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Citation: Nashida, Yusuke, and Dam T. Son. "." Physical Review A 82.4 (2010): 043606. © 2010 The American Physical Society

As Published: <http://dx.doi.org/10.1103/PhysRevA.82.043606>

Publisher: American Physical Society

Persistent URL: <http://hdl.handle.net/1721.1/60874>

Version: Final published version: final published article, as it appeared in a journal, conference proceedings, or other formally published context

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Universal four-component Fermi gas in one dimension

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(Received 20 August 2009; revised manuscript received 26 August 2010; published 11 October 2010)

A four-component Fermi gas in one dimension with a short-range four-body interaction is shown to exhibit a one-dimensional analog of the BCS-BEC crossover. Its low-energy physics is governed by a Tomonaga-Luttinger liquid with three spin gaps. The spin gaps are exponentially small in the weak coupling (BCS) limit where they arise from the charge-density-wave instability, and become large in the strong coupling (BEC) limit because of the formation of tightly bound tetramers. We investigate the ground-state energy, the sound velocity, and the gap spectrum in the BCS-BEC crossover and discuss exact relationships valid in our system. We also show that a one-dimensional analog of the Efimov effect occurs for five bosons while it is absent for fermions. Our work opens up a very rich field of universal few-body and many-body physics in one dimension.

DOI: [10.1103/PhysRevA.82.043606](https://doi.org/10.1103/PhysRevA.82.043606)

PACS number(s): 03.75.Ss, 05.30.Fk, 67.85.Lm, 71.10.Pm

I. INTRODUCTION

Experiments using ultracold atomic gases have achieved striking progress in realizing and studying various many-body systems previously regarded as purely theoretical models. One example is the Tonks-Girardeau gas in one dimension, proposed 50 years ago [1] and realized experimentally in 2004 [2,3]. Another example is the BCS-BEC crossover in a two-component Fermi gas with a short-range two-body interaction [4–6]. It was predicted about 40 years ago [7–9] and has been subject to extensive studies after recent experimental realization [10–12]. In the weak coupling limit, the system is a BCS superfluid where fermionic excitations have an exponentially small gap, while at strong coupling it becomes a dilute Bose-Einstein condensate (BEC) of tightly-bound dimers with a large gap for the fermionic excitations. These two limits are smoothly connected by varying a single parameter, the scattering length. When the scattering length is much larger than the range of the interaction potential, the properties of such a system are independent of the potential shape. This universality makes the study of the BCS-BEC crossover extremely worthwhile because the same properties are shared by many different systems [13].

In this paper, we propose a purely one-dimensional analog of the BCS-BEC crossover in a four-component Fermi gas with a short-range four-body interaction. The short-range four-body interaction in one dimension is characterized by the scattering length exactly in the same way that it characterizes the short-range two-body interaction in three dimensions [14], and therefore, leads to the universal “BCS-BEC” crossover in one dimension. We note that while the BCS-BEC crossover of a two-component Fermi gas in a quasi-one-dimensional geometry has been studied before [15,16], the crossover studied in this paper has the distinction of being universal, i.e., independent of the confinement potential. We also note that four-component (spin-3/2) Fermi gases with two-body interactions have been studied and reviewed in Ref. [17].

In Sec. II A, we start with a lattice model that realizes the BCS-BEC crossover in one dimension. The universal regime in the vicinity of a four-body resonance is described by a continuum theory derived in Sec. II B. We show in Sec. II C that a one-dimensional analog of the Efimov effect occurs for

five bosons, while it is absent for fermions which is necessary for the stability of the many-body system studied in Sec. III. We investigate the sound velocity and the gap spectrum in the BCS limit (Sec. III A) and in the BEC limit (Sec. III B) and hypothesize that these two limits are smoothly connected without phase transitions just as in three dimensions. In the unitarity limit, a one-dimensional Bertsch parameter and its connection to the Tomonaga-Luttinger parameter are introduced in Sec. III C, whose value can be estimated in principle by using ϵ expansions. Exact relationships involving a contact density are derived in Sec. III D and the contact density is determined from the ground-state energy density in the BCS and BEC limits. Finally, Sec. IV is devoted to the summary of this paper.

II. FEW-BODY PROBLEMS

A. Lattice model

We start with a system of fermions with four components labeled by $\sigma = a, b, c, d$ living on a one-dimensional lattice. We assume that each lattice site can accommodate one, two, or three particles with no change in energy, but an introduction of a fourth particle into a site with three particles releases a finite amount of energy. The lattice Hamiltonian for such a system is

$$H = -t \sum_{\langle xy \rangle, \sigma} c_{x\sigma}^\dagger c_{y\sigma} - U \sum_x c_{xa}^\dagger c_{xb}^\dagger c_{xc}^\dagger c_{xd}^\dagger c_{xd} c_{xc} c_{xb} c_{xa}. \quad (1)$$

We will be interested in the dilute limit where the average number of particles per site is small. To find the universal regime, we consider the scattering among all different components of fermions. Such a four-body problem is described by the Schrödinger equation

$$\left[-t \sum_{\sigma} \Delta_{\sigma} + V(\mathbf{x}) \right] \Psi(\mathbf{x}) = E \Psi(\mathbf{x}), \quad (2)$$

where $\mathbf{x} = (x_a, x_b, x_c, x_d)$ is a set of coordinates of four particles and Δ_{σ} is the discrete Laplacian with respect to x_{σ} ; $\Delta_{\sigma} \Psi(x_{\sigma}) \equiv \Psi(x_{\sigma} + l) + \Psi(x_{\sigma} - l) - 2\Psi(x_{\sigma})$ with l being

the lattice spacing. The four-body interaction potential is given by $V(\mathbf{x}) = -U$ when all x_σ are equal and $V(\mathbf{x}) = 0$ otherwise.

