Collusive Dominant-Strategy Truthfulness
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Abstract
Fifty years ago, Vickrey published his famous mechanism for auctioning a single good in limited supply. The main property of Vickrey’s mechanism is efficiency in dominant strategies. In absence of collusion, this is a wonderful efficiency guarantee. We note, however, that collusion is far from rare in auctions, and if some colluders exist and have some wrong beliefs, then the Vickrey mechanism dramatically loses its efficiency. Accordingly, we put forward a new mechanism that, despite unconstrained collusion, guarantees efficiency by providing a richer set of strategies and ensuring that it is dominant for every player to reveal truthfully not only his own valuation, but also with whom he is colluding, if he is indeed colluding with someone else.

Our approach meaningfully bypasses prior impossibility proofs.

1 Introduction
The presence of collusion and wrong beliefs can clearly frustrate the aims of a mechanism designer. Beliefs do not come into play in a dominant-strategy mechanism, but collusion might continue to be a problem. Indeed, any equilibrium, including a dominant-strategy one, only guarantees that no individual player has any incentive to deviate from his envisaged strategy. However, two or more players may have plenty of incentive to jointly deviate from their equilibrium strategies.

An Unconstrained Collusion Model Sometimes the ability of the players to collude is constrained by suitable assumptions. For instance, one may assume an upper bound to the number of possible colluders, the colluders’ inability to keep secret their cooperation, their inability to make “side payments” to each other, and their mutual distrust. It has also been assumed that who colludes with whom may be known to the players. The collusion model envisaged in this paper is instead quite unconstrained.

In essence, after a mechanism is announced, we assume that the players secretly partition themselves into arbitrarily many coalitions of arbitrary size. (An independent player forms a coalition of cardinality 1.) The members of the same coalition can make side payments to each other, and perfectly coordinate their actions—for instance, thanks to their ability to enter secret binding agreements. The only constraint is that all coalitions are rational. That is, the members of the same coalition $C$ act so as to maximize the sum of their individual utilities. Since they can separately compensate each other, this is indeed the rational thing for them to do.
Vulnerability of the Vickrey Auction  The Vickrey mechanism [21] is dominant strategy, but is it resilient to collusion? When a single copy of the good is available, the Vickrey auction coincides with the second-price one. The second-price auction is so simple that its efficiency is actually “automatically immune to collusion.” Informally, if the member of coalition $C$ with the highest valuation for the good is player $i$, and if his value for the good is $v$, then the “best collective strategy” for $C$ is to have $i$ bid $v$, and all other members bid 0. By so doing $C$ may affect the auction’s revenue, but not its efficiency. On the other hand, an auction of a single good in limited supply can be viewed as a special case of a combinatorial auction, and the Vickrey mechanism as a special case of the general VCG mechanism [21, 7, 12], which indeed guarantees efficiency in dominant strategies for combinatorial auctions. However, Ausubel and Milgrom [1] have already shown that the efficiency of the VCG can be totally destroyed by just two (sufficiently informed) collusive players.

The Vickrey auction does not fall in either extreme. However, its efficiency is highly vulnerable to a combination of collusion and wrong beliefs. Let us illustrate this point by the following example, assuming some familiarity with the Vickery mechanism, which is anyway recalled in Subsection 3.2.

Example  Consider a Vickrey auction (with ties broken at random) for a good available in two identical copies. As usual, we assume that a player’s marginal value for a second copy is no greater than his value for a first copy: that is, all valuations are of the form $(x, y)$, with $x \geq y$, where $x$ represents a player’s value for a first copy, and $y$ his marginal value for a second one. There are 4 players: $a$ and $b$, who form a coalition, and $c$ and $d$, who are independent. Their respective valuations are $(100,0), (100,0), (1,0)$, and $(1,0)$. Accordingly, the maximum social welfare possible in this context is 200, and can be realized only by allocating one copy to $a$ and the other to $b$.

The two collusive players know each other’s valuations, and their beliefs are as follows: “$c$ and $d$ are independent and their respective valuations are $(1000,0)$ and $(x,0)$, where $x \leq 100$.” Let us now argue that, with such beliefs, bidding truthfully is not a joint dominant strategy for $a$ and $b$.

