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# Dynamic Pricing and Stabilization of Supply and Demand in Modern Electric Power Grids

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**Abstract**—The paper proposes a mechanism for real-time retail pricing of electricity in smart power grids, with price stability as the primary concern. In previous articles, the authors argued that relaying the real-time wholesale market prices to the end consumers creates a closed loop feedback system which could be unstable or lack robustness, leading to extreme price volatility. In this paper, a mathematical model is developed for characterization of the dynamic evolution of supply, (elastic) demand, and market clearing (locational marginal) prices under real-time pricing. It is assumed that the real-time prices for retail consumers are derived from the Locational Marginal Prices (LMPs) of the wholesale balancing markets. The main contribution of the paper is in presenting a stabilizing pricing algorithm and characterization of its effects on system efficiency. Numerical simulations conform with our analysis and show the stabilizing effect of the mechanism and its robustness to disturbances.

## I. INTRODUCTION AND MOTIVATION

Demand response and dynamic retail pricing in electricity transmission and distribution (T&D) networks have been long advocated for improving system efficiency, mitigating wholesale price volatility, reducing system peak load, and the need to hold excessive reserve capacity. Reducing the annual system peak load lowers the required system-wide maximum capacity and lessens the need for expensive mega power plants (peakers) that are brought online for only a couple of hours per year. The impact on the required level of near real-time reserve capacity margins could be even greater. As the contribution of renewable resources to the supply of electricity grows in magnitude and extent, increased supply-side stochasticity challenges the system operators. System reliability constraints then compel the system operators to maintain additional reserve capacity to deal with the increased uncertainty. Dynamic pricing is one of the mechanisms that could be used for mitigating the effect of these uncertainties by allowing the consumers to react—in their own monetary or environmental interest—to the wholesale market prices, which reflect the real-time fluctuations in the system's capacity. While this real-time price-based coupling between supply and demand, should, at least in principle, mitigate the effects of stochastic fluctuations on the supply side, it challenges system operators in new ways. The first challenge concerns the endogenous uncertainty introduced by the demand side due to uncertainty in consumer behavior, preferences, and reactions to real-time price variations. The second challenge

concerns system stability (both price and supply/demand stability) as we will explore in the sequel. Investigating the trade-offs between the endogenous uncertainties induced by consumer behavior and mitigation of exogenous system uncertainties is outside the scope of this paper. Herein, we will focus entirely on system stability and efficiency questions that arise when the retail prices are tied to the spot prices of the wholesale markets.

The literature on dynamic pricing in communication or transportation networks—so that certain system objectives are met—is extensive, see for instance [8], [7], [4], [3] and the references therein. However, the specific characteristics of power grids arising from the distinctively close interplay between physics, market operation, and economics, along with the safety-critical nature of the system pose new and unique challenges to be addressed. The implications and consequences of various forms of dynamic pricing such as *Critical Peak Pricing*, *Time-of-Use Pricing*, and *Real-Time Pricing*, have been investigated by many economists and regulatory agencies. They mostly argue in favor of real-time pricing, characterized by passing on a price, that best reflects the wholesale market prices, to the end consumers. See for instance the papers by Borenstein et. al. [2], Hogan [6], technical report by the IEA-DSMP [5] and the report on California's 2011 deadline for mandatory real-time pricing [14]. In [10], the authors argued that simply relaying the wholesale market spot prices to the end consumers may create system instabilities. In particular, it appears that real-time pricing as defined above, in the absence of well-designed financial instruments for hedging, could potentially aggravate price volatility in wholesale markets. Whether real-time pricing will mitigate, or aggravate wholesale price volatility depends on many factors including implementation details, contract types, and most importantly, the mathematical relations between the cost functions of the producers and the value functions of consumers.

The framework developed in this paper considers the consumer as an autonomous agent myopically adjusting her usage in response to price signals, based on maximization of a quasi-linear smooth concave utility function. It is assumed that supply follows demand precisely, in the following sense: at each instant of time, any amount of electricity demanded by the consumers must be matched by the producers and the per unit price associated with this exchange is the exanté price corresponding to the marginal cost of supplying the predicted demand. The consumers then (myopically) adjust their usage by maximizing their utility functions for the next time period

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based on the new given price. This adjusted demand is in turn, a feedback signal to the wholesale market and affects the prices for the next time step. This framework favors the consumers in that it isolates them from the risk of large discrepancies between *ex-ante* and *ex-post* prices. We analyze stability and efficiency properties of the closed loop system arising from this setup and provide a subgradient-based [1] stabilizing algorithm for pricing.

