Plug Repairs of Marine Glass Fiber / Vinyl Ester Laminates
Subjected to In-plane Shear Stress or In-plane Bending Moment

by

Roberto Urrutia Valenzuela

B. S. in Naval Electrical Engineering
Naval Polytechnic Academy, Chile 1997

Submitted to the Department of Mechanical Engineering
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Signature of Author

Department of Mechanical Engineering
May 7, 2010

Certified by

James H. Williams, Jr.
School of Engineering Professor of Teaching Excellence, Emeritus
Thesis Supervisor

Accepted by

David E. Hardt
Ralph E. and Eloise F. Cross Professor of Mechanical Engineering
Chairman, Department Committee on Graduate Students
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Abstract

Glass fiber / vinyl ester composite laminates represent an important class of modern
fiber composites being proposed or used in state-of-the-art shipbuilding. This thesis
examined the effectiveness of chopped strand mat (CSM) plug repairs of glass fiber /
v vinyl ester woven roving laminates subjected to in-plane shear and bending. An
advantage of this type of repair scheme is its simplicity when compared to more
traditional schemes such as scarf or step repairs.

The stress concentrations around circular holes in glass fiber / vinyl ester woven roving laminates subjected to in-plane shear and bending were calculated before and after repairs
using CSM plugs, also of glass fiber / vinyl ester, having varying fiber volume fractions.
The laminates were orthotropic and ranged from balanced to unidirectional woven
roving, and the CSM plug fiber volume fractions ranged from 0 to 0.40.

For in-plane shear stress, as the plug fiber volume fraction increased from 0 to 0.40, the
maximum stress concentration along the circular holes in the laminate was reduced from
about 25% to 61%. For in-plane bending, as the plug fiber volume fraction increased
from 0 to 0.40, the maximum stress concentration in the laminate was reduced from about
25% to 45%.

Thesis Supervisor: James H. Williams, Jr.
Titles: School of Engineering Professor of Teaching Excellence,
Emeritus, Charles F. Hopewell Faculty Fellow, Professor of
Writing and Humanistic Studies
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1. Introduction

1.1 Background and motivation

Shipbuilding is more than 3,000 years old, but only a few important changes have been made to ship materials during this period. Ancient small boats were built with animal leathers; then, wood was adopted as the primary shipbuilding material for 2,000 years. In the 18th century, an important and revolutionary change took place: the adoption of steel. More recently, in the late 1940s, different types of composite materials were introduced into shipbuilding.

Increasing oil prices are a driving force behind important research in the field of materials for transportation machines. This research is yielding good results in the search for lighter construction materials. Some industries, like the aeronautical industry have taken advantage of the development of these materials of higher specific strength and stiffness. In commercial airplanes, where there is a clear correlation between weight and fuel performance, composite materials have been widely used in structural components since the 1980s [1].

The use of composite materials in the shipbuilding industry has been extensive in small boats since the 1950s. In the 1970s composites were introduced into larger ships for some specific naval applications such as minehunters and ship superstructures for reduced radar signatures. By the year 2000, the development of more advanced composite materials allowed the full hull construction of corvettes measuring 72 meters at a displacement of 650 tonnes (Swedish Visby Class corvette). As newer and more capable composite materials have been developed, the size of ship hulls built completely of composites has also been increased. Thus far, most of the major developments have been made in naval ships. It is anticipated that a more extensive use of composite materials will be initiated when their production cost and mechanical properties become suitable for bigger commercial ships.

The direct impact of rising fuel costs in maritime transportation is a powerful motivation for introducing composite materials into commercial ships. Due to better corrosion resistance and lower fuel consumption, Goubalt and Mayers [2] predict that the operational costs of a composite patrol boat are 7% less than for a steel boat of the same
size. For a corvette-size ship, this life cycle cost reduction could reach 15% [3]. Another motivation for building lightweight commercial ships is the growing restriction in emissions of greenhouse gases and pollution. For instance, a lighter commercial ship consuming less fuel during a 25-day trip across the Pacific Ocean will also reduce the amount of greenhouse gases and pollution released into the atmosphere.

Regarding fire safety and combustibility, there are new concerns for using composite materials in commercial shipbuilding. Since 2002 a new rule promulgated by the International Convention for the Safety of Life at Sea (SOLAS) allows construction using other materials than steel provided that they can render the same fire safety levels as steel. Although many composite materials are flammable, they have a low thermal conductivity relative to metals [11]. In recent years, successful empirical tests have been carried out to demonstrate that composite materials can have a high fire resistance performance when they are properly protected [4].

Another major consideration in the use of composite materials is the cost-effectiveness of their repair. The principal motivation of this thesis is to extend the work initiated by Michelis [16] on plug repair schemes in marine glass fiber / vinyl ester. The work by Michelis investigated the plug repair scheme as a permanent structural repair, as a temporary repair, and as a cosmetic restoration.

The simplicity of the plug repair scheme makes it attractive for the marine industry where high costs and long time frames are usually required to execute more traditional and complex structural repairs such as patch or scarf schemes.

Using analytical expressions, Michelis calculated the stress reductions that are achievable using the plug repair scheme in glass fiber / vinyl ester laminates subjected to uniaxial tension. In this thesis, the calculations are taken one step further: the stress reductions are calculated for the plug repair scheme using the same laminates and plug materials as Michelis, but for plates subjected to pure in-plane shear stress and pure in-plane bending moment.
1.2 Glass fiber / vinyl ester in ship structures

This section introduces the physical properties of the constituents of glass fiber / vinyl ester and compares them with the physical properties of the constituents of other composite materials in use in the marine industry. Glass fiber / vinyl ester is composed of E-glass as the reinforcing elements and vinyl ester as the matrix material. This composite is a cost-effective solution with characteristics that are suitable for marine applications and specifically for shipbuilding.

1.2.1 The reinforcing fiber: E-glass

E-glass fiber is the most extensively used fiber material used in the marine industry [12]. A dominant factor in the choice of materials in shipbuilding is cost. The selection of more expensive fibers with better properties, notably carbon and aramid, are restricted to specific applications such as high-performance sailing yachts. Examples of these applications are some expensive prototypes of yachts built to race the America’s Cup sailing regatta, where higher cost is accepted in exchange for higher performance. Apart from such special cases, glass fibers are used in 95% of marine applications [5].

The underlying reason for the use of glass fiber in large-volume applications is its high specific strength. When the specific strength of different materials is compared, the combination of glass fiber with polyesters, epoxies, or vinyl esters outperforms metals. Nevertheless, if the specific stiffness is compared among materials, only carbon fiber composites can compete with metals. The more flexible nature of glass fibers makes them suitable for applications where high strength and low weight are required but not high stiffness.

Glass fibers are classified into two groups: S-glass and E-glass. S-glass has better mechanical properties but costs 20 to 30 times E-glass. For this reason S-glass is mostly restricted to aerospace applications where higher performance is required. A cheaper version of S-glass is S-2 glass, which costs about 8 times E-glass and provides intermediate mechanical properties. For cost reasons, S-2 glass has suppressed the use of S-glass [12].
Thus, E-glass fiber offers high strength, low stiffness, and low cost relative to S-glass and S-2 glass. A quantitative comparison of cost and typical properties of E-glass, S-glass, aramid, and carbon fibers is presented in Table 1.1 [12, 13].

Table 1.1. Comparison of typical properties of reinforcing fibers. E-glass is the most extensively used in the marine industry because of its cost-effective properties [12, 13].

<table>
<thead>
<tr>
<th>Fiber</th>
<th>Young's modulus [GPa]</th>
<th>Poisson's ratio</th>
<th>Tensile strength [MPa]</th>
<th>Failure strain [%]</th>
<th>Relative cost to E-glass</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-glass</td>
<td>72</td>
<td>0.2</td>
<td>2.4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>S2-glass</td>
<td>88</td>
<td>0.2</td>
<td>3.4</td>
<td>3.5</td>
<td>8</td>
</tr>
<tr>
<td>S-glass</td>
<td>90</td>
<td>-</td>
<td>4.5</td>
<td>-</td>
<td>20 to 30</td>
</tr>
<tr>
<td>Aramid</td>
<td>124</td>
<td>-</td>
<td>2.8</td>
<td>2.5</td>
<td>15</td>
</tr>
<tr>
<td>(Kevlar 49)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Carbon</td>
<td>297</td>
<td>-</td>
<td>4.1</td>
<td>1.4</td>
<td>50</td>
</tr>
<tr>
<td>(Thornel T-40)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1.2.2 The matrix resin: Vinyl ester

The matrix materials used in the marine industry can be classified into three main groups: unsaturated polyesters, epoxies, and vinyl esters. The matrix material affects some important characteristics of the composite such as:

- Off-axis strength
- Cost
- Damage tolerance
- Corrosion resistance
- Resistance to water absorption
- Thermal stability
- Toxic fumes emissions
- Matrix-fiber bonding
- Resistance to delamination.
Polyester resins are widely used in the marine industry because they provide an attractive combination of mechanical strength and resistance to the marine environment [5]. At the same time, they offer ease of fabrication and lower cost compared to epoxies and vinyl esters.

Epoxy resins offer the best strength, toughness, and corrosion resistance of the three matrix types, but epoxies have curing requirements that make the fabrication processes more difficult and hazardous than polyesters and vinyl esters [12]. Another important drawback of epoxies is their cost; they are the most expensive of the three types of matrices with a cost range that can be 2 to 3 times higher than polyesters [12].

Vinyl ester resins are a cost-effective solution between polyesters and epoxies. A quantitative comparison of cost and typical properties of resins from Smith [12] is presented in Table 1.2.

<table>
<thead>
<tr>
<th>Resin</th>
<th>Young's modulus [GPa]</th>
<th>Poisson's ratio</th>
<th>Tensile strength [MPa]</th>
<th>Tensile failure strain [%]</th>
<th>Compressive strength [MPa]</th>
<th>Relative cost to polyester (isophthalic)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polyester (isophthalic)</td>
<td>3.6</td>
<td>0.36</td>
<td>60</td>
<td>2.5</td>
<td>130</td>
<td>1</td>
</tr>
<tr>
<td>Vinyl ester (Derakane 411-45)</td>
<td>3.4</td>
<td>-</td>
<td>83</td>
<td>5</td>
<td>120</td>
<td>1.8</td>
</tr>
<tr>
<td>Epoxy (DGEBA)</td>
<td>3</td>
<td>0.37</td>
<td>85</td>
<td>5</td>
<td>130</td>
<td>2.3</td>
</tr>
</tbody>
</table>

Ship construction using vinyl esters is as easy as ship construction using polyesters, but vinyl esters offer better mechanical properties. For instance, relative to (isophthalic) polyester, Smith [12] mentions the following advantages of vinyl esters:

- Superior resistance to water and chemical attack
- Superior retention of strength and stiffness at elevated temperature
• Greater toughness associated with a higher failure strain (up to 6%, compared with 2-3% for typical polyesters)

The higher strength of glass/vinyl ester relative to glass/polyester is not only due to the stronger vinyl ester matrix. Adhesion defects can occur between fibers and the matrix resin in glass/polyester composites after curing. Such defects are less likely in glass/vinyl ester composites. Adhesion defects between fibers and the polyester matrix are caused by the relatively high curing shrinkage of polyester resins, which results in internal stresses [13]. Polyester resins shrink about 5-8% during the curing process.

