Applying AdS/CFT to Many Body Systems

by

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Abstract

In this thesis, we model a many body system by a conformal field theory, and calculate the correlation function of a scalar operator in this theory using the AdS/CFT correspondence. We describe numerical techniques for calculating the retarded Green function of the operator. The results suggest that further improvement in the robustness of the codes is needed.

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Chapter 1

Introduction

The AdS/CFT duality allows physical observables in a conformal field theory (CFT) to be computed using string theory in an anti-de-Sitter (AdS) space. When interactions in the CFT are strong, the corresponding string theory approaches its Einstein gravity limit. In this limit, correlation functions of a generic operator in the CFT can be calculated using the Einstein field equations. These correlation functions are interesting because they encode information about excitations coupled to the operator in the many-body system. They also tell us whether the system exhibits some instability. Possible instabilities include onset of a superconducting phase. In this thesis, we study a two-point function of a scalar operator in CFT, and develop numerical techniques for finding instabilities via the correlation function.

Chapter two sets up the AdS model corresponding to the dual CFT, and explains how the correlation function is found. To facilitate later numerical calculations, the equations are cast into dimensionless form, and the asymptotic behaviors of the solutions are determined.

Chapter three develops numerical techniques for finding divergencies in the correlation functions.

Chapter four shows the results of numerical computations. The purpose is to compare the virtues of different numerical techniques, and to check if the results are consistent with analytic calculations in the infrared limit.
1.1 Motivation

One of the greatest mysteries in condensed matter physics is the underlying mechanism for high-$T_c$ superconductors. In 1957, the BCS theory successfully explained superconductivity at low temperatures [1, 5, 6]. However, the BCS theory is unsatisfactory because it assumes weak coupling. For decades, no model has been able to give a satisfying description of superconductivity at high temperatures. Even brute force computer simulation is unrealistic because of the computational complexity.

In recent years, new hope has been found with the realization that a system described by a strongly coupled conformal field theory is equivalent to a geometry described by Einstein’s gravity theory. This connection is given by the the AdS/CFT correspondence.

The AdS/CFT correspondence was first proposed by Maldacena in 1997 [7]. It says that a string theory in a $(d + 1)$-dimensional anti de Sitter space is equivalent to a field theory defined on the $d$-dimensional conformal boundary of that space. Therefore, a many-body system described by a conformal field theory can be recast into a geometry described by a string theory. Moreover, when the interactions in the many-body system are strong, the corresponding string theory approaches its classical limit described by Einstein's field equations.

We are often interested in a many-body system at finite temperature and charge density. In order to specify these conditions in the gravity theory, a black hole background is added to the AdS space. The correspondence between the properties of the many-body system and those of the black hole is explained in the next section.

Later sections explain how solutions in the AdS dual give the correlation function of a generic operator in the CFT. In particular, they explain how poles of the correlation function in the complex frequency plane reveal the dispersion relation of the many-body system.
1.2 Black Hole Background in AdS Space

According to the no-hair theorem, the three quantities characterizing a classical black hole are its mass, charge, and angular momentum. However, because Einstein's gravitational field equations do not allow a black hole to transfer any mass outside its event horizon, a classical black hole is considered to have zero temperature.

In 1974, Stephen Hawking argued that, due to quantum mechanical effects, a black hole actually emits radiation [4]. This feature allows a black hole to thermally equilibrate with other thermodynamic systems, and therefore it must have nonzero temperature. In fact, Hawking showed that all the thermodynamic quantities of a black hole are well defined, and depend only on its three properties - mass, charge, and angular momentum.

In the AdS/CFT many-body theory, a black hole with certain mass, charge, and angular momentum in the AdS geometry imposes thermodynamic constraints on the corresponding CFT. To be specific, the charge of the black hole fixes the charge density of the many-body system, and the temperature (fixed by mass, charge, and angular momentum) of the black hole fixes the temperature of the many-body system.

One interesting case of a black hole background is the extremal black hole background. An black hole is extremal when its mass is minimized under fixed charge and angular momentum. Thermodynamically, it also corresponded to a black hole of zero temperature. This black hole then corresponds to the zero temperature limit of the many-body system described by the dual CFT.

1.3 Retarded Green Functions in CFT

In this thesis, the correlation functions computed are momentum space retarded Green functions. While other correlation functions encode similar physics, the boundary conditions for retarded Green functions are the easiest to impose in the dual ADS string theory. The imaginary part of the retarded Green function of an operator is the spectral function, which represents the density of states coupled to the operator.
The spectral function is interesting since we are interested in the excitations of the
many-body system.

If we impose the in-falling boundary condition at the horizon, then the solution
near the boundary of the AdS space gives the retarded Green function [8].

1.4 Poles in the Complex Frequency Plane

The retarded Green function depends on parameters including temperature, mass,
and frequency. The retarded Green function is most conveniently regarded as a
function of frequency alone with other parameters fixed. Moreover, we would like
the frequency parameter to encode both the physical frequency and the amplitude
growth. Hence the frequency is taken to lie in the complex plane.