## II. PRELIMINARIES

### A. Market Participants

We develop a market model with three participant types: 1. The consumers, 2. The producers, and 3. An independent system operator (ISO). Below, we describe the characteristics of these agents in detail.

1) *The Consumers and the Producers:* Let  $D = \{1, \dots, n\}$  and  $S = \{1, \dots, m\}$  denote the index sets of consumers and producers respectively. We associate a value function  $v_j(x)$  to each consumer  $j \in D$ , where  $v_j(x)$  can be thought of as the dollar value that consumer  $j$  derives from consuming  $x$  units of electricity. Similarly, to each producer  $i \in S$ , we associate a cost function  $c_i(x)$  representing the dollar cost of producing  $x$  units of the resource.

*Assumption 1:* For all  $i \in S$ , the cost functions  $c_i(\cdot)$  are in  $C^2[0, \infty)$ , strictly increasing, and strictly convex. For all  $j \in D$ , the value functions  $v_j(\cdot)$  are in  $C^2[0, \infty)$ , strictly increasing, and strictly concave.

Given a clearing price  $\lambda$ , the utility function of supplier  $i \in S$  is given by  $u_i(\lambda, x) \stackrel{\text{def}}{=} \lambda x - c_i(x)$ . Similarly, the utility function of consumer  $j \in D$  is given by  $u_j(\lambda, x) \stackrel{\text{def}}{=} v_j(x) - \lambda x$ . Let  $d_j: \mathbb{R}_{++} \rightarrow \mathbb{R}_+$ ,  $j \in D$ , and  $s_i: \mathbb{R}_{++} \rightarrow \mathbb{R}_+$ ,  $i \in S$ , denote  $C^1$  functions mapping price to consumption and production respectively. Then

$$d_j(\lambda) = \arg \max_{x \in \mathbb{R}_+} v_j(x) - \lambda x \stackrel{(a)}{=} \dot{v}_j^{-1}(\lambda), \quad j \in D \quad (1)$$

$$s_i(\lambda) = \arg \max_{x \in \mathbb{R}_+} \lambda x - c_i(x) \stackrel{(b)}{=} \dot{c}_i^{-1}(\lambda), \quad i \in S \quad (2)$$

where the rightmost equalities ((a) and (b)) in (1) and (2) are valid if we extend the inverse functions to define  $\dot{v}_j^{-1}(\lambda) = 0$ ,  $\forall \lambda > \dot{v}(0)$ , and  $\dot{c}_i^{-1}(\lambda) = 0$ ,  $\forall \lambda < \dot{c}(0)$ .

2) *The Independent System Operator:* The independent system operator (ISO) is a non-for-profit organization responsible for operating the wholesale markets and the transmission grid. The ISO's primary function is to optimally match supply and demand (adjusted for reserve requirements) subject to network constraints. In particular, operation of the real-time balancing markets involves solving a constrained optimization problem with the objective of maximizing the aggregate benefits of the consumers and producers. The constraints include power flow constraints (Kirchhoff's current and voltage laws (KCL and KVL)), transmission line constraints, generation capacity constraints, and local and system-wide reserve capacity requirements, and possibly a few other ISO-specific constraints. We refer the interested readers to [12], [9], [13] for more details.

For real-time system and market operation, the constraints are linearized near the steady state operating point and the ISO's problem is reduced to a convex optimization problem—usually a linear program—often referred to as the *Economic Dispatch Problem (EDP)*. A set of Locational Marginal Prices (LMPs) emerge as the dual variables corresponding to the nodal power balance constraints of this optimization problem. When the transmission lines are congested, the prices may vary from location to location as they represent the marginal cost of supplying electricity at each particular node [11], [12].

