Two-Dimensional Numerical Modeling of Radio-Frequency Ion Engine Discharge

by

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ABSTRACT

Small satellites are gaining popularity in the space industry and reduction in spacecraft size requires scaling down its propulsion system. Low-power electric propulsion poses a unique challenge due to various scaling penalties. Of high-performance plasma thrusters, the radio-frequency ion engine is most likely to succeed in scaling as it does not require an externally applied magnetic field and is structurally simple to construct.

As part of a design package an original two-dimensional simulation code for radio-frequency ion engine discharge is developed. The code models the inductive plasma with fluid assumption and resolves the electromagnetic wave in the time domain. Major physical effects considered include magnetic field diffusion and coupling, plasma current induction and ambipolar plasma diffusion.

The discharge simulation is benchmarked with data from an experimental thruster. It shows excellent performance in predicting the load power and the internal power loss of the plasma. Predictability of anode current depends on the operating power but is generally adequate. Optimum skin depth on the order of half of chamber radius is suggested by the simulation. The code also demonstrates excellent scaling ability as it successfully predicts the performance of a smaller thruster with errors less than 10%. Using the code a brief optimization study was conducted and the results suggest the maximum thrust efficiency does not necessarily occur at the same frequency that maximizes the power coupling efficiency of the matching circuit.

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Title: Professor of Aeronautics and Astronautics
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### Nomenclature

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<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>$\vec{A}$</td>
<td>magnetic field vector potential, tesla-m</td>
</tr>
<tr>
<td>$E$</td>
<td>electric field strength, volts/m</td>
</tr>
<tr>
<td>$\vec{j}$</td>
<td>electric current density, amperes/m$^2$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>electric potential, volts</td>
</tr>
<tr>
<td>$e$</td>
<td>electron charge, 1.6x10^{-19} coulombs</td>
</tr>
<tr>
<td>$k$</td>
<td>Boltzmann constant, 1.38x10^{-23} J/K</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>electrical conductivity, siemens/m</td>
</tr>
<tr>
<td>$\delta$</td>
<td>skin depth, m</td>
</tr>
<tr>
<td>$\omega$</td>
<td>driving frequency, radian/s</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>permeability of vacuum, 4$\pi$x10^{-7} N/m$^2$</td>
</tr>
<tr>
<td>$\dot{n}_e$</td>
<td>ionization rate, 1/m$^3$/s</td>
</tr>
<tr>
<td>$n_g$</td>
<td>number density of species g, 1/m$^3$</td>
</tr>
<tr>
<td>$m_g$</td>
<td>mass of species g, kilogram</td>
</tr>
<tr>
<td>$T_e$</td>
<td>electron temperature, electron-volt</td>
</tr>
<tr>
<td>EEDF</td>
<td>electron energy distribution function</td>
</tr>
<tr>
<td>CFD</td>
<td>computational fluid dynamics</td>
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Small satellites are gaining popularity in the space industry as recent technology advancement allows manufacturers to build smaller and more cost-effective spacecraft without sacrificing functionality. Reduction in spacecraft size requires scaling down its subsystems and reducing their power consumption. Of the subsystems, low-power electric propulsion (EP) poses a unique challenge due to particularly severe scaling penalties in both efficiency and power-to-mass ratio. While non-plasma-based EP such as colloid [1,2] and field-emission electric propulsion (FEEP) thrusters [3] can maintain high thrust efficiency regardless of scale, they are emerging technologies that have not been proven for space flight. Plasma-based EP such as ion engines and Hall-effect thrusters are flight proven and can offer high specific impulse (Isp), but their efficiency decreases at reduced scale and power due to the energy expenditure necessary for generating the plasma source and compensating ion wall loss. In addition, the often-required magnetic structure of these thrusters does not scale well and consequently prevents significant mass reduction.

The radio-frequency (RF) ion engine, also known as an RF ion thruster, is a type of plasma-based electrostatic EP device that does not rely on externally applied magnetic fields to create its plasma source. Isp performance is typically high, similar to other gridded ion engines. Their uniqueness comes from simplicity, as no permanent magnetic structure is required and the thruster can easily be scaled down. Such feature makes RF ion engines ideal for small satellites that desire high-performance EP but have strict budget in volume, mass and power. Although simple to construct, designing an RF ion engine requires thorough knowledge of the inductively-coupled plasma (ICP) discharge and its matching circuit in order to efficiently generate the plasma source. This thesis research aims to tackle such problem with the goal of developing a design tool for RF ion engines.
1.1 RF Ion Engines

RF ion engines are related to the traditional direct-current (DC), electron-bombardment ion engines in the sense that they both rely on electrostatic force between grids to accelerate positively-charged heavy particles to produce thrust. High Isp and moderate efficiency are the norm. Like most EP thrusters, performance of RF ion engines varies widely with input power and size. Flight heritage systems exist with grid diameters up to 25 cm and power up to 6 kW[4]. Total efficiencies (thrust efficiency \times propellant utilization) can easily exceed 50% for 400W-class systems and above [5]. Depending on power level, thrust can vary from 10 mN to 200 mN with Isp ranging between 2500 and 5500 sec [6,7].

The first RF ion engine tested in space was the German RIT-10 thruster launched by the Space Shuttle Atlantis in 1992. It served as a technology demonstrator onboard the European Retrievable Carrier (EURECA) spacecraft. RIT-10 operated for 240 hours at 5-10 mN thrust level and 3000 sec Isp before succumbing to an overheating problem on the RF coil. This problem was later resolved and RIT-10 was commissioned again in 2001 onboard the Advanced Relay Technology Mission Satellite (ARTEMIS). It operated for 7,500 hours and helped raise ARTEMIS into its operational geostationary orbit [8,9]. Figure 1 shows the RIT-10 thruster configured for ARTEMIS.

![RIT-10 Thruster by EADS Astrium](image)

Figure 1 RIT-10 Thruster by EADS Astrium [9]

1.1.1 Concept and Description

An RF ion engine typically consists of an axisymmetric, cylindrical or conical discharge chamber made of dielectric material. A helical coil, energized at low mega-hertz radio frequency, is used to generate and sustain the plasma discharge. RF ion engines can be equipped with two or three grids. The first grid extracts the ions and sometimes serves as
anode to collect electrons from the discharge. The voltage on the first grid is normally 1500-2000 V. The second grid focuses and accelerates the ion beam, while at the same time sets up a negative potential field to prevent electrons back-streaming from the external neutralizer. Typical voltage on the second grid is 100-200 V. The third grid, when used, decelerates the ion beam and brings the potential field up to spacecraft ground. A conceptual arrangement of a typical two-grid RF ion engine is presented in Figure 2.

To start the thruster, electrons emitted by an external cathode are drawn into the chamber by switching on the positive high voltage to the screen grid [8]. These electrons bombard the neutral particles and release some more free electrons. They are then accelerated azimuthally by the oscillating electromagnetic field generated by the RF coil. In the path of acceleration, the electrons collide with neutral particles and ionize them, and the plasma discharge is ignited. Unlike typical plasma thrusters RF ion engines do not rely on externally applied magnetic field to sustain the discharge. This is because the main plasma current is induced by the azimuthal electric field, which directs most of the electrons in the discharge toward a circular path away from the walls. Applied magnetic fields are actually undesirable as they would hinder the path of the induced plasma current. Magnetic structures can also cause significant RF power coupling loss as they are typically made of conductive material. Though lacking active electron confinement, the inductive plasma in an RF ion engine is sustainable for a wide range of flow rate, RF power and grid extraction voltage.
The most common propellant used in RF ion engines is the noble gas xenon. Other propellants have been experimented with, including mercury, argon and krypton. Xenon is favorable for its low first ionization energy (12.1 eV) and high atomic weight (131.3 g/mole). The low ionization energy contributes to more efficient usage of discharge power, and the heavy mass would minimize loss factors for a given specific impulse [11]. Due to the heaviness, xenon ions are seldom magnetized in the presence of permanent or induced magnetic field and therefore can be accelerated electrostatically without magnetic interference. Xenon is chemically inert and easy to handle, making it a prime propellant for EP.

1.1.2 Advantages and Applications

Gridded ion engines have long history of operation and their reliability has been proven. Though high efficiency is achievable, thrust density is intermediate at best since extractable ion beam is limited by the space-charge current between the grids. They do, however, possess significant Isp advantage over chemical rockets as well as other EP thrusters because of its purely electrostatic acceleration mechanism. Among various ion engine designs the RF type is the simplest as it does not require any magnetic structure. The absence of applied magnetic field also means that the thruster can be operated away from a specific discharge condition, resulting in a wider thrust and power range.

Durability is another favorable trait of RF ion engines. Service life can be greatly extended by eliminating the need for an internal cathode, which is highly susceptible to erosion problems. Long service life makes RF ion engines perfect for long-term missions such as interplanetary travel and station-keeping/drag make-up at high-altitude orbit. Eliminating the internal cathode can also translate to reduction of ion production cost as high-energy (30-40 eV) primary electrons no longer exist, so energy is less likely spent in creating double ions. The combining effect of no magnetic structure and no internal cathode helps reduce the required volume of the thruster and saves mass, making miniaturization of the thruster much simpler.

RF ion engines excel in precision propulsion and rapid thrust response because they can vary thrust in as quickly as one milli-second by simply throttling the discharge power. Busek Co., Inc. has demonstrated such ability with its experimental RF ion engine in an interesting experiment where the input RF power was driven by the amplitude waveform of a music clip while the grid voltage and flow rate were kept constant. The music of choice was “The Imperial March (Darth Vader’s Theme Song)” composed by John Williams. The thrust
response was recorded and the result is shown in Figure 3. The original waveform of the music signal is shown at the top and the resultant thrust, estimated from the anode current, is shown at the bottom. The recorded thrust waveform resembles to the driving music signal. In fact, by playing back the resultant thrust waveform as music, the original melody can be clearly identified. The demonstration proves that RF ion engines can indeed change thrust rapidly and precisely by programming the RF power [10]. This unique feature can be important for formation flight missions that require fine thrust control. Potential applications of such include the Space Interferometer Mission (SIM), Space Astronomy Far Infrared Telescope (SAFIR), Laser Interferometer Space Antenna (LISA), Micro-Arcsecond X-ray Imaging Mission (MAXIM) and Submillimeter Probe of the Evolution of Cosmic Structure (SPECS) [12]. The highly responsive and programmable thrust is also perfect for drag-cancellation missions. Drag force encountered in the low-Earth orbit (LEO) can be actively damped by measuring instantaneous deceleration of the spacecraft and commanding the onboard propulsion device to generate an equal amount of thrust force. This method has been validated during the recent flight of the European GOCE satellite [13]. A perfect drag-less flight is difficult to achieve as sensors cannot anticipate the degree of deceleration. However, with a fast sensor and a responsive thruster like an RF ion engine, drag-less-ness can be simulated to a high extent.

![Waveform and Thrust Waveform](image)

**Figure 3** Rapid and Precise Change of Thrust by Modulating RF Power Programmed through a Music Waveform; Performed by Busek’s BRFIT-7 RF Ion Engine [10]
1.1.3 Issues

Despite their numerous advantages, RF ion engines have not been fully accepted by the U.S. space industry. One barrier involves the development of a highly efficient RF amplifier. Although good efficiency is not difficult to achieve because of the low operating frequency (order of MHz rather than GHz), problems still exist regarding scaling down the amplifier components. Developing high-efficiency RF power electronics at smaller scales is an active field of research but is not within the scope of this thesis.

Other issues of RF ion engines mainly concern the complex design methodology. The RF circuit needs to be designed for high power coupling efficiency so the source does not dissipate power within the transmission circuit. This is referred to as impedance matching and requires the knowledge of the plasma’s electrical properties, which are not known a priori. The selected operating frequency needs to produce the proper skin depth for RF power deposition. Such mechanism is related to the plasma conductivity, gas pressure, chamber geometry and coil geometry. All these parameters intertwine together and form a very complex design problem. On top of that, parasitic power loss such as ohmic heating of the coil adds extra design constraints as its electrical resistance could vary with temperature as well as driving frequency.

1.2 Busek RF Ion Engines

Two experimental RF ion engines developed by Busek Co., Inc. were provided for this thesis work. Pictures of the two thrusters, designated BRFIT-7 and BRFIT-3, are shown in Figure 4 to Figure 7. Detail hardware description and critical dimensions cannot be published for proprietary reasons. BRFIT-7, whose name stands for “Busek RF Ion Thruster” with a 7-cm-diameter grid, is throttleable between 0.7 and 14 mN thrust and can operate efficiently over 3,300 to 5,300 sec Isp. Thrust efficiency of 75% (thrust power-to-total power ratio, not including cathode and DC/RF conversion) can be obtained in nominal operation. Total power consumption ranges from 150 to 400 W depending on the thrust output [14]. BRFIT-7 was developed under the guidance of NASA/JPL and was empirically optimized to some extent. Its performance data were used to validate the simulation code of this thesis.
BRFIT-3 is a direct scale-down version of BRFIT-7 in terms of chamber and coil geometry. Operating frequency was adjusted but no optimization work was involved. Nominal power consumption is 100 W total (beam + RF) and thrust output is between 1.0 and 2.5 mN. Isp performance ranges from 1,300 to 3,000 seconds with nominal thrust efficiency of 46% [14]. Performance data of BRFIT-3 were used to test the scaling capability of the simulation code. The code is also used to verify the adequacy of BRFIT-3 design, since it was not empirically optimized.

Figure 4  Busek 7-cm RF Ion Engine (BRFIT-7)

Figure 5  BRFIT-7 Operating at 400W Total Power on Xenon

Figure 6  Busek 3-cm RF Ion Engine (BRFIT-3)

Figure 7  BRFIT-3 Operating at 100W Total Power on Xenon

Photo courtesy of Busek [14]
1.3 Previous Numerical Work

Experimental and theoretical work on inductively-coupled plasma (ICP) discharge has been around for decades and is still an ongoing effort. A brief literature review is presented in this section. This review focuses on theory development, specifically on numerical work related to this thesis.

Inductive plasma has two distinct configurations pertaining to coil geometry. In addition to the helical coil structure found on RF ion engines, plasma can also be generated by a planar coil wound on one side of a discharge chamber. Planar-coil ICP has been extensively researched in the semiconductor industry for plasma etching and surface treatment of silicon wafer. It is seldom applied to RF ion engines as planar-coil ICPs operate at 100s of MHz frequency and the RF power supply is not very efficient as the result. Despite different matching circuit designs, physics on the plasma coupling mechanism are similar between the two coil types. For general ICP discharge regardless of coil geometry, Piejak [15] developed an zero-th order analytical model by considering the plasma discharge as a one-turn secondary of an air-core transformer. Power deposition into the plasma was solved by finding equivalent electrical parameters of spatially-averaged plasma properties on the primary circuit. Maxwell’s equations were not solved using this method. Following Piejak’s work, Vahedi [16] from the Lawrence Livermore National Laboratory constructed a more detailed analytic model to describe power deposition in a planar-coil ICP discharge. He focused on the electromagnetic wave interaction with the plasma, especially the spatial decay of the electric field in the “skin effect” region. Both Piejak and Vahedi’s models show good agreement with published experimental data and can be used for scaling the applied frequency and input power.

Higher-order computational models of ICP plasma do exist, but mostly tailored for planar-coil type of discharge. Lymberopoulos [17], Kumar [18] and Lee [19] all developed two-dimensional fluid simulations in which full set of Maxwell’s equations were implemented. A two-dimension particle-in-cell/Monte Carlo (PIC/MC) model for carbon tetrafluoride was developed recently by Takekida [20] to study selective plasma etching of silicon oxide for manufacturing integrated circuits.

