Holographic Superconductor with Multiple Competing Condensates

by

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Abstract

Holography is a novel approach to modeling strongly interacting many-body systems. By reorganizing the quantum many-body problem into an equivalent problem in classical gravity, holography makes it relatively easy, for example, to study linear response and compute transport coefficients. These techniques have recently been used to build toy models of superconductivity known as "Holographic Superconductors". In this thesis, we will be applying these same holographic principles to a strongly-interacting superconducting system in which multiple condensates compete. In the gravitational description, our system begins with two non-interacting Abelian Higgs multiplets, each comprised of a vector and a charged scalar. This describes a system with two independent condensates. Coupling the scalars via a quartic interaction induces a competition between the two condensates, with a condensate of one operator suppressing or enhancing the other depending on the sign of the coupling.

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Chapter 1

Introduction

One of the reoccurring issues in physics is the problem of describing systems that are strongly coupled. For weakly coupled system, where small perturbations lead to nearly negligible changes, one has many methods, based on perturbation theory, to calculate the state of the system. These methods are no longer applicable to strongly coupled systems, where the same small perturbations result in large changes to the state of the system. Exploring strongly coupled systems is particularly challenging in the context of quantum field theories where perturbation theory, also called Feynman diagrammatics, is often the only simple tool available.

This matters because many of the systems we care about – from nuclear physics to high-temperature superconductivity – are strongly coupled many-body systems governed by strongly-coupled quantum field theories at finite density and temperature. Over the years, numerous strategies and techniques have been developed (large-N, the ϵ-expansion, etc) to explore various specific strongly coupled systems; the cost of this progress it that these techniques tend to be difficult to implement and are often tightly wed to specific systems. For example, Lattice QCD is a non-perturbative technique used to study various thorny problems in quantum chromodynamics, the strongly-interacting theory of quarks and gluons. While a powerful tool for various problems, Lattice QCD is extremely computationall demanding and is poorly suited to several of the most interesting physical questions about QCD, such as transport properties of the quark-gluon plasma.

Holography is a novel approach to modeling strongly interacting many-body systems. By reorganizing the quantum many-body problem into an equivalent problem in classical gravity (as will be described in more detail below), holography makes it relatively easy to study linear response and compute transport coefficients in a host of strongly-coupled many-body systems. The drawback to this method is that, at this point, holographic methods are only applicable to very specific systems: we do not yet have a general rule for mapping a given quantum field theory into a specific gravitational system.

The equivalence between a system without gravity and a dual description with gravity may be motivated in the following way. Consider a set of electrons falling into a black hole in 3+1 dimensions. To an observer co-moving with the electrons as they approach the horizon, the appropriate description of the system is of particles interacting with gravity (they are all falling toward the center of the black hole) in 3+1 dimensions. However, to an asymptotically distant observer who does not fall toward the black hole, the situation looks very different: as the electrons approach the horizon, they appear to slow down, and indeed the distant observer never sees the electrons actually fall into the horizon. Rather, what they see is the electrons freezing onto the horizon as their wavefunctions spread over the horizon. The salient description to the asymptotic observer, then, is of particles confined to the 2-dimensional black hole horizon and not falling into the black hole – ie, not responding to gravity. Since these two very different descriptions are both valid representations of the same physical process, they must encode the same information.

One may think intuitively of this correspondence in terms of a hologram. A hologram is a 2-dimensional representation of a 3-dimensional object; this 2-dimensional representation retains all the information that is present in the original 3-dimensional object. In the same way, all the information present in our 2+1 dimensional particle system without gravity is also present in the 3+1 dimensional gravitational description, and vice versa. This is what we mean by Holography.

The AdS/CFT Correspondence is a precise version of this holographic equivalence in which a strongly-interacting quantum field theory in d dimensions is related, in a very precise way, to weakly-coupled classical gravity expanded around Anti-de Sitter space (AdS). Roughly speaking, AdS can be thought of as a covariant box for gravity - it is homogeneous and isotropic, but has constant curvature such that anything you throw away from you in AdS will, if it follows a geodesic, return to you in finite time (hence box: nothing can get away). It is useful to describe the "points at infinity" as "the boundary" and all points at finite coordinate position as "the bulk". Thus, our asymptotic observer, who described the electrons using a 2+1 QFT, would be living on the boundary, while the gravitational description is appropriate to observers moving inertially in the bulk of the 3+1 AdS spacetime.

In this correspondence, the temperature of the QFT is mapped to the temperature of a black hole placed in the center of the AdS spacetime. The temperature of the black hole is due to Hawking Radiation, a quantum effect which arises from the impossibility of defining invariant creation and annihilation operators.[1]. A black hole that is only radiating and not absorbing will eventually evaporate, as its energy is being radiated away. As AdS spacetime acts like a box, any energy radiated away from the black hole returns back to it. Effectively, AdS spacetime allows us to put our black hole and entire gravity theory in a black body cavity where the walls are in thermal equilibrium with the black hole. Restricting ourselves to AdS spacetime thus not only allows us to heat the boundary of our spacetime but also allows us to do so in a time-independent manner by putting the black hole and the wall in thermal equilibrium.

As mentioned, the connection between the bulk gravity and the boundary quantum field theory is precise. The basic equivalence is between the partition functions for the bulk and boundary theories, with the sources of operators $\mathcal{O}$ in the boundary theory given by the asymptotic values of the bulk fields ψ,

$$Z_{\text{bulk}}\Big|_{\psi\to\delta\psi_{(0)}} = \langle \exp\Big(i \int d^d x \sqrt{-g_{(0)}}\delta\psi_{(0)}\mathcal{O} \Big)\rangle_{\text{F.T.}} \tag{1.1}$$

where Z is the partition function, ψ is a dynamical field in the bulk, $\delta\psi_{(0)}$ is its

[1]See Appendix for an explanation of Hawking Radiation

value at the boundary, $g_{(0)}$ is the value of the metric on the boundary and $\mathcal{O}$ is the boundary operator. [4] For an asymptotically AdS spacetime, we can calculate the falloff for the scalar field and its expectation value. For these conditions, the equation of motion for small z, where the boundary lies at $z = 0$, is

$$z^2\partial_z^2\psi - (d-1)z\partial_z\psi = m^2\psi \tag{1.2}$$

where m^2 is the mass of the scalar field, due to the potential $V(z) = \frac{1}{2}m^2\psi^2 \ldots$. From this, we find that the falloff behavior of the scalar field goes as

$$\psi = z^{d-\Delta}\psi_{(0)} + z^d\psi_{(1)} + \cdots \tag{1.3}$$

where Δ solves the equation

$$\Delta(\Delta - d) = m^2. \tag{1.4}$$

In AdS/CFT, we generally have that $\phi_{(0)}$ is non-normalizable and $\phi_{(1)}$ is normalizable. We consider $\phi_{(0)}$ to be the source of our field and $\phi_{(1)}$ to be the expectation value.

From the form of this falloff, we wish to calculate the vacuum expectation value of the operator in the specific case where $\psi_{(0)}$ is non-normalizable and $\psi_{(1)}$ is normalizable. This will not be true in the specific case of holographic superconductors but we will detail the differences later. We start with the relation between an expectation value and the partition function and the action on the boundary:

$$\langle\mathcal{O}\rangle = -i\frac{\partial_z Z_{\text{Bulk}}(\psi_0)}{\partial\psi_0} = \frac{\partial_z S(\psi_0)}{\partial\psi_0} \tag{1.5}$$

and

$$S_{\text{Bdy}} = -\int_{z\to 0} d^dx\sqrt{\gamma}\Big(\psi n^\mu\nabla_\nu\psi + \Delta\psi^2\Big) \tag{1.6}$$

where Z is the partition function, S is the action, γ is the induced metric on the boundary and n^μ is the outward pointing unit normal vector to the boundary. In writing the second part of Eq. 1.5, we have taken the semi-classical limit $N \to \infty$ in order to write $Z_{\text{Bulk}} = e^{iS}$. Inserting Eq. 1.3 into Eqs. 1.5 and 1.6, we find that

$$\langle \mathcal{O} \rangle = \frac{2\Delta - d}{L} \psi_{(1)} \tag{1.7}$$

where

$$\dim[\mathcal{O}] = \Delta. \tag{1.8}$$

This is an important result for $\phi_{(0)}$ non-normalizable and $\phi_{(1)}$ normalizable and will guide us through our calculations for holographic superconductors.