Since $V(\mathbf{x})$ is translationally invariant, it is convenient to introduce new coordinates $X = (x_a + x_b + x_c + x_d)/4$, $r_1 = (x_a + x_b - x_c - x_d)/2$, $r_2 = (x_a - x_b + x_c - x_d)/2$, and $r_3 = (x_a - x_b - x_c + x_d)/2$ and assume $\Psi(\mathbf{x})$ to be independent of the center-of-mass coordinate X . The Schrödinger equation (2) in terms of the remaining three relative coordinates $\mathbf{r} = (r_1, r_2, r_3)$ becomes

$$\left[-t \sum_{i=1}^4 \Delta_i - \delta_{r,0} U \right] \Psi(\mathbf{r}) = E \Psi(\mathbf{r}), \quad (3)$$

where $\Delta_i \Psi(\mathbf{r}) \equiv \Psi(\mathbf{r} + \mathbf{e}_i) + \Psi(\mathbf{r} - \mathbf{e}_i) - 2\Psi(\mathbf{r})$ with $\mathbf{e}_1 = \frac{l}{2}(1, 1, 1)$, $\mathbf{e}_2 = \frac{l}{2}(1, -1, -1)$, $\mathbf{e}_3 = \frac{l}{2}(-1, 1, -1)$, and $\mathbf{e}_4 = \frac{l}{2}(-1, -1, 1)$. Equation (3) is equivalent to the Schrödinger equation describing one particle moving in a body-centered cubic lattice with an attractive potential of the magnitude U concentrated at one lattice site.

One can see from Eq. (3) that the zero-energy wave function at a long distance has the form

$$\Psi(|\mathbf{r}| \rightarrow \infty)|_{E=0} \propto \frac{1}{|\mathbf{r}|} - \frac{1}{a}, \quad (4)$$

where $|\mathbf{r}|^2 = \sum_\sigma (x_\sigma - X)^2 = \frac{1}{4} \sum_{\sigma < \tau} (x_\sigma - x_\tau)^2$ is the hyperradius of four particles in one dimension. The form (4) is familiar in two-body scattering problems in three dimensions, which can be understood from the fact that the continuum limit of Eq. (3) with $l^2 t \equiv \hbar^2/(2m)$ is exactly the Schrödinger equation in three dimensions. Here a is an arbitrary real parameter characterizing the long-distance physics and referred to as the scattering length. By matching the solution of Eq. (3) [18]:

$$\begin{aligned} \frac{\Psi(\mathbf{r})}{\Psi(\mathbf{0})} \Big|_{E=0} &= 1 - \frac{\Gamma(\frac{1}{4})^4 U}{32\pi^3 t} \\ &+ \frac{U}{8t} \int_{-\pi}^{\pi} \frac{d\mathbf{k}}{(2\pi)^3} \frac{e^{2i\mathbf{k}\cdot\mathbf{r}/l}}{1 - \cos k_1 \cos k_2 \cos k_3} \\ &\rightarrow 1 - \frac{\Gamma(\frac{1}{4})^4 U}{32\pi^3 t} + \frac{U}{8\pi t} \frac{l}{|\mathbf{r}|} \quad (|\mathbf{r}| \rightarrow \infty) \end{aligned} \quad (5)$$

with the asymptotic form (4), we find a in units of the lattice spacing l to be

$$\frac{l}{a} = \frac{\Gamma(\frac{1}{4})^4}{4\pi^2} - \frac{8\pi t}{U}. \quad (6)$$

The scattering length a can be fine-tuned to infinite corresponding to the four-body resonance by choosing

$$\frac{U}{t} = \frac{32\pi^3}{\Gamma(\frac{1}{4})^4} \approx 5.742. \quad (7)$$

This value of U/t separates the weak coupling regime ($a < 0$) with no bound state from the strong coupling regime ($a > 0$) in which there exists a four-body bound state (tetramer). The wave function and binding energy of the tetramer for $a \gg l$

are given by the universal formulas independent of the lattice parameters:

$$\Psi(|\mathbf{r}| \rightarrow \infty) \propto \frac{e^{-|\mathbf{r}|/a}}{|\mathbf{r}|} \quad \text{and} \quad E_0 = -\frac{\hbar^2}{2ma^2}. \quad (8)$$

The long-distance physics near the critical value of U/t should be universal and, in particular, scale and conformal invariance are achieved in the unitarity limit $a \rightarrow \infty$ [14].

B. Field-theoretical formulation

The physics in the universal regime can be described by the following continuum-limit Hamiltonian density (hereafter $\hbar = 1$):

$$\mathcal{H} = - \sum_{\sigma} \frac{\psi_{\sigma}^{\dagger} \nabla^2 \psi_{\sigma}}{2m} - c_0 \psi_d^{\dagger} \psi_b^{\dagger} \psi_c^{\dagger} \psi_d^{\dagger} \psi_d \psi_c \psi_b \psi_a. \quad (9)$$

Throughout this paper, we neglect two-body and three-body interactions and interactions involving the same components of fermions. In addition to the translational and Galilean symmetries, the Hamiltonian density has global U(1) and SU(4) symmetries,

$$\psi_{\sigma} \rightarrow e^{i\theta} \psi_{\sigma} \quad \text{and} \quad \psi_{\sigma} \rightarrow U_{\sigma\sigma'} \psi_{\sigma'}, \quad (10)$$

corresponding to the conservations of charge and SU(4) spins, respectively.

