

Towards Birational Aspects of Moduli Space of Curves

by

Shashank S. Dwivedi

B.Tech., Indian Institute of Technology (2007)

Submitted to the Department of Electrical Engineering and Computer Science

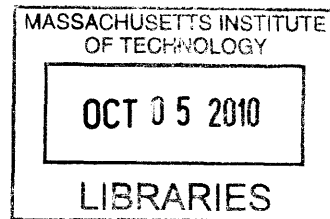
in partial fulfillment of the requirements for the degree of

Master of Science in Electrical Engineering

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2010



© Massachusetts Institute of Technology 2010. All rights reserved.

ARCHIVES

Author
Department of Electrical Engineering and Computer Science
August 10, 2010

Certified by
James M^cKernan
Professor
Thesis Supervisor

Accepted by
Terry P. Orlando
Professor
Chairman, Department Committee on Graduate Students

Towards Birational Aspects of Moduli Space of Curves

by

Shashank S. Dwivedi

Submitted to the Department of Electrical Engineering and Computer Science
on August 10, 2010, in partial fulfillment of the
requirements for the degree of
Master of Science in Electrical Engineering

Abstract

The moduli space of curves has proven itself a central object in algebraic geometry. The past decade has seen substantial progress in understanding its geometry. This has been spurred by a flurry of ideas from geometry (algebraic, symplectic, and differential), topology, combinatorics, and physics. One way of understanding its birational geometry is by describing its cones of ample and effective divisors and the dual notion of the Mori cone (the closed cone of curves).

This thesis aims at giving a brief introduction to the moduli space of n -pointed stable curves of genus g , $\overline{M}_{g,n}$, and some intuition into it and its structure. We do so by surveying what is currently known about the ample and the effective cones of $\overline{M}_{g,n}$, and the problem of determining the closed cone of curves $\overline{NE}_1(\overline{M}_{g,n})$.

The emphasis in this exposition lies on a partial resolution of the Fulton-Faber conjecture (the F-conjecture). Recently, some positive results were announced and the conjecture was shown to be true in a select few cases. Conjecturally, the ample cone has a very simple description as the dual cone spanned by the F-curves. Faber curves (or F-curves) are irreducible components of the locus in $\overline{M}_{g,n}$ that parameterize curves with $3g - 4 + n$ nodes. There are only finitely many classes of F-curves. The conjecture has been verified for the moduli space of curves of small genus. The conjecture predicts that for large g , despite being of general type, \overline{M}_g behaves from the point of view of Mori theory just like a Fano variety. Specifically, this means that the Mori cone of curves is polyhedral, and generated by rational curves. It would be pleasantly surprising if the conjecture holds true for all cases. In the case of the effective cone of divisors the situation is more complicated.

F-conjecture. *A divisor on $\overline{M}_{g,n}$ is ample (nef) if and only if it intersects positively (nonnegatively) all 1-dimensional strata or the F-curves. In other words, every extremal ray of the Mori cone of effective curves $\overline{NE}_1(\overline{M}_{g,n})$ is generated by a one dimensional stratum.*

The main results presented here are:

- (i) the Mori cone $\overline{NE}_1(\overline{M}_{g,n})$ is generated by F-curves when $g \leq 8, n = 1$ or $g = 6, n = 2$.

- (ii) the F-conjecture is true for \overline{M}_g for $g \leq 24$.
- (iii) the F-conjecture holds if and only if it holds for $g = 0$.

Thesis Supervisor: James McKernan

Title: Norbert Wiener Professor of Mathematics

Acknowledgments

I would like to thank my advisor, Prof. James M^cKernan, first and foremost for believing in me to be capable enough to conduct research in mathematics. He has been very patient answering my more often than not silly questions, and allowing me to attend conferences, that have proved to be a boost to my learning curve. I am grateful for the contributions he made in my ongoing development as a mathematician. It has been equally entertaining listening to him explain ideas from the basic theory of schemes to shifty notions of birational geometry.

Yoonsuk, thanks for being there to help me out with simpler problems in my research. I made a bunch of wonderful friends at the summer school in Utah, especially Han-Bom with whom I have had lengthy discussions on birational geometry both in class and while hiking. Arpit, Shreerang and Shreevatsa, thanks for dragging me out on weekends for some excursion or another, and making my not so interesting days fun.

I would also like to thank my previous advisor, Pablo Parrilo, to whom I'll be ever indebted to accept my decision to switch my focus to pure mathematics. He has always been patient with me and has always given me the latitude to pursue my interests. Last but not the least, thanks to my undergraduate thesis advisor, Harish Pillai, with whom it all started. It was for the first time that I worked on a deep problem in algebraic geometry, and have since been falling in love with the subject.

Finally I would like to express my deepest gratitude to my family for their ever-growing love and support.

Contents

1	Moduli Space of Curves	9
1.1	Introduction	9
1.1.1	Basic facts and Notations	9
1.1.2	The moduli space of curves	12
1.1.3	Basic divisor classes of $\overline{M}_{g,n}$	15
1.2	The Big Picture	18
2	Effective, Mori and nef cones	21
2.1	Effective cone	21
2.2	Mori and nef cones	26
2.3	MF -conjecture	33
3	Bridge and Ray theorems: proofs	37
3.1	Bridge theorem: proof	37
3.2	Ray theorem: proof	41
	References	43

Chapter 1

Moduli Space of Curves

1.1 Introduction

This section is dedicated to the statements of basic definitions and some well known theorems. The first subsection introduces various cones of interest. The next subsection is a fast-paced journey through the moduli space of stable curves. The subsection after that introduces various (naturally occurring) divisor classes and some of the relations between these classes.

1.1.1 Basic facts and Notations

In this section, we review some basic terminology and facts. Let X be a normal, \mathbb{Q} -factorial variety or more generally a scheme.

Definition 1.1.1. *A prime divisor is an irreducible and reduced subscheme of X of codimension one. A Weil divisor on X is a formal linear combination $D = \sum_i d_i D_i$ of prime divisors where $d_i \in \mathbb{Z}$. A \mathbb{Q} -divisor is a formal linear combination $D = \sum_i d_i D_i$ of prime divisors where $d_i \in \mathbb{Q}$. D is called effective if $d_i \geq 0$ for every i . Two divisors D_1, D_2 are linearly equivalent, written $D_1 \sim D_2$, if $D_1 - D_2$ is principal.*

A Cartier divisor is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$ [20]. A \mathbb{Q} -divisor D is \mathbb{Q} -Cartier if for some $m \in \mathbb{Z}$, the multiple mD is a Cartier divisor. Since X is

normal, a \mathbb{Q} -Cartier divisor is a \mathbb{Q} -Weil divisor [20]. The next definition gives a weaker equivalence relation on divisors, which will be of more interest to us.

Definition 1.1.2. *Two divisors D_1, D_2 are called numerically equivalent, written $D_1 \equiv_{\text{num}} D_2$, if the intersection numbers $D_1 \cdot C = D_2 \cdot C$ are equal for every irreducible curve $C \subset X$. Two curves C_1, C_2 in X are called numerically equivalent if $D \cdot C_1 = D \cdot C_2$ for every codimension one subvariety $D \subset X$.*

Numerical equivalence naturally extends to \mathbb{Q} -Cartier or \mathbb{R} -Cartier divisors. The intersection pairing (shown below) gives a duality between curves and divisors.

Definition 1.1.3. *The Néron-Severi space of divisors, $N^1(X)_{\mathbb{R}}$ is the vector space of numerical equivalence classes of \mathbb{R} -divisors. Dually, $N_1(X)_{\mathbb{R}}$ denotes the vector space of curves up to numerical equivalence.*

The Néron-Severi spaces are finite-dimensional real vector spaces. The dimension $\dim_{\mathbb{R}}(N^1(X)_{\mathbb{R}}) =: \rho(X)$ of $N^1(X)_{\mathbb{R}}$ is called the Picard number of X . The vector spaces $N^1(X)_{\mathbb{R}}$ and $N_1(X)_{\mathbb{R}}$ are dual under the intersection pairing:

$$N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \rightarrow \mathbb{R}, \quad (\Delta, \gamma) \mapsto \Delta \cdot \gamma \in \mathbb{R}.$$

The vector spaces $N^1(X)$ and $N_1(X)$ contain several natural cones that control the birational geometry of X . (We'll drop the subscript, as we always consider them as real vector spaces.)

Definition 1.1.4. *A line bundle L on X is called very ample if there exists a closed embedding $X \subset \mathbf{P}$ of X into some projective space $\mathbf{P} = \mathbb{P}^N$ such that $L = \mathcal{O}_{\mathbb{P}^N}(1)|_X$. A line bundle L is called ample if a positive multiple of L is very ample. A divisor D on X is ample if the line bundle associated to it is ample.*

The *Nakai-Moishezon criterion* says that a divisor D on a projective variety is ample if and only if $D^{\dim(V)} \cdot V > 0$ for every irreducible, positive dimensional subvariety V of X . In particular, being ample is a numerical property. The tensor product of two ample line bundles is again ample. Moreover, the tensor product of any line bundle

with a sufficiently high multiple of an ample line bundle is ample. Consequently, the classes of ample divisors form an open convex cone called the *ample cone*, $\text{Ample}(X)$, in the Néron-Severi space.

Definition 1.1.5. *A divisor D is called nef if $D \cdot C \geq 0$ for every irreducible curve $C \subset X$.*

The property of being nef is a numerical property. Since the sum of two nef divisors is nef, the set of nef divisors on X forms a closed convex cone in $N^1(X)$ called the *nef cone* of X . The nef cone, $\text{Nef}(X)$, contains the ample cone. In fact, a theorem due to Kleiman characterizes the ample cone as the interior of the nef cone and the nef cone as the closure of the ample cone.

Definition 1.1.6. *A line bundle L on X is called big if its Iitaka dimension¹ is equal to the dimension of X . A divisor D is big if the associated line bundle is so.*

A smooth, projective variety X is called of *general type* if and only if its *canonical divisor* K_X is big. A singular variety is called of general type if a desingularization is of general type.

A characterization of big divisors, that is often useful, is as those divisors that are numerically equivalent to the sum of an ample and an effective divisor [27]. From the previous definitions, it is clear that the property of being big is a numerical property as well. Since the sum of two big divisors is again big, the set of big divisors forms an open, convex cone called the *big cone* in the Néron-Severi space. The closure of the big cone consists of all divisor classes that are limits of divisor classes that are effective. This closed convex cone is called the *pseudo-effective cone*, $\overline{NE}^1(X)$ (sometimes also denoted by $\overline{\text{Eff}}(X)$).

Definition 1.1.7. *The closed cone of curves or Mori cone, $\overline{NE}_1(X)$ in $N_1(X)$ is the closure of the cone of classes that can be represented by non-negative linear combinations of classes of effective curves.*

¹Refer to [27] for the definition of Iitaka dimension.