In this paper, we work with a simplified DC version of the EDP and make the following assumptions: 1. Resistive losses in the T&D lines are negligible. 2. Voltages are fixed and all generators and loads are ideal current sources/sinks. 3. There are no reserve capacity requirements. These assumptions are made in the interest of keeping the technical development in this paper focused and manageable within the limited space. They could be relaxed at the expense of heavier notation and a somewhat more involved technical analysis. The following DCOPF problem is the EDP problem considered in this paper:

$$\min_{s, d, I} -\mathcal{W}(s, d) \quad (3)$$

$$\begin{bmatrix} K & E \\ 0 & R \end{bmatrix} \begin{bmatrix} s \\ I \end{bmatrix} = \begin{bmatrix} d \\ 0 \end{bmatrix}$$

$$\text{s.t.} \quad -I_{\max} \leq I \leq I_{\max}$$

$$s_{\min} \leq s \leq s_{\max}$$

where  $d = [d_1, \dots, d_n]^T$  is the demand vector,  $s = [s_1, \dots, s_m]^T$  is the supply vector,  $E \in \{-1, 0, 1\}^{n \times r}$  is the graph incidence matrix ( $n$  is the number of nodes (buses) and  $r$  the number of branches (transmission lines)),  $R \in \mathbb{R}^{p \times r}$  is the loop-impedance matrix ( $RI = 0$  accounts for the KVL),  $K \in \{0, 1\}^{n \times m}$  is a matrix aggregating the output of several generators connected to one node <sup>1</sup> ( $Ks + EI = d$  accounts for the KCL), and  $\mathcal{W}(s, d)$  is the social welfare function:

$$\mathcal{W}(s, d) = \sum_{j \in D} v_j(d_j) - \sum_{i \in S} c_i(s_i)$$

The decision parameters of (3) are  $s$ ,  $d$ , and line current flows  $I$ . When the demand is fixed  $d_j = \tilde{d}_j$ ,  $\forall j$  the EDP reduces to satisfying a given fixed demand at minimum cost. In this case we have

$$\mathcal{W}(s, \tilde{d}) = -\sum_{i \in S} c_i(s_i)$$

and the decision variables of (3) are  $s$  and  $I$ .

### B. Dynamic Supply-Demand Model

In this sub-section we develop a model for the dynamic evolution of spot prices in electricity provisioning networks with price sensitive consumers. In order to prevent a large deviation from the nominal value in the system frequency, the

<sup>1</sup>The demand is assumed homogeneous, hence, the effect of all the loads connected to one node is treated as one flow. Therefore, an analog of the  $K$  matrix for demand is not included in the formulation.



Fig. 1.  $\lambda_t^{ea}$  ( $\lambda_t^{ep}$ ) is the exanté (ex-post) LMP for  $[t, t+1]$ ,  $\pi_t$  is the retail price for  $[t, t+1]$  and it is announced at the beginning of the interval,  $d_t$  is consumption during  $[t, t+1]$ , and will be known at time  $t+1$ .

aggregate supply needs to match the aggregate demand at each instant of time. Therefore, in real-time, supply must follow the demand in the sense that the amount of power requested by the consumers has to be supplied by the producers. The model developed here is based on this assumption and is consistent with the current practice in real-time balancing markets in the United States. The differences are: 1. The consumers adjust their usage based on the real-time price (which is assumed to reflect the real-time wholesale market price). 2. There are no ex-post adjustments to the (exanté) price passed to the end consumers. Since this model represents the price dynamics in smart grids with real-time retail pricing, it is reasonable to assume that the ISO (or the retail pricing entity) does not know the utility functions of the consumers, hence, a model of how the consumers will respond to price signals is not available.<sup>2</sup>

Before we proceed, we clarify our notation. Let  $N = \{1, \dots, n\}$  denote the set of all nodes (equivalently buses) in the network and  $e_l$  the  $l$ -th standard unit vector in  $\mathbb{R}^n$ . We use  $\mathcal{E}(c, d)$  to refer to the EDP problem (3) with fixed demand vector  $d$ . The inputs of  $\mathcal{E}(c, d)$  are the cost functions  $c_i(\cdot)$ ,  $i \in S$ , and the fixed demand  $d$ . The decision variables (and the outputs) of  $\mathcal{E}(c, d)$  are the vectors of generation  $s$  and the line current flows  $I$ . All other parameters  $E, K, I_{\max}, s_{\max}, s_{\min}$  are also assumed fixed in  $\mathcal{E}(c, d)$ . The *partial* dual problem to  $\mathcal{E}(c, d)$  is defined as:

$$\mathcal{D}(\lambda|c, d) = \min_{s, I} \sum_{i \in S} c_i(s_i) - \lambda(Ks + EI - d) \quad (4)$$

$$\begin{aligned} RI &= 0 \\ \text{s.t.} \quad -I_{\max} &\leq I \leq I_{\max} \\ s_{\min} &\leq s \leq s_{\max} \end{aligned}$$