Much of the physics of a many-body system is determined by the poles of the
retarded Green function in the complex frequency plane. The dispersion relation can
be determined by tracing the position of the poles under as the wavenumber is varied.
If a pole goes to zero as the wavenumber approaches specific values, then it suggests
the presence of gapless excitations.

A pole in the upper half-plane leads to causality violation and instability. The
occurrence of such an instability at zero momentum for a charged scalar field may
suggest the onset of superconductivity [3]. Other cases of instabilities may also imply
the onset of other phenomena.

1.5 Emergent Behaviors

At zero temperature, a \((d + 1)\)-dimensional black hole geometry degenerates into the
direct product of a two-dimensional AdS and a \((d - 1)\)-dimensional Euclidean space
\((\text{AdS}_2 \times \mathbb{R}^{d-1})\) in the near horizon region. As suggested by [9], emergent behaviors
in the \(\text{AdS}_{d+1}\) geometry only depends on properties of the \(\text{AdS}_2\) geometry [9]. The
form of the retarded Green function in \(\text{AdS}_2 \times \mathbb{R}^{d-1}\) was analytically derived, and
qualitative conclusions about the positions of the poles and the stability of the system
were drawn.

While the emergent qualitative properties are interesting, this thesis develops numerical techniques for calculating the quantitative properties. We work with Einstein’s field equations in the black hole geometry. Both zero and finite temperatures are considered. We discuss two methods, the integration method and the matrix method. The integration method gives the the retarded Green function as a function of frequency. Unfortunately, it does not work when the frequency has a negative imaginary part. The matrix method applies to the entire complex frequency plane, but it only finds the poles of the retarded Green function rather than the function itself.

1.6 Results

The behavior of the retarded Green function is obtained using the integration and matrix methods. Its behavior under varying wavenumber and frequency is studies at both finite and zero temperature cases. We find no instability under any set of parameters. At zero temperature, this result disagrees with the emergent behaviors of [9]. Furthermore, the integration method and matrix methods are inconsistent. It is most likely that the codes used to perform numerical calculations are not stable, thereby producing anamolous results.
Chapter 2

Scalar Field in Charged Black Hole Geometry

In this chapter, we present a theoretical model for a strongly coupled many-body system. We derive the Einstein field equations in AdS$_{d+1}$ with a charged black hole background. We then rescale the equations to make the variables dimensionless. Finally, we discuss how the retarded Green function is obtained from the equations, and how it determines the stability and dispersion relation of the system.

We use Greek letters to index the $d$ coordinates in CFT$_d$, uppercase Roman letters to index the $d+1$ coordinates in AdS$_{d+1}$, and lowercase Roman letters to index the $\vec{x}$ coordinates.

2.1 Black Hole Geometry

Consider a $d$-dimensional conformal field theory (CFT$_d$) with global $U(1)$ symmetry. Under the AdS/CFT correspondence, the current $J_\mu$ in this CFT maps to a $U(1)$ gauge field $A_M$ in the dual AdS$_{d+1}$. We consider the strong coupling limit, so that the AdS$_{d+1}$ gravity theory is described by Einstein’s field equations.

The action for a vector field coupled to AdS$_{d+1}$ gravity is [9]

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} \left[ \mathcal{R} + \frac{d(d+1)}{R^2} - \frac{R^2}{g^2_F} F_{MN} F^{MN} \right]$$ (2.1)
where \( \kappa \) is a weighting constant, \( g \) is the metric determinant, \( \mathcal{R} \) is the Ricci scalar, \( R \) is the radius of curvature, \( g_F \) is a gauge coupling constant, and \( F_{MN} = \partial_M A_N - \partial_N A_M \).

The equations of motion obtained from Equation (2.1) is solved by a charged black hole geometry with charge \( Q \) and mass \( M \). The metric is given by [9]

\[
ds^2 \equiv g_{MN}dx^M dx^N = \frac{r^2}{R^2}(-f dt^2 + dx^2) + \frac{R^2}{r^2} \frac{dr^2}{f} \tag{2.2}
\]

where

\[
f = 1 + \frac{Q^2}{r^{2d-2}} - \frac{M}{r^d}. \tag{2.3}
\]

The \( t \)-component of the gauge field is (other components zero) [9]

\[
A_t = \mu \left( 1 - \frac{r_0^{d-2}}{r^{d-2}} \right) \tag{2.4}
\]

where \( r_0 \) is the largest root of \( f(r) \), and \( \mu \) is the chemical potential of the black hole given by

\[
\mu = \frac{g_F Q}{c_d R^2 r_0^{d-2}}, \quad c_d = \sqrt{\frac{2(d-2)}{d-1}}. \tag{2.5}
\]

The temperature of the black hole is [4]

\[
T = \frac{dr_0}{4\pi R^2} \left( 1 - \frac{(d-2)Q^2}{dr_0^{2d-2}} \right). \tag{2.6}
\]