Under the intersection pairing, the closed cone of curves is dual to the nef cone. For surfaces, the nef cone and the pseudo-effective cone are dual to each other under the intersection pairing.

Definition 1.1.8. *An irreducible curve C on a projective variety X is called a moving or moving curve if C is a member of an algebraic family of irreducible curves which covers a dense subset of X .*

The cone of moving curves of a projective variety X is defined as the closure of the convex cone in $N_1(X)$ which is spanned by numerical equivalence classes of moving curves. A moving curve has non-negative intersection with every irreducible divisor, a property that'll often be used in the proofs presented in Chapters 2 and 3.

Theorem 1.1.9 ([28]). *The cone of moving curves is dual to the pseudo-effective cone.*

A K_X -negative extremal ray of the Mori cone, which has a fibration as corresponding extremal contraction, is spanned by the class of a rational moving curve.

1.1.2 The moduli space of curves

In this section, we recall some of the basic facts about the Deligne-Mumford moduli space of stable curves. Fix non-negative integers g and n such that $2g - 2 + n > 0$.

Definition 1.1.10. *An n -pointed, genus g stable curve (C, p_1, \dots, p_n) is a reduced, connected, projective, nodal curve C of arithmetic genus g together with n distinct, ordered, smooth points $p_i \in C$ such that the canonical bundle, $\omega_C(\sum_{i=1}^n p_i)$, is ample.*

The stability condition mentioned above is equivalent to requiring that on the normalization of C , every rational component has at least three distinguished points².

Following standard notation, we denote by $\overline{M}_{g,n}$ the moduli space of stable n -pointed genus g curves, and by \overline{M}_g the moduli space of stable genus g curves with

²A distinguished point of the normalization of C is any point that lies over a marked point p_i or a node of C .

no marked points. More generally, if P is a set with n elements, we denote by $\overline{M}_{g,P}$ the moduli space of stable genus g curves whose marked points are indexed by P .

Let S be a scheme of finite type over a field. The moduli functor

$$\overline{\mathcal{M}}_{g,n} : \{\text{Schemes}/S\} \rightarrow \{\text{Sets}\}$$

associates to an S -scheme of finite type X , the set of isomorphism classes of families $f : Y \rightarrow X$ flat over X with n sections $s_1, \dots, s_n : X \rightarrow Y$ such that for every closed point $x \in X$, $(f^{-1}(x), s_1(x), \dots, s_n(x))$ is an n -pointed genus g stable curve.

Theorem 1.1.11. (*Deligne, Mumford, Knudsen*) *The functor $\overline{\mathcal{M}}_{g,n}$ is coarsely represented by an irreducible, normal, \mathbb{Q} -factorial projective variety $\overline{M}_{g,n}$ of dimension $3g - 3 + n$ with only quotient singularities.*

The locus of stable curves that have a node has codimension one in \overline{M}_g . This locus has $\lfloor g/2 \rfloor + 1$ irreducible components, each of codimension one. These components are called boundary divisors, since their union constitutes the boundary of \overline{M}_g , that is, $\overline{M}_g - M_g$. The locus of curves that have a non-separating node (i.e., a node p such that $C - p$ is connected) forms an irreducible component denoted by Δ_{irr} . The locus of curves that have a separating node p such that $C - p$ has two components one of genus i and one of genus $g - i$, where $1 \leq i \leq \lfloor g/2 \rfloor$, also forms an irreducible component denoted by Δ_i .

There is a stratification of $\overline{M}_{g,n}$, called the *topological stratification*, where the strata are indexed by the dual graphs. To each graph of a stable curve, we associate the subset of curves, a stratum in $\overline{M}_{g,n}$ with that dual graph.

Definition 1.1.12. *The dual graph of a stable curve C is a decorated graph such that*

1. *The vertices are in one-to-one correspondence with the irreducible components of C . Each vertex is marked by a non-negative integer equal to the geometric genus of the corresponding component.*
2. *For every node of C there is an edge connecting the corresponding vertices.*

3. For every marked point p_i , there is a half-edge emanating from the vertex corresponding to the component containing p_i .

There are only finitely many graphs that can occur as the dual graphs corresponding to n -pointed genus g stable curves. The codimension of a stratum is the number of nodes of a curve contained in the stratum (equivalently, the number of edges in the dual graph). In particular, the strata consisting of curves with $3g - 4 + n$ nodes form curves in $\overline{M}_{g,n}$ called F -curves (in honor of Faber and Fulton). Every ample divisor has positive degree on each F -curve.

We next describe some natural maps between these moduli spaces of curves.

Definition 1.1.13 (Forgetful morphism). *Given any n -pointed genus g curve (where $(g, n) \neq (0, 3), (1, 1), n > 0$), we can forget the n^{th} point, to obtain an $(n - 1)$ -pointed nodal curve of genus g : $\overline{M}_{g,n} \rightarrow \overline{M}_{g,n-1}$*

The curve in the image may not be stable, but it can be ‘stabilized’ by contracting all components that are 2-pointed genus 0 curves. The universal curve³ over $\overline{M}_{g,n}$ is given by the forgetful map: $\overline{M}_{g,n+1} \rightarrow \overline{M}_{g,n}$.

Definition 1.1.14 (Gluing morphism). *The following two maps are usually called gluing maps. Given an $(n_1 + 1)$ -pointed curve of genus g_1 , and an $(n_2 + 1)$ -pointed curve of genus g_2 , we can glue the first curve to the second along the last point of each, resulting in an $(n_1 + n_2)$ -pointed curve of genus $g_1 + g_2$. This gives a map*

$$\overline{M}_{g_1, n_1+1} \times \overline{M}_{g_2, n_2+1} \rightarrow \overline{M}_{g_1+g_2, n_1+n_2}.$$

Similarly, we could take a single $(n + 2)$ -pointed curve of genus g , and glue its last two points together to get an n -pointed curve of genus $g + 1$. This gives a map

$$\overline{M}_{g, n+2} \rightarrow \overline{M}_{g+1, n}.$$

³Universal family of n -pointed genus g curves.

1.1.3 Basic divisor classes of $\overline{M}_{g,n}$

Let's begin with the moduli space M_g of ordinary smooth curves which comes equipped with one obvious line bundle. The universal curve $\pi : C_g \rightarrow M_g$ comes equipped with a relative dualizing sheaf ω_{C_g/M_g} which we can naively think of as the bundle whose restriction to each fiber C is the canonical bundle. Taking the direct image of this bundle gives a bundle of rank g on M_g whose fiber over $[C]$ is just $H^0(C, K_C)$.

Definition 1.1.15. *We call this bundle the Hodge bundle and denote it by Λ . We set $\lambda_i = c_i(\Lambda)$, where $1 \leq i \leq g$, and call the divisor class $\lambda = \lambda_1$ the Hodge class.⁴*

Also there is a second way to use ω to produce classes on M_g . Instead of first pushing down to M_g and then taking the Chern class, we can reverse the order of these operations. Define $\gamma = c_1(\omega)$ which is a divisor class on C_g and then set $\kappa_1 = \pi_*\gamma^2$ (the squaring produces a class in codimension 2 on C_g which then pushes down to one of codimension 1 in M_g).

Definition 1.1.16. *The Mumford-Morita-Miller κ -class is defined as $\kappa = \kappa_1$.*

More generally, we can define the κ -classes as $\kappa_i = \pi_*\gamma^{i+1}$ for $i \geq 0$, where κ_i has codimension i .

Last but not the least, we have boundary divisors as defined in the previous subsection. We will adopt the usual convention that the divisor class defined by a boundary divisor Δ (possibly with decoration) is denoted by δ (similarly decorated).

Theorem 1.1.17 ([16]). *The Picard group $\text{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ is generated by λ and the classes of the boundary divisors, $\delta_{irr}, \delta_1, \dots, \delta_{\lfloor g/2 \rfloor}$.*

Let the total boundary class be denoted by $\delta = \delta_{irr} + \delta_1 + \dots + \delta_{\lfloor g/2 \rfloor}$.

Theorem 1.1.18. *The canonical divisor class of the coarse moduli scheme \overline{M}_g is given by*

$$[K_{\overline{M}_g}] = 13\lambda - 2\delta - \delta_1.$$

⁴ $c_i(V)$ denotes the i^{th} Chern class of the vector bundle V .

Theorem 1.1.19. *The divisor class λ is big and nef.*

Note that λ itself is not ample, but since it is big it is the sum of an ample and an effective divisor. Consequently, to show that the canonical bundle $K_{\overline{M}_g}$ is big, it suffices to express it as a sum of λ and an effective divisor. This is one approach to show that the projective variety \overline{M}_g is of general type.

Now, let's switch our attention to divisor classes on $\overline{M}_{g,n}$. Let's consider the forgetful morphism $\pi : \overline{M}_{g,n+1} \rightarrow \overline{M}_{g,n}$, also called the universal curve over $\overline{M}_{g,n}$. The forgetful map π is endowed with canonical sections $\sigma_i : \overline{M}_{g,n} \rightarrow \overline{M}_{g,n+1}$ indexed by n . The section σ_i attaches to any n -pointed curve (C, p_1, \dots, p_n) a copy of \mathbb{P}^1 by identifying p_i and $0 \in \mathbb{P}^1$, and labeling the points 1 and ∞ by p_i and p_{n+1} . Let Σ_i , the image of σ_i , be the corresponding divisor in $\overline{M}_{g,n+1}$. The relative dualizing sheaf ω_π yields the following definition.

Definition 1.1.20. *The Hodge class and κ -class of $\overline{M}_{g,n}$ are obtained by setting*

$$\lambda = c_1(\pi_*\omega_\pi) \quad \text{and} \quad \kappa = \pi_*(c_1(\omega_\pi(\sum_{i \leq n} \Sigma_i))^2).$$

A new set of classes, the tautological classes, ψ_i for $1 \leq i \leq n$, is defined as

$$\psi_i = \sigma_i^*(c_1(\omega_\pi)).$$

Next, we define the boundary divisors on $\overline{M}_{g,n}$, along the same lines as that on \overline{M}_g . Before proceeding a couple of remarks are in order. First, the locus Δ_{irr} is empty if $g = 0$. Second, when $n > 0$, the Δ_i can be further decomposed. A deformation which preserves the node must also preserve the partition of N , where $N = \{1, \dots, n\}$, into two subsets corresponding to which side of the node each marked point lies on. For $0 \leq i \leq g$ and $P \subset N$, we denote by $\Delta_{i,P}$ the locus of curves C with a node which divides C into a component of genus i containing the points indexed by P and a component of genus $g - i$ containing the points indexed by $N \setminus P$. Of course, $\Delta_{i,P} = \Delta_{g-i, N \setminus P}$. Often, we think of ψ_i as an anti-boundary and define $\delta_{0,\{i\}} = -\psi_i$.

