Optimal Estimation of Ionosphere-Induced Group Delays of Global Positioning Satellite Signals during Launch, Orbit and Re-Entry

by

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Abstract

There are many sources of range error in a Global Positioning Satellite (GPS) signal that has traveled to a receiver near the earth’s surface. Among these is the ionospheric group delay. In the past, a single-state, dual-frequency filter has been used to estimate the ionospheric delay for authorized users. Although sufficient for terrestrial receivers for which the ionospheric delay changes very slowly, such a filter is inadequate for space-based missions in which a receiver passes rapidly through the ionosphere. Various Kalman filters are examined and simulation results presented.

The most robust Kalman filter considered was a seven-state filter. This filter utilizes four measurements: dual-frequency pseudo-range differencing, dual-frequency delta-range differencing, and single-frequency rate measurements for both frequencies (L1 and L2). Two states are necessary for the model dynamics plus five constant states necessary for processing rate measurements.

The process model selected for the seven-state filter was the integral of a first-order Markov process. The filter was used to estimate both the ionospheric group delay and the deviation of the delay from a given reference model. When used to estimate the deviation of the delay from a reference model, the group delay transitioned from “estimated” to “modeled” smoothly in the absence of measurements. In the absence of measurements, the estimated group delay tends to a bias from the reference model provided.

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Figure 5.2.8-2: PRN 6, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.8-3: PRN 6, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.8-4: PRN 6, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.8-5: PRN 6, Ionospheric Delay 2\(^{nd}\) Derivative for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.8-6: PRN 6, Ionospheric Delay 2\(^{nd}\) Derivative Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, No Signal Loss
Figure 5.2.8-7: PRN 6, Ionospheric Delay for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, No Delay Model

Figure 5.2.8-8: PRN 6, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, No Delay Model

Figure 5.2.8-9: PRN 6, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, No Delay Model

Figure 5.2.8-10: PRN 6, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, No Delay Model

Figure 5.2.8-11: PRN 6, Ionospheric Delay 2\textsuperscript{nd} Derivative for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, No Delay Model

Figure 5.2.8-12: PRN 6, Ionospheric Delay 2\textsuperscript{nd} Derivative Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, No Delay Model

Figure 5.2.8-13: PRN 6, Ionospheric Delay for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, with Delay Model

Figure 5.2.8-14: PRN 6, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, with Delay Model

Figure 5.2.8-15: PRN 6, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, with Delay Model

Figure 5.2.8-16: PRN 6, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, with Delay Model

Figure 5.2.8-17: PRN 6, Ionospheric Delay 2\textsuperscript{nd} Derivative for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, with Delay Model

Figure 5.2.8-18: PRN 6, Ionospheric Delay 2\textsuperscript{nd} Derivative Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry,
Signal Loss at $t = 200$ Seconds, with Delay Model

Figure 5.3-1: PRN 18, Ionospheric Delay for Filter with 3 Dynamic States, 6/6 Channels, Launch, No Signal Loss

Figure 5.3-2: PRN 18, Ionospheric Delay Error for Filter with 3 Dynamic States, 6/6 Channels, Launch, No Signal Loss

Figure 5.3-3: PRN 18, Ionospheric Delay Rate for Filter with 3 Dynamic States, 6/6 Channels, Launch, No Signal Loss

Figure 5.3-4: PRN 18, Ionospheric Delay Rate Error for Filter with 3 Dynamic States, 6/6 Channels, Launch, No Signal Loss

Figure 5.3-5: PRN 18, Ionospheric Delay 2$^{nd}$ Derivative for Filter with 3 Dynamic States, 6/6 Channels, Launch, No Signal Loss

Figure 5.3-6: PRN 18, Ionospheric Delay 2$^{nd}$ Derivative Error for Filter with 3 Dynamic States, 6/6 Channels, Launch, No Signal Loss

Figure 5.3-7: PRN 18, Ionospheric Delay for Filter with 2 Dynamic States, 6/6 Channels, Launch, No Signal Loss

Figure 5.3-8: PRN 18, Ionospheric Delay Error for Filter with 2 Dynamic States, 6/6 Channels, Launch, No Signal Loss

Figure 5.3-9: PRN 18, Ionospheric Delay Rate for Filter with 2 Dynamic States, 6/6 Channels, Launch, No Signal Loss

Figure 5.3-10: PRN 18, Ionospheric Delay Rate Error for Filter with 2 Dynamic States, 6/6 Channels, Launch, No Signal Loss

Figure 5.3-11: PRN 18, Ionospheric Delay for Filter with 1 Dynamic State, 6/6 Channels, Launch, No Signal Loss

Figure 5.3-12: PRN 18, Ionospheric Delay Error for Filter with 1 Dynamic State, 6/6 Channels, Launch, No Signal Loss

