1. Complex Exponentials as Eigenfunctions of LTI Systems
2. Fourier Series representation of CT periodic signals
3. How do we calculate the Fourier coefficients?
4. Convergence and Gibbs’ Phenomenon
Portrait of Jean Baptiste Joseph Fourier

Image removed due to copyright considerations.
Desirable Characteristics of a Set of “Basic” Signals

a. We can represent large and useful classes of signals using these building blocks

b. The response of LTI systems to these basic signals is particularly simple, useful, and insightful

Previous focus: Unit samples and impulses

Focus now: Eigenfunctions of all LTI systems
The eigenfunctions $\phi_k(t)$ and their properties

(Focus on CT systems now, but results apply to DT systems as well.)

$\phi_k(t)$ → System → $\lambda_k \phi_k(t)$

Eigenfunction in $\rightarrow$ same function out with a “gain”

From the superposition property of LTI systems:

$$ x(t) = \sum_k a_k \phi_k(t) \rightarrow \text{LTI} \rightarrow y(t) = \sum_k \lambda_k a_k \phi_k(t) $$

Now the task of finding response of LTI systems is to determine $\lambda_k$. 
Complex Exponentials as the Eigenfunctions of any LTI Systems

\[ x(t) = e^{st} \rightarrow h(t) \rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \]

\[ = \left[ \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \right] e^{st} \]

\[ = H(s) e^{st} \]

\[ x[n] = z^n \rightarrow h[n] \rightarrow y[n] = \sum_{m=-\infty}^{\infty} h[m]z^{n-m} \]

\[ = \left[ \sum_{m=-\infty}^{\infty} h[m]z^{-m} \right] z^n \]

\[ = H(z) z^n \]
\begin{align*}
x(t) & \xrightarrow{e^{st}} h(t) \xrightarrow{H(s)e^{st}} y(t) \\
H(s) &= \int_{-\infty}^{\infty} h(t) e^{-st} \, dt \\
x(t) &= \sum_{k} a_{k} e^{s_{k} t} \quad \rightarrow \quad y(t) = \sum_{k} H(s_{k}) a_{k} e^{s_{k} t} \\
\text{DT:}
\quad x[n] & \xrightarrow{z_{k}^{n}} h[n] \xrightarrow{H(z_{k}) z_{k}^{n}} y[n] \\
H(z) &= \sum_{n=-\infty}^{\infty} h[n] z^{-n} \\
x[n] &= \sum_{k} a_{k} z_{k}^{n} \quad \rightarrow \quad y[n] = \sum_{k} H(z_{k}) a_{k} z_{k}^{n}
\end{align*}
What kinds of signals can we represent as “sums” of complex exponentials?

For Now: Focus on restricted sets of complex exponentials

CT: \( s = j \omega \) – purely imaginary, i.e., signals of the form \( e^{j \omega t} \)

DT: \( z = e^{j \omega} \), i.e., signals of the form \( e^{j \omega n} \)

\( \downarrow \)

CT & DT Fourier Series and Transforms

Magnitude 1

Periodic Signals
Fourier Series Representation of CT Periodic Signals

\[ x(t) = x(t + T) \quad \text{for all } t \]

- smallest such \( T \) is the fundamental period
- \( \omega_0 = \frac{2\pi}{T} \) is the fundamental frequency

\[ e^{j\omega t} \text{ periodic with period } T \iff \omega = k\omega_0 \]

\[ \Downarrow \]

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T} \]

- periodic with period \( T \)
- \( \{a_k\} \) are the Fourier (series) coefficients
- \( k = 0 \) DC
- \( k = \pm 1 \) first harmonic
- \( k = \pm 2 \) second harmonic
**Question #1:** How do we find the Fourier coefficients?

First, for simple periodic signals consisting of a few sinusoidal terms

Ex: \( x(t) = \cos 4\pi t + 2 \sin 8\pi t \)

Euler's relation (memorize!)