Now consider a scalar field \( \phi \) in this charged black hole geometry, with charge \( q \) and mass \( m \), that is dual to an operator \( \mathcal{O} \) in CFT\(_d\) of charge \( q \) and dimension

\[
\Delta = \frac{d}{2} + \sqrt{m^2 R^2 + \frac{d^2}{4}}. \tag{2.7}
\]

The action for \( \phi \) is [9]

\[
S = -\int d^{d+1}x \sqrt{-g} \left[ (D_M \phi)^* (D^M \phi) + m^2 \phi^* \phi \right] \tag{2.8}
\]

where

\[
D_M = \partial_M - iqA_M. \tag{2.9}
\]
Under a Fourier transform in $t$ and $\bar{x}$ coordinates,

$$\phi(r, k_\mu) = \int d^dx \phi(r, x^\mu) \exp(-i k_\mu x^\mu), \quad (2.10)$$

the equation of motion given by Equation (2.8) becomes

$$-\frac{1}{\sqrt{-g}} \partial_r \left( \sqrt{-g} g^{rr} \partial_r \phi \right) + \left( g^{ii} \left( k^2 - u^2 \right) + m^2 \right) \phi = 0 \quad (2.11)$$

where

$$u \equiv \frac{g_{ii}}{g_{tt}} \left( \omega + \mu_q \left( 1 - \frac{r_0^{d-2}}{r^{d-2}} \right) \right), \quad \mu_q = \mu q, \quad k^2 = |\bar{k}|^2. \quad (2.12)$$

### 2.2 Transformations

In this section, we rescale Equation (2.11) to make it dimensionless.

We make the following transformations:

$$r \mapsto r r_0, \quad (2.13)$$

$$x^\mu \mapsto x^\mu \frac{R^2}{r_0}, \quad (2.14)$$

$$Q \mapsto Q r_0^{d-1}, \quad (2.15)$$

$$M \mapsto M r_0^d, \quad (2.16)$$

$$m \mapsto m \frac{1}{R}, \quad (2.17)$$

$$(\omega, k_\mu) \mapsto (\omega, k_\mu) \frac{r_0}{R^2}, \quad (2.18)$$

$$\mu \mapsto \frac{\mu}{R^2}, \quad (2.19)$$

$$T \mapsto T \frac{r_0}{R^2}. \quad (2.20)$$
Then Equations (2.3)-(2.6) become

\begin{align*}
\frac{g_{t t}}{g_{i i}} &= 1 + \frac{Q^2}{r^{2d-2}} - \frac{M}{r^d}, \quad M = 1 + Q^2, \\
A &= \mu \left(1 - \frac{1}{r^{d-2}}\right), \\
\mu &= \frac{g_F Q}{c_d}, \quad c_d = \sqrt{\frac{2(d-2)}{d-1}}, \\
T &= \frac{d}{4\pi} \left(1 - \frac{(d-2)Q^2}{d}\right),
\end{align*}

and Equation (2.2) becomes

\[\text{ds}^2 = R^2 \left[ r^2 \left(-f\text{d}t^2 + d\text{d}^2\right) + \frac{1}{r^2} \frac{d\text{d}^2}{f} \right].\]

The metric coefficients can be read off from Equation (2.25):

\begin{align*}
-g_{t t} &= R^2 r^2 f, \\
g_{i i} &= R^2 r^2, \\
g_{r r} &= R^2 \frac{1}{r^2 f},
\end{align*}

giving

\[\sqrt{-g} = \frac{1}{R^{d-1}} r^{d-1}.\]

Substituting Equations (2.13)-(2.29) into Equation (2.11) and multiplying through by \(R^2\) give the dimensionless equation of motion

\[-\frac{1}{r^{d-1}} \partial_r (r^{d+1} f \partial_r \phi) + \left(\frac{1}{r^2} (k^2 - u^2) + m^2\right) \phi = 0,\]

where

\[u = \sqrt{\frac{1}{f}} \left(\omega + \mu_q \left(1 - \frac{1}{r^{d-2}}\right)\right).\]

The solution to Equation (2.30) is a complex function in \(r\) on the interval \((1, \infty)\). Since numerical analysis is more conveniently done on a compact interval, we introduce a
new variable:

\[ z = 1 - \frac{1}{r}. \]  

(2.32)

The horizon is located at \( z = 0 \) \((r = r_0)\) and the boundary at \( z = 1 \) \((r = \infty)\). Hence \( z \in (0, 1) \).

Substituting Equation (2.32) into Equation (2.30) gives the equation of motion in \( z \) \((\partial \equiv \partial_x)\)

\[-(1 - z)^{d+1} \partial \left( ((1 - z)^{-d+1} f \partial \phi + ((1 - z)^2 (k^2 - u^2) + m^2) \phi = 0 \right. \]  

(2.33)

where

\[ u = \sqrt{\frac{T}{f}} (\omega + \mu_q (1 - (1 - z)^{d-2})) \], \quad f = 1 + Q^2 (1 - z)^{2d-2} - M (1 - z)^d. \]  

(2.34)

### 2.3 Asymptotic Solutions

In this section, we solve Equation (2.33) asymptotically near the horizon and the boundary. These asymptotic solutions are used in Section 2.4 to define the in-falling boundary condition, and in Section 3.3 to give a convergent power series expansion.