Because the Vickery mechanism is dominant-strategy, the collusive players expect $c$ and $d$ to bid truthfully: that is, $(1000,0)$ and $(x,0)$. Therefore, $a$ and $b$ also expect that, if they too bid truthfully, then at most one of them —say $a$— will win a copy of the good, in which case the bid of $b$ will set $a$’s price to $100$. In sum, the colluders expect that, by bidding truthfully, their “collective utility” will be 0 in any case. This is not an attractive prospect for the two colluders. In accordance to their beliefs, a better —indeed a “joint weakly dominant”— strategy for $a$ and $b$ is for one of them to bid $(100,0)$ and the other $(0,0)$. If they bid so, however, and if $c$ and $d$ rationally —and thus truthfully— bid $(1,0)$ and $(1,0)$, then the Vickrey mechanism must allocate one copy to a collusive player and the other to an independent one, thus realizing a social welfare of 101 rather 200. That is, the Vickrey mechanism does not guarantee efficiency in the presence of unconstrained collusion.

Our Contribution  We modify the Vickrey auction so as to make it resilient to beliefs and collusion. Our mechanism actually *welcomes* colluders to the auction by providing them with special collusive bids, so that it becomes dominant for a coalition to report its presence and the true valuations of its members. Since independent players are just coalitions of size 1, truthful revelation is the best strategy for independent players and coalitions alike. In a sense, our *collusive dominant-strategy truthfulness* tries to harmonize the cooperative and the non-cooperative settings.

So far, by forbidding and prosecuting collusion we have not eradicated it, we have just pushed it underground where it continues to be quite disruptive. Perhaps it is time, at least in some applications, to try a new course: namely, bringing colluders into the open and incentivizing them too to help us achieve our social goals.
2 Related Work

Coalition Incentive Compatibility  Our solution concept is closely related to coalition incentive compatibility, as put forward by Green and Laffont [11]. A coalition incentive-compatible auction mechanism requires that all strategies consist of valuations, and that every agent, whether an individual player or a coalition, has a dominant “course of action”: an individual strategy for an independent player and a subprofile of individual strategies for a coalition. As it turns out, however, requiring strategies to coincide with valuations is a severe restriction, and causes the main results about coalition incentive compatibility to consist of impossibility proofs. In particular, Green and Laffont [11] prove the impossibility of maximizing social welfare via coalition incentive compatible mechanisms. Our collusive dominant-strategy mechanism bypasses such impossibility proofs by endowing each individual strategy with an additional coalitional component.

Other Notions of Resiliency to Collusion  Assuming, as we do, that coalitions are rational, Goldberg and Hartline exhibit mechanisms that are collusion resilient so long as the cardinality of each coalition is suitably bounded [9]. Namely, using again dominant strategies as the underlying solution concept, their notion of c-truthful mechanism ensures that a coalition of at most c collusive players cannot “collectively gain more than they could by bidding individually.” (They exemplify their notion for auctions of multiple goods, and prove that, to be c-truthful, a mechanism M must, for any subset of the goods S and player i, fix a price \( p_{S,i} \) and offer S to i for that price. The same authors also investigate a weaker variant of their notion, c-truthful with high probability.)

For rational coalitions again, Laffont and Martimort [14] and Che and Kim [4] study collusion resiliency on a variety solution concepts, ultimately all equilibrium-based. (The latter authors further allow the utility of a coalition to be the weighted sum of the individual utilities of its members.)

Collusion resiliency has also been studied when (1) each coalition prefers an outcome \( \omega \) to an outcome \( \omega' \) if and only if each of its members prefers \( \omega \) to \( \omega' \); and (2) players cannot guarantee side-payments to one another. In this model, a mechanism can be considered resilient to collusion if it ensures that any gain for a member of a coalition is accompanied by a loss for another member of the same coalition. Such mechanisms have been constructed under different solution concepts: by [15, 17, 20] using equilibrium, and by [16, 3, 13, 18, 8, 19] using group (or coalition) strategy-proofness.