*Remark 1:* In general, even when the cost functions  $c_i(\cdot)$ ,  $i \in S$  are strictly convex, the optimal solution  $(s^*, I^*)$  of  $\mathcal{D}(\lambda|c, d)$  need not be unique, though, when  $c_i(\cdot)$  are strictly convex,  $s^*$  will be unique for  $\lambda > 0$ . It can be verified that (4) is separable into two independent optimization problems: a convex optimization problem in terms of  $s$ , and an linear optimization problem (LP) in  $I$ . For a generic loop impedance matrix  $R$ , the feasible set of the LP ( $\{I \mid RI = 0, |I| \leq I_{\max}\}$ ) may have degenerate basic feasible solutions. Hence, the dual

<sup>2</sup>In this paper we merge the role of the ISO and that of an entity in charge of real-time retail pricing. Whether in reality this will or will not be the case, does not change the message and the results that we intend to deliver.

optimal solution

$$\lambda^* = \arg \max_{\lambda} \mathcal{D}(\lambda|c, d) \quad (5)$$

need not be unique. This, however, is not a major issue as: 1. the degenerate cases are rare (in a measure-theoretic sense), 2. one can remove the degeneracies by reformulating (4) in a reduced dimension, possibly as small as  $r - p$ , i.e., the dimension of subspace defined by  $R$ . Within the scope of this paper, we will assume that the dual optimal solution  $\lambda^*$  is unique.

*Remark 2:* If  $(s^*, I^*)$  is an optimal solution of  $\mathcal{D}(\lambda|c, d)$  then  $-Ks^* - EI^* + d$  is a subgradient of  $\mathcal{D}(\lambda|c, d)$  at  $\lambda$ .

The following sequence of actions taken by the ISO, the consumers, and the producers define the dynamics of the system:

- 1) At time  $t$ , the ISO announces the (exanté) wholesale LMP vector  $\lambda_t = [\lambda_{1,t}, \dots, \lambda_{n,t}]^T$  and the retail (locational) price vector  $\pi_t = [\pi_{1,t}, \dots, \pi_{n,t}]^T$  corresponding to the time interval  $[t, t+1]$ :

$$\begin{aligned} \pi_t &= \Pi_t(\tilde{\lambda}_t, \tilde{\pi}_{t-1}) \\ \tilde{\lambda}_t &= [\lambda_t, \dots, \lambda_{t-T}], \quad \tilde{\pi}_{t-1} = [\pi_{t-1}, \dots, \pi_{t-1-T}] \end{aligned}$$

where  $\lambda_{l,t}$  is the (exanté) LMP at location  $l \in N$  corresponding to  $[t, t+1]$ , and  $\Pi_t : \mathbb{R}^{2T+2} \rightarrow \mathbb{R}$  is the pricing function and  $T \in \mathbb{Z}_+$  is its memory. The LMP price  $\lambda_{l,t}$  is calculated by solving  $\mathcal{E}(c_t, \hat{d}_t)$  based on forecast demand for the time interval and computing the corresponding dual variables. Alternatively:

$$\lambda_t = \arg \max_{\omega} \mathcal{D}(\omega|c_t, \hat{d}_t) \quad (6)$$

We assume that the forecast function is of the form

$$\hat{d}_{l,t} = \hat{D}_t(d_{l,t-1}, \dots, d_{l,t-1-T})$$

for some function  $\hat{D}_t : \mathbb{R}^{T+1} \rightarrow \mathbb{R}$ .

- 2) The consumers adjust their usage for this time period according to (1):

$$d_t^l = \arg \max_{x \in \mathbb{R}_+} v_l(x) - \pi_{l,t}x = \dot{v}_l^{-1}(\pi_{l,t})$$

However, the quantity  $d_{l,t} \equiv \dot{v}_l^{-1}(\pi_{l,t}^l)$  will not be realized and revealed until time  $t+1$ , that is, the beginning of the next pricing interval. Total payments by the consumers for this time period is  $\sum_{l \in N} \pi_{l,t} d_{l,t} \equiv \sum_{l \in N} \pi_{l,t} \dot{v}_l^{-1}(\pi_{l,t})$ .