Superconductors are conductors that, below a certain temperature, have a zero resistance. We can look to Landau's explanation about superfluids, a fluid with zero energy dissipation, for guidance in understanding superconductors. In this argument, we find that the dispersion relation for the fluid is linear. Using the fact that massless particles have linear dispersion relationship and Goldstone's Theorem, we can conclude that superconductivity, which is the electromagnetic correspondent to superfluidity, is caused by spontaneous symmetry breaking.[2]

Before we move on to discuss the effect of the spontaneous symmetry break in the bulk, a word about superfluids versus superconductors. Superconductors are electrodynamical systems described by Quantum Electrodynamics, which has local U(1) symmetry. However, AdS/CFT links gauged symmetries in the bulk with global symmetries in the boundary. Therefore, the system at which we actually will be looking is a superfluid, which has global, not local, U(1) symmetry. In order to look at a superconducting system on the boundary, we have to assume that the system is "weakly gauged", which is different than weakly coupled. In a "weakly-gauged" theory, the local symmetry is approximatively equivalent to the global symmetry. In our system, "weakly-gauged" will restrict our charge q to be small.

We now know what the onset of superconductivity will look like in a quantum system: as the system goes through the critical temperature, one of the fields will take on a non-zero vacuum expectation value and a spontaneous symmetry break

[2] See appendix for Landau's explanation for superfluidity and a discussion of the logic of this argument. Also included in the footnote is a discussion of the Abelian-Higgs mechanism, a simple examples of spontaneous symmetry breaking

will occur. Accompanying this change will be the creation of a massless Goldstone Boson. If we look back at Eq. 1.1, we see that there is a link between an operator in the field theory and a field in the gravity theory. When the global U(1) symmetry is broken on the boundary of our space, an operator condenses. Since this operator is related to a field in the bulk, when one condenses, so does the other; this leads to a symmetry break in the gauged U(1) symmetry in the bulk. The field in the bulk condenses on the edge of the event horizon of the black hole, which in the language of General Relativity, gives the black hole hair. Let us take a moment to discuss the phenomenon of hairy black holes. A theorem called the No-Hair Theorem states that, in an idealized, flat, homogeneous and isotropic 4-dimensional spacetime, a black hole may only be classified by its mass, charge and angular momentum. Said more simply, the vacuum expectation values of fields must vanish on the event horizon. The No-Hair Theorem does not apply to AdS spacetime; they are, however, unstable and thus usually energetically disfavored in comparison to bald black holes. Below a certain radius, this switches: a hairy black hole in AdS becomes energetically favored in comparison to a bald black hole. As temperature is proportional to the radius of a black hole, as we will see below, we have found that the hair on a black hole mirrors the condensate on the boundary: above T_c, is is not present, below T_c it is.

Previous work with holographic superconductors has proven the existence of a critical temperature in both non-backreacting and backreacting cases. It has also calculated the conductivity of the condensate with regards to the frequency of the perturbation, i.e. the current density. A lot of work remains to be done, as at this point there is no systematic was of predicting what gravity system will give the desired result. The main method of doing calculations is to take a system that has demonstrated superconducting properties and then add a small change to the system. If the superconducting behavior is still present, one can compare the two systems or compare the new system to experimental data.

In this thesis, we will take the original setup for a holographic superconductor with no back-reaction and have two condensates with two different types of charges. We will also include in the potential a coupling between the two scalar fields. Our

Lagrangian is

$$\mathcal{L} = R + 6 + \mathcal{L}_a + \mathcal{L}_b + V(|\psi_a|, |\psi_b|) \tag{1.9}$$

where

$$\mathcal{L}_i = -\frac{1}{4} F_{(i)}^{\ \mu\nu} F_{(i)\mu\nu} - |\psi_i - iq_i A_i \psi_i|^2 \tag{1.10}$$

and

$$V(|\psi_a|, |\psi_b|) = -(m^2|\psi_a|^2 + \mu^2|\psi_b|^2) + \lambda|\psi_a|^2|\psi_b|^2 \tag{1.11}$$

where $m^2 > 0$, $\mu^2 > 0$ and λ is any real number. Using this Lagrangian, we hope to calculate the critical temperature for different values of q_a, q_b and the coupling constant λ and show the interdependence of the two condensates, where the amount of interdependence depends on the sign and strength of the coupling.

Chapter 2

Background Work in Holographic Superconductors

Before looking at multiple condensates, let us first look at the simplest superconductor system, where we only have one charged complex scalar field and one vector field. In this system, we will assume that the fields themselves do not affect the spacetime metric; this is called the non-backreacting limit. [1] We will go through the process of calculating the phase diagram of the system in careful detail, in order to prepare ourselves for looking at more complicated superconducting systems.

The general process of solving our system will be as follows. We will first derive the equations of motion for our two fields, which are second order non-linear differential equations. In order to solve these equations, we will first expand and solve them around the boundary and the horizon; this will give us ψ_{Hor}, ϕ_{Hor}, ψ_{Bdy} and ϕ_{Bdy}. These four solutions will be in the form of a power series expansion. We will then numerically integrate the equations of motion, using ψ_{Hor}, ψ'_{Hor}, ϕ_{Hor} and ϕ'_{Hor} as the initial conditions. Finally, to find the boundary values of the two fields, we will set the numerical solutions of ψ and ϕ equivalent to the ψ_{Bdy} and ϕ_{Bdy} and their derivatives to find the value of the leading terms in ψ_{Bdy} and ϕ_{Bdy}.

2.1 Lagrangian and Equations of Motion

There are two approaches to AdS/CFT Correspondence. The first uses the string theory description of the system that includes all the dimensions and then truncates the theory into lower energy and dimension. The second approach, which is usually used with Holographic Superconductors is more phenomenological. In this approach, one uses one's intuition to guess at a Lagrangian to describe the system and then, applying holographic principles, checks whether the results describe a physical system of interest. We will be taking the phenomenological approach in this work.

When guessing at the Lagrangian, we will use the simplest Lagrangian that satisfies all of the requirements of the system. The system in which we are interested is a relativistic system comprised of a large number of charged particles that condenses at low temperature. The system must be Lorentz invariant if it is to behave relativistically. The stress energy tensor $T_{\mu\nu}$ is conserved, as we are working in the semi-classical limit. There also has to be a global U(1) symmetry on the boundary by the rules of AdS/CFT Correspodence; we, however, will be looking at a weakly gauged U(1) system. The global U(1) symmetry gives rise to a conserved current J^μ, by Noether's Theorem. We also need to be able to spontaneously break this symmetry in order to have a condensate on the boundary. Lastly, our Lagrangian must represent all of the fields and operators that are related by the holographic principle. The three operators on the boundary are the stress energy tensor $T_{\mu\nu}$, the current J^μ and the operator $\langle\mathcal{O}\rangle$, which are dual to the metric $g_{\mu\nu}$, the Maxwell vector field A_μ and the charged scalar field ψ in the bulk. The charged scalar field ψ is necessarily complex in order to allow for the Abelian-Higgs Mechanism. Putting all of these requirements together, we find the minimal Lagrangian density to be

$$\mathcal{L} = R + 6 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - |\psi - iqA\psi|^2 - V(|\psi|) \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, R is the Ricci scalar and we are free to choose $V(|\psi|)$. We are also setting the AdS length scale $L = 1$.