#### 2.3.1 Near Horizon

At finite \( T \), Taylor expanding the coefficients of Equation (2.33) at the horizon gives

\[ \partial^2 \phi + \left( \frac{1}{z} + O(z^0) \right) \partial \phi + \left( \frac{\omega^2}{16 \pi^2 T^2 z^2} + O(z^{-2}) \right) \phi = 0. \]  

(2.35)

The solution to Equation (2.35) has the form

\[ \phi(z \to 0) = C_i(\omega, k^2) \exp \left( -\frac{i \omega}{4 \pi T} \log z \right) + C_o(\omega, k^2) \exp \left( \frac{i \omega}{4 \pi T} \log z \right). \]  

(2.36)

We call the term with a negative exponent “in-falling” and the term with a positive exponent “out-going”. To understand the meaning of these names, we perform an
inverse Fourier transform on $\phi$

$$\phi(r, t, k^2) = \int \frac{d\omega}{2\pi} \phi(r, \omega, k^2) \exp(-i\omega t)$$

$$= \phi_i(r, t, k^2) + \phi_o(r, t, k^2), \quad (2.37)$$

where

$$\phi_i(r, t, k^2) = \int \frac{d\omega}{2\pi} C_i(\omega, k^2) \exp\left( -\frac{i\omega}{4\pi T} \log z - i\omega t \right), \quad (2.38)$$

$$\phi_o(r, t, k^2) = \int \frac{d\omega}{2\pi} C_o(\omega, k^2) \exp\left( \frac{i\omega}{4\pi T} \log z - i\omega t \right). \quad (2.39)$$

To keep the phases constant, as $t$ increases, $z$ must decrease in $\phi_i(r, t, k^2)$ and increase in $\phi_o(r, t, k^2)$. Therefore, the first term represents a wave falling into the black hole, and the second represents a wave going out of the black hole.

At zero $T$, the Taylor expansion of Equation (2.33) gives

$$\partial^2 \phi + O(z^{-1}) \partial \phi + \left( \frac{\omega^2}{d^2(d-1)^2 z^4} + O(z^{-3}) \right) \phi = 0. \quad (2.40)$$

The leading order solution is

$$\phi(z) = C_i(\omega, k^2) \exp\left( \frac{i\omega}{d(d-1)z} \right) + C_o(\omega, k^2) \exp\left( \frac{-i\omega}{d(d-1)z} \right). \quad (2.41)$$

With similar reasoning we call the first term in-falling and the second out-going.

### 2.3.2 Near Boundary

Taylor expanding the coefficients of Equation (2.33) at the boundary gives

$$\partial^2 \phi + \left( \frac{d-1}{1-z} + O\left( (1-z)^0 \right) \right) \partial \phi - \left( \frac{m^2}{(1-z)^2} + O\left( (1-z)^{-1} \right) \right) \phi = 0. \quad (2.42)$$

The solution is

$$\phi(z \to 1) = A(\omega, k^2)(1-z)^{-\Delta+d} + B(\omega, k^2)(1-z)^{\Delta} \quad (2.43)$$
where

\[ \Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}. \]  

(2.44)

## 2.4 Retarded Green Function

The momentum space retarded Green function of \( O \) is obtained by imposing the in-falling condition at the horizon and looking at the solution near the boundary \[8\]. That is, we set \( C_i(\omega, k^2) = 1 \) and \( C_o(\omega, k^2) = 0 \) in Equation (2.36). Since Equation (2.33) is second order, these two conditions fix the full solution. Near the boundary the solution is given by Equation (2.43). Then the retarded Green function is

\[ G_R(\omega, k^2) = K \frac{B(\omega, k^2)}{A(\omega, k^2)} \]  

(2.45)

where \( K \) is a positive constant.

The imaginary part of \( G_R(\omega, k^2) \) is the spectral function representing the density of states coupled to \( O \). We note that \( G_R(\omega, k^2) \) develops a divergency when \( A(\omega, k^2) = 0 \). Then, Equation (2.43) reduces to \( \phi(z \to 1) = B(\omega, k^2)z^{-\Delta} \), and so \( \phi \) is normalizable. Thus, divergencies in \( G_R(\omega, k^2) \) correspond to excitable modes of the system. To study these modes, we fix \( k^2 \) and look at poles of \( G_R(\omega, k^2) \) in the complex \( \omega \) plane.

While the real part of \( \omega \) represents the physical frequency, the imaginary part represents the variation of the amplitude over time. In particular, a pole in the upper half-plane corresponds to an exponentially growing amplitude, which makes the system unstable.