Besides its good mechanical properties and moderate cost, a further advantage of vinyl esters is that they can be produced in a fire retardant resin. This advantage makes vinyl esters very attractive for shipbuilding applications. The fire retardant version of the vinyl ester, called Brominated, is used in all U.S. Navy craft [13].

The use of matrix resins has been affected by legal regulations restricting the emissions of styrene [14]. Styrene is a chemical component used as a solvent to provide adequate resin viscosity. Styrene accounts for 40-50% of the weight of conventional resins and it is evaporated at the surface of the composite during the curing process. Some resin manufacturers are offering formulations with low styrene content. Because the reduction of styrene content increases the viscosity, other changes are required to keep the viscosity within acceptable limits. Vinyl ester seems to be the most attractive resin in which to reduce the styrene content. Table 1.3 presents some low-styrene resins where three of them are based on vinyl ester.

Table 1.3. Commercial matrix resins with low-styrene content.

<table>
<thead>
<tr>
<th>Commercial Name</th>
<th>Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derakene 441-400</td>
<td>Vinyl ester</td>
</tr>
<tr>
<td>CoRezyn 8360-34</td>
<td>Vinyl ester</td>
</tr>
<tr>
<td>COR-75-AA-034</td>
<td>Isophthalic polyester</td>
</tr>
<tr>
<td>Hydrex 33254</td>
<td>Vinyl ester</td>
</tr>
</tbody>
</table>
1.2.3 The Fabrics: Chopped strand mat and woven roving

Strands of glass fiber can be laid in different configurations to form fabrics with diverse mechanical properties. By changing the configuration of the reinforcement, the mechanical properties of a composite can be adjusted to meet varying strength and stiffness requirements throughout the structure. The types of fabrics studied in this thesis are chopped strand mat (CSM) and woven rovings (WR), the latter ranging from balanced WR to biased unidirectional configurations.

CSM is formed by fiber strands that are chopped into lengths of 5 to 10 [mm] and randomly distributed in the lamina. The traditional use of CSM is in a spray-up laminating system, where the chopped fiber, resin, catalyst, and accelerator are sprayed together over a mold. This glass-fiber mixture is consolidated by manual rolling, producing a laminate with a fiber weigh fraction of about 0.25 to 0.3 [12]. In this thesis CSM is studied as a plug material for composite repair.

WR fabrics are formed by orthogonal interwoven fiber strands. This type of fabric is the most extensively used in the marine industry [19]. There are different ways to set up the woven configuration; examples of WR are presented in Figure 1.1. Plain weave is the most common set-up [16].

![Figure 1.1. Different weaves of orthogonal woven roving configuration. Plain weave woven roving is the most common set up [5].](image)
The mechanical properties of the laminate can be adjusted by biasing the amount of fibers laid in the orthogonal directions. High fiber content is desirable in some applications and specific regions of ships. Unidirectional rovings allow the highest fiber volume fraction and higher strength in the direction of the fibers. On the other hand, balanced woven rovings have equal amounts of fibers in both directions; thus, providing the same mechanical properties in both. The bias is expressed in percentage and ranges from 50% (balanced) to 100% (unidirectional) along a specific direction or axis.

1.3 Origins of shear stress and bending moments in ship structures

The objective of this section is to present a qualitative description of the loading affecting ship structures and the standard approaches that have been used to estimate these forces and corresponding stresses. The derivation of formulas and definitions of concepts used to estimate these forces are out of the scope of this thesis. Detailed derivations of these formulas and definitions can be found in many textbooks on the mechanics of materials and ship structural design. Nevertheless, it is these forces and stresses that lead to the loadings on the damaged and repaired structures that are analyzed in Chapter 2.

The structure of a ship floating on the sea surface is subjected to loads of different origins. The shape of the immersed hull produces a non-uniform distribution of the buoyancy force along the length of the ship. The weight of the ship structure, equipment, and cargo are also generally non-uniformly distributed along the length, which produces uneven loading. Further, when the ship is underway, the motion of crests and troughs of the waves along the hull change the distribution of the buoyancy force. The change of the buoyancy distribution can be considered as a random process, depending largely on the wave length, wave height, and angle at which the ship encounters the moving waves. Due to the relative speed between the ship and wave systems, and the changing nature of the buoyancy distribution, a ship in a seaway is subjected to both static and dynamic forces. The determination of the loading and response of the structure in such conditions constitutes a complex problem. One way to deal with this problem is to reduce it to a series of individual problems that can be superposed [5]. For the purposes of structural
design, it is common to reduce the ship loading problem to a static one [6]. When inertial forces due to ship motion are excluded, the main sources of loading are gravity and buoyancy.

1.3.1 Standard shear and moment calculation using the hull girder approximation

In this approach the hull is idealized as a hollow box referred to as a “hull girder” and it is assumed to act in accordance with beam theory [7]. The hull girder approach gives approximate results that are meant to be purely notional and comparative; nevertheless, this approach has been used to design ship structures for almost 100 years and it has proven to be a safe calculation procedure [5].

The standard calculation procedure for forces and moments considers the ship to be positioned statically in a wave system. It is customary to assume two wave profiles acting on the hull of the ship. These wave profiles are standardized and produce two loading conditions in the hull that are known as “sagging” and “hogging”. Figure 1.2 presents hogging and sagging waves acting on a hull.

Figure 1.2. Hogging and sagging waves act on the hull girder, producing deflections and stresses [7].
The ship is in a hogging condition when the wave crest is at or near the central point of the hull length and the troughs are at the forward and aft perpendiculars. The ship is in a sagging condition when the wave has its trough at the central point of the hull length and the crests are at the forward and aft perpendiculars. The length of both waves corresponds to the length of the ship at the waterline.

To calculate the forces and moments in the hogging and sagging conditions, the ship and the wave are assumed to be stationary and the total weight of the ship is balanced by the total buoyancy. In this condition, curves of weight per unit length and buoyancy per unit length are deduced. These non-uniform distributions of weight and buoyancy along the length of the ship produce a curve of net loading, which generates bending moments and shear forces on the hull girder. Mathematically, the shearing force is obtained by the integration of the loading curve along the length; and the bending moment is obtained by the integration of the shearing force curve along the length. Figure 1.3 presents a typical set of curves of net loading, shearing force and bending moment.

Figure 1.3. Distributions of net loading, shearing force, and bending moment along the length of the hull structure [5].
1.3.2 Bending moment and maximum normal stresses

The axial bending moment distribution along the length of the ship enables the calculation of the normal stresses at each longitudinal cross section. The assessment of the maximum normal stresses in the design stage is an important way to verify that the stress levels expected in the structure are within the limits of the allowable stress of the materials being used.

Once the bending moment distribution is obtained, the bending stress at any longitudinal location \( x \) can be calculated from the following equation as

\[
\sigma_x = -\frac{M_x y}{I_x}
\]

where \( M_x \) is the axial bending moment at longitudinal position \( x \)
\( y \) is the vertical distance from the neutral axis
\( I_x \) is the second moment of area about the neutral axis.

The maximum normal stress generally occurs in the portion of the structure most remote from the neutral axis. In ships, this portion of the structure corresponds to the deck or bottom plating of the hull [9]; longitudinally, this portion is usually near the mid-length of the hull. Figure 1.4 presents a schematic of the stresses at different vertical positions from the neutral axis on a ship. In this figure the maximum stress occurs in the deck plating at a distance \( c \) from the neutral axis.

The beam equation above is derived from strength of materials where it is assumed that plane cross sections remain plane in a beam subjected to a pure bending moment. Even though the structures on ships are much more complex than a beam, this equation generally gives useful results [10].
1.3.3 Shear stress in the hull structure

When a beam is loaded by transverse vertical forces, there is a vertical shearing force $S$ acting on the cross section [7]. The shearing force tends to move one section of the structure vertically relative to the adjacent section. This effect is a consequence of the non-uniform loading distribution along the hull length. Figure 1.5 represents (a) the shearing force between two adjacent portions of the hull and (b) the shear stress distributed on the cross-sectional area.

The presence of the shearing force produces a corresponding shear stress. Due to the complementary property of shear [8], associated shear stresses act on the longitudinal and vertical planes of the hull components.

The shear stress does not have a constant value over the entire area of the cross section. In Figure 1.5 (b) the intensity of the stress is schematically represented by the thickness of the solid black arrows. For instance, in the case of a solid rectangular beam the shear stress varies along the vertical direction and is a function of the shearing force $S$, the moment of inertia about the neutral axis, and the width of the beam. In such a case, the maximum shear stress occurs at the neutral axis. In real ships, the geometry of the
cross section is much more complex than in a rectangular solid beam. In the ships case, the calculation of the shear stress becomes more complicated and is usually done with the aid of finite element analysis software.

Figure 1.5. (a) Shearing force $S$ between two adjacent sections of the hull and (b) the schematic of shear stress distribution over the cross-sectional area.
1.4 **Repair schemes of composite materials used in ship structures**

This section introduces a brief description of the most common repair schemes for composite materials in marine structures. The objective is to summarize the basic concept of each scheme, so their advantages and disadvantages can be compared against the plug repair scheme that is studied in this thesis.

Composite structures at sea can be damaged by the impact of debris in the water, repeated sliding of the hull against a mooring or wharf, or the impact of falling objects inside compartments, for instance, in a cargo bay or machinery space. These damage loads can produce surface scratches, matrix cracking, fiber failure or rupture, and in severe cases penetration of the hull.

Depending on the type and severity of the mechanical damage, different types of repair schemes may be applied. The role of nondestructive evaluation (NDE), whether elementary such as visual with the unaided eye or sophisticated, is to characterize the damage to assist a repair decision. Though NDE is an important phase of the entire repair procedure, it will be not discussed in this thesis.

The objective of the repair can vary, depending on numerous mission requirements and circumstances, but it is likely to include [17] the following:

- Restoration of static strength, stiffness, and/or elastic stability
- Restoration of fatigue performance and long-term durability
- Restoration of surface profile

A standardized description of the types of damage and repair classifications for marine composite materials is proposed by the American Society for Materials (ASM) [17]. This is presented in Table 1.4.
Table 1.4. Description of the types of damage and repair classifications for marine composite materials [17].