Theorem 1.1.21 ([1]). $\text{Pic}(\overline{M}_{g,n})$ is generated by the basic classes λ , κ , δ_{irr} ⁵, $\delta_{i,P}$ and ψ_i for all g and n . For $g \geq 3$, the only relations on these classes are⁶:

1. $\kappa = 12\lambda + \psi - \delta$, where $\psi = \sum_{i \leq n} \psi_i$, and $\delta = \delta_{irr} + \sum_{i \geq 0, P} \delta_{i,P}$.

2. The symmetries $\delta_{i,P} = \delta_{g-i, N \setminus P}$.

Theorem 1.1.22 ([29]). The canonical divisor class of $\overline{M}_{g,n}$ is given by,

$$[K_{\overline{M}_{g,n}}] = 13\lambda + \psi - 2\delta - \sum_P \delta_{1,P}.$$

In genus 0, the locus δ_{irr} is empty (a curve with a non-disconnecting node has positive genus) as is the class λ (in fact the bundle Λ is zero). In $\overline{M}_{0,4}$, the three points of the boundary (which correspond to $\delta_{0,\{1,2\}} = \delta_{0,\{3,4\}}$, $\delta_{0,\{1,3\}} = \delta_{0,\{2,4\}}$, $\delta_{0,\{1,4\}} = \delta_{0,\{2,3\}}$) are all linearly equivalent. Pulling these back to $\overline{M}_{0,n}$, we get the *four point relations*: for any subset $Q = \{i, j, k, l\}$ of $\{1, \dots, n\}$ of order 4, the class $\delta_Q := \sum_{i,j \in P, k,l \notin P} \delta_{0,P}$ depends, as the notation suggests, only on Q and not on the choice of the pair of elements i and j . Keel [23] proves that all relations in genus 0 are consequences of these.

For future reference, it's worth noting a few relations expressing the classes κ and ψ_i in terms of boundary classes. The basic case is $\overline{M}_{0,3}$ (a point) where all these classes are 0. Pulling back these relations from $\overline{M}_{0,3}$ to $\overline{M}_{0,n}$ gives the relations:

- $\kappa = \sum_{q,r \notin P} \delta_{0,P}$ for any q, r in $\{1, \dots, n\}$
- $\kappa = \sum_P \left(\frac{|P|(n-|P|)}{n-1} - 1 \right) \delta_{0,P}$
- $\psi_i = \sum_{s \in P, q,r \notin P} \delta_{0,P}$ for any q, r in $\{1, \dots, n\}$ distinct from s .

⁵Switching notation from divisor Δ to its divisor class δ .

⁶In smaller genera, there are some extra relations.

1.2 The Big Picture

For a complex projective variety X , one way of understanding its birational geometry is by describing its cones of ample ($\text{Ample}(X)$) and effective divisors ($NE^1(X)$)

$$\text{Ample}(X) \subset NE^1(X) \subset N^1(X).$$

The closure in $N^1(X)$ of $\text{Ample}(X)$ is the cone $\text{Nef}(X)$ of numerically effective divisors. The interior of the closure $\overline{NE^1}(X)$ is the cone of big divisors on X . Loosely speaking, one can think of the nef cone as parameterizing regular contractions⁷ from X to other projective varieties, whereas the effective cone accounts for rational contractions of X . For arbitrary varieties of dimension ≥ 3 there is little connection between $\text{Nef}(X)$ and $NE^1(X)$. For surfaces though, there is the Zariski decomposition which provides a unique way of writing an effective divisor as a combination of a nef and a negative part [27], and this relates the two cones. Most questions in higher dimensional geometry can be phrased in terms of the ample and effective cones. For instance, a smooth projective variety X is of general type precisely when $K_X \in \text{int}(\overline{NE^1}(X))$, that is, K_X is big.

We often wish to explore the relationship between the canonical class K_X of a smooth projective variety X and rational curves on X . The Cone and Contraction theorems [26] are our main tools in understanding this relationship. The importance of the extremal rays⁸ in the K_X -negative part of the cone of curves is that they can be contracted. If R is an extremal ray of the cone of curves satisfying $K_X \cdot R < 0$, then there exists a morphism $\text{cont}_R : X \rightarrow Y$ such that any curve whose class lies in the ray R is contracted. Furthermore, the class of any curve contracted by cont_R lies in the ray R . A variant of this idea is used in the proof of the MF-conjecture. The Contraction Theorem provides a very important way of constructing new birational models of X . Unfortunately, we do not understand the K_X -positive part of the cone of curves. Even the K_X -negative part of the cone can be very complicated.

⁷morphisms with geometrically connected fibers to projective varieties.

⁸An extremal ray of a cone is a ray that cannot be expressed as a conic (non-negative) combination of any ray directions in the cone distinct from it.

The question of describing the ample and the effective cone of \overline{M}_g has a long and rich history. Moduli spaces of curves with their inductive structure given by the boundary stratification are good test cases for many problems coming from higher dimensional birational geometry. We briefly review how the landscape looks like as of today. These are very deep results, and we make no attempt here to present an insight into these. Historical references can be found in [11].

- For genus, $g \leq 14$, \overline{M}_g is unirational, cf. [34].
- For $g = 15$, \overline{M}_g is rationally connected, and its Kodaira dimension, $\kappa(\overline{M}_{15}) = -\infty$, cf. [4].
- For $g = 16$, \overline{M}_g is uniruled, and $\kappa(\overline{M}_{16}) = -\infty$, cf. [5].
- For $17 \leq g \leq 21$, its an open question whether $\kappa(\overline{M}_g) = -\infty$.
- For $g = 22$, \overline{M}_g is of general type, cf. [11].
- For $g = 23$, only partial results are known, $\kappa(\overline{M}_{23}) \geq 2$, cf. [8].
- For $g \geq 24$, \overline{M}_g is of general type, cf. [17].

Chapter 2

Effective, Mori and nef cones

In this chapter, we deal in depth with various cones of interest. The first section is dedicated to the effective cone of $\overline{M}_{g,n}$. The next section presents the main theorems and some state of art results regarding the Mori and nef cones of $\overline{M}_{g,n}$, leading to the grand finale: reducing the F-conjecture to the more tractable MF-conjecture. We now state the F-conjecture in its various equivalent forms. For sake of brevity, and as per contemporary notation, we will denote the F-conjecture on the moduli space $\overline{M}_{g,n}$ by $F_1(\overline{M}_{g,n})$.

F-Conjecture 2.0.1 ($F_1(\overline{M}_{g,n})$). *A divisor on $\overline{M}_{g,n}$ is ample (nef) if and only if it has positive (nonnegative) intersection with all 1-dimensional strata or the F-curves.*

Put differently, any effective curve in $\overline{M}_{g,n}$ is numerically equivalent to an effective combination of 1-strata. In other words, every extremal ray of the Mori cone of effective curves $\overline{NE}_1(\overline{M}_{g,n})$ is generated by a one dimensional stratum.

2.1 Effective cone

In this section, we'll first study the effective cone of $\overline{M}_{0,n}/S_n$. The moduli space of n -pointed genus g curves admits a natural action of the symmetric group S_n , where the symmetric group acts by permuting the marked points. The Neron-Severi space of $\overline{M}_{0,n}/S_n$ is generated by the classes of the boundary divisors $\Delta_2, \Delta_3, \dots, \Delta_{\lfloor n/2 \rfloor}$.

Theorem 2.1.1. (Keel-McKernan, [22]). *The effective cone of $\overline{M}_{0,n}/S_n$ is the cone spanned by the classes of the boundary divisors Δ_i for $2 \leq i \leq \lfloor n/2 \rfloor$.*

Proof. Let D be an effective prime divisor different from a boundary divisor. We would like to show that the class of D is a non-negative linear combination of boundary divisors. Write $D \equiv_{num} \sum_{i=2}^{\lfloor n/2 \rfloor} a_i \Delta_i$. We show that $a_i \geq 0$ by induction on i .

Let C_2 be the curve obtained in $\overline{M}_{0,n}/S_n$ by fixing $n-1$ points on \mathbb{P}^1 and varying the n^{th} point on \mathbb{P}^1 . $\Delta_2 \cdot C_2 = n-1$ (there is one intersection each time the n^{th} point crosses one of the other fixed $n-1$ points) and $\Delta_i \cdot C_2 = 0$ for $i > 2$ (C_2 is disjoint from the other Δ_i 's). Moreover, C_2 is a moving curve. Since the closed cone of moving curves is dual to the pseudo-effective cone, we conclude that $a_2 \geq 0$.

Now suppose $a_j \geq 0$ for $2 \leq j < i \leq \lfloor n/2 \rfloor$. Fix a \mathbb{P}^1 with i distinct fixed points p_1, \dots, p_{i-1} and q_1 (call this curve C'_i). Fix another \mathbb{P}^1 with $n-i+1$ fixed points p_i, \dots, p_n and one variable point q_2 (call this curve C''_i). Glue the two \mathbb{P}^1 's along q_1 and q_2 . Let C_i be the curve in $\overline{M}_{0,n}/S_n$ obtained by letting q_2 vary. Then $\Delta_i \cdot C_i = n-i+1$ (one intersection each time q_2 crosses one of the $n-i+1$ fixed points p_i, \dots, p_n). Curves with class C_i cover the boundary divisor Δ_{i-1} . Note that C_i lies in Δ_{i-1} . Consider the family over C_i as $C'_i \times C''_i$. The section corresponding to q_1 has self-intersection 0 (since q_1 is fixed on C'_i). Now consider the family $C''_i \times C''_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ blown up at the points where the constant sections corresponding to the p_j , where $i \leq j \leq n$, meet the diagonal section corresponding to q_2 and hence the proper transform of that section has self-intersection $2 - (n-i+1)$. Hence, $\Delta_{i-1} \cdot C_i = 2 - (n-i+1) = -n+i+1 < 0$, and $\Delta_j \cdot C_i = 0$ for $j \neq i-1, i$ (C_i is disjoint from the other Δ_j 's). Since D is an irreducible divisor different from the boundary divisors, we conclude that a general curve with class C_i cannot be contained in D . Hence, $D \cdot C_i \geq 0$. It follows that $a_i \geq 0$ concluding the induction step. \square

In contrast to $\overline{M}_{0,n}/S_n$, the effective cone of $\overline{M}_{0,n}$ seems to be very complicated. Already for $\overline{M}_{0,6}$ the boundary divisors do not generate the effective cone. It is known that the effective cone of $\overline{M}_{0,5}$ is the cone spanned by the boundary divisors. There are several ways of generating effective divisors on $\overline{M}_{0,n}$. First, there are natural

gluing maps

$$\theta : \overline{M}_{0,2n} \rightarrow \overline{M}_n$$

obtained by gluing the points marked p_{2i-1}, p_{2i} to obtain an n -nodal genus n curve. One can pull-back effective divisors that do not contain the image of θ to obtain effective divisors on $\overline{M}_{0,2n}$. There are several other such gluing maps that one may consider. For example, one may attach a fixed one-pointed elliptic curve at each of the marked points to obtain a map

$$\theta' : \overline{M}_{0,n} \rightarrow \overline{M}_n.$$

Pulling back effective divisors not containing the image of θ' produces effective divisors on $\overline{M}_{0,n}$. Next, given an effective divisor in $\overline{M}_{0,n-k}$, one may pull-back this divisor via the forgetful maps

$$\pi_{i_1, \dots, i_k} : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n-k}$$

to obtain effective divisors in $\overline{M}_{0,n}$. More interestingly, by appropriately choosing the forgetful morphisms, one may construct birational morphisms from $\overline{M}_{0,n}$ to a product of \overline{M}_{0,n_i} 's. Again by choosing the numerics carefully, one sometimes obtains divisorial contractions [3]. The exceptional divisor in that case is an extremal ray of the effective cone.