According to \[9\], at zero temperature, when

\[ k^2 < k_0^2 \equiv \left(1 - \frac{4\pi T}{d}\right) \frac{1}{d(d-1)} \left[\left(\frac{d-2}{2}\right) - m^2 - \frac{1}{4}\right], \]  

(2.46)

all poles lie in the upper half-plane and the system is unstable.

We may also study the dispersion relation of the system by varying \( k^2 \) and tracing the movement of the poles.
Chapter 3

Numerical Analysis

In this chapter, we explain how to obtaining numerical solutions to Equation (2.33) with the in-falling boundary condition. To find poles in the complex $\omega$ plane, we introduce two methods, the direct integration method and the matrix method. And to determine the location of a pole to high accuracy, we introduce the complex Newton’s method and the Nelder-Mead method. We also provide the Mathematica command for every given step.

The following discussion assumes finite temperature. The zero temperature case is obtained by using Equation (2.41) instead of Equation (2.36).

3.1 Direct Integration Method

To obtain $\phi$ with the in-falling condition, the most naive method is to numerically integrate Equation (2.33) from the horizon to the boundary. Since Equation (2.33) is second-order, the solution is determined by specifying $\phi$ and its first derivative at a starting point $z_0 \approx 0$. By Equation (2.45), $G_R(\omega, k^2)$ is unchanged under constant scaling of $\phi$. Hence only the ratio $\partial\phi(z_0) / \phi(z_0)$ has physical meaning. This ratio is determined by the in-falling term of Equation (2.35):

$$\frac{\partial\phi(z_0)}{\phi(z_0)} = -\frac{i\omega}{4\pi T} \frac{1}{z_0}. \quad (3.1)$$
With \( \omega \) and \( k^2 \) fixed, we integrate from \( z_0 \) to some \( z_1 \approx 1 \) by the “ND Solve” command. To determine the coefficients \( A(\omega, k^2) \) and \( B(\omega, k^2) \) in Equation (2.43), we take data points of \( \phi \) near \( z_1 \) using the “Table” command, and fit the points to Equation (2.43) using the “Find Fit” command. Then \( G_R(\omega, k^2) \) is determined from the fitted values of \( A(\omega, k^2) \) and \( B(\omega, k^2) \) by Equation (2.45). We take \( K = 1 \) since its value does not affect the divergent behavior of \( G_R(\omega, k^2) \).

To visualize \( G_R(\omega, k^2) \) in the complex \( \omega \) plane, we repeat the above procedure for a lattice of \( \omega \) values. These values are plotted over the complex \( \omega \) plane with the “DensityPlot” command.

To obtain a high resolution for the pole positions by this method alone, we require very fine grids in the \( \omega \) plane. Then the computational complexity is \( O((1/\delta)^2) \) where \( \delta \) is the resolution of \( \omega \). This is not very efficient, so below we introduce two other methods, the complex Newton’s method and the Nelder-Mead method. They improve the resolution with logarithmic complexity.

### 3.1.1 Complex Newton’s Method

The complex Newton’s method is a generalization of Newton’s method to complex functions. Ideally, when a good initial guess is given, this method approaches a root step by step. Since the poles of \( G_R(\omega, k^2) \) in the complex \( \omega \) plane are the zero’s of \( 1/G_R(\omega, k^2) \), we locate the poles by applying the complex Newton’s method to \( 1/G_R(\omega, k^2) \).

We must be careful about this method’s convergence. Let a pole of \( G_R(k_\mu) \) in the complex \( \omega \) plane be

\[
G_R(\omega, k^2) = \omega^{-\nu}, \quad \nu \in \mathbb{R}_{\geq 1}.
\] (3.2)

We visualize the geometry by plotting \( |1/G_R(\omega, k^2)| \) over \( \omega \). The plots for \( \nu = 1, 1.5, 2 \) are given in Figures 3.1.1-3.1.1. We see that the geometry around \( \omega = 0 \) is like a bowl, for which the complex Newton’s method converges.

In general, \( G_R(\omega, k^2) \) involves terms additional to the divergent term of Equation (3.2). Hence \( 1/G_R(\omega, k^2) \) itself may have poles, as illustrated by Figure 3.1.1.
Figure 3-1: Plot of $|1/f(\omega)|$ over complex $\omega$ where $f = \omega^{-1}$.

Figure 3-2: Plot of $|1/f(\omega)|$ over complex $\omega$ where $f = \omega^{-1.5}$. 
3.1.2 Nelder-Mead Method

The Nelder-Mead method finds a local minimum of a real function with a complex variable. The idea is to enclose a region in the complex plane by a polygon, and determine where a nearby minimum is located in reference to this polygon. If a minimum lies inside, then we shrink the polygon; otherwise, we move the polygon “downhill” to approach the minimum. Since a pole of $G_R(\omega, k^2)$ corresponds to a local
minimum of $-|G_R(\omega, k^2)|$, we apply the method to the target function $-|G_R(\omega, k^2)|$.

Below we describe the method for a triangle. For simplicity of notation, let $g(\omega) = G_R(\omega, k^2)$.

1. **Initialization:** Start with a triangle with vertices $\omega_1, \omega_2, \omega_3$.