<table>
<thead>
<tr>
<th>Damage Type</th>
<th>Repair Classification</th>
<th>Damage Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-structural</td>
<td>Mark and Monitor</td>
<td>This approach is undertaken when the nature of the damage does not warrant any form of repair. With this technique, it is necessary to record all information about the nature of the damage in order to be able to compare any worsening of the defect while the structure is subjected to operational loading.</td>
</tr>
<tr>
<td>Temporary</td>
<td>Cosmetic</td>
<td>Cosmetic repairs are designed to repair localized surface defects to the original profile and to prevent moisture ingress.</td>
</tr>
<tr>
<td>Temporary</td>
<td></td>
<td>Temporary repairs are to be conducted only when there are operational reasons that prevent the recommended repair approach. These types of repairs must be made permanent at the next designated maintenance period.</td>
</tr>
<tr>
<td>Structural</td>
<td>Minor</td>
<td>Structural minor restoration of the structure to full serviceability, requiring increased inspection frequencies to ensure that the repair remains effective.</td>
</tr>
<tr>
<td>Structural</td>
<td>Major</td>
<td>Major restoration of the composite structure to its original integrity.</td>
</tr>
</tbody>
</table>

Regarding the repairing materials, ABS rules [20] require the following:

- The resin has to be capable of bonding to the cured sound laminate of the craft being repaired.
- The fabric has to be of the original, primary fiber reinforcement whenever practical. Alternative fabrics are allowed if they are similar in type and weight to those being replaced.

As stated above, the assessment of the size and depth of the damaged area are not within the scope of this thesis, but determining the extent of the damage inside the laminate is always the first step of any repair procedure. Specific information about nondestructive evaluation techniques used in composite ship structures is available in Ref. [21].
1.4.1 Patch repair

The patch repair scheme is used for minor structural damage of small boats and is often adequate for hulls thinner than 10 [mm] [17]. This scheme can also be used to repair decks and superstructures in large composite ships. The geometry of this scheme is presented in Figure 1.6. In this scheme the load is taken over and around the damaged area; therefore, the patch must be capable of withstanding the high peel and shear stresses that develop at the edges of the overlap [18]. The edges of the patch are tapered by an angle of less than 5° to minimize the edge peel. The sound material needs to be sanded and cleaned with a solvent (acetone) before the application of the patch; good bonding of the patch over the damaged laminate is critical to achieving an effective path.

Ease of application is the main advantage of the patch repair. It can be performed as a field repair when the boat is dockside. Nevertheless, there are two main drawbacks; it can be used only for minor structural damage and patches rarely recover more than 70% of the original strength due to uneven shear stress distribution along the adhesive layer [17].

![Figure 1.6. Geometry of the patch repair scheme used in marine applications. In cases of minor damage, the patch is usually applied on one side of the laminate [17].](image)

1.4.2 Scarf repair

The scarf scheme is used to repair major structural damage and is recommended whenever it is practical [17]. Depending on the thickness of the damaged laminate and on the size and depth of the damaged area, three types of scarf schemes are used: single-
scarf, double scarf, and partial-thickness scarf. The geometry of these schemes is presented in Figure 1.7.

![Geometry of the three types of scarf repair schemes](image)

**Figure 1.7. Geometry of the three types of scarf repair schemes used in marine applications. The scheme to be applied depends on the thickness of the sound laminate and size of the damage [17].**

The single-scarf repair is used when there is no access to both sides of the damaged laminate, or when laminates are no thicker than several millimeters. A backing plate with the same profile as the surrounding structure is needed to mold the scarf. This ensures that the repair shape coincides with the geometry of the laminate. Figure 1.8 (a) and (b) present a single-scarf repair with the molding backing plate installed and the finished repair, respectively.
The double-scarf repair is strongly recommended for repairing damage that severely compromises the mechanical properties of the structure, such as holes through the hull near the waterline. Usually, a molding backing plate is also required in this type of repair. The backing plate may be bolted provisionally while the first half of the scarf is being installed. Figure 1.9 (a) presents the backing plate bolted to the laminate, and the finished double-scarf repair is shown in Figure 1.9 (b).

Partial-thickness scarf is suitable to repair thick composites that are damaged close to the surface of the laminate, such as shallow delaminations or gouges that penetrate partway through the thickness.

The main advantage of the scarf repairs is that they are considered to be adequate for permanent structural repair. They are durable and can restore up to 90% of the original strength and stiffness of the undamaged structure, although this depends on the quality of the repair and whether it is loaded in compression, tension, or bending.

The disadvantage of these schemes is the complexity of the repair procedure. It is affected by factors such as matching plies, properties of the adhesive, scarf angle and, quality of workmanship. Therefore, in some cases much less than 90% of the undamaged mechanical properties are restored [17]. The 5° scarf angle deserves special attention in
this respect. For instance, application of the single-scarf repair to a circular area of 50 [mm] diameter in a laminate 60 [mm] thick\(^1\), requires removal of sound material within a diameter of 1,422 [mm] from the center of the damaged area. This relatively large area that is tapered back (1.58 \([\text{m}^2]\)) requires adequate surface preparation and high quality workmanship to achieve good restoration of the mechanical properties. These conditions make scarf repairs expensive and time consuming.

1.4.3 Step repair

The step repair is also used to repair major structural damage. This scheme has a similar geometry to the scarf scheme. In step repair, the tapered area is shaped to a stepped surface. This is made by using a cut and peeling-back process. This process is repeated within the peeled-out area, down to the depth of the damage. Figure 1.10 presents the schematics of the step repair in a single-skin composite.

---

\(^1\) The laminate thickness required in the hull of 40 to 60 [m] minehunters lies in the range 20 to 50 [mm] and can be as high as 150 [mm]. Ref.[12]
The step repair scheme is more complex than the scarf repair, but it is often selected in preference of the scarf scheme when the laminate is subjected to bending loads (which is the case for ship structures) [17]. Major structural repairs of the glass-reinforced plastic (GRP) minehunters in the UK Navy are undertaken using this scheme. Nevertheless, due to the time consuming and expensive nature of the step repair, the UK Navy decided to investigate the damage tolerance of the GRP to attempt to reduce the number of such repairs. This investigation enabled a 50% reduction in the number of hull repairs undertaken, with consequent savings in cost and time [24].

1.4.4 Plug repair

Currently, the plug repair scheme is used to repair laminates with damage produced by localized impact or through-bolt failure. In this scheme, the damaged area of the laminate is cut in a circular shape and removed. The circular hole is then plugged with a filling material that depends on the type of structure and the remaining sound laminate material. Authors recommend plug repairs for minor damage in both sandwich [23] and single-skin composites [20, 22].

The process and geometry of a plug repair in a single skin laminate are presented in Figure 1.11. The interior walls of the hole must be cleaned out to remove grease and dust using a detergent. The area must be fully dry before the application of the plug.
material. A backing plate following the contour of the structure should be secured to the laminate. A wax parting agent should also be used on the backing plate to ensure that a smooth, polished laminate surface is retained. This plate is removed after the plug material is cured. The outer surface of the repair may be sanded as necessary.

![Figure 1.11. Plug repair process and geometry for localized impact damage. (a) Damaged material is removed. (b) A backing plate is used to mold the plug material and is removed after the plug material is cured [20].](image)

A big advantage of a plug repair is its simplicity. Compared to the scarf or step schemes, little undamaged material needs to be removed by tapering back, and surface preparation must be done only on the interior of the hole. The scarf and plug schemes can be compared using the same example presented above in section 1.4.2. The application of a plug repair to a circular area of 50 [mm] diameter in a laminate 60 [mm] thick does not require the removal of sound material. The surface preparation needs to be made on an area of 0.009 [m²], which is considerably less than the 1.58 [m²] required to repair an area of the same size using the scarf scheme.

Regarding the application of the plug repair scheme, a comprehensive literature review is presented by Michelis [16]. In his review, the plug scheme seems to be limited
to minor non-structural repairs up to hole diameters of 50[mm]. In the Design Standard of the UK Navy [22], plug repairs are classified as non-structural repairs since “they are contained within the 300 [mm] diameter tolerance envelope”. In this document, all damage smaller than 300 [mm] is classified as “Mark and monitor” or “Cosmetic”; therefore, non-structural.
Analyses of plug repair scheme

2.1 Constitutive properties of glass fiber / vinyl ester woven roving (WR) and glass fiber / vinyl ester chopped strand mat (CSM)

Table 2.1 presents a summary of the constitutive properties of laminates made of glass fiber / vinyl ester woven roving (WR) calculated by Michelis [16]. The properties of all the laminates presented in Table 2.1 correspond to a 29.2% fiber volume fraction \((V_f)\). The fiber volume fractions in each material direction, \(V_{fx}\) and \(V_{fy}\), are also presented.

Table 2.1. Constitutive properties for different biased WR orthotropic laminates with constant volume fraction \(V_f = 29.2\%\) [16].

<table>
<thead>
<tr>
<th>WR bias percentage in x-direction</th>
<th>50% (Balanced)</th>
<th>60%</th>
<th>70%</th>
<th>80%</th>
<th>90%</th>
<th>100% (Unidirectional)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_x) [GPa]</td>
<td>23.42</td>
<td>26.45</td>
<td>29.49</td>
<td>32.53</td>
<td>35.56</td>
<td>38.60</td>
</tr>
<tr>
<td>(V_{fx}) [%]</td>
<td>14.6</td>
<td>17.52</td>
<td>20.44</td>
<td>23.36</td>
<td>26.28</td>
<td>29.2</td>
</tr>
<tr>
<td>(E_y) [GPa]</td>
<td>23.42</td>
<td>20.38</td>
<td>17.34</td>
<td>14.30</td>
<td>11.27</td>
<td>8.23</td>
</tr>
<tr>
<td>(V_{fy}) [%]</td>
<td>14.6</td>
<td>11.68</td>
<td>8.76</td>
<td>5.84</td>
<td>2.92</td>
<td>0</td>
</tr>
<tr>
<td>(G_{xy}) [GPa]</td>
<td>8.86</td>
<td>8.86</td>
<td>8.86</td>
<td>8.86</td>
<td>8.86</td>
<td>8.86</td>
</tr>
<tr>
<td>(v_{xy})</td>
<td>0.09</td>
<td>0.11</td>
<td>0.12</td>
<td>0.15</td>
<td>0.19</td>
<td>0.26</td>
</tr>
</tbody>
</table>

Table 2.2 presents the constitutive properties of plug materials made of chopped strand mat (CSM) as calculated by Michelis [16] for plugs with different volume fraction \(V_f\). This calculation assumes that fibers of length < 20 [mm] are randomly oriented in the two-dimensional plane of the WR orthotropic laminate [12]. \(E\) is the elastic modulus, \(G\) is the shear modulus, and \(v\) is the Poisson’s ratio.
Table 2.2 Constitutive properties of glass fiber / vinyl ester chopped strand mat (CSM) plugs [16].

