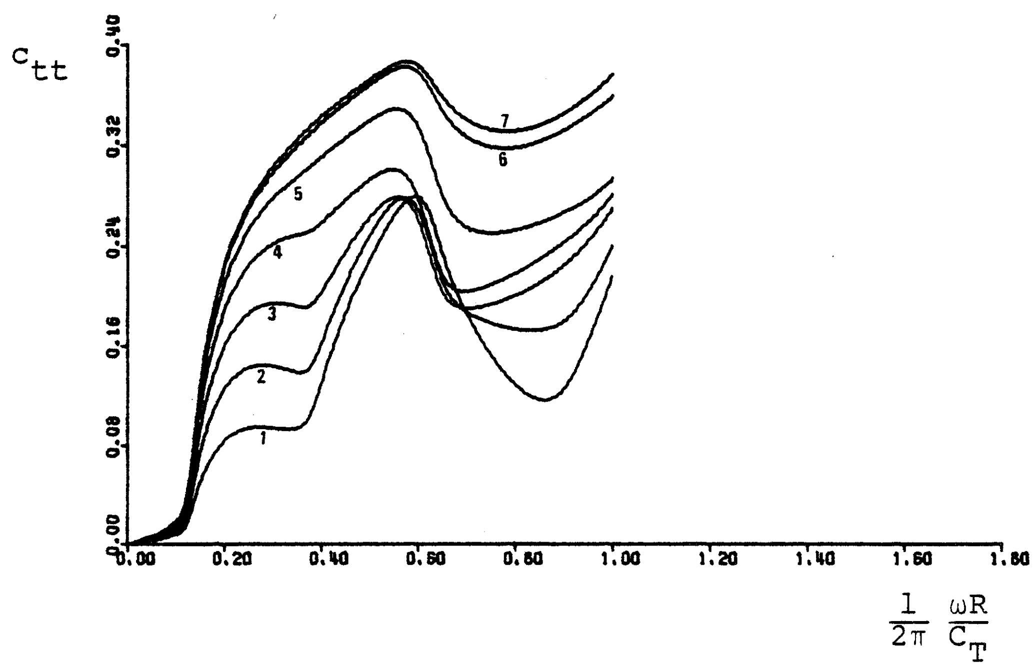
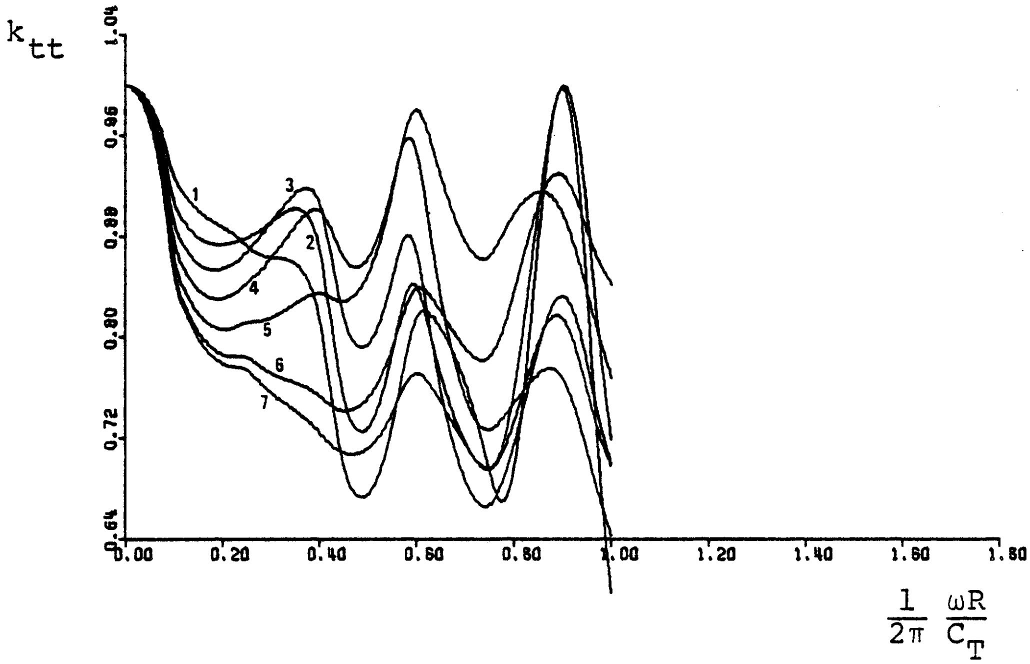
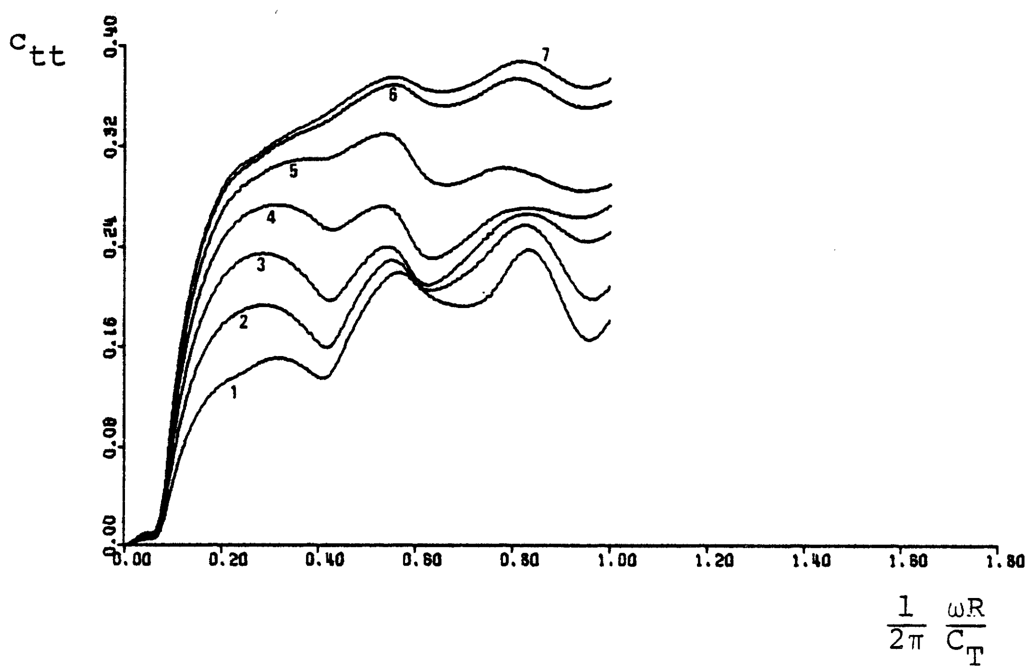


Figure 5.12 - Torsional stiffness coefficients (E/R = 1 , h/R = 2 , β = 0.05).





<u>Figure 5.13</u> - Torsional stiffness coefficients (E/R = 1 , h/R = 3 , β = 0.05).



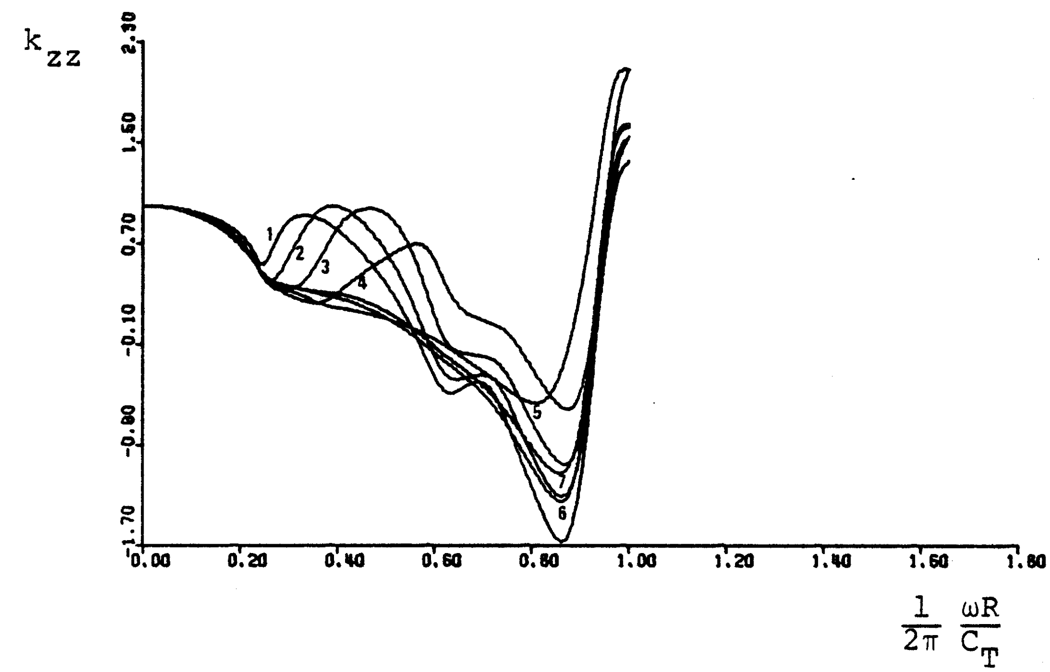
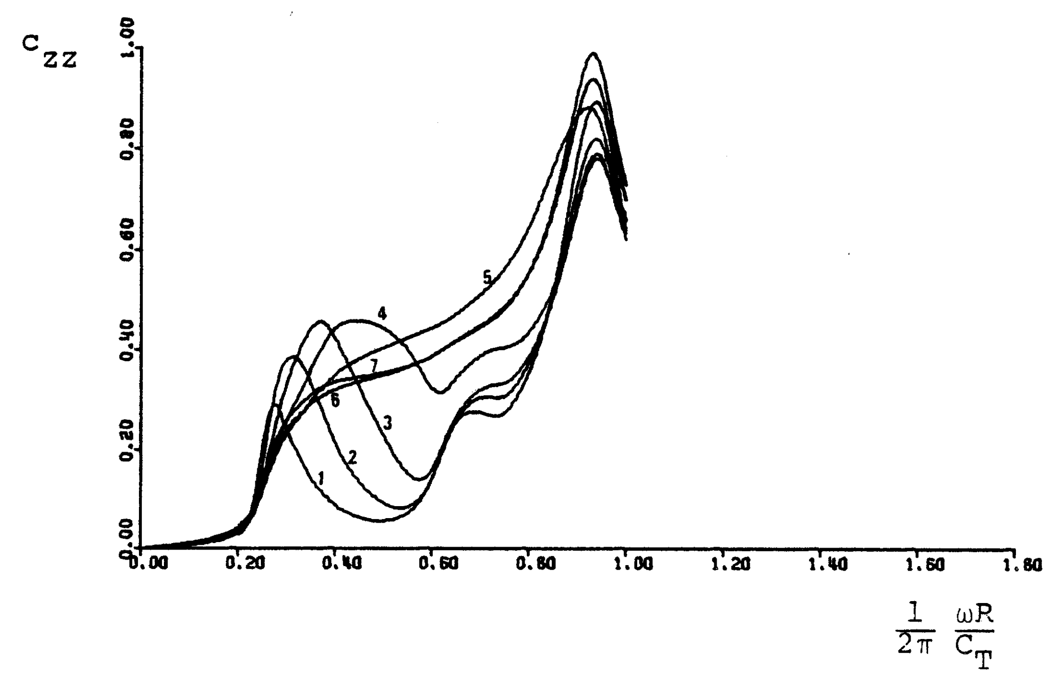


Figure 5.14 - Vertical stiffness coefficients
(E/R = 1 , h/R = 2 , ν = 1/3 , β = 0.05).