Few computational models have been developed for the plasma discharge found in RF ion engines. Oh [21] and Froese [22] have both worked on one-dimensional versions of PIC/MC simulations. Arzt [23] presented a two-dimensional fluid code in which ambipolar
diffusion, presheath potential formulation and an ion mobility term for momentum transport are included. RF power deposition and electromagnetic field coupling were however not described and a simple linear relationship between plasma density and RF power was used. Mistoco [24] took a similar approach and developed a fluid model for an RF ion thruster discharge using COMSOL multiphysics software. Mistoco’s work includes a transformer model for calculating RF plasma dissipation. Coupling of electromagnetic field between the coil and the plasma was again not described. Both Arzt and Mistoco’s codes assume steady-state fluid flows and do not solve Maxwell’s equations except for the electric potential distribution.

It is worth mentioning that Closs [25] of German Astrium GmbH (the manufacturer of RIT ion thrusters) has developed a practical design software for RF ion engines. This software considers many aspects of thruster operation, including thermal transfer to the thruster housing, optimization of the coil, and ion optics. The included discharge model is however very simplified and lacks spatial resolution. As the result the plasma discharge is not fully characterized.

1.4 Research Objective

The objectives of this research are two-fold. The primary objective is to increase the understanding of important aspects regarding ICP discharge. This includes how the plasma reacts to the coil-induced electromagnetic field, electron confinement (or lack of) supplied by the induced magnetic field, and scaling of the power deposition region that is related to the skin effect. The second objective is to develop a systematic approach to designing and optimizing the ICP source for an RF ion engine. Limitations of the theory should be identified and explained if possible.

1.5 Thesis Outline

The thesis is organized in a roughly chronological order that represents how the research progressed. Chapter 2 describes the premises of the simulation code, basic assumptions and limitations. Chapter 3 derives the physics behind the code. It starts with a transformer model that is used to calculate coil current and presents solution to Maxwell’s equations for
electromagnetic coupling between the coil and the plasma. The full set of fluid equations for ions and electrons is detailed for momentum and particle transport. The system is closed by equations of global energy balance and global neutral particle balance. Chapter 4 presents simulation results and comparisons with a benchmark thruster, the Busek thruster BRFIT-7. Comparison with the BRFIT-3 is also conducted to demonstrate scaling capability of the code. A brief optimization study for the BRFIT-3 is presented at the end of Chapter 4. Finally, Chapter 5 summarizes the accomplishments and contributions of this research, and recommends future work.
Chapter 2
MODEL DESCRIPTION

A two-dimensional simulation is developed for the ICP discharge inside an RF ion engine. Both electrons and singly-charged ions are treated as fluid continuum and neutral atoms are assumed as stationary background. Quasi-neutrality is imposed throughout the discharge. Differential forms of Maxwell’s equations are solved along with full sets of fluid equations. A time-domain approach is used to resolve the electromagnetic wave propagation and plasma current induction. Ion momentum is also solved with time dependency to allow plasma to reach its potential and density distribution in steady-state. The code considers cylindrical coordinates with 2D in space ($v_r, v_z$) and 3D in velocity ($v_r, v_\theta, v_z$). Xenon is the default propellant but other gases can be easily adopted by incorporating the necessary cross section data.

A previously-developed transformer model is implemented for finding the coil current that serves as discharge source. It takes spatially-averaged plasma properties and calculates their equivalent impedance in the primary circuit. By prescribing RF forward power, source impedance (typically 50 $\Omega$) and circuit capacitance, the model can approximate the amount of current passing through the RF coil. Since the plasma properties change over time, this transformer sub-model is accessed at every time step for finding the coil current. The time-dependent coil current is used to compute the coil-induced magnetic field, which can be coupled to the plasma current through solution of Maxwell’s equations.

The code is written in serial form and executed in MATLAB, a software package developed by MathWorks, Inc. The MATLAB language is very similar to C++ and the software can be compiled on any PC or work station. Convergence is determined by cyclic steady-state observation of anode current, electron temperature, and various dissipated powers. Convergence time using a 4 GHz, 64-bit processor and 4 GB memory is approximately two hours.
2.1 Fluid Approximation

Modeling charged particles as fluid in a high-frequency plasma discharge has been widely accepted. This assumption is valid for pressures $>100$ mTorr where collisions dominate [18,26]. RF ion engines however operate at much lower pressure, typically $1-10$ mTorr, so verification is needed before applying this assumption. Mikellides [27] suggests the onset of collisional plasma occurs when the mean-free-path of charged particle is significantly less than the characteristic length of the discharge vessel. He uses the following inequality to justify the fluid assumption, which can be adopted for general use.

\[ Kn = \frac{\lambda_{mfp}}{L} < 0.4 \]  

where $Kn$ is the Knudsen number defined as the ratio between mean-free-path $\lambda_{mfp}$ and characteristic length $L$. Although in an RF ion engine the Knudsen number for electrons can be high, it is less meaningful because electrons can be repelled by the sheath potential which increases their chance for collision. Ions, on the other hand, have no confinement mechanism and speed toward the walls. So the ion-neutral collision is the important issue here concerning the validity of the continuum assumption. The mean-free-path for ion-neutral collision is defined as

\[ \lambda_{mfp} = \frac{1}{n_n Q_{in}} \]  

where $n_n$ is the neutral number density and $Q_{in}$ is the ion-neutral scattering cross section modeled from Banks’ formula [28],

\[ Q_{in} = \frac{8.28072 \times 10^{-16}}{c_r}, \quad [\text{m}^2] \]  

The denominator term $c_r = \sqrt{16kT_i/(\pi m)}$ is the relative thermal velocity between singly-charged ions and background neutrals and is defined with reduced mass. The neutral number density can be approximated by a Maxwellian-averaged flux equation for neutral flow,
\[ \dot{m} = m_n \left( \frac{n_n \bar{c}_n}{4} \cdot A_{\text{grid}} \phi_n \right) \]  

(4)

with \( \bar{c}_n = \sqrt{\frac{8kT_e}{\pi m_i}} \) being the mean thermal velocity of neutrals, \( A_{\text{grid}} \) the grid area and \( \phi_n \) the effective grid transparency to the neutrals. The grid transparency to neutrals is difficult to define because there is a slight confinement from the geometry of the grid aperture. However, a first-order estimation can be reached for a two-grid system,

\[ \phi_n = \frac{1}{(1/\phi_{\text{screen}}) + (1/\phi_{\text{accel}}) - 1} \]  

(5)

where \( \phi_{\text{screen}} \) and \( \phi_{\text{accel}} \) represent the physical opening of screen grid and accelerator grid respectively. Finally, by assuming thermal equilibrium among ions, neutrals and the wall (cold plasma approximation), the Knudsen number can be calculated. Figure 8 shows the Knudsen number estimated for BRFIT-7 thruster in its operational flow rate range. Neutral pressure, calculated based on ideal gas law, is also plotted. Figure 8 suggests that Knudsen number is small enough for the fluid assumption of ions to be valid.

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**Figure 8**  Knudsen Number Estimated for Busek BRFIT-7 Ion Engine
2.2 Basic Assumptions

The requirement for a self-sustaining plasma is that sufficient ionization must take place to compensate for wall losses. Though time-dependency is considered, this code does not model transient effects and must be started with sufficient amount of plasma density to overcome the immediate ion fluxes to the walls. The code also considers the case of pure ICP discharge and cannot handle capacitively-coupled plasma (CCP) discharges. Models with such combining effect, also known as E-H mode transition, can be found in Ref. 29. Other mathematical and physical assumptions are enumerated below for clarity. They are generally accepted for modeling inductively-coupled plasmas with heavy noble gas.

1) In cylindrical axis-symmetry, all quantities are functions of coordinates $r$ and $z$ (except for a globally-defined value), but independent of $\theta$. Symmetry condition also requires $\partial/\partial \theta = 0$ throughout the computational domain and $\partial/\partial r = 0$ at the axis of symmetry.

2) The plasma contains electrons, singly-charged ions and neutrals in quantities $n_e$, $n_i$, and $n_n$. Quasi-neutrality is imposed throughout the plasma so $n_i = n_e$. Due to low ionization percentage in an RF ion engine, neutrals are assumed as stationary background. Neutral density $n_n$ is therefore a global variable, but time-dependent. Charge-exchange between ions and neutrals is not accounted for.

3) The electrons are assumed to be in thermal equilibrium at a volume averaged value $T_e$. The ions are near equilibrium with the neutrals at a significantly lower temperature such that $T_i \sim T_n \ll T_e$. Ions and neutral particles are then given a temperature close to the wall temperature, estimated as 450 K from experimental observation. The effect of wall temperatures between 400 and 600 K was explored and the changes in simulation results are miniscule.

4) Because electrons possess higher thermal velocity than ions, they rush out of the plasma toward the walls and leave behind excess of positive charge behind. The resulting positive charge density gives rise to an electrostatic field that repels the electrons and accelerates the ions [23]. This is referred to as pre-sheath region where space-charge field links together the trajectory of electrons and ions such that ambipolar diffusion can be assumed. In the code, ambipolar diffusion is carried out to the boundary, where the ions reach sonic speed and the Bohm criterion is satisfied. Ions that reach this boundary are eventually returned to the
plasma as neutrals for mimicking wall recombination. The plasma sheath, which is a thin layer in front of the walls where the ion density is much greater than the electron density, is not resolved.

5) Azimuthal ion current is negligible as the induced plasma current is mainly contributed by the fast-moving electrons. The direction of the induced electric field is alternating at such high frequency (order of MHz) that the heavy ions do not have sufficient time to react to it. The azimuthally induced electron current is assumed dominant over meridional-plane (r-z plane) electron diffusion currents. This assumption is based on the ICP working mechanism. As such, meridional-plane electron currents do not contribute to magnetic field generation and coupling.

6) The helical coil is approximated as separate rings of concentrated current source. This allows completion of the 2D axisymmetric assumption. The validity of this claim is examined in the next section. Electrical resistance of the coil is a required input for the code for estimating ohmic loss and is assumed constant regardless of the operating condition.

7) The plasma discharge is treated as a "black box" in the sense that the influence of the extraction system on the source plasma is neglected. The model however employs a known ion transparency function of the grids for a specific operating condition. The ion transparency is used to calculate the extractable ion beam current given the ion flux arriving to the screen grid. This is only meant for benchmarking the code against a measurable quantity and is not intended for broader purpose. The physics of ion optics and the computation of ion trajectories can be found in Ref. 30

2.3 The Spatial 2D Assumption

For the model to consider only 2D in space, rather than 3D, it requires the assumption of an axisymmetric RF coil. Under such assumption, the coil generates electric field purely in the azimuthal (θ) direction and magnetic field purely in the radial and axial (r and z) directions. The plasma current is subsequently induced only in the θ direction. By neglecting magnetic contributions from the diffusion current in the meridional plane, the plasma-induced magnetic field becomes significant only in r and z directions. The spatial 2D assumption then holds true for magnetic field coupling between the plasma and the coil.
Such assumption is generally adequate because the cylindrical discharge chamber found in an RF ion engine is usually short axially for the purpose of minimizing wall loss, which renders a short coil with its winding diameter being longer than its axial length. This configuration naturally suggests axis-symmetry from the geometrical point of view. In fact, over the years many studies on ICP discharge have adopted 2D axisymmetric approach for various coil geometries [31].

The spatial 2D assumption was verified experimentally with the use of Busek BRFIT-7 thruster. A specially-made axisymmetric coil (Figure 9) was tested and compared with a typical helical coil (Figure 10). Thrust results estimated from the anode current, shown in Figure 11 and Figure 12, display ~5% performance difference between the two types of coil. However, the experimental error was deemed on the order of 5%, contributed by both flow measurement error and slight inconsistency of coil location that could affect the self-inductance of the coil. If taking the experimental error into account, one can conclude that there is no significant difference between helical and axisymmetric coil. The spatial 2D assumption is therefore considered adequate.
Figure 11  BRFIT-7 Performance Comparison Between Helical and Axisymmetric Coils with 3 sccm Xenon Flow

Figure 12  BRFIT-7 Performance Comparison Between Helical and Axisymmetric Coils with 4 sccm Xenon Flow
2.3 Computation Domain

2.4.1 Mesh

The main plasma code takes a computational fluid dynamics (CFD) approach with the addition of Maxwell's equations. A finite-difference algorithm with 2nd order accuracy differencing scheme is implemented on a structured and orthogonal mesh. The geometry of the discharge chamber and the location of the RF coil resemble the configuration found in the Busek BRFIT-7 thruster. A very simplified, not-to-scale cross sectional view of BRFIT-7 is shown in Figure 13. The computation mesh, shown in Figure 14, represents the upper half of the discharge chamber and extends some distance away into the vacuum background. The inner mesh (discharge zone) employs 441 nodes and the outer mesh employs 2921 nodes.

The outer mesh is required because of the boundary condition for magnetic field. The magnetic field solution needs to be terminated away from the discharge. In this case, the outer boundary of the outer mesh represents zero magnetic field. The absence of a computational mesh downstream of the discharge chamber is related to the shorting of magnetic field on the grid surface. Because the grids are very conductive (order of $10^7$ Si/m), the skin depth for MHz-frequency electromagnetic wave penetration is on the order of 0.1 mm, which for BRFIT-7 and BRFIT-3 is less than 10% of the total grid thickness. It is therefore safe to assume that RF field is absent downstream of the grids and the plume is non-magnetized.

![Conceptual Arrange of BRFIT-7 RF Ion Engine](image-url)
2.4.1 Time Step

Because of the quasi-neutrality assumption, the plasma frequency does not need to be resolved. The time step is determined with the general computational fluid dynamics (CFD) stability criterion,

\[ \Delta t \leq \frac{1}{2} \left( \frac{\Delta x}{\alpha} \right)^2 \quad \text{and} \quad \omega \Delta t \ll 1 \quad (6) \]

where \( \Delta x \) is the smallest mesh spacing, \( \alpha \) is a general diffusivity, and \( \omega \) is the angular frequency of quantity variation (in this case the RF frequency). Since the magnetic field coupling between the plasma and the coil needs to be resolved in the time-domain and requires the smallest time scale, \( \alpha \) is taken to be the magnetic field diffusivity \( D_m \),

\[ \alpha = D_m = \frac{1}{\mu_0 \sigma} \quad (7) \]
where \( \mu_0 \) is the permeability of free space and \( \sigma \) is the plasma conductivity. From post-examining simulation results \( \sigma \) is found to be 500-1000 Si/m under various operating conditions of BRFIT-7. Given the thruster geometry and the computational mesh used, the maximum allowable \( \Delta t \) is approximately \( 10^{-9} \) seconds. In the code, \( \Delta t \) is found by prescribing the number of points that each wave period is resolved,

\[
\Delta t = \frac{1/f}{N_{\text{resolution}}}
\]

where \( f \) is the driving frequency in hertz and \( N_{\text{resolution}} \) is the number of points discretizing the RF wave in time. The default \( N_{\text{resolution}} \) is 2000, rendering a \( \Delta t \) on the order of \( 10^{-10} \) seconds.
Chapter 3

THEORY

The theory section is divided into four major parts. Section 3.1 presents a transformer model that is used to calculate the RF coil current. Section 3.2 is the main ICP discharge model, which includes the physics of inductive coupling, plasma current generation, formulation of plasma conductivity and a fluid model that addresses the diffusion of charged particles. Following the discharge model is a description of the numerical method in Section 3.3. This section details the computation of the coil-induced magnetic field and contains important boundary conditions and the finite-difference CFD scheme. Lastly, a simple ion optics model is presented for the purpose of benchmarking the code with measured experimental data. The ion optics model is case-specific and is not intended for general use.