The second term in Eq. (9) describes the four-body contact interaction among all different components of the fermionic field ψ_{σ} . c_0 is a cutoff-dependent coupling constant and can be related to the above-introduced scattering length a by matching in the four-body problem. The four-body scattering amplitude $\mathcal{A}(E, p)$ is obtained by summing the Feynman diagrams in Fig. 1 into a geometric series:

$$\begin{aligned} [i\mathcal{A}(E, p)]^{-1} &= \frac{1}{ic_0} + i \int_{-\infty}^{\infty} \frac{dk_1 dk_2 dk_3}{(2\pi)^3} \\ &\times \frac{2m}{k_1^2 + k_2^2 + k_3^2 + (k_1 + k_2 + k_3)^2 + p^2/4 - 2mE - i0^+}. \end{aligned} \quad (11)$$

Here the integrations over momenta k_1 , k_2 , and k_3 are linearly divergent. Introducing a momentum cutoff $\sqrt{k_1^2 + k_2^2 + k_3^2} < \Lambda$ and choosing the cutoff dependence of c_0 as

$$\frac{1}{c_0} = \frac{m\Lambda}{3\sqrt{3}\pi} - \frac{m}{4\pi a}, \quad (12)$$

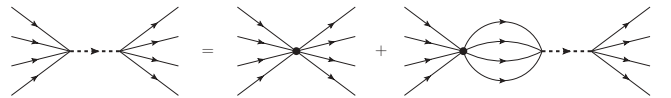


FIG. 1. Feynman diagrams describing the four-body scattering in vacuum. The dot represents the bare vertex ic_0 and the dashed line represents the scattering amplitude $i\mathcal{A}(E, p)$.

we obtain the following cutoff-independent scattering amplitude in the limit $\Lambda \rightarrow \infty$:

$$\mathcal{A}(E, p) = \frac{4\pi}{m} \frac{1}{-1/a + \sqrt{p^2/4 - 2mE - i0^+}}. \quad (13)$$

In particular, when $a > 0$, $\mathcal{A}(E, 0)$ has a pole at a real and negative E , indicating the existence of the four-body bound state. Because its binding energy is given by $E_0 = -1/(2ma^2)$, we can identify a in Eq. (12) with the scattering length introduced in Eq. (4).

C. Five-body problem and Efimov effect

The above arguments equally apply to four-component bosons in one dimension. However, many-body systems of attractive bosons tend to be unstable to collapse, in contrast to the case of fermions where the Pauli exclusion principle acts against such a collapse. We now show how such a difference already appears in a five-body problem: five bosons develop deep bound states while five fermions do not. This can be seen by studying a scaling dimension of five-body composite operator $\phi\psi_\sigma$ in the unitarity limit $a \rightarrow \infty$, where $\phi \equiv c_0\psi_a\psi_b\psi_c\psi_d$ is a tetramer field. If this operator has a real scaling dimension, the corresponding five-body system is scale invariant and thus does not support bound states. However, if the scaling dimension is complex, the full scale invariance is broken down to a discrete one [19], which indicates the formation of an infinite tower of bound states. Such a connection between the complex scaling dimension and the infinite tower of bound states has been observed in resonantly interacting three bosons or mass-imbalanced fermions in three dimensions [19], two particles interacting with a $1/r^2$ potential [20], and the nonrelativistic anti-de Sitter spacetime and conformal field theory (AdS-CFT) correspondence [21].

Feynman diagrams to renormalize $\phi\psi_\sigma$ are depicted in Fig. 2. The vertex function $z(p)$ satisfies the following integral equation:

$$z(p) = 1 + \lambda \int_{-\infty}^{\infty} \frac{dq}{2\pi} z(q) \frac{8\pi}{\sqrt{5}|q|} \times \int_{-\infty}^{\infty} \frac{dk_1 dk_2}{(2\pi)^2} \frac{1}{\frac{2p^2 + 2q^2 + pq}{3} + k_1^2 + k_2^2 + k_1 k_2}, \quad (14)$$

where $\lambda = \pm 1$ for four-component bosons or fermions. Because of the scale invariance, we can assume $z(p) \propto (|p|/\Lambda)^\gamma$. Performing the integrations in Eq. (14), we find that the

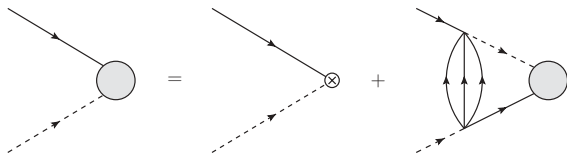


FIG. 2. Feynman diagrams to renormalize five-body composite operators. The dashed line is a resummed propagator of ϕ field which is equal to $-i\mathcal{A}$. The shaded bulb represents the vertex function $z(p)$.

anomalous dimension γ satisfies [22]

$$1 = -\frac{4\lambda}{\sqrt{15}} \frac{\cos(\gamma \arctan \frac{1}{\sqrt{15}})}{\gamma \sin(\frac{\pi\gamma}{2})}. \quad (15)$$

The anomalous dimensions of even-parity operators satisfy the same equation (15) and their scaling dimensions are given by

$$\Delta_\phi + \Delta_\psi + \gamma = \frac{3}{2} + \gamma. \quad (16)$$

For fermions ($\lambda = -1$), we can find a series of real solutions; $\gamma = 1.59, 4.08, 5.99, \dots$. According to the operator-state correspondence [23–25], each solution corresponds to the energy of resonantly interacting five fermions in a one-dimensional harmonic potential by

$$E = (\frac{3}{2} + \gamma) \omega. \quad (17)$$

On the other hand, for bosons ($\lambda = +1$), in addition to real solutions $\gamma = 2.25, 3.91, 6.01, \dots$, we can find a pair of complex solutions $\gamma = \pm 0.735i$. This is a signal of the formation of an infinite tower of five-body bound states (pentamers) whose spectrum exhibits the discrete scaling symmetry [26]:

$$\frac{E_n}{E_{n+1}} = e^{2\pi/|\text{Im}\gamma|} = (71.8)^2. \quad (18)$$

For identical bosons with the four-body resonant interaction, the anomalous dimension of the five-body composite operator $\phi\psi$ satisfies Eq. (15) with $\lambda = 4$. In addition to real solutions $\gamma = 3.43, 6.02, 8.15, \dots$, it has complex solutions $\gamma = \pm 1.25i$, and therefore, the spectrum of pentamers is much denser [26];

$$\frac{E_n}{E_{n+1}} = e^{2\pi/|\text{Im}\gamma|} = (12.4)^2. \quad (19)$$

This is an analog of the Efimov effect for three identical bosons in three dimensions [27]. Note that the ordinary Efimov effect can occur only in a spatial dimension $2.30 < d < 3.76$ and not in one dimension [28]. For comparison, the scaling factor for Efimov trimers in three dimensions is known to be $e^{2\pi/|\text{Im}\gamma|} = (22.7)^2$.