<table>
<thead>
<tr>
<th>$V_f$ [%]</th>
<th>$E$ [GPa]</th>
<th>$G$ [GPa]</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.4</td>
<td>1.25</td>
<td>0.36</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1.48</td>
<td>0.35</td>
</tr>
<tr>
<td>5</td>
<td>4.66</td>
<td>1.73</td>
<td>0.35</td>
</tr>
<tr>
<td>6</td>
<td>5.38</td>
<td>2</td>
<td>0.34</td>
</tr>
<tr>
<td>10</td>
<td>6.16</td>
<td>2.29</td>
<td>0.34</td>
</tr>
<tr>
<td>13</td>
<td>6.99</td>
<td>2.61</td>
<td>0.34</td>
</tr>
<tr>
<td>16</td>
<td>7.9</td>
<td>2.95</td>
<td>0.34</td>
</tr>
<tr>
<td>20</td>
<td>8.88</td>
<td>3.31</td>
<td>0.34</td>
</tr>
<tr>
<td>23</td>
<td>9.95</td>
<td>3.71</td>
<td>0.34</td>
</tr>
<tr>
<td>27</td>
<td>11.13</td>
<td>4.15</td>
<td>0.34</td>
</tr>
<tr>
<td>31</td>
<td>12.42</td>
<td>4.64</td>
<td>0.34</td>
</tr>
<tr>
<td>36</td>
<td>13.86</td>
<td>5.18</td>
<td>0.34</td>
</tr>
<tr>
<td>40</td>
<td>15.48</td>
<td>5.79</td>
<td>0.34</td>
</tr>
</tbody>
</table>

2.2 Analytical solutions of Lekhnitskii

Lekhnitskii [33] developed a group of analytical expressions to calculate stress concentrations along the contour of a circular opening in anisotropic two-dimensional plates. These expressions consider the effect of different loading conditions applied along the edges of the plate. The plate was assumed to be infinitely large, so any effects of the external edges were disregarded. This section provides the calculation of stress concentration in orthotropic laminates with circular openings, when they are subjected to two different loading conditions: 1) in-plane shear stress and 2) in-plane bending moment. In each of these loading conditions, the stress concentration is calculated firstly when the opening is unfilled, and secondly when the opening is filled with CMS after it has been repaired using a plug scheme. The results of the calculations are presented in
plots, which facilitate the comparison of stress distributions between the filled and unfilled cases for each of the loading conditions.

2.3 Coordinate system and stress transformation equations

All the analytical solutions provided by Lekhnitskii are expressed in polar coordinates; and the characteristics of the orthotropic glass fiber / vinyl ester woven roving are easily expressed in Cartesian coordinates. The reference system for both the polar and Cartesian coordinates is presented in Figure 2.1.

![Figure 2.1. Two-dimensional laminate with circular opening reference systems [33].](image)

The center of the opening is the origin for both coordinate systems. In orthotropic laminates the directions of the fibers are 90° apart. These fiber directions are the principal directions of elasticity of the plate material, and they are laid in the x – y axes. In the polar coordinate system, θ is the angle measured counterclockwise from the x-axis and r is the radius measured from the center of the circular hole. The radius of the hole is a.
The plate is considered to be an elastic body obeying the generalized Hooke’s law. The far-applied loading stresses are parallel to the plane \(x - y\); therefore, the stress condition is plane stress. The normal stress \(\sigma_{zz}\) and the two out-of-plane shear stresses are zero.

For general cases, the stress transformation from the polar to the Cartesian coordinate system is given by Timoshenko [34] in eqns. (1), (2), and (3)

\[
\begin{align*}
\sigma_{xx} &= \sigma_{rr} \cos^2 \theta + \sigma_{\theta \theta} \sin^2 \theta - 2 \tau_{r \theta} \sin \theta \cos \theta \\
\sigma_{yy} &= \sigma_{rr} \sin^2 \theta + \sigma_{\theta \theta} \cos^2 \theta + 2 \tau_{r \theta} \sin \theta \cos \theta \\
\tau_{xy} &= (\sigma_{rr} - \sigma_{\theta \theta}) \sin \theta \cos \theta + \tau_{r \theta} (\cos^2 \theta - \sin^2 \theta)
\end{align*}
\]

2.4 Glass fiber / vinyl ester composite laminate loaded by in-plane shear stress

The reduction of stress concentration in the opening contour is achieved by plugging the circular opening with isotropic CSM. To calculate the reduction of the stresses, we can compare the stress distribution along the opening contour in the two conditions, without the plug and with the plug installed.

2.4.1 Stress distribution along the contour of a circular opening without plug

This calculation can be done using one of the analytical solutions developed by Lekhnitskii [33]. Eqn. (4) gives the normal tangential stress \(\sigma_{\theta \theta}\) along the contour of the opening in an orthotropic plate loaded by in-plane shear stress as a function of the angle \(\theta\). Figures 2.1 and 2.2 present the variables and reference axes for eqn. (4). The angle \(\theta\) is measured from the \(x\) axis in the counterclockwise direction. The loading shear stress is referred to the \(x’-y’\) axes and the laminate fibers are laid in the \(x-y\) axes. The angle between the \(x’-y’\) and the \(x-y\) axes is represented by \(\phi\) and is measured in the clockwise direction from the \(x’\) axis. Based on this definition of the reference frames, the loading shear stress is applied at \(\phi = 0^\circ\) for all the calculations and results in subsections 2.4.1 and 2.4.2; and the laminate fibers along the \(x\)-axis are laid at an angle \(\phi\) with respect to the loading direction.
Figure 2.2. Reference axes for Lekhnitskii’s analytical solution of plate loaded by shear stresses $\tau_{x'y'}$ [33].

\[
\sigma_{\theta\theta} = \tau_{x'y'} \frac{E_\theta}{2E_x} (1 + k + n) \left\{ \frac{- n \cdot \cos(2\phi) \cdot \sin(2\theta)}{((1 + k) \cdot \cos(2\theta) + k - 1)} \cdot \sin(2\phi) \right\}
\]

(4)

where

\[
\frac{1}{E_\theta} = \frac{\sin^4 \theta}{E_x} + \left( \frac{1}{G_{xy}} - \frac{2v_{xy}}{E_x} \right) \cdot \sin^2 \theta \cdot \cos^2 \theta + \frac{\cos^4 \theta}{E_y}
\]

(5)

\[
k = \sqrt{\frac{E_x}{E_y}}
\]

(6)

\[
n = \sqrt{2 \left( \frac{E_x}{E_y} - v_{xy} \right) + \frac{E_x}{G_{xy}}}
\]

(7)

In this case, since the opening is unfilled, the other two polar components of the stress along the contour are $\sigma_{rr} = 0$, and $\tau_{r\theta} = 0$.

As Lekhnitskii notes, the highest stress concentrations along the contour are produced when $\phi = 45^\circ$, regardless of the WR bias percentage of the laminate material.
When $\phi = 45^\circ$, the highest compressive stresses $\sigma_{\theta\theta}$ are obtained at points B and B$_1$, and the highest tensile stresses $\sigma_{\theta\theta}$ are obtained at points A and A$_1$. Figure 2.3 and Figure 2.4 present the distribution of stresses $\sigma_{\theta\theta}$, $\sigma_{xx}$ and $\sigma_{yy}$ for a laminate loaded by pure shear stress $\tau_{x'y'}$ when $\phi = 45^\circ$ and $\phi = 0^\circ$, respectively. Note that $\sigma_{xx}$ and $\sigma_{yy}$ are calculated using eqns. (1) and (2), respectively, for each value of $\sigma_{\theta\theta}$ along the contour. The comparison of these two results illustrates the effect of $\phi$ on the stress concentration. In these two cases, the laminate is made of glass fiber / vinyl ester with balanced WR. To present the distribution of stress concentration as a fraction of the applied stress, the stresses are normalized by the applied shear stress $\tau_{x'y'}$. Due to the symmetry of the stress distribution along the contour, only the stresses along the arc AB ($\theta = 0^\circ$ to $\theta = 90^\circ$) are plotted.

![Figure 2.3](image_url)

Figure 2.3. Stresses $\sigma_{\theta\theta}$, $\sigma_{xx}$ and $\sigma_{yy}$ around contour of unfilled circular opening in orthotropic laminate of glass fiber / vinyl ester with balanced WR due to external shear stress $\tau_{x'y'}$. The x-axis principal direction of elasticity is at $\phi = 45^\circ$. 

38
When $\phi = 45^\circ$ (see Figure 2.3), the highest tensile stress is produced at $\theta = 0^\circ$ (point A), and the highest compressive stress is produced at $\theta = 90^\circ$ (point B). At these points the normalized stresses are $\sigma_{\theta \theta} = 4.11$ and $\sigma_{\theta \theta} = -4.11$, respectively. When $\phi = 0^\circ$ (see Figure 2.4), the highest compressive stress is produced at $\theta = 45^\circ$ and corresponds to $\sigma_{\theta \theta} = -3.88$ times the applied $\tau_{xy}$. Due to symmetry, the highest tensile stress is produced at $\theta = 135^\circ$ with a value of $\sigma_{\theta \theta} = 3.88$. As we noted above, $\sigma_{\theta \theta}$ is higher when $\phi = 45^\circ$; therefore, this orientation of the fibers with respect to the external stress can be considered as the least favorable for a laminate with balanced WR.
We can use eqns. (4) – (7) and the constitutive properties given in Table 2.1 to calculate stresses in biased WR laminates. Figure 2.5 presents the stress distribution $\sigma_{\theta\theta}$ around the contour of a circular opening for the woven roving laminates of Table 2.1 when $\phi = 45^\circ$; that is, the direction of the x-axis with respect to the $x'$-axis. The stress at $\theta = 90^\circ$ increases as the bias fraction of the WR increases in the x-direction. The opposite occurs at $\theta = 0^\circ$, where the stress decreases as the bias fraction of the WR increases in the x-direction.

Figure 2.5. Stresses $\sigma_{\theta\theta}$ around contour of unfilled circular opening for different woven roving laminates, loaded by shear stress $\tau_{x'y'}$ and when $\phi = 45^\circ$.

Figure 2.6 presents the stress $\sigma_{\theta\theta}$ around the contour of a circular opening for the woven roving laminates of Table 2.1, but when $\phi = 0^\circ$; that is, the direction of the x-axis
with respect to the $x'$-axis. In both cases ($\phi = 45^\circ$ and $\phi = 0^\circ$), we observe a similar stress behavior when the WR bias is changed; the stress $\sigma_{\theta\theta}$ increases when the bias of the WR laminate is increased, in these cases toward the $x$-direction.