Collusion leveraging have been investigated by Chen, Micali, and Valiant [6] for generating revenue in combinatorial auctions. Their mechanism incentivizes players and coalitions alike to help the auctioneer to sell the goods, and elicits coalitional information to do so. They rely on a solution concept weaker than dominant strategies, and envisage a setting of incomplete information where a player knows that the valuation subprofile of his opponents is restricted to a given list of candidates.

Collusion resiliency has also been studied in settings of complete information, by Chen, Hassidim, and Micali [5], using unique subgame-perfect equilibrium as the solution concept and assuming that who colludes with whom is common knowledge among the players (but totally unknown to the designer). The latter restriction has been removed by Azar, Chen, and Micali [2], who show how to maximize social welfare in a budget-balanced way in very general markets, using a solution concept only slightly weaker than dominant strategies. Their results too assume a setting of complete information.
3 Preliminaries

All games considered in this paper are auctions of a single good, available in fixed number of identical copies. Thus, for simplicity only, we shall define/recall the notions we need just for such auctions.

3.1 Collusive Auctions

Each game \( G \) can be decomposed into a context \( \mathcal{C} \) and a mechanism \( \mathcal{M} \), \( G = (\mathcal{C}, \mathcal{M}) \), where \( \mathcal{C} \) specifies the players, the outcomes, and the players’ preferences over the outcomes,\(^1\) while \( \mathcal{M} \) specifies the available strategies and how they lead to outcomes. Our contexts and mechanisms are as follows.

Collusive Auction Contexts  A collusive auction context \( \mathcal{C} \) has the following components

- The players, a finite set \( N \), whose cardinality is consistently denoted by \( n \).
- The number of copies, a positive number \( m \).
- The outcomes, the set of all pairs \( (A, P) \) where \( A \) is a allocation, a vector in \( \{0, \ldots , m\}^N \) such that \( \sum_i A_i = m \), and \( P \) a price profile, a profile of reals. For \( i \in N \), \( A_i \) is the number of copies allocated to \( i \), and \( A_\perp \) is the number of copies left unallocated. Each \( P_i \) is \( i \)’s price.
- The valuation bound, a number \( B \) upper bounding any player’s marginal value for any copy.
- The valuations, the set of all vectors \( v : \{1, \ldots , m\} \rightarrow [0, B] \) such that \( v(1) \geq \cdots \geq v(m) \).
- The true valuations, a valuation profile \( \theta \), where each \( \theta_i(k) \) is \( i \)’s marginal value for a \( k \)th copy.
- The collusion structure, a partition \( \mathcal{C} \) of the players.

We refer to context \( \mathcal{C} \) as non-collusive if \( \mathcal{C} = \{\{i\} : i \in N\} \). For each player \( i \), \( \mathcal{C}_i \) denotes the set in \( \mathcal{C} \) containing \( i \). We refer to a player \( i \) as independent if \( \mathcal{C}_i = \{i\} \), and as collusive otherwise; and to a set \( C \in \mathcal{C} \) as a coalition.

In a context \( \mathcal{C} \), \( N \) and \( m \) are common knowledge to everyone. For each coalition \( C \in \mathcal{C} \), the set \( C \) itself and the subprofile \( \theta_C \) are common knowledge to the players in \( C \).

For an outcome \((A, P)\), the individual utility of a player \( i \) is \( u_i(A, P) \triangleq \sum_{k=1}^{A_i} \theta_i(k) - P_i \), and the collective utility of a coalition \( C \) is \( u_A(A, P) \triangleq \sum_{i \in C} u_i(A, P) \). We refer to \( u_i \) as \( i \)’s individual utility function, and to \( u_C \) as \( C \)’s collective utility function. Each player \( i \) acts so as to maximize \( u_C \).

The social welfare of an allocation \( A \) for \( \mathcal{C} \), \( SW(A) \), is \( \sum_i \sum_{k=1}^{A_i} \theta_i(k) \). An allocation \( A \) for \( \mathcal{C} \) is efficient if for all allocations \( A' \) for \( \mathcal{C} \), \( SW(A) \geq SW(A') \).