Figure 5.3.1-1: PRN 18, Ionospheric Delay for First-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.1-2: PRN 18, Ionospheric Delay Error for First-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.1-3: PRN 18, Ionospheric Delay for First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model
Figure 5.3.1-4: PRN 18, Ionospheric Delay Error for First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.1-5: PRN 18, Ionospheric Delay for First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.1-6: PRN 18, Ionospheric Delay Error for First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.2-1: PRN 18, Ionospheric Delay for Second-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.2-2: PRN 18, Ionospheric Delay Error for Second-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.2-3: PRN 18, Ionospheric Delay Rate for Second-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.2-4: PRN 18, Ionospheric Delay Rate Error for Second-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.2-5: PRN 18, Ionospheric Delay for Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.2-6: PRN 18, Ionospheric Delay Error for Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.2-7: PRN 18, Ionospheric Delay Rate for Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.2-8: PRN 18, Ionospheric Delay Rate Error for Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.2-9: PRN 18, Ionospheric Delay for Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.2-10: PRN 18, Ionospheric Delay Error for Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.2-11: PRN 18, Ionospheric Delay Rate for Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.2-12: PRN 18, Ionospheric Delay Rate Error for Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.3-1: PRN 18, Ionospheric Delay for Third-Order Markov Process, 10/2
Figure 5.3.3-2: PRN 18, Ionospheric Delay Error for Third-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.3-3: PRN 18, Ionospheric Delay Rate for Third-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.3-4: PRN 18, Ionospheric Delay Rate Error for Third-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.3-5: PRN 18, Ionospheric Delay 2\textsuperscript{nd} Derivative for Third-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.3-6: PRN 18, Ionospheric Delay 2\textsuperscript{nd} Derivative Error for Third-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.3-7: PRN 18, Ionospheric Delay for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.3-8: PRN 18, Ionospheric Delay Error for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.3-9: PRN 18, Ionospheric Delay Rate for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.3-10: PRN 18, Ionospheric Delay Rate Error for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.3-11: PRN 18, Ionospheric Delay 2\textsuperscript{nd} Derivative for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.3-12: PRN 18, Ionospheric Delay 2\textsuperscript{nd} Derivative Error for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.3-13: PRN 18, Ionospheric Delay for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.3-14: PRN 18, Ionospheric Delay Error for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.3-15: PRN 18, Ionospheric Delay Rate for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model
Figure 5.3.3-16: PRN 18, Ionospheric Delay Rate Error for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.3-17: PRN 18, Ionospheric Delay 2$^{nd}$ Derivative for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.3-18: PRN 18, Ionospheric Delay 2$^{nd}$ Derivative Error for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.4-1: PRN 18, Ionospheric Delay for Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.4-2: PRN 18, Ionospheric Delay Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.4-3: PRN 18, Ionospheric Delay Rate for Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.4-4: PRN 18, Ionospheric Delay Rate Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.4-5: PRN 18, Ionospheric Delay for Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.4-6: PRN 18, Ionospheric Delay Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.4-7: PRN 18, Ionospheric Delay Rate for Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.4-8: PRN 18, Ionospheric Delay Rate Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.4-9: PRN 18, Ionospheric Delay for Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.4-10: PRN 18, Ionospheric Delay Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model
Figure 5.3.4-11: PRN 18, Ionospheric Delay Rate for Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.4-12: PRN 18, Ionospheric Delay Rate Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.5-1: PRN 18, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.5-2: PRN 18, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.5-3: PRN 18, Ionospheric Delay Rate for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.5-4: PRN 18, Ionospheric Delay Rate Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.5-5: PRN 18, Ionospheric Delay $2^{nd}$ Derivative for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.5-6: PRN 18, Ionospheric Delay $2^{nd}$ Derivative Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.5-7: PRN 18, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.5-8: PRN 18, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.5-9: PRN 18, Ionospheric Delay Rate for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.5-10: PRN 18, Ionospheric Delay Rate Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.5-11: PRN 18, Ionospheric Delay $2^{nd}$ Derivative for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model
Figure 5.3.5-12: PRN 18, Ionospheric Delay $2^{nd}$ Derivative Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.5-13: PRN 18, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.5-14: PRN 18, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds

Figure 5.3.5-15: PRN 18, Ionospheric Delay Rate for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds

Figure 5.3.5-16: PRN 18, Ionospheric Delay Rate Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds

Figure 5.3.5-17: PRN 18, Ionospheric Delay $2^{nd}$ Derivative for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds

Figure 5.3.5-18: PRN 18, Ionospheric Delay $2^{nd}$ Derivative Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch

Figure 5.3.6-1: PRN 18, Ionospheric Delay for Integral of First-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.6-2: PRN 18, Ionospheric Delay Error for Integral of First-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.6-3: PRN 18, Ionospheric Delay Rate for Integral of First-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.6-4: PRN 18, Ionospheric Delay Rate Error for Integral of First-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.6-5: PRN 18, Ionospheric Delay for Integral of First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model

Figure 5.3.6-6: PRN 18, Ionospheric Delay Error for Integral of First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds,
Delay Model

Figure 5.3.6-7: PRN 18, Ionospheric Delay Rate for Integral of First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.6-8: PRN 18, Ionospheric Delay Rate Error for Integral of First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.6-9: PRN 18, Ionospheric Delay for Integral of First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.6-10: PRN 18, Ionospheric Delay Error for Integral of First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.6-11: PRN 18, Ionospheric Delay Rate for Integral of First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.6-12: PRN 18, Ionospheric Delay Rate Error for Integral of First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.7-1: PRN 18, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.7-2: PRN 18, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.7-3: PRN 18, Ionospheric Delay Rate for Integral of Second-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.7-4: PRN 18, Ionospheric Delay Rate Error for Integral of Second-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.7-5: PRN 18, Ionospheric Delay 2nd Derivative for Integral of Second-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.7-6: PRN 18, Ionospheric Delay 2nd Derivative Error for Integral of Second-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.7-7: PRN 18, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, No Delay Model
Figure 5.3.7-8: PRN 18, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.7-9: PRN 18, Ionospheric Delay Rate for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.7-10: PRN 18, Ionospheric Delay Rate Error for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.7-11: PRN 18, Ionospheric Delay 2nd Derivative for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.7-12: PRN 18, Ionospheric Delay 2nd Derivative Error for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.7-13: PRN 18, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.7-14: PRN 18, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.7-15: PRN 18, Ionospheric Delay Rate for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.7-16: PRN 18, Ionospheric Delay Rate Error for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.7-17: PRN 18, Ionospheric Delay 2nd Derivative for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.7-18: PRN 18, Ionospheric Delay 2nd Derivative Error for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.8-1: PRN 18, Ionospheric Delay for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss
Figure 5.3.8-2: PRN 18, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.8-3: PRN 18, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.8-4: PRN 18, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.8-5: PRN 18, Ionospheric Delay 2nd Derivative for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.8-6: PRN 18, Ionospheric Delay 2nd Derivative Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.8-7: PRN 18, Ionospheric Delay for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.8-8: PRN 18, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.8-9: PRN 18, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.8-10: PRN 18, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.8-11: PRN 18, Ionospheric Delay 2nd Derivative for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.8-12: PRN 18, Ionospheric Delay 2nd Derivative Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.8-13: PRN 18, Ionospheric Delay for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.8-14: PRN 18, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100
Seconds, with Delay Model

Figure 5.3.8-15: PRN 18, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.8-16: PRN 18, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.8-17: PRN 18, Ionospheric Delay 2nd Derivative for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.8-18: PRN 18, Ionospheric Delay 2nd Derivative Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.4-1: PRN 1, Ionospheric Delay for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4-2: PRN 1, Ionospheric Delay Error for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4-3: PRN 1, Ionospheric Delay for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4-4: PRN 1, Ionospheric Delay Error for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4-5: PRN 1, Ionospheric Delay Rate for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4-6: PRN 1, Ionospheric Delay Rate Error for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4-7: PRN 1, Ionospheric Delay Rate for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4-8: PRN 1, Ionospheric Delay Rate Error for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4-9: PRN 1, Ionospheric Delay 2nd Derivative for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4-10: PRN 1, Ionospheric Delay 2nd Derivative Error for Filter with 3 Dynamic
Figure 5.4-11: PRN 1, Ionospheric Delay 2$^{nd}$ Derivative for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4-12: PRN 1, Ionospheric Delay 2$^{nd}$ Derivative Error for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4-13: PRN 1, Ionospheric Delay for Filter with 2 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4-14: PRN 1, Ionospheric Delay Error for Filter with 2 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4-15: PRN 1, Ionospheric Delay for Filter with 2 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4-16: PRN 1, Ionospheric Delay Error for Filter with 2 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4-17: PRN 1, Ionospheric Delay Rate for Filter with 2 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4-18: PRN 1, Ionospheric Delay Rate Error for Filter with 2 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4-19: PRN 1, Ionospheric Delay Rate for Filter with 2 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4-20: PRN 1, Ionospheric Delay Rate Error for Filter with 2 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4-21: PRN 1, Ionospheric Delay for Filter with 1 Dynamic State, 6/6 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4-22: PRN 1, Ionospheric Delay Error for Filter with 1 Dynamic State, 6/6 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4-23: PRN 1, Ionospheric Delay for Filter with 1 Dynamic State, 6/6 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4-24: PRN 1, Ionospheric Delay Error for Filter with 1 Dynamic State, 6/6 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.1-1: PRN 1, Ionospheric Delay for First-Order Markov Process, 10/2 Channels,
LEO, No Signal Loss, from \( t = 0 \) to \( t = 190 \) Seconds