From this Lagrangian density, we derive three equations of motion by varying ψ,

A_μ, and $g_{\mu\nu}$. These equations are the scalar equation, found by varying ψ,

$$-(\nabla_\mu - iqA_\mu)(\nabla^\mu - iqA^\mu)\psi + \frac{1}{2}\frac{\psi}{|\psi|}V'(|\psi|) = 0, \tag{2.2}$$

Maxwell's equations, found by varying A_μ,

$$\nabla^\mu F_{\mu\nu} = iq[\psi^*(\nabla_\nu - iqA_\nu)\psi - \psi(\nabla_\nu + iqA_\nu)\psi^*] \tag{2.3}$$

and Einstein's equations, found by varying $g_{\mu\nu}$,

$$\begin{aligned} R_{\mu\nu} &- \frac{g_{\mu\nu}R}{2} - 3g_{\mu\nu} = \frac{1}{2}F_{\mu\alpha}F_\nu{}^\alpha - \frac{g_{\mu\nu}}{8}F^{\alpha\zeta}F_{\alpha\zeta} - \frac{g_{\mu\nu}}{2}V(|\psi|) - \frac{g_{\mu\nu}}{2}|\nabla\psi - iqA\psi|^2 \\ &+ \frac{1}{2}[(\nabla_\mu\psi - iqA_\mu\psi)(\nabla_\nu\psi^* + iqA_\nu\psi^*) + \mu \leftrightarrow \nu] \end{aligned} \tag{2.4}$$

2.2 Ansatz for Metric, Scalar and Vector Field

With the Lagrangian density defining our system, we now have to make a guess at the solution of the equations of motion. The form of our spacetime metric will help guide our ansatz for the scalar and vector field. The metric has to satisfy three important properties: it has to be asymptotically Anti-de Sitter, it must have a black hole in the center to allow our boundary to be heated and it has to transition from a bald black hole to a hairy black hole at T_c. The simplest metric that satisfies these conditions in 3+1 dimensions, is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(dx^2 + dy^2) \tag{2.5}$$

where

$$f(r) = r^2\left(1 - \frac{r_+^3}{r^3}\right); \tag{2.6}$$

r_+ is the radius of the black hole, also known as the location of the event horizon. Note that as we go to the boundary, $f(r) \to 0$, where the boundary is located at infinity. With no hair, this black hole is a Schwarzschild black hole in AdS spacetime.

This metric is called the Poincaré coordinate patch, which is not a complete covering of AdS space, as it only covers the spacetime for $z > 0$. Also, this patch is valid only when the length scale is large. One can intuitively see why this coordinate patch works by imaging AdS space as a sphere. The surface of a sphere is not identically flat. However, when the radius of the sphere is large, compared to the radius of the black hole, the surface of the sphere looks flat. The use of the Poincaré coordinate patch is only correct when the system is non-backreacting and not in the low temperature regime, as when we are at a low temperature, the horizon is near the boundary and we can no longer assume that the space spanned by x and y is flat.

Using Eqs. 2.5 and 2.6, we find that the temperature is

$$\begin{aligned} T &= \frac{3M^{1/3}}{4\pi} \\ &= \frac{3}{4\pi}r_+. \end{aligned} \tag{2.7}$$

In order to guess the form of our scalar and vector fields, we look at the symmetries of our metric. Since it is symmetric in x and y and we want time-independent solutions, ψ and A_μ should only depend on the radial coordinate. This gives us the vector potential

$$A = \phi(r)dt \tag{2.8}$$

as we only want the electric component of the vector field at this point in time. Looking at the r component of Eq. 2.3, we find that

$$0 = iq\left(\psi^*\partial_r\psi - \psi\partial_r\psi^*\right) \tag{2.9}$$

which gives us that the phase of ψ is constant. [1]. For simplicity, let us set ψ to be real. To summarize, our ansatz for the metric is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(dx^2 + dy^2) \tag{2.10}$$

[1]We know that the phase was time independent, as we are necessarily working on a time independent problem

where

$$f(r) = r^2\left(1 - \frac{r_+^3}{r^3}\right) \tag{2.11}$$

and our ansatz for the scalar and vector field is

$$\psi = \psi(r) \quad \text{and} \quad A = \phi(r)dt. \tag{2.12}$$

2.3 Simplified Equations of Motion and Potential

Using Eqs. 2.10, 2.11 and 2.12, we can simplify our equations of motion to

$$\begin{aligned} \psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\psi' + \frac{q^2\phi^2}{f^2}\psi - \frac{1}{2f}V'(\psi) &= 0 & (2.13) \\ \phi'' + \frac{2}{r}\phi' - \frac{2q^2\psi^2}{f}\phi &= 0 & (2.14) \end{aligned}$$

Notice that the ψ term in the equation of motion for ϕ has a mass term that has a r dependence. This is the mass term that indicates that spontaneous symmetry breaking has occured. We will take a simple form of the potential,

$$\begin{aligned} V(\psi) &= \tilde{m}^2\psi^2 \\ &= -m^2\psi^2 \end{aligned} \tag{2.15}$$

where $\tilde{m}$ is the conformal mass of the scalar field in AdS_4. We rewrite the potential in terms of m^2, as it is more natural to think of a squared number as positive. The choice of m will depend on two things. First, the potential has to lie above the Breitenlohner-Freedman bound for stability. Second, the form of the falloff depends on the mass and so we want to pick a mass such that our falloff is simple. For the moment, let us leave m as a free parameter; we will set it to a specific value after we talk about the scaling symmetries of our metric.

Before we continue onwards in our calculations, let us return once more to the question of how symmetry breaking occurs in our holographic system. Classically, mass is not allowed to be complex. Quantum mechanically, a positive mass indicates a stable solution and a negative mass indicates an unstable solution. For the moment, set $\phi = 0$ and look at Eq. 2.13:

$$\psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\psi' + \frac{m^2}{f}\psi = 0. \tag{2.16}$$

If we were in flat space, the solutions to this equation would be unstable. However, in curved spacetime, mass terms are allowed to be complex as long as they are above the Breitenlohner-Freedman bound for stability

$$m_{BF}^2 = -\frac{d^2}{4} \tag{2.17}$$

where d is the number of spacetime dimensions. As long as $\phi = 0$ and our mass terms is above the bound, we will have only stable solutions and therefore no condensate. As soon as we make $\phi \neq 0$, our second equation of motion turns on. In this case, the mass term is positive, which drives ϕ to stable solutions. However, this also turns on another negative mass term in our first equation of motion, thus driving our effective mass closer to the Breitenlohner-Freedman bound. These two equations act as a counter on each other that under the correct initial conditions, drive the ψ solution unstable and thus produces a non-zero vacuum expectation value.

2.4 Scaling Symmetries

The equations of motion have two scaling symmetries. They are

$$r \to ar, \quad t \to at, \quad L \to aL, \quad q \to q/a \tag{2.18}$$

which leads to a rescaling of the metric by a and the vector field A by a

$$r \to ar, \quad \{t, x, y\} \to \{t, x, y\}/a, \quad f \to a^2 f, \quad \phi \to \phi/a \tag{2.19}$$

which leaves both the metric and the vector field unchanged. Using the first scaling symmetry, we eliminated the length scale of the AdS spacetime; this is why the lenght scale L does not appear anywhere in these calculations. The second symmetry allows us to greatly simplify our calculations by setting $r_+ = 1$. Referring back to Eq. 2.7, notice that the temperature depends on the location of the horizon. Every time we wish to change the temperature of the system, we would have to change the value of r_+. However, using these symmetries, we can do all the calculations at one value of r_+ and then only look at dimensionless parameters when analyzing our results. For example, later in this section, we will present a graph of the value of the operator of mass dimension one and versus temperature. We will return to the second scaling to that point in time to demonstrate the use of this scaling in simplifying our calculations.