3.1 Transformer Model

Using a transformer analogy to calculate circuit properties of an RF plasma is not new. Many people have developed similar models as presented in Ref. 15, 24 and 32. While such a zero-th order model is typically used to estimate the power dissipated by the plasma, it is employed in the code only for finding the coil current. This model is needed because prescribing the external loop current or loop potential for an ICP discharge simulation is not an intuitive method. Often these quantities are not known to the user and therefore do not serve well as inputs for the discharge simulation.

The transformer model used in the code is based on previous work described in Ref. 32. The basic idea is to model the whole RF circuit as a transformer with the plasma being a one-turn secondary coil. Figure 15 illustrates the premise. The circuit representation shown in Figure 15 is known as a parallel matching circuit since the capacitive element is placed in parallel to the plasma load. After all equivalent circuit parameters are defined by the transformer model, total load impedance is calculated and with it the coil current is found.
Referring to the nomenclature shown in Figure 15, the geometric self-inductance of the coil of a short-solenoid is expressed as [32]

\[ L_e = 0.002\pi(D_w \times 100)\left(N^2\ln\left(\frac{4D_w}{L_e}\right) - \frac{1}{2}\right) \times 10^{-6}, \]  \[ \text{[H]} \]  \[ (9) \]

where \( D_w \) is the winding diameter of the coil (typically the outer diameter of the discharge chamber) in meters, \( l_e \) is the effective axial length of the coil in meters, and \( N \) is the number of coil turns. The plasma inductance is expressed similarly with the addition of an inertial inductance term that accounts for the phase lag between the RF electric field and the RF induction current [24,34],

\[ L_p = 0.002\pi(D_p \times 100)\left[\ln\left(\frac{4D_p}{L}\right) - \frac{1}{2}\right] \times 10^{-6} + \left(\frac{R_p}{v_{eff}}\right), \]  \[ \text{[H]} \]  \[ (10) \]

where \( D_p \) is the plasma “winding diameter” in meters, \( L \) is the discharge chamber length in meters, \( R_p \) is the plasma resistance and \( v_{eff} \) is the effective collision frequency of electrons. The winding diameter of plasma is a peculiar term as plasma has no physical shape. The code treats this term as 2/3 of the chamber’s inner diameter and it seems to match experiment data for most cases. The effective collision frequency used here is a spatially averaged term taken from the main discharge model. The plasma resistance is also an averaged value with inputs from the discharge model. It takes the form of
\[ R_p = \frac{2\pi R}{\sigma L \delta}, \quad \text{[Ohms]} \quad (11) \]

where \( R \) is the radius of the discharge in meters, \( \sigma \) is the averaged plasma conductivity in Si/m, and \( \delta = \sqrt{2/(\omega \mu_0 \sigma)} \) is a globally defined skin depth. Finding the equivalent inductance and resistance on the primary circuit for the plasma requires the knowledge of a mutual inductance. From the transformer theory regarding a solenoid with a coaxial coil, mutual inductance is written with parameters presented above [35],

\[ L_m = 0.0095N \frac{(D_p \times 100)^2}{\sqrt{(D_w \times 100)^2 + (l_c \times 100)^2}}, \quad \text{[H]} \quad (12) \]

and the transformed plasma inductance and resistance become

\[ L_2 = \frac{-\omega^2 L_m^2 L_p}{R_p^2 + (\omega L_p)^2}, \quad \text{[H]} \quad (13) \]

\[ R_2 = \frac{\omega^2 L_m^2 R_p}{R_p^2 + (\omega L_p)^2}, \quad \text{[Ohms]} \quad (14) \]

After all the elements on the equivalent circuit are defined, total load impedance is simply

\[ Z = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{1}{j \omega C} \left( R_c + R_2 + j \omega L_c + j \omega L_2 \right) \]

\[ \left( \frac{1}{j \omega C} \right) + \left( R_c + R_2 + j \omega L_c + j \omega L_2 \right) \]

\[ (15) \]

The total load impedance can be used to characterize the quality of the matching circuit using a parameter known as “percentage of power reflection,”

\[ PR = \frac{|Z - Z_0|^2}{|Z + Z_0|} \times 100\% \quad (16) \]

where \( Z_0 \) is the source impedance typically 50 \( \Omega \). A low PR value is desirable because it indicates high power coupling efficiency between the source and the plasma. The power coupling efficiency formally is defined as
Power Couple Efficiency = 1 – Power Reflection % \quad (17)

The load impedance parameter from Eq. 15, more importantly, is used to calculate the peak current drawn from the source,

\[ I_{\text{peak}} = \sqrt{\frac{2Z_0P_{\text{forward}}}{Z_0}} \left(1 - \frac{Z - Z_0}{Z + Z_0}\right) \quad (18) \]

which leads to the RMS coil current using current divider law,

\[ I_{\text{coil,RMS}} = \left| I_{\text{peak}} \left(\frac{Z_1}{Z_1 + Z_2}\right)\right|/\sqrt{2} \quad (19) \]

Note that the coil current is actually treated as a sine wave in the main discharge code. The RMS coil current found here is utilized to change the amplitude of that sinusoidal function at each time step, which subsequently excites the discharge. This method takes into account the interaction among plasma, RF coil and power source without having to solve the backward EM wave induced on the primary circuit by the plasma.

3.2 ICP Discharge Model

The discharge model is presented first with solutions to Maxwell’s equations, followed by formulations of plasma parameters that are unique for ICP discharge and the fluid equations describing charged particle diffusion. Diffusion of plasma is considered up to the sheath edge with ions reaching sonic speed at the computational boundary. The plasma sheath is not resolved by this code. The discharge model concludes with equations regarding the neutral particle conservation and the electron power balance within the discharge. Xenon plasma is considered and its cross section data are default in the code.

3.2.1 Magnetic Field Coupling and Plasma Current Induction

Magnetic field coupling is a natural phenomenon in ICP discharge that occurs when the coil-induced magnetic field is distorted from the induction of plasma current. Physics of this coupling effect can be realized by solving Maxwell’s equations of electromagnetism. Eq.
20-23 show the full set of Maxwell’s equations assuming the magneto-quasi-static (MQS) condition which neglects the displacement current. This is generally valid for low MHz frequency.

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{20}
\]

\[
\nabla \times \frac{\vec{B}}{\mu_0} = \vec{j} \tag{21}
\]

\[
\nabla \cdot \vec{B} = 0 \tag{22}
\]

\[
\nabla \cdot \vec{E} = \frac{\rho_{\text{en}}}{\varepsilon_0} \tag{23}
\]

Because quasi-neutral plasma is assumed, the electric potential is solved from fluid equations rather than from charge distribution and Eq. 23 is no longer needed. In the model, the solution for the magnetic field is represented by the magnetic vector potential \( \vec{A} \); it is defined as,

\[
\vec{B} = \nabla \times \vec{A} \quad \text{and} \quad \nabla \cdot \vec{A} = 0 \tag{24}
\]

From the definition of \( \vec{A} \), Eq. 22 is automatically satisfied. As explained in Section 2.2 regarding the spatial 2D assumption, the magnetic fields considered here are only \( B_r \) and \( B_z \), which means that only the \( \theta \) component of the vector potential exists. Thus, the Coulomb gauge condition (\( \nabla \cdot \vec{A} = 0 \)) is also automatically satisfied,

\[
\nabla \cdot \vec{A} = \frac{\partial A_r}{\partial r} + \frac{A_r}{r} + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} = 0 \tag{25}
\]

because the first, second and fourth term drops out, and the \( \partial / \partial \theta \) term is zero due to axis-symmetry. Since \( \nabla \cdot \vec{A} = 0 \) is valid, the following vector property holds true,

\[
\nabla \times \nabla \times \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = -\nabla^2 \vec{A} \tag{26}
\]
Faraday’s law represented by Eq. 20 can be written with the use of $\vec{A}$,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \vec{A} = -\nabla \times \frac{\partial \vec{A}}{\partial t} \Rightarrow \nabla \times \left( \vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0 \quad (27)$$

from which a scalar electric potential $\phi$ can be defined,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi \Rightarrow \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t} \quad (28)$$

By combining Eq. 21 and the derived properties of $\vec{A}$ field, the equation governing the induced plasma current with respect to the magnetic vector potential is reached,

$$\nabla \times \frac{\vec{B}}{\mu_0} = \vec{j} \Rightarrow \nabla \times \nabla \times \vec{A} = \mu_0 \vec{j} \Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{j} \quad (29)$$

Because the induced ion current is neglected (Section 2.2), the current density term in Eq. 29 is purely an electron current and is modeled as

$$\vec{j} = \vec{j}_e = \sigma \left( \vec{E} + \frac{\nabla P_e}{e n_e} \right) \quad (30)$$

where $\nabla P_e$ is the electron pressure gradient representing body force and $\sigma$ is the plasma conductivity that will be derived in the next section. The electron pressure gradient is expressed with the assumption of ideal gas and constant electron temperature, so that

$$\nabla P_e \approx k T_e \nabla n_e \quad (31)$$

Now substituting Eq. 30 and Eq. 28 into Eq. 29, the following equation is obtained,

$$\nabla^2 \vec{A} = -\mu_0 \left[ \sigma \left( \vec{E} + \frac{\nabla P_e}{e n_e} \right) \right] = \mu_0 \sigma \left( \nabla \phi + \frac{\partial \vec{A}}{\partial t} - \frac{\nabla P_e}{e n_e} \right) \quad (32)$$

From the spatial 2D assumption, the $A$ field retains only the $\theta$ component. Eq. 32 therefore considers solely the $\theta$ component of the partial derivatives. The gradient terms regarding
electric potential $\phi$ and electron pressure $P_e$ are immediately dropped because of axis-symmetry. The only terms left are

$$\nabla^2 A_\theta - \frac{A_\theta}{r^2} = \mu_0 \sigma \left( \frac{\partial A_\theta}{\partial t} \right) \tag{33}$$

Since both the coil and the plasma contribute to the induced $A_\theta$, it is considered a superposition of these two contributions

$$A_\theta = (A_\theta)_{co} + (A_\theta)_{pl} \tag{34}$$

where the subscripts “co” and “pl” denote coil and plasma respectively. Substituting the above expression into Eq. 33 and rearranging, Eq. 35 is obtained as following,

$$\nabla^2 (A_\theta)_{pl} - \frac{(A_\theta)_{pl}}{r^2} = \left[ \nabla^2 (A_\theta)_{co} - \frac{(A_\theta)_{co}}{r^2} \right] + \mu_0 \sigma \frac{\partial (A_\theta)_{pl}}{\partial t} + \mu_0 \sigma \frac{\partial (A_\theta)_{co}}{\partial t} \tag{35}$$

The large bracket on the right-hand-side of Eq. 35 refers to the vector Laplacian of the coil-induced $A$ field in the $\theta$ direction, and its value should be zero. This can be verified by taking an example of time-invariant coil current that generates a $(A_\theta)_{co}$ field but $\partial (A_\theta)_{co}/\partial t = 0$ (a DC field). Because there is no electromagnetic wave, no plasma current is induced and subsequent no $(A_\theta)_{pl}$ would exist. Eq. 35 in this example would have zeros all across except for the large bracket, which must equal to zero as the result. With this realization, Eq. 35 is reduced to the final form for governing magnetic field diffusion,

$$\nabla^2 (A_\theta)_{pl} - \frac{(A_\theta)_{pl}}{r^2} = \mu_0 \sigma \left[ \frac{\partial (A_\theta)_{pl}}{\partial t} + \frac{\partial (A_\theta)_{co}}{\partial t} \right] \tag{36}$$

Eq. 36 is an important result because it shows that although the coil field and the plasma field are physically coupled, the coil field can be treated as a decoupled source term in this diffusion equation. By pre-computing the coil-induced source field $(A_\theta)_{co}$ in the background with respect to space and time, $(A_\theta)_{pl}$ field can be computed everywhere at each time step. More
importantly, once the total field $A_\theta$ is found, the azimuthally-induced plasma current can be calculated from the combined form of Eq. 33 and Eq. 29,

$$\nabla^2 A_\theta - \frac{A_\theta}{r^2} = \mu_0 \sigma \left( \frac{\partial A_\theta}{\partial t} \right) = -\mu_0 j_\theta \quad \Rightarrow \quad j_\theta = -\sigma \left( \frac{\partial A_\theta}{\partial t} \right) \quad (37)$$

The importance of this Eq. 37 is twofold. First, it describes the induction of azimuthal plasma current in the presence of a propagating magnetic field. This plasma current dissipates power into the plasma through local ohmic heating, which is the working mechanism of ICP discharge. The azimuthal plasma current is also required to compute the ionization and excitation rate because electrons are modeled by a shifted Maxwellian energy distribution function with drift energy in the azimuthal direction.

### 3.2.2 Plasma Conductivity

Plasma conductivity in the discharge model is considered a local quantity and is derived with a classical approach. Consider a non-magnetized uniform plasma in the presence of a background gas that is driven by a time-varying electric field with frequency $\omega$,

$$\bar{E}_x(t) = \text{Re}(E_x e^{i \omega t}) \quad (38)$$

where $E_x$ is the electric field amplitude. Motion of ions is neglected by assuming an infinite mass. Take a 1D approach and assume the electrons accelerate along the electric field with impedance from collisionality. The equation of motion for electrons is

$$m \frac{d\bar{u}_x}{dt} = -e\bar{E}_x - m v_{\text{eff}} \bar{u}_x \quad (39)$$

where $v_{\text{elastic}}$ is the scattering frequency that contains electron-neutral and electron-ion elastic collision. Letting the electron velocity oscillate with the same frequency,

$$\bar{u}_x(t) = \text{Re}(u_x e^{i \omega t}) \quad (40)$$

and using the expression along with Eq. 38 and Eq. 39, the velocity is obtained with a complex amplitude,
\[ u_x j \omega = -\frac{e}{m} E_x - v_{\text{elastic}} u_x \Rightarrow u_x = -\frac{e}{m(v_{\text{elastic}} + j\omega)} E_x \quad (41) \]

Since the plasma current, under the absence of ion contribution, is written as

\[ j_e = -e u_x n_e \quad (42) \]

Eq. 41 can be substituted in to obtain

\[ j_e = \frac{e^2 n_e}{m(v_{\text{elastic}} + j\omega)} E_x \quad (43) \]

The conductivity term can therefore be defined in the complex sense as

\[ \sigma = \text{Re} \left[ \frac{e^2 n_e}{m(v_{\text{elastic}} + j\omega)} \right] \quad \text{with} \quad v_{\text{elastic}} = v_{\text{el}} + v_{\text{en}} = \bar{c}_e n_i Q_{ei} + \bar{c}_e n_n Q_{en} \quad (44) \]

where \( v_{\text{el}} \) is the Coulomb collision frequency, \( v_{\text{en}} \) is the electron-neutral scattering frequency, and the mean velocity is the thermal velocity of electrons \( \bar{c}_e = \sqrt{8kT_e/(\pi m_e)} \). An effective collision frequency for electrons can be defined for Eq. 44 in complex notation,

\[ v_{\text{eff}} = v_{\text{elastic}} + j\omega \quad (45) \]

The Coulomb collision in low-pressure ICP discharge is not as significant because ionization fraction is typically less than 5% and electrons are less likely to interact with ions than neutrals. Nevertheless the Coulomb term is included in the model. The Coulomb cross section is modeled numerically [36],

\[ Q_{ei} = \frac{4\pi}{9} \frac{e^4 \ln \Lambda}{(4\pi e_0 T_e)^2} = \frac{2.87 \times 10^{-14} \ln \Lambda}{(T_e, eV)} \quad [\text{cm}^2] \quad (46) \]

with the Coulomb logarithm approximated as
\[ \ln \Lambda = \ln \left[ \frac{3}{2\sqrt{\pi}} \frac{(4\pi e_0 T_e)^{3/2}}{e^3 n_e^{3/2}} \right] = 13.57 + 1.5 \log(T_e,eV) - 0.5 \log(n_e,cm^{-3}) \] (47)

The energy-averaged electron-neutral scattering cross section \( Q_{en} \) is discussed in the next section.