Figure 2.6. Stresses $\sigma_{\theta\theta}$ around contour of unfilled circular opening for different woven roving laminates, loaded by shear stress $\tau_{x'y'}$ and when $\phi = 0^\circ$.

2.4.2 Stress distribution along the contour of a circular opening plugged with isotropic CSM

Lekhnitskii offers an analytical solution for plugged circular openings, but only for the case when $\phi = 0^\circ$. As mentioned above, the highest stress concentration occurs when $\phi = 45^\circ$; therefore, it is necessary to use a different method to find the stress concentration for the case where $\phi = 45^\circ$. Such a solution can be found using a linear
superposition of two cases of uniaxial loading 90° degrees apart. The explanation of the linear superposition method used to find this stress distribution in a laminate with balanced WR is presented in the Appendix.

For the case when \( \phi = 0^\circ \), the stress concentration can be found in a straightforward manner using Lekhnitskii’s [33] analytical solution given by eqn. (8)

\[
\sigma_{\theta\theta} = -\tau_{r\theta} \left\{ \frac{1}{2\Delta_1} \left[ \frac{q_{66} - q_{66}'}{E_x} \cdot \left( (1 + k)n + (n^2 - 2)\sin^4 \theta + \frac{2k\sin^2 \theta \cdot \cos^2 \theta + (n^2 - 2k^2)\cos^4 \theta}{2} \right) \right] \cdot \sin(2\theta) \right\}
\]

\[
\sigma_{rr} = \frac{\tau_{r\theta}}{\Delta_1} \left\{ q_{11}kn + (2q_{12} + q_{66}')k + q_{22}(2 + n) \right\} \cdot \sin(2\theta)
\]

\[
\tau_{r\theta} = \sigma_{rr} \cdot \cot(2\theta)
\]

where

\[
\frac{E_{\theta}}{E_x} = \left( \sin^4 \theta + \frac{2q_{12} + q_{66}'}{q_{11}} \cdot \sin^2 \theta \cdot \cos^2 \theta + \frac{q_{22}}{q_{11}} \cdot \cos^4 \theta \right)^{-1}
\]

The parameters \( k \) and \( n \) are given in eqns. (6) and (7). The parameter \( \Delta_1 \) is given by

\[
\Delta_1 = q_{11}kn + (2q_{12} + q_{66}')k + q_{22}(2 + n)
\]

\( q_{11}, q_{12}, q_{22}, \) and \( q_{66} \) are independent elastic compliances of the laminate; and they are expressed in terms of the constitutive properties of the orthotropic plate \( E_x, E_y, \nu_{xy}, \) and \( G_{xy} \) as

\[
q_{11} = \frac{1}{E_x}
\]

\[
q_{22} = \frac{1}{E_y}
\]

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\[ q_{12} = \frac{v_{xy}}{E_x} \]  
\[ q_{66} = \frac{1}{G_{xy}} \]  

\( q_{66}' \) is the shear elastic compliance of the plug; it can be expressed in terms of the shear modulus \( G \) of the isotropic CSM plug as

\[ q_{66}' = \frac{1}{G} \]  

Figures 2.7 and 2.8 present the stress concentrations \( \sigma_{xx} \) and \( \sigma_{yy} \) along the contour of the circular opening in a balanced WR laminate of glass fiber / vinyl ester plugged with isotropic CSM plugs of different \( V_f \). In this case \( \phi = 45^\circ \); therefore, the superposition method was applied. The data that are plotted in Figures 2.7 and 2.8 are obtained by substituting eqns. (A-9) from the Appendix into eqns. (1) and (2), respectively. Figures 2.9 and 2.10 present the stress concentrations \( \sigma_{xx} \) and \( \sigma_{xy} \) for the same WR laminate and CSM plugs when \( \phi = 0^\circ \); therefore, eqns. (8) were applied. The data that are plotted in Figures 2.9 and 2.10 are obtained by substituting eqns. (8) into eqns. (1) and (2), respectively. In both cases, the constitutive properties for the CSM plugs are given in Table 2.2.
Figure 2.7. Stress $\sigma_{xx}$ along circular contour of balanced WR laminate filled with CSM plugs of different fiber volume fractions, loaded by external shear stress and when the laminate orientation is $\phi = 45^\circ$. 
Figure 2.8. Stress $\sigma_{yy}$ along circular contour of balanced WR laminate filled with CSM plugs of different fiber volume fractions, loaded by external shear stress and when the laminate orientation is $\phi = 45^\circ$.

In Figures 2.7 and 2.8, the stress distribution without plug was calculated directly using Lekhnitskii's analytical solution given by eqn. (4). The stress distributions for plug
materials ranging from $V_f = 0\%$ to $V_f = 40\%$ were calculated using the linear superposition method described in the Appendix. Notice that as the fiber volume fraction of the CSM plug decreases (i.e. the stiffness of the plug decreases), the stress distribution approaches Lekhnitskii's solution for the case without a plug. This behavior reinforces the validity of the linear superposition method when used in materials with balanced WR and when $\phi = 45^\circ$.

Figure 2.9. Stress $\sigma_{xx}$ along circular contour of balanced WR laminate filled with CSM plugs of different fiber volume fractions, loaded by external shear stress and when the laminate orientation is $\phi = 0^\circ$. 

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Figure 2.10. Stress $\sigma_{yy}$ along circular contour of balanced WR laminate filled with CSM plugs of different fiber volume fractions, loaded by external shear stress and when the laminate orientation is $\phi = 0^\circ$.

Figures 2.7 through 2.10 reveal the beneficial effect of the CSM plug in reducing the maximum stresses around the contour of the opening. The introduction of the plug reduces the magnitude of both the maximum tensile stress and maximum compressive stress. As the fiber volume fraction of the plug increases, the effect of the plug’s presence in stress reduction also increases. The same effect was found by Michelis [16] for a plate loaded by uniaxial stress.
When \( \phi = 45^\circ \) and due to the symmetry between the external shear and the principal directions of the laminate, the magnitudes of the maximum compressive and tensile stresses are equal. Figure 2.7 shows that the stress \( \sigma_{xx} \) is compressive along the entire contour and has a maximum at \( \theta = 90^\circ \). Figure 2.8 shows that the stress \( \sigma_{yy} \) is tensile along the entire contour and has a maximum at \( \theta = 0^\circ \). Figure 2.11 presents the effect of the increase of glass fiber volume fraction in the CSM plug on the maximum compressive stress \( \sigma_{xx} \) for a balanced WR laminate. Figure 2.12 presents the effect of the increase of glass fiber volume fraction in the CSM plug on the maximum compressive stress \( \sigma_{yy} \) for a balanced WR laminate. It is clear that the increase of the fiber volume fraction in the CSM plug contributes to the reduction of the stresses along the contour of the filled opening.

![Graph showing the effect of glass fiber volume fraction on the maximum compressive stress](image)

**Figure 2.11.** Normalized maximum compressive stress \( \sigma_{xx} \) around circular contour of balanced WR laminate filled with CSM plugs of different volume fraction, loaded by external shear stress and when the laminate orientation is \( \phi = 45^\circ \).
Figure 2.12. Normalized maximum tensile stress $\sigma_{yy}$ around circular contour of balanced WR laminate filled with CSM plugs of different volume fractions, loaded by external shear stress and when the laminate orientation is $\phi = 45^\circ$.

A summary of the stress reductions achieved by using CSM plugs in balanced WR laminates is presented in Table 2.3.

Table 2.3. Stress reductions in balanced WR laminates repaired with CSM plugs of different $V_f$, loaded by external shear stress and when $\phi = 45^\circ$. [Refer to the ordinate in Figure 2.7 at $\theta = 90^\circ$ for $\sigma_{xx}$ and to the ordinate in Figure 2.8 at $\theta = 0^\circ$ for $\sigma_{yy}$].

<table>
<thead>
<tr>
<th>CSM $V_f$ [%]</th>
<th>Stress concentration without plug for both $\sigma_{xx}$ and $\sigma_{yy}$</th>
<th>Stress concentration with plug for both $\sigma_{xx}$ and $\sigma_{yy}$</th>
<th>Reduction of maximum $\sigma_{xx}$ and $\sigma_{yy}$ [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.113</td>
<td>3.08</td>
<td>25.1</td>
</tr>
<tr>
<td>10</td>
<td>4.113</td>
<td>2.536</td>
<td>38.3</td>
</tr>
<tr>
<td>20</td>
<td>4.113</td>
<td>2.16</td>
<td>47.5</td>
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<td>56.1</td>
</tr>
<tr>
<td>40</td>
<td>4.113</td>
<td>1.578</td>
<td>61.6</td>
</tr>
</tbody>
</table>
The plots presented in Figures 2.11 and 2.12 can be expanded to represent laminates of different WR biases.

Figures 2.13 and 2.14 present the normalized maximum compressive stress $\sigma_{xx}$ and the normalized maximum tensile stress $\sigma_{yy}$ for laminates with different WR biases and CSM plugs having different fiber volume fractions ($V_f$). As the WR bias increases in the x-direction, the compressive stress ratio $\sigma_{xx}$ increases and the tensile stress ratio $\sigma_{yy}$ decreases. The effect of increasing $V_f$ of the CSM plug has the same effect in the two principal directions of the laminate: a higher $V_f$ decreases both $\sigma_{xx}$ and $\sigma_{yy}$.

![Diagram](image)

**Figure 2.13.** Normalized maximum compressive stress $\sigma_{xx}$ around circular contour for laminates with different WR biases and CSM plugs with different fiber volume fractions, loaded by external shear stress and when $\phi = 45^\circ$. 

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Figure 2.14. Normalized maximum tensile stress $\sigma_{yy}$ around circular contour for laminates with different WR biases and CSM plugs with different fiber volume fractions, loaded by external shear stress and when $\phi = 45^\circ$. 
2.5 Glass fiber / vinyl ester composite laminate loaded by in-plane bending moment.

The stress concentration along the contour of a circular opening in an orthotropic laminate subjected to an in-plane bending moment will be calculated using a similar approach as for the pure shear loading case. The stress distribution along the contour will be compared for two conditions, an unfilled hole and a hole filled with various CSM plugs.

2.5.1 Stress distribution along the contour of a circular opening without plug

These calculations are conducted using an analytical solution developed by Lekhnitskii [33]. Eqn. (16) gives the normal tangential stress $\sigma_{0\theta}$ along the contour of the circular opening of radius $a$ in an orthotropic plate loaded by an in-plane bending moment as a function of the angle $\theta$. Figures 2.1 and 2.15 present the variables and reference axes for eqn. (16). The angle $\theta$ is measured from the $x$ axis in the counterclockwise direction and locates the stress of interest. The applied in-plane bending moment $M$ is referred to the $x'-y'$ axes and the orthotropic laminate fibers are laid along the $x$-$y$ axes. The angle between the $x'$-$y'$ and the $x$-$y$ axes is represented by $\phi$ and is measured in the clockwise direction from the $x'$ axis.