Note that \( \mathcal{C} \) is totally identified by \( N, m, B, \theta \), and \( \mathcal{C} \) alone: that is, \( \mathcal{C} = (N, m, B, \theta, \mathcal{C}) \). If \( \mathcal{C} \) is non-collusive, then \( \mathcal{C} = (N, m, B, \theta) \).

The set of all contexts with player set \( N \), number of goods \( m \) and valuation bound \( B \) is \( \mathcal{C}^{N,m,B} \).

Directly Collusive Auction Mechanisms  A directly collusive auction mechanism \( M \) for \( \mathcal{C}^{N,m,B} \) is a normal-form mechanism in which the set of (pure) strategies for a player \( i \), \( \Sigma_i \), is so defined: \( \Sigma_i = \{(v, C) : v : \{1, \ldots , m\} \rightarrow [0, B], C \in 2^N\} \). Accordingly, \( M \) is fully described by its allocation function, denoted by \( M_\alpha \), and its price function, denoted by \( M_\pi \). That is, \( M = (M_\alpha, M_\pi) \).

\(^1\)Since our solution concept is (stronger than) dominant strategies, we have no need to specify the players’ beliefs.
Remarks Let us emphasize that the designer of a collusively direct mechanism $M$ for $\mathcal{C}^{N,m,B}$ has no information about (not only the true valuation profile $\theta$, but also about) the collusion structure $C$ of the context $\mathcal{C} = (N, m, B, \theta, C)$ in which $M$ will be played. Indeed $C$ may arise after $M$ has been chosen. Each of our contexts formally includes its collusion structure, but in all our theorems

"contexts are universally quantified after mechanisms."

Because collusively direct mechanisms only specify strategies for individual players, the “collusive strategies” available to coalition $C$ are the subprofiles of strategies $s_C$, where $s_j \in \Sigma_i$ for each $j \in C$.

3.2 Collusive Dominant-Strategy Truthful Auction Mechanisms

A collusively direct auction mechanism $M$ for $\mathcal{C}^{N,m,B}$ is collusively dominant-strategy truthful if for all collusive contexts $\mathcal{C} = (N, m, B, \theta, C) \in \mathcal{C}^{N,m,B}$, all coalitions $C \in C$, and all strategy subprofiles $s_C$ and $s_{-C}$

$$\sum_{i \in C} u_i(t_C, s_{-C}) \geq \sum_{i \in C} u_i(s_C, s_{-C})$$

where $t = (\theta_1, C_1), \ldots, (\theta_n, C_n)$ and $u_i(x) \Delta= u_i(M_\alpha(x), M_\pi(x))$ for any strategy profile $x$.

We refer to $t$ as a the truthful strategy profile of the game $(\mathcal{C}, M)$: in symbols, $t = truth(\mathcal{C}, M)$.

3.3 The Vickrey Mechanism (with lexicographically broken ties)

The Vickrey mechanism for a player set $N$ and a number of copies $m$, $\text{Vick}^{N,m}$, is so defined. The pure strategies of each player $i$ consist of all possible valuations. Given a profile of valuations $V$, the mechanism orders the set $\{(i, V_i(k)) : i \in N, k = 1, \ldots, m\}$ according to $>_V$, where $(i, k) >_V (i', k')$ if and only if

$$V_i(k) > V_i(k') \quad \text{or} \quad V_i(k) = V_i(k') \quad \text{and} \quad i < i' \quad \text{or} \quad V_i(k) = V_i(k'), \ i = i', \ and \ k < k'.$$

Then, it computes $F^m_V$, the sequence of the first $m$ pairs in the above order, and for all $i \in N$ sets $\text{Vick}^{N,m}_\alpha(V)_i = A_i$, where $A_i$ is the number of pairs of $F^m_V$ whose first component is $i$, and

$$\text{Vick}^{N,m}_\pi(V)_i = \begin{cases} 0 & \text{if } A_i = 0, \text{ else} \\ \sum_{k=1}^{A_i} p_i^k & \text{where } (i', p_i^k) \text{ is the } (A_i - k + 1)\text{th pair not in } F^m_V \text{ such that } i' \neq i \end{cases}$$