Similarly, the anomalous dimensions of odd-parity operators [e.g., $4\phi(\nabla\psi_\sigma) - (\nabla\phi)\psi_\sigma$] are found to satisfy [22]

$$1 = -\frac{4\lambda}{\sqrt{15}} \frac{\sin(\gamma \arctan \frac{1}{\sqrt{15}})}{\gamma \cos(\frac{\pi\gamma}{2})} \quad (20)$$

with the scaling dimensions given by Eq. (16). In this channel, both fermions and bosons have real solutions only; $\gamma = 0.833, 3.15, 4.87, \dots$ for $\lambda = -1$, $\gamma = 1.17, 2.85, 5.12, \dots$ for $\lambda = +1$, and $\gamma = 2.04, 5.53, 6.57, \dots$ for $\lambda = 4$. Therefore, the corresponding states in a harmonic potential are universal and their energies are given by Eq. (17) [23–25].

III. MANY-BODY PROBLEMS

Since bosons with the four-body resonant interaction in one dimension develop deep five-body bound states, the corresponding many-body system cannot be stable to collapse. Therefore, we will study the many-body physics of four-component Fermi gas in one dimension as a function of

the dimensionless parameter $k_F a$ characterizing the short-range four-body interaction. Here $k_F \equiv \pi n/4$ is the Fermi momentum defined by the total number density n . In analogy with the BCS-BEC crossover in three dimensions, we will refer to the weak (strong) coupling limit $k_F a \rightarrow -(+)0$ as the ‘‘BCS’’ (‘‘BEC’’) limit. We caution that this terminology should not be taken literally, since we do not have the spontaneous symmetry breaking in one dimension. We shall see below that the properties of our system in these two limits are consistent with the crossover hypothesis that they are smoothly connected without phase transitions.

A. BCS limit

The many-body physics is conveniently described by introducing the chemical potential term $-\mu \psi_\sigma^\dagger \psi_\sigma$ to the Hamiltonian density in Eq. (9). In the weak coupling (BCS) limit $k_F a \rightarrow -0$, the system develops two Fermi points $k = \pm k_F$ and low-energy degrees of freedom are excitations around them. Therefore, assuming $|k \mp k_F| \ll k_F$, we can linearize the dispersion relation and express the fermionic field in terms of two slowly varying fields describing excitations around the Fermi points:

$$\psi_\sigma(x) \simeq e^{ik_F x} \psi_\sigma^R(x) + e^{-ik_F x} \psi_\sigma^L(x). \quad (21)$$

The low-energy effective theory consistent with the original symmetries (10) can be written as

$$\begin{aligned} \mathcal{H}_{\text{BCS}} = & -i v_F \psi_\sigma^{R\dagger} \nabla \psi_\sigma^R + i v_F \psi_\sigma^{L\dagger} \nabla \psi_\sigma^L \\ & + g_1 \psi_\sigma^{L\dagger} \psi_\tau^{R\dagger} \psi_\tau^L \psi_\sigma^R + g_2 \psi_\sigma^{R\dagger} \psi_\tau^{L\dagger} \psi_\tau^L \psi_\sigma^R \\ & + \frac{g_4}{2} (\psi_\sigma^{R\dagger} \psi_\tau^{R\dagger} \psi_\tau^R \psi_\sigma^R + \psi_\sigma^{L\dagger} \psi_\tau^{L\dagger} \psi_\tau^L \psi_\sigma^L), \end{aligned} \quad (22)$$

where $v_F \equiv k_F/m$ is the Fermi velocity and summations over $\sigma(\tau) = a, b, c, d$ are implicitly understood. The g_1 term describes the backward scattering and the g_2 and g_4 terms describe the forward scatterings. The low-energy parameters g_1, g_2 , and g_4 are determined by matching two-body scattering amplitudes at the Fermi points with those from the microscopic theory (9). To leading order in $k_F a$, we find

$$g_1 = g_2 = g_4 = -\frac{4v_F}{\pi} k_F |a| + O[(k_F a)^2]. \quad (23)$$

The spectrum of the low-energy effective theory \mathcal{H}_{BCS} can be obtained exactly by bosonization. We introduce charge current operators

$$J_0^{\text{R(L)}} \equiv \psi_\sigma^{\text{R(L)\dagger}} \psi_\sigma^{\text{R(L)}} \quad (24)$$

and spin current operators

$$J_\alpha^{\text{R(L)}} \equiv \psi_\sigma^{\text{R(L)\dagger}} (t_\alpha)_{\sigma\sigma'} \psi_{\sigma'}^{\text{R(L)}}, \quad (25)$$

where t_α with $\alpha = 1, \dots, 15$ are generators of SU(4) Lie algebra normalized as $\text{Tr}(t_\alpha t_\beta) = \delta_{\alpha\beta}/2$. Using these current operators, \mathcal{H}_{BCS} can be separated into two mutually commuting parts (spin-charge separation) [29,30]; $\mathcal{H}_{\text{BCS}} = \mathcal{H}_{\text{ch}} + \mathcal{H}_{\text{sp}}$ with

$$\mathcal{H}_{\text{ch}} = \frac{2\pi v_F + 3g_4}{8} (J_0^R J_0^R + J_0^L J_0^L) + \frac{4g_2 - g_1}{4} J_0^R J_0^L \quad (26)$$

and

$$\mathcal{H}_{\text{sp}} = \sum_{\alpha=1}^{15} \left[\frac{2\pi v_F - g_4}{5} (J_\alpha^R J_\alpha^R + J_\alpha^L J_\alpha^L) - 2g_1 J_\alpha^R J_\alpha^L \right]. \quad (27)$$