2.5 Expansion around Horizon and Boundary

In order to fully solve our equations of motion, we will expand and solve them around the horizon and boundary and then link the two solutions by numerically integrating the equations of motion from the bulk to the boundary. Since we have two second order differential equations, we should have a four parameter family on the horizon and boundary. This is not always true though; in the case of the horizon, due to physical and mathematical restrictions, we will only have a two parameter family

Let us first look at the horizon. There are two arguments for setting $\phi(r_+) = 0$. The first argues that A_μ has to normalizable, which means that it has to be zero on the horizon in order for $g^{\mu\nu}A_\mu A_\nu$ to be finite. However, A_μ is gauge-dependent and so a diverging vector potential is not automatically problematic. The second argument is that the source for the Maxwell equations in the bulk is guage invariant. Choosing a guage in which the scalar field ψ is real, the current is $\psi^2 A_\mu$. In order for the current to be finite on the horizon, so does A_μ. Using these arguments, we reduced our four parameter family to a three parameter family. Next, looking at ψ equation of motion and using the fact that $\phi(r_+) = 0$, we are left with

$$\psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\psi' + \frac{2}{L^2}\psi = 0. \tag{2.20}$$

There are two divergent terms in Eq. 2.20; these are the two terms that will dominate our equation of motion very close to the boundary. Luckily, both of them are inversely proportional to $f(r)$. From this we find the equation of motion very close to the horizon reduces to

$$\frac{3}{2L^2}\psi' + \psi = 0 \tag{2.21}$$

which is a first order differential equation; therefore, we only have one parameter for ψ on the boundary. In total, for the horizon, we have a two parameter family.

Moving onto the boundary, we will repeat the above procedure for ψ and ϕ. In calculating the boundary expansion, let us finally set the value of m to a definite value to fix the form of our falloffs. To solve the equations of motion for ψ and ϕ, we will have a four parameter family, as there is no physical or mathematical reason to reduce the number of parameters. Let us look at the ψ equation of motion to describe the procedure for calculating the form of the falloff. Since we know that we have a second order differential equation, we know that our solution is going to be a linear combination of solutions. We can assume that it will be of the form of

$$\psi(r) = \frac{\psi^{(1)}(r)}{r^\alpha} + \frac{\psi^{(2)}(r)}{r^\beta} \tag{2.22}$$

where α and β are non-negative constants, in order to have no divergent terms, and $\psi^{(1)}(r)$ and $\psi^{(2)}(r)$ are both normalizable functions. If we plug in Eq. 2.22 into our equation of motion for ψ, we find that the leading term is

$$\frac{1}{r^{\alpha-2}}(m^2 - 3\alpha + \alpha^2); \tag{2.23}$$

we find the same leading relation for β. As we want our equation of motion to be zero, this leading term must be zero; therefore, we find a restriction of α in terms of the mass:

$$\alpha = \frac{1}{2}(3 \pm \sqrt{9 - 4m^2}). \tag{2.24}$$

Since we want the simplest falloff possible, we choose $m^2 = -2$ and find that $\alpha = 1$ and $\beta = 2$. Notice that this agrees with Eq. 1.4 for $m^2 = -2$. Our potential is now

$$V(\psi) = -2\psi^2 \tag{2.25}$$

and our scalar field falloff is

$$\psi(r) = \frac{\psi^{(1)}}{r} + \frac{\psi^{(2)}}{r^2} \cdots \tag{2.26}$$

where $\psi^{(1)}$ and $\psi^{(2)}$ are the leading terms of the expansion for $\psi^{(1)}(r)$ and $\psi^{(2)}(r)$. Repeating for the vector field, we find that the falloff of ϕ goes as

$$\phi(r) = \mu - \frac{\rho}{r} + \cdots \tag{2.27}$$

where μ is the chemical potential and ρ is the charge density. The identification of μ and ρ and the chemical potential and the charge density arises from looking at the action. The component of th action on the boundary that depends on the vector field is

$$\begin{aligned} \Delta S &\approx \int_\partial A^\partial_\mu J^\mu \\ &\approx \int_\partial A^\partial_t J^t \\ &\approx \mu \int_\partial J^t \\ &\approx \mu Q \end{aligned} \tag{2.28}$$

where Q is the total charge of the system. We only expanded A_t to leading order and, assuming a homogeneous system, pull μ in front of the integral and integrate to get the total charge enclosed. Our result, $S = \mu Q$, tells us that μ has to be the chemical potential. A similar argument will lead to the conclusion that ρ is the charge density.

Returning back to our earlier derivation of the boundary operators in AdS/CFT Correspondence, we now have that both of $\psi^{(1)}$ and $\psi^{(2)}$ are normalizable, which leads to issues with quantizing the field in the bulk. Mainly, with two normalizable fields, one can use either the fields themselves or their canonical momenta to define creation and annihilation operators. Therefore, both quantizations must be considered equally valid. Our two cases are that $\psi^{(1)}$ is the source and $\psi^{(2)}$ is the expectation value or vice versa. Regardless of which we set as the expectation value, the source field has to be zero on the boundary, as we want the source of the condensate to be from the bulk, not the boundary.[3] Therefore, the two sets of solutions we are going to find are

$$\mathcal{O}^{(1)} = \sqrt{2}\psi^1 \quad \text{and} \quad \mathcal{O}^{(2)} = 0$$

or

$$\mathcal{O}^{(2)} = \sqrt{2}\psi^2 \quad \text{and} \quad \mathcal{O}^{(1)} = 0$$

2.6 Phase Diagram

We have used the scaling symmetries expressed in Eq. 2.19 to study our system at $r_+ = 1$ and thus a temperature $T = \frac{3}{4\pi}$. However, the phenomena of superconductivity is intimately linked with a changing temperature. It may seem that we have committed a fallacy by setting $r_+ = 1$ and not allowing the temperature to change; however, if one looks at the results of the numerics, one notices that ρ, the charge density is changing as the value of $\mathcal{O}^{(1)}$ or $\mathcal{O}^{(2)}$ is changing. We have the reverse of the situation that we would like to study - instead of keeping the charge density constant and varying temperature, we are varying the charge density and keeping the temperature constant. It turns out that this is all due to the scaling symmetries of the equations of motion. Doing one variation scheme or the other is equivalent; we just have to know how to unpack the information. Looking back at Eq. 2.19, we see that

$$T \to aT \quad \textbf{and} \quad \rho \to a^2\rho \tag{2.29}$$

where we know that T scales as $1/t$. From this, we see that $T \propto \sqrt{\rho}$. Therefore, we can do calculations with $r_+ = 1$ and then evaluate $\frac{T}{\sqrt{\rho}}$ to eliminate the scaling symmetries that we used to set $r_+ = 1$.