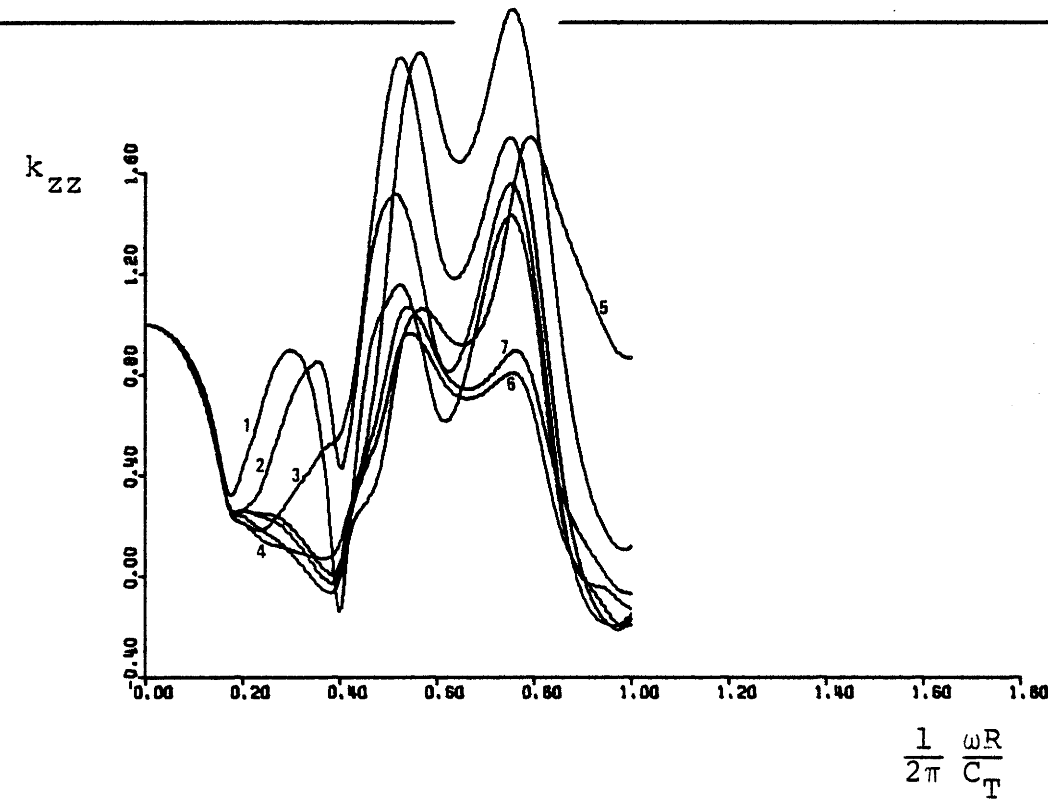
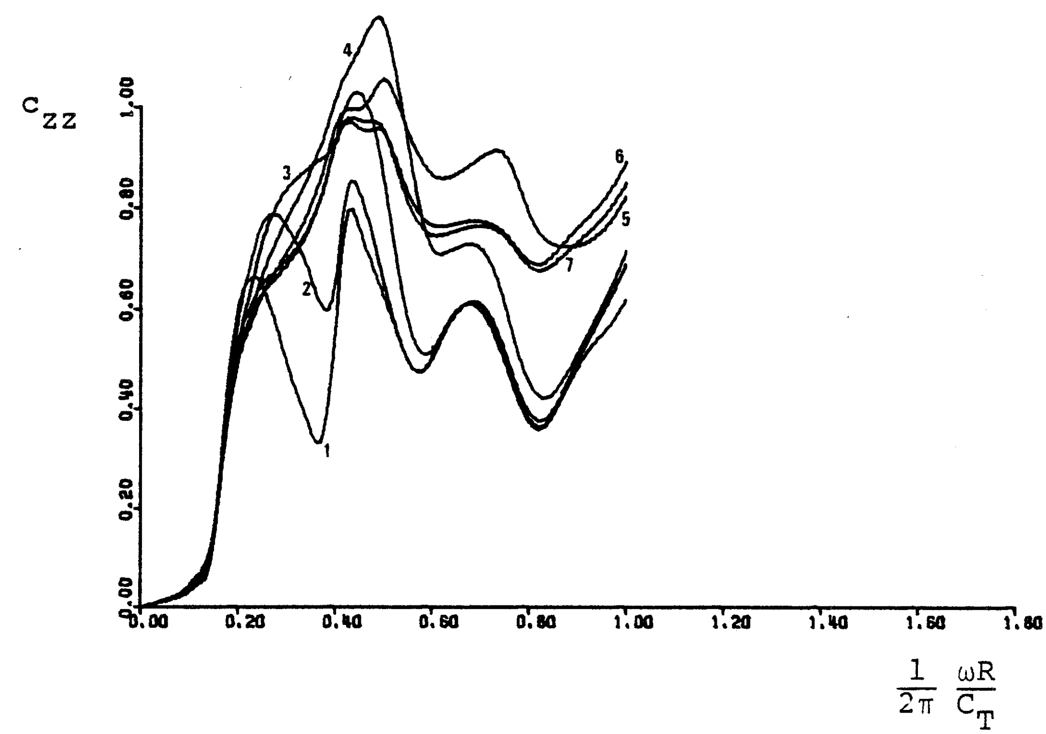
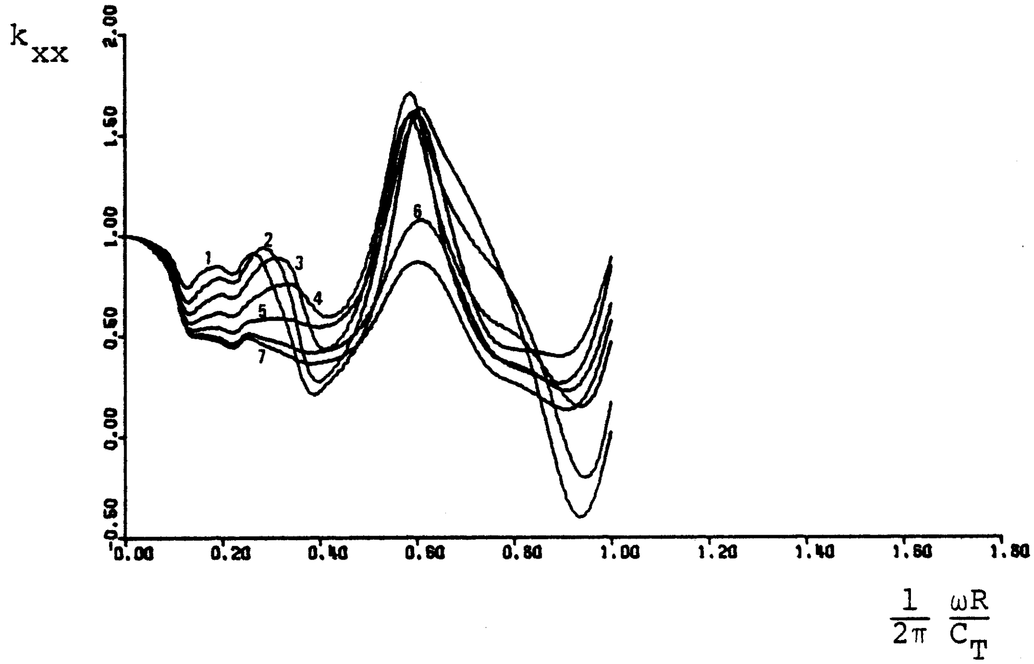
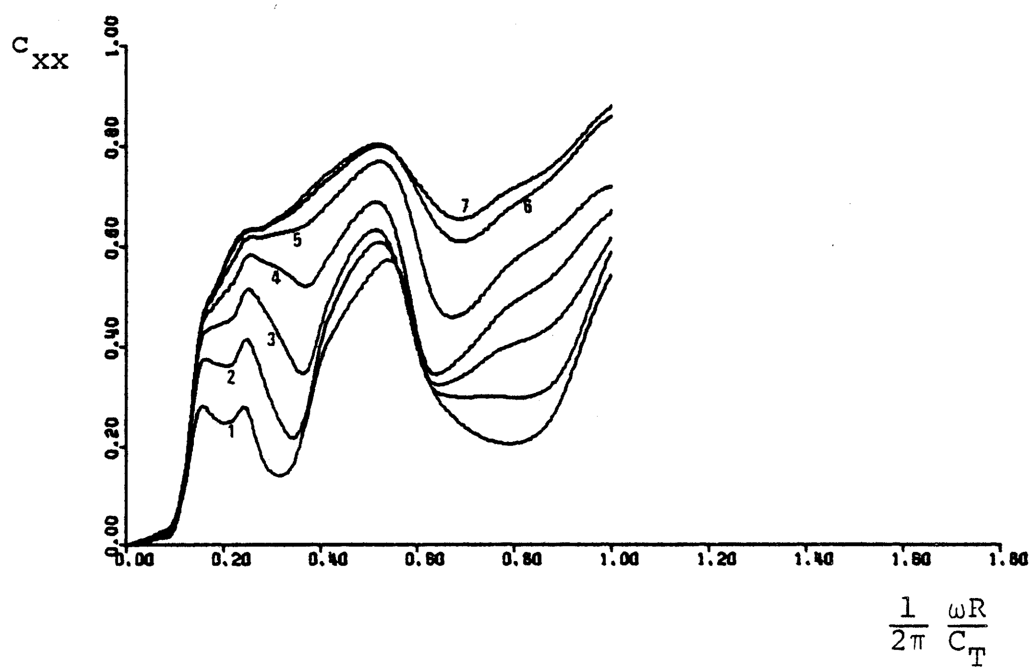


Figure 5.15 - Vertical stiffness coefficients
(E/R = 1 , h/R = 3 , ν = 1/3 , β = 0.05).





<u>Figure 5.16</u> - Horizontal stiffness coefficients
(E/R = 1 , h/R = 2 , ν = 1/3 , β = 0.05).



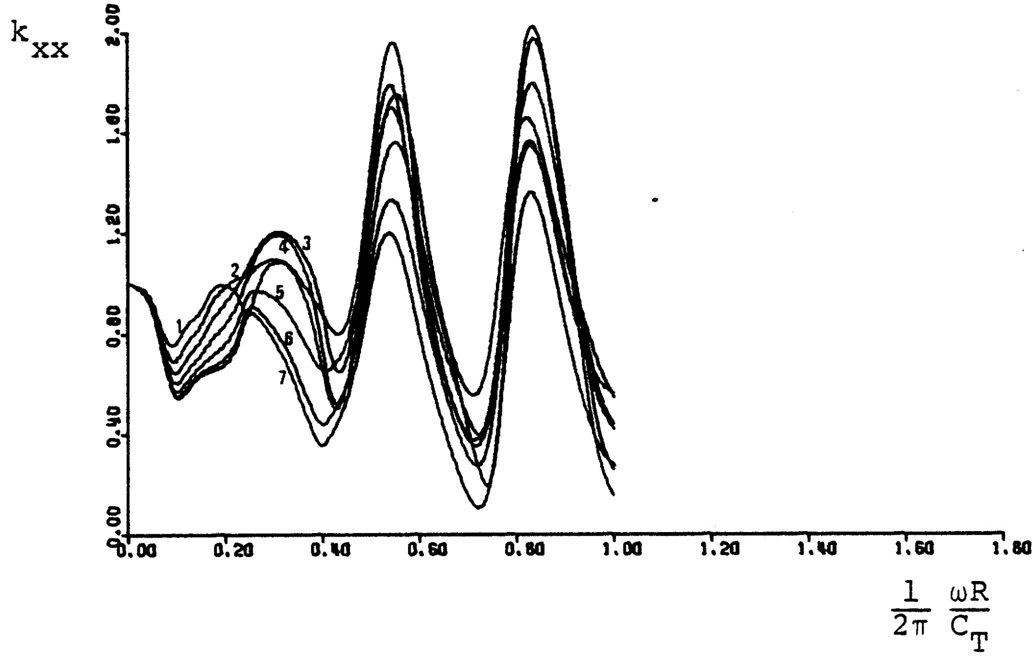
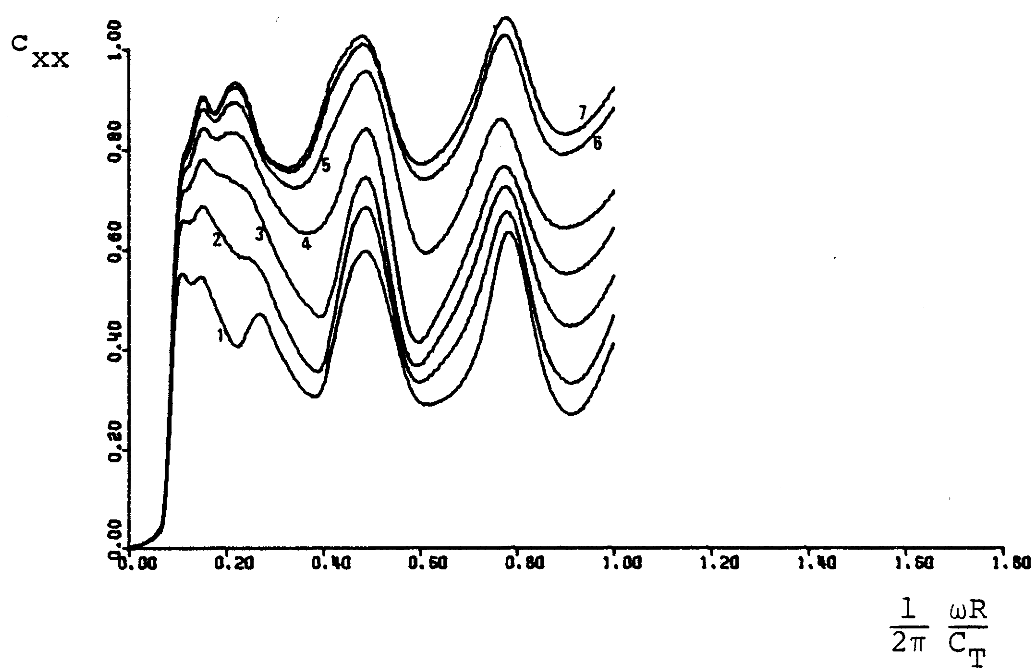
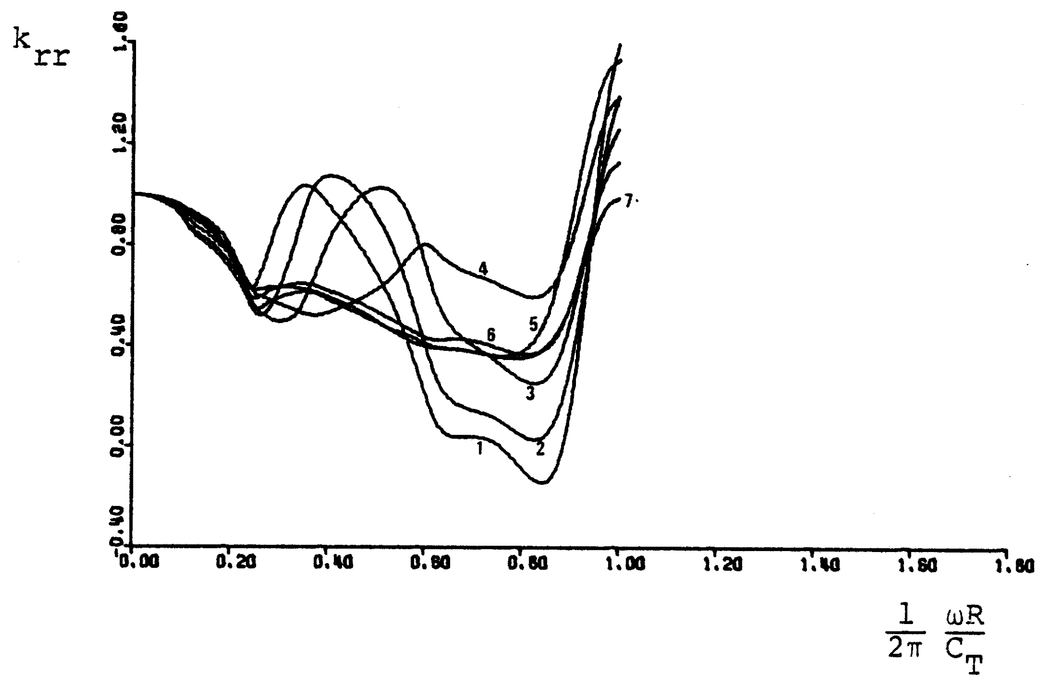
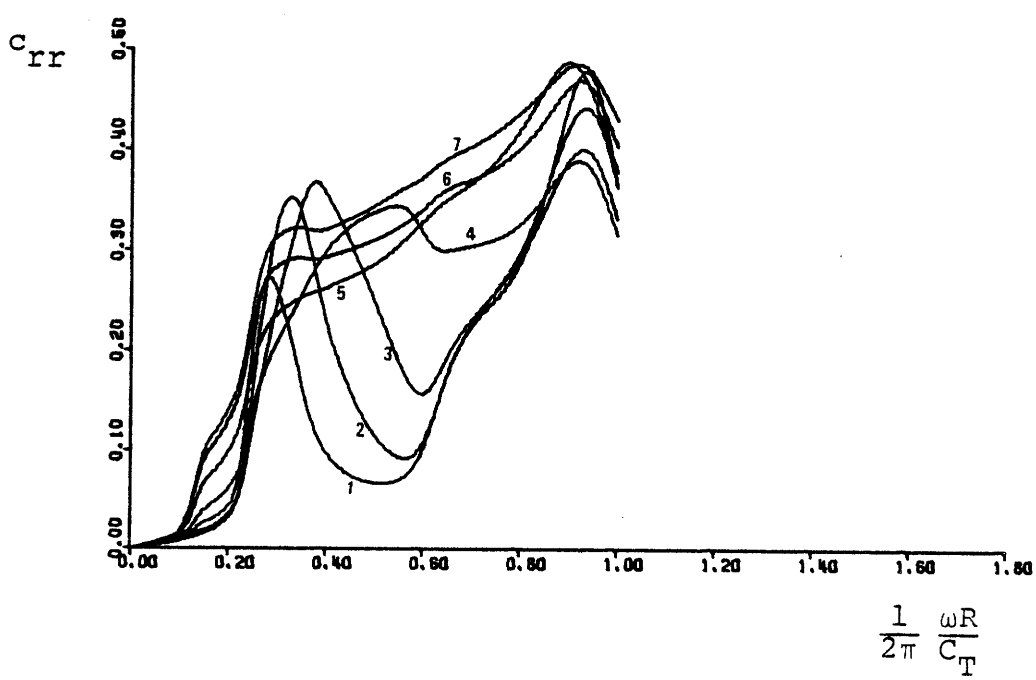