The charge part \mathcal{H}_{ch} is easily diagonalized by the Bogoliubov transformation and equivalent to the Tomonaga-Luttinger liquid. Introducing a bosonic field $\varphi_0(x)$ by

$$\partial_x \varphi_0 \equiv \frac{\pi}{2} (J_0^R + J_0^L) \quad (28)$$

and its canonical conjugate by

$$\Pi_0 \equiv -\frac{1}{2} (J_0^R - J_0^L), \quad (29)$$

the Hamiltonian density can be brought into the standard form

$$\mathcal{H}_{\text{ch}} = \frac{\pi K v_s}{2} \Pi_0^2 + \frac{v_s}{2\pi K} (\partial_x \varphi_0)^2, \quad (30)$$

which describes a gapless excitation transporting a particle number with the linear dispersion relation $E = \pm v_s k$. Here the Tomonaga-Luttinger parameter K and the sound velocity v_s in the BCS limit $k_F a \rightarrow -0$ are given by

$$K = \sqrt{\frac{2\pi v_F + 3g_4 - 4g_2 + g_1}{2\pi v_F + 3g_4 + 4g_2 - g_1}} \rightarrow 1 + \frac{6k_F |a|}{\pi^2} \quad (31)$$

and

$$v_s = \sqrt{\left(v_F + \frac{3g_4}{2\pi} \right)^2 - \left(\frac{4g_2 - g_1}{2\pi} \right)^2} \rightarrow v_F \left(1 - \frac{6k_F |a|}{\pi^2} \right). \quad (32)$$

We note that the relationship $K v_s = v_F$ is guaranteed by Galilean invariance [31]. The Tomonaga-Luttinger parameter also appears in other physical observables [29,32], for example, in the compressibility

$$\kappa \equiv \frac{\partial n}{\partial \mu} = \frac{4K}{\pi v_s} \quad (33)$$

and the long-distance asymptotics of correlation functions [see Eqs. (44) and (45) below].

On the other hand, the coupling $g_1 < 0$ in the spin part \mathcal{H}_{sp} is marginally relevant and thus opens up gaps in the spectrum. This can be seen by studying the renormalization group flows of the couplings in Eq. (22). The straightforward one-loop calculations result in

$$\frac{dg_1}{ds} = -\frac{4g_1^2}{2\pi v_F}, \quad \frac{dg_2}{ds} = -\frac{g_1^2}{2\pi v_F}, \quad \frac{dg_4}{ds} = 0, \quad (34)$$

where $\Lambda_s = e^{-s} \Lambda_0$ is the momentum scale at which the couplings $g_i(s)$ are defined. g_4 and $4g_2 - g_1$ are exactly marginal as is consistent with the fact that Eq. (26) can be diagonalized, while g_1 evolves as

$$g_1(s) = \frac{1}{\frac{1}{g_1(0)} + \frac{4s}{2\pi v_F}}. \quad (35)$$

When $g_1(0) < 0$, $g_1(s)$ reaches the Landau pole at $s = -\frac{2\pi v_F}{4g_1(0)} = \frac{\pi^2}{8k_F |a|}$, indicating that the second term in Eq. (27) develops spin gaps whose magnitude is set by

$$\Delta \sim v_F \Lambda_s \sim v_F k_F e^{-\pi^2/(8k_F |a|)}. \quad (36)$$

The exact gap spectrum can be obtained from the Bethe-ansatz solution if we recognize \mathcal{H}_{sp} as the non-Abelian part of the SU(4) chiral Gross-Neveu model [33–35]:

$$\Delta_f \propto v_F k_F e^{-\pi^2/(8k_F|a|)} \sin\left(\frac{f\pi}{4}\right). \quad (37)$$

Here $f = 1, 2, 3$ is a number of excited fermions, and accordingly, there are three distinct gaps which are exponentially small in the BCS limit $k_F a \rightarrow -0$. The degeneracy of Δ_1 and Δ_3 can be traced back to an accidental particle-hole symmetry in \mathcal{H}_{BCS} under $\psi_\sigma^{\text{R(L)}} \leftrightarrow \psi_\sigma^{\text{R(L)\dagger}}$. This symmetry is broken by quadratic derivative terms $\psi_\sigma^{\text{R(L)\dagger}} \nabla^2 \psi_\sigma^{\text{R(L)}} / (2m)$ neglected in \mathcal{H}_{BCS} . Because the characteristic momentum scale is set by $\Lambda_s \sim k_F e^{-\pi^2/(8k_F|a|)}$, the small splitting of the degeneracy is estimated to be

$$\Delta_3 - \Delta_1 \sim \frac{\Lambda_s^2}{2m} \sim v_F k_F e^{-\pi^2/(4k_F|a|)}. \quad (38)$$

B. BEC limit

In the strong coupling (BEC) limit $k_F a \rightarrow +0$, four fermions with all different components form a tightly-bound tetramer and thus the many-body system will be a dilute Bose gas of such tetramers. In this limit, fermionic excitations are largely gapped because of the binding energy of the tetramer $E_0 = -1/(2ma^2)$. The gap spectrum with the fermion number $f = 1, 2, 3$ is simply given by

$$\Delta_f \rightarrow \frac{f}{8ma^2}. \quad (39)$$

Interestingly, we find that the ordering of the three spin gaps is $\Delta_1 < \Delta_2 < \Delta_3$ in the BEC limit while it is $\Delta_1 \simeq \Delta_3 < \Delta_2$ in the BCS limit [see Eqs. (37) and (38)]. Therefore, there has to be a crossing between two gaps Δ_2 and Δ_3 as a function of $-\infty < (k_F a)^{-1} < \infty$ in the BCS-BEC crossover.