As already mentioned, the effective cone of $\overline{M}_{0,6}$ is not generated by the boundary divisors. In fact, Keel and Vermeire [33] constructed an effective divisor that is not in the non-negative span of the boundary. The idea is to look at the locus of curves that are invariant under the element $(i_1, i_2)(i_3, i_4)(i_5, i_6) \in S_6$. Since there are 15 such triplets of pairs, there are 15 such divisors. An alternative way to think about it would be to consider the gluing map; $\theta : \overline{M}_{0,6} \rightarrow \overline{M}_3$ and considering the divisor obtained by taking the closure of the preimage under θ of the locus of hyperelliptic curves. Again there are $\binom{6}{2}$ different possible identifications of pairs of points giving 15 such different divisor classes. Hassett and Tschinkel [21] later proved that the effective cone of $\overline{M}_{0,6}$ is generated by the boundary divisors and the 15 Keel-Vermeire divisors

whose constructions were sketched above. We do not yet know the effective cone of $\overline{M}_{0,n}$ for $n > 6$. It is not even known whether the effective cone of $\overline{M}_{0,n}$ has finitely many extremal rays.

Definition 2.1.2. *Suppose that the group of line bundles $\text{Pic}(X)$ is a finitely generated abelian group. We may pick a set of divisors D_1, \dots, D_k so that the line bundles $\mathcal{O}_X(D_1), \dots, \mathcal{O}_X(D_k)$ generate $\text{Pic}(X)$. The Cox ring of X , denoted by $\text{Cox}(X)$ is given by $R(X, D_1, \dots, D_k) = \bigoplus_{m \in \mathbb{Z}^k} H^0(X, \mathcal{O}_X(D))$ where $D = \sum m_i D_i$.*

A Mori dream space is a \mathbb{Q} -factorial¹, projective variety X with $\text{Pic}(X) \otimes \mathbb{R} = N^1(X)$ and whose Cox ring is finitely generated. They satisfy many other nice properties. For example, on a Mori dream space, one can run Mori's program for every divisor. The nef cone of a Mori dream space is generated by finitely many semi-ample divisors². The effective cone of a Mori dream space is polyhedral. One of the big open problems is to prove that the cones of ample and effective divisors are polyhedral cones, which is implied by the following conjecture.

Conjecture 2.1.3. *$\overline{M}_{0,n}$ is a Mori dream space for all n .*

Compared to Mori and nef cones, our knowledge of the effective cone of $\overline{M}_{0,n}$ is even more limited. Each time one constructs an effective divisor ([13], [9] and others) one determines part of the effective cone. Approaching from the other way around, one can bound the cone of effective divisors by constructing moving curves. Each time one constructs a moving curve ([19], [6], [30] and others), the effective cone has to lie on one side of the hyperplane in $N^1(\overline{M}_{0,n})$ determined by that moving curve (using the duality of the cone of moving curves and the effective cone). One thus obtains a cone containing the effective cone. The former approach approximates the effective cone from the inside, and the later from the outside.

Theorem 2.1.4. *The effective cone of \overline{M}_2 is generated by the boundary divisors δ_{irr} and δ_1 .*

¹A variety is called \mathbb{Q} -factorial if every \mathbb{Q} -divisor is \mathbb{Q} -Cartier.

²A divisor D is semi-ample if the corresponding line bundle is so, and a line bundle L is semi-ample if $L^{\otimes m}$ is globally generated for some $m > 0$. In other words a divisor such that the linear system of some positive multiple is base-point free.

Proof. Since in genus 2, the divisors δ_{irr} , δ_1 and λ satisfy the linear relation

$$10\lambda = \delta_{irr} + 2\delta_1,$$

the Néron-Severi space has dimension two. We need to determine the two rays bounding the effective cone. Write the class of an effective divisor D as $a\delta_{irr} + b\delta_1$. We would like to show that $a, b \geq 0$. We can assume that D is an irreducible divisor that does not contain any of the boundary divisors. Take a general pencil of $(2, 3)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$. This pencil induces a moving curve C in \overline{M}_2 . Since none of the curves in this pencil is reducible and 20 members of the family are singular, we conclude $D \cdot C = 20a \geq 0$. Hence, $a \geq 0$. Let B be the curve in \overline{M}_2 obtained by taking a fixed elliptic curve E with a fixed point $p \in E$ and identifying a variable point $q \in E$ with p to form a genus two nodal curve. Note that B is a moving curve in the boundary divisor Δ_{irr} . (Refer to [18] for the self-intersection calculations.) Since

$$\delta_{irr} \cdot B = -2, \quad \delta_1 \cdot B = 1$$

we conclude that $-2a + b \geq 0$. Hence $b \geq 2a \geq 0$. □

It can be shown that D is ample if and only if $12a > b > 2a > 0$. Hence we conclude that the effective cone properly contains the ample cone which is bounded by the rays $\delta_{irr} + 2\delta_1$ and $12\delta_{irr} + \delta_1$.

Theorem 2.1.5. [31] *The effective cone of \overline{M}_3 is generated by the classes of the divisor of hyperelliptic curves D_{hyp} and the boundary divisors δ_{irr} and δ_1 .*

Proof. The class of D_{hyp} is given by $[D_{hyp}] = 18\lambda - 2\delta_{irr} - 6\delta_1$. Let D be a prime divisor that is not a boundary divisor or D_{hyp} . As usual, express the class of the divisor D as $a[D_{hyp}] + b_0\delta_{irr} + b_1\delta_1$. Take a general pencil of quartic curves in \mathbb{P}^2 . This pencil induces a moving curve C_1 in the moduli space which is disjoint from Δ_1 and D_{hyp} , and has intersection number $\delta_{irr} \cdot C_1 = 27$ (note also that $\lambda \cdot C_1 = 3$). It follows that $b_0 \geq 0$. Fix a genus 2 curve A and a pointed genus one curve (E, p) . Let

C_2 be the curve in moduli space induced by attaching (E, p) to A at a variable point $q \in A$. We have the intersection numbers

$$\lambda \cdot C_2 = 0, \quad \delta_{irr} \cdot C_2 = 0, \quad D_{hyp} \cdot C_2 = 12, \quad \delta_1 \cdot C_2 = -2.$$

Since the class of C_2 is a moving curve class in Δ_1 , we conclude that $12a - 2b_1 \geq 0$. Next fix a genus 2 curve A and a point $p \in A$. Let C_3 be the curve induced in the moduli space by the one-parameter family of nodal genus 3 curves obtained by gluing p to a variable point $q \in A$. The intersection numbers of C_3 are

$$\lambda \cdot C_3 = 0, \quad \delta_{irr} \cdot C_3 = -4, \quad D_{hyp} \cdot C_3 = 2, \quad \delta_1 \cdot C_3 = 1.$$

Since C_3 is a moving curve class in Δ_{irr} , we have that $2a - 4b_0 + b_1 \geq 0$. Rewriting, we get $2a + b_1 \geq 4b_0 \geq 0$. Since $a \geq \frac{1}{6}b_1$, we conclude that a has to be non-negative. Finally, to see that b_1 is non-negative, restrict the class of D to D_{hyp} . D_{hyp} is ample in the Satake compactification of \overline{M}_g . Hence, D_{hyp} intersects D in an effective divisor. So we conclude that $b_1 \geq 0$. □

2.2 Mori and nef cones

This section reviews what is known and what is conjectured about the cone of nef divisors on $\overline{M}_{g,n}$ and its dual, the Mori cone. As we saw in section 1.2, it is quite hard to compute these cones for a general projective variety. But it seems possible to do so for the moduli space of curves because of the following developments. The first is a conjectural geometric description of the extremal rays of the Mori cone, the F-Conjecture 2.0.1. The second is that the form of the F-conjecture allows us to make the *inductive structure* of the set of all these spaces, as expressed in the forgetful and gluing maps, a powerful tool. Essentially, the general case on $\overline{M}_{g,n}$ can be reduced to statements in genus 0. Unfortunately, settling the conjecture on $\overline{M}_{0,n}$ seems quite hard as well.

The F-Conjecture is motivated by a question originally asked by Fulton only in

the case $g = 0$ and whose analogue for $n = 0$ was later considered by Faber, whence the name. For our convenience, let's make a few working definitions.

Definition 2.2.1. *The Faber cone of curves is the subcone of the Mori cone spanned by the curve strata (F-curves). The Faber cone of divisors is the dual of the Faber cone of curves. A Faber curve or Faber divisor is one that lies in the corresponding cone.*

We will make use of the standard product decomposition for strata as a finite image of products of various \overline{M}_{g_i, n_i} 's [24]. We describe this decomposition by the following finite, proper, surjective gluing maps, where the first one acts by attaching curves coming from the two factors along p and q respectively (N denotes the set $\{1, \dots, n\}$ and $S \subset N$):

$$\tilde{\Delta}_{i,S} := \overline{M}_{i, S \cup p} \times \overline{M}_{g-i, (N \setminus S) \cup q} \rightarrow \Delta_{i,S} \subset \overline{M}_{g,n}$$

and the second identifies points p and q of a curve

$$\tilde{\Delta}_{irr} := \overline{M}_{g-1, N \cup p \cup q} \rightarrow \Delta_{irr} \subset \overline{M}_{g,n}.$$

Lemma 2.2.2. [15] *The pullback to $\tilde{\Delta}_{i,S}$ of any line bundle is numerically equivalent to a tensor product of unique line bundles from the two factors. The given line bundle is nef on $\Delta_{i,S}$ iff each of the line bundles on the factors is nef. Dually, let C be any curve on the product, and C_i, C_{g-i} be its images on the two factors (with multiplicity for the pushforward of cycles) which we also view as curves in $\overline{M}_{g,n}$ by the usual device of gluing on a fixed curve. Then, C and $C_i + C_{g-i}$ are numerically equivalent.*

Also, it follows that every curve in $\overline{M}_{g,n}$ is numerically equivalent to an effective combination of curves whose moving subcurves are all generically irreducible.