Figure 5.17 - Horizontal stiffness coefficients (E/R = 1 , h/R = 3 , ν = 1/3 , β = 0.05).





<u>Figure 5.18</u> - Rocking stiffness coefficients (E/R = 1 , h/R = 2 , ν = 1/3 , β = 0.05).



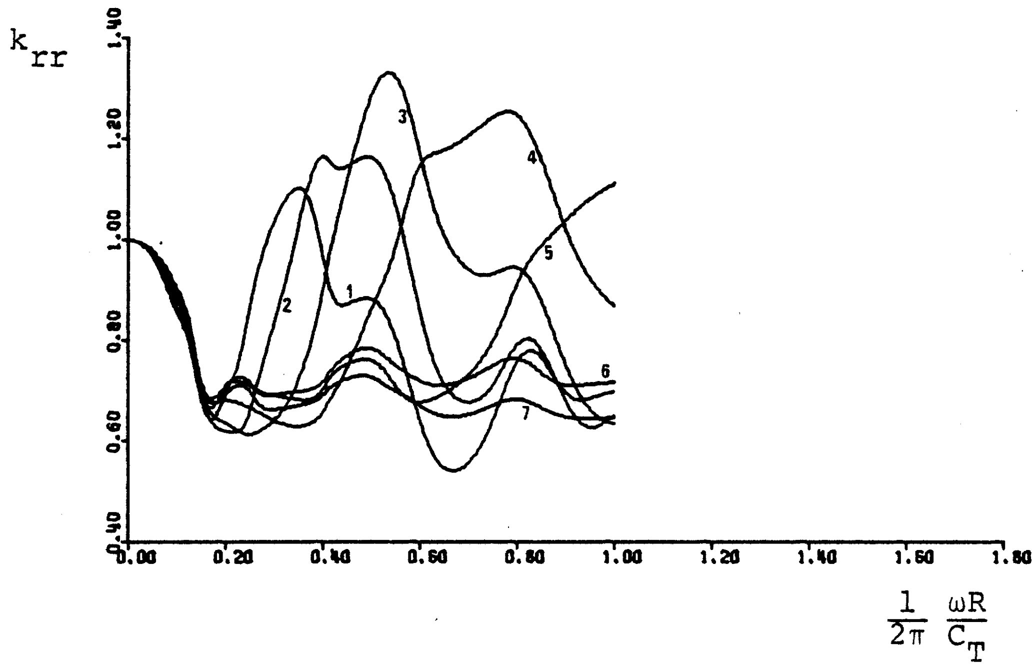
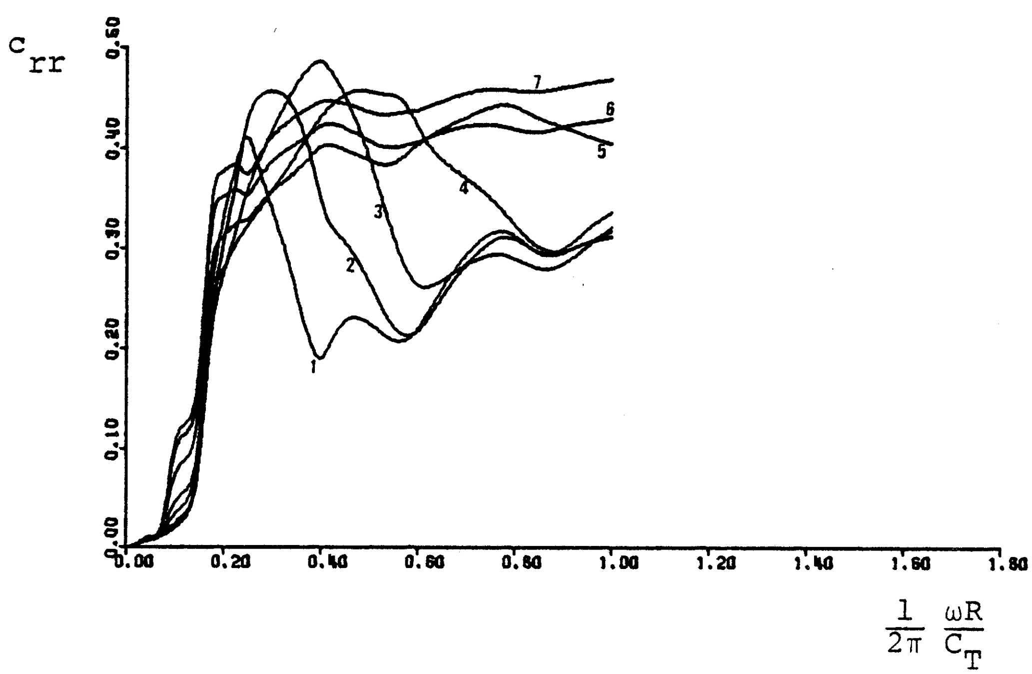


Figure 5.19 - Rocking stiffness coefficients (E/R = 1 , h/R = 3 , ν = 1/3 , β = 0.05).



damping present in the system. Part of the damping is due to energy dissipation ($\beta = 0.05$) in the stratum. The rest is due to energy radiation into the far-field. As would be expected, the height of the side-wall has little effect on the stiffness coefficients c_{zz}, c_{rr}. This is because, for vertical vibrations and rocking, damping is primarily due to energy dissipation in the region of the stratum surrounding the footing, since energy gets "trapped" in that region (because of multiple reflections between the footing and the fixed base). However, the effect of the height of the side-wall on the stiffness coefficients c_{tt} and c_{xx} is significant. It is clearly seen in figures 5.12, 5.13, 5.16, 5.17 that these stiffness coefficients increase with increasing $\frac{e}{E}$ ratio over the entire frequency range considered ($0 \leq \frac{1}{2\pi} \frac{\omega R}{C_T} \leq 1$). This is because energy radiation into the far-field is very important for damping of torsional and horizontal vibrations and it is greatly enhanced by the attachment of the side-wall on the region of the stratum above the footing. We note that, as previously discussed, the static stiffnesses increase with increasing height of the side-wall and, therefore, the imaginary part of the dynamic stiffnesses is further increased.

5.2 THE STIFFNESS OF RIGID AND ROUGH RING FOOTINGS ON THE SURFACE OF A STRATUM

Let us now consider a ring footing of inside radius R_1, outside radius R_2 on the surface of a stratum of depth h (see figure 5.20). We assume that the footing is rigid and rough. The stratum is taken homogeneous. We write the boundary conditions

$$\left.\begin{aligned} u(r,\theta,0) &= \Delta_x \cos\theta \\ w(r,\theta,0) &= \Delta_z - r\phi_r \cos\theta \\ v(r,\theta,0) &= r\phi_t - \Delta_x \sin\theta \end{aligned}\right\} \qquad R_1 \leq r \leq R_2$$

$$\left.\begin{aligned} \sigma_z\Big|_{z=0} &= 0 \\ \tau_{zr}\Big|_{z=0} &= 0 \\ \tau_{\theta z}\Big|_{z=0} &= 0 \end{aligned}\right\} \qquad 0 \leq r < R_1,\ R_2 < r \ .$$

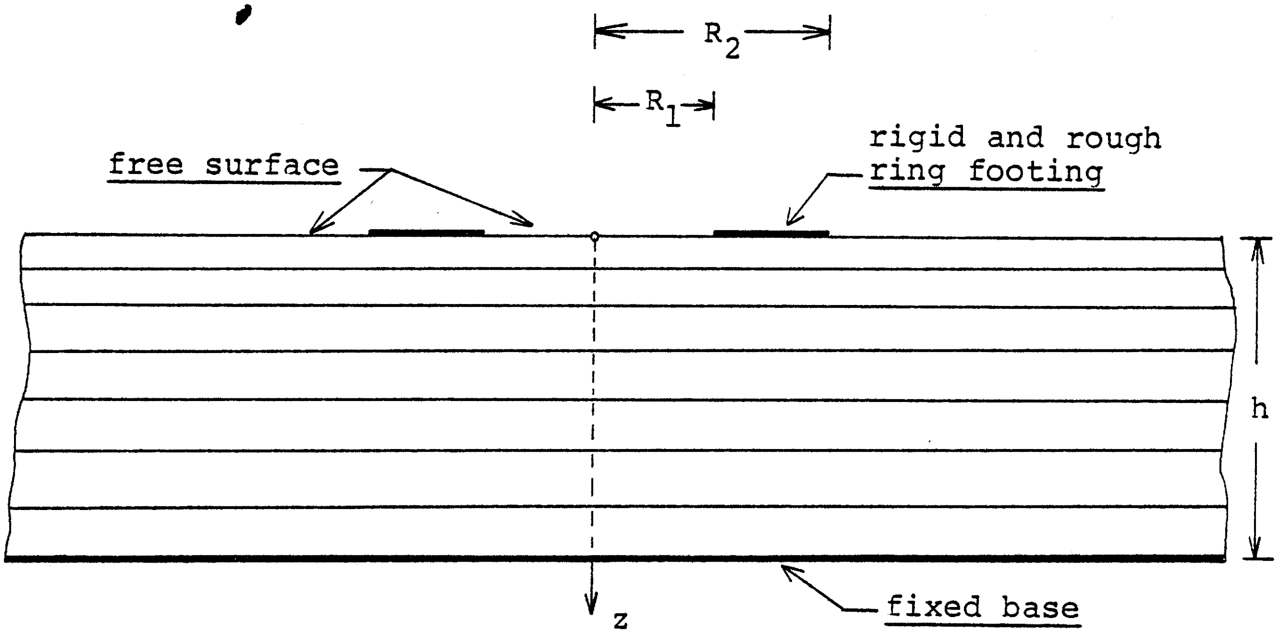


Figure 5.20 - A ring footing on a layered stratum.