4 Our Mechanism

4.1 Intuition

Our mechanism, $\mathcal{M}$, allows each player to report not only his true valuation, but also the coalition he belongs to (if he is collusive). Assume for a moment that each player reports truthfully. Then, $\mathcal{M}$ considers each coalition $C$ as a single fictional player, whose fictional valuation $V_C$ is obtained by merging the valuations of $C$’s members. Having done this, $\mathcal{M}$ runs the Vickrey mechanism for $m$ copies at hand. We can however consider a valuation $v$ as mapping $\{1, 2, \ldots, m\}$, rather than $\{1, \ldots, m\}$, to $[0, B]$, where $v(k)$ is the marginal value of a $k$th copy of the good, if available. The Vickrey mechanism remains well-defined and dominant-strategy with such “infinite” valuations.

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2The Vickrey mechanism does not envisage a valuation bound. We describe it from a price-per-copy perspective.

3Notice that if there are $m$ copies of the good for sale and $C$ consists of $c$ players, then $C$’s fictional valuation will consist of a decreasing sequence of $cm$ numbers. That is, the number of elements in $V_C$ is higher than the number of copies at hand. We can however consider a valuation $v$ as mapping $\{1, 2, \ldots, m\}$, rather than $\{1, \ldots, m\}$, to $[0, B]$, where $v(k)$ is the marginal value of a $k$th copy of the good, if available. The Vickrey mechanism remains well-defined and dominant-strategy with such “infinite” valuations.
copies, with the fictional players and their fictional valuations, so as to compute an allocation of the copies of the good to the fictional players and the price paid by each fictional player. Having figured out this way that \( C \) should collectively receive —say— \( m_C \) copies and make a payment \( P_C \), \( M \) must now decide how to distribute these copies and this payment to the members of \( C \). Conceptually, \( M \) asks \( C \) how to do so. Practically, it uses the originally reported (non-fictional) valuations to compute the right answer. (It is quite clear that if the members of \( C \) report truthfully \( C \), then it is in their interest to report truthfully also their individual valuations. Not doing so would be quite irrational of them, since they try to maximize the sum of their individual utilities.) As for prices, since again what matters is \( C \)'s collective utility, \( M \) is at liberty of subdividing \( P_C \) in any way it wishes among the members of \( C \). (For simplicity, we charge just one player for each coalition.)

Note that, if the coalitional components of the bids were ignored, and the Vickrey mechanism ran on just the reported valuations, then the produced allocation would be the same as that computed by \( M \) in the manner specified above, but the sum of the prices paid by the players in \( C \) might be higher. This is so because, taking for a moment an individual/per-copy perspective, in \( M \) “the valuations of all the players in \( C \) are not used to set the price of a player \( i \) in \( C \)” while in the Vickrey case “only the valuation of \( i \) himself is not used to set \( i \)'s price.” In a sense, therefore, \( M \) guarantees efficiency while generating potentially less revenue: essentially, it gives a discount to coalitions of multiple players.

Discounts, of course, are attractive to every one. It is thus unclear whether it is dominant for —say— an independent player \( i \) to report truthfully \( \{i\} \) as his coalitional component. Indeed, if a coalition \( C \) were somehow kind enough, in its report, to include him as a member (i.e., if \( C \)'s members reported \( C \cup \{i\} \) instead of \( C \)), then he might be better off reporting \( C \cup \{i\} \) instead of \( \{i\} \).

To guarantee collusive dominant-strategy truthfulness, \( M \) must therefore add incentives preventing such eventualities from being rationally entertained. Ideally, \( M \) should check the “consistency” of all reported coalitional components, and punish all “misreporters.” The problem, however, is that this is not obviously implementable, and in a sense it is actually impossible.

For instance, if players \( a, b, c, \) and \( d \) form a coalition \( C \), but \( a \) and \( b \) consistently report their coalition to be \( \{a, b\} \), while \( c \) and \( d \) consistently report their coalition to be \( \{c, d\} \), then there is no way for \( M \) to figure out that this is not true. The only thing that \( M \) can do, and actually does, is to ensure that it will not be in \( C \)'s interest to “pretend to be two smaller coalitions”, or a smaller coalition plus one or two independent players.