- 3) During  $[t, t+1]$ , the producers supply the total demand ( $\sum_{l \in N} d_{l,t}$ ), for this time period and they get paid according to their actual marginal cost of production. The total payment to the generators for this time period is<sup>3</sup>  $\sum_{l \in N} \sum_{j \in S(l)} \lambda_{l,t}^{\text{ep}} s_{j,l,t}$ , where  $\lambda_{l,t}^{\text{ep}}$  is calculated by

<sup>3</sup>This is a specific and simplified ex-post pricing rule for reimbursement of the generators and is mentioned here only to point out to risk issue. In reality, the ex-post pricing rules are more complicated and vary across ISOs. They are designed to encourage generators to follow dispatch instructions while minimizing the ISO's exposure.

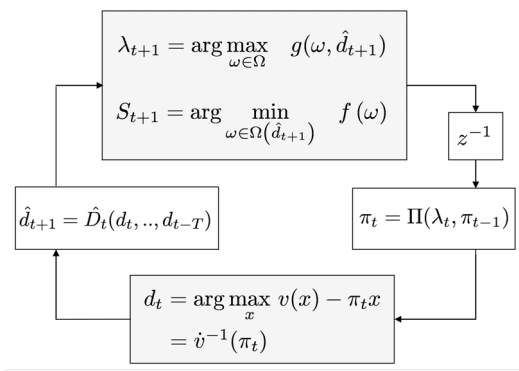


Fig. 2. Closed Loop System under Dynamic Pricing. The  $z^{-1}$  block represents the unit delay element.

solving  $\mathcal{E}(c_t, d_t)$ , and computing the dual variable. The ISO's risk or revenue differential is:

$$\Delta_t = \sum_{l \in N} (\pi_{l,t} d_{l,t} - \sum_{j \in S(l)} \lambda_{l,t}^{\text{ep}} s_{j,l,t})$$

It can be verified that in the absence of congestion in transmission lines, if  $\hat{d}_{l,t} = d_{l,t}$  and  $\pi_{l,t} = \lambda_{l,t}$ ,  $\forall l$ , then  $\Delta_t = 0$ .

- 4) The ISO re-solves the dispatch problem for the next time period:

$$\mathcal{E}(c_{t+1}, \hat{d}_{t+1}) \rightarrow \min_s \quad (7)$$

and dispatches the generation units accordingly. The process iterates as in Step 1.

*Remark 3:* Under a different pricing model, that is, ex-post pricing, the retail price charged for consumption of one unit of electricity during the interval  $[t, t+1]$  is calculated and declared only at the end of the interval, when the total consumption has materialized. In this case, the price uncertainty and the associated risks are all bore by the consumer, who must estimate the prices for the next time interval. A price prediction function  $\hat{P}_t$  will take the role of the demand prediction function  $\hat{D}_t$ . Assuming that the consumer's price prediction function has the same form as the demand prediction function, i.e.,  $\hat{\pi}_{t+1} = \hat{P}_t(\pi_t, \dots, \pi_{t-T})$ , it is not difficult to see that the emerging price dynamics is essentially the same as the case with ex-ante pricing, and hence, with some minor changes our framework would still be applicable. However, if this assumption is not satisfied, or if the function  $\hat{P}_t$  is very complicated, the system's dynamics could become very complex and unpredictable from the ISO's perspective.

### III. MAIN RESULTS

In this section we present the main result of the paper which is a subgradient-based stabilizing pricing algorithm. Consider the dynamic model developed in the previous section and suppose for simplicity, that

$$\Pi_t(\tilde{\lambda}_t, \tilde{\pi}_{t-1}) = \Pi(\lambda_t, \pi_{t-1})$$

and

$$\hat{D}_t(d_{t-1}, \dots, d_{t-T}) = \hat{D}(d_{t-1}) = d_{t-1}$$

Then we obtain the following price dynamics

$$\pi_{t+1} = \Pi(\lambda_{t+1}, \pi_t) \quad (8)$$

$$\lambda_{t+1} = \arg \max_{\omega} \mathcal{D}(\omega, c_{t+1}, d_t) \quad (9)$$