![Figure 2.15. Reference axes for Lekhnitskii’s [33] analytical solution for a plate loaded by in-plane bending moment $M$.](image-url)
Based on this definition of the reference frames, the loading bending moment is applied at $\phi = 0^\circ$ and the orthotropic laminate fibers are laid along the $x$ and $y$ axes, at an angle $\phi$ with respect to the loading axis.

The normal tangential stress $\sigma_{\theta \theta}$ is given by

$$
\sigma_{\theta \theta} = \frac{Ma}{2l_{zz}} \frac{E_\theta}{E_x} [1 - k - (1 + k + n) \cos(2\theta)] \sin \theta
$$

(16)

In eqn. (16) $E_\theta$ is given by eqn. (5), $k$ is given by eqn. (6), and $n$ is given by eqn. (7).

In this case, since the opening is unfilled, the other two polar components of the stress along the contour are $\sigma_{rr} = 0$, and $\tau_{r\theta} = 0$.

To present the distribution of stress concentration as a fraction of the in-plane bending moment $M$, the stresses are normalized by the quotient $Ma/I_{zz}$, where $a$ is the radius of the opening and $I_{zz}$ is the moment of inertia of the plate with respect to the out-of-plane $z$-axis. The dimensional analysis of this normalization gives

- Bending moment: $M = [N \cdot m] = \left[ \frac{kg \cdot m}{s^2} \cdot m \right]$
- Opening radius: $r = a [m]$
- Transverse moment of inertia: $I_{zz} = [m^4]$
- Non-dimensional stress: $\frac{\sigma_{\theta \theta}}{\frac{Ma}{I_{zz}}} = \left[ \frac{kg \cdot m}{s^2 \cdot m^4} \right] = [non-dim]$

In all the following calculations, the quotient $Ma/I_{zz}$ has been set to unity. This allows us to obtain the results as non-dimensional stresses along the contour of openings of any radius $r = a$, and as a fraction of the in-plane bending moment $M$. 

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As Lekhnitskii notes, the highest stress concentration along the contour occurs when $\phi = 0^\circ$, regardless of the WR bias percentage of the laminate. When $\phi = 0^\circ$, the highest compressive stress $\sigma_{\theta\theta}$ is obtained at point B$_1$; and the highest tensile stress $\sigma_{\theta\theta}$ is obtained at point B. Therefore, throughout the calculations that follow, $\phi = 0^\circ$ and thus the $x'-y'$ and $x-y$ axes are coincident.

Figure 2.16 presents the distribution of stresses $\sigma_{\theta\theta}$, $\sigma_{xx}$, and $\sigma_{yy}$ for a laminate made of glass fiber / vinyl ester with balanced WR and loaded by in-plane bending moment $M$ when $\phi = 0^\circ$. Due to the symmetry of the stress distribution along the contour, only the stresses along the arc B$_1$B ($\theta = -90^\circ$ to $\theta = 90^\circ$) are plotted.

![Stress distribution graph](image)

**Figure 2.16.** Stresses $\sigma_{\theta\theta}$, $\sigma_{xx}$ and $\sigma_{yy}$ around contour of unfilled circular opening in orthotropic laminate of glass fiber / vinyl ester with balanced WR due to external bending moment $M$. The $x$-axis principal direction of elasticity is at $\phi = 0^\circ$. 

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The highest tensile stress occurs at \( \theta = 90^\circ \) (point B), and the highest compressive stress occurs at \( \theta = -90^\circ \) (point B_1). At points B and B_1, the normalized stresses are \( \sigma_{\theta\theta} = 2.05 \) and \( \sigma_{\theta\theta} = -2.05 \), respectively.

Figure 2.17 presents the variation of the stress distribution \( \sigma_{\theta\theta} \) around the contour of a circular opening for all the woven roving laminates of Table 2.1. Increasing the WR bias in the x-direction increases the stresses at \( \theta = 90^\circ \) and \( \theta = -90^\circ \). For unidirectional WR the maximum stress at points B and B_1 are \( \sigma_{\theta\theta} = 2.82 \), and \( \sigma_{\theta\theta} = -2.82 \), respectively. These are 37.5\% higher than the corresponding maximum stress in the balanced WR.

![Figure 2.17. Stresses \( \sigma_{\theta\theta} \) around contour of unfilled circular opening for different woven roving laminates, loaded by in-plane bending moment M.](image-url)
2.5.2 Stress distribution along the contour of a circular opening plugged with isotropic CSM

When the laminate presented in Figure 2.15 has the opening filled with an isotropic CSM plug, the stress concentration around the contour can be found using Lekhnitskii’s [33] analytical solution, which is expressed in eqns. (17):

\[
\begin{align*}
\sigma_{\theta\theta} &= \frac{Ma}{I_{zz} d} E_y \sin \theta \left[ (d - c_1) \sin^6 \theta + [d(n^2 - 2k) + c_1(1 + k + 2n)] \sin^4 \theta \cos^2 \theta + [dk^2 - c_1k - c_2(1 + 2k)(1 + k + 2n) + c_3(2 + k)(2k + n)] \sin^2 \theta \cos^4 \theta + [c_2 k(1 + 2k) - c_4(2k + n)] \cos^6 \theta \right]
\end{align*}
\]

\[
\begin{align*}
\sigma_{rr} &= \frac{Ma}{2I_{zz} d} [d + 2c_3 + (d - 2c_2 - 2c_3) \cos(2\theta)] \sin \theta
\end{align*}
\]

\[
\begin{align*}
\tau_{r\theta} &= \frac{Ma}{2I_{zz} d} [-d + 2c_3 + (d - 2c_2 - 2c_3) \cos(2\theta)] \cos \theta
\end{align*}
\]

where \(E_y/E_x\) is given by eqn. (9). The parameters \(k\) and \(n\) are given by eqns. (6) and (7), respectively. The parameters \(d, c_1, c_2, c_3\) and \(c_4\) are given by

\[
d = \left[ 2q_{22}(q_{66} + 2q_{11} + 2q_{12}) + 2[q_{11}q_{22} + q_{11} + (2q_{12} + q_{22} + q_{66})]k \right] + 4q_{22}q_{11} + 4q_{11}(q_{11} + q_{22} + q_{66})k - 2(q_{12} - q_{12})^2 k
\]
\[
c_1 = \left\{ \begin{array}{l}
(q_{11} - q_{11}')[2q_{22}(n^2 - k) + (2q_{12} + q_{22}' + q_{66}')kn] \\
- (q_{12} - q_{12}') (q_{11}kn + 2q_{22}) + (q_{12} - q_{12}')^2 kn
\end{array} \right. \\

c_2 = (q_{11} - q_{11}') (q_{11}kn + 2q_{22}) + q_{11} (q_{12} - q_{12}') k (2 + n)
\]
\[
c_3 = \left\{ \begin{array}{l}
(q_{11} - q_{11}')[q_{22}(1 + 2n) + (2q_{12} + q_{12}' + q_{66}')k] \\
+ (q_{12} - q_{12}') (q_{22} - q_{11}k) + (q_{12} - q_{12}')^2 k
\end{array} \right.
\]
\[
c_4 = \left\{ \begin{array}{l}
(q_{11} - q_{11}')[q_{22} + (q_{66} - q_{22}' + q_{66}')k] \\
+ (q_{12} - q_{12}') [q_{22} + (q_{11} + 2q_{12} + q_{66})k + 2q_{11}kn]
\end{array} \right.
\]
\[
- (q_{12} - q_{12}')^2 k
\]
\]

The parameters \(q_{11}, q_{22}, q_{12},\) and \(q_{66}\) are independent elastic compliances of the laminate material and they are given by eqns. (11) through (14). The parameters \(q_{11}', q_{12}',\) and \(q_{66}'\) are elastic compliances of the isotropic chopped strand mat (CSM) plug material. They are expressed in terms of the constitutive properties of the isotropic CSM plugs as

\[
\left\{ \begin{array}{l}
q_{11}' = q_{22}' = \frac{1}{E} \\
q_{12}' = -\frac{\nu}{E} \\
q_{66}' = \frac{1}{G}
\end{array} \right.
\]

Figures 2.18 and 2.19 present the stress concentrations \(\sigma_{xx}\) and \(\sigma_{yy}\), respectively, along the contour of the circular opening in a balanced WR laminate of glass fiber / vinyl ester filled with isotropic CSM plugs of different fiber volume fractions (Vf). The data that are plotted in Figures 2.18 and 2.19 are obtained by substituting eqns. (17) into eqns. (1) and (2), respectively. The constitutive properties of the CSM plugs are given in Table 2.2.
Figure 2.18. Stress $\sigma_{xx}$ along circular contour of balanced WR laminate filled with CSM plugs of different fiber volume fractions, loaded by in-plane bending moment $M$. 
Figure 2.19. Stress $\sigma_{yy}$ along circular contour of balanced WR laminate filled with CSM plugs of different fiber volume fractions, loaded by in-plane bending moment $M$. 
In Figures 2.18 and 2.19 we observe that as the fiber volume fraction of the CSM plug decreases (that is, as the stiffness of the plug decreases), the stress distributions approach Lekhnitskii’s solutions for the case without a plug.

Figures 2.18 and 2.19 reveal the beneficial effect of the CSM plug in reducing the maximum stresses around the contour of the opening. The introduction of the plug reduces the magnitude of both the maximum tensile stress and maximum compressive stress. As the fiber volume fraction of the plug increases, the effect of the plug’s presence in stress reduction also increases. The same beneficial effect was found in the previous section 2.4.2, when the plate was loaded by pure shear stress; and in the case studied by Michelis [16], when the plate was loaded by uniaxial stress.

The plots of the stress distributions $\sigma_{xx}$ and $\sigma_{yy}$ show odd symmetry about $\theta = 0^\circ$; therefore, the maximum tensile stress in arc AB (see Figure 2.15) has the same absolute value of the maximum compressive stress in arc $B_1A$.

Figure 2.20 presents the effect of the increase of glass fiber volume fraction in the CSM plug on the maximum tensile and compressive stresses $\sigma_{xx}$ for a balanced WR laminate. Due to the odd symmetry, both tensile and compressive stresses have the same maximum value. Figure 2.21 presents the effect of the increase of glass fiber volume fraction in the CSM plug on the maximum tensile and compressive stresses $\sigma_{yy}$ for a balanced WR laminate. We can observe that the increase of the fiber volume fraction in the CSM plug contributes to the reduction of the maximum stresses along the contour of the filled opening.
Figure 2.20. Normalized maximum tensile and compressive stresses $\sigma_{xx}$ around circular contour of balanced WR laminate filled with CSM plugs of different volume fraction, loaded by in-plane bending moment M.
Figure 2.21. Normalized maximum tensile and compressive stresses $\sigma_{yy}$ around circular contour of balanced WR laminate filled with CSM plugs of different volume fractions, loaded by in-plane bending moment $M$. 
A summary of the maximum stress reductions achieved by using CSM plugs in balanced WR laminates is presented in Table 2.4.