2. **Ordering:** Relabel $\omega_i$ such that $g(\omega_1) \leq g(\omega_2) \leq g(\omega_3)$.

3. **Reference Point:** Calculate $\omega_0 = (\omega_1 + \omega_2)/2$.

4. **Reflection:** Calculate $\omega_r = \omega_0 + (\omega_0 - \omega_3)$. If $g(\omega_1) \leq g(\omega_r) < g(\omega_2)$, then set $\omega_3 = \omega_r$, and go back to step 2. Else if $g(\omega_2) < g(\omega_r)$, go to step 5. Otherwise, go to step 6.

5. **Expansion:** Calculate $\omega_e = \omega_0 + 2(\omega_0 - \omega_3)$. If $g(\omega_e) < g(\omega_r)$, then set $\omega_3 = \omega_e$; otherwise, set $\omega_3 = \omega_r$. Go back to step 2.

6. **Contraction:** Calculate $\omega_c = (\omega_0 + \omega_3)/2$. If $g(\omega_c) < g(\omega_3)$, then set $\omega_3 = \omega_c$, and go to step 2. Otherwise, go to step 7.
7. **Reduction**: Set $\omega_2 = \omega_1 + (\omega_2 - \omega_1)/2$ and $\omega_3 = \omega_1 + (\omega_3 - \omega_1)/2$. Go back to step 2.

Although the Nelder-Mead method is more robust than the complex Newton’s method, it is also computationally more expensive.

### 3.2 Instability of the In-Falling Condition

The integration method described in Section 3.1 does not work when $\omega$ is in the lower half-plane. We study this difficulty in this section.

In the previous section, we specified the in-falling condition by Equation (3.1). At finite $z_0$, this condition is not exact because Equation (2.36) is only asymptotic. This error introduces an out-going component in Equation (2.36).

Let $\omega = \omega_r + i\omega_i$ where $\omega_r$ and $\omega_i$ are real. When $\omega_i < 0$, the out-going component grows exponentially as we integrate away from the horizon. In other words, the in-falling condition is unstable. Applying Equation (2.45) to this solution then gives the advanced Green function instead of the retarded.

Suppose that at $z = z_0$, the ratio between the out-going and in-falling amplitudes is

$$
\left| \frac{\phi_o(z_0)}{\phi_i(z_0)} \right| = \frac{C_o(\omega, k^2)}{C_i(\omega, k^2)} \exp \left( -\frac{\omega_i}{2\pi T} \log z_0 \right) = \epsilon > 0
$$

Then at $z \approx z_0$,

$$
\left| \frac{\phi_o(z)}{\phi_i(z)} \right| = \epsilon \exp \left( -\frac{\omega_i}{2\pi T} (\log z - \log z_0) \right).
$$

We see that $\phi_o$ dominates $\phi_i$ at $z$ if

$$
\log z - \log z_0 > \frac{2\pi T}{\omega_i} \log \epsilon.
$$

For $z = O(1)$, Equation (3.5) is approximately

$$
\frac{\log z_0}{\log \epsilon} > \frac{-\omega_i}{2\pi T}.
$$
There is a relation between $\epsilon$ and $z_0$. To derive it, consider the first order correction to Equation (2.36). In the $\omega \to 0$ limit, the correction factor is

$$1 + \frac{k^2 + m^2}{4\pi T} z + O(z^2). \quad (3.7)$$

Let

$$\phi(z \to 0) = C_i(\omega, k^2) \left[ \exp \left( -\frac{i\omega}{4\pi T} \log z \right) + \tilde{\epsilon} \exp \left( \frac{i\omega}{4\pi T} \log z \right) \right] \quad (3.8)$$

$$\times \left( 1 + \frac{k^2 + m^2}{4\pi T} z + O(z^2) \right) \quad (3.9)$$

where $\tilde{\epsilon} = C_0(\omega, k^2)/C_i(\omega, k^2) \in \mathbb{C}$. Then to satisfy Equation (3.1), we need

$$2\tilde{\epsilon} \exp \left( -\frac{i\omega}{2\pi T} \log z_0 \right) \approx \frac{k^2 + m^2}{i\omega} z_0. \quad (3.10)$$

Taking the absolute value on both sides of Equation (3.10) in the limit $\omega \to 0$ gives ($\epsilon = |\tilde{\epsilon}|$)

$$2\epsilon \approx \left| \frac{k^2 + m^2}{\omega} \right| z_0. \quad (3.11)$$

Assuming $|\omega| < |k^2 + m^2|$, Equation (3.11) gives a lower bound for $\log z_0/\log \epsilon$:

$$\frac{\log z_0}{\log \epsilon} \geq 1. \quad (3.12)$$

Equality occurs only in the limit $z_0 \to 0$. By Equations (3.6) and (3.12), we see that the out-going component outgrows the in-falling if

$$\omega_i < -2\pi T. \quad (3.13)$$

Even if we use a ratio of $\partial \phi(z_0) / \phi(z_0)$ that includes the first order correction given by Equation (3.7), the second order term introduces a new error. To suppress the instability, a long power series expansion at the horizon is needed.