#### A. Stabilization

*Theorem 1:* Suppose that there exist  $d_{\max} > 0$  and strictly convex functions  $c_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,  $j \in S$ , such that  $d_t \in [0, d_{\max}]$  and  $\forall j \in S : c_{j,t} = c_j$ ,  $\forall t$ . Let the pricing functions  $\Pi^\lambda$  and  $\Pi^{\mathcal{G}}$  be defined as

$$\Pi^\lambda(\lambda, \pi) = (1 - \gamma)\pi + \gamma\lambda \quad (10)$$

$$\Pi^{\mathcal{G}}(\pi) = \pi + \gamma\mathcal{G}(\pi) \quad (11)$$

where  $\gamma > 0$ , and  $\mathcal{G}(\pi)$  is a subgradient direction given by

$$\mathcal{G}(\pi) = -K\tilde{s} - E\tilde{I} + d \equiv -K\tilde{s} - E\tilde{I} + \dot{v}^{-1}(\pi)$$

and  $\tilde{s}$ ,  $\tilde{I}$ ,  $d$  together with  $\pi$  solve  $\mathcal{D}(\pi, c, d)$ . Then for sufficiently small  $\gamma$ , both  $\Pi^\lambda$  and  $\Pi^{\mathcal{G}}$  stabilize the price dynamics (8):

$$\pi_{t+1} = \Pi^\lambda(\lambda_{t+1}, \pi_t) = \pi_t + \gamma(\lambda_{t+1} - \pi_t) \quad (12)$$

$$\pi_{t+1} = \Pi^{\mathcal{G}}(\pi_t) = \pi_t + \gamma\mathcal{G}(\pi_t) \quad (13)$$

in the sense that  $(\pi_t, d_t, s_t)$  converges to a small neighborhood of  $(\pi^*, s^*, d^*)$  where

$$\pi^* = \arg \max_{\lambda} \mathcal{D}(\lambda | c, d^*)$$

and  $(s^*, d^*)$  solves (3).

*Proof:* The proof is based on standard Lyapunov techniques in convergence analysis of subgradient methods, and is omitted for brevity. ■

*Remark 4:* There are two stabilizing mechanisms proposed in Theorem 1. The first, (12), is based on the LMPs which are already available at the time when a new retail price must be computed and communicated. The second, (13), however, requires solving optimization problem (4) for computing the subgradient. There are various trade-offs in choosing one mechanism over another. In (10), the coupling between the LMPs and the retail prices is relatively tighter. This implies that wholesale market disturbances would reflect more quickly and more aggressively in the retail market. Our simulations (not shown in this paper due to lack of space) conform with this analysis and it was observed that for the same speed of convergence, (12) typically leads to more volatile retail prices than (13). In contrast, in the presence of external disturbances in demand and supply cost, (13) leads to larger discrepancies between average wholesale and retail prices. These discrepancies (which exist for both (12) and (13)) lead to excess or shortage of the pricing entity's revenue, which possibly could be adjusted through billing and accounting at slower time scales. Another trade-off is in that (13) can be implemented in a semi-distributed manner, whereas (10) requires knowledge of the LMPs which are computed centrally by the ISO. If

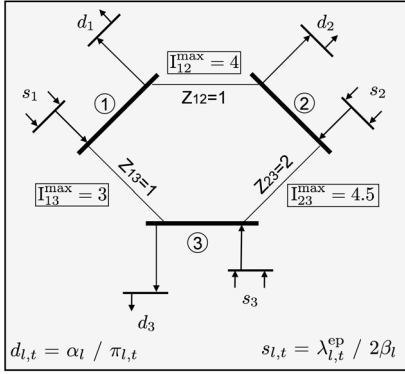


Fig. 3. The three bus system studied in the Numerical Simulations Section.

the network parameters  $R$ ,  $E$ , and  $I_{\max}$  are known to all the agents, the subgradient direction can be computed at each node (by the producers) and real-time retail pricing can be done independently at each node. While this distributed feature may seem unnecessary at the transmission system operation level, it is an attractive option for pricing and management of resources in the electricity distribution grid. Similar subgradient-based algorithms have in the past been reported for system decomposition and obtaining distributed algorithms for congestion control and scheduling in communication networks [4]. The specific differences and challenges in electricity networks lie mostly in the coupling and interaction between wholesale and retail markets, and exposure of the market participants to the risk of monetary loss (or benefit) from dynamic pricing.