\[
\begin{align*}
\text{Ex: } x(t) &= \frac{1}{2} [e^{j4\pi t} + e^{-j4\pi t}] + \frac{2}{2j} [e^{j8\pi t} - e^{-j8\pi t}] \\
\omega_0 &= 4\pi \quad T = \frac{2\pi}{\omega_0} = \frac{2\pi}{4\pi} = \frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
a_0 &= 0 \quad \text{no dc component} \\
a_1 &= \frac{1}{2} \\
a_{-1} &= \frac{1}{2} \\
a_2 &= \frac{1}{j} \\
a_{-2} &= -\frac{1}{j} \\
a_3 &= 0 \\
a_{-3} &= 0 \\
\vdots
\end{align*}
\]
• For real periodic signals, there are two other commonly used forms for CT Fourier series:

\[ x(t) = a_0 + \sum_{k=1}^{\infty} [\alpha_k \cos k\omega_0 t + \beta_k \sin k\omega_0 t] \]

or

\[ x(t) = a_0 + \sum_{k=1}^{\infty} [\gamma_k \cos (k\omega_0 t + \theta_k)] \]

• Because of the eigenfunction property of \( e^{j\omega t} \), we will usually use the complex exponential form in 6.003.

- A consequence of this is that we need to include terms for both positive and negative frequencies:

\[ e^{jk\omega_0 t}, \quad e^{-jk\omega_0 t} \]

Remember

\[ \cos (k\omega_0 t) = \frac{1}{2}(e^{jk\omega_0 t} + e^{-jk\omega_0 t}) \]

and

\[ \sin (k\omega_0 t) = \frac{1}{2j}(e^{jk\omega_0 t} - e^{-jk\omega_0 t}) \]
Now, the complete answer to Question #1

Suppose

1) multiply by $e^{-jn\omega_0 t}$

2) integrate over one period

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

1) multiply by $e^{-jn\omega_0 t}$

2) integrate over one period

\[
\int_T x(t)e^{-jn\omega_0 t} dt = \int_T \left( \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) e^{-jn\omega_0 t} dt
\]

\[
= \sum_{k=-\infty}^{\infty} a_k \left( \int_T e^{j(k-n)\omega_0 t} dt \right)
\]

(Here $\int_T$ denotes integral over any interval of length $T$ (one period).)

Next, note that

\[
\int_T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}
\]

\[
= T\delta[k - n] \quad \text{Orthogonality}
\]
\[ \int_{-\infty}^{\infty} x(t) e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \left( \int_{-\infty}^{\infty} e^{j(k-n)\omega_0 t} dt \right) = \sum_{k=-\infty}^{\infty} a_k \cdot T \delta[k-n] \]

\[ \int_{-\infty}^{\infty} x(t) e^{-jn\omega_0 t} dt = a_n T \]

\[ \Downarrow \]

CT Fourier Series Pair \((\omega_0 = \frac{2\pi}{T})\)

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \] (Synthesis equation)

\[ a_k = \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt \] (Analysis equation)
Ex: Periodic Square Wave

For $k = 0$

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt = \frac{2T_1}{T}$$

DC component is just the average

For $k \neq 0$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt$$

$$= -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 T_1} \big|_{-T_1}^{T_1} = \frac{\sin k\omega_0 T_1}{k\pi} \left( \omega_0 = \frac{2\pi}{T} \right)$$

$T = 8T_1$
Convergence of CT Fourier Series

• How can the Fourier series for the square wave possibly make sense?
• The key is: What do we mean by

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t} \]

• One useful notion for engineers: there is no energy in the difference

\[ e(t) = x(t) - \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t} \]

\[ \int_{T} |e(t)|^2 dt = 0 \]

(just need \( x(t) \) to have finite energy per period)

\[ \int_{T} |x(t)|^2 dt < \infty \]
Under a different, but reasonable set of conditions (the Dirichlet conditions)

Condition 1. \( x(t) \) is *absolutely integrable* over one period, i.e.
\[
\int_{T} |x(t)| dt < \infty
\]

And

Condition 2. In a finite time interval, \( x(t) \) has a *finite* number of maxima and minima.

Ex. An example that violates Condition 2.
\[
x(t) = \sin \left( \frac{2\pi}{t} \right) \quad 0 < t \leq 1
\]

And

Condition 3. In a finite time interval, \( x(t) \) has only a *finite* number of discontinuities.

Ex. An example that violates Condition 3.
• Dirichlet conditions are met for the signals we will encounter in the real world. Then

- The Fourier series $= x(t)$ at points where $x(t)$ is continuous

- The Fourier series = “midpoint” at points of discontinuity

• Still, convergence has some interesting characteristics:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- As $N \to \infty$, $x_N(t)$ exhibits Gibbs’ phenomenon at points of discontinuity

**Demo:** Fourier Series for CT square wave (Gibbs phenomenon).