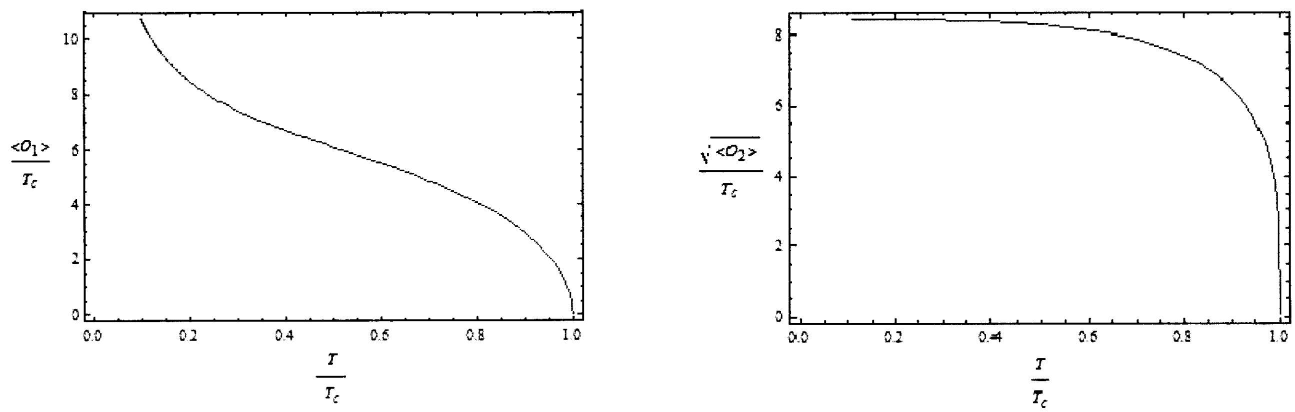

Figure 2-1: Phase Diagram for the ψ operators mass dimension one and two. Notice that both curves clearly indicate a transition from zero expectation value to non-zero expectation value.

In Fig. 2-1, we reproduce the results for the single condensate system, with $q = 1$. Both diagrams display superconducting characteristics. Mainly, as $T \to T_c$ from below, the value of the condensate goes to zero; there is also no condensate for temperatures higher than T_c. The right hand figure, looking at the operator of mass dimension two, looks similar to experimental curves for superconducting materials. One may worry about the low-temperature divergences in the left hand figure. However, this is the result of us overextending our system - here, we are working in the non-backreacting case and so our results are not valid at low temperatures.

Chapter 3

Multiple Condensates

Taking what we learned from the previous chapter, working with only one condensate, we will now look at a system that has two condensates with two different types of charges. The condensates will be coupled by a quartic interaction, but each will only couple to one vector field. The multiple condensate Lagrangian will have the same requirements as the Lagrangian for the single condensate system. Namely, we need a conserved stress energy tensor $T_{\mu\nu}$ and conserved currents J_i^ν, global U(1) symmetry in the boundary, gauged symmetry in the bulk and the ability to spontaneously break the symmetry of the system. The Lagrangian also needs to have terms that relate our boundary operators, $T_{\mu\nu}$, J_i^μ and the operators $\langle \mathcal{O}_i \rangle$, and the bulk fields, $g_{\mu\nu}$, the Maxwell vector fields $A_\mu^{(i)}$ and the charged scalar fields ψ_i. Using the one condensate Lagrangian as the model for the multiple condensate Lagrangian, we write the most general simple Lagrangian available to us:

$$\begin{aligned} \mathcal{L} &= R + 6 - \frac{1}{4}(F^{\mu\nu}F_{\mu\nu})_a - |\psi_a - iq_a A_a \psi_a|^2 \\ &- \frac{1}{4}(F^{\mu\nu}F_{\mu\nu})_b - |\psi_b - iq_b A_b \psi_b|^2 - V(|\psi_a|, |\psi_b|) \end{aligned} \tag{3.1}$$

where

$$(F^{\mu\nu}F_{\mu\nu})_i = (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{3.2}$$

The potential is going to have a similar form to the single condensate case, but with a quartic interaction

$$V(|\psi_a|, |\psi_b|) = -m_a^2|\psi_a|^2 - m_b^2|\psi_b|^2) + \lambda|\psi_a|^2|\psi_b|^2 \tag{3.3}$$

where the conformal masses of the fields are $-m_a^2$ and m_b^2 and λ is a real constant. We will keep the conformal masses of the fields as free parameters of the system for the time being.

3.1 Equations of Motion and Fields

We can derive the general equations of motion for Eq. 3.1 using the Lagrange's equation in curved spacetime. We will have four equations of motion, as we have two scalar fields and two vector fields; we are working in the non-backreacting case and so have a fixed metric. The four general equations of motion are

$$-(\nabla_\mu - iq_iA^i_\mu)(\nabla^\mu - iq_iA^\mu_i)\psi_i + \frac{1}{2}\frac{\psi_i}{|\psi_i|}V'(|\psi_i|, |\psi_k|) = 0, \tag{3.4}$$

for the scalar fields and

$$\nabla^\mu F^i{}_{\mu\nu} = iq[\psi_i^*(\nabla_\nu - iq_iA^i_\nu)\psi_i - \psi_i(\nabla_\nu + iq_iA^i_\nu)\psi_i^*] \tag{3.5}$$

for the vector potentials.

We can further simplify these equations of motion by providing more information about our scalar and vector fields and our metric. Our metric has to be plane symmetric and also has to grow hair below T_c. As these are the same criteria that we used for the single condensate case and two Lagrangians are very similar, we will guess that we have the same spacetime metric.

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(dx^2 + dy^2) \tag{3.6}$$

where

$$f(r) = r^2 \left(1 - \frac{r_+^3}{r^3}\right) \tag{3.7}$$

where r_+ is the radius of the black hole. Our two scalar and vector fields can only depend on the r coordinate, as we require that the system is symmetric in x and y and independent of time. The two vectors will only be in the t direction and independent of each other. Thus, our two scalar fields will be

$$A_a = \phi_a dt \quad \text{and} \quad A_b = \phi_b dt. \tag{3.8}$$

Using these vector fields, we find that the r components of the Maxwell equations give the following restrictions on the scalar fields

$$0 = iq_i \left(\psi_i^* \partial_r \psi_i - \psi_i \partial_r \psi_i^*\right) \tag{3.9}$$

which allows ψ_i to have a constant phase. For simplicity, we choose the scalar fields to always be real. Thus, our two fields are

$$\psi_a = \psi_a(r) \quad \text{and} \quad \psi_b = \psi_b(r) \tag{3.10}$$

where ψ_i is a real function. Using these simplifications for the scalar fields, vector fields and metric, along with the potential given in Eq. 3.3, the equations of motion reduce to

$$\psi_i'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\psi_i' + \frac{q_i^2\phi_i^2}{f^2}\psi + \frac{4\psi_i - 2\lambda\psi_i\psi_k^2}{2f} = 0 \tag{3.11}$$

$$\phi_i'' + \frac{2}{r}\phi_i' - \frac{2q_i^2\psi_i^2}{f}\phi_i = 0. \tag{3.12}$$

Notice that the ψ equations of motion have an r dependent mass term; this mass term is due to the spontaneous symmetry breaking.

3.2 Horizon and Boundary Solutions

We will follow the same procedure for solving the equations of motion at the horizon and the boundary as we did in the single condensate procedure. In the same way as before, let us first look at the number of parameters that we need to specify on the horizon and the boundary.

We have four equations of motion, each one being a second order differential equation, and so we should have an eight parameter family on the horizon and on the boundary. However, we can reduce the number of parameters on the horizon by looking at normalization requirements and the equations of motion themselves. Both vector fields are gauge-dependent and so the arguments presented in the single condensate case are still valid. Therefore, we eliminate one parameter from the family by requiring that $\phi_a(r_+) = 0$ and $\phi_b(r_+) = 0$. Looking at the equation of motion for ψ_i, there are three divergent terms in each equation; one term is dependent on ϕ_i and so the equation of motion reduces to

$$f'\psi_a' + 2\psi_a - \lambda\psi_a\psi_b^2 = 0 \tag{3.13}$$

$$f'\psi_b' + 2\psi_b - \lambda\psi_b\psi_a^2 = 0 \tag{3.14}$$

on the horizon. As these equations are now coupled first order differential equations of motion, the four ψ parameters reduce to two parameters. Therefore, our parameter family on the horizon is a four parameter family.