The base of the stratum is fixed. Δ_x, Δ_z, ϕ_t, ϕ_r denote the amplitudes of horizontal vibrations, vertical vibrations, torsional vibrations and rocking respectively. The nondimensional torsional stiffness

$$\frac{K_{tt}}{GR_2^3}$$

is a function of the nondimensional quantities $\frac{\omega R_2}{C_T}$ (nondimensional frequency), $\frac{h}{R_2}$, $\frac{R_1}{R_2}$, β. The nondimensional stiffnesses

$$\frac{K_{zz}}{GR_2}, \quad \frac{K_{xx}}{GR_2}, \quad \frac{K_{rr}}{GR_2^3}, \quad \frac{K_{xr}}{GR_2^2}$$

depend, in addition, on Poisson's ratio ν. The static stiffnesses and the stiffness coefficients are now defined as

$$\frac{K_{tt}}{GR_2^3} = \frac{\overset{o}{K}_{tt}}{GR_2^3}\left[k_{tt} + i\,\frac{\omega R_2}{C_T}\,c_{tt}\right](1 + 2i\beta)$$

$$\frac{K_{zz}}{GR_2} = \frac{\overset{o}{K}_{zz}}{GR_2}\left[k_{zz} + i\,\frac{\omega R_2}{C_T}\,c_{zz}\right](1 + 2i\beta)$$

$$\frac{K_{xx}}{GR_2} = \frac{\overset{o}{K}_{xx}}{GR_2}\left[k_{xx} + i\,\frac{\omega R_2}{C_T}\,c_{xx}\right](1 + 2i\beta)$$

$$\frac{K_{rr}}{GR_2^3} = \frac{\overset{o}{K}_{rr}}{GR_2^3}\left[k_{rr} + i\,\frac{\omega R_2}{C_T}\,c_{rr}\right](1 + 2i\beta) \quad .$$

The dynamic stiffnesses were calculated for two different depths ($\frac{h}{R_2} = 2$, $\frac{h}{R_2} = 3$) and several values of the ratio $\frac{R_1}{R_2}$ of the radii of the footing. In all cases $\nu = 1/3$ and $\beta = 0.05$ were used. The elements developed in Chapter 4 (modeling the region under the footing) were combined (see figure 5.21) with the elements developed by Kausel and Roësset [9] (modeling the region $0 \leq r \leq R_1$) and the transmitting boundaries developed by Waas [23] and Kausel [6] (these elements are described in Section 2.4). The stratum was divided into twelve sublayers of equal depth for $\frac{h}{R_2} = 2$, while for $\frac{h}{R_2} = 3$ the first six sublayers were of depth $R_2/6$, the next four of depth $R_2/4$ and the last two of depth $R_2/2$ (the accuracy achieved by this spacing in the range $0 \leq \frac{1}{2\pi} \frac{\omega R_2}{C_T} \leq \frac{1}{2}$ is comparable to that obtained using eighteen sublayers of equal depth). Let us consider the static stiffnesses. They are given in Tables 5.5, 5.6, 5.7, 5.8 and plotted in figures 5.22, 5.23, 5.24, 5.25. As one would expect, for increasing $\frac{R_1}{R_2}$ the static stiffnesses decrease.

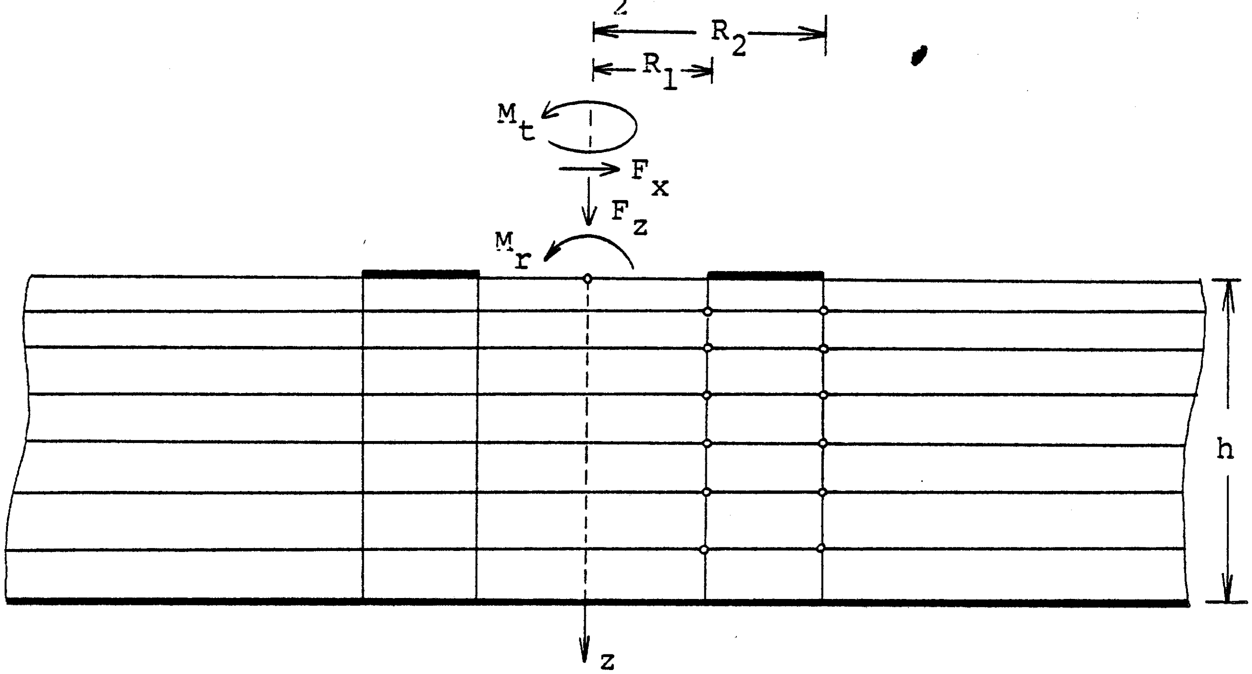


Figure 5.21 - Scheme for the calculation of the stiffness of a ring footing on a layered stratum.

Table 5.5 - The nondimensional torsional static stiffness (ring footing).

$\frac{R_1}{R_2}$	$\frac{K^0_{tt}}{GR_2^3}$	
	$h/R_2 = 2$	$h/R_2 = 3$
0	5.79	5.75
0.1	5.79	5.75
0.2	5.79	5.75
0.3	5.79	5.75
0.4	5.78	5.74
0.5	5.77	5.73
0.6	5.75	5.71
0.7	5.68	5.64
0.8	5.55	5.50
0.9	5.25	5.22
0.95	4.98	4.95

Table 5.6 - The nondimensional vertical static stiffness (ring footing, ν = 1/3).

$\frac{R_1}{R_2}$	$\frac{K^o_{zz}}{GR_2}$	
	h/R_2 = 2	h/R_2 = 3
0	10.37	8.78
0.1	10.37	8.78
0.2	10.35	8.76
0.3	10.31	8.74
0.4	10.22	8 68
0.5	10.06	8.57
0.6	9.80	8.40
0.7	9.41	8.13
0.8	8.84	7.71
0.9	7.94	7.27
0.95	7.25	6.50

Table 5.7 - The nondimensional horizontal static stiffness (ring footing, $\nu = 1/3$).

$\frac{R_1}{R_2}$	$\frac{K_{xx}^o}{GR_2}$	
	$h/R_2 = 2$	$h/R_2 = 3$
0	6.36	5.88
0.1	6.36	5.88
0.2	6.35	5.87
0.3	6.34	5.86
0.4	6.31	5.84
0.5	6.25	5.79
0.6	6.16	5.72
0.7	6.02	5.60
0.8	5.81	5.42
0.9	5.47	5.12
0.95	5.22	4.91

Table 5.8 - The nondimensional rocking static stiffness (ring footing, $\nu = 1/3$).

$\frac{R_1}{R_2}$	$\frac{K^0_{rr}}{GR_2^3}$	
	$h/R_2 = 2$	$h/R_2 = 3$
0	4.63	4.48
0.1	4.63	4.48
0.2	4.63	4.48
0.3	4.63	4.48
0.4	4.62	4.47
0.5	4.61	4.46
0.6	4.58	4.43
0.7	4.51	4.37
0.8	4.38	4.24
0.9	4.10	3.98
0.95	3.83	3.73

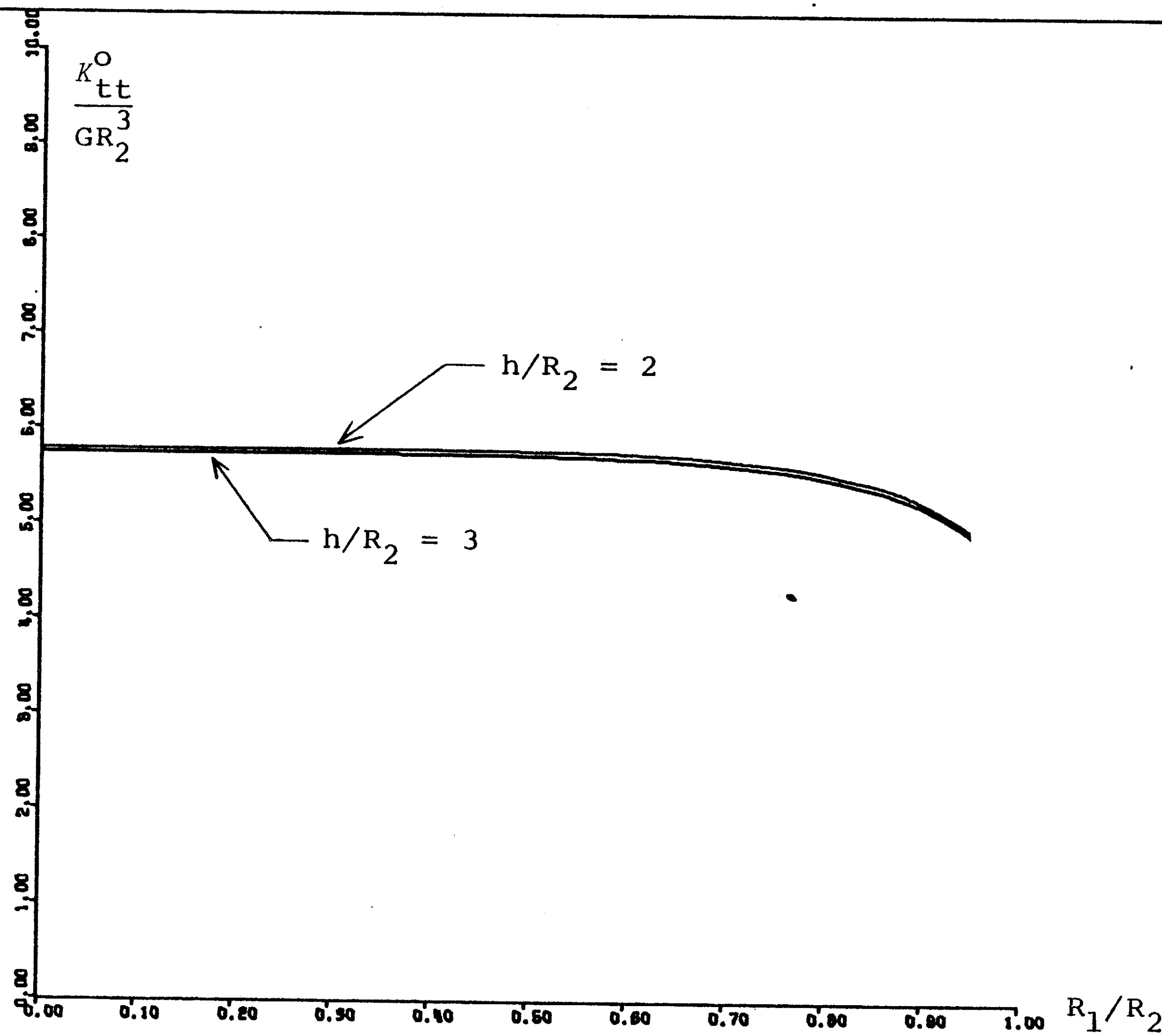


Figure 5.22 - The nondimensional torsional static stiffness (ring footing).