Surprisingly enough, it is quite easy to describe the curve strata of $\overline{M}_{g,n}$, up to numerical equivalence, and to compute the degrees of the standard divisor classes on them. The following gives a listing of numerical possibilities for 1-dimensional strata, giving explicit representatives for each numerical equivalence class. The parts

in 2.2.4 and 2.2.3 correspond, that is, for each family X listed in parts (2-6) of 2.2.3, the inequality that expresses the condition that a divisor D , as given in 2.2.4, meets X non-negatively is given in the corresponding part of 2.2.4.

Theorem 2.2.3. *Let $X \subset \overline{M}_{g,n}$ be a 1-dimensional stratum. Then X is either*

1. *For $g \geq 1$, a family of elliptic tails, that is the image of the map $\overline{M}_{1,1} \rightarrow \overline{M}_{g,n}$, obtained by attaching a fixed $n + 1$ -pointed curve of genus $g - 1$ to the moving pointed elliptic curve. Any two families of elliptic tails are numerically equivalent. Let's call such a family E . Except for E , all the following curve strata are numerically equivalent to families of rational curves, the image of $\overline{M}_{0,4} \rightarrow \overline{M}_{g,n}$ defined by one of the attaching procedures below.*
2. *For $g \geq 3$, attach a fixed $n + 4$ pointed curve of genus $g - 3$.*
3. *For $g \geq 2$, $I \subset N$, $0 \leq i \leq g - 2$, $i + |I| > 0$, attach a fixed $I + 1$ -pointed curve of genus i and a fixed $I^c + 3$ -pointed curve of genus $g - 2 - i$.*
4. *For $g \geq 2$, $I \subset N$, $0 \leq i \leq g - 2$, attach a fixed $I + 2$ -pointed curve of genus i and a fixed $I^c + 2$ -pointed curve of genus $g - 2 - i$.*
5. *For $g \geq 1$, $i, j \geq 0$, $I, J \subset N$, $I \cap J = \emptyset$, $i + j \leq g - 1$ and $i + |I|, j + |J| > 0$, attach a fixed $I + 1$ -pointed curve of genus i , a fixed $J + 1$ -pointed curve of genus j and a fixed $(I + J)^c + 2$ -pointed curve of genus $g - 1 - i - j$.*
6. *For $g \geq 0$, $i, j, k, l \geq 0$, $i + j + k + l = g$, I, J, K, L a partition of N , and $i + |I|, j + |J|, k + |K|, l + |L| > 0$, attach $I + 1$, $J + 1$, $K + 1$ and $L + 1$ -pointed curves of genus i, j, k, l respectively.*

From the above description, we see that all 1-dimensional strata except for E are numerically equivalent to families of rational curves, meaning that all its irreducible components are rational. As we will soon see, strata defined by 2.2.3.6 play distinctly different roles, both geometrically and combinatorially, from those defined by 2.2.3.1-2.2.3.5. Every moving component of a curve strata of type (2.2.3.6) is smooth and rational.

The next result describes the Faber cone of divisors as an intersection of half spaces. Following the notation set earlier, $\delta_{0,\{i\}} = -\psi_i$.

Theorem 2.2.4. *Fix a divisor D on $\overline{M}_{g,n}$ and express its class as $a\lambda - b_{irr}\delta_{irr} - \sum_{\mathcal{I}_{g,n}} b_{i,I}\delta_{i,I}$, with the convention that $\delta_{irr} = 0$ if $g = 0$. Here $\mathcal{I}_{g,n} = \{(i, I) | 0 \leq i \leq \lfloor g/2 \rfloor, I \subset \{1, \dots, n\}, |I| \geq 1 \text{ for } i = 0\}$. $b_{i,I}$ is defined to be b_{g-i, I^c} for $i > \lfloor g/2 \rfloor$. Consider the inequalities*

1. $a - 12b_{irr} + b_{1,\emptyset} \geq 0$.
2. $b_{irr} \geq 0$.
3. $b_{i,I} \geq 0$ for $0 \leq i \leq g - 2$.
4. $2b_{irr} \geq b_{i+1,I}$ for $0 \leq i \leq g - 2$.
5. $b_{i,I} + b_{j,J} \geq b_{i+j, I \cup J}$ for $i, j \geq 0, i + j \leq g - 1, I \cap J = \emptyset$.
6. $b_{i,I} + b_{j,J} + b_{k,K} + b_{l,L} \geq b_{i+j, I \cup J} + b_{i+k, I \cup K} + b_{i+l, I \cup L}$ for $i, j, k, l \geq 0, i + j + k + l = g$, and I, J, K, L a partition of N .

Then D is a Faber divisor if and only if

- when $g \geq 3$, all of (1 - 6) hold.
- when $g = 2$, (1) and (3 - 6) hold.
- when $g = 1$, (1) and (5 - 6) hold.
- when $g = 0$, (6) holds.

The proof of the above theorem is evident from Theorem 2.2.3, and the duality of the Néron-Severi spaces of divisors and curves (as shown by the intersection pairing in the previous chapter).

We now give a brief description of the curve strata 2.2.3.1 - 2.2.3.6. As seen earlier, the closure of every stratum is the finite image of a product of spaces \overline{M}_{g_i, n_i} 's. Only $\overline{M}_{0,3}$ is 0-dimensional, and only $\overline{M}_{0,4}$ and $\overline{M}_{1,1}$ are 1-dimensional, so we must have

one factor of the latter type and several of the former. This still leaves an enormous number of combinatorial possibilities for the dual graph but we can effectively ignore these by viewing curve strata B as test curves in which a moving component (either $\overline{M}_{0,4}$ or $\overline{M}_{1,1}$) is attached to a fixed curve (the various $\overline{M}_{0,3}$'s). Applying Lemma 2.2.2, we see that the numerical equivalence class of B is unchanged if we replace the fixed curve by any smoothing of it at the set of nodes not on the moving component.

Thus, in the $\overline{M}_{1,1}$ -case where the moving component C has genus 1, we can assume that the fixed component is a smooth curve of genus $g - 1$ and we have n additional marked points on the fixed curve. Thus, $\overline{M}_{1,1}$ -case corresponds to a family of elliptic tails (see 2.2.3.1) and gives the first inequality 2.2.4.1. In the $\overline{M}_{0,4}$ -case, things are a bit more complicated. Now the numerical type of the stratum depends on the genus of each connected component and the marked points lying on each connected component of the fixed curve and the number of points at which each is attached to the moving component. The connected components of the fixed curve thus determine one of the 5 partitions of the 4 marked points. The partitions $\{4\}$, $\{3, 1\}$, $\{2, 2\}$, $\{2, 1, 1\}$ and $\{1, 1, 1, 1\}$ give rise to the 1-dimensional strata 2.2.3.2 - 2.2.3.6 respectively, and the corresponding inequalities in 2.2.4. For details, see [15]. As shown above it is straightforward to list all F-curves on a given $\overline{M}_{g,n}$. For instance, F-curves on $\overline{M}_{0,n}$ are in 1 : 1 correspondence with partitions (n_1, n_2, n_3, n_4) of n , the corresponding F-curve being the image of the gluing map which takes a rational 4-pointed curve (C, p_1, p_2, p_3, p_4) to a rational n -pointed curve obtained by attaching a fixed rational $(n_i + 1)$ -pointed curve at the point p_i .

Next, we review partial results in the direction of proving that Faber divisors are nef. We begin with an important technical result.

Theorem 2.2.5. *If $g \geq 2$ or $g = 1, n \geq 2$, a divisor D in $\text{Pic}(\overline{M}_{g,n})$ is nef if and only if its restriction to Δ , the boundary of $\overline{M}_{g,n}$, is nef.*

Using this, the F-conjecture for general g reduces to the genus 0-case (see section 3.1). $\overline{M}_{0,g+n}/S_g$ is the quotient of $\overline{M}_{0,g+n}$ by the action of S_g on the last g marked points. A point of $\overline{M}_{0,g+n}/S_g$ has n ordinary ordered marked points and g unordered

marked points. We get a map $f_{g,n} : \overline{M}_{0,g+n}/S_g \rightarrow \overline{F}_{g,n} \subset \overline{M}_{g,n}$, by attaching a fixed pointed curve of genus 1 at each of the g unordered points and the map $f_{g,n}$ is the normalization of the image $\overline{F}_{g,n}$. As per current usage, we call $\overline{F}_{g,n}$ the *flag locus* and call curves whose moduli points lie in it *flag curves*. The flag locus contains curve strata of type (2.2.3.6). They are the only strata in genus 0, and there is a face of $\overline{NE}_1(\overline{M}_{0,n})$ that contains exactly these strata. Next, we state the most important theorem which is the key to all the results presented here.

Theorem 2.2.6 ([15]). *A divisor D on $\overline{M}_{g,n}$ is nef iff D has non-negative intersection with all curve (1-dimensional) strata and its restriction to $\overline{F}_{g,n}$ is nef. Conversely, every nef line bundle on $\overline{M}_{0,g+n}/S_g$ is the pullback of a nef line bundle on $\overline{M}_{g,n}$. In particular, $F_1(\overline{M}_{0,g+n}/S_g)$ is equivalent to $F_1(\overline{M}_{g,n})$.*

Using the proof of Theorem 2.2.6, one can produce a nef divisor class D on $\overline{M}_{g,n}$ which has degree 0 on all curve strata of type (2.2.3.6) and has strictly positive degree on all other curve strata. In fact D is trivial on $\overline{F}_{g,n}$. This shows that the Mori cone of $\overline{F}_{g,n}$ is a face of the Mori cone of $\overline{M}_{g,n}$. Next, we state the following strengthening of Theorem 2.2.6.