For some forms of inconsistent coalitional reporting, \( M \) identifies a guilty player and severely fines him. For others, it figures out a “correct coalitional report” and acts as if that were what the players actually reported.

For instance, if player \( a \) reports his coalition to be \( \{a, b\} \); player \( b \) that his coalition is \( \{a, b, c\} \); player \( c \) that her coalition is \( \{b, c, d\} \); and player \( d \) that his coalition is \( \{c, d\} \) (and if all other coalitional reporting seems to be in good order), then the mechanism proceeds to compute allocation, prices and fictional players as if \( a, b, c, \) and \( d \) all reported the coalition \( \{a, b, c, d\} \), although such a coalition was not reported by anyone. Yet, collusive dominant-strategy truthfulness will be guaranteed.

### 4.2 Formalization

We denote by \( M^{N,m,B}_N \) the collusively direct mechanism for \( C^{N,m,B} \) defined as follows.

Let \( s = (V_1, C_1), \ldots, (V_n, C_n) \) be a strategy profile. A disagreement (with respect to \( s \)) is a pair of players \( (i, j) \) such that \( j \in C_i \) but \( i \notin C_j \). If there exists a disagreement, then \( M^{N,m,B}_N \) and \( M^{N,m,B}_\pi \) are defined via the “punishing procedure” below, where \( x := y \) denotes the operation that assigns value \( y \) to variable \( x \). Else, they are defined via the subsequent “standard procedure.”
Punishing Procedure: Set $P_x = 0$ for each player $x$. Then, for each disagreement $(i, j)$ do:

$P_i := P_i + 2mB$ and $P_j := P_j - mB$ (i.e., $i$ pays $j$ and the seller a fine of $mB$).

Finally, set $\mathcal{M}^{N,m,B}_\alpha(s)_\perp = m$ (i.e., all copies remain unallocated) and $\mathcal{M}^{N,m,B}_\pi(s)_i = P_i \forall i \in N$.

Standard Procedure: Let $\mathbb{P}$ be the partition of the players consisting of the connected components of the graph having the players as nodes, and having an edge $(i, j)$ whenever $j \in C_i$ — and thus $i \in C_j$.

Order the set $\{(i, V_i(k)) : i \in N, k = 1, \ldots, m\}$ according to $> V$, compute $F^m_V$, the set of the first $m$ pairs, and then for each $i \in N$ set

$\mathcal{M}^{N,m,B}_\alpha(s)_i = A_i$, the number of pairs of $F^m_V$ whose first component is $i$, as for Vick$^{N,m}_\alpha$, and

$\mathcal{M}^{N,m,B}_\pi(s)_i = \begin{cases} P_\mathbb{P}_i & \text{if } i = \min\{i' \in \mathbb{P}_i\}, \\
0 & \text{otherwise}
\end{cases}$

where, for each $C \in \mathbb{P}$, letting $A_C = \sum_{i \in C} A_i$,

$P_C = \begin{cases} 0 & \text{if } A_C = 0, \\
\sum_{k=1}^{A_C} p^k_C & \text{otherwise, where } (i', p^k_C) \text{ is the } (A_C - k + 1)\text{th pair not in } F^m_V \text{ such that } i' \notin C.
\end{cases}$

5 Analysis of Our Mechanism

Theorem 1. For all $N$, $m$, and $B$,

- $\mathcal{M}^{N,m,B}$ is a collusive dominant-strategy truthful mechanism for $\mathcal{C}^{N,m,B}$ and

- For all $\mathcal{C} \in \mathcal{C}^{N,m,B}$, $\mathcal{M}^{N,m,B}_\alpha(t)$ is an efficient allocation for $\mathcal{C}$, where $t = \text{truth}(\mathcal{C}, \mathcal{M}^{N,m,B})$.

Proof. Let us more simply denote $\mathcal{M}^{N,m,B}$ by $\mathcal{M}$, and assume that $\mathcal{M}$ is played in a collusive context $(N, m, B, \theta, \mathcal{C})$.