The dilute Bose gas of tetramers to leading order in $k_F a$ is described by the Hamiltonian density

$$\mathcal{H}_{\text{BEC}} = -\frac{\phi^\dagger \nabla^2 \phi}{2M} - \frac{1}{M a_{tt}} \phi^\dagger \phi^\dagger \phi \phi, \quad (40)$$

where $M \equiv 4m$ is the tetramer mass and the tetramer density is $n_t \equiv n/4$. a_{tt} is a tetramer-tetramer scattering length (analogous to the dimer-dimer scattering length in three dimensions [36]) characterizing the scattering of two tetramers in one dimension. Because the scattering length a is the only scale of the system in vacuum, a_{tt} has to be proportional to a :

$$a_{tt} = -\eta a. \quad (41)$$

The coefficient η is a universal number obtained by solving the eight-body problem of fermions nonperturbatively. η is expected to be positive because tetramers should repel each other due to the fermionic statistics of the constituents. If $\eta > 0$, then the many-body system of tetramers is stable. Here we shall assume $\eta > 0$ and leave the determination of the exact value of η as a future problem.

The effective theory of tetramers \mathcal{H}_{BEC} is nothing but bosons with a δ -function interaction in one dimension. In contrast to the dilute Bose gas in three dimensions, that in one dimension is strongly interacting because the tetramer-tetramer coupling in Eq. (40) is inversely proportional to the

scattering length. As a consequence, the tetramers in the limit $k_F a \rightarrow +0$ behave as noninteracting spinless “fermions” and the thermodynamic properties of our system in the BEC limit are equivalent to those of a noninteracting Fermi gas with the same mass M and density n_t [1]. Beyond such a hard-core limit, the ground-state energy and the excitation spectrum of \mathcal{H}_{BEC} have been obtained exactly in Ref. [37]. In particular, its low-energy physics is described by the Tomonaga-Luttinger liquid (30) again [31]. The Tomonaga-Luttinger parameter K and the sound velocity v_s in the BEC limit $k_F a \rightarrow +0$ are given by

$$K \rightarrow 4(1 - 2n_t a_{tt}) = 4 \left(1 + \eta \frac{2k_F a}{\pi} \right) \quad (42)$$

and

$$v_s \rightarrow \frac{\pi n_t}{M} (1 + 2n_t a_{tt}) = \frac{v_F}{4} \left(1 - \eta \frac{2k_F a}{\pi} \right). \quad (43)$$

In the expression for K , we have taken into account the fact that the particle number of a tetramer is four [38]. The BCS-BEC crossover hypothesis indicates that K in Eqs. (31), (42) and v_s in Eqs. (32), (43) are smoothly connected, and therefore, there has to be a maximum (minimum) in K (v_s) as a function of $-\infty < (k_F a)^{-1} < \infty$.

The Tomonaga-Luttinger parameter determines the long-distance asymptotics of correlation function. Because the spin degrees of freedom are gapped in the BCS-BEC crossover, only SU(4) singlet operators can have quasi-long-range orderings. Two such examples are the density-density correlation function:

$$\langle \delta n(x) \delta n(0) \rangle |_{x \rightarrow \infty} \rightarrow -\frac{2K}{\pi^2 x^2} + A \frac{\cos(2k_F x)}{|k_F x|^{K/2}} + \dots \quad (44)$$

and the tetramer-tetramer correlation function:

$$\langle \phi(x) \phi^\dagger(0) \rangle |_{x \rightarrow \infty} \rightarrow \frac{B}{|k_F x|^{2/K}} + \dots, \quad (45)$$

where A, B are unknown parameters and both $x \gg k_F^{-1}$ and $x \gg v_F/\Delta_f$ are assumed. We can see that the $2k_F$ -charge-density wave is the dominant order for $K \lesssim 2$ (BCS side), while the tetramer quasicondensation is the dominant order for $K \gtrsim 2$ (BEC side), and there is a crossover in between. We note that these correlation functions have been studied in the context of spin-3/2 Fermi gases with two-body interactions [17,39–43].

C. Unitarity limit

It would be difficult to compute K and v_s away from the BCS or BEC limit. However, in the unitarity limit $k_F a \rightarrow \infty$, we can derive exact relationships between K and thermodynamic quantities. Because the density n is the only scale of the system, the ground-state energy density of the unitary Fermi gas can be written as

$$\mathcal{E}_{\text{unitary}}(n) \equiv \xi \mathcal{E}_{\text{free}}(n), \quad (46)$$

where the ground-state energy density of a noninteracting Fermi gas is

$$\mathcal{E}_{\text{free}}(n) = \frac{\pi^2}{96m} n^3. \quad (47)$$

Here ξ , which measures how much energy is gained due to the attractive interaction, is a universal number to characterize the strongly interacting unitary Fermi gas and analogous to the Bertsch parameter in three dimensions [13]. From the thermodynamic relationships, we obtain the pressure as $P(n) = 2\mathcal{E}(n)$, and thus, the sound velocity is given by

$$v_s^2 = \frac{1}{m} \frac{\partial P}{\partial n} = \xi v_F^2. \quad (48)$$

Because $K v_s = v_F$ is guaranteed by Galilean invariance [31], we find that the Tomonaga-Luttinger parameter is related to the one-dimensional Bertsch parameter by

$$K = \frac{1}{\sqrt{\xi}}. \quad (49)$$

This relationship implies $K > 1$ in the unitarity limit because $0 < \xi < 1$ is expected for the attractive interaction. It is a challenging many-body problem to determine the exact value of ξ .