Theorem 2.2.7 ([15]). *Let $g \geq 1$, and let $N \subset \overline{NE}_1(\overline{M}_{g,n})$ be the subcone generated by the curve strata of types (2.2.3.1 - 2.2.3.5) as defined in genus g . Then N is the subcone generated by curves $C \subset \overline{M}_{g,n}$ whose associated family of curves has no moving smooth rational components. Equivalently, $\overline{NE}_1(\overline{M}_{g,n}) = N + \overline{NE}_1(\overline{M}_{0,g+n}/S_g)$.*

Let us say, by slight abuse of notation, that a curve in $\overline{M}_{g,n}$ is rational if all the components of its normalization are rational. These form a locus $\overline{R}_{g,n} \subset \overline{M}_{g,n}$ which is the closure of the locus of irreducible g -nodal curves and is the image of the quotient of $\overline{M}_{0,2g+n}$ by the group $G \subset S_{2g}$ of permutations commuting with the product $(12)(34) \dots (2g-1, 2g)$ of g transpositions by the map $r_{g,n}$, which identifies the corresponding pairs of marked points (and again normalizes $\overline{R}_{g,n}$). In other words, the map $r_{g,n} : \overline{M}_{0,2g+n}/G \rightarrow \overline{R}_{g,n}$ is the normalization of the image $\overline{R}_{g,n}$. By degenerating all the fixed components, we can find representatives of all the curve strata (2.2.3.2-2.2.3.6) lying inside $\overline{R}_{g,n}$. Hence,

Corollary 2.2.8. *A divisor D on $\overline{M}_{g,n}$ is nef iff its restriction to $\overline{R}_{g,n} \cup E$ is nef, where E is as described in 2.2.3.1.*

The arguments above show that the Mori cone is generated by E together with curves in $\overline{R}_{g,n}$. Using the fact that the nef cone is dual to the Mori cone, the above corollary follows easily. However, there is no converse here, nor do the curve classes in $\overline{R}_{g,n}$ form a face of $\overline{NE}_1(\overline{M}_{g,n})$.

The reductions above make it natural to ask how one might attack $F_1(\overline{M}_{0,n})$. A natural question to ask would be:

Question 2.2.9. *Is every Faber divisor D on $\overline{M}_{0,n}$ an effective sum of boundary divisors?*

This would imply the F-conjecture for $\overline{M}_{0,n}$ and hence for every $\overline{M}_{g,n}$ by an induction on the inductive structure of the set of all spaces $\overline{M}_{g,n}$. Indeed, a positive answer would reduce the problem of showing that a Faber divisor D is nef on $\overline{M}_{0,n}$ to a simpler problem of showing that D is nef on every boundary component $\Delta_{0,S} = \Delta_{0,S^c}$. Each $\Delta_{0,S}$ is the finite image of a product of $\overline{M}_{0,i}$'s with $i < n$. Also, D would restrict to a Faber divisor D' on each factor of the above product which by induction on n would be nef. Since a given line bundle is nef on a boundary component iff each of the line bundles on the factors is nef (Lemma 2.2.2), D itself would be nef. Moreover, this question is purely combinatorial and can be restated as whether one explicit polyhedral cone is contained in another. The combinatorial formulation is as follows:

Question 2.2.10. *Let \mathbb{V} be the \mathbb{Q} -vector space that is spanned by symbols δ_S for each subset $S \subset \{1, \dots, n\}$ subject to the relations*

1. $\delta_S = \delta_{S^c}$ for all S .

2. $\delta_S = 0$ for $|S| \leq 1$.

3. for each 4 element subset $\{i, j, k, l\} \subset \{1, \dots, n\}$,
$$\sum_{i,j \in S; k,l \in S^c} \delta_S = \sum_{i,k \in S; j,l \in S^c} \delta_S.$$

(These relations come from the four point relations described in the first chapter.)

Let $N \subset \mathbb{V}$ be the set of elements $\sum b_S \delta_S$ satisfying $b_{I \cup J} + b_{I \cup K} + b_{I \cup L} \geq b_I + b_J + b_K + b_L$, for each partition of $\{1, \dots, n\}$ into 4 disjoint subsets I, J, K, L , and let $A \subset \mathbb{V}$ be the affine hull of δ_S . Then, is $N \subset A$?

2.3 MF-conjecture

In this section, we show that the F -Conjecture can be reduced to what has been termed the MF -Conjecture (informally called the Modified Fulton conjecture) which asserts that F -divisors on $\overline{M}_{0,n}$ are the sum of the canonical divisor and an effective divisor.

MF-Conjecture 2.3.1. *Every F -divisor on $\overline{M}_{0,n}$ is of the form $cK_{\overline{M}_{0,n}} + E$ where $c \geq 0$ and E is an effective sum of boundary classes.*

Reducing the F -conjecture to 2.3.1 provides a better numerical criterion and algorithm to verify whether a divisor is nef, as shown later in the section. This is because showing that a divisor class is in the convex hull of boundary classes is more difficult than showing it is in the convex hull of boundary classes and the canonical divisor. We say that the MF -conjecture behaves better numerically as it makes a considerably weaker combinatorial assertion than that posed in the Question 2.2.10, and yet we have the following wonderful theorem.

Theorem 2.3.2 ([14]). *If the MF -Conjecture is true on $\overline{M}_{0,N}$ for $N \leq g + n$, then the F -conjecture is true on $\overline{M}_{g,n}$. In particular, if the MF -conjecture is true then the F -conjecture is true.*

Two key ingredients are needed to explain how the MF -Conjecture implies the F -conjecture. Their proofs are presented in the next chapter.

The first is that if the F -conjecture is true on $\overline{M}_{0,g+n}$ then it is true on $\overline{M}_{g,n}$. More precisely, let $f : \overline{M}_{0,g+n} \rightarrow \overline{M}_{g,n}$ be the morphism associated to the map given by attaching pointed elliptic tails at each of the first g marked points.

Theorem 2.3.3 (Bridge theorem). *A divisor D on $\overline{M}_{g,n}$ is nef if and only if*

1. D is an F -divisor, and
2. f^*D is a nef divisor on $\overline{M}_{0,g+n}$.

The second key theorem is as follows.

Theorem 2.3.4 (Ray theorem). *If R is an extremal ray of the cone of curves of $\overline{M}_{0,N}$ and if $(K_{\overline{M}_{0,N}} + G) \cdot R < 0$ where G is any effective sum of boundary divisors for which $\Delta - G$ is also effective, then R is spanned by an F -curve.*

The symbol Δ denotes the sum of boundary classes. So the condition mentioned above is that $G = \sum_S a_S \Delta_S$ such that $0 \leq a_S \leq 1$ for all S . The above theorem is an extension of the work by Keel and McKernan [22] which states that if R is an extremal ray of $NE_1(\overline{M}_{0,N})$ and if $R \cdot (K_{\overline{M}_{0,N}} + G) \leq 0$ for $G = \sum_S a_S \Delta_S$ such that $0 \leq a_S < 1$, then R is spanned by an F -curve.

Proof of Theorem 2.3.2. Suppose that whenever one has an F -divisor D on $\overline{M}_{0,N}$, there exists a constant $c \geq 0$ for which $D = cK_{\overline{M}_{0,N}} + E$, where E is an effective sum of boundary classes. We will show that this assumption implies that the F -conjecture is true on $\overline{M}_{g,n}$. By Theorem 2.3.3, in order to prove the F -conjecture on $\overline{M}_{g,n}$, it is enough to show that any F -divisor on $\overline{M}_{0,g+n}$ is nef. Hence if we show that our assumption implies that D is nef, then the theorem is proved. By definition, if D intersects non-negatively all the extremal rays of the cone of curves, then D is nef. Suppose R is an extremal ray of the cone of curves. The first thing to note is that since D is an F -divisor, and if R is spanned by an F -curve, then D intersects R non-negatively. We will prove that there are no other kinds of extremal rays. We do this by induction on the number of marked points. As base case we take $N = 7$ since the F -conjecture is true for $N \leq 7$ [22].

(The cone of curves is the closure of $NE_1(\overline{M}_{0,N})$ in the real vector space $N_1(\overline{M}_{0,N})$. So every extremal ray R is either spanned by an irreducible curve or is the limit of rays spanned by irreducible curves.)

Suppose that R is a D -negative extremal ray of the cone of curves of $\overline{M}_{0,N}$, for

$N > 7$, that isn't spanned by an F -curve. In other words, suppose that

$$D \cdot R = (cK_{\overline{M}_{0,N}} + E) \cdot R < 0.$$

Choose d such that $0 < d < c$ and $d < 1$. Set $G = \frac{d}{c}E$ and $E' = \frac{1-d}{c}E$. Then $D = cK_{\overline{M}_{0,N}} + E = c((K_{\overline{M}_{0,N}} + G) + E')$. Since D is F -divisor, $D \cdot R < 0$ implies that R is not a Faber ray, that is R is not spanned by a F -curve. By Theorem 2.3.4, we have $(K_{\overline{M}_{0,N}} + G) \cdot R \geq 0$. Hence, $E' \cdot R < 0$. In particular, we have $E \cdot R < 0$.

Let's say R is spanned by a curve B . Since E is an effective sum of boundary classes and $E \cdot R < 0$, the curve B must lie in one of the boundary components in the support of E . To get a contradiction, it is enough to show that D is nef when restricted to a boundary divisor in the support of E . Pick Δ_T in the support of E such that $\Delta_T \cdot R < 0$. Its easy to see that such a boundary divisor can be found, otherwise $E \cdot R$ would be non-negative. Restricting D to Δ_T results in pulling D back to a space $\overline{M}_{0,n}$, for $n < N$ along a gluing morphism. Applying Lemma 2.2.2, the pullback of an F -divisor along a gluing morphism is an F -divisor. One can repeat this argument until ending up in $\overline{M}_{0,n}$ for $n \leq 7$, and we are done. Before proceeding with the case when R is a limit of curves, let's state the following lemma (communicated by James McKernan).

Lemma 2.3.5 (Extremal ray lemma). *Let X be a smooth projective variety and let Y be a smooth prime divisor. Let $i : Y \rightarrow X$ be the natural inclusion morphism. If R is an extremal ray of the closed cone of curves of X such that $Y \cdot R < 0$ then there is an extremal ray S of the closed cone of curves of Y such that $i_*S = R$, where $i_* : \overline{NE}_1(Y) \rightarrow \overline{NE}_1(X)$ is the natural map.*

Proof. Pick a sequence of irreducible curves C_1, C_2, \dots such that the rays $\mathbb{R}_+[C_i]$ approach the ray R . By continuity, possibly passing to a tail of this sequence, we may assume that $Y \cdot C_i < 0$. As Y and C_1, C_2, \dots are irreducible, it follows that $C_i \subset Y$. Let $S_i = \mathbb{R}_+[C_i] \subset \overline{NE}_1(Y)$. Possibly passing to a subsequence, we may assume that S_i converges to a ray S . We have, $i_*S_i = R_i$, so that by continuity $i_*S = R$.