On the boundary, there is no equivalent reduction in the number of parameters. Our ϕ equations of motion are identical to the ϕ equations of motion for the single condensate case; the falloffs will also be identical:

$$\phi_a(r) = \mu_a - \frac{\rho_a}{r} \tag{3.15}$$

$$\phi_b(r) = \mu_b - \frac{\rho_b}{r} \tag{3.16}$$

where μ_i and ρ_i are the chemical potential and the charge density of the vector field ϕ_i. The ψ equations of motion are different than the single condensate case, due to the quartic coupling term. However, if we repeat the procedure for calculating the falloff, we see that the leading order terms are not dependent on λ. This is due to the fact that the quartic coupling term is a higher order term and thus negligible in comparison to the other terms on the boundary. Thus, our falloff is

$$\psi_a(r) = \frac{\psi_a^{(1)}}{r} + \frac{\psi_a^{(2)}}{r^2} \tag{3.17}$$

$$\psi_b(r) = \frac{\psi_b^{(1)}}{r} + \frac{\psi_b^{(2)}}{r^2} \tag{3.18}$$

where both $\psi_i^{(1)}$ and $\psi_i^{(2)}$ are normalizable terms. Since both of these terms are normalizable, either the first term or the second term can be considered the source, as explained in the single condensate case. Therefore, we can either look for combinations of input parameters such that

$$\mathcal{O}^{(1)} = \sqrt{2}\psi^{(1)} \quad \text{and} \quad \mathcal{O}^{(2)} = 0$$

or

$$\mathcal{O}^{(2)} = \sqrt{2}\psi^{(2)} \quad \text{and} \quad \mathcal{O}^{(1)} = 0$$

Since we have two different sets of operators, one for ψ_a and one for ψ_b, we have a total of four possible boundary solutions which we can explore.

3.3 Scaling and Numerics

This system will have the same type of symmetries as the single condensate case, namely

$$r \to ar, \quad t \to at, \quad L \to aL, \quad q_i \to q_i/a \tag{3.19}$$

which rescales the metric by a and the vector field A_i by a and

$$r \to ar, \quad \{t, x, y\} \to \{t, x, y\}/a, \quad f \to a^2 f, \quad \phi_i \to \phi_i/a \tag{3.20}$$

which does not rescale the metric, nor the vector fields. We can use these scalings to simplify our numerics, by finding the boundary parameters at fixed temperature and variable charge density and then looking at the ratio between the temperature and the charge density to describe our system, realizing that this relation is independent of the scaling parameter a.

In the case of multiple condensates, we were hoping to demonstrate that ability of one scalar field to condense was either inhibited or encouraged by the presence of the condensate; the coupling constant λ would control the amount and direction of dependence. We were unable to explicitly show this dependence, though in performing the calculations, we did subjectively notice that the values of the charge density would change with a change in lambda and a change in the charge density of the other condensate. We believe that the failure to demonstrate explicitly the impedance or encouragement of one condensate due to the other has to do with the rigorousness of the numerical methods. In looking to calculate the critical charge density of one field, where the critical charge density is the minimum charge density needed to be able to have a condensate, we also needed to keep the charge density of the other scalar field consant; In terms of the system, this would correspond to looking at the behavior of one condensate on the background of another fixed condensate. This calculation required allowing three of the four parameters on the horizon to vary which perhaps gave too much freedom to the system.

We were able to reproduce the high temperature results of the single condensate system when $\lambda = 0$. As soon as λ was allowed to vary and the restriction on the value of the charge condensate was applied, the script failed to correctly reproduce results. By this we mean the following: the family $\{\psi_+^a, E_+^a, \psi_+^b, E_+^b\}$, for specific values of the four parameters, gives the boundary solutions $\{0, \mathcal{O}_a^{(2)}, 0, \mathcal{O}_b^{(2)}, \mu_a, \rho_a, \mu_b, \rho_b\}$. When the script was run to find the values of the two vector fields and the second scalar field evaluated on the horizon for the specific value, ψ_+, such that the charge density due to ϕ_b was ρ_b, the script would not return $\{E_+^a, \psi_+^b, E_+^b\}$. We believe that this is due to the limited accuracy and precision in the script, especially as more parameters are allowed to vary; this is possibly due to the nonlinearity of the equations of motion. We were able to create a three dimenional plot of the values of the two charge densities versus the value of the mass one and mass two operators for some values of lambda; there was not enough resolution to be able to conclusively say what the effect of having two interacting condensates. In order to run the script effectively, the equations that match the falloff terms with the results of the numerical integration were solved analytically; this was only possible to do for small values of the coupling constant. If the magnitude of λ was allowed to be larger, it might be possible to conclusively say the effect of one condensate on the other.

Chapter 4

Summary of Results and Future Work

4.1 Summary of Results

We cannot say qualitatively what the effect one condensate has on another, as at this point in time, our script is not rigorous enough to do the level of computation necessary to handle the problem. The first step in trying to come to a complete conclusion is to extend the script's range of accuracy to higher values of λ. This should let us accurately follow the trend of critical charge density values.

4.2 Future Work

As the field of Holographic Superconductors is relatively new, there is still many different systems that have not yet been explored. With multiple condensates, there are three main types of systems that we could study.

The first thing to study is the effect of extending the reach of our model. We used as many simplifications as possible. We gave both fields the same charge and mass and looked at the non-backreacting case. It would be interesting to see how different charge values or different masses affect the degree to which one condensates impedes or encourages the other condensate to form. Looking at the backreacting case would also allow us to see if there is any change in the degree of impedance or encouragement at low temperatures.

Another interesting question is the effect of an uncharged condensate. At this point in time, no one has looked at how a condensate from an uncharged real field would affect a charged condensate if the two shared a quartic interaction. The Lagrangian for this system would be

$$\begin{aligned} \mathcal{L} &= R+6-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-|\psi-iqA\psi|^2 \\ &- \eta^2 - V(|\psi|,\eta) \end{aligned} \tag{4.1}$$

where

$$V(|\psi|,\eta) = -m^2|\psi|^2 - \mu\eta^2 + \lambda|\psi|^2\eta^2 \tag{4.2}$$

where m^2 and μ^2 are both positive. In this system, we could also explore the effects of differing charge, mass and backreaction.

The last and perhaps most interesting multiple condensate system at which to look is one that only has one vector potential. The Lagrangian would be

$$\begin{aligned} \mathcal{L} &= R+6-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-|\psi_a-iq_aA\psi_b|^2 \\ &- |\psi_b-iq_bA\psi_b|^2 - V(|\psi_a|,|\psi_b|) \end{aligned} \tag{4.3}$$

where

$$V(|\psi_a|,|\psi_b|) = -m^2|\psi_a|^2 - \mu|\psi_b|^2 + \lambda|\psi_a|^2|\psi_b|^2 \tag{4.4}$$

This system is perhaps the most difficult of the three to study, as the r component of Maxwell's equation no longer gives that the phase of the scalar field is constant; instead, it gives a relationship between the derivatives of the phases of the two scalar fields. In order to eliminate this difficulty, we would rewrite the system in terms of four field instead of three. The four fields would be the vector field ϕ, the real scalar field $\tilde{\psi}_a$ and $\tilde{\psi}_b$ and the Stueckelberg field, θ.