*Remark 5:* With a properly defined notion of stochastic convergence, the result can be extended to cover cases with time-varying costs when  $\{c_t\}$  is an i.i.d. or Markovian process. For arbitrary time varying costs, the result does not hold.

#### IV. NUMERICAL SIMULATION

In this section we present numerical simulations that confirm and complement the results presented in the previous sections.

Consider the three bus system shown in Figure 3, where we have elected to aggregate all the consumers connected to one bus—which are assumed to be homogeneous—via a representative agent model. Hence, for our purposes this model has one consumer connected to each bus. We further assume that the consumers' value functions are logarithmic:

$$v_l(x) = \alpha_l \log x, \quad l = 1, 2, 3.$$

This implies that given a price  $\pi_{l,t}$ , we have  $d_{l,t} = \dot{v}_l^{-1}(\pi_{l,t}) = \alpha_l / \pi_{l,t}$ . Different parameters  $\alpha_l$  represent different consumer types based on income level, geographical, and demographic characteristics. Note that under this model, the total payment by the consumers over any time period of fixed length  $T$ , e.g., one month, is constant:

$$\frac{1}{T} \sum_{t=k}^{k+T} \pi_{l,t} d_{l,t} = \alpha_l$$

The interpretation is that regardless of the price, consumer  $l$  always consumes an amount which makes the total payment for one time period equal to  $\alpha_l$ . While this might appear as an over-simplification of consumer behavior, it is consistent with the casual observation that many consumers prefer to allocate a fixed budget to their utility bills, and while they might welcome some savings, their total willingness to pay is more or less constant. To make the simulation more realistic, a small perturbations noise  $a_l \varepsilon_t$  where  $\varepsilon_t \sim N(0, \sigma_d^2)$  is added to  $\alpha_l / \pi_{l,t}$ , and lower and upper bounds ( $d_{\min}$  and  $d_{\max}$ ) are imposed on the consumption:

$$d_{l,t} = \max(d_{\min}, \min(d_{\max}, \alpha_l / \pi_{l,t} (1 + \varepsilon_t))).$$

(We could instead, or as well, impose price caps and floors). It is assumed that there are several generators connected to each bus and that their aggregate cost is a piecewise linear approximation of the cost function of a single generator with quadratic cost  $c_l(x) = \beta_l x^2$ . The following set of parameters achieve this piecewise linear approximation with uniformly sized pieces of length  $q$ :

$$s_{j,l \min} = 0, \quad s_{j,l \max} = q, \quad dc_{j,l}(x)/dx = q\beta_l(2j-1), \quad \forall l.$$

We could instead work with a single generator with quadratic cost and solve the economic dispatch problem as a constrained quadratic optimization problem. In fact, that would magnify and aggravate possible system instabilities when the retail prices are the same as the LMPs. However, we chose the piecewise linear approximation model for two reasons. First, to be consistent with the current practice of most ISOs who formulate the EDP as a linear program. Second, to demonstrate—via simulations—that system instabilities can occur even with piecewise constant marginal costs. Once again, in these simulations, we added a small perturbation noise to the marginal costs:

$$dc_{j,lt}(x)/dx = q\beta_l(2j-1)(1 + \delta_t), \quad \delta_t \sim N(0, \sigma_s^2).$$

The network parameters corresponding to (3) are both displayed in Figure 3 and summarized in Table I.