Suppose that α and $\beta \in \overline{NE}_1(Y)$ such that $\alpha + \beta \in S$. If we set $\gamma = i_*\alpha$ and $\delta = i_*\beta$, then $\gamma + \delta \in R$, and $\gamma \in \overline{NE}_1(X)$ and $\delta \in \overline{NE}_1(X)$ so that γ and $\delta \in R$. It follows that $F = i_*^{-1}R$ is a non-empty face of $\overline{NE}_1(Y)$, so that we may always find $S \subset F$ an extremal ray. \square

Coming back to our proof, let's say the extremal ray R is a limit of curves. Since E is an effective sum of boundary classes, pick a prime divisor Δ_T in its support, such that $\Delta_T \cdot R < 0$. Now applying Lemma 2.3.5, we obtain an extremal ray S in $\overline{NE}_1(\Delta_T)$. Since, S is a limiting ray on the boundary, it is spanned by a F-curve (by induction hypothesis). Also, restricting D to Δ_T results in pulling D back along a gluing morphism. Hence $D|_{\Delta_T}$ is an F-divisor and intersects S non-negatively. Hence the contradiction. \square

Theorem 2.3.6 ([14]). *The F-conjecture is true on $\overline{M}_{0,g}/S_g$ for $g \leq 24$.*

Remark: The above result is counter-intuitive since for $g \geq 22$, the Kodaira Dimension of \overline{M}_g is positive (in fact, for $g = 22$ and $g \geq 24$, the moduli space is of general type).

Corollary 2.3.7. *The F-conjecture is true on \overline{M}_g for $g \leq 24$.*

Proof. The corollary follows from Theorem 2.2.6. In particular, $F_1(\overline{M}_g)$ is equivalent to $F_1(\overline{M}_{0,g}/S_g)$. \square

Corollary 2.3.8. *The F-conjecture is true on $\overline{M}_{g,n}$ for $g \leq 8, n = 1$; $g = 6, n = 2$*

Proof. In either of the cases $g + n \leq 24$. Hence, applying Theorem 2.2.6, the result easily follows. \square

Chapter 3

Bridge and Ray theorems: proofs

In the following pages, we sketch proofs of Theorems 2.3.3 and 2.3.4, the key ingredients involved in reducing the F-conjecture to the MF-conjecture.

3.1 Bridge theorem: proof

In this section, we sketch a proof of Theorem 2.3.3, which follows from Theorem 2.2.6. Theorem 2.2.6 not only states the former theorem, but also its converse. So, now all we need to do is prove Theorem 2.2.6 (or its strengthening Theorem 2.2.7). The proof is combinatorially intensive and uses various relations amongst divisor classes on $\overline{M}_{g,n}$. We omit these symbolic manipulations. The proof presented here is along the lines of [15].

We begin with a proof of Theorem 2.2.5. Assume $g \geq 3$. Fix the class of a divisor D on $\overline{M}_{g,n}$ as $a\lambda - b_{irr}\delta_{irr} - \sum_{\mathcal{I}_{g,n}} b_{i,I}\delta_{i,I}$. We claim:

Lemma 3.1.1. *If $g \geq 1$ and D meets all curve strata of types (2.2.3.1 - 2.2.3.5), that are relevant for g , non-negatively, then $D \cdot B \geq 0$ for any curve B not lying in Δ . If $g = 1$, then such a D is linearly equivalent to an effective sum of boundary divisors.*

First, we deal with $g = 1$. In this case, using the relations between divisor classes, we can assume a and all the $b_{0,\{i\}}$ (coefficients of ψ_i) are all 0. Then applying the inequalities 2.2.4.5 inductively, it follows that $b_{0,I} \geq 0$ for any I and hence that D is

equivalent to an effective sum of boundary divisors.

Now assume $g \geq 2$ and define the class of an associated divisor D' on \overline{M}_g by $a\lambda - b_{irr}\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} \beta_i \delta_i$, where $\beta_i = \max_{I \subset N} b_{i,I}$. Observe that, because we defined β_i 's to be maxima, if the coefficients of D satisfy any one of the sets of inequalities of types (2.2.4.1 - 2.2.4.5), then the coefficients of D' satisfy the corresponding inequalities for \overline{M}_g . Using the relation $\psi_i = \omega_i + \sum_{i \in I} \delta_{0,I}$, we can express D as $\sum_{i \in N} b_{0,\{i\}} \omega_i + \pi^* D' + E$ where π is the forgetful map to \overline{M}_g . ω_i on $\overline{M}_{g,n}$ stands for the pullback of the relative dualizing sheaf of the universal curve $\overline{M}_{g,1} \rightarrow \overline{M}_g$ by the projection given by dropping all but the i^{th} point, and E is the effective sum of boundary divisors given by

$$E = \sum_{I \subset N, |I|, |I^c| \geq 2} \left(\sum_{i \in I} b_{0,\{i\}} - b_{0,I} \right) \delta_{0,I} + \sum_{(i,I) \in \mathcal{I}_{g,n}, i > 0} (\beta_i - b_{i,I}) \delta_{i,I}.$$

Note that the sum defining E is effective. The non-negativity of the coefficient of $\delta_{0,I}$ in E follows from an induction using the inequalities 2.2.4.5, and that of $\delta_{i,I}$ for $i > 0$ from the definition of D' . It can be shown that D' is nef outside Δ [15] and each ω_i is nef [24]. Hence, all three terms in the expression of D meet a curve $C \subset \Delta$ non-negatively. This proves the lemma.

From this Corollary 2.2.8 follows by an easy induction. We must show that the class of any divisor D satisfying the inequalities of types (2.2.4.1 - 2.2.4.5) must meet non-negatively any curve B whose moving components are not rational and we proceed by simultaneous induction on g and n . When $g = 0$ or $g = 1, n = 1$ there is nothing to prove, so we may suppose $g \geq 2$ or $g = 1, n \geq 2$. By Lemma 3.1.1, if B lies outside Δ , $D \cdot B \geq 0$. Hence we may assume that B lies in Δ . First suppose the component containing B is some $\Delta_{i,I}$. Applying induction using Lemma 2.2.2, this induces a decomposition of both B and D and it suffices to show that $D' \cdot B' \geq 0$. But D' is again a Faber divisor and B' again has no rational moving component. If $B \subset \Delta_{irr}$, we apply the same argument to the normalization at one of the irreducible nodes. Finally, suppose that D is also nef restricted to $\overline{R}_{g,n}$. We can repeat the induction as above, maintaining this extra hypothesis. The argument holds as we

may assume that D' is nef on $\overline{R}_{i,I\cup\star}$ (where $i \leq g$, $I \subset \{1, \dots, n\}$), since we may choose the fixed curve B'' , that we attach, to have all components rational without changing its numerical equivalence class.

Next, let's sketch the proof of the reduction to the flag locus, Theorem 2.2.6. Here the key technical result is

Lemma 3.1.2. *If D is a \mathbb{Q} -Cartier divisor satisfying the hypotheses of Lemma 3.1.1 and let X be a stratum of $\overline{R}_{g,n}$ whose generic member is a stable curve with no disconnecting nodes, then $D|_X$ is linearly equivalent to an effective sum of boundary divisors and nef divisors.*

Corollary 3.1.3. *If D is a divisor on $\overline{M}_{g,n}$ meeting non-negatively all curve strata of types (2.2.3.1 - 2.2.3.5) and B is a curve on $\overline{M}_{g,n}$ whose general member has no moving component that is smooth and rational, then $D \cdot B \geq 0$.*

The corollary follows by induction as in the above argument proving Corollary 2.2.8 above. Applying Lemma 2.2.2, we may assume that the general member of B is irreducible and applying Corollary 2.2.8 that all components are rational. Then the above lemma applied to the smallest stratum S whose closure contains B (so that B does not lie in the boundary of S) expresses D as a sum of classes that meet B effectively. Applying Lemma 2.2.2 again, the corollary yields Theorem 2.2.7.

The proof of Lemma 3.1.2 proceeds in two steps. As in the proof of Lemma 3.1.1, we can express D as $\sum_{i \in N} b_{0,\{i\}} \omega_i + \pi^* D' + G$ where π is the forgetful map to \overline{M}_g . G is now an effective sum of boundary divisors parameterizing degenerations with a disconnecting node. (If $g = 1$, G is empty and D' is a multiple of δ_{irr}) Thus, it suffices to prove the lemma replacing D by D' and X by its image under π and we may assume $n = 0$. The image of X is a point if $g = 1$, and is either a point or a curve stratum of type (2.2.3.4) if $g = 2$ (since there are, generically, no disconnecting nodes). So we can assume that $g \geq 3$.

Let C be the stable curve corresponding to a general point of X and let C' be a component of C . By hypothesis, C' comes from a point in the open stratum of some $\overline{M}_{0,k}$ by a gluing map. The k marked points come equipped with a partition P

whose parts are the pairs of points lying over nodes of C' and the subsets lying on the intersection of C' with each connected component of $C \setminus C'$. At least two points lie in each such subset, since there are no disconnecting nodes. Observe that each boundary class δ_i on \overline{M}_g will pull back on $\overline{M}_{0,k}$ to an effective sum of classes $\delta_{0,I}$ where I is a union of parts of P , which we'll denote by $I \succ P$. The coefficient b_{irr} of δ_{irr} in D must be non-negative by inequality 2.2.4.2. If it is 0, applying inequalities 2.2.4.3 and 2.2.4.4 we are done. If it is positive, we may rescale D so that $b_{irr} = 1$. Recall that $\delta_{irr} + 12\lambda = \kappa + \sum_{i>0} \delta_i$ and that λ is trivial on $\overline{M}_{0,k}$. Thus, we see that D pulls back to $\kappa + \sum_{I \succ P} a_I \delta_I$, with each $a_I \geq -1$, which is an effective sum of boundary divisors (follows from [15], Lemma 4.4). Hence the lemma.

To get the converse of Theorem 2.2.6, consider the class of the divisor D on $\overline{M}_{g,n}$ as

$$a\lambda - b_{irr}\delta_{irr} - \sum_{(i,I)} (g+n - (i+|I|))(i+|I|)\delta_{i,I}.$$

It can be shown that the intersection of D with any stratum of type (2.2.3.6) is 0. These strata are exactly those inherited from $\overline{M}_{0,g+n}$. Since, by [23], the strata of $\overline{M}_{0,g+n}$ generate its Chow group, this shows that D has trivial pullback to $\overline{M}_{0,g+n}$ and hence to $\overline{M}_{0,g+n}/S_g$. If we now require that $a > 12b_{irr} - (g+n-1)$, then D meets strata defined by 2.2.3.1 positively, and if we also require that $b_{irr} > \left(\frac{g+n}{2}\right)^2$, then it meets strata defined by 2.2.3.2 - 2.2.3.5 positively. By the forward direction of Theorem 2.2.6, such a D is nef.