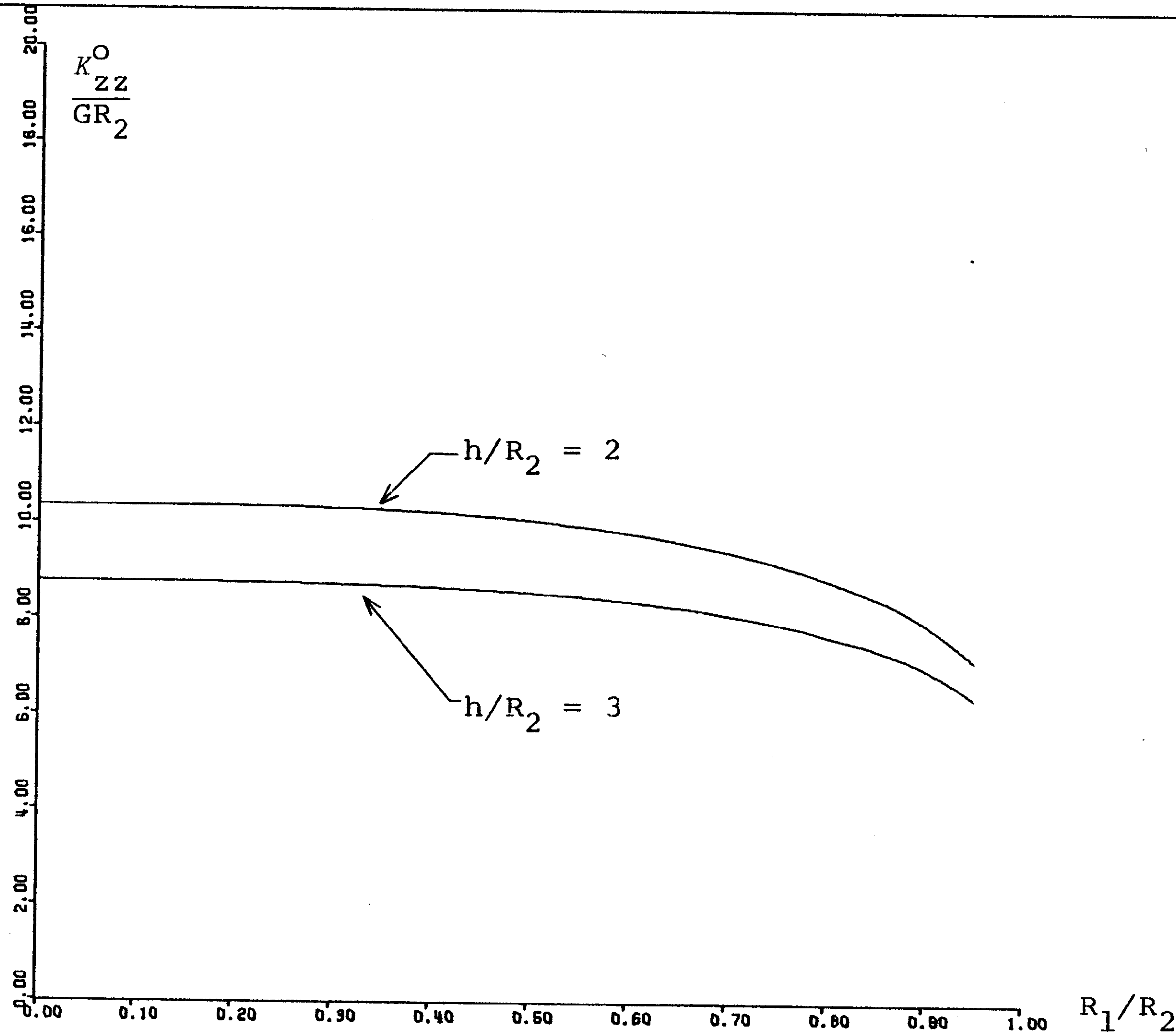


Figure 5.23 - The nondimensional vertical static stiffness (ring footing , $\nu = 1/3$).

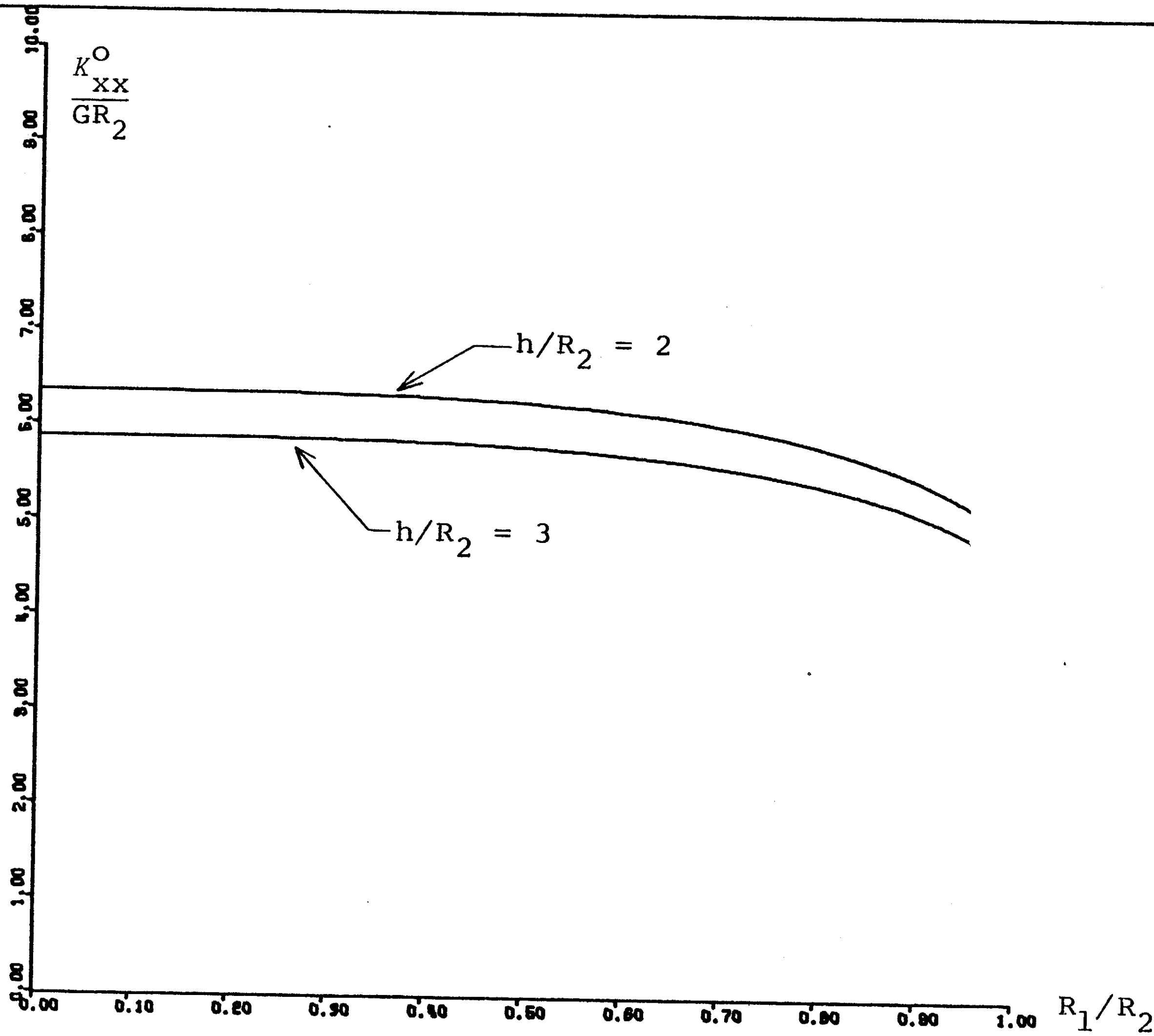


Figure 5.24 - The nondimensional horizontal static stiffness (ring footing , $\nu = 1/3$).

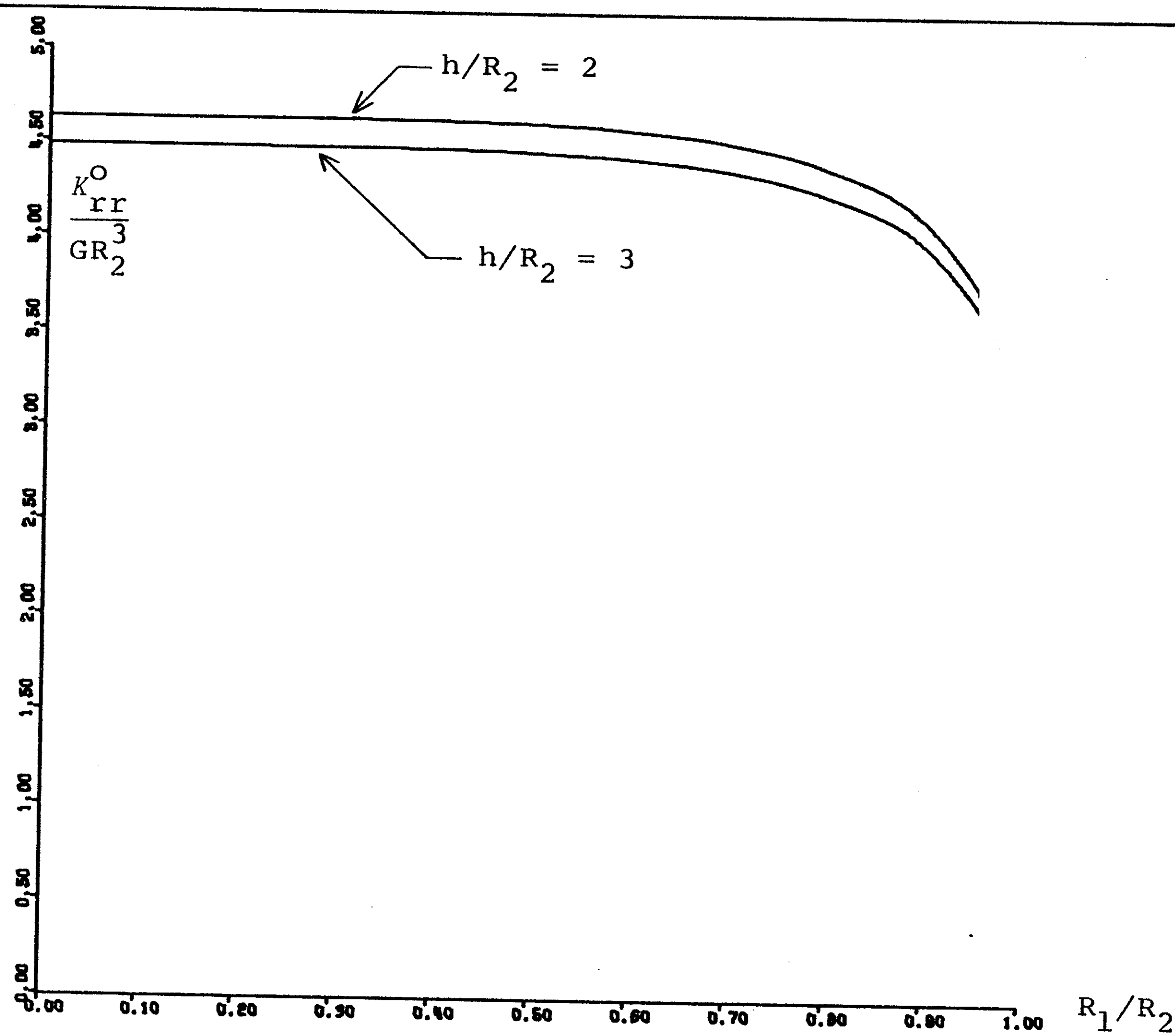


Figure 5.25 - The nondimensional rocking static stiffness (ring footing , $\nu = 1/3$).

In fact, they vanish in the limit $\frac{R_1}{R_2} \to 1$. The torsional and rocking static stiffnesses deviate considerably from the value corresponding to $R_1 = 0$ (i.e. the circular footing) only for values of the ratio $\frac{R_1}{R_2}$ greater than about 0.75. This is explained by the fact that for torsion and rocking the stiffness is mainly provided by the area near the outside edge of the footing. However, the vertical and horizontal static stiffnesses are more sensitive to the ratio of the radii of the footing. They deviate considerably from the static stiffness of the circular footing for values of $\frac{R_1}{R_2}$ greater than about 0.60.

The stiffness coefficients for $\frac{h}{R_2} = 2$ are plotted versus the nondimensional frequency $\frac{1}{2\pi} \frac{\omega R_2}{C_T}$ for $\frac{R_1}{R_2} = 0.5, 0.8, 0.9$ in figures 5.26, 5.27, 5.28, 5.29. There is little change in the coefficients for the three values of the ratio $\frac{R_1}{R_2}$. Figures 5.30, 5.31, 5.32, 5.33 show the stiffness coefficients for $\frac{h}{R_2} = 2$, $\frac{h}{R_2} = 3$, and $\frac{R_1}{R_2} = 0.9$ versus the nondimensional frequency. The range of values of the stiffness coefficients is for all practical purposes the same for the two depths. As would be expected, the peaks and troughs of the curves are simply shifted.

Finally, we note that the inner region (up to a radius of, say, $0.60\ R_2$) of a circular footing does not influence significantly the dynamic behavior of the footing in the frequency