Table 2.4. Maximum stress reductions in balanced WR laminates repaired with CSM plugs of different fiber volume fractions $V_f$, loaded by in-plane bending moment and when $\phi = 0^\circ$. [Refer to Figures 2.20 (or 2.22) and 2.21 (or 2.23)].

<table>
<thead>
<tr>
<th>CSM $V_f$ [%]</th>
<th>$\sigma_{xx}$ concentration without plug</th>
<th>$\sigma_{xx}$ concentration with plug</th>
<th>Reduction of maximum $\sigma_{xx}$ [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.05</td>
<td>1.54</td>
<td>24.8</td>
</tr>
<tr>
<td>0</td>
<td>2.05</td>
<td>1.368</td>
<td>33.2</td>
</tr>
<tr>
<td>20</td>
<td>2.05</td>
<td>1.265</td>
<td>38.3</td>
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<tr>
<td>31</td>
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</tr>
<tr>
<td>40</td>
<td>2.05</td>
<td>1.12</td>
<td>45.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CSM $V_f$ [%]</th>
<th>$\sigma_{yy}$ concentration without plug</th>
<th>$\sigma_{yy}$ concentration with plug</th>
<th>Reduction of maximum $\sigma_{yy}$ [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.45</td>
<td>0.184</td>
<td>59.1</td>
</tr>
<tr>
<td>10</td>
<td>0.45</td>
<td>0.114</td>
<td>74.6</td>
</tr>
<tr>
<td>20</td>
<td>0.45</td>
<td>0.079</td>
<td>82.4</td>
</tr>
<tr>
<td>31</td>
<td>0.45</td>
<td>0.054</td>
<td>88.0</td>
</tr>
<tr>
<td>40</td>
<td>0.45</td>
<td>0.042</td>
<td>90.6</td>
</tr>
</tbody>
</table>

The plots presented in Figures 2.20 and 2.21 can be expanded to represent laminates of different WR biases. Figures 2.22 and 2.23 present the normalized maximum stresses $\sigma_{xx}$ and $\sigma_{yy}$, respectively, for laminates with different WR biases and CSM plugs of different fiber volume fractions ($V_f$). In Figure 2.22 we observe that the maximum stress $\sigma_{xx}$ increases as the laminate WR bias increases in the $x$-direction; and that $\sigma_{xx}$ decreases as the glass fiber volume fraction of the CSM plug increases. Figure 2.23 presents the stress $\sigma_{yy}$. 
Figure 2.22. Normalized maximum stress $\sigma_{xx}$ around circular contour for laminates with different WR biases and CSM plugs with different fiber volume fractions, loaded by in-plane bending moment.
Figure 2.23. Normalized maximum stress $\sigma_{yy}$ around circular contour for laminates with different WR biases and CSM plugs with different fiber volume fractions, loaded by in-plane bending moment.
3 Conclusions and recommendations

3.1 Conclusions

The stresses are calculated in damaged glass fiber / vinyl ester orthotropic laminates that are repaired with the plug scheme for two independent loading cases: in-plane shear and in-plane bending. In the pure shear loading case, the shear was applied at angles of 0° and 45° relative to the x-direction (one of the fiber directions) of the material. The highest stress concentrations were found when the shear was applied at 45°. In the pure bending loading case, the bending moment was applied only at 0° relative to the x-direction of the material, which corresponded to the case where the highest stress concentrations are produced. Tables 3.1 and 3.2 present summaries of the stress reductions for balanced woven roving (WR) laminates repaired with chopped strand mat (CSM) plugs of different fiber volume fractions (Vf), subjected to shear and bending, respectively. In both loading cases, as the fiber volume fraction of the CSM plug increases, the stress reduction increases. Thus, the higher the Young’s modulus of the plug, the greater the beneficial effect of the plug in reducing the stress concentration.

Table 3.1 Stress reductions in balanced WR laminates repaired with different CSM plugs and subjected to in-plane shear. [Refer to Table 2.3].

<table>
<thead>
<tr>
<th>Laminate subjected to in-plane shear</th>
<th>Stress concentration without plug for both $\sigma_{xx}$ and $\sigma_{yy}$</th>
<th>Stress concentration with plug for both $\sigma_{xx}$ and $\sigma_{yy}$</th>
<th>Reduction of maximum stress concentration [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSM $V_f$ [%]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
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</table>
Table 3.2 Stress reductions in balanced WR laminates repaired with different CSM plugs and subjected to in-plane bending moment. [Refer to Table 2.4].

<table>
<thead>
<tr>
<th>Laminate subjected to in-plane bending moment</th>
<th>( \sigma_{xx} ) concentration without plug</th>
<th>( \sigma_{xx} ) concentration with plug</th>
<th>Reduction of maximum ( \sigma_{xx} ) [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSM Vf [%]</td>
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</tr>
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<td>0.042</td>
<td>90.6</td>
</tr>
</tbody>
</table>

The effect of the plug in reducing the stress concentration was also calculated for biased WR laminates. Figures 2.13, 2.14, 2.22, and 2.23 present the variation of the stress reduction as the bias percentage of the laminate varies from balanced to unidirectional fabric (i.e., 100% bias in the x-direction) and as the volume fraction plug varies from 0% to 40%. The same effect of the plug in reducing the stress concentration was found by Michelis [16] in a plate loaded by uniaxial stress.

Regarding the validation of Lekhnitskii’s analytical model [33], several authors have found good agreement between Lekhnitskii’s model, finite element analysis (FEA), and experimental data. For instance, Hindman and Horn [29] calculated the stresses in
composite plates with a filled hole under uniaxial tension. They compared the results of Lekhnitskii’s model against FEA computations. Both predictions for the stress field in the plate were in good agreement. In their calculations, the presence of the plug reduced the stress around the hole up to 50% relative to the unplugged condition. Berbinau, Filiou and Soutis [30] mention that experiments also showed reductions of the stresses when the plug was installed, but the effectiveness and thickness of the adhesive layer between the plug and sound laminate greatly influenced the experimental results.

The two-dimensional analytical approach does not account for the effect that plate thickness has on stress concentration. Previous studies on three-dimensional stress concentrations around a circular hole in an isotropic plate of arbitrary thickness showed that the stress concentration near the surface is lower than near the mid-thickness plane of the hole [25, 26]. Results from a two-dimensional model were compared to results from a three-dimensional model. When the plate was subjected to uniaxial stress, the maximum stress at the surface as predicted by the three-dimensional model was 7% less than the maximum stress predicted by the two-dimensional model. On the other hand, the maximum stress at the mid-thickness plane was 3% higher than the maximum predicted by the two-dimensional model. In a later three-dimensional analysis of a thick isotropic plate subjected to in-plane shear, Youngdahl and Sternberg [28] found that the maximum stress on the plate’s surface at the hole was 23% lower, and the maximum stress in the interior of the hole was 3% higher, both relative to the value normally predicted for a thin plate. W. Pilkey and D. Pilkey [27] stated that “the usual two-dimensional stress concentration factors are sufficiently accurate for design applications to elements of arbitrary thickness”. Based on the results of the three-dimensional experiments mentioned above, a conservative approach for design is to correct the stress values from two-dimensional models by +3%. It seems that even though the two-dimensional analytical solutions do not account for the effect of plate thickness, they provide useful design guidelines.

---

2 The stress concentration factor for thin isotropic plates with unplugged holes subjected to shear stress is 4 [27].
A literature review by Michelis [16] showed that for the design of composite structures, authors generally recommend keeping the allowable stress at 10% to 30% of the ultimate strength of the material. These allowable stress values imply safety factors ranging from 3.3 to 10. Baley, Davies, Grohens and Dolto [14] mentioned that the lack of precise information about the behavior of composites has resulted in the design of over-designed marine structures with safety factors varying from 4 to 6 for static loads and up to 10 for dynamic loads.

Given the large safety factors that are used in the design of marine structures, it appears that the plug repair scheme can be applied to laminates subjected to pure in-plane shear or pure in-plane bending moment while maintaining a reasonable safety factor in the repaired structure. For instance, a plate subjected to shear and repaired with a plug of \( V_f = 40\% \) has a stress concentration of 1.578 (see Table 3.1). If this repair is undertaken in a laminate designed with safety factor of 3.3 (worst case scenario), the theoretical maximum stress in the contour of the hole will be 47.8% of the ultimate stress. That is, the new safety factor in the repaired section of the structure would be 2.09.

In a study about damage tolerance of marine composite materials by Elliott and Trask [24], the authors mentioned the need for reducing the high cost and time involved in traditional composite structural repairs, such as the step repair scheme (Figure 1.10). In this study, they investigated the damage tolerance of the glass fiber reinforced plastic (GFRP) used in minehunters to attempt to reduce the number of repairs. This investigation enabled a 50% reduction in the number of hull repairs undertaken, with consequent savings in cost and time.

In the context of cost reduction in the maintenance of ship structures, the plug repair scheme seems to be a good alternative; but, the stress concentration around the damaged area needs to be assessed prior to application. Such assessments are the requirements of nondestructive evaluation, which should also ensure that the safety factor of the repaired structure satisfies the requirements of the technical authority that is responsible for certifying the vessel.

The calculated stress reductions in this thesis assume a perfect bonding between the laminate and the CSM plug. Berbinau, Filiou and Soutis [30] mentioned the
importance of the bonding between sound material of the laminate and the plug in reducing the stress concentration in the repaired region. They concluded that “for the plug repair of damaged composite plates to be truly beneficial, the bonding between the core (plug) and the plate is of paramount importance.” The less effective the bonding and the thicker the adhesive layer, the less the stress reduction, relative to the predictions of the theoretical results. Experiments with carbon fiber-epoxy laminates [31] showed that the stress concentration in plug repairs increases as the adhesive thickness increases.

Based on these results, the bond between the CMS plug and glass fiber / vinyl ester laminate is an important feature in achieving the stress reductions predicted in this thesis.

3.2 Recommendations

The work initiated by Michelis [16] and the work done in this thesis are the initial steps towards a better understanding of the behavior of plug repairs in glass fiber / vinyl ester laminates. So far, the work has investigated plug repairs in uniaxial tension, in-plane shear stress and in-plane bending loading conditions. In order to fully validate this repair scheme, the stress field around such plug repair needs to be further investigated, both theoretically and experimentally, including combined loading conditions. Further, the effects of the adhesive thickness between the plug and the laminate should be studied and optimized.