Figure 5.4.1-2: PRN 1, Ionospheric Delay Error for First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 0 \) to \( t = 190 \) Seconds

Figure 5.4.1-3: PRN 1, Ionospheric Delay for First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.1-4: PRN 1, Ionospheric Delay Error for First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.1-5: PRN 1, Ionospheric Delay for First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at \( t = 4800 \) Seconds, No Delay Model, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.1-6: PRN 1, Ionospheric Delay Error for First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at \( t = 4800 \) Seconds, No Delay Model, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.1-7: PRN 1, Ionospheric Delay for First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at \( t = 4800 \) Seconds, with Delay Model, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.1-8: PRN 1, Ionospheric Delay Error for First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at \( t = 4800 \) Seconds, with Delay Model, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.2-1: PRN 1, Ionospheric Delay for Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 0 \) to \( t = 190 \) Seconds

Figure 5.4.2-2: PRN 1, Ionospheric Delay Error for Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 0 \) to \( t = 190 \) Seconds

Figure 5.4.2-3: PRN 1, Ionospheric Delay for Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.2-4: PRN 1, Ionospheric Delay Error for Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.2-5: PRN 1, Ionospheric Delay Rate for Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 0 \) to \( t = 190 \) Seconds

Figure 5.4.2-6: PRN 1, Ionospheric Delay Rate Error for Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 0 \) to \( t = 190 \) Seconds

Figure 5.4.2-7: PRN 1, Ionospheric Delay Rate for Second-Order Markov Process, 10/2
Figure 5.4.2-8: PRN 1, Ionospheric Delay Rate Error for Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.2-9: PRN 1, Ionospheric Delay for Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.2-10: PRN 1, Ionospheric Delay Error for Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.2-11: PRN 1, Ionospheric Delay Rate for Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.2-12: PRN 1, Ionospheric Delay Rate Error for Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.2-13: PRN 1, Ionospheric Delay for Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, with Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.2-14: PRN 1, Ionospheric Delay Error for Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, with Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.2-15: PRN 1, Ionospheric Rate Delay for Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, with Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.2-16: PRN 1, Ionospheric Delay Rate Error for Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, with Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.3-1: PRN 1, Ionospheric Delay for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.3-2: PRN 1, Ionospheric Delay Error for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.3-3: PRN 1, Ionospheric Delay for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds
Figure 5.4.3-4: PRN 1, Ionospheric Delay Error for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.3-5: PRN 1, Ionospheric Delay Rate for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.3-6: PRN 1, Ionospheric Delay Rate Error for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.3-7: PRN 1, Ionospheric Delay Rate for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.3-8: PRN 1, Ionospheric Delay Rate Error for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.3-9: PRN 1, Ionospheric Delay 2nd Derivative for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.3-10: PRN 1, Ionospheric Delay 2nd Derivative Error for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.3-11: PRN 1, Ionospheric Delay 2nd Derivative for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.3-12: PRN 1, Ionospheric Delay 2nd Derivative Error for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.3-13: PRN 1, Ionospheric Delay for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.3-14: PRN 1, Ionospheric Delay Error for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.3-15: PRN 1, Ionospheric Delay Rate for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.3-16: PRN 1, Ionospheric Delay Rate Error for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.3-17: PRN 1, Ionospheric Delay 2nd Derivative for Third-Order Markov Process,
10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.3-18: PRN 1, Ionospheric Delay 2\textsuperscript{nd} Derivative Error for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.3-19: PRN 1, Ionospheric Delay for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.3-20: PRN 1, Ionospheric Delay Error for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.3-21: PRN 1, Ionospheric Delay Rate for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.3-22: PRN 1, Ionospheric Delay Rate Error for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.3-23: PRN 1, Ionospheric Delay 2\textsuperscript{nd} Derivative for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.3-24: PRN 1, Ionospheric Delay 2\textsuperscript{nd} Derivative Error for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.4-1: PRN 1, Ionospheric Delay for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.4-2: PRN 1, Ionospheric Delay Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.4-3: PRN 1, Ionospheric Delay for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.4-4: PRN 1, Ionospheric Delay Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds
Figure 5.4.4-5: PRN 1, Ionospheric Delay Rate for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.4-6: PRN 1, Ionospheric Delay Rate Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.4-7: PRN 1, Ionospheric Delay Rate for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.4-8: PRN 1, Ionospheric Delay Rate Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.4-9: PRN 1, Ionospheric Delay for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.4-10: PRN 1, Ionospheric Delay Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.4-11: PRN 1, Ionospheric Delay Rate for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.4-12: PRN 1, Ionospheric Delay Rate Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.4-13: PRN 1, Ionospheric Delay Rate for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.4-14: PRN 1, Ionospheric Delay Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.4-15: PRN 1, Ionospheric Delay Rate for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.4-16: PRN 1, Ionospheric Delay Rate Error for Two Cascades of First-Order
Figure 5.4.5-1: PRN 1, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.5-2: PRN 1, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 5407$ Seconds

Figure 5.4.5-3: PRN 1, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-4: PRN 1, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-5: PRN 1, Ionospheric Delay Rate for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.5-6: PRN 1, Ionospheric Delay Rate Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.5-7: PRN 1, Ionospheric Delay Rate for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-8: PRN 1, Ionospheric Delay Rate Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-9: PRN 1, Ionospheric Delay 2nd Derivative for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.5-10: PRN 1, Ionospheric Delay 2nd Derivative Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.5-11: PRN 1, Ionospheric Delay 2nd Derivative for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds
Figure 5.4.5-12: PRN 1, Ionospheric Delay 2nd Derivative Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-13: PRN 1, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-14: PRN 1, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-15: PRN 1, Ionospheric Delay Rate for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-16: PRN 1, Ionospheric Delay Rate Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-17: PRN 1, Ionospheric Delay 2nd Derivative for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-18: PRN 1, Ionospheric Delay 2nd Derivative Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-19: PRN 1, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-20: PRN 1, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-21: PRN 1, Ionospheric Delay Rate for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-22: PRN 1, Ionospheric Delay Rate Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-23: PRN 1, Ionospheric Delay 2nd Derivative for Three Cascades of First-
Order Markov Processes, 10/2 Channels, LEO, Signal Loss at \( t = 4800 \) Seconds, with Delay Model, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.5-24: PRN 1, Ionospheric Delay 2nd Derivative Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at \( t = 4800 \) Seconds, with Delay Model, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.6-1: PRN 1, Ionospheric Delay for Integral of First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 0 \) to \( t = 190 \) Seconds