### 3.2.3 Energy-Averaged Cross Sections

The cross sections for electron-neutral scattering, single ionization and excitation are averaged in the energy space with a special electron energy distribution function (EEDF) that considers a drift energy \( E_D \) in the direction of electric field [37],

\[
f_D(T_e, E_D, E) = \frac{1}{2\sqrt{\pi T_e E_D}} e^{-\frac{(\sqrt{E-E_D})^2}{kT_e}} \left( 1 - e^{-\frac{4\sqrt{E-E_D}}{kT_e}} \right) \] (48)

The special EEDF, graphically illustrated in Figure 16 for the case of 5 eV electron temperature, is derived from the zero-th spherical harmonic of a shifted Maxwellian EEDF in the velocity space,

\[
f_D(T_e, v_D, \tilde{w}) = \left( \frac{m_e}{2\pi kT_e} \right)^{\frac{3}{2}} e^{-\frac{m_e \left( w^2 + w_v^2 + (w_0 - v_D)^2 \right)}{2kT_e}} \] (49)

where \( \tilde{w} \) is the velocity-space vector and \( v_D \) is the drift velocity in the azimuthal direction. This derivation can be found in Appendix A. The method of decomposing a non-symmetric distribution function (Eq. 49) into its zero-th spherical harmonic is valid here because cross sections are always spherically symmetric and proportional to only the zero-th harmonic. Thus for averaging purposes all the higher harmonics of the distribution function cancel out as they are defined to be orthogonal to each other.

The cross section data were obtained from Szabo’s collection and curve fits shown in Ref. 38. They are presented in Figure 17. Tabulated form of ionization and excitation cross sections can be found in Appendix B and Appendix C. First ionization energy and first excitation energy for xenon are 12.1 eV and 8.32 eV respectively. The data of excitation cross section used here does not differentiate levels of excitation and lumped-sum estimation is used.
Drifted Maxwellian Energy Distribution Function, $T_e = 5\text{eV}$

Figure 16   Illustration of the Special EEDF with Drift Energy, $T_e = 5\text{eV}$

Figure 17   Cross Section Data for Xenon, Collected by Szabo [38]
The cross sections are averaged in the energy space with the special EEDF (Eq. 48),

$$
\bar{Q}(T_e, E_D) = \frac{\int c \cdot f_D(T_e, E_D, E) \cdot Q(E) \cdot dE}{\bar{c}_e}
$$

(50)

where \( \bar{c}_e \) is approximated by the electron mean thermal velocity and \( Q \) is a type of cross section. Notice that if \( Q \) represents ionization or excitation, \( \bar{Q} \) is actually the rate coefficient normalized by the electron mean thermal velocity. This normalization is for visual illustration only and does not affect the averaging results because in the code the same mean thermal velocity is multiplied back to obtain the originally computed rate coefficients. Since the EEDF depends both on the drift energy and electron temperature, no analytical expression can be found for Eq. 50 and numerical integration is required. The code pre-computes and stores these values in a 3D form with respect to \( T_e, E_D, \) and averaged cross sections. The numerical integration takes the form

$$
\bar{Q}(T_e, E_D) = \frac{1}{\bar{c}_e} \sum_{j=1}^{n} f(E_j, T_e, E_D) Q(E) \sqrt{\frac{2E_j}{m_e}} \Delta E
$$

(51)

where the energy is averaged up to 100 eV. Results of the numerical integration are shown in Figure 18 to Figure 20. The range of drift energy considered here is sufficient for low-pressure ICP discharge. In fact, post-examining the simulation results reveals that the drift energy does not exceed 2 eV.
Figure 18  Energy-Averaged Electron-Neutral Scattering Cross Section for Xenon

Figure 19  Energy-Averaged Ionization Cross Section (Ionization Rate Coefficient Normalized by Mean Thermal Velocity) for Xenon
Figure 20 Energy-Averaged Excitation Cross Section (Excitation Rate Coefficient Normalized by Mean Thermal Velocity) for Xenon

### 3.2.4 Fluid Equations

The governing equations for plasma diffusion in the meridional plane are derived from a two-fluid isentropic Euler-Poisson system [39]. Magnetic force on the electron is also considered in the momentum equation:

\[
m_{n_i} \left( \frac{\partial \vec{v}_{i}}{\partial t} + \vec{v}_{i} \cdot \nabla \vec{v}_{i} \right) + \nabla P_{i} = en_{i} \vec{E} - m_{i} n_{i} \nu_{is} \vec{v}_{i}
\]  \hspace{1cm} (52)

\[
m_{e} n_{e} \left( \frac{\partial \vec{v}_{e}}{\partial t} + \vec{v}_{e} \cdot \nabla \vec{v}_{e} \right) + \nabla P_{e} = -en_{e} \vec{E} - en_{e} \vec{v}_{e,\theta} \times \vec{B} - m_{e} n_{e} \nu_{\text{eff}} \vec{v}_{e}
\]  \hspace{1cm} (53)

\[
\frac{\partial n_{i}}{\partial t} + \nabla \cdot (n_{i} \vec{v}_{i}) = \dot{n}_{i}
\]  \hspace{1cm} (54)

\[
\frac{\partial n_{e}}{\partial t} + \nabla \cdot (n_{e} \vec{v}_{e}) = \dot{n}_{e}
\]  \hspace{1cm} (55)

where \( \dot{n} \) is the rate of ionization in terms of particle density. The effective elastic collision frequency for electrons \( \nu_{\text{eff}} \) is previously defined in Eq. 36 and the ion-neutral scattering
frequency $\nu_n$ can be calculated from the cross section model presented in Eq. 3. Recombination is treated by a global neutral particle balance, which is discussed in the next section.

The electron inertial terms are neglected in Eq. 53 to result in a drift-diffusion approximation for the electron flux,

$$\Gamma_e = n_e \vec{v}_e = -D_e \nabla n_e - \mu_e n_e \left[ \nabla \phi + \frac{\partial \vec{A}}{\partial t} - \vec{v}_e \times \vec{B} \right]$$  \hspace{1cm} (56)$$

where $D_e = \frac{kT_e}{m_e v_{eff}}$ is the electron diffusivity and $\mu_e = \frac{-eD_e}{kT_e}$ is the electron mobility. The $\vec{A}$ field actually does not have $r$ and $z$ components because of the axis-symmetric argument so it does not contribute to the electron flux in the meridional plane.

A quasi-neutral condition is introduced to the fluid equations so the computation domain does not need to resolve the Debye length and the plasma frequency. The quasi-neutrality constraint $n_i = n_e$ can be expressed by taking the difference of the continuity equations (54) and (55) and leads to the divergence-free constraint of current,

$$\nabla \cdot \vec{j} = \nabla \cdot (n_e \vec{v}_e - n_e \vec{v}_e) = 0$$ \hspace{1cm} (57)

Since $\partial / \partial \theta = 0$, Eq. 57 also implies that divergence of current in the meridional plane is zero. Eq. 57 is used to find the electric potential in the absence of Poisson’s equation. This is done by substituting in Eq. 56 and rearranging,

$$\nabla \cdot \left( n_e \vec{v}_e - \frac{en_e}{m_e v_{eff}} \nabla \phi + \frac{n_e \nabla P_e}{m_e n_e v_{eff}} + \frac{en_e}{m_e v_{eff}} \left( \vec{v}_e \times \vec{B} \right) \right) = 0$$ \hspace{1cm} (58)

Multiply electron charge $e$ across and substitute in the expressions for plasma conductivity, electron pressure (from ideal gas law) and magnetic vector potential to obtain the governing equation of electric potential,

$$\nabla \cdot (\sigma \nabla \phi) = e \nabla \cdot (n_e \vec{v}_e) + \nabla \cdot \left( \frac{ekT_e}{m_e v_{eff}} \nabla n_e \right) + \nabla \cdot \left\{ \sigma \left[ \vec{v}_e \times (\nabla \times \vec{A}) \right] \right\}$$ \hspace{1cm} (59)
which is a Poisson-like equation and considers only the azimuthal components of $\mathbf{v}_e$ and $\mathbf{A}$.

The azimuthal electron velocity $v_{e,\theta}$ is defined by the induced azimuthal plasma current density,

$$j_\theta = -e \cdot v_{e,\theta} \cdot n_e$$  \hspace{1cm} (60)

Under quasi-neutrality the diffusion of plasma is ambipolar as long as the relative drift $\mathbf{v}_i - \mathbf{v}_e$ is small compared to the mean thermal velocity of electrons. This means the motion of electrons and ions is linked together by a space-charge field. The trajectory of ions is found by substituting the solution to electric potential (Eq. 59) into Eq. 52,

$$\frac{\partial \mathbf{v}_i}{\partial t} = -\mathbf{v}_i \cdot \nabla \mathbf{v}_i - \frac{kT_i}{m_i} \nabla n_e - \frac{e}{m_i} \nabla \phi - v_{in} \mathbf{v}_i$$  \hspace{1cm} (61)

where $\nabla \mathbf{v}_i$ describes the tensor derivative of ion velocity. In the code Eq. 61 is used to update the ion velocity at each time step. The change of plasma density is then tracked by the ion conservation equation shown in Eq. 54. The time derivative of ion velocity turns out to be quite important when modeling the discharge in time-domain. Since plasma is generated locally and ionization rate has time-dependency, gradual ion diffusion must be allowed so the plasma can shape into a steady-state density distribution when the balance between ionization and diffusion is reached.

### 3.2.5 Neutral Conservation

Because the neutrals are treated as a uniform background, the conservation of neutrals must be done in a global sense. The neutral density is updated in time by considering the difference between the rate of generation and the rate of loss,

$$\frac{\partial}{\partial t} (n_n V) = \frac{\dot{m}}{m_{n_{\text{inj}}}} + \iint_{\text{wall}} \Gamma_{\text{wall}} dA + \iint_{\text{grid}} (1 - \phi_i) \Gamma_{\text{grid}} dA - \phi_k A_{\text{grid}} \frac{n_{n_{\text{escaped}}}}{4} - \iiint_{\text{ionized}} n_{n_{\text{escaped}}} dV$$  \hspace{1cm} (62)

where $\dot{m}$ is the flow rate in kg/s, $\Gamma$ is the ion flux normal to the wall (or grid webbing), $\phi_i$ is the grid transparency to ions, $\phi_k$ is the grid transparency to neutrals, $\bar{c}$ is the average
thermal velocity of the neutrals, and $A_{\text{wall}}$ and $A_{\text{grid}}$ are the wall and grid areas. On the right-hand-side of Eq. 62, the first term describes flow injection rate into the chamber, the second term describes the neutrals being introduced back to the chamber after ions recombined at the wall, the third term describes the rate of neutrals escaping through grid apertures, and the fourth term describes the consumption of neutrals due to ionization.

### 3.2.6 Discharge Energy Balance

The electron temperature is taken to be a spatially-averaged value in the discharge model. The energy balance for electrons therefore must be considered on a global scale. The governing equation is formulated by tracking the averaged change of internal energy in the discharge. The change of internal energy is the difference between the power absorbed by the plasma and the energy expended within. Absorbed power, or power dissipated into the discharge, is contributed mainly by the induced current $j_{e,\theta}$ through ohmic heating. Power dissipation by the meridional-plane electron current is ignored. Energy expenditure within the discharge consists of ionization, excitation, and kinetic energy loss in the sheath and to the walls. The energy balance model is presented in Eq. 63. Previous investigation concluded that the energy loss due to Coulomb collision is negligible and it is therefore not included.

$$\frac{\partial}{\partial t} \left( \frac{3}{2} kT_e \iint n_e dV \right) = \iint j_{e,\theta}^2 \sigma dV - \iint \dot{n}_i eV_i dV - \iint \dot{n}_{\text{exc}} eV_{\text{exc}} dV$$

$$- \iint (2kT_e + e\phi_{\text{sheath}}) n_{\text{wall}} dA$$

(63)

In Eq. 63, $j_{e,\theta}$ is the azimuthal electron current, $\Gamma_{\text{wall}}$ is the electron flux normal to the walls, $\dot{n}_i$ is the ionization rate with $V_i = 12.1$ V being the first ionization energy and $\dot{n}_{\text{exc}}$ is the excitation rate with $V_{\text{exc}} = 8.32$ V being the total excitation energy. The ionization rate and excitation rate are computed from the rate coefficients discussed in Section 3.2.3. Specifically,

$$\dot{n}_e = n_e \dot{n}_e \cdot (\overline{c_e Q_{\text{ionization}}})$$

(64)
The last integral of Eq. 63 represents energy loss of electrons due to the presence of a confined area. On average, each electron surrenders $2kT_e$ of energy to the walls due to its high thermal velocity [40], but before reaching the walls electrons have already lost energy by the repelling sheath potential. The drop of potential across the sheath can be found by equating the ion and electron fluxes to the walls,

$$
\frac{n_{e,\text{sheath EDGE}} U_B}{n_{e,\text{sheath EDGE}} U_B} = \frac{n_{e,\text{sheath EDGE}} e c}{4 \exp \left( \frac{e \Delta \phi}{kT_e} \right)}
$$

The normal ion flux is represented by the sonic velocity $u_B = \sqrt{kT_e/m_i}$, which occurs at the sheath edge. This satisfies the Bohm criterion of sheath formation. According to Eq. 66 the sheath potential is only a function of electron temperature,

$$
(\Delta \phi)_{\text{sheath}} = -\frac{kT_e}{e} \ln \left( \sqrt{\frac{m_i}{2\pi m_e}} \right)
$$

For xenon, this potential drop is approximately 5.27 $T_e$.

### 3.2.7 Coil-Induced Magnetic Field

The theory of magnetic field diffusion in the ICP discharge relies on the knowledge of a coil-induced magnetic field as shown in Eq. 36. It is possible, and desirable, to pre-compute such coil field as it is not mathematically coupled to the solution of the plasma-induced magnetic field. Knowing the coil field at every mesh node and at every time step allows the field to serve as a “driving source” for Eq. 36.