For marine applications, the ultimate strength of glass fiber / vinyl ester subjected to compressive loads needs to be determined by experiment. Studies in modern composite systems have shown that their ultimate compressive strengths are limited to 60 to 70% of their tensile strengths [32]. In ship structures, the highest primary load may be compression in the decks above the neutral axis of the hull. Therefore, if glass fiber / vinyl ester is the proposed material for ship decks, the ultimate strength in compression must be determined for both the initial design and the repaired structures.
References


Appendix

Applying Superposition for Stresses in Composite Laminate Repaired by Chopped Strand Mat (CSM) Plugs

The state of stress in an orthotropic laminate loaded with in-plane shear stress $\tau_{x'y'}$ is presented in Figure A-1. The square mesh inside the rectangle represents the fiber directions and therefore the principal directions of elasticity of the orthotropic material. The circle with the black dots represents the isotropic plug made of CSM. The $x$-$y$ axes are fixed along the principal directions of the material, and the $x'$-$y'$ axes are the reference directions for the external shear stress $\tau_{x'y'}$. The angle between the principal directions of the material and the external shear stress is $\phi = 45^\circ$.

![Figure A-1. State of stress of an orthotropic laminate loaded with in-plane shear stress $\tau_{x'y'}$ at an angle $\phi = 45^\circ$ with respect to the principal directions of elasticity of the laminate material.](image)

Given the external shear stress $\tau_{x'y'}$, associated with the $x'$-$y'$ axes, we can find the equivalent stress components associated with the $x$-$y$ axes using stress transformation equations. Crandall, Dahl and Lardner [35] provide these equations, which in this case take the form of eqns. (A-1) - (A-3).
\[
\sigma_{xx} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cdot \cos(2(-\phi)) + \tau_{x'y'} \sin(2(-\phi)) \tag{A-1}
\]
\[
\sigma_{yy} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2} \cdot \cos(2(-\phi)) - \tau_{x'y'} \sin(2(-\phi)) \tag{A-2}
\]
\[
\tau_{xy} = -\frac{\sigma_{xx} - \sigma_{yy}}{2} \cdot \sin(2(-\phi)) + \tau_{x'y'} \cos(2(-\phi)) \tag{A-3}
\]

The loading condition in the \(x' - y'\) axes is produced by pure shear; therefore, we have

\[
\sigma_{x'x'} = 0 \tag{A-4}
\]
\[
\sigma_{y'y'} = 0 \tag{A-5}
\]

If equations (A-4) and (A-5) are substituted into equations (A-1) – (A-3), the stress components in the \(x - y\) reference system are obtained

\[
\sigma_{xx} = -\tau_{x'y'} \tag{A-6}
\]
\[
\sigma_{yy} = \tau_{x'y'} \tag{A-7}
\]
\[
\tau_{xy} = 0 \tag{A-8}
\]

We can use the results (A-6) - (A-8) to represent the original shear loading condition using the superposition of two uniaxial normal stresses. The uniaxial normal stresses \(\sigma_{xx} = -\tau_{x'y'}\) and \(\sigma_{yy} = \tau_{x'y'}\) are shown in Figure A-2. Figure A-2 presents the decomposition of the loading shear stress into a tensile stress component \(\sigma_{yy}\) in the \(y\)-direction plus a compressive stress component \(\sigma_{xx} = -\tau_{x'y'}\) in the \(x\)-direction.
\[ \sigma_{xx} = -\tau_{x'y'} = \sigma_{o \text{ comp}} \]

\[ \sigma_{yy} = \tau_{x'y'} = \sigma_{o \text{ tens}} \]

**Figure A-2.** Shear stress represented by a linear superposition of two uniaxial stresses.

\( \theta \) is the polar angle measured counterclockwise from the x axis. This angle is the reference for the resultant tangential stresses \( \sigma_{\theta \theta}, \sigma_{rr}, \) and \( \tau_{r\theta} \). The reference polar angles for the tensile and compressive components are \( \theta_{\text{tens}} \) and \( \theta_{\text{comp}} \), respectively. Figure A-2 shows that the two normal components of the shear stress are aligned with the principal directions of the material. The tensile stress component \( \sigma_{yy} \) is aligned with the fibers running in the y-direction, which is one of the principal directions of elasticity of the orthotropic plate. Similarly, the compressive stress component \( \sigma_{xx} \) is aligned with the x-direction of the laminate.

Lekhnitskii [33] provides an analytical solution for the stresses along the contour of plugged circular opening in an orthotropic plate, and loaded with uniaxial normal stress. Equations (A-9) give the stresses along the contour of the opening as a function of the angle \( \theta \), as shown in Figure A-3. The angle \( \theta \) is measured from the x axis in the counterclockwise direction. Equations (A-9) can be used when \( \sigma_o \) is either tensile or compressive.
Figure A-3. Uniaxial stress $\sigma_o$ applied on orthotropic plate with plugged circular opening of $r = a$ [33].

$$
\sigma_{\theta\theta} = \frac{\sigma_o}{\frac{E_x}{E_y} + \frac{E_y}{E_z} + \frac{G_{xy}}{E_z}} \left\{ \frac{(\Delta - q_1)\sin^6 \theta + [(\Delta(n^2 - 2k) + (k + n)q_1}{\Delta \cdot E_x} \right. \\
+ (1 + 2k)q_2 - (2 + k)(1 + n)q_3]\sin^4 \theta \cdot \cos^2 \theta + \frac{[k^2\Delta - (1 + 2k)(k + n)q_2 + k(2 + k)q_3 + k(1 + n)q_4]}{\sin^2 \theta \cdot \cos^4 \theta - k^2 q_4 \cos^6 \theta} \\
\left. \right\} \\
\sigma_{rr} = \frac{\sigma_o}{2\Delta} [\Delta + q_3 + q_2 + (\Delta + q_3 - q_2)\cos(2\theta)] \\
\tau_{r\theta} = -\frac{\sigma_o}{2\Delta} (\Delta + q_3 - q_2)\sin(2\theta)
$$

where

$$
\frac{E_\theta}{E_x} = \left( \frac{\sin^4 \theta + \frac{2q_{12} + q_{66}}{q_{11}} \cdot \sin^2 \theta \cdot \cos^2 \theta + \frac{q_{32}}{q_{11}} \cos^4 \theta}{q_{11}} \right)^{-1}
$$

$$
k = \sqrt{\frac{E_x}{E_y}}
$$

$$
n = \sqrt{2 \left( \frac{E_x}{E_y} - \nu_{xy} \right) + \frac{E_x}{G_{xy}}}
$$
In equation (A-9) $\Delta$, $q_1$, $q_2$, $q_3$ and $q_4$ are coefficients that depend on the elastic constants of the plate material and the plug material. These parameters are given by

\begin{align*}
\Delta &= (q_{11}q_{22} + q_{11}'q_{22}')k + q_{22}(q_{66} + 2q_{12}') + (q_{11}q_{22}'k + q_{22}q_{11}')n - (q_{12} - q_{12}')^2 k \tag{A-13} \\
q_1 &= (q_{11} - q_{11}')q_{22}'(n^2 - k) + [(q_{11} - q_{11}')q_{22} + (q_{12} - q_{12}')^2]kn - q_{22}(q_{12} - q_{12}') \tag{A-14} \\
q_2 &= (q_{11} - q_{11}')q_{22} + q_{11}(q_{12} - q_{12}')k(1 + n) \tag{A-15} \\
q_3 &= (q_{11} - q_{11}')(q_{22}n + q_{22}'k) + q_{22}(q_{12} - q_{12}') + (q_{12} - q_{12}')k \tag{A-16} \\
q_4 &= -(q_{11} - q_{11}')q_{22}' + (q_{12} - q_{12}')[2q_{12} + q_{66} + q_{11}(k + n)] - (q_{12} - q_{12}')^2 \tag{A-17}
\end{align*}

$q_{11}$, $q_{12}$, $q_{22}$, $q_{16}$, $q_{26}$, and $q_{66}$ are independent elastic compliances of the plate material and they are expressed in terms of the constitutive properties of the orthotropic plate $E_x$, $E_y$, $\nu_{xy}$, and $G_{xy}$

\begin{align*}
q_{11} &= \frac{1}{E_x} \tag{A-18} \\
q_{22} &= \frac{1}{E_y} \tag{A-19} \\
q_{12} &= \frac{\nu_{xy}}{E_x} \tag{A-20} \\
q_{66} &= \frac{1}{G_{xy}} \tag{A-21} \\
q_{16} = q_{26} &= 0 \tag{A-22}
\end{align*}

$q_{11}'$, $q_{12}'$, $q_{22}'$, $q_{16}'$, $q_{26}'$, and $q_{66}'$ are the elastic compliances of the plug material and they can be expressed in terms of the constitutive properties of the isotropic CSM plug $E$, $\nu$, and $G$ as

\begin{align*}
q_{11}' &= q_{22}' = \frac{1}{E} \tag{A-23} \\
q_{12}' &= -\frac{\nu}{E} \tag{A-24}
\end{align*}
Therefore, for an applied shear stress of $\tau_{xy'}$, the stresses $\sigma_{\theta\theta}$, $\sigma_{rr}$, and $\tau_{r\theta}$ are obtained by adding the corresponding results of equations (A-9) for both the tensile and compressive components shown in Figure A-2. Note that the tensile component is rotated 90° degrees with respect to the reference axes of equations (A-9). The superposition must be done with respect to the same reference axes; therefore, the contribution to $\sigma_{\theta\theta}$, $\sigma_{rr}$, and $\tau_{r\theta}$ produced by the tensile and compressive components should be calculated using the following substitutions for $\sigma_o$ and $\theta$ in equations (A-9) and (A-10):

Substitution to calculate the tensile component

$$\sigma_{yy} = \sigma_o \text{ tens} = \tau_{xy'} = \sigma_o$$  \hspace{1cm} (A-27)
$$\theta_{\text{tens}} = \theta - 90^\circ$$ \hspace{1cm} (A-28)

Substitution to calculate the compressive component

$$\sigma_{xx} = \sigma_o \text{ comp} = -\tau_{xy'} = -\sigma_o$$  \hspace{1cm} (A-29)
$$\theta_{\text{comp}} = \theta$$ \hspace{1cm} (A-30)

Although the reference frame is rotated for the two components, the constitutive properties and elastic compliances of the laminate are the same for both components when the laminate is made of balanced WR. For laminates with biased WR, the constitutive properties are different in the two principal material directions. In this case, equations (A-9) require the substitution of the constitutive properties along the corresponding axes for the calculation of the compressive and tensile stress components.