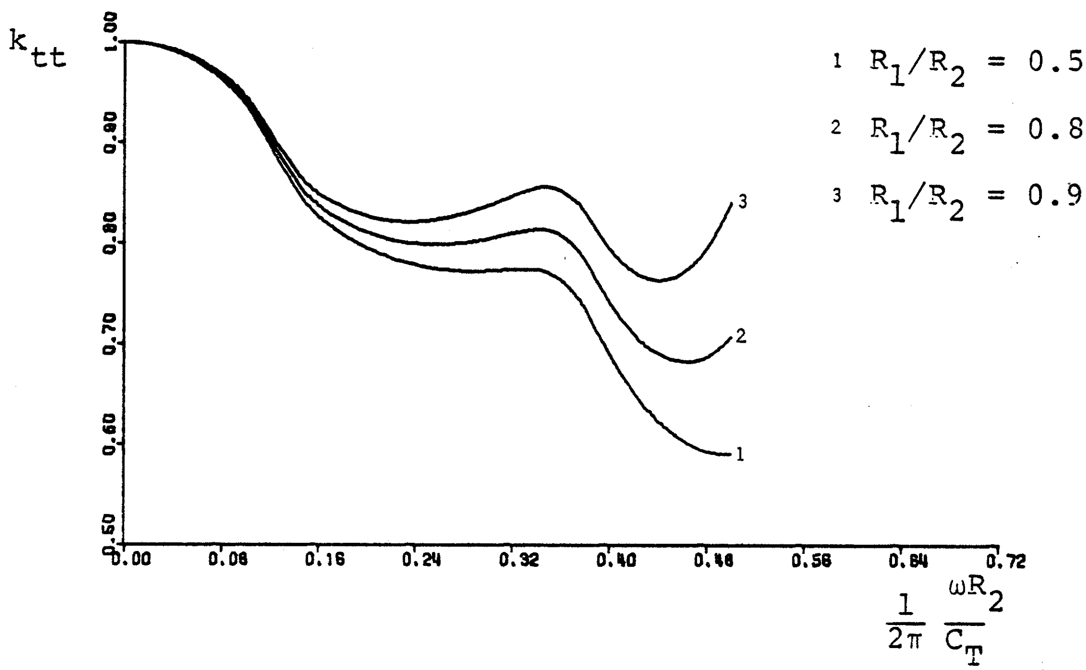
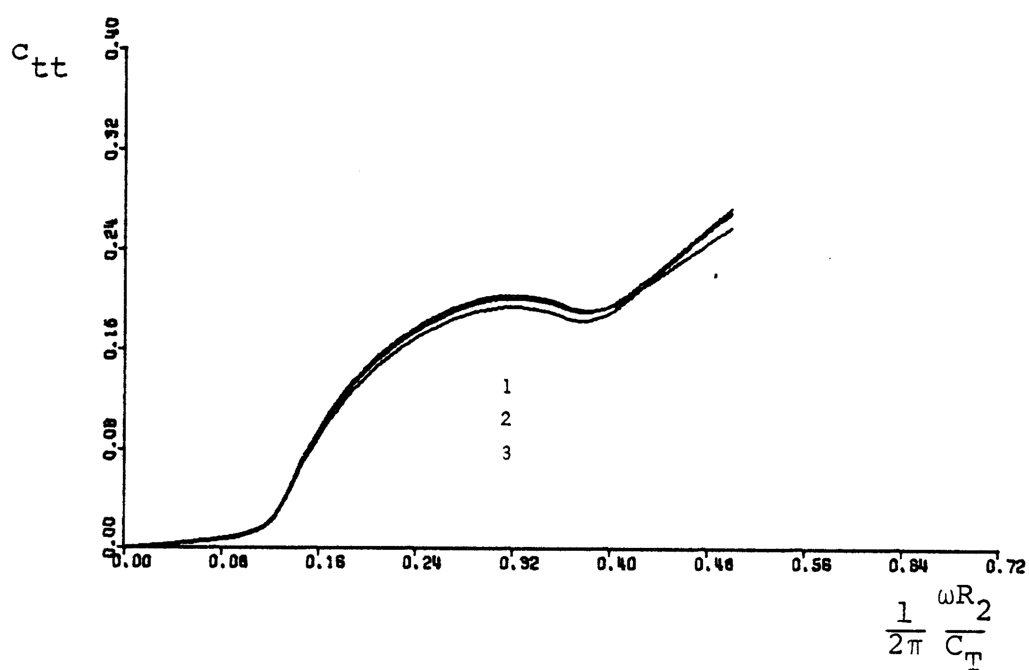


Figure 5.26 - Torsional stiffness coefficients (ring footing , $h/R_2 = 2$, $\beta = 0.05$).



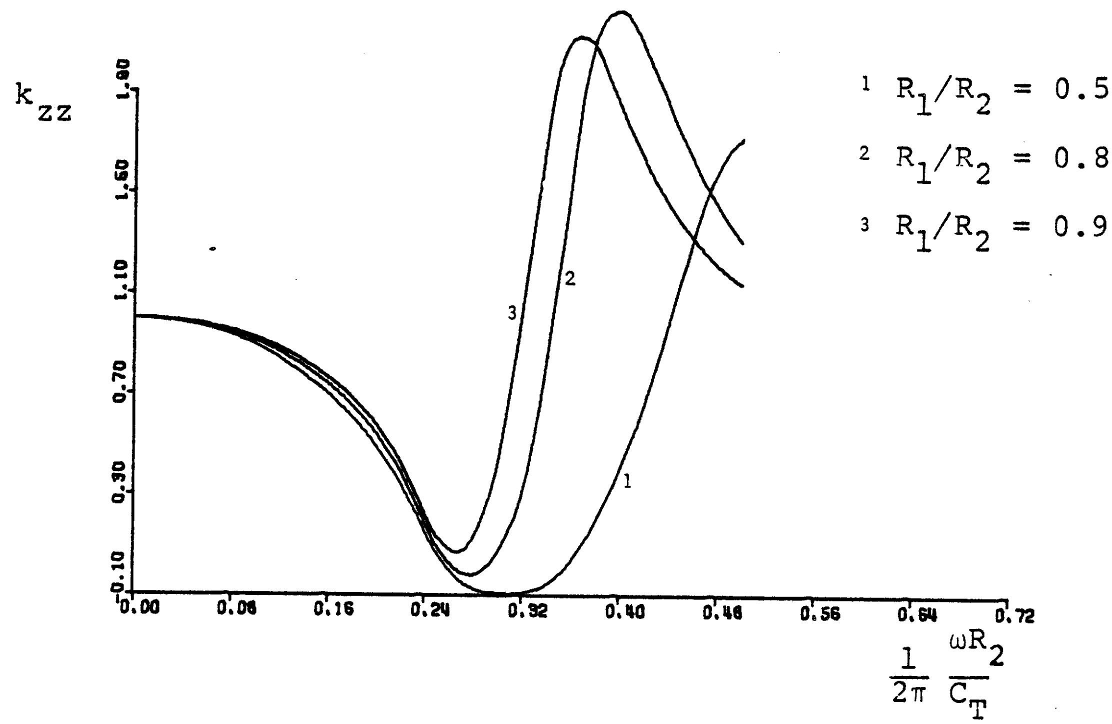
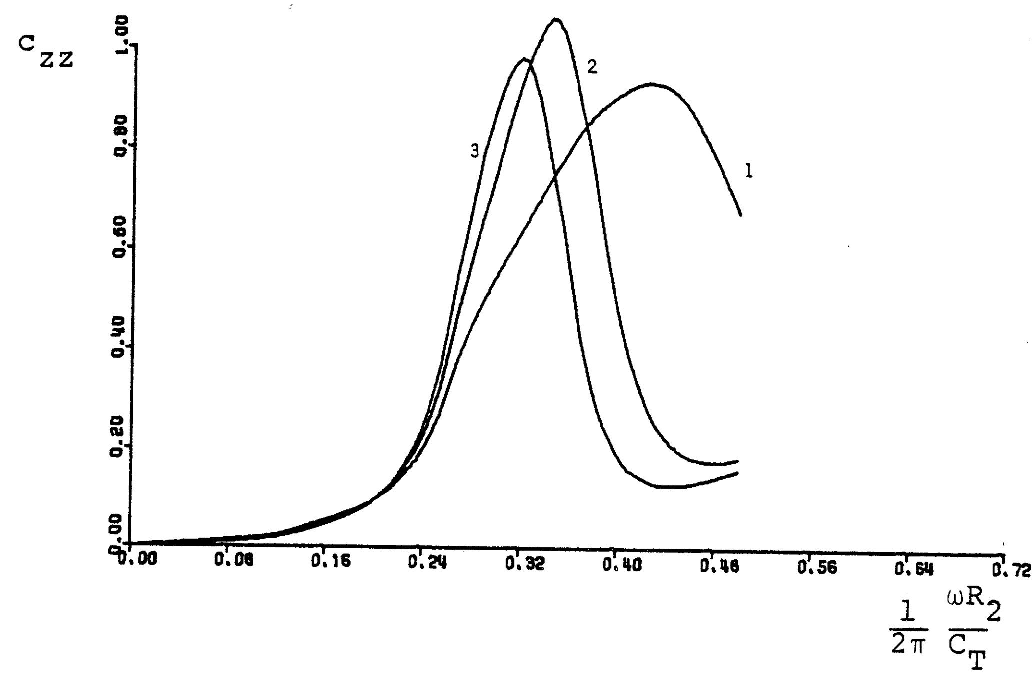


Figure 5.27 - Vertical stiffness coefficients (ring footing , $h/R_2 = 2$, $\nu = 1/3$, $\beta = 0.05$).



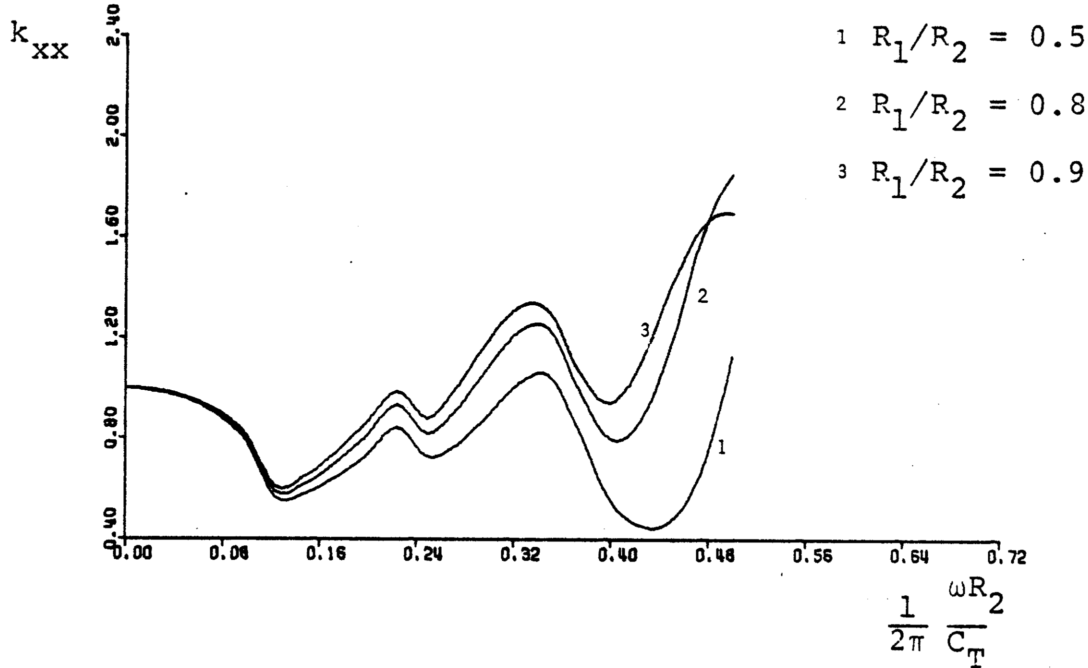
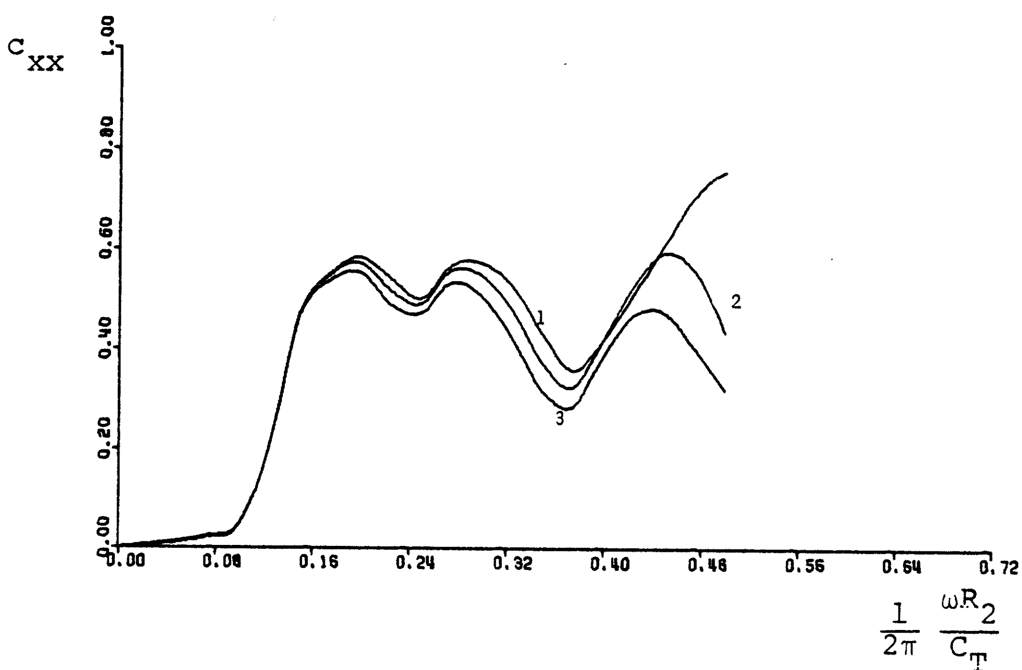


Figure 5.28 - Horizontal stiffness coefficients (ring footing , $h/R_2 = 2$, $\nu = 1/3$, $\beta = 0.05$).