Figure 5.4.6-2: PRN 1, Ionospheric Delay Error for Integral of First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 0 \) to \( t = 190 \) Seconds

Figure 5.4.6-3: PRN 1, Ionospheric Delay for Integral of First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.6-4: PRN 1, Ionospheric Delay Error for Integral of First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.6-5: PRN 1, Ionospheric Delay Rate for Integral of First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 0 \) to \( t = 190 \) Seconds

Figure 5.4.6-6: PRN 1, Ionospheric Delay Rate Error for Integral of First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 0 \) to \( t = 190 \) Seconds

Figure 5.4.6-7: PRN 1, Ionospheric Delay Rate for Integral of First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.6-8: PRN 1, Ionospheric Delay Rate Error for Integral of First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.6-9: PRN 1, Ionospheric Delay for Integral of First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at \( t = 4800 \) Seconds, No Delay Model, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.6-10: PRN 1, Ionospheric Delay Error for Integral of First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at \( t = 4800 \) Seconds, No Delay Model, from \( t = 4200 \) to \( t = 5407 \) Seconds

Figure 5.4.6-11: PRN 1, Ionospheric Delay Rate for Integral of First-Order Markov
Figure 5.4.6-12: PRN 1, Ionospheric Delay Rate Error for Integral of First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.6-13: PRN 1, Ionospheric Delay for Integral of First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.6-14: PRN 1, Ionospheric Delay Error for Integral of First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.6-15: PRN 1, Ionospheric Delay Rate for Integral of First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.6-16: PRN 1, Ionospheric Delay Rate Error for Integral of First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.7-1: PRN 1, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.7-2: PRN 1, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.7-3: PRN 1, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.7-4: PRN 1, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.7-5: PRN 1, Ionospheric Delay Rate for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.7-6: PRN 1, Ionospheric Delay Rate Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.7-7: PRN 1, Ionospheric Delay Rate for Integral of Second-Order Markov
Figure 5.4.7-8: PRN 1, Ionospheric Delay Rate Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.7-9: PRN 1, Ionospheric Delay 2nd Derivative for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.7-10: PRN 1, Ionospheric Delay 2nd Derivative Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.7-11: PRN 1, Ionospheric Delay 2nd Derivative for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.7-12: PRN 1, Ionospheric Delay 2nd Derivative Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.7-13: PRN 1, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.7-14: PRN 1, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.7-15: PRN 1, Ionospheric Delay Rate for Integral of Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

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A  Wave amplitude
A  Non-frequency-dependent ionospheric delay
A  Continuous model dynamics matrix
B  Clock bias in units of distance
B  Magnetic field vector
B_{L2}  L1-L2 inter-frequency bias
B_d  Continuous model deterministic input matrix
B_s  Continuous model stochastic input matrix
C  Common mode term
C  Continuous model observation matrix
c  Speed of light
D  Electric displacement
D_d  Continuous model deterministic coupling matrix
D_s  Continuous model stochastic coupling matrix
E  Expected value
E  Electric field vector
e  State error vector
f  Frequency in units of Hertz
G_d  Discrete model deterministic input matrix
G_s  Discrete model stochastic input matrix
H  Discrete model observation matrix
Magnetic field
J  Source current density
K  Kalman gain matrix
k  Discrete time reference
L  Geometric line-of-sight
L_1  Reference to GPS signal at 1575.42 MHz
L_2  Reference to GPS signal at 1227.60 MHz
N  Continuous stochastic input covariance matrix
N_e  Local electron density
n  Index of refraction
P  State covariance matrix
Q  Process covariance matrix
Q_{obs}  Observability matrix
Q_{SS}  Steady-state process covariance matrix
R  Measured range
R  Measurement covariance matrix
t  Continuous time reference
t1  L1 rate measurement reference time
t2  L2 rate measurement reference time
u  Deterministic input vector
V  Variance
v  Velocity
v  Measurement noise vector
w  Gaussian white noise
w  Process noise vector
w_{L2}  L1-L2 frequency conversion term
x  Position
x  State vector
y  Output vector
z  Measurement vector
\beta  Reciprocal of Markov time constant
\Gamma_d  Discrete model deterministic coupling matrix
\Gamma_s  Discrete model stochastic coupling matrix
\Delta A  Atmospheric signal delay in units of distance
\Delta I  Ionospheric signal delay in units of distance
\Delta M  Modeled ionospheric signal delay in units of distance
\Delta t  Time increment
\Delta \phi  Carrier phase advance
\delta  Delta-range
\epsilon  Dielectric constant
\theta  Angle between incident ray and normal to media boundary
\mu  Magnetic permeability
\xi  Gaussian white noise
\rho  Pseudo-range
\sigma  Standard deviation
\Phi  State transition matrix
\phi  Carrier phase
\omega  Frequency in radians per second
\Omega  Permutation matrix
\Lambda  Additional process covariance matrix
\cdot  Derivative
\sim  Simulated or modeled value
\wedge  Estimated value
\cdot  Estimated state value before measurement
\cdot  Estimated state value after measurement
\cdot  Alternate
\top  Transpose
Chapter 1

Introduction

There are many sources of range error in a Global Positioning Satellite (GPS) signal that has traveled to a receiver near the earth’s surface. Such a signal is refracted and slowed by the neutral atmosphere and the ionosphere, causing the modulated signal to arrive at the receiver later than it would have had it traveled through a vacuum. Left uncorrected, this would cause the navigation software to estimate the receiver’s location as being farther away from the satellite than it actually was. Relativistic effects, imperfect clocks and multi-pathing (the detection of a signal multiple times: directly from the satellite and reflected off of objects or terrain) can also result in an erroneous navigation solution.

Among the many sources of error, the ionosphere’s effect on a GPS signal is of particular interest. For satellites at low elevation angles, ionosphere-induced range errors can be greater than 50 meters. The ionosphere is also notoriously difficult to model. The most complicated and robust models cannot estimate the delay/range error caused by the ionosphere to better than 75% for a terrestrial receiver [1].