The coil-induced magnetic field is highly non-uniform as the coil is typically short and the edge effect is strong. To properly compute the induced field, spatial integration of the Biot-Savart law is required. Referring to the representation shown in Figure 21, the Biot-Savart law states

$$
\dot{n}_{e,\text{exc}} = n_a n_e \cdot (\overline{c_e Q_{\text{excitation}}})
$$

The parentheses in Eq. 64 and 65 represent the rate coefficients directly computed from the energy integrals and not normalized by the electron thermal velocity (Section 3.2.3).
that the differential magnetic field vector at any given point in space is related to the source geometry and magnitude of the driving current, as well as the distance from such source,

\[ \frac{d\vec{B}(P)}{dP} = \frac{\mu_0 I d\vec{I} \times \vec{e}_r}{4\pi \frac{R^2}{R}} \quad \vec{e}_r = \frac{\vec{R}}{R} \]  

(68)

where \( I \) is the amplitude of the loop current. Considering a single-turn, axisymmetric coil, the coil-induced magnetic field \( B_r \) and \( B_z \) at any given point \( P(r,z) \) can be found by integrating Eq. 68 through the loop,

\[ B_r = -\frac{\mu_0 I a}{2\pi} \frac{\partial}{\partial z} \int_{-\pi/2}^{\pi/2} \sin \theta d\theta \]  

(69)

\[ B_z = -\frac{\mu_0 I a}{2\pi} \frac{\partial}{\partial a} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{(z^2 + a^2 + r^2 - 2ar \sin \theta)^{1/2}} \]  

(70)

From the definition of vector potential, \( (A_\theta)_{co} \) is then found,

\[ (A_\theta)_{co} = -\frac{\mu_0 I a}{2\pi \sqrt{z^2 + (r + a)^2}} \left[ -\frac{z^2 + r^2 + a^2}{ar} \frac{K(m)}{ar} + \frac{z^2 + (r + a)^2}{ar} E(m) \right] \]  

(71)

where

\[ m = \frac{4ar}{z^2 + (r + a)^2} \]  

(72)

\[ K(m) = \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - m \sin^2 \phi}} \]  

(73)

\[ E(m) = \int_{0}^{\pi/2} \sqrt{1 - m \sin^2 \phi} d\phi \]  

(74)

The functions \( K \) and \( E \) are known as the complete elliptic integral of first and second kind, respectively. Detailed calculation of the induced magnetic field and the corresponding vector potential can be found in Appendix A.

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In the discharge code, a subroutine is devoted to calculating the magnetic field vector potential induced by multiple single-turn coils with the use of superposition and offsetting origins. This subroutine is written to automatically handle any number of coil turns and user-specified coil and chamber geometry. Because the magnitude of coil current \( I \) is time-dependent and can change with respect to plasma properties, a “baseline” \((A_\theta)_{co}\) field is pre-computed using a sinusoidal, 1-Amp RMS current for a single wave period. As the main iteration progresses, this baseline \((A_\theta)_{co}\) field is extracted from memory and augmented by the value of the RMS coil current computed by the transformer model at each time step. This generates a real-time background source field for computing the magnetic field diffusion. Figure 22 illustrates this baseline \((A_\theta)_{co}\) field with the geometry of the BRFIT-7 thruster. It shows the field when the current reaches its positive peak (1/4 wave period for a sine function).
Figure 22  Baseline Coil-Induced Magnetic Field $(A_\theta)_{co}$ with BRFIT-7 Geometry; Coil Current is at Positive Peak (1/4 Wave)

3.3 Numerical Method

This section first discusses critical boundary conditions for the computation. It then follows by a discretization method for solving the differential equations.

3.3.1 Boundary Conditions for Magnetic Field

The governing equation for the magnetic vector potential described by Eq. 36 is valid only within the discharge region. To establish magnetic boundary conditions, the plasma-induced field $(A_\theta)_{pl}$ must be solved continuously outside the discharge chamber. In the free-space region $(A_\theta)_{pl}$ is not affected by the coil-induced field and propagates/decays without distortion. The governing equation in free space is similar to Eq. 36, but without the forcing terms,

$$\nabla^2 (A_\theta)_{pl} - \frac{(A_\theta)_{pl}}{r^2} = 0$$  \hspace{1cm} (75)

The boundary condition $(A_\theta)_{pl} = -(A_\theta)_{co}$ (zero total field) is approximately valid on the outermost boundaries. Eq. 36 is also displayed here in the form that was used for solution,
Ideally Eq. 75 and Eq. 76 need to be solved simultaneous as they describe the same quantity \( (A_0)_{pl} \) in time. This is difficult to achieve as it is most convenient to solve Eq. 76 by time-advancing while Eq. 75 requires an exact solution at each time step. The remedy is a scheme that attempts to generates a piece-wise continuous function across the discharge boundary. The basis of this scheme is the known absence of concentrated induced current layers on the dielectric boundary.

Figure 23 shows how this scheme works. Eq. 75, using node #3 as the boundary condition, has an exact solution on the outer mesh at time \( t-1 \). A slope “m” calculated by node #2 and #3 is found at time \( t-1 \). At time \( t \), the inner mesh solution is updated by time-advancing Eq. 76, but the boundary node (node #3) is left untouched. The new value of node #3 comes from extrapolating node #4 with a slope that is “m” (computed in time \( t-1 \)) multiplied by a correction factor. The new node #3 is then used as the boundary condition for finding the exact outer mesh solution at time \( t \). For coarse meshes, large numerical instability will result if the outer boundary slope “m” is forced on the first inner mesh segment because “m” is very steep if mesh is coarse. The slope correction factor mentioned above accounts for such mesh coarseness and it should approach to unity when the mesh across the discharge boundary is infinitely small. Determining the correction factor of a mesh takes a little trial and error, but once selected it does not need to be changed. The correction factor used in BRFIT-7 simulations is 0.2.
The magnetic boundary condition on the discharge/grid interface is derived from the assumption that the grid is a perfect conductor. This is valid as the conductivity for metallic grids is typically in the $10^7 \text{ Si/m}$ regime while the plasma conductivity is on the order of $10^3 \text{ Si/m}$. The perfect conductor condition requires the $B$-field normal to the grid ($B_z$) to be zero. This translates to the radial derivative of the “total” magnetic vector potential $A_\theta$ must be zero,

$$\frac{1}{r} \left[ \frac{\partial (r A_\theta)}{\partial r} \right] = 0$$ (77)

In addition, using infinite conductivity in Eq. 76 will result in a solution of $A_\theta$ with no time-dependency,

$$\frac{\partial (A_\theta)_{pl}}{\partial t} = -\frac{\partial (A_\theta)_{co}}{\partial t} \Rightarrow \frac{\partial A_\theta}{\partial t} = \frac{\partial (A_\theta)_{pl}}{\partial t} + \frac{\partial (A_\theta)_{co}}{\partial t} = 0$$ (78)

thus, the natural boundary condition of $A_\theta$ on the discharge/grid boundary is zero, and $(A_\theta)_{pl} = -(A_\theta)_{co}$ is used on this boundary when Eq. 76 is marched in time. The absence of $A_\theta$ field downstream of the grids can be argued from the concept of skin depth. Because the conductivity of grids is on the order of $10^7 \text{ Si/m}$ and the driving frequency is in the MHz range, the skin depth for electromagnetic wave penetration is on the order of 0.1 mm, which for BRFIT-7 and BRFIT-3 is less than 10% of the total grid thickness. This implies that the RF field downstream of the grids is very weak and can be ignored.

### 3.3.2 Bohm Condition

One difficulty associated with any kind of plasma simulation is to resolve the pre-sheath and sheath near the walls. In the code, the sheath is not resolved, and while the pre-sheath is in principle resolved, it steepens to a very large slope as the sheath is approached, and the resolution is in practice poor near the wall. Because of this, the pre-sheath has to be resolved artificially unless an extremely fine mesh is utilized in that region. Rather than relying on ion-acoustic waves to properly shape the potential near the sheath edge so that ions can accelerate to the Bohm velocity, the pre-sheath sonic condition is imposed at the discharge boundary and the Bohm criterion for sheath formation is forced [41],

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There are two ways the plasma density at the boundary can be updated. One is to rely on the
time-advancing feature of the ion conservation equation (Eq. 54) without the ionization term.
The other is to force flux conservation normal to the boundary as described by Ref. 41,

\[ n_{e,boundary} = \frac{\Gamma_{\text{normal}}}{u_{Bohm}} \]  

Both methods are adequate. However, utilizing Eq. 54 to update the boundary density takes
slightly longer time to converge. On the other hand, forcing the normal flux conservation is a
faster method but can cause numerical instability at low gas pressures as it would tend to “pull”
the plasma toward the wall at a faster rate. The code employs Eq. 80 as default for updating
plasma density at the boundary. This is to take advantage of its speedier convergence.

Based on the sonic condition at the presheath edge, zero electric potential is set at the numerical
discharge boundary that represents the sheath edge [23]. This corresponds to the wall being
charged negatively with respect to the plasma, which is a physical phenomenon.

3.3.3 Numerical Solution

A five-point finite-differencing scheme (Figure 24) is used throughout the discharge model for
discretizing the differential fluid equations. This approach is simple and typical of CFD
simulation on a structured orthogonal mesh. The code was not optimized for computing speed
as this was not within the scope of this research. Elliptic differential equations after
discretization are solved by MATLAB’s direct solver via Gaussian Elimination. Due to the
relative coarseness of the mesh, the direct solver actually performs pretty fast and no iterative
solution method is required. A successive over-relaxation iterative method will need to be
used if the mesh is further refined. The discretized equations shown in this section use
subscript \([S,W,C,E,N]\) to represent positions in space and superscript \([t, t-1]\) for positions in
time.
Solution to the magnetic field coupling and diffusion in the ICP discharge is found by expanding and discretizing Eq. 76,

\[
\frac{\partial (A_\theta)_{pl}}{\partial t} = \frac{1}{\mu_0 \sigma} \left[ \frac{\partial^2 (A_\theta)_{pl}}{dr^2} + \frac{\partial^2 (A_\theta)_{pl}}{dz^2} + \frac{1}{r} \frac{\partial (A_\theta)_{pl}}{dr} - \frac{(A_\theta)_{pl}}{r^2} \right] - \frac{\partial (A_\theta)_{co}}{\partial t}
\]  

(81)

The discretization begins in the time-domain with the use of time-step \( \Delta t \),

\[
\langle (A_\theta)_{pl} \rangle' = \langle (A_\theta)_{pl} \rangle^{t-1} + \frac{\Delta t}{\mu_0 (\sigma)^{i}} \left\{ \frac{\partial^2 \langle (A_\theta)_{pl} \rangle^{t-1}}{dr^2} + \frac{\partial^2 \langle (A_\theta)_{pl} \rangle^{t-1}}{dz^2} + \frac{1}{r} \frac{\partial \langle (A_\theta)_{pl} \rangle^{t-1}}{dr} \right\} - \left\{ \langle (A_\theta)_{co} \rangle' - \langle (A_\theta)_{co} \rangle^{t-1} \right\}
\]  

(82)

and is followed by spatial-discretization,
\[
\begin{aligned}
\langle (A_0)_{\rho l} \rangle_{C}^{t} &= \langle (A_0)_{\rho l} \rangle_{C}^{t-1} + \frac{\Delta t}{\mu_0 \langle \sigma \rangle_C} \left[ \langle (A_0)_{\rho l} \rangle_{N}^{t-1} - 2 \langle (A_0)_{\rho l} \rangle_{C}^{t-1} + \langle (A_0)_{\rho l} \rangle_{S}^{t-1} \right] \\
+ \langle (A_0)_{\rho l} \rangle_{E}^{t-1} - 2 \langle (A_0)_{\rho l} \rangle_{C}^{t-1} + \langle (A_0)_{\rho l} \rangle_{W}^{t-1} + \frac{1}{r_c^2} \left[ \langle (A_0)_{\rho l} \rangle_{N}^{t-1} - \langle (A_0)_{\rho l} \rangle_{C}^{t-1} \right] - \frac{1}{r_c^2} \langle (A_0)_{\rho l} \rangle_{E}^{t-1} \\
- \langle (A_0)_{\rho l} \rangle_{C}^{t-1} - \langle (A_0)_{\rho l} \rangle_{C}^{t-1} 
\end{aligned}
\] (83)

Rearrange to Eq. 83 and collect like-terms in space to get

\[
C_1 \langle (A_0)_{\rho l} \rangle_{S}^{t-1} + C_2 \langle (A_0)_{\rho l} \rangle_{W}^{t-1} + C_3 \langle (A_0)_{\rho l} \rangle_{E}^{t-1} + C_4 \langle (A_0)_{\rho l} \rangle_{N}^{t-1} + C_5 \langle (A_0)_{\rho l} \rangle_{C}^{t-1} - C_6 = \langle (A_0)_{\rho l} \rangle_{C}^{t} 
\] (84)

with

\[
C_1 = \frac{\Delta t}{\mu_0 \langle \sigma \rangle_C^t} \left[ \frac{1}{(\Delta r)^2} - \frac{1}{(2\Delta r)r_C} \right]
\]

\[
C_2 = \frac{\Delta t}{\mu_0 \langle \sigma \rangle_C^t} \left[ \frac{1}{(\Delta z)^2} \right]
\]

\[
C_3 = \left\{1 + \frac{\Delta t}{\mu_0 \langle \sigma \rangle_C^t} \left[ \frac{-2}{(\Delta r)^2} - \frac{2}{(\Delta z)^2} \frac{1}{r_C^2} \right]\right\}
\]

\[
C_4 = \frac{\Delta t}{\mu_0 \langle \sigma \rangle_C^t} \left[ \frac{1}{(\Delta z)^2} \right]
\]

\[
C_5 = \frac{\Delta t}{\mu_0 \langle \sigma \rangle_C^t} \left[ \frac{1}{(\Delta r)^2} + \frac{1}{(2\Delta r)r_C} \right]
\]

\[
C_6 = \left\{\langle (A_0)_{\rho l} \rangle_{C}^{t-1} - \langle (A_0)_{\rho l} \rangle_{C}^{t-1}\right\}
\]

In matrix operations with a \( N \times N \) mesh set up in the discharge zone, Eq. 84 is represented as...
\[ \vec{A}' = Q \vec{A}'^{-1} - \vec{C}_6 \quad (85) \]

where \( Q \) is a \( N^2 \times N^2 \) sparse matrix containing the coefficients \( C_i \) to \( C_5 \), and \( \vec{A}', \vec{A}'^{-1} \) and \( \vec{C}_6 \) are \( N^2 \times 1 \) column vectors. The \( \vec{C}_6 \) vector represents the coil-induced background field discussed previously.