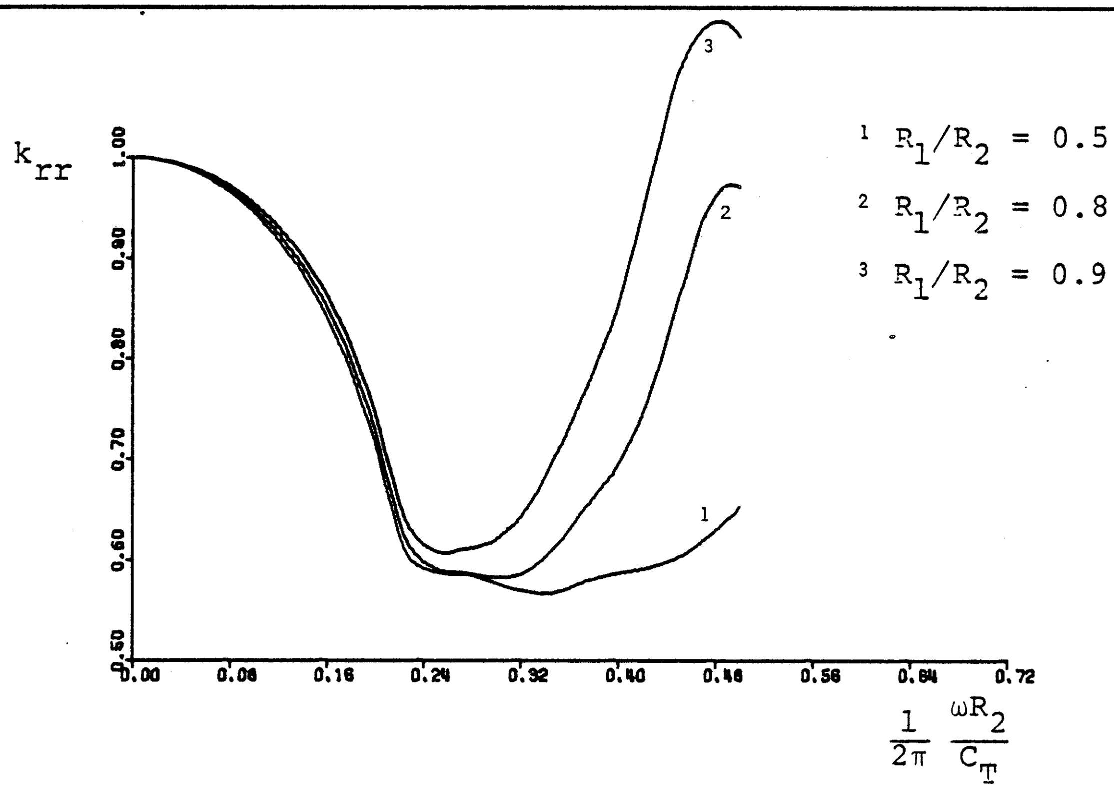
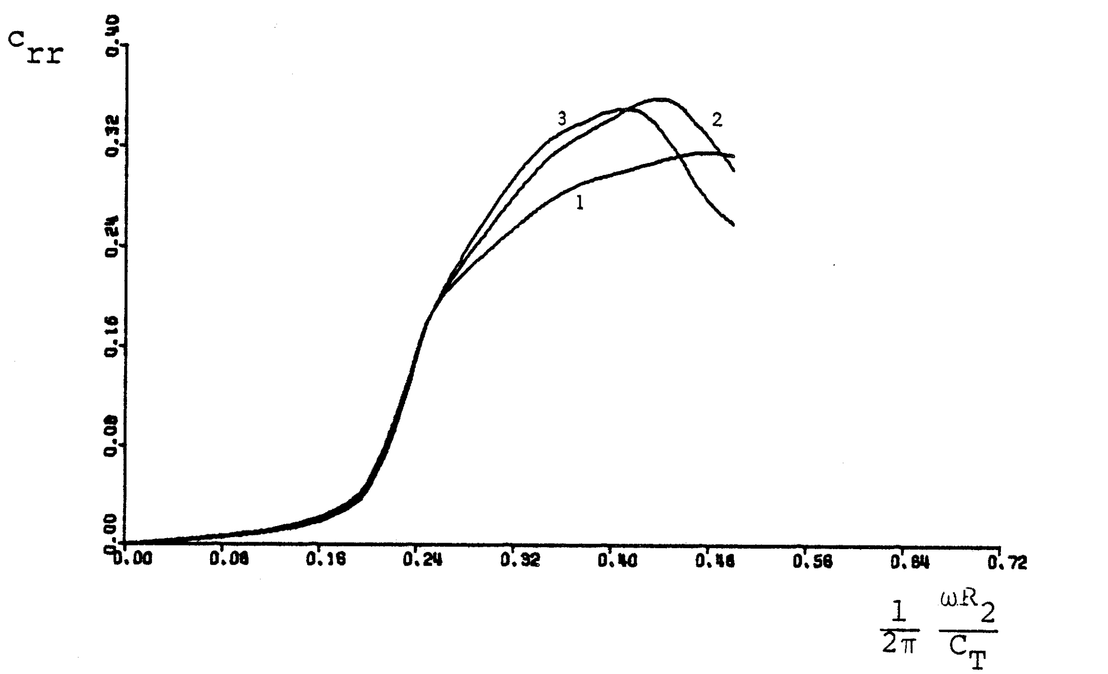


Figure 5.29 - Rocking stiffness coefficients (ring footing , $h/R_2 = 2$, $\nu = 1/3$, $\beta = 0.05$).



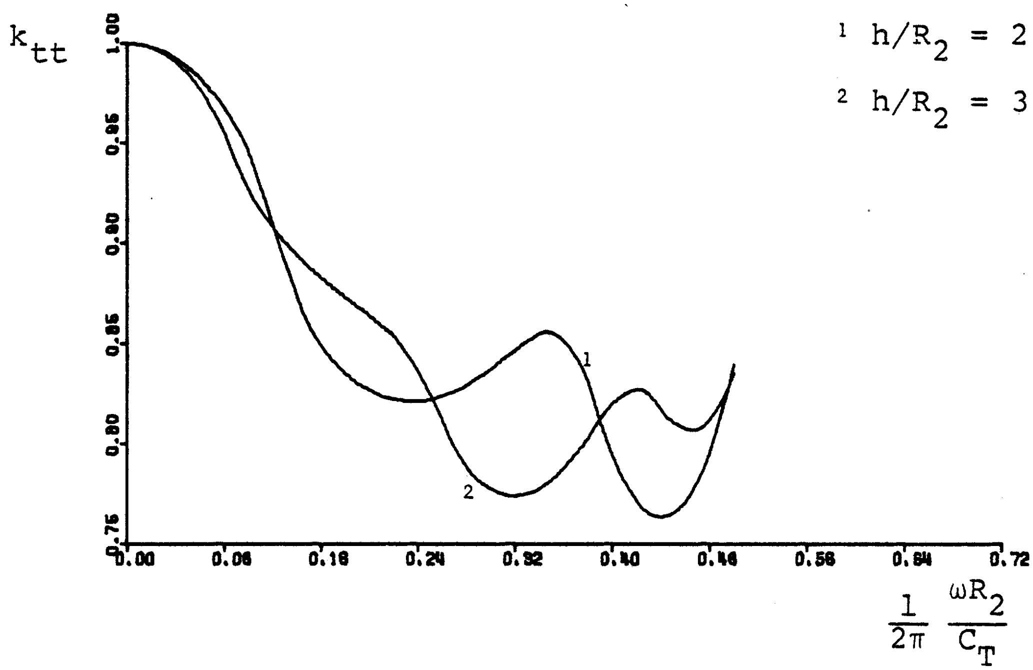
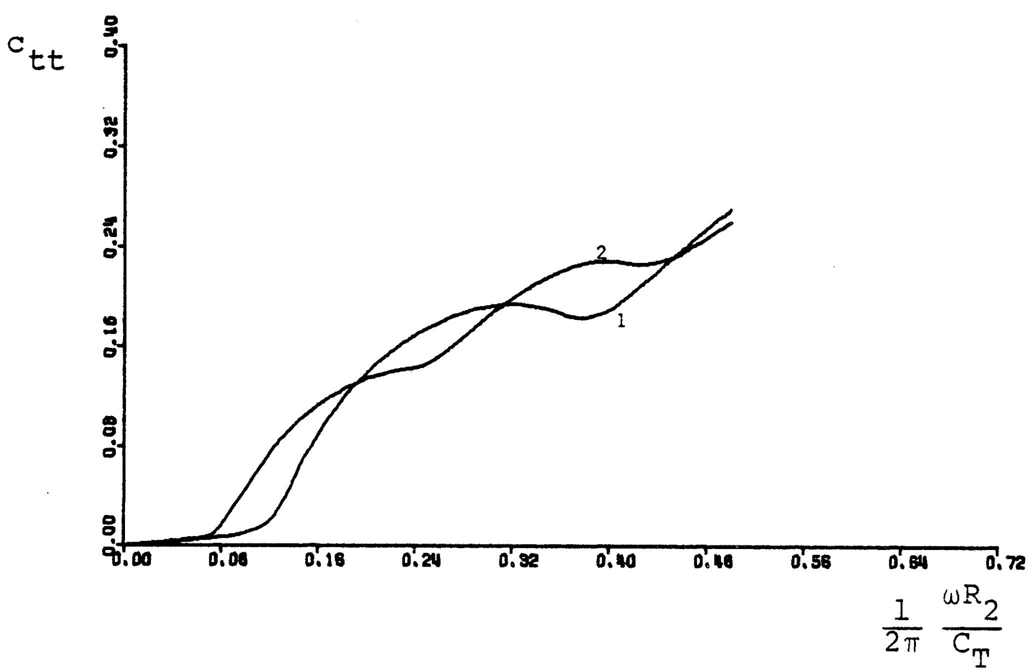


Figure 5.30 - Torsional stiffness coefficients (ring footing , $R_1/R_2 = 0.9$, $\beta = 0.05$).



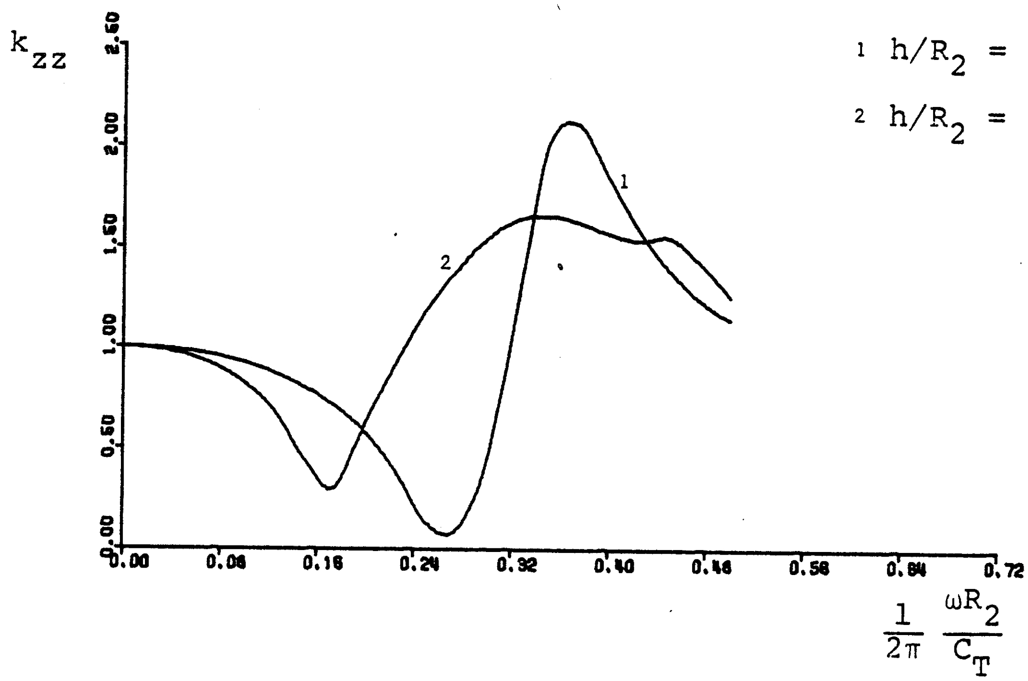
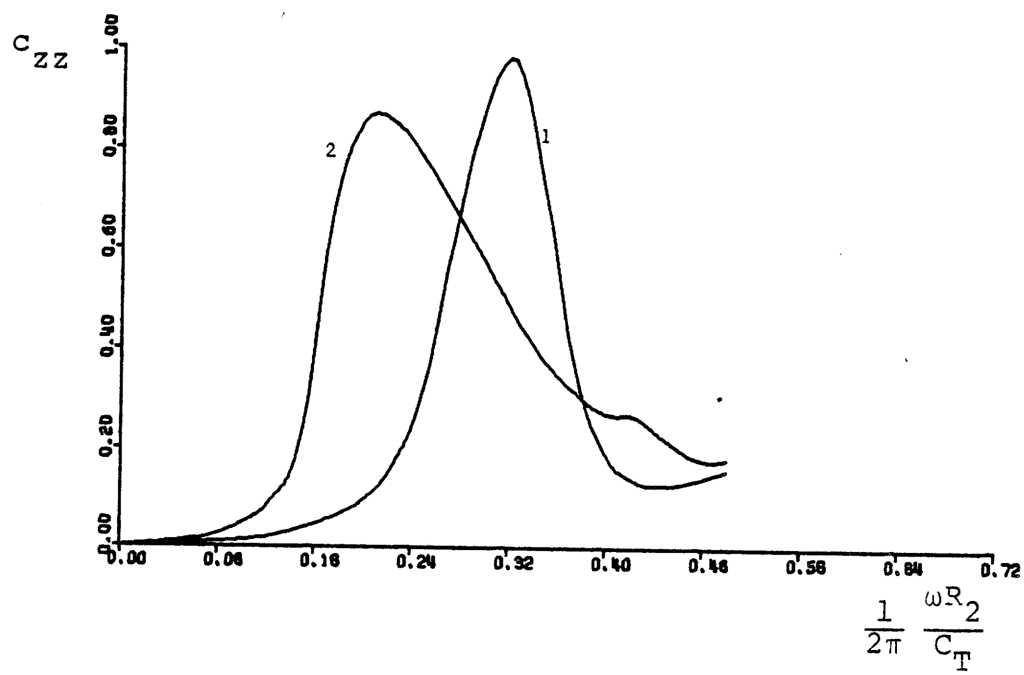
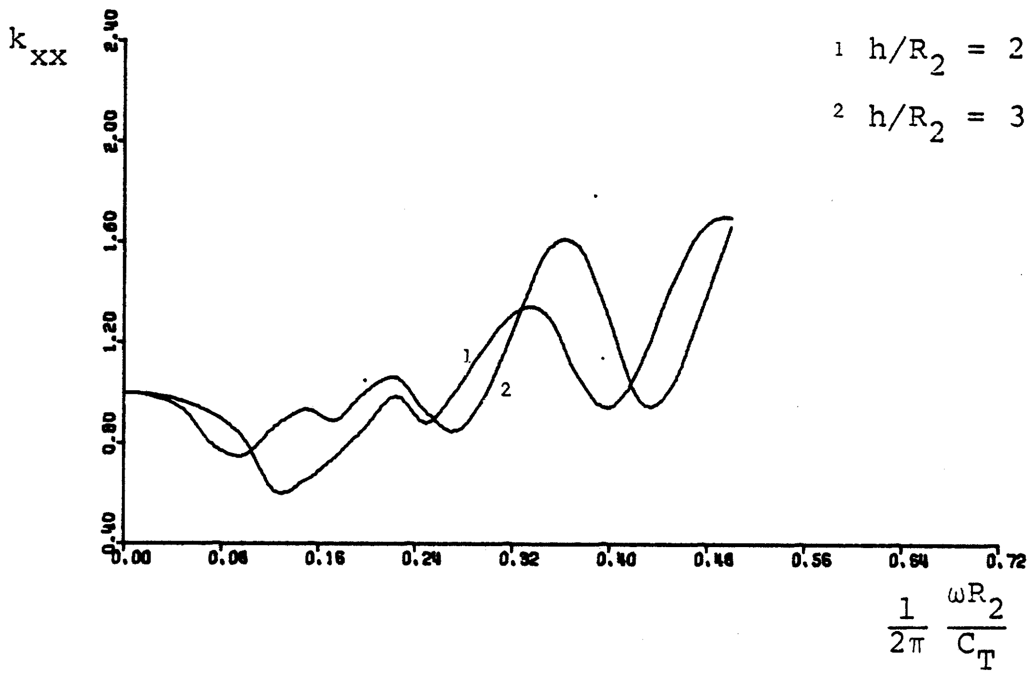
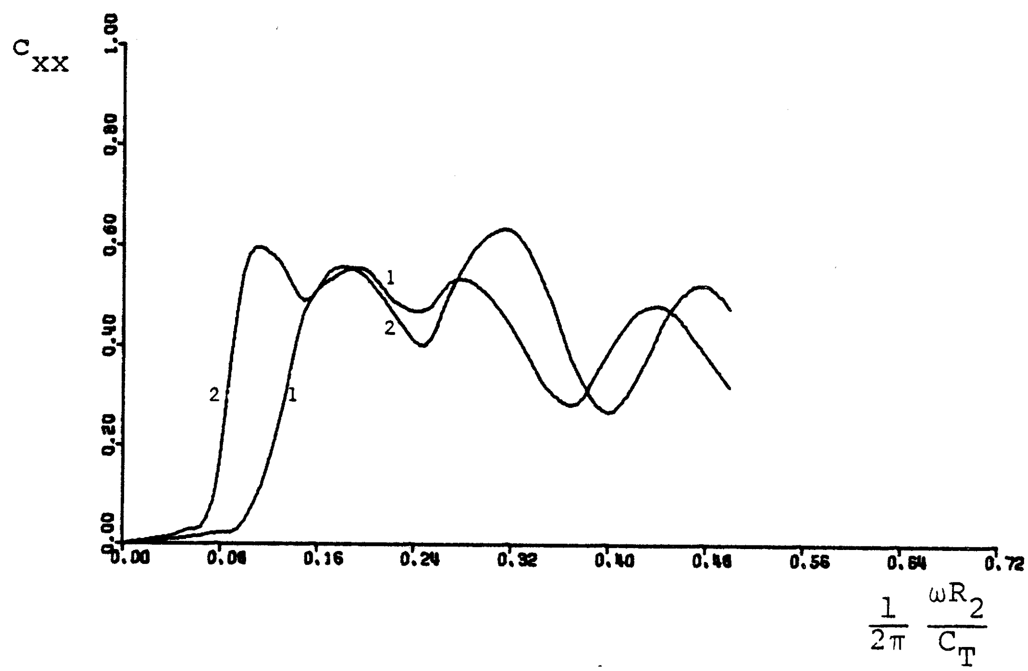