Because of regional and temporal variations in the ionosphere, the use of a model to correct for ionospheric signal delay may not be the best choice. An alternative solution is to estimate the delay using a Kalman filter or similar estimator. The most common method of estimating the ionosphere-induced delay involves using a single-state filter with a smoother and a large time constant to suppress noise [1]. Although sufficient for terrestrial receivers for which the rate of change of the ionospheric group delay over a few minutes is very small, this filter would be unable to handle the rapidly-changing delays experienced by launching, orbiting or re-entering vehicles.

Figure 1-1 shows the true ionospheric group delay (blue) on the same plot as a sample and hold algorithm (green) for PRN 27 on a simulated re-entry trajectory. The channel allocation (12 channels with 10 allocated to the primary frequency and 2 to the secondary with 5 seconds of dwell time per satellite) is such that the receiver has 5 seconds of dual-frequency data followed by 20 seconds of single-frequency data (single-frequency rate measurements are not used). Clearly, this method is inadequate.
Figure 1-1: Results of Sample and Hold Algorithm (green) Compared to Truth (blue) for PRN 27 on a Re-Entry Trajectory
Chapter 2

GPS Signal Propagation

2.1 Phase and Group Velocities

First, it is important to understand the physical effects the ionosphere has on GPS signals. The ionosphere is a dispersive medium, meaning signals of different frequencies travel at different velocities through it. Modulated signals or signals consisting of two or more frequencies overlaid may appear to behave strangely in dispersive media if certain velocity terms are not well-defined (or well-understood). A thorough explanation of dispersive phenomena would fill several volumes, so only a brief and very simplified description is included here. Such a superficial explanation of such a complex topic is regrettable but necessary for the purposes of this thesis.

Starting with Maxwell’s equations in macroscopic media:

\[ \nabla \cdot \mathbf{D} = 4\pi \rho \]  
(Coulomb’s law) (2.1-1)

\[ \nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c} \mathbf{J} \]  
(Ampere’s law) (2.1-2)

\[ \nabla \cdot \mathbf{B} = 0 \]  
(no free magnetic poles) (2.1-3)

\[ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \]  
(Faraday’s law) (2.1-4)

For which \( \rho \) is the source charge density, \( \mathbf{J} \) is the source current density, \( \mathbf{B} \) is the magnetic field vector, \( \mathbf{E} \) is the electric field vector and \( c \) is the speed of light. \( \mathbf{D} \) and \( \mathbf{H} \) are known as the electric displacement and magnetic field, respectively. In a vacuum, \( \mathbf{D} = \mathbf{E} \) and \( \mathbf{H} = \mathbf{B} \) and for linear, isotropic media:

\[ \mathbf{D} = \varepsilon \mathbf{E} \]  
(2.1-5)
\[
\mathbf{H} = \frac{1}{\mu} \mathbf{B}
\]  

(2.1-6)

\(\mathbf{B}\) is then known as the magnetic induction. \(\mu\) is magnetic permeability and \(\varepsilon\) is the dielectric constant. These relationships are only valid for fields in which the characteristic time scale of field variations is longer than the relaxation time of the medium. For more quickly varying fields, the electric displacement depends on the electric field and the past history of the field.

Returning to the relationships given by equations 2.1-1 through 2.1-6, Maxwell’s equations in the absence of sources in linear, isotropic media are:

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= 0 \\
\n\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} &= 0 \\
\n\nabla \cdot \mathbf{B} &= 0 \\
\n\nabla \times \frac{1}{\mu} \mathbf{B} - \frac{1}{c} \frac{\partial (\varepsilon \mathbf{E})}{\partial t} &= \nabla \times \mathbf{B} - \frac{\mu \varepsilon}{c} \frac{\partial \mathbf{E}}{\partial t} = 0
\end{align*}
\]

(2.1-7)  
(2.1-8)  
(2.1-9)  
(2.1-10)

Since different waves (the components of an electromagnetic wave, constituents of a modulated waveform or both) can be analyzed separately and superimposed, only the plane wave solution to the wave equation will be considered for simplicity:

\[
\nabla^2 u - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = 0
\]

(2.1-11)

For which:

\[
v = \frac{c}{\sqrt{\mu \varepsilon}}
\]

(2.1-12)

Ignoring absorption (considering only the real part of the index of refraction):

\[
n = \sqrt{\mu \varepsilon}
\]

(2.1-13)

Considering waves in the dimension \(x\), the solution to the wave equation becomes:

\[
u(x,t) = A e^{ikx - i\omega t}
\]

(2.1-14)

For which \(\omega\) is frequency, \(k\) is the wave number and:

\[
k = \frac{\omega}{v} = \sqrt{\mu \varepsilon} \frac{\omega}{c}
\]

(2.1-15)
If we consider $\omega$ as a general function of $k$, $\omega(k)$ (allowing the possibility of dispersion), it must be an even function [$\omega(k) = \omega(-k)$] since dispersive properties cannot depend on whether the wave travels to the left or right. From equation 2.1-14, the general solution is of the form:

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)e^{ikx - i\omega(t)k} dk$$

(2.1-16)

Using the Fourier integral notation:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)e^{ikx} dk$$

(2.1-17)

For which:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)dx$$

(2.1-18)

For the problem considered here:

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,0)dx$$

(2.1-19)

$u(x,0)$ is a harmonic wave, $e^{i k_0 x}$, for all $x$. The orthogonality condition:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k - k')x} dx = \delta(k - k')$$

(2.1-20)

Gives:

$$A(k) = \sqrt{2\pi} \delta(k - k_0)$$

(2.1-21)

This is a monochromatic traveling wave:

$$u(x,t) = e^{ikx - i\omega(t)k_0}$$

(2.1-22)

If $u(x,0)$ is a wavetrain with finite length of the order $\Delta x$, $A(k)$ is not a delta function. It is a peaked function with a width of the order $\Delta k$, centered around $k_0$, the dominant wave number in $u(x,0)$. Long sinusoidal wave trains are almost monochromatic and $A(k)$ is sharply peaked at $k_0$. If this is the case, the frequency can then be expanded about $k$: 57
\[ \omega(k) = \omega_0 + \frac{d\omega}{dk} \bigg|_0 (k - k_0) + \cdots \] (2.1-23)

\[ u(x,t) = e^{i \left[ \int (\omega(k) - \omega_0) x dk \right]} \]

\[ = e^{i \left[ \int (\omega(k) - \omega_0) x dk \right]} e^{i \left[ \int (\omega(k) - \omega_0) t dk \right]} \] (2.1-24)

\[ u(x,t) = \frac{e^{i \left[ \int (\omega(k) - \omega_0) x dk \right]}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{i \left[ \int (\omega(k) - \omega_0) t dk \right]} dk \] (2.1-25)

From \( A(k) \) in equation 2.1-19 and its inverse, it is apparent that the integral in equation 2.1-25 is \( u(x',0) \) for which:

\[ x' = x - \frac{d\omega}{dk} \bigg|_0 t \] (2.1-26)

Therefore:

\[ u(x,t) = u \left( x - \frac{d\omega}{dk} \bigg|_0 t,0 \right) e^{i \left[ \int (\omega(k) - \omega_0) t dk \right]} \] (2.1-27)

Apart from an overall phase factor, the pulse travels along undistorted (ignoring higher-order terms in equation 2.1-23) with a group velocity of:

\[ v_g = \frac{d\omega}{dk} \bigg|_0 \] (2.1-28)

For light waves:

\[ \omega(k) = \frac{ck}{n(k)} \] (2.1-29)

The phase velocity is:

\[ v_p = \frac{\omega(k)}{k} = \frac{c}{n(k)} \] (2.1-30)

And is greater or less than \( c \), depending on \( n \). For most frequencies in most substances, \( n \) is greater than unity.