One possible way to estimate the value of ξ is to use the ϵ expansion [19,44]. Considering the same Hamiltonian density (9) in an arbitrary spatial dimension d , we find that the dimension of the coupling constant c_0 is given by $\nu = 2 - 3d$, which also determines the behavior of the four-body wave function at a short distance:

$$\Psi(|\mathbf{r}| \rightarrow 0) \rightarrow |\mathbf{r}|^\nu. \quad (50)$$

In the limit $d \rightarrow 2/3$, we have $\nu \rightarrow 0$ and the singularity in the wave function disappears. This means that the contact interaction among four fermions disappears and thus the unitary Fermi gas reduces to a noninteracting Fermi gas [45]. On the other hand, in the limit $d \rightarrow 4/3$, we have $\nu \rightarrow -2$ so that the normalization integral of the wave function

$$\int d^{3d} \mathbf{r} |\Psi(\mathbf{r})|^2 \sim \int_0^\infty dr r^{3-3d} \quad (51)$$

diverges at the origin $|\mathbf{r}| \rightarrow 0$. This means that four fermions behave as a pointlike composite boson and thus the unitary Fermi gas reduces to a noninteracting Bose gas of such tetramers [45,46]. Therefore, ξ defined as in Eq. (46) is found to be

$$\xi|_{d \rightarrow \frac{2}{3}} \rightarrow 1 \quad \text{and} \quad \xi|_{d \rightarrow \frac{4}{3}} \rightarrow 0. \quad (52)$$

It is possible to formulate appropriate perturbation theories around these critical dimensions [47]. Interpolations of two systematic expansions in terms of $\bar{\epsilon} = d - \frac{2}{3}$ and $\epsilon = \frac{4}{3} - d$ would provide a reasonable estimate of ξ in $d = 1$ as in three dimensions [19,44].

D. Exact relationships

Another characteristic of our system (9) that resembles the BCS-BEC crossover in three dimensions is the large-momentum tail of the momentum distribution of fermions and its relationships to other properties of the system [48]. In

order to see this, we consider the following operator product expansion (no sum over $\sigma = a, b, c, d$):

$$\begin{aligned} & \psi_\sigma^\dagger\left(x - \frac{y}{2}\right) \psi_\sigma\left(x + \frac{y}{2}\right) \\ &= \psi_\sigma^\dagger \psi_\sigma(x) + \frac{y}{2} \psi_\sigma^\dagger \overleftrightarrow{\nabla} \psi_\sigma(x) \\ & \quad - \frac{\sqrt{3}|y|}{8\pi} (mc_0)^2 \psi_a^\dagger \psi_b^\dagger \psi_c^\dagger \psi_d^\dagger \psi_d \psi_c \psi_b \psi_a(x) + O(y^2). \end{aligned} \quad (53)$$

This can be confirmed by evaluating expectation values of the both sides for a state consisting of four fermions with all different components [49]. The nonanalytic term $\sim |y|$ indicates that the momentum distribution of fermions

$$\rho_\sigma(k) \equiv \int dy e^{-iky} \left\langle \psi_\sigma^\dagger\left(x - \frac{y}{2}\right) \psi_\sigma\left(x + \frac{y}{2}\right) \right\rangle \quad (54)$$

falls off by a power of $|k| \rightarrow \infty$ as

$$\rho_\sigma(k) \rightarrow \frac{\sqrt{3}}{4\pi} \frac{\mathcal{C}}{k^2}. \quad (55)$$

The coefficient is given by the so-called contact density:

$$\mathcal{C} \equiv \langle (mc_0)^2 \psi_a^\dagger \psi_b^\dagger \psi_c^\dagger \psi_d^\dagger \psi_d \psi_c \psi_b \psi_a \rangle. \quad (56)$$

The same result can be obtained by the method used in Ref. [50] (see Fig. 3).

From Eqs. (9), (12), and (56), we find that the energy density of the system $\mathcal{E} \equiv \langle \mathcal{H} \rangle$ can be expressed by

$$\mathcal{E} = \sum_\sigma \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{k^2}{2m} \left(\rho_\sigma(k) - \frac{\sqrt{3}}{4\pi} \frac{\mathcal{C}}{k^2} \right) + \frac{\mathcal{C}}{4\pi ma}. \quad (57)$$

This relationship is valid for any state of the system and for any value of the scattering length a . Derivations of the pressure

$$P = 2\mathcal{E} + \frac{\mathcal{C}}{4\pi ma}, \quad (58)$$

the adiabatic relationship

$$\frac{d\mathcal{E}}{da} = \frac{\mathcal{C}}{4\pi ma^2}, \quad (59)$$

and the generalized virial theorem in the presence of a harmonic potential $V_\omega = \int dx \frac{1}{2} m \omega^2 x^2 \sum_\sigma \psi_\sigma^\dagger(x) \psi_\sigma(x)$:

$$E = 2\langle V_\omega \rangle - \int dx \frac{\mathcal{C}(x)}{8\pi ma} \quad (60)$$

are straightforward from the above results by using the methods in Refs. [48,49].

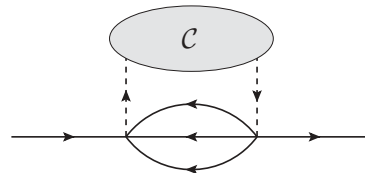


FIG. 3. Contribution of the contact density \mathcal{C} to the fermion propagator $iG_\sigma(k_0, k)$. The integration over k_0 leads to the momentum distribution in Eq. (55).