Generators	Consumers	Network	Network
$\beta_1 = 0.5$	$\alpha_1 = 1500$	$z_{12} = 1$	$I_{\max}^{12} = 4$
$\beta_2 = 2$	$\alpha_2 = 1000$	$z_{23} = 2$	$I_{\max}^{23} = 4.5$
$\beta_3 = 1$	$\alpha_3 = 500$	$z_{13} = 1$	$I_{\max}^{13} = 3$
$s_{i \max}^j = q = 5$	$d_{\min} = 1$		
$s_{i \min}^j = 0$	$d_{\max} = 100$		

Table I

The results of the simulations are shown in Figures 4 and 5. The simulation in Figure 4 corresponds to the case where the LMPs are directly passed on to the consumers. In this case both price and consumption at all three buses are very oscillatory and unstable. If the supply costs grew faster than quadratic (i.e. with strictly convex marginal costs) the instabilities would be

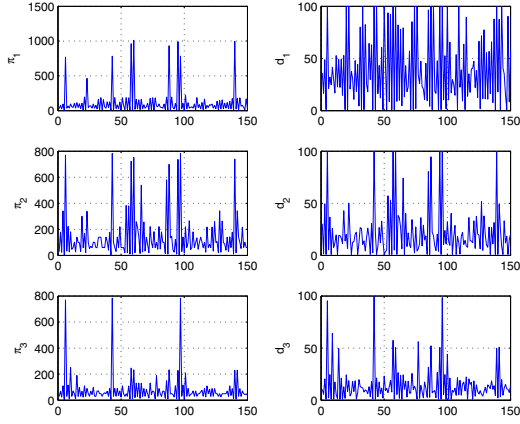


Fig. 4. Stochastic evolution of prices (left column) and demand (right column) when the LMPs are directly relayed to consumers ( $\pi_{l,t} = \lambda_{l,t}$ ). Both prices and consumption are extremely volatile. (simulation parameters: noise SD:  $\sigma_d = \sigma_s = 0.01$ ).

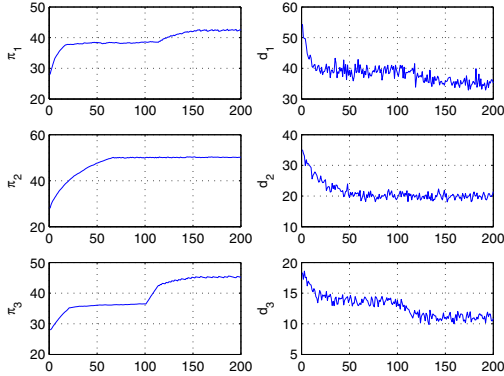


Fig. 5. Evolution of prices (left column) and demand (right column) under pricing model (11) with stepsize  $\gamma = 0.05$ . At time  $t = 100$  the cost of production at node 3 is doubled to simulate a disturbance causing tighter supply. The demand responds by lowering consumption and the system reaches a new equilibrium. (simulation parameters: noise SD:  $\sigma_d = \sigma_s = 0.001$ ).

much more pronounced. For the simulation shown in Figure 5, the subgradient algorithm (11) was implemented with constant step size. At time  $t = 100$  the parameters of the cost functions at node 3 were increased to simulate a situation where supply suddenly becomes tight. It can be observed that the system converges to a new equilibrium with lower consumption. Moreover, the retail prices are smooth which gives time to the consumers to gradually adjust, whereas the LMPs rise suddenly and sharply. This suggests that the mechanism is also robust to disturbances.

## V. CONCLUSIONS AND FUTURE WORK

We argued that directly passing on the wholesale balancing market's *locational marginal prices* to the end consumers may create an unstable closed loop system, and presented

a pricing algorithm that stabilizes the system. Adaptation of the proposed framework to more realistic models suitable for practical implementation is an important direction of future research. These include AC power flow models, market clearing models with reserve capacity constraints, and consumer behavior models. The algorithm presented in this paper does not require knowledge of the utility functions of the consumers, it instead, uses the consumers' response to price signals to find a subgradient direction. It would be however, both interesting and important to create more realistic models of consumer behavior. For instance, an energy management software might be in charge of optimization, decision-making, and scheduling of loads for large consumers. Such software might then use historical price patterns and predictions along with load-shifting and (possibly) storage to optimize usage and minimize energy costs. Several interesting questions arise here. First, how does presence of storage, load shifting, price-anticipating consumers, and dynamic optimization affect system stability and in particular, the algorithm presented in this paper? Is it possible that in aggregate, such complex demand response can still be modeled as a concave utility-maximization problem? Lastly, several fairness and efficiency questions arise. The first is the effects of price-anticipating consumers in the overall system efficiency. The second concerns fairness. If only a portion of the consumer base is subject to dynamic pricing, the nonparticipating population could conveniently consume energy at a time when supply is tight and subject the other consumers to excessive risk or inconvenience.

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