To see that every nef bundle on $\overline{M}_{0,g+n}/S_g$ comes from one on $\overline{M}_{g,n}$, first note that, by computations of Faber in [7], the pullback map $f_{g,n}^* : \text{Pic}(\overline{M}_{g,n}) \rightarrow \text{Pic}(\overline{M}_{0,g+n}/S_g)$ is surjective and, by construction, D has trivial pullback. Given a class $G \in \text{Pic}(\overline{M}_{0,g+n}/S_g)$, choose a class \tilde{E} on $\overline{M}_{g,n}$ and such that $G = f_{g,n}^*(\tilde{E})$. If G is nef then, for large m , $\tilde{E} + mD$ will meet any effective curve not pulled back from $\overline{M}_{0,g+n}/S_g$ positively (because D does) and hence must itself be nef. Hence the converse in Theorem 2.2.6 holds.

3.2 Ray theorem: proof

Here, we sketch a proof of Theorem 2.3.4. The sketch here follows the original plan of Keel and McKernan [22] as modified by Farkas and Gibney [12]. The key step in the proof is to show that R lives on some boundary component. Suppose not. Then $\Delta_S \cdot R \geq 0$ for every $S \subset \{1, \dots, n\}$ and $K_{\overline{M}_{0,n}} \cdot R < 0$. Now κ is an ample divisor class with support the full boundary Δ (as shown by the relations amongst divisor classes in the first chapter). By an application of the Cone and Contraction theorems [26], the ray R must be spanned by a contractible curve C not lying in Δ . The associated contraction $f : \overline{M}_{0,n} \rightarrow X$ must be finite on Δ and the relative Picard number of f is one. Moreover, by [23] each Δ_S has anti-nef normal bundle, that is $\Delta_S \cdot A \leq 0$ for any curve $A \subset \Delta_S$.

Now we claim that the exceptional locus of f must be the curve C . Given this we reach a contradiction if $n \geq 7$. Applying Theorem 1.14 in [25], we estimate the dimension of the space of deformations of C inside $\text{Hilb}(\overline{M}_{0,n})$ as $(-(K_{\overline{M}_{0,n}} \cdot C) + n - 6) \geq 1$. It shows that C moves in $\overline{M}_{0,n}$. Deformations of C must also lie in the exceptional locus of f . This contradicts the fact that the exceptional locus of f is a curve. Of the remaining cases, $n = 4$ and $n = 5$ are trivial and $n = 6$ is handled in [12] by a direct verification that the Faber and Mori cones coincide. (Recall that the Faber cone is the cone spanned by F-divisors.)

To see the claim, we follow the lines of Proposition 2.5 in [22]. For a proof by contradiction, assume instead that some irreducible surface E gets mapped by f onto a curve or point. Since $\Delta = \sum_S \Delta_S$ has ample support and $f|_{\Delta}$ is finite, $I := \Delta \cap E$ is non-empty and each $\Delta_S \cap I = \Delta_S \cap E$ is an effective \mathbb{Q} -Cartier divisor of E which is either empty or a union of components of I . Furthermore, $f|_I$ is finite and f contracts E to an irreducible curve $U \subset f(E)$. Now choose an irreducible component B of I lying in a maximal number of Δ_S . Since the Δ_S have anti-nef normal bundles and Δ has ample support, there must be a Δ_J not containing B and such that $\Delta_J \cdot B > 0$. If B' is a component of $\Delta_J \cap E (= \Delta_J \cap I)$, by maximality of B , there must be a Δ_S containing B and $B' \not\subset \Delta_S$. Now Δ_S and Δ_J both meet fibers of f . So choosing a

suitable $\gamma > 0$, we must have $\Delta_S - \gamma\Delta_J$ pulled back from X . Now let $V : (\Delta_S - \gamma\Delta_J)|_E$. Our choices mean that $V \cdot B < 0$ and $V \cdot B' \geq 0$. But V is pulled back from U , and B, B' are multi-sections of f so this is a contradiction. Thus we have proved the claim.

Going back to our proof, we now know that R lives on some boundary component. Let's first define a gluing map, $\theta : \overline{M}_{0,T\cup l} \rightarrow \overline{M}_{0,n}$, which acts by choosing a fixed curve in $\overline{M}_{0,(N\setminus T)\cup m}$, where $N = \{1, \dots, n\}$, and gluing the point p_m on the fixed curve to the point p_l on the moving curve in $\overline{M}_{0,T\cup l}$. Now since we know that R is pulled back along θ and is contained in some boundary divisor Δ_T , we replace G (assumed to be an effective sum of boundary divisors, ie. $G = \sum_S a_S \Delta_S$, such that $0 \leq a_S \leq 1$) by the divisor $G' := G + (1 - a_T)\Delta_T$. Note that G' is an effective sum of boundary divisors as well, with $0 \leq a'_S \leq 1$. Since Δ_T has anti-nef normal bundle, $(K_{\overline{M}_{0,n}} + G') \cdot R$ is again non-positive. Pulling back divisors, we get $\theta^*(G') = G'' - \psi_l$ where G'' is also an effective sum of boundary divisors, with $0 \leq a''_S \leq 1$. By adjunction $\theta^*(K_{\overline{M}_{0,n}}) = K_{\overline{M}_{0,T\cup l}} + \psi_l$. So $\theta^*(K_{\overline{M}_{0,n}} + G') = K_{\overline{M}_{0,T\cup l}} + G''$. Thus $(K_{\overline{M}_{0,T\cup l}} + G'') \cdot R \leq 0$, that is, we have exactly the initial situation on a lower dimensional moduli space and we can now conclude by induction that R is an extremal ray.

Bibliography

- [1] E. Arbarello and M. Cornalba, Calculating cohomology groups of moduli spaces of curves via algebraic geometry, *Inst. Hautes études Sci. Publ. Math.* **88** (1998), 97127 (1999).
- [2] A. Bruno and A. Verra, M_{15} is rationally connected. Projective varieties with unexpected properties, 51-65, *Walter de Gruyter* (2005).
- [3] A.-M. Castravet and J. Tevelev. Exceptional loci on $\overline{M}_{0,n}$ and hypergraph curves, preprint.
- [4] M. C. Chang and Z. Ran, The Kodaira dimension of the moduli space of curves of genus 15, *J. Differential Geom.* **24** (1986), 205-220.
- [5] M. C. Chang and Z. Ran, On the slope and Kodaira dimension of \overline{M}_g for small g , *J. Differential Geom.* **34** (1991), 267-274.
- [6] D. Chen, Covers of elliptic curves and the moduli space of stable curves, to appear *J. reine angew. Math.*
- [7] C. Faber, Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians, *New trends in algebraic geometry* (Warwick, 1996), *London Math. Soc. Lecture Note Ser.*, vol. 264, Cambridge Univ. Press, Cambridge, 1999, 93109.
- [8] G. Farkas, The geometry of the moduli space of curves of genus 23, *Math. Annalen* **318** (2000), 43-65.

- [9] G. Farkas, Syzygies of curves and the effective cone of \overline{M}_g . *Duke Math. J.* **135** (2006), 53-98.
- [10] G. Farkas, The Global Geometry of the Moduli Space of Curves. *Proceedings of Symposia in Pure Mathematics* **80** (2009), 125-147.
- [11] G. Farkas, Birational Aspects of the Geometry of \overline{M}_g . *Surveys in Differential Geometry* Vol. **14** (2010), 57-111.
- [12] G. Farkas and A. Gibney, The Mori cones of moduli spaces of pointed curves of small genus. *Trans. Amer. Math. Soc.* **355** no.3 (2003), 1183-1199.
- [13] G. Farkas and M. Popa, Effective divisors on \overline{M}_g , curves on K3 surfaces, and the slope conjecture. *J. Algebraic Geom.* **14** (2005), 241-267.
- [14] A. Gibney, Numerical criteria for the divisors on \overline{M}_g to be ample. *Compos. Math.* **145** no. 5 (2009), 1227-1248.
- [15] A. Gibney, S. Keel and I. Morrison, Towards the ample cone of the moduli space of curves, *J. Amer. Math. Soc.* **15** no.2 (2002), 273-294.
- [16] J. Harer, The second homology group of the mapping class group of an orientable surface. *Invent. Math.* **72** (1983), 221-239.
- [17] J. Harris and D. Mumford, On the Kodaira dimension of the moduli space of curves, *Inventiones Math.* **67** (1982), 23-88.
- [18] J. Harris and I. Morrison, *Moduli of Curves*, Graduate Texts in Mathematics, 187, Springer, New York, 1998.
- [19] J. Harris and I. Morrison, Slopes of effective divisors on the moduli space of stable curves. *Invent. Math.* **99** (1990), 321-355.
- [20] R. Hartshorne, *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York, 1977.

- [21] B. Hassett and Y. Tschinkel, On the effective cone of the moduli space of pointed rational curves, *Topology and geometry: commemorating SISTAG 83-96*, *Contemp. Math.* **314**, Amer. Math. Soc., Providence, RI, 2002.
- [22] S. Keel and J. McKernan, Contractible extremal rays on the moduli space of n -pointed genus zero curves. preprint.
- [23] S. Keel, Intersection theory of moduli space of stable n -pointed curves of genus zero, *Trans. Amer. Math. Soc.* **330** (1992), no. 2, 545574.
- [24] S. Keel, Basepoint freeness for nef and big line bundles in positive characteristic, *Ann. of Math. (2)* **149** (1999), no. 1, 253286.
- [25] J. Kollár, Rational curves on algebraic varieties, *A Series of Modern Surveys in Mathematics*, vol. 32, Springer-Verlag, Berlin, 1996.
- [26] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge University Press, Cambridge, 1998.
- [27] R. Lazarsfeld, *Positivity in Algebraic Geometry I. Classical setting: line bundles and linear series*, *Ergebnisse der Mathematik* **48**, Springer 2004.
- [28] R. Lazarsfeld, *Positivity in Algebraic Geometry II. Positivity for vector bundles and multiplier ideals*. *Ergebnisse der Mathematik* **49**, Springer 2004.
- [29] A. Logan, The Kodaira dimension of moduli spaces of curves with marked points, *Amer. J. Math.* **125** (2003), no. 1, 105138.
- [30] R. Pandharipande. Descendent bounds for effective divisors on the moduli space of curves. preprint.
- [31] W. Rulla, The birational geometry of \overline{M}_3 and $\overline{M}_{2,1}$, 2001 Ph.D. Thesis, University of Texas, Austin.
- [32] R. Vakil, The Moduli Space of Curves and its Tautological Ring. *Notices of the Amer. Math. Soc.* **50** no. 6, 2003, 647658.

- [33] P. Vermeire, A counterexample to Fulton's conjecture, *J. of Algebra*, **248**, (2002), 780-784.
- [34] A. Verra, The unirationality of the moduli space of curves of genus ≤ 14 , *Compositio Mathematica* **141** (2005), 1425-1444.