The solution scheme for \( (A_0)_{pl} \) in free space is a little different as it cannot be advanced in time. Based on Eq. 75, the discretization becomes

\[
\left\langle (A_0)_{pl} \right\rangle_N - 2\left\langle (A_0)_{pl} \right\rangle_C + \left\langle (A_0)_{pl} \right\rangle_S + \left\langle (A_0)_{pl} \right\rangle_E - 2\left\langle (A_0)_{pl} \right\rangle_C + \left\langle (A_0)_{pl} \right\rangle_W
+ \frac{1}{r_c} \left[ \left\langle (A_0)_{pl} \right\rangle_N - \left\langle (A_0)_{pl} \right\rangle_S \right] - \frac{1}{r_c} \left\langle (A_0)_{pl} \right\rangle_C = 0
\quad (86) \]

which can be reduced to matrix operation \( Q \vec{A} = \vec{f} \) with \( \vec{f} \) being a null vector and \( Q \) being the sparse matrix containing the coefficients \( C_1 \) to \( C_5 \),

\[
C_1 = \frac{1}{(\Delta r)^2} - \frac{1}{(2\Delta r)r_c} \\
C_2 = \frac{1}{(\Delta z)^2} \\
C_3 = \frac{-2}{(\Delta r)^2} - \frac{2}{(\Delta z)^2} - \frac{1}{r_c^2} \\
C_4 = \frac{1}{(\Delta z)^2} \\
C_5 = \frac{1}{(\Delta r)^2} + \frac{1}{(2\Delta r)r_c}
\]

60
Numerical solution scheme to the electric potential equation (Eq. 59) is similar to the above. Discretization on the left-hand-side involves treating the conductivity as a local parameter,

\[ LHS = \nabla \cdot (\sigma \nabla \phi) \]

\[ = \left[ \frac{\sigma_N - \sigma_S}{2\Delta r} + \frac{\sigma_C}{r_c} \right] \left[ \frac{\phi_N - \phi_S}{2\Delta r} \right] + \left[ \frac{\sigma_E - \sigma_W}{2\Delta z} \right] \left[ \frac{\phi_E - \phi_W}{2\Delta z} \right] + \sigma_C \left( \frac{\phi_N - 2\phi_C + \phi_S}{\Delta r^2} \right) + \sigma_C \left( \frac{\phi_E - 2\phi_C + \phi_W}{\Delta z^2} \right) \]

and the matrix coefficients become

\[ LHS \equiv C_1 \phi_e + C_2 \phi_w + C_3 \phi_C + C_4 \phi_E + C_5 \phi_N \]

\[ C_1 = -\frac{1}{2\Delta r} \left[ \frac{\sigma_N - \sigma_S}{2\Delta r} + \frac{\sigma_C}{r_c} \right] + \frac{\sigma_C}{\Delta r^2} \]

\[ C_2 = -\frac{1}{2\Delta z} \left[ \frac{\sigma_E - \sigma_W}{2\Delta z} \right] + \frac{\sigma_C}{\Delta z^2} \]

\[ C_3 = \frac{2\sigma_C}{(\Delta r)^2} - \frac{2\sigma_C}{(\Delta z)^2} \]

\[ C_4 = \frac{1}{2\Delta z} \left[ \frac{\sigma_E - \sigma_W}{2\Delta z} \right] + \frac{\sigma_C}{\Delta z^2} \]

\[ C_5 = \frac{1}{2\Delta r} \left[ \frac{\sigma_N - \sigma_S}{2\Delta r} + \frac{\sigma_C}{r_c} \right] + \frac{\sigma_C}{\Delta r^2} \]

The right-hand-side of Eq. 59 is discretized in a similar fashion and the resultant values serve as the forcing term in the matrix equation.
3.4 Ion Optics

The ICP discharge code does not have a built-in ion optics simulation. Rather, a grid ion transparency function published in the literature is implemented for the purpose of verifying the code with experimental data. Because both the BRFIT-7 and the BRFIT-3 thrusters utilize a similar grid design as the NEXT ion engine developed by NASA, simulation results for the NEXT ion optics can be adopted. Farnell [30] has published such a simulation with Colorado State University’s three-dimensional charge-exchange ion optics code named ffx. Details of the ffx code can be found in Ref. 30. One set of the simulation parameters closely resembles the experimental condition of BRFIT-7 and BRFIT-3. These parameters are listed in Table 1. The effective grid transparency to ions as calculated by the ffx code under these conditions is shown in Figure 25. Figure 25 also contains the results of other ion optics code (igx, CEX2D, and CEX3D), but they are not considered in this thesis because the ffx code has demonstrated very good agreement with the experimental data of NASA’s NSTAR ion engine [30]. Since the ion optics of the NEXT thruster is very similar to that of the NSTAR thruster, the ffx code is considered reliable in this instance.

The discharge code incorporates a curve-fit function of the ffx simulation result. The anode current (approximately equal to the ion beam current) is then found by multiplying the ion flux reaching the grid with the grid ion transparency and integrating over the grid area,

\[ J_{anode} = e \int_{grid} (n_{e,boundary} \cdot u_{Bohm} \cdot \phi_{ion}) dA \]  

(88)

where \( n_{e,boundary} \) is the plasma density at the discharge boundary in front of the grid, \( u_{Bohm} \) is the sonic velocity of ions, and \( \phi_{ion} \) is the ion transparency function found in Figure 25.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Parameters for Grid Ion Transparency Simulation [30]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Screen Grid Voltage</td>
<td>1800 V</td>
</tr>
<tr>
<td>Accelerator Grid Voltage</td>
<td>-210 V</td>
</tr>
<tr>
<td>Beam Plasma Potential</td>
<td>10 V</td>
</tr>
<tr>
<td>Source Electron Temperature</td>
<td>6 eV</td>
</tr>
<tr>
<td>Grid Spacing-to-Screen Aperture Diameter Ratio</td>
<td>0.347</td>
</tr>
<tr>
<td>Physical Open Fraction of Screen Grid</td>
<td>67%</td>
</tr>
</tbody>
</table>
Figure 25  Ion Transparency Simulation Published by Farnell [30]
Chapter 4
RESULTS

Results of the ICP discharge code are presented in this chapter. The transformer sub-model is first shown being able to predict the operating frequency and the circuit capacitance required to maximize RF power coupling efficiency. This part of the design work can be done without executing the main discharge simulation, but does require a guess on the averaged plasma conductivity. Since plasma conductivity is a function of operating condition, the discharge simulation becomes useful in verifying the optimum driving frequency.

BRFIT-7 experiment data were used to compare with the simulation results. This benchmark verification is detailed in Section 4.2. Scaling-capability of the code was demonstrated by data comparison with the smaller BRFIT-3 thruster. This result is presented in Section 4.3. Section 4.4 is an optimization study that aims to improve the performance of BRFIT-3 with the use of the discharge simulation. In this chapter, proprietary data of BRFIT-7 and BRFIT-3 are intentionally obscured; these include their operating frequencies and circuit capacitance values. The discharge geometries depicted by the simulation results also do not represent the exact physical dimensions of these two thrusters.

A flowchart of the code can be found in Figure 26. It illustrates how the computation progresses at each time-step, although some parameters do not need to be solved in the specified order. Computation of the baseline background coil-induced magnetic field must be carried out every time when there is a change in wave resolution (number of points discretizing the wave period) or a change in geometry (i.e. coil location, number of turns, size of the discharge chamber, etc.). This baseline background field is generated with 1A RMS sinusoidal coil current and is independent of operating frequency. Frequency is resolved from the definition of time-step (Eq. 8). The real-time coil-induced magnetic field is found within each time-step by accessing the pre-computed baseline data and augmenting it with the coil current obtained from the transformer model.
4.1 Matching Circuit Design

A variation of the transformer model employed by the ICP discharge code can be executed independently to aid the matching circuit design. The transfer model by itself does not require RF power as input because its main function is to compute impedance of circuit elements. The impedance due to plasma loading can be simulated by prescribing an averaged plasma conductivity. As long as proper coil and chamber geometries are defined, the transformer model can optimize the power coupling efficiency of the matching circuit by sweeping through frequency and parallel capacitance value. Again, the power coupling efficiency is defined as 

\[ (1 - \text{power reflection %}) \]

Figure 27 shows the circuit design for the BRFIT-7 thruster,
assuming a plasma conductivity of 500 Si/m. The minimum power reflection, or the maximum power coupling efficiency, occurs between 1.2 and 1.3 MHz frequency with corresponding parallel capacitance between 32,000 and 38,000 pico-farad. These values closely resemble to the BRFIT-7 circuit, design which was empirically optimized. The catch of this circuit design method is that plasma conductivity must be estimated correctly. For example, Figure 28 shows the same design approach, but using a higher conductivity of 1000 Si/m. The optimum frequency is pushed up to the 1.9-2.1 MHz range and the optimum capacitance drops below 27,000 pf. Experimentally BRFIT-7 cannot operate in this regime efficiently, so the 1000 Si/m is definitely an over-estimate. This analysis shows that while the transformer model by itself can approximate the desirable operating frequency and its corresponding circuit capacitance, true optimization must be performed by the discharge simulation as the true plasma conductivity is not known a priori. In practice the capacitance and the frequency are designed for a nominal operating condition that matches to a certain plasma conductivity. As discharge conditions change, one can either vary the frequency or accept the penalty in coupling efficiency. Varying capacitance is also feasible, but the bulkiness of a variable capacitor is not practical for flight systems.

![Figure 27 Matching Circuit Design for BRFIT-7 with the Transformer Model; Plasma Conductivity is Assumed 500 Si/m](image-url)
4.2 Performance of ICP Discharge Code

The performance of the discharge code is characterized by three categories: convergence, validity, and scaling ability. The code convergence is described in Section 4.2.1. Section 4.2.2 validates the code with the experimental data of BRFIT-7. Lastly, Section 4.2.3 verifies the scaling capability of the code by data comparison with BRFIT-3.

4.2.1 Convergence

The convergence of a time-dependent RF plasma simulation is difficult to define due to the nature of high-frequency oscillation. For the ICP discharge code, convergence is defined by observing steady-state (or cyclic steady-state) outputs for all the following quantities: 1) electron temperature, 2) plasma dissipation power calculated from \( \int \left( j_o^2 / \sigma \right) dV \), 3) plasma internal power losses due to ionization, excitation and ion wall loss, 4) circuit ohmic heating loss calculated from RMS coil current and circuit resistance, and 5) anode current. The dissipated power ideally must equal the sum of the three internal power losses. And the sum
of the dissipated power and the circuit ohmic heating loss must equal the RF load power (forward power \times power coupling efficiency). These two equalities have proven to be difficult, if not impossible, to preserve as all the power terms are calculated independently and the RF forward power input basically prescribes only a voltage source. The equality requirement of the two statements is therefore relaxed to allow slight convergence error.

A test case using the geometry of BRFIT-7 thruster is presented here. The forward RF power is 52.2 W and the flow rate is 4 sccm Xe. The RF load power is 52.0 W according to the experimental data, measured by the forward RF power multiplying with (1-power reflection %). Figure 29 to Figure 33 show the convergence plots for the various quantities of interest. The frequency of oscillations seen in Figure 29 to Figure 31 corresponds to the operating frequency. In BRFIT-7 simulation the RF wave is discretized by 1000 points. The code convergence is thought to occur at around 12,000 iterations (12 wave periods), which corresponds to a physical time of approximately 9.5 µsec. Table 2 lists the converged values, which are averaged through the final wave period.

Convergence errors are inevitable as discussed in the previous paragraphs. For this test case the sum of internal power losses is 31.11 W while the dissipated power is 30.57 W. The 1.8% difference is small enough that convergence can be claimed. In addition, the sum of dissipated power and circuit ohmic heating loss is 51.88 W, very close to the RF load power of 52.0 W.

The code is stable in general except for cases of very low pressure and very high RF power. Both conditions contribute to large initial imbalance between the ionization rate and the diffusion rate, causing solutions to diverge rapidly. Although the initial imbalance can be partially blamed on the mesh coarseness, this observation does resemble a true discharge phenomenon as under these extreme conditions the plasma cannot be sustained experimentally. The code may be used to predict the stability boundary of operations. But to do so, the computation mesh will need to be refined so the solutions are not subjected to extra numerical instability when the discharge is becoming physically unstable.
### Table 2  Converged Values from Discharge Simulation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_e$</td>
<td>electron temperature</td>
<td>3.2 eV</td>
</tr>
<tr>
<td>Dissipated Power</td>
<td>$\int (\frac{j_o^2}{\sigma}) dV$</td>
<td>30.57 W</td>
</tr>
<tr>
<td>Ionization Power</td>
<td>$\int \dot{n}_i eV_i dV$</td>
<td>6.31 W</td>
</tr>
<tr>
<td>Excitation Power</td>
<td>$\int n_{exc} eV_{exc} dV$</td>
<td>9.91 W</td>
</tr>
<tr>
<td>Wall Loss Power</td>
<td>$\int (2kT_e + e \phi_{sheath}) \Gamma_{wall} dA$</td>
<td>14.89 W</td>
</tr>
<tr>
<td>Circuit Ohmic Heating Loss</td>
<td>$J_{RMS,coil}^2 \cdot R_{circuit}$</td>
<td>21.31 W</td>
</tr>
<tr>
<td>Anode Current</td>
<td>$e \cdot \int (n_{e,boundary} \cdot u_{Bohm} \cdot \phi_{ion}) dA$</td>
<td>143.7 mA</td>
</tr>
</tbody>
</table>

![Figure 29](image.png)

**Figure 29  Convergence of Electron Temperature in the BRFIT-7 Simulation; Each Iteration Represents $7.7 \times 10^{-10}$ Seconds in Real Time**
Figure 30  Convergence of Dissipated Power in the BRFIT-7 Simulation

Figure 31  Convergence of Various Power Loss Terms in the BRFIT-7 Simulation
Figure 32  Convergence of Circuit Ohmic Heating Loss

Figure 33  Convergence of Anode Current in the BRFIT-7 Simulation
4.2.2 BRFIT-7 Simulation

Referring to the same test case as shown in Section 4.2.1, the ICP discharge in the BRFIT-7 thruster is graphically illustrated in this section. First the magnetic field diffusion and the induction of plasma current are examined. The total magnetic vector potential $A_\theta$ across the computational domain is shown in Figure 34 and Figure 35. The former represents the instance when the coil current reaches positive peak (1/4 wave period) and the latter represents the case of negative peak current (3/4 wave). Both figures demonstrate that the solution scheme for $A_\theta$, especially on the discharge/free space boundary, is adequate. This is one achievement by itself as preserving the piece-wise continuity of $A_\theta$ across such boundary is a very difficult problem. Such discussion can be found in Section 3.2.2. The distortion of $A_\theta$ inside the discharge also suggests the presence of a conductive medium (in this case the plasma). The degree of distortion can be examined by comparing Figure 34 with Figure 22, which is the coil-induced magnetic field without plasma presence. In fact, if the plasma is replaced by a perfect conductor, $A_\theta$ will not penetrate into this region at all as the current is induced purely on the surface of the conductor. This is known as a strong skin effect.

![Figure 34 Total Magnetic Vector Potential at 1/4 Wave](image-url)
Closely related to the diffusion of the magnetic field is the induction of the azimuthal plasma current. Just like inside an AC motor, the induced current flows in the opposite direction of the exciting current source in the attempt to cancel out the mutually-induced magnetic field. The induced plasma current at 1/4, 1/2 and 3/4 wave periods are shown in Figure 36, Figure 37, and Figure 38 respectively. The azimuthal plasma current density seen in Figure 36 is purely negative, which is the correct direction since at 1/4 wave period the coil current is at the positive peak. Figure 37 shows that because of wave propagation, the induced current never reaches zero even when the coil current is instantaneously zero. This mechanism allows the plasma to sustain itself. Figure 38 is perhaps the most important figure here as it clearly illustrates the concept of skin depth. The "skin depth" term in AC circuit theory refers to the spatial decay of the induced current density within a conductor. The convention is the depth where the current density drops to $1/e \approx 0.37$ of its peak value at the boundary. In inductive plasma skin depth is a very important issue. If the skin layer is too large, RF power cannot couple to the plasma effectively. On the other hand, if the skin is too small, power is deposited very close to the walls and the plasma generated there is quickly lost. BRFIT-7 operates at a regime where skin depth is about $1/2$-$2/3$ of the chamber radius, as suggested by Figure 38.
This range seems to be optimum and can be used as a guideline for designing an efficient ICP discharge.