Appendix A

Detailed Explanation of Relevant Phenomena

A.1 Hawking Radiation

Hawking Radiation is a quantum gravity effect due to Bogoliubov transformations. Let us take a bit more time to understand from where this phenomena arises. In flat space time, a massless field that obeys the equation

$$-\eta^{\mu\nu}\partial_\mu\partial_\nu\psi = 0 \tag{A.1}$$

where η is the Minkowski metric, can be decomposed into positive frequency modes and negative frequency modes,

$$\psi = \sum_i^\infty (f_i\mathbf{a}_i + \bar{f}_i\mathbf{a}_i^\dagger) \tag{A.2}$$

where f_i and $\bar{f}$ form a complete orthogonal basis for all positive and negative modes, respectively, and $\mathbf{a}_i$ and $\mathbf{a}_i^\dagger$ are creation and annihilation operators for the ith state. We define the vacuum state $|0\rangle$ by

$$\mathbf{a}_i|0\rangle = 0 \quad \text{for all } i. \tag{A.3}$$

The story is quite different in curved spacetime. Now, our scalar has the equation of motion

$$\frac{-1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\psi) = 0 \tag{A.4}$$

where $g^{\mu\nu}$ is the inverse of the metric of the spacetime and $\sqrt{-g}$ is the square root of the determinant of the metric. In this case, ψ cannot be decomposed in the manner of Eq. A.2, as positive and negative frequencies are not invariant in curved spacetime. This means that there is not one invariant vacuum state in curved spacetime. Thus, as Hawking states in his seminal paper, "One can interpret this as implying that the time dependent metric or gravitational field has caused the creation of a certain number of particles of the scalar field."

A.2 Landau's Superfluid Model

The system which we are exploring is a superfluid or superconductor. When a fluid is in a superfluid state, there is no dissipation of energy as various parts of the material flow through the container. A superconductor is similar, in that the resistance of the material goes to zero. L.D. Landau's theory of the superfluid state of ^{4}He provides a good intuitive explanation of how superfluids work in the quantum mechanical world. Imagine that you have a tube of fluid through which you are tracking the motion of a small component of the fluid, of mass M; this small component is moving with velocity v. As it moves through the fluid, it may create waves in the remaining fluid; these waves have momentum $\hbar k$. Due to the momentum transfer and potential energy dissipation, the small component of fluid slows down to a velocity v_f. Using energy and momentum conservation we can relate the initial and final velocity to the mass, the wave number and the frequency.

$$\frac{1}{2}Mv^2 \geq \frac{1}{2}Mv_f^2 + \hbar\omega(k) \tag{A.5}$$

$$Mv = Mv_f + \hbar k \tag{A.6}$$

where $\omega(k)$ is the frequency of the excited waves. Eliminating v_f we find the relationship

$$v \geq \frac{\omega}{k} \tag{A.7}$$

where we have used the assumption that the $mv >> \hbar k$. Eq. A.7 says that in order for the fluid to be a superfluid, which is characterized by zero energy dissipation, the dispersion relationship has to be linear. Using this result and Goldstone's Theorem we can come to a quantum explanation of superfluidity. Goldstone's Theorem states that whenever there is a spontaneous symmetry break, massless fields, called Goldstone bosons, are created. This spontaneous symmetry break is caused by the vacuum expectation value of a field changing from zero to non-zero. As massless fields are described by linear dispersion relationships, we see that superfluidity can be explained in terms of a non-zero vacuum expectation value, otherwise known as a condensate.

A.3 Abelian-Higgs Mechanism

Let us look at a simple example of the Abelian-Higgs mechanism to explore spontaneous symmetry breaking.[4]

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - |\psi - iqA\psi|^2 - V(|\psi|) \tag{A.8}$$

which is invariant under local U(1) transformation

$$\psi(x) \rightarrow e^{i\alpha(x)}\psi(x) \quad A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x). \tag{A.9}$$

Now, we choose the very specific potential

$$V(\psi) = -\mu^2\psi^*\psi + \frac{\lambda}{2}(\psi^*\psi)^2 \tag{A.10}$$

where $\mu^2 > 0$ and $\lambda > 0$. Remembering that we have a complex scalar field, we see that $\psi = 0$ is an unstable solution while there is an infinite number of stable solutions in the circle of minima; The expectation value of ψ is

$$\langle\psi\rangle = \left(\frac{\mu^2}{\lambda}\right)^{1/2} \tag{A.11}$$

which is non-zero and therefor results in a spontaneous symmetry break.

In order to look at the effect that this symmetry breaking has had on our two fields, let us expand our system around the minima circle, assuming we are at low energy. As our field is a complex field, we may write it as

$$\psi(x) = \psi_0 + \frac{1}{\sqrt{2}}(\psi_1(x) + i\psi_2(x)) \tag{A.12}$$

where $\psi_0 = \langle\psi\rangle$, as given in Eq. A.11. The potential may now be rewritten as

$$V(\psi) \quad = \quad -\frac{1}{2\lambda}\mu^4 + \mu^2\psi_1^2 + \mathcal{O}(\psi_i^3) \tag{A.13}$$

where $i = 1, 2$. Eq. A.13 tells us that the real field ψ_1 has a mass $m = \sqrt{2}\mu$ and the field ψ_2 is massless. This is in agreement with Goldstone's Theorem. [1]

Next, look at the effect the spontaneous symmetry break has on the kinetic term of ψ, $|\psi - iqA\psi|^2$. Inserting our expansion for ψ, we find

$$|\psi - iqA\psi|^2 = \frac{1}{2}(\partial_\mu\psi_1)^2 + \frac{1}{2}(\partial_\mu\psi_2)^2 + e^2\psi_0^2 A_\mu A^\mu + \cdots . \tag{A.14}$$

Notice that one of the terms, $e^2\psi_0^2 A_\mu A^\nu$ is actually a photon mass term,

$$\Delta\mathcal{L} = \frac{1}{2}m_A^2 A_\mu A^\mu \tag{A.15}$$

where

$$m_A^2 = \frac{2e\mu^2}{\lambda}. \tag{A.16}$$

Let me summarize this most surprising discovery: by allowing a non-zero vacuum expectation value for ψ, our vector field A_μ, which is normally a massless field, has

[1] nutshell

acquired a mass. Quantum theorists refer to this phenomenon as the vector field eating the Goldstone Boson.

Appendix B

Backreacting Case

In both the original superconductor system and our new system with multiple condensates, we assumed that the perturbations due to the fields were small enough that they did not cause any deformation in the metric. It is also possible to do a similar analysis, except now allowing backreaction on the metric. [2] Our Lagrangian is the same as the original Lagrangian, as we only want to work with only one condensate

$$\mathcal{L} = R + 6 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - |\psi - iqA\psi|^2 - V(|\psi|) \tag{B.1}$$

B.1 Ansatz

We have to derive our equations of motion from the original, general equations of motion, given in Eqs. 2.2, 2.3 and 2.4. In order to do this, we first provide an ansatz for our metric, scalar and vector field. We still want to maintain planar symmetry and time invariance, so we can immediately write down that

$$\psi = \psi(r) \quad \text{and} \quad A = \phi(r)dt. \tag{B.2}$$

We are going to have to be a bit more careful about defining our metric. Let us start with a similar metric to the one expressed in Eq. 2.5.

$$ds^2 = -h(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2(dx^2 + dy^2) \tag{B.3}$$

where $h(r) \to f(r)$ on the boundary and $f(r) \to 0$ at the horizon. These restrictions make our metric asymptotically AdS, which is one of the restrictions we are forced to place on our system. The other restriction that we can use to guide our ansantz is that, above the critical temperature, the black hole must turn into a bald, electrically charged hole. This is called the Reissner-Nordstrom-AdS black hole and is described by the metric

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(dx^2 + dy^2) \tag{B.4}$$

where

$$f(r) = r^2 - \frac{1}{r}\left(r_+^3 + \frac{\rho^2}{4r_+}\right) + \frac{\rho^2}{4r^2} \tag{B.5}$$

and ρ is the charge density. In this spacetime, our scalar and vector fields are

$$\psi = 0 \quad \text{and} \quad \phi = \rho\left(\frac{1}{r_+} - \frac{1}{r}\right) \tag{B.6}$$

Using the above T_c solution to guide our solution, let us write the below T_c metric as

$$ds^2 = -g(r)e^{-\chi(r)}dt^2 + \frac{1}{g(r)}dr^2 + r^2(dx^2 + dy^2) \tag{B.7}$$

where $\chi \to 0$ on the boundary and both $g(r)$ and $\chi(r)$ depend on ψ and χ. We use an exponential form, $e^{-\chi}$, as it always remains positive, which forces g_{tt} to always be negative. There are several reasons for why we place it on g_{tt} instead of g_{rr}, g_{xx} or g_{yy}. We cannot place it on only g_{xx} or g_{yy} as the system has to be symmetric in x and y. We place the exponential form on g_{tt} and not g_{rr} is because $g_t t$ determines the leading gravitation effects in the Newtonina weak-field approximation and is also the metric component that appears in the Kubo formulas, which are used to determine transportation coefficients.