Let us prove first that $\mathcal{M}$ indeed returns an allocation maximizing social welfare when each player $i$ truthfully bids $t_i = (\theta_i, C_i)$. To this end, notice that, when the strategy profile is $t$, no disagreement exists, and thus $\mathcal{M}$ does not execute the punishing procedure, but the standard one. Notice too that, on input $t = (\theta_1, C_1), \ldots, (\theta_n, C_n)$, the standard procedure computes the same allocation $A$ as Vick$^{N,m}_\alpha(\theta)$. Thus, since we are assuming $\theta$ to be the true-value profile, $A$ has maximum welfare.

Let us now prove that, if $\mathcal{C}$ is a coalition in $\mathcal{C}$, then $t_C$ is a (very weakly) dominant (collective) strategy for $\mathcal{C}$. We do so by proving two separate propositions. The first is as follows.

(1) For all $C \in \mathcal{C}$, all valuation subprofiles $V_C$, and all strategy subprofiles $\mathcal{S}_C = \{(V_j, C_j) : j \in C\}$, $\mathcal{S}_C = \{(V_j, C_j) : j \in C\}$, and $s_{-C}$: $u_C(\mathcal{S}_C, s_{-C}) \geq u_C(\mathcal{S}_C, s_{-C})$.

To prove Proposition 1, we consider the following four exhaustive cases.

Case 1: The Punishing Procedure is invoked both for $(\mathcal{S}_C, s_{-C})$ and $(\mathcal{S}_C, s_{-C})$.

In this case, no copy is allocated, and thus $C$’s utility coincides with $IN_C - OUT_C$, where $IN_C$ is the sum of all fines paid to members of $C$ by outside players, and $OUT_C$ is the sum of all fines paid by members of $C$ to outside players or the seller.

Recall that the punishing procedure is invoked only if there exists a disagreement $(i, j)$ — that is, only if $i$ declares $j$ to collude with him, but $j$ does not “reciprocate” — and that such a disagreement in particular results in $i$ paying $j$ and the seller a fine of $mB$. Thus, under the strategy profile $(\mathcal{S}_C, s_{-C})$, $\text{Case 1: The Punishing Procedure is invoked both for } (\mathcal{S}_C, s_{-C}) \text{ and } (\mathcal{S}_C, s_{-C})$.

4The absence of disagreements in fact implies that $i \in C_j$ whenever $j \in C_i$, and thus that $\mathbb{P}$ is a partition.
\( \text{OUT}_C = 0; \) while under \((\mathcal{S}_C, s_{-C})\), \( \text{OUT}_C \geq 0 \). On the other hand, if a disagreement \((i, j)\) between an outsider \(i\) and a member \(j\) of \(C\) exists under \((\mathcal{S}_C, s_{-C})\), then it exists also under \((\mathcal{S}_C, s_{-C})\). Thus the total value of \( \text{IN}_C \) under \((\mathcal{S}_C, s_{-C})\) is greater or equal to its value under \((\mathcal{S}_C, s_{-C})\).

Accordingly Proposition (1) holds in Case 1.

**Case 2: The Punishing Procedure is invoked for \((\mathcal{S}_C, s_{-C})\) but not for \((\mathcal{S}_C, s_{-C})\)**

In this case, there is a disagreement under \((\mathcal{S}_C, s_{-C})\) and all utilities are determined by fines. Note that, for each such a disagreement \((i, j)\) it cannot be \(i \notin C\), else \((i, j)\) would be a disagreement under \((\mathcal{S}_C, s_{-C})\) too. Thus, \(i \in C\) and pays each of \(j\) and the seller, a fine of \(mB\). Accordingly, the collective utility of \(C\) decreases by at least \(mB\). Thus, \(u_C(\mathcal{S}_C, s_{-C}) \leq -mB\).

On the other side, under the standard procedure all utilities are due to the allocation of the copies and the prices paid for them. And whenever a member \(j\) of \(C\) receives his \(k\)th copy of the good, then the collective utility of \(C\) increases by \(\theta_j(k) \geq 0\) and decreases by \(p_j^k\), where \(p_j^k \leq V_j(k) \leq B\). Since the are \(m\) copies altogether, we have \(u_C(\mathcal{S}_C, s_{-C}) \geq -mB\).