Figure 36  Induced Plasma Current at 1/4 Wave

Figure 37  Induced Plasma Current at 1/2 Wave
The skin depth can also be observed through the distribution of the dissipated power, which is found by the ohmic heating of the induced current density within a discretized volume, \((j_o^2/\sigma)V\). This is illustrated in Figure 39 to Figure 41. Through this mechanism the plasma receives the energy for supporting ionization. The spatial decay of the power dissipation is an extension to the skin depth description. From Figure 39 and Figure 41, a skin depth of 1/2 chamber radius can be concluded. It is also noted that maximum power deposition occurs every half wave period (1/4 and 3/4 wave).
Figure 39  Dissipated Power at 1/4 Wave

Figure 40  Dissipated Power at 1/2 Wave
The steady-state solution for the electric potential is plotted in Figure 42 and Figure 43. It is interesting to see the maximum potential does not occur at the axis of symmetry. This is because the potential is mainly correlated with the ionization rate, which is a function of the local plasma density and the induced plasma current. The plasma density has the lowest value at the discharge boundary due to diffusion while the induced current reaches its maximum at the same boundary. The resulting balance between the density and the induced current drives the ionization rate, and subsequently the potential, to the steady-state distribution shown in Figure 42 and Figure 43.

The plasma density at steady state is shown in Figure 44 and Figure 45. The density and potential exhibits resemblance to the Boltzmann relation, that is \( n_e \propto e^{\frac{e\phi}{kT}} \). This relationship does not hold true in the "corner" regions where the wall lost is extra intense as ions are forced to reach the sonic velocity in both the \( r \) and \( z \) directions.
Figure 42  Electric Potential Distribution

Figure 43  Electric Potential Distribution (Planar View)
Figure 44 Plasma Density Distribution

Figure 45 Plasma Density Distribution (Planar View)
Figure 46 is the axial (z-direction) velocity profile for ions. It illustrates how the ions accelerate from the center of discharge and reach Mach 1 at the boundaries. Note that since the boundary is imposed with the Bohm velocity $u_B = \sqrt{\frac{kT_e}{m_i}}$, it is subjected to periodic variation due to oscillations in electron temperature. The radial velocity distribution looks similar to the axial velocity profile.

![Figure 46 Ion Axial Velocity Profile](image)

The last output from the discharge simulation is the profile of ion beam current density at the immediate grid exit (Figure 47). The beam density is calculated with the ion transparency function discussed in Section 3.4. The profile shows a "hump" of ion flux between the center axis and the wall, which corresponds to the location of maximum potential and maximum ionization. This ion beam profile is in somewhat an agreement with experimental observation. Figure 48 displays an ion beam profile of BRFIT-7 measured by a Faraday probe placed 6 cm downstream of the accelerator grid. The hump of ion flux density is also observed. Although this is not a quantitative comparison, it does add validity to the discharge code. One thing to notice is that the experiment data does not have a sharp drop of current density toward the walls as predicted by the code. This might be explained by 1) the probe is not placed at the immediate grid exit, so ion beam divergence could contribute to the higher measurements near
the walls, and 2) the collector area of the Faraday probe is finite, which could average out the ion flux over a relatively large area.

![Simulated Ion Beam Current Density Profile](image)

**Figure 47**  Simulated Ion Beam Current Density Profile

![Ion Beam Flux Profile Measured by a Faraday Probe](image)

**Figure 48**  Ion Beam Flux Profile Measured by a Faraday Probe

Three additional experimental data points from BRFIT-7 were used for comparison. The results are presented in Table 3. The anode current data are plotted in Figure 49. Two observations can be made. First, the load power (plasma dissipated power + circuit ohmic heating loss) is correctly simulated. This has been a difficult task as there are many parameters involved in finding the load power. Getting a correct match on the power means all the sub-models regarding ionization, excitation, wall loss, and current induction are valid to
some extent. Relative errors however might exist, but there are no data on the internal losses that can be compared with. Second, the anode current prediction is pretty accurate (2% error or less) at high RF power, but falters at low power (20% error). The source of this discrepancy has not been found as for the BRFIT-3 case shown in the next section the opposite is true. The employed ion transparency function was deemed a non-factor in this issue.

### Table 3 Results Comparison for BRFIT-7 Simulation

<table>
<thead>
<tr>
<th>Flow, sccm</th>
<th>$P_{\text{forward}}, \text{W}$</th>
<th>$P_{\text{load}}, \text{W}$</th>
<th>$J_{\text{anode}}, \text{mA}$</th>
<th>$P_{\text{load}}, \text{W}$</th>
<th>$J_{\text{anode}}, \text{mA}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>42.4</td>
<td>42.2</td>
<td>109.4</td>
<td>42.6</td>
<td>131.0</td>
</tr>
<tr>
<td>4</td>
<td>52.2</td>
<td>52.0</td>
<td>133.0</td>
<td>51.9</td>
<td>143.7</td>
</tr>
<tr>
<td>4</td>
<td>62.6</td>
<td>62.4</td>
<td>155.8</td>
<td>61.6</td>
<td>156.9</td>
</tr>
<tr>
<td>4</td>
<td>67.8</td>
<td>67.6</td>
<td>166.4</td>
<td>66.3</td>
<td>163.3</td>
</tr>
</tbody>
</table>

**Figure 49  Anode Current Comparison for BRFIT-7**

### 4.2.3 BRFIT-3 Simulation

Based on good agreements with the experimental data from BRFIT-7, the discharge code is applied to BRFIT-3 to verify its scaling capability. The code is not altered in any way except for geometrical changes. The Knudsen number is first examined to ensure the fluid
assumption is valid. According to Figure 50, at the flow rate of interest (~1 sccm) the Knudsen number is approximately 0.33 and the ICP discharge code is considered loosely valid.

![Figure 50 Knudsen Number Estimated for BRFIT-3](image)

Four cases of simulation were performed for BRFIT-3, and the results are shown in Table 4 and Figure 51. Again, the load power is in good agreement with the experimental data, albeit slightly higher values this time. The simulated load power being slightly higher than the forward power is not a great concern because the forward power is essentially a voltage source and the power is not a constraint. Anode current simulation for BRFIT-3 is in much better agreement with experiment data than that for BRFIT-7. In fact, the largest error does not exceed 10%. The difference in anode current is seen larger at higher RF power, which is the opposite case compared to the BRFIT-7 results. Convergence and stability of the BRFIT-3 simulation are unchanged. Figure 52 to Figure 54 are examples of data convergence for the case of 54 W forward power and 1 sccm xenon. The RF wave is discretized in 2000 points for the simulation.

**Table 4 Results Comparison for BRFIT-3 Simulation**

<table>
<thead>
<tr>
<th>Flow, sccm</th>
<th>$P_{\text{forward}}$, W</th>
<th>$P_{\text{load}}$, W</th>
<th>$J_{\text{anode}}$, mA</th>
<th>$P_{\text{load}}$, W</th>
<th>$J_{\text{anode}}$, mA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>43.7</td>
<td>43.6</td>
<td>25.7</td>
<td>43.8</td>
<td>25.5</td>
</tr>
<tr>
<td>1</td>
<td>47</td>
<td>46.7</td>
<td>27.4</td>
<td>47.2</td>
<td>26.4</td>
</tr>
<tr>
<td>1</td>
<td>50.5</td>
<td>50.3</td>
<td>28.7</td>
<td>50.5</td>
<td>27.4</td>
</tr>
<tr>
<td>1</td>
<td>54</td>
<td>53.8</td>
<td>30.9</td>
<td>54.4</td>
<td>28.4</td>
</tr>
</tbody>
</table>

83
Figure 51  Anode Current Comparison for BRFIT-3

Figure 52  Convergence of Electron Temperature in the BRFIT-3 Simulation; Each Iteration Represents $2.8 \times 10^{-10}$ Seconds in Real Time
Figure 53  Convergence of Dissipated Power in the BRFIT-3 Simulation

Figure 54  Convergence of Anode Current in the BRFIT-3 Simulation
Simulations of the two Busek RF ion engines using the ICP discharge code offer a different look on how the total electric power is distributed. Table 5 and Figure 55 show such comparison. The ion beam power listed here is the actual beam power (ion beam current × net acceleration voltage) and the thrust efficiency is defined as the ratio of ion beam power to the total power. The ionization, excitation and wall loss turns out not as significant in causing the scaling penalty of BRFIT-3. Rather, circuit ohmic loss is the big issue as illustrated in Figure 55.

The higher circuit loss in BRFIT-3 is contributed by the higher resonating coil current, generated by the lower parallel capacitance given approximately the same load impedance. This relationship can be examined by Eq. 19. The low parallel capacitance corresponds to the higher matching frequency, which is needed for scaling down the skin depth to accommodate the reduced chamber dimension. As the result if the overall circuit resistance is not reduced, the circuit ohmic loss will only increase when scaling down the thruster. For BRFIT-3, thrust efficiency can easily be increased to 60% if the circuit loss is reduced by 50%. Wall loss is though to be a hidden limitation barring the development of micro RF ion engines; it is difficult to reduce and can dominate the loss mechanism if the thruster is further miniaturized.

<table>
<thead>
<tr>
<th></th>
<th>BRFIT-7: 52.2W, 4 sccm</th>
<th>BRFIT-3: 54W, 1sccm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ionization, W</td>
<td>6.3</td>
<td>4.5</td>
</tr>
<tr>
<td>Excitation, W</td>
<td>9.9</td>
<td>6.6</td>
</tr>
<tr>
<td>Wall Loss, W</td>
<td>14.9</td>
<td>10.7</td>
</tr>
<tr>
<td>Circuit Ohmic Loss, W</td>
<td>21.3</td>
<td>33.0</td>
</tr>
<tr>
<td>Ion Beam Power, W</td>
<td>239.4</td>
<td>55.6</td>
</tr>
<tr>
<td>Total Electric Power, W</td>
<td>291.8</td>
<td>110.3</td>
</tr>
<tr>
<td>Thrust Efficiency</td>
<td>82.0%</td>
<td>50.4%</td>
</tr>
</tbody>
</table>
4.3 Thruster Optimization

Optimization of RF ion engines has always been thought a difficult problem due to the combining effect of skin depth, operating frequency, and power coupling efficiency. While the power coupling efficiency can be optimized relatively easy via analytical model or empirical work, the frequency where the optimum power coupling occurs might not be optimum for the plasma discharge. Thus, if a discharge condition prefers a certain skin depth, the corresponding operating frequency may require slight tradeoff of power coupling.

An optimization study of BRFIT-3 is conducted. Thrust efficiency, defined as the ratio between the ion beam power and the total power (ion beam + RF forward power), is optimized here. BRFIT-3 was originally designed to achieve high power coupling efficiency and operates at around 1.8 MHz. The matching circuit design theory described in Section 4.1 agrees with such design frequency as illustrated in Figure 56. However, an ideal scaling theory published by Horst [42] suggests that frequency needs to be scaled inversely proportional to the chamber size and 2.7 MHz should instead be used for the BRFIT-3 geometry (Figure 57). The ICP discharge code is applied here to investigate such claim by sweeping through a range of operating frequency. For the various frequencies tested, the
parallel capacitance is scaled according to $\omega \propto \sqrt{1/C}$. The simulation results are shown in Figure 58.

**Figure 56** Matching Circuit Analysis for BRFIT-3

**Figure 57** Ideal Scaling for Optimizing Thrust Efficiency of an RF Ion Engine [42]
The optimum frequency of 2.5 MHz as suggested by Figure 58 maximizes the anode current, which in turn maximizes thrust efficiency given the same RF power. The 2.5MHz, however, is not where the power coupling efficiency is maximized. The rationale is that since a large portion of the RF power is lost to the ohmic heating load of the circuit, chasing after the power coupling efficiency does not necessarily guarantee the highest anode current. The plasma dissipated power instead should be the only important parameter. Figure 59 shows the plasma dissipated power actually increases with frequency despite the maximum power coupling efficiency occurring at 1.8 MHz. Figure 60 and Figure 61 illustrate the dissipated power for the 1.8 MHz and 2.5 MHz simulation cases, and the latter is clearly seen higher. The reason why the anode current (and thrust efficiency) does not increase without bound can simply be explained by the scaling of skin depth. As frequency keeps increasing, more and more power is deposited close to the wall, and at some point the ion wall loss will become significant. This can be correlated to the argument of the optimum skin depth that is about 1/2-2/3 of the chamber radius. Although this analysis suggests the maximum thrust efficiency does not necessarily occur at the same frequency that maximizes the power coupling efficiency, one should realize that deviating too far away from the matching frequency will result in large standing wave in the circuit and the plasma will not ignite as the result.

Figure 58  Thrust Efficiency Optimization of BRFIT-3
Figure 59  Plasma Dissipated Power as Function of Frequency

Figure 60  Distribution of Dissipated Power with 1.8 MHz
Figure 61  Distribution of Dissipated Power with 2.5 MHz
Chapter 5

CONCLUSIONS

5.1 Results Summary

An original two-dimensional simulation code for RF ion engine discharge is developed. The code models the interaction between the ICP discharge and the electromagnetic wave in the time-domain, resulting in both spatial and temporal resolution of the discharge. Major physical effects considered include magnetic field diffusion and coupling, plasma current induction and ambipolar plasma diffusion. The discharge is excited by RF coil current, which is calculated by a transformer model that relates time-varying plasma properties to changes of impedance on the primary circuit. As the result, users of the code only need to specify the forward RF power and circuit properties to carry out the discharge simulation.

The transformer model is shown being able to predict the optimum frequency and the corresponding parallel capacitance for maximizing the RF power coupling efficiency. This design work can be done independently without executing the main discharge code, but does require an educated guess on the plasma conductivity. A more systematic approach is to use such frequency prediction as a guideline and utilize the discharge code to find the true optimum frequency.

The discharge simulation is benchmarked with the experimental data from the Busek RF ion engine BRFIT-7. The prediction of load power is in very good agreement with the experiment. In addition, the power dissipated by the plasma is shown equal to the internal energy loss that includes ionization, excitation and ion wall loss. Optimum skin depth of 1/2-2/3 of chamber radius is suggested by the simulation. Data comparison of anode current shows very good results at high RF power with 2% error, but the results are not as good at lower operating power.

After validation with the BRFIT-7 thruster, the code is applied to the smaller BRFIT-3 thruster for examining scaling ability. Very good results were obtained in this case, with errors in anode current prediction ranging from 0 to 10%. The simulation also shows the main scaling
penalty for BRFIT-3 is the circuit ohmic heating loss, which if reduced by 50% can help the thruster achieve 60% thrust efficiency.

An optimization of thrust efficiency is conducted for BRFIT-3 with respect to the operating frequency. The discharge code is employed to test the tradeoff between the power coupling efficiency and the anode current. The results suggest the maximum thrust efficiency does not necessarily occur at the same frequency that maximizes the power coupling efficiency, as the ohmic heating loss in the circuit and the plasma skin depth both need to be considered. The predicted optimum frequency is 2.5 MHz for maximizing the thrust efficiency, similar to the value obtained from a published ideal scaling theory.

5.2 Contributions

The main contribution of this thesis work is the systematic approach to design an RF ion engine. The simulation code serves a tool package that unifies the designs of matching circuit and ICP discharge. The code is user-friendly in terms of user specified inputs and fast convergence. The theory of magnetic field coupling and diffusion using a background coil-induced field as the source term is also considered innovative. The two-dimensional solution of RF ion engine discharge in time-domain is another innovation that was not found in literature research.

5.3 Recommendations for Future Work

Future work on this project includes the implementation of a self-consistent ion optics code. The ion transparency function used in the model is very grid and voltage-specific and is not intended for general purpose. Adaptation of a published ion optics code is possible, but most likely it will be easier to write the code from scratch using the same MATLAB platform.