Figure 5.31 - Vertical stiffness coefficients
(ring footing , R_1/R_2 = 0.9 , ν = 1/3 , β = 0.05).





<u>Figure 5.32</u> - Horizontal stiffness coefficients (ring footing , $R_1/R_2 = 0.9$, $\nu = 1/3$, $\beta = 0.05$).



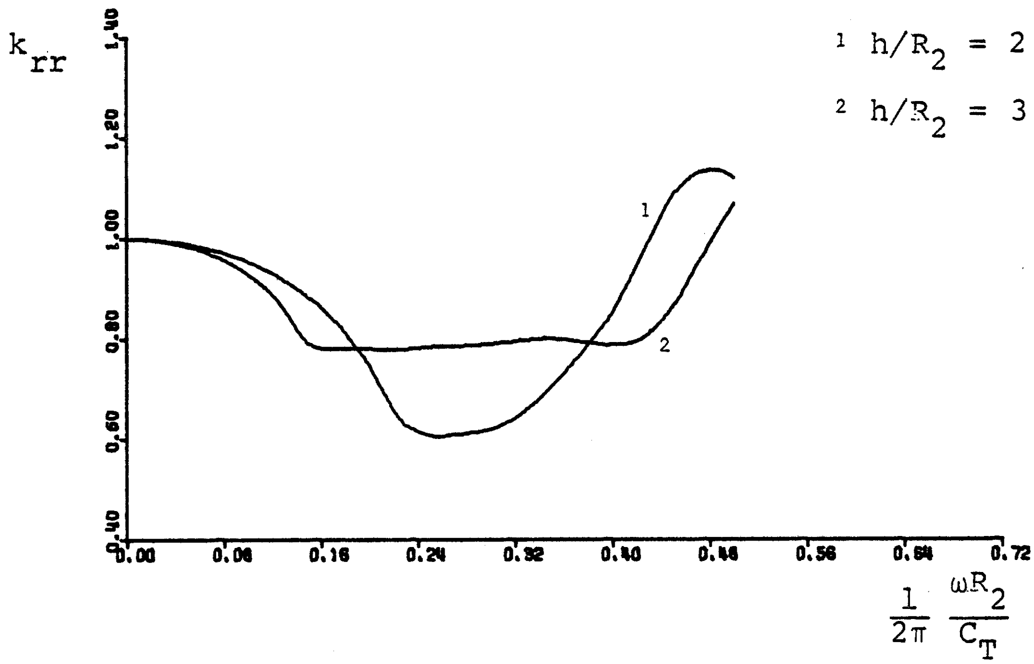
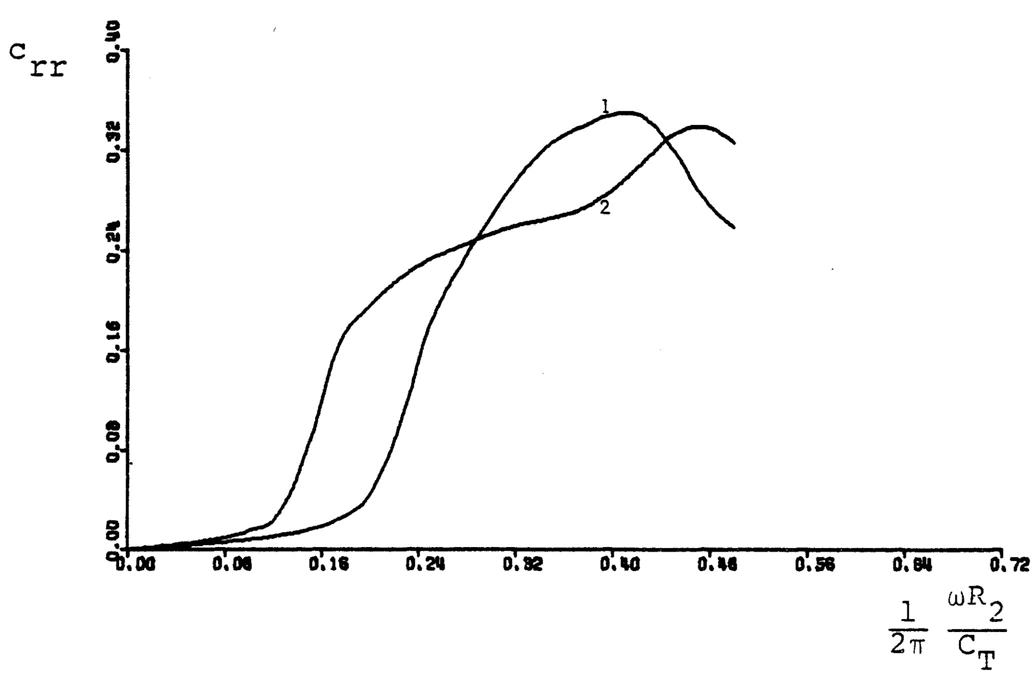


Figure 5.33 - Rocking stiffness coefficients
(ring footing , $R_1/R_2 = 0.9$, $\nu = 1/3$, $\beta = 0.05$).



range $0 \leq \frac{1}{2\pi} \frac{\omega R_2}{C_T} \leq \frac{1}{2}$, provided that the outer region may be considered rigid. This is because the static stiffnesses of the ring footing do not differ considerably from those of the circular footing but for values of $\frac{R_1}{R_2}$ greater than about 0.60 and, moreover, the stiffness coefficients in the range $0 \leq \frac{1}{2\pi} \frac{\omega R_2}{C_T} \leq \frac{1}{2}$ do not change appreciably with $\frac{R_1}{R_2}$.

CHAPTER 6

CONCLUSIONS

In this work we presented elements for the numerical analysis of wave motion in layered media. Plane elements which were developed in full detail include those modeling the rectangular region $x_1 \leq x \leq x_2$, $0 \leq z \leq h$, of a layered stratum in plane strain or antiplane shear. The boundary conditions specified at the surface of the region correspond to a rigid and rough strip footing. Other plane elements for which the boundary conditions at the base are inhomogeneous (for example, base motion) were also described (see Chapter 3). Axisymmetric elements which were considered are those modeling the regions $0 \leq r \leq r_o$, $0 \leq z \leq h$, and $r_1 \leq r \leq r_2$, $0 \leq z \leq h$, with boundary conditions at the surface corresponding to rigid and rough circular and ring footings respectively. The development of other axisymmetric elements was outlined (see Chapter 4). Applications were presented which illustrate the use of the elements. Significant improvements with respect to the computational effort are the low storage requirements and the fact that the number of arithmetic operations is independent of the length of the elements (plane elements) or their thickness in the radial direction (axisymmetric elements).

The technique developed in this work relies upon the calculation of semidiscrete solutions. It was demonstrated that semidiscrete particular solutions may be found for a variety of inhomogeneous boundary conditions. Moreover, semidiscrete modes satisfying the corresponding homogeneous boundary conditions are easily calculated. In previous works[23, 6], a semidiscrete solution (satisfying homogeneous boundary conditions) for part of the region was combined with a fully discrete solution (satisfying inhomogeneous boundary conditions) for the rest of the region. It is now possible to obtain a semidiscrete solution for the entire region. We note that an advantage of the technique beyond those emphasized previously is the fact that the displacements, stresses and strains at any point in the region may be expressed in terms of relatively few parameters, namely, the participation factors of the semidiscrete modes and particular solutions.

The method may be extended to the analysis of wave motion in regions other than a layered stratum. Consider, for example, a bar of rectangular cross section. Let the z-direction be along the longitudinal axis of the bar. Boundary conditions must be specified on the four sides of the bar. The modes are of the form

$$u(x,y,z) = U(x,y)\exp(-ikz)$$

$$v(x,y,z) = V(x,y)\exp(-ikz)$$

$$w(x,y,z) = W(x,y)\exp(-ikz)$$

Some analytical results fot the case of traction-free boundaries are given by Mindlin and Fox [19]. Discrete solutions for the amplitudes U, V, W and approximate wave numbers k may be obtained using the finite element method. The resulting algebraic eigenvalue problem is of the same form as the ones encountered in this work (it may be derived as in section 2.2). An extension of the method to the analysis of wave motion in a cylindrical rod is also possible. The modes are given by

$$u_n(r,\theta,z) = U_n(r) \begin{Bmatrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{Bmatrix} \exp(-ikz)$$

$$w_n(r,\theta,z) = W_n(r) \begin{Bmatrix} \cos(n\theta) \\ \\ \sin(n\theta) \end{Bmatrix} \exp(-ikz)$$

$$v_n(r,\theta,z) = V_n(r) \begin{Bmatrix} -\sin(n\theta) \\ \\ \cos(n\theta) \end{Bmatrix} \exp(-ikz)$$

$$n = 0,\ 1,\ 2,\ \ldots$$

Details may be found in [2]. For a rod of circular cross section and traction-free boundaries extensive results may be found in the works of Mindlin and McNiven [20] and Onoe, McNiven and Mindlin [21]. They treat the case of axially symmetric waves (note that the eigenvalue problem which yields k, U_n, W_n, V_n depends upon the value of n). Again, discrete solutions for the amplitudes U_n, W_n, V_n and approximate wave numbers k may be calculated using the finite element method. The rod may be hollow and material properties may vary with r (but not with θ or z).

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