From the adiabatic relationship (59) together with the ground-state energy density in the BCS limit (up to the mean-field correction):

$$\mathcal{E}_{\text{BCS}} = \mathcal{E}_{\text{free}} + \frac{4\pi a}{m} \left(\frac{n}{4}\right)^4 + O[k_{\text{F}}^5 a^2] \quad (61)$$

and in the BEC limit [37]:

$$\mathcal{E}_{\text{BEC}} = E_0 n_t + \frac{\pi^2}{6M} n_t^3 + \frac{\pi^2}{3M} n_t^4 a_{tt} + O[k_{\text{F}}^5 a^2], \quad (62)$$

we find that the contact density \mathcal{C} is given by

$$\frac{\mathcal{C}}{k_{\text{F}}^2} \rightarrow \frac{16k_{\text{F}}^2 a^2}{\pi^2} \quad (k_{\text{F}} a \rightarrow -0) \quad (63)$$

and

$$\frac{\mathcal{C}}{k_{\text{F}}^2} \rightarrow \frac{4}{k_{\text{F}} a} - \eta \frac{k_{\text{F}}^2 a^2}{3\pi} \quad (k_{\text{F}} a \rightarrow +0). \quad (64)$$

The BCS-BEC crossover hypothesis indicates that both \mathcal{E} and \mathcal{C} smoothly evolve as functions of $-\infty < (k_{\text{F}} a)^{-1} < \infty$.

References [48,49] have also shown that the contact density in three dimensions is related to the local pair density, which is the number of pairs of spin-up and -down fermions with small separations. Similarly, the contact density in our one-dimensional system is related to the local quadruplet density $\mathcal{N}_4(R)$, which is the number of sets of four-component fermions with small hyperradii. This can be seen from the following operator product expansion:

$$\begin{aligned} & \psi_a^\dagger \psi_a(x_a) \psi_b^\dagger \psi_b(x_b) \psi_c^\dagger \psi_c(x_c) \psi_d^\dagger \psi_d(x_d) \\ &= \frac{(m c_0)^2 \psi_a^\dagger \psi_b^\dagger \psi_c^\dagger \psi_d^\dagger \psi_a \psi_b \psi_c \psi_d(X)}{16\pi^2 (r_1^2 + r_2^2 + r_3^2)} + O(|\mathbf{r}|^{-1}). \end{aligned} \quad (65)$$

The integral of the left-hand side over the three relative coordinates $|\mathbf{r}| < R$ counts the number of sets of four-component

fermions at the fixed center-of-mass coordinate X but with the hyperradius smaller than R . Therefore, we find that the short-distance asymptotics of the local quadruplet density is related to the contact density by

$$\mathcal{N}_4(R \rightarrow 0) \rightarrow \frac{\mathcal{C}}{4\pi} R. \quad (66)$$

IV. CONCLUSIONS

In summary, we have demonstrated that the four-component Fermi gas in one dimension exhibits the one-dimensional analog of the BCS-BEC crossover as a function of the scattering length characterizing the short-range four-body interaction. We investigated the ground-state energy, the sound velocity, the gap spectrum, and the exact relationships in the BCS-BEC crossover and found that the gap spectrum has the rich structure because of the existence of three distinct gaps. We also showed that the one-dimensional analog of the Efimov effect occurs for five bosons while it is absent for fermions. This work extends our perspectives on the universal few-body and many-body physics to one dimension and possibly opens up a very rich research area. Finally, we note that the system considered in this paper is highly fine-tuned: not only the four-body interaction is tuned to the resonance, two-body and three-body interactions have to be tuned to vanish. Its experimental realization would be challenging.

ACKNOWLEDGMENTS

The authors thank Shina Tan for discussions. Y.N. was supported by MIT Department of Physics Pappalardo Program and DOE Office of Nuclear Physics under Grant No. DE-FG02-94ER40818. This work was supported, in part, by DOE Grant No. DE-FG02-00ER41132.

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$$\frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{dk_1 dk_2 dk_3}{1 - \cos k_1 \cos k_2 \cos k_3} = \frac{\Gamma(\frac{1}{4})^4}{4\pi^3}.$$

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- $$Z_{\pm}(p) = \lambda \int_{-\Lambda}^{\Lambda} \frac{dq}{2\pi} Z_{\pm}(q) \frac{4\pi}{-\frac{1}{a} + \sqrt{\frac{5q^2}{4} + 2m|E|}} \\ \times \int_{-\Lambda}^{\Lambda} \frac{dk_1 dk_2}{(2\pi)^2} \frac{1}{\frac{2p^2 + 2q^2 + pq}{3} + k_1^2 + k_2^2 + k_1 k_2 + m|E|},$$
- where $Z_{\pm}(-p) = \pm Z_{\pm}(p)$ correspond to the even- and odd-parity channels, respectively.
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$$\mathcal{A}(E, \mathbf{p})|_{a=\infty} = - \left(\frac{2\pi}{m} \right)^{\frac{3d}{2}} \frac{2^d}{\Gamma \left(1 - \frac{3d}{2} \right) \left[\frac{p^2}{8m} - E - i0^+ \right]^{\frac{3d}{2} - 1}}.$$

When $d \rightarrow 2/3$ or $d \rightarrow 4/3$, we find $\mathcal{A} \rightarrow 0$ corresponding to the noninteracting limit.

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$$\xi|_{d=\frac{2}{3}+\bar{\epsilon}} = 1 - \frac{81\bar{\epsilon}}{2^{1/3}\Gamma\left(\frac{1}{3}\right)^3} + O(\bar{\epsilon}^2).$$

Due to the large $O(\bar{\epsilon})$ correction, the naive extrapolation to $d \rightarrow 1$ does not work ($\xi \rightarrow -0.115$) and needs to be combined with the expansion in terms of $\epsilon = \frac{4}{3} - d$.

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