The next method discussed uses this idea of power series expansion. However,
rather than expanding at \( z = 0 \), we expand at \( z = 1/2 \) so that the radius of convergence covers the range \((0, 1)\). Integration is then unnecessary.

### 3.3 Matrix Method

The matrix method approximates \( \phi(z) \) by a series expanded at \( z = 1/2 \) after factoring out the asymptotic behaviors.

\[
\phi(z) = (1 - z)^{-A} \exp \left( \frac{i \omega}{4\pi T} \log z \right) h(z)
\]

where

\[
h(z) = \sum_{n=0}^{N} c_i (z - \frac{1}{2})^n.
\]

Note that the divergent factors in Equation (3.14) are chosen to be the in-falling term at the horizon and the normalizable term at the boundary. \( h(z) \) converges nicely only if \( \phi(z) \) satisfies the two conditions.

We substitute Equation (3.14) into Equation (2.33) and clear out the \( z \)-dependent denominators. Write the resulting equation as

\[
p(z) \partial^2 h(z) + q(z) \partial h(z) + r(z) h(z) = 0.
\]

Then

\[
h(u) = \sum_{n=0}^{\infty} h_n u^n,
\]

\[
p(u) = \sum_{n=0}^{4d} p_n u^n,
\]

\[
q(u) = \sum_{n=0}^{4d-1} q_n u^n,
\]

\[
r(u) = \sum_{n=0}^{4d-2} r_n u^n.
\]

The series for \( p(u), q(u), r(u) \) are finite because the \( z \)-dependent terms in the denomi-
nators are cleared out. The coefficients in Equations (3.18)-(3.20) are obtained by the “Coefficient” command. Substituting Equations (3.17) to (3.20) into Equation (3.16) gives

\[
0 = \sum_{m=0}^{4d} p_m u^m \sum_{n=0}^{\infty} (n+1)(n+2) h_{n+2} u^n + \sum_{m=0}^{4d-1} q_m u^m \sum_{n=0}^{\infty} (n+1) h_{n+1} u^n \\
+ \sum_{m=0}^{4d-2} r_m u^m \sum_{n=0}^{\infty} h_n u^n
\]  

(3.21)

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[ (m+1)(m+2)p_{n-m} h_{m+2} + (m+1)q_{n-m} h_{m+1} + r_{n-m} h_m \right] u^n
\]  

(3.22)

where \( p_l = 0 \) for \( l < 0 \) or \( l > 4d \), \( q_l = 0 \) for \( l < 0 \) or \( l > 4d - 1 \), and \( r_l = 0 \) for \( l < 0 \) or \( l > 4d - 2 \). Hence we have a recursion relation for \( h_i \) that involves at most \( 4d \) terms:

\[
\sum_{m=0}^{n} (m+1)(m+2)p_{n-m} h_{m+2} + (m+1)q_{n-m} h_{m+1} + r_{n-m} h_m = 0
\]  

(3.23)

Enforcing this relation for \( n = 0, \ldots, N \) and assuming \( h_i \approx 0 \) for \( i > N \) gives a matrix equation

\[
A \begin{bmatrix} h_0 \\ \vdots \\ h_N \end{bmatrix} = 0
\]  

(3.24)

where \( A \) is \((N+1)\)-by-\((N+1)\). Then for there to exist a nontrivial set of \( h_i \) satisfying Equation (3.24), we need

\[
\det A = 0.
\]  

(3.25)

According to [2], when \( N \) is large, Equation (3.25) ensures that Equation (3.17) is convergent.

Because each row of \( A \) is a finite \((4d)\) term recursion, \( \det A \) may be calculated efficiently by the “Determinant” command. To visualize \( \det A \) in the complex \( \omega \) plane, we use the “DensityPlot” command. The locations of poles are then found to high
accuracy by applying the complex Newton’s method (§ 3.1.1) to \( \det A \), or applying the Nelder-Mead method (§ 3.1.2) to \(|\det A|\).

The Nelder-Mead method has two main advantages over the integration method. First, the instability of the in-falling condition when \( \text{Im} \omega < 0 \) can be suppressed by choosing a large \( N \). Second, since the boundary condition is implied in the factorization of Equation (3.14), we avoid the errors from fitting \( \phi(z) \) near \( z = 1 \). A drawback to the Nelder-Mead method is that it obtains only the location of poles of \( G_R(\omega, k^2) \) but not the physical \( G_R(\omega, k^2) \).
Chapter 4

Results

This chapter presents the numerical results. We study the behavior of \( G(\omega, k^2) \) as \( k^2 \) and \( T \) are varied, and also compare the integration method and the matrix method. The following values of parameters are used throughout this chapter:

\[
d = 3, \quad \mu_q = 2, \quad m = \sqrt{\frac{1}{12}}. \tag{4.1}
\]

At zero temperature, the critical momentum given by Equation (2.46) is \( k_0^2 = 1/3 \). At a small but finite temperature, \( k_0^2 \) is slightly above \( 1/3 \).