The group velocity in 2.1-28 is then:
\[
V_g = \frac{c}{n(\omega) + \omega (dn/d\omega)}
\]  (2.1-31)

For “normal” dispersion, \(dn/d\omega > 0\) and \(n > 1\), so group velocity is less than phase velocity is less than \(c\) [2].

Superluminal phase velocities and group velocities appear to occur when signals are passed through a medium near the medium’s resonance frequency. Figure 2.1-1 shows two phenomena which occur near these resonant frequencies (there may be more than one resonant frequency). The first is the index of refraction can drop below one. With indices of refraction below one, phase velocities appear to be superluminal. GPS signals in the ionosphere exhibit such “phase advances.”

![Index of Refraction Versus Frequency with Resonant Frequency at \(\omega_{\text{res}}\)](image)

**Figure 2.1-1: Illustration of the Variability of the Index of Refraction with Frequencies Near Optical Resonance Frequencies**

From figure 2.1-1, it is clear that \(dn/d\omega\) can also be large and negative. Dispersion in this region of frequencies is generally referred to as “anomalous” dispersion. The group velocity as defined in equation 2.1-31 can then become larger than the speed of light or even negative. For these cases, the assumptions made in 2.1-23 are no longer valid [2]. Group velocity as defined in equation 2.1-31 no longer holds physical meaning. Although anomalous dispersion is not an issue for GPS signals in the ionosphere, it was considered important to note that claims of superluminal group velocities are not violations of Einstein’s causality, but rather a misuse of the term “group velocity”.

For the purposes of this thesis, group velocity will be defined as the velocity of information and is always less than or equal to \(c\). The ionospheric group delay of a GPS signal refers to the delay of information reception caused by the ionosphere with respect to the time of reception had the signal traveled through a vacuum.

It has been stated that phase velocities can appear to be superluminal in some situations. Feynman [3] gives a qualitative explanation of what is actually happening. Feynman states that an index of refraction which can be greater than or less than one simply means the resulting
phase shift can either be positive or negative. The actual beginning of a signal is not advanced [3].

Many simplifying assumptions were made in the analysis outlined in equations 2.1-11 through 2.1-31. These assumptions are acceptable for GPS signals passing through the ionosphere, but a much more general proof comes from using the method of stationary phase. A proof using this method was much preferred, but a thorough explanation would have simply proved too lengthy for a thesis in which dispersion is not the focus. Jackson [2] has a good explanation of stationary phase as does Whitham [4]. Again, Feynman gives a very good description of dispersion in more qualitative terms. For a thorough understanding of dispersion, an investigation into quantum mechanics is required.

2.2 Propagation in the Neutral Atmosphere

For the frequencies considered here, the neutral atmosphere is a non-dispersive medium (there is clearly dispersion at optical frequencies). For the analysis contained herein, group and phase velocity in the atmosphere are the same. The GPS signal is delayed and refracted according to Snell’s Law:

\[ n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad (2.2-1) \]

And:

\[ \frac{\sin \theta_1}{\sin \theta_2} = n_2 \frac{v_1}{n_1 v_2} \quad (2.2-2) \]

\( \theta \) is the angle between the direction of wave propagation and the normal to the medium. The effects of the neutral atmosphere on GPS signals, although a large source of range error, are not considered in this thesis. Section 3.2.1 shows the neutral atmosphere delay term falls out of the measurement equations, so it is not of concern for this analysis.

2.3 Propagation in the Dispersive Ionosphere

The relationship between the velocity of a waveform in a vacuum and the velocity of the same waveform in a given medium is defined as the index of refraction:

\[ n = \frac{c}{v} \quad (2.3-1) \]

The phase refractive index for the ionosphere was derived by Appleton and Hartree to be:
\[ n_p^2 = 1 - \frac{X}{1 - iZ - \frac{Y_T^2}{2(1 - X - iZ)} \pm \sqrt{\frac{Y_T^4}{4(1 - X - iZ)^2} + Y_L^2}} \]  

(2.3-2)

For which:

\[ X = \frac{N_e e^2}{\varepsilon_0 m_e \omega^2} = \frac{f_p^2}{f^2} \]

\[ Y_L = \frac{e B_L}{m_e \omega} = \frac{f_H \cos \theta}{f} \]

\[ Y_T = \frac{e B_T}{m_e \omega} = \frac{f_H \sin \theta}{f} \]  

(2.3-3 : 2.3-7)

\[ Z = \frac{f_v}{\omega} \]

\[ \omega = 2\pi f \]

Ne is the local electron density, \( \varepsilon_0 \) is the permittivity of free space (8.854e-12 Farad/m), \( m_e \) is the mass of an electron at rest (9.107e-31 kg), \( f_p \) is the plasma frequency, \( f \) is the signal frequency in Hz, \( e \) is the charge of an electron (-1.602e-19 Coulomb), \( B_L \) is the longitudinal component of the magnetic field, \( f_H \) is the electron gyro frequency, \( \theta \) is the angle of the incoming signal with respect to the Earth’s magnetic field, \( B_T \) is the transverse component of the magnetic field and \( f_v \) is the electron-neutral collision frequency.

Klobuchar states that the electron gyro frequency is generally about 1.5MHz, the plasma frequency is rarely greater than 20 MHz and the electron-neutral collision frequency, \( f_v \), is about 100 kHz. With these assumptions, equation 2.3-2 can be approximated as:

\[ n_p \approx 1 - \left( \frac{X}{2} \right) \]  

(2.3-8)

For GPS frequencies:

\[ X \approx \frac{80.6 N_e}{f^2} \]  

(2.3-9)

Equation 2.3-8 is reportedly accurate to better than 1% [1].