B.2 Equations of Motion

For the equations of motion for this system, we start off with the general equations of motion, derived directly from the Lagrangian, and insert our ansatz for the scalar field, the vector field and our metric; in total, we have four equations of motion. The equations of motion for $\chi(r)$ and $g(r)$ are derived from taking linear combinations of the various components of Einstein's equations. Our four equations of motion are

$$\psi'' + \left(\frac{g'}{g} - \frac{\chi'}{2} + \frac{2}{r}\right)\psi' + \frac{q^2\phi^2 e^{\chi}}{g^2}\psi - \frac{1}{2g}V'(\psi) = 0 \quad \text{(B.8)}$$

$$\phi'' + \left(\frac{\chi'}{2} + \frac{2}{r}\right)\phi' - \frac{2q^2\psi^2}{g}\phi = 0 \quad \text{(B.9)}$$

$$\chi' + r\psi'^2 + \frac{rq^2\phi^2\psi^2 e^{\chi}}{g^2} = 0 \quad \text{(B.10)}$$

$$\frac{1}{2}\psi'^2 + \frac{\phi'^2 e^{\chi}}{4g} + \frac{g'}{rg} + \frac{1}{r^2} - \frac{3}{g} + \frac{V(\psi)}{2g} + \frac{q^2\psi^2\phi^2 e^{\chi}}{2g^2} = 0 \quad \text{(B.11)}$$

Again, we will use a potential that gives a simple falloff for the scalar field and is also above the Breitenlohner-Freedman bound:

$$V(|\psi|) = -2\psi^2 \quad \text{(B.12)}$$

Again, we have suppressed the length scale L, as we can set $L = 1$ due to one of the three scaling symmetries present in our system.

B.3 Solving the Equations of Motion using Scaling Symmetries

We will solve the equations of motion for this system in the same manner that we solved the non-backreacting case. As this system is a bit more complicated than the previous one, there are a couple of subtleties that we would like to address and discuss directly.

B.3.1 Mapping from Horizon to Boundary

In non-backreacting case, we had two parameters on the horizon and four parameters on the boundary. There were only two parameters on the boundary due to the normalization requirement on A_μ and the reduction of the ψ equation of motion from a second order differential equation to a first order differential equation. These reductions in the number of parameters at the horizon still hold for the backreacting case. In terms of χ and g, naively, we have one parameter for χ, as Eq. B.10 is a first order differential equation and one parameters for g, as Eq. B.11 is a first order differential equation. The number of parameters is further restricted by the definition of the horizon, $g(r_+) = 0$. This leads us with a three parameter family for the horizon:

$$\psi(r_+) = \psi_+, \quad \phi'(r_+) = E_+, \quad \text{and} \quad \chi(r_+) = \chi_+ \tag{B.13}$$

where we have set r_+ to be some value. On the boundary, naively, we have a six parameter family: two parameters each from ψ and ϕ and one each from χ and g. Since our two fields have the same falloff as in the non-backreacting case

$$\psi = \frac{\psi^{(1)}}{r} + \frac{\psi^{(2)}}{r^2} \tag{B.14}$$

$$\phi = \mu - \frac{\rho}{r} \tag{B.15}$$

we have four parameters total from ψ and ϕ. In terms of χ and g, we use several restrictions to eliminate them from the family of parameters. We require that the metric is an asymptotically AdS metric, which fixes the value of g at the boundary and forces $\chi \to 0$. This restriction eliminates the two potential parameters for g and χ and so we are left with a four parameter family on the boundary. Thus, the equations of motion give us a following mapping from the horizon to the boundary:

$$\{\psi_+, E_+\} \to \{\psi^{(1)}, \psi^{(2)}, \mu, \rho\} \tag{B.16}$$

for some radius of the black hole r_+

B.3.2 Temperature and Scaling Symmetries

The temperature of the black hole given by Eq. B.7 is

$$T = \frac{r_+}{16\pi}\left((12 + 4\psi_+^2)e^{-\chi_+/2} + E_+ e^{\chi_+/2}\right) \tag{B.17}$$

which can be found, for example, by requiring the regularity of the Euclidean solution. In the non-backreacting case, we had the temperature of the black hole was identical to the temperature of the boundary. In this case, this is no longer true, due to the exponential factor in g_{tt}. As our temperature is related to the time component of the metric, we find that the relationship between the horizon temperature T_H and the boundary temperature T_B, is

$$T_B = e^{-\chi_B} T_H \tag{B.18}$$

where $\lim_{r\to\infty} \chi(r) = \chi_B$ and we have eliminated the requirement that $\chi_B = 0$ for the moment. We have relaxed this condition due to the fact that, while performing numerics, it is very difficult to pick the correct input parameters, namely ψ_+ and E_+, such that $\chi_B = 0$. Here, the three scaling symmetries come into play and make the task of performing numerics much easier.

We have three scaling symmetries for this system. The first two are identical to the scaling symmetries for the non-backreacting case, given in Eqs. 2.18 and 2.19:

$$r \to ar, \quad t \to at, \quad L \to aL, \quad q \to q/a \tag{B.19}$$

and

$$r \to ar, \quad \{t, x, y\} \to \{t, x, y\}/a, \quad g \to a^2 g, \quad \phi \to \phi/a. \tag{B.20}$$

These allow us to set $L = 1$ and $r_+ = 1$. The new scaling symmetry is

$$e^{\chi} \to a^2 e^{\chi}, \quad t \to at \quad \phi \to \phi/a. \tag{B.21}$$

We will use this last symmetry in the following way. First, we will find a solution to our equations of motion for some family $\{\psi_H, E_H\}$ without the restriction that

$\chi_B = 0$. We will then use Eq. B.21 to rescale χ_B such that $\chi_B = 0$. We will apply this same rescaling to the rest of our solution to find the parameters ψ_H and E_+ that give us a solution to our equations of motion for $T_B = T_H$.

B.4 Solutions on the Boundary

As in the non-backreacting case, we will be looking for two solutions to our equations of motion:

$$\mathcal{O}^{(1)} = \sqrt{2}\psi^{(1)} \quad \text{and} \mathcal{O}_2 = 0$$

or

$$\mathcal{O}^{(2)} = \sqrt{2}\psi^{(2)} \quad \text{and} \mathcal{O}_1 = 0,$$

as this will give us the non-zero vacuum expectation value of ψ due to spontaneous symmetry breaking and not a source on the boundary.

B.5 Phase Diagram

In plotting the phase diagram for the backreacting case, one follows a similar procedure as for the non-backreacting case. However, more caution must be taken with plotting quantities that do not depend on scale, as the constant of proportionality between T and ρ is now dependent on the value of the charge q. We will not reproduce the results for the backreacting case.[1]

[1]For the complete treatment of the backreacting single condensate case, see *Holographic Superconuctors* by Hartnoll, Herzog and Horowitz

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