Accordingly Proposition (1) holds also in Case 2.

**Case 3: The Punishing Procedure is invoked for \((\mathcal{S}_C, s_{-C})\) but not for \((\mathcal{S}_C, s_{-C})\)**

In this case, under \((\mathcal{S}_C, s_{-C})\) there must exist a disagreement. Moreover, for each such disagreement \((i, j)\), we must have (a) \(i \notin C\), because \(C_x = C\) for all \(x \in C\), and (b) \(j \in C\), because otherwise \((i, j)\) would be a disagreement also under \((\mathcal{S}_C, s_{-C})\), contrary to our hypothesis. Therefore, \(i \notin C\) pays \(j \in C\) a fine of \(mB\). This implies \(u_C(\mathcal{S}_C, s_{-C}) \geq mB\).

On the other side, because under the standard procedure all utilities are due to allocation and payments, and because whenever a player \(i \in C\) receives his \(k\)th copy his value received is \(\theta_i(k) \leq B\) and is payment \(p_i^k \geq 0\), we have \(u_C(\mathcal{S}_C, s_{-C}) \leq mB\).

Accordingly, Proposition (1) holds also in Case 3 too.

**Case 4: The Standard Procedure is invoked for both \((\mathcal{S}_C, s_{-C})\) and \((\mathcal{S}_C, s_{-C})\)**

In this case, all copies are allocated, both under \((\mathcal{S}_C, s_{-C})\) and \((\mathcal{S}_C, s_{-C})\). Moreover, since the underlying valuation profile \(V\) is the same for both strategy profiles, so is the allocation, \(A\). However, the player partitions computed by \(\mathcal{M}\), respectively denoted by \(\mathcal{P}\) and \(\mathcal{P}\), may be different: \(\mathcal{P}\) and \(\mathcal{P}\) must coincide outside \(C\), but while \(C \in \mathcal{P}\), \(C\) may not be a member of \(\mathcal{P}\). However, if \(C \notin \mathcal{P}\), then \(C_1, \ldots, C^k \in \mathcal{P}\), where \(\{C_1, \ldots, C^k\}\) is a partition of \(C\). (This is so otherwise a disagreement should have existed under either \((\mathcal{S}_C, s_{-C})\) or \((\mathcal{S}_C, s_{-C})\).) Thus, if \(i \in C\) and \(A_i \neq \emptyset\), then —letting without loss of generality \(i \in C_j\), and setting \(A_C = \sum_{i \in C} A_i\) and \(A_{C_i} = \sum_{i \in C_j} A_i\)— the price that \(i\) pays for the \(k\)th copy he actually receives is \(p_i^k\) under \((\mathcal{S}_C, s_{-C})\), and \(p_i^k\) under \((\mathcal{S}_C, s_{-C})\), where

- \((i', p_i^k)\) is the \((A_C - k + 1)\)th pair not in \(F^m_{i'}\) such that \(i' \notin C\), and
- \((i', p_i^k)\) is the \((A_{C_i} - k + 1)\)th pair not in \(F^m_{i'}\) such that \(i' \notin C_j\).

Thus, \(p_i^k \leq p_i^k\), and the collective price of \(C\) is no greater under \((\mathcal{S}_C, s_{-C})\) than under \((\mathcal{S}_C, s_{-C})\).

Accordingly, Proposition (1) holds in this last case too, and has thus been proven.

Having established that a rational player \(i\) truthfully declares the coalition \(C\) he belongs to, let us now point out that he must also be truthful about his valuation. More formally,

(2) For all strategy subprofiles \(t_C' = \{(V_j, C_j) : j \in C\}\), and all strategy profiles \(s_{-C}\), \(u_C(t_C, s_{-C}) \geq u_C(t_C', s_{-C})\).
This is indeed an immediate corollary of the dominant strategy truthfulness of the Vickrey mechanism, and of the fact that, when all players in C truthfully reveal C, our mechanism guarantees that C is essentially treated as a single player in the Vickrey mechanism.

References