The solution for magnetic field should be extended to the plume region. Currently the plume is assumed un-magnetized from the argument that the grids are conductive enough to shield the electromagnetic wave penetration. The plume is also assumed non-divergent so the cylindrical boundary between the plume and vacuum has zero magnetic field. Although these assumptions are valid theoretically, extending the computation mesh to the plume region will be a more rigorous approach. The grids will need to receive a fine mesh to resolve the RF wave penetration.
The scheme that generates a piece-wise continuous solution of magnetic field across the discharge/vacuum boundary should be revised. Though the current method works adequately, it is perhaps best not to rely on the “slope correction factor” which requires some initial trial-and-error every time when a new computation mesh is utilized. Other possible slope-matching schemes should be pursued. One suggestion is to reduce the plasma conductivity on the discharge boundary nodes to create a smooth transition function.

Further examination is needed to explain the error margins of the anode current simulation for BRFIT-7 and BRFIT-3. As illustrated previously, the BRFIT-7 simulation has a maximum 20% error when the RF power is low and the BRFIT-3 simulation has a maximum 10% error when the power is high. The trends are opposite and the errors are significant. More work is needed to iron out the discrepancies.

Lastly, it is desirable to rewrite the code in another platform that can easily increase the mesh resolution without having memory issue as with MATLAB. A finer mesh can increase the accuracy of computation. It can also potentially ease a lot of numerical problems encountered at low gas pressure and high RF power.
Appendix A

Derivation of EEDF in Energy Space

This section derives the special energy-space EEDF presented in Section 3.2.3. The EEDF is shown in Eq. 48 and again in Eq. A1,

\[ f_E = f_D(T_e, E_D, E) = \frac{1}{2\sqrt{\pi k T_e E_D}} e^{\frac{(\sqrt{E-E_D})}{\sqrt{kT_e}}} \left( 1 - e^{\frac{4\sqrt{E-E_D}}{kT_e}} \right) \quad (A1) \]

The first realization is that any velocity-space EEDF can be decomposed into spherical harmonics. In this case, only the zero-th harmonic is important from the argument presented in Section 3.2.3. The EEDF in the energy space is therefore derived from the zero-th spherical harmonic of the velocity-space EEDF using the following relationship,

\[ \int f_E \, dE = \iiint f_v \, dw \, dw_\theta \, dw_\phi \quad (A2) \]

where

\[ f_v = f_D(T_e, v_D, \vec{w}) = \left( \frac{m_e}{2\pi k T_e} \right)^{\frac{3}{2}} e^{-\frac{m_e}{2kT_e} \left[ v_\perp^2 + \frac{(w_\parallel - v_{\parallel 0})^2}{2kT_e} \right]} \quad (A3) \]

is the non-symmetric, shifted Maxwellian EEDF. The triple integral on the right-hand-side of Eq. A2 is the zero-th spherical harmonic because the distribution function is multiplied by unity (the angular factor for the zero-th harmonic) and integrates over the whole velocity sphere. Computation of the integral starts by transforming the cylindrical coordinates \((r, \theta, z)\) into some spherical coordinate \((w, \psi, \phi)\). The transformed velocity components are expressed as
\[ w_r = w \sin \psi \cos \phi \]
\[ w_z = -w \sin \psi \sin \phi \]
\[ w_\theta = w \cos \psi \quad (A4) \]

And the velocity volume element is

\[ dw_r dw_\theta dw_z = [2\pi (w \sin \psi)] \nu d\psi dw = 2\pi w^2 \sin \nu d\psi dw \quad (A5) \]

Also,

\[ w_z^2 + w_r^2 + (w_\theta - v_\theta)^2 = w^2 \sin^2 \psi \sin^2 \phi + w^2 \sin^2 \psi \cos^2 \phi + w^2 \cos^2 \psi + v_\theta^2 - 2w \cos \nu v_\theta \]
\[ = w^2 - 2v_\theta w \cos \psi + v_\theta^2 \quad (A6) \]

Using the new coordinate system, the triple integral is re-written as

\[ \iiint f_\nu(T_r, \psi, \nu) dw_r dw_\theta dw_z = \int_{\psi=0}^{\pi/2} \int_{\theta=0}^{\pi} \left( \frac{m_r}{2\pi k T_\nu} \right)^{3/2} e^{-\frac{m_r (v_r^2 + v_\theta^2)}{2k T_\nu}} 2\pi w^2 \sin \psi d\psi dw \quad (A7) \]

which becomes the following with a little arrangement,

\[ \iiint f_\nu(T_r, \psi, \nu) dw_r dw_\theta dw_z = \int_{\psi=0}^{\pi/2} \int_{\theta=0}^{\pi} \left( \frac{m_r}{2\pi k T_\nu} \right)^{3/2} e^{-\frac{m_r J_0^2}{2k T_\nu}} w^2 e^{2k T_\nu} dw \int_{\theta=0}^{\pi} e^{\frac{m_r \nu \cos \psi}{k T_\nu}} \sin \psi d\psi \quad (A8) \]
Now, let \( t = \frac{m_v \cos \psi}{kT_v} \) and the last integral turns into

\[
\int e^{\frac{m_v \cos \psi}{kT_v}} \sin \psi d\psi = -d \cos \psi = -d \left( \frac{kT_v}{m_v v_0 w} t \right)
\]

\[
\int e^{\frac{-m_v \cos \psi}{kT_v}} \sin \psi d\psi = -\frac{kT_v}{m_v v_0 w} \int e^{\frac{-m_v \cos \psi}{kT_v}} \left( \frac{e^{\frac{m_v \cos \psi}{kT_v}}}{kT_v} - e^{\frac{-m_v \cos \psi}{kT_v}} \right) d\psi.
\]

Substitute Eq. A9 into Eq. A8 into to get

\[
\int \int \int f_s(T_s, v_0, \tilde{w}) dv_0 dw_0 dw_v = \int 2\pi \left( \frac{m_v}{2\pi kT_v} \right)^\frac{3}{2} e^{\frac{-m_v \cos \psi}{2kT_v}} \sin \psi d\psi = \int 2\pi \left( \frac{m_v}{2\pi kT_v} \right)^\frac{3}{2} e^{\frac{-m_v \cos \psi}{2kT_v}} \sin \psi d\psi
\]

\[
\int \int \int f_s(T_s, v_0, \tilde{w}) dv_0 dw_0 dw_v = \int 2\pi \left( \frac{m_v}{2\pi kT_v} \right)^\frac{3}{2} e^{\frac{-m_v \cos \psi}{2kT_v}} \sin \psi d\psi
\]

By the definition of energy \( E = \frac{1}{2} m_v w^2 \), \( dE = m_v wdw \) and drift energy \( E_D = \frac{1}{2} m_v v_0^2 \), Eq. A9 can be written with energy notations,

\[
\int \int \int f_s(T_s, v_0, \tilde{w}) dv_0 dw_0 dw_v
\]

\[
= \int 2\pi \left( \frac{m_v}{2\pi kT_v} \right)^\frac{3}{2} e^{\frac{-E}{2kT_v}} \left( \frac{2\sqrt{kT_v E}}{kT_v} - e^{\frac{-E}{kT_v}} \right) dE
\]

\[
= \int 2\pi \left( \frac{m_v}{2\pi kT_v} \right)^\frac{3}{2} e^{\frac{-E}{2kT_v}} \left( \frac{-E}{kT_v} - e^{\frac{-E}{kT_v}} \right) dE
\]

\[
= \int 2\pi \left( \frac{m_v}{2\pi kT_v} \right)^\frac{3}{2} e^{\frac{-E}{2kT_v}} \left( \frac{-E}{kT_v} - e^{\frac{-E}{kT_v}} \right) dE
\]

From the definition shown in Eq. A2, the EEDF in the energy space can be realized from Eq. A11 as
\[ f_D(T_c, E_D, E) = \frac{1}{2\sqrt{\pi kT_c E_D}} e^{-\frac{(\sqrt{E_L} - \sqrt{E_D})^2}{4kt_cE_D}} \left( 1 - e^{-\frac{4\sqrt{E_L}E_D}{kt_cE_D}} \right) \quad (A12) \]

which completes the derivation of Eq. A1.
Appendix B

Ionization Cross Section of Xenon

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Appendix C

Excitation Cross Section of Xenon

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Appendix D
Derivation of Coil-Induced Magnetic Field

The coil-induced magnetic field $B_r$ and $B_z$ are derived at point $P$. They are induced by a loop current with amplitude $I$ in a single-turn coil with radius $a$.

A Cartesian coordinate is chosen for simplicity, but instead of $[x, y, z]$ a $[z, y, r]$ coordinate notation. This is for presenting the final result in $[r, z]$ coordinate as in a cylindrical system. This approach is valid because the component of the $B$ field is not considered.

From the designated coordinate system the following vectors are presented,

$$OP = \begin{pmatrix} z \\ 0 \\ r \end{pmatrix} \quad OQ = \begin{pmatrix} 0 \\ a \cos \theta \\ a \sin \theta \end{pmatrix} \quad \vec{R} = OP - OQ = \begin{pmatrix} z \\ -a \cos \theta \\ r - a \sin \theta \end{pmatrix}$$

where $R^2 = z^2 + a^2 \cos^2 \theta + (r - a \sin \theta)^2 = z^2 + a^2 + r^2 - 2ar \sin \theta$
The Biot-Savart law governing the current-induced magnetic field is

\[ dB(P) = \frac{\mu_0 I d\vec{l} \times \vec{e}_r}{4\pi R^2} \]

with \( d\vec{l} = \begin{pmatrix} \frac{z}{R} \\ -a \cos \theta \sin \phi \\ a \sin \theta \cos \phi \end{pmatrix} \) and \( \vec{e}_r = \begin{pmatrix} -\frac{a \cos \theta}{R} \\ r - a \sin \theta \end{pmatrix} \)

( D1 )

Substitute the vector notations into Eq. D1 to get

\begin{align*}
\frac{z}{R} a \cos \theta d\theta \\
\frac{z}{R} a \sin \theta d\theta
\end{align*}

( D2 )

Substitute \( R = \left( z^2 + a^2 + r^2 - 2ar \sin \theta \right)^{1/2} \) into Eq. D2,

( D3 )
Integrate Eq. D3 to get

$$\bar{B}(P) = \begin{cases} B_x(P) & \frac{\mu_0 I a^2 \pi}{4\pi} \left( \frac{a - r \sin \theta}{(z^2 + a^2 + r^2 - 2ar \sin \theta)^{\frac{3}{2}}} \right) \\ B_y(P) & \text{not considered} \\ B_z(P) & \frac{z \sin \theta}{(z^2 + a^2 + r^2 - 2ar \sin \theta)^{\frac{3}{2}}} \end{cases} d\theta$$

(D4)

Note:

$$\frac{\partial}{\partial \alpha} \left[ \frac{1}{(z^2 + a^2 + r^2 - 2ar \sin \theta)^{\frac{3}{2}}} \right] = \frac{a - r \sin \theta}{(z^2 + a^2 + r^2 - 2ar \sin \theta)^{\frac{3}{2}}}$$

$$\frac{\partial}{\partial z} \left[ \frac{1}{(z^2 + a^2 + r^2 - 2ar \sin \theta)^{\frac{3}{2}}} \right] = \frac{z}{(z^2 + a^2 + r^2 - 2ar \sin \theta)^{\frac{3}{2}}}$$

and the equations for \(B_x\) and \(B_z\) are arrived,

$$B_x = -\frac{\mu_0 I a}{2\pi} \frac{\partial}{\partial z} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \theta d\theta \left( z^2 + a^2 + r^2 - 2ar \sin \theta \right)^{\frac{3}{2}}$$

(D5)

$$B_z = -\frac{\mu_0 I a}{2\pi} \frac{\partial}{\partial \alpha} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \left( z^2 + a^2 + r^2 - 2ar \sin \theta \right)^{\frac{3}{2}}$$

(D6)

A limiting case can be checked for verification. For \((r, z) = (0, 0)\) at the axis of symmetry, \(B_r = 0\) and \(B_z = \frac{\mu_0 I}{2a}\) (field induced by a long solenoid). Indeed,

$$B_z@(0,0) = \frac{\mu_0 I a}{2\pi} \frac{\partial}{\partial \alpha} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = -\frac{\mu_0 I a}{2\pi} \left( -\frac{1}{a^2} \right) \pi = \frac{\mu_0 I}{2a}$$

and the derivation is verified.
The analytic form of vector potential $\vec{A}$ that satisfies $\vec{B} = \nabla \times \vec{A}$ still needs to be found. Here the only interest is in $A_\theta$ from the axisymmetric assumption of the code. There are two ways that $A_\theta$ can be found as the curl of $\vec{A}$ yields two equations: 

$$\frac{-\partial A_\theta}{\partial z} = B_r \quad \text{and} \quad \frac{1}{r} \frac{\partial}{\partial r} (rA_\theta) = B_z;$$

and the former is chosen. To proceed, the equation of $B_r$ is first transformed into analytical form. The procedure is as follows. Let

$$m = \frac{4ar}{z^2 + (a + r)^2} \quad \text{and} \quad \varphi = \frac{\theta + \pi}{2} \quad (D7)$$

$$B_r = -\frac{\mu_0 I_a}{2\pi} \int_0^\varphi \frac{-\left(1 - 2\sin^2 \varphi\right)}{\sqrt{z^2 + (a + r)^2}} \frac{2d\varphi}{\sqrt{1 - 4ar}}$$

$$= \frac{\mu_0 I_a}{2\pi \sqrt{z^2 + (a + r)^2}} \int_0^\varphi \frac{2 - 4\sin^2 \varphi}{\sqrt{1 - m\sin^2 \varphi}} d\varphi \quad (D8)$$

Note: \( D = 1 - m\sin^2 \varphi \rightarrow \sin^2 \varphi = \frac{1 - D}{m} \)

$$2 - 4\sin^2 \varphi = 2 - 4 \left(1 - \frac{1 - m\sin^2 \varphi}{m}\right) = 2 - \frac{4}{m} + \frac{4}{m}(1 - m\sin^2 \varphi)$$

Rearrangement Eq. D8 to obtain
\[ B_r = \frac{\mu_c I_a}{2\pi \sqrt{z^2 + (a+r)^2}} \frac{\partial}{\partial z} \left[ \int_0^{\pi/2} \left( 2 - \frac{4}{m} \right) d\varphi + \frac{4}{m} \int_0^{\pi/2} \frac{\sin^2 \varphi d\varphi}{\sqrt{1 - m \sin^2 \varphi}} \right] \]

\[ = \frac{\mu_c I_a}{2\pi \sqrt{z^2 + (a+r)^2}} \frac{\partial}{\partial z} \left[ 2 - \frac{4}{m} \right] K(m) + \frac{4}{m} E(m) \]  

where \( K(m) \) and \( E(m) \) are the complete elliptic integral of the first and second kind. With this result, return to the differential equation \( \frac{-\partial A_\theta}{\partial z} = B_r \) to find the analytical form of \( A_\theta \).

\[ A_\theta = \frac{-\mu_c I_a}{2\pi \sqrt{z^2 + (a+r)^2}} \left[ 2 - \frac{4}{m} \right] K(m) + \frac{4}{m} E(m) \]  

where

\[ m = \frac{4ar}{z^2 + (a+r)^2}. \]

which is defined zero at the centerline.
References


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Busek Brochure: Ion Thruster Systems.


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