Kaplan represents the phase and group refractive indices by:

\[ n_p = 1 + \frac{C_2}{f^2} + \frac{C_3}{f^3} + \frac{C_4}{f^4} + \cdots \]  

(2.3-10)
\[ n_g = 1 - \frac{c_2}{f^2} - \frac{2c_3}{f^3} - \frac{3c_4}{f^4} - \ldots \]  
(2.3-11)

The first-order approximations are:

\[ n_p = 1 + \frac{c_2}{f^2} \quad \text{and} \quad n_g = 1 - \frac{c_2}{f^2} \]  
(2.3-12 : 2.3-13)

For GPS signals, \( c_2 \approx -40.3N_e \) Hz\(^2\), yielding the same first-order approximation for the phase refractive index as in equation 2.3-8:

\[ n_p \approx 1 - \frac{40.3N_e}{f^2} \quad \text{and} \quad n_g = 1 + \frac{40.3N_e}{f^2} \]  
(2.3-14 : 2.3-15)

The measured range and geometric line-of-sight range are (respectively):

\[ R = \int_{\text{Sat}}^{R_{\text{Rcvr}}} ndr \quad \text{and} \quad L = \int_{\text{Sat}}^{R_{\text{Rcvr}}} dl \]  
(2.3-16 : 2.3-17)

The delay/advance is then the difference between the measured range and the line-of-sight range. The group delay and phase advance are then:

\[ \Delta l = \int_{\text{Sat}}^{R_{\text{Rcvr}}} \left(1 + \frac{40.3N_e}{f^2}\right) dr - \int_{\text{Sat}}^{R_{\text{Rcvr}}} dl \quad \text{and} \quad \Delta \phi = \int_{\text{Sat}}^{R_{\text{Rcvr}}} \left(1 - \frac{40.3N_e}{f^2}\right) dr - \int_{\text{Sat}}^{R_{\text{Rcvr}}} dl \]  
(2.3-18 : 2.3-19)

Since the delay will be very small compared to the range, equations 2.3-18 and 2.3-19 can be simplified by changing the integration from that along the refracted path to that along the line-of-sight path (dr becomes dl). This simplifies the equations to:

\[ \Delta l = \frac{40.3}{f^2} \int_{\text{Sat}}^{R_{\text{Rcvr}}} N_e dl \quad \text{and} \quad \Delta \phi = -\frac{40.3}{f^2} \int_{\text{Sat}}^{R_{\text{Rcvr}}} N_e dl \]  
(2.3-20 : 2.3-21)

\( N_e \) is the electron density in electrons/m\(^3\) and \( \int N_e dl \) is the total electron content (TEC) in units of electrons/m\(^2\). To a first-order approximation, the group delay and carrier phase advance are “equal and opposite.” This relationship will be an important assumption used in the formulation of the Kalman filter [5]. According to Klobuchar [1], neglecting the second-order term (f\(^3\) term) results in about 1.6 cm of error and neglecting the third-order term (f\(^4\) term) results in approximately 0.9 mm of error for a TEC of \(10^{18}\) electrons/m\(^2\). He references Bassiri and Hajj [6] for these numbers.
Munekane [7] gives the second-order approximation of the ionospheric group delay and carrier phase advance:

\[
\Delta l = \frac{40.3}{f^2} \int_{\text{Sat}}^{\text{Rcvr}} N_e dl + \frac{7527c}{f^3} \left( \mathbf{U}_R \cdot \mathbf{B} \right) \int_{\text{Sat}}^{\text{Rcvr}} N_e dl \tag{2.3-22}
\]

\[
\Delta \phi = -\frac{40.3}{f^2} \int_{\text{Sat}}^{\text{Rcvr}} N_e dl - \frac{7527c}{2f^3} \left( \mathbf{U}_R \cdot \mathbf{B} \right) \int_{\text{Sat}}^{\text{Rcvr}} N_e dl \tag{2.3-23}
\]

For which \( c \) is the speed of light, \( \mathbf{U}_R \) is the unit vector in the direction of signal propagation and \( \mathbf{B} \) is the earth’s magnetic field vector. To be completely accurate, the \( \mathbf{U}_R \cdot \mathbf{B} \) term should also be integrated along the path, but for simplicity, it is moved outside of the integral and only the magnetic field vector at the receiver’s location is used.
Chapter 3

Current Methods of Handling Ionosphere Delay

3.1 Modeled Ionosphere

There are many models designed to estimate the ionosphere TEC and/or ionospheric group delay of a signal. Among the most well known are the Klobuchar model and the International Reference Ionosphere (IRI). As these two models were the only ones utilized for the simulations described in this thesis, they are the only common models described below.

3.1.1 Klobuchar Model

One of the most commonly used terrestrial models of the ionosphere is the Klobuchar model. It was designed to provide the single-frequency, terrestrial GPS receiver with “good” corrections for ionosphere-induced signal delay. The Klobuchar model attempts to account for the receiver’s latitude and longitude as well as varying elevation and azimuth angles from receiver to satellite. The eight coefficients used in the model are provided as part of the GPS broadcast navigation message in order to give the receiver timely information on world-wide ionospheric conditions. The Klobuchar model, however, is a fairly simple model and can only correct for approximately 60% (rms) of the ionospheric time delay/range error in a terrestrial receiver [1].

3.1.2 International Reference Ionosphere

The International Reference Ionosphere (IRI) is an empirical model, developed from actual data taken from a variety of sources. Most topside data came from incoherent scatter observations and topside sounder profiles. Lower ionospheric data comes primarily from rocket to ground radio propagation measurements [8] [9] [10]. The IRI provides monthly averages of electron density, relative ion composition, temperatures and other characteristics. It provides data for 50 km – 2000 km altitudes in magnetically quiet, non-auroral conditions. Originally entirely empirical, the IRI now contains some physical modeling to fill in data gaps and to ensure output
consistency. Annual workshops continue to improve and add to the model [11] [12]. Detailed information can be found in [8], [9], [10] and [12] and the latest updates can be found in [11].

3.1.3 Neutral Atmosphere/Ionosphere Signal Delay Model

The Neutral Atmosphere/Ionosphere Signal Delay Model (NAISD) is the combination of the IRI-2001 and the GRAM-99 (Global Reference Atmospheric Model - 1999). It ray-traces a GPS signal through the neutral atmosphere and ionosphere between a satellite and a receiver. The ray-tracing is performed on each model separately, the delays are summed and the results are printed to a text file. The results were validated against several other models and methods, some of the details of which are presented in [13]. Detailed documentation is available in [14] and [15]. The NAISD is the property of the Charles Stark Draper Laboratory and is not currently available to the general public.