Section 4.1 studies the behavior of \( G(\omega, k^2) \) under varying \( k^2 \) at finite \( T \). Section 4.2 studies this behavior at zero \( T \). In those two sections, we are interested in the whole complex plane, so we use the matrix method. In Section 4.3, we redo a finite \( T \) calculation using the integration method and compare the result with the matrix method result.

4.1 Varying \( k^2 \) at Finite Temperature

We fix \( T = 0.01 \), and choose the following four values for \( k^2 \): 1/12, 1/3, 1. These values are interesting because they scan over \( k_0^2 \). The results of applying the matrix method with \( N = 500 \) are shown in Figures 4.1-4.1.

We do not find any pole in the upper half-plane. We also do not see any significant
change in the figure as $k^2$ is varied. For $\text{Im}\omega > -0.3$, there are poles lying nicely on the negative $y$-axis. These poles form a branch cut. However, for $\text{Im}\omega < -0.3$, there is a triangular region with very complicated geometry.

For $k^2 = 1$, the result with $N = 300$ and $N = 150$ are shown in Figures 4.1-4.1. We see that the complicated geometry disappears are we decrease $N$. This is peculiar because a larger $N$ supposedly gives a finer result. The problem is perhaps that our way of calculating the determinant is not reliable when the matrix is very large.

### 4.2 Varying $k^2$ at Zero Temperature

We use the same values of $k^2$ but at $T = 0$. The results of applying the matrix method with $N = 500$ are shown in Figures 4.2-4.2. We see some poles lining up near the negative $y$-axis and forming a branch cut. Again, no pole is found in the upper half-plane, and there is no significant change in the figure under varying $k^2$.

According to [9], for $k^2 = 0, 1/12$, there are poles in the upper half-plane close to the origin. This is not observed in our results. The problem may also lie in the instability of determinant calculation.

### 4.3 The Integration Method

We apply the integration method for $T = 0.01$ and $k^2 = 1$. The result is shown in Figure 4.3. There is a pole on the negative $x$-axis. Because of the instability of the in-falling condition, we expect this method to fail below $-2\pi T \approx -0.1$. However, this pole has a clear structure within a radius of $\approx 2$, contradicting this instability. The result is also inconsistent with the matrix method result in Figure 4.1. It is possible that the integration method is unstable due to the errors from fitting $\phi(z)$ near the boundary.

To resolve the problems in the above results, future improvement in the robustness of the codes is needed.
Figure 4-1: Density plot of \( \log |\text{det} A| \) over complex \( \omega \) for \( T = 0.01, k^2 = 0 \) and \( N = 500 \).
Figure 4-2: Density plot of $\log |\det A|$ over complex $\omega$ for $T = 0.01$, $k^2 = 1/12$ and $N = 500$. 
Figure 4-3: Density plot of $\log |\det A|$ over complex $\omega$ for $T = 0.01$, $k^2 = 1/3$ and $N = 500$. 
Figure 4-4: Density plot of $\log |\det A|$ over complex $\omega$ for $T = 0.01$, $k^2 = 1$ and $N = 500$. 

\[ \begin{array}{cccc} 
1.0 & -1.0 & -0.5 & 0.0 & 0.5 & 1.0 \\
-1.0 & -0.5 & 0.0 & 0.5 & 1.0 & \\
-0.5 & 0.0 & 0.5 & 1.0 & -0.5 & \\
0.0 & 0.5 & 1.0 & -0.5 & 0.0 & \\
0.5 & 1.0 & -0.5 & 0.0 & 0.5 & \\
1.0 & -0.5 & 0.0 & 0.5 & 1.0 & \\
\end{array} \]
Figure 4-5: Density plot of $\log |\det A|$ over complex $\omega$ for $T = 0.01$, $k^2 = 1$ and $N = 300$. 
Figure 4-6: Density plot of $\log|\det A|$ over complex $\omega$ for $T = 0.01$, $k^2 = 1$ and $N = 150$. 

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1.0  0.5  0.0  -0.5  -1.0  
-1.0  -0.5  0.0  0.5  1.0
Figure 4-7: Density plot of $\log \det A$ over complex $\omega$ for $T = 0$, $k^2 = 0$ and $N = 500$. 
Figure 4-8: Density plot of $\log|\det A|$ over complex $\omega$ for $T = 0$, $k^2 = 1/12$ and $N = 500$. 
Figure 4-9: Density plot of $\log |\det A|$ over complex $\omega$ for $T = 0$, $k^2 = 1/3$ and $N = 500$. 
Figure 4-10: Density plot of $\log |\det A|$ over complex $\omega$ for $T = 0$, $k^2 = 1$ and $N = 500$
Figure 4-11: Density plot of $\log|G(\omega, k^2)|$ over complex $\omega$ for $T = 0.01$ and $k^2 = 1$. 
Bibliography