3.1.4 Augmented Klobuchar Model

The Klobuchar model has two significant drawbacks for a space-based mission. The most salient drawback of the Klobuchar model for a GPS receiver in space is the explicit assumption that the receiver is below the entire ionosphere; the model has no altitude dependency. Secondly, the Klobuchar model is less and less accurate at lower elevation angles and fails completely for negative line-of-sight elevation angles. Although sufficient for terrestrial GPS use, such a model is inappropriate for boosting, orbiting or re-entering vehicles.

To “augment” the Klobuchar model for use in a space-based mission, J. Arnold Soltz created an exponential altitude mapping function for the model. This mapping function was designed to match data generated by the NAISD and to smoothly transition to the original Klobuchar model at 200 km and tends to zero at high altitudes.

\[
\Delta \tilde{T} = f(h) \Delta T_{\text{Klobuchar}}
\]

\[
f(h) = \begin{cases} 
  e^{A_2 x^2 + A_4 x^4 + A_6 x^6} & h > h_0 \\
  1 & h \leq h_0 
\end{cases}
\]

\[
x = \frac{h - h_0}{H}
\]

For which \( h \) is the receiver altitude in kilometers and the other parameters are defined in table 3.1.4-1.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>h₀</td>
<td>Mapping function floor</td>
<td>200 km</td>
</tr>
<tr>
<td>H</td>
<td>Scaling factor</td>
<td>1000</td>
</tr>
<tr>
<td>A₂</td>
<td>Second-order exponential coefficient</td>
<td>-15.3327</td>
</tr>
<tr>
<td>A₄</td>
<td>Fourth-order exponential coefficient</td>
<td>35.2893</td>
</tr>
<tr>
<td>A₆</td>
<td>Sixth-order exponential coefficient</td>
<td>-32.9825</td>
</tr>
</tbody>
</table>

Table 3.1.4-1: Augmented Klobuchar Parameters

Figures 3.1.4-1 and 3.1.4-2 show the performance of the augmented Klobuchar model compared to the original Klobuchar and the NAISD "truth." Note the plots are of ionospheric delay in meters versus altitude.

![Figure 3.1.4-1: Klobuchar Model, Augmented Klobuchar Model and True Ionospheric Group Delay Versus Altitude](image1)

![Figure 3.1.4-2: Klobuchar Model Error and Augmented Klobuchar Model Error Versus Altitude](image2)

Figures 3.1.4-3 and 3.1.4-4 show the NAISD and the augmented Klobuchar ionospheric delay first and second derivatives versus altitude.

![Figure 3.1.4-3 and 3.1.4-4](image3)
The NAISD artificially drops to zero at 1050 km, causing the discontinuity in rate at this point in figure 3.1.4-3. The augmented Klobuchar transitions to the original model at 200 kilometers, causing the flattening of the rate at this point in figure 3.1.4-3 (in some cases, there is also a slight discontinuity in the modeled delay itself because the transition to the Klobuchar model at 200 km isn’t completely smooth). The flattening of the rate causes the discontinuity in the modeled second derivative seen in figure 3.1.4-4. The “noise” observed in the NAISD delay second derivatives is due to computer round-off. The NAISD prints the ionospheric delays to a text file which are subsequently read and used to calculate delay rates and delay second derivatives. The round-off error is visible in the derivatives, but small enough to not be noticeable in the delays themselves.

3.2 Estimated Ionospheric Delay

Because of regional and temporal variations in the ionosphere, the use of a model to correct for ionospheric signal delay may not be the best choice. An alternative solution is to estimate the delay using a Kalman filter or similar estimator. The most common method of estimating the ionosphere involves using a single-state filter with a smoother and a large time constant to suppress noise. Although sufficient for terrestrial receivers for which the rate of change of the ionospheric group delay over a few minutes is very small, this filter would be unable to handle the rapidly-changing delays experienced by launching, orbiting or re-entering vehicles.
3.2.1 Measurements from Pseudo-Range and Delta-Range

Confusion can be created when referring to “measured” pseudo-ranges and delta-ranges and the Kalman filter definition of “measurements.” For this reason, pseudo-range and delta-range measurements will be referred to as raw data. The Kalman filter measurement vector will consist of linear combinations of these data, the elements of which will be called “measurements.”

Raw pseudo-range is defined as the difference between time of reception and time of transmission of a signal. This difference multiplied by the speed of light gives pseudo-range in units of distance [16]:

\[ \rho = c(t_{\text{received}} + \Delta t_{\text{rcvr}}) - (t_{\text{sent}} + \Delta t_{\text{sat}}) \]  
(3.2.1-1)

For which \( c \) is the speed of light, \( t_{\text{received}} \) is the true time of reception of the GPS signal, \( \Delta t_{\text{rcvr}} \) is the receiver clock offset, \( t_{\text{sent}} \) is the true time of transmission of the GPS signal and \( \Delta t_{\text{sat}} \) is the satellite’s clock offset.

Defining:

\[ B_{\text{rcvr}} = c\Delta t_{\text{rcvr}} \quad \text{and} \quad B_{\text{sat}} = c\Delta t_{\text{sat}} \]  
(3.2.1-2)

We have:

\[ \rho = c(t_{\text{received}} - t_{\text{sent}}) + B_{\text{rcvr}} - B_{\text{sat}} \]  
(3.2.1-3)

There are many physical phenomena which delay the time of reception from what it would have been had the signal traveled through a vacuum, relative acceleration and velocity of satellite and receiver were zero, etc. Quantifying these effects in terms of distance, we have:

\[ c(t_{\text{received}} - t_{\text{sent}}) = R + \Delta A + \Delta I + \Delta E + \cdots + \xi \rho \]  
(3.2.1-4)

For which \( R \) is the true range, \( \Delta A \) is the atmospheric delay of the signal in units of distance, \( \Delta I \) is the ionospheric group delay of the signal, \( \Delta E \) is the relativist contribution and \( \xi \rho \) is modeled as Gaussian white noise. Several other phenomena could be included such as antenna and other hardware effects, but are not itemized for the sake of brevity.

Raw pseudo-range has several terms which are common (approximately) across the GPS frequencies considered here. These common mode terms are lumped for our purposes:

\[ C = R + \Delta A + \Delta E + B_{\text{rcvr}} - B_{\text{sat}} + \cdots \]  
(3.2.1-5)

Pseudo-range can then be defined as:

\[ \rho = C + \Delta I + \xi \rho \]  
(3.2.1-6)
Differencing pseudo-ranges taken from two different frequencies at the same time removes the common mode terms and leaves the difference between the ionospheric delays and two noise terms:

\[ \rho_{L2} - \rho_{L1} = \Delta I_{L2} - \Delta I_{L1} + \xi \rho_{L2} - \xi \rho_{L1} \] 

(3.2.1-7)

The ionospheric delay is approximately inversely proportional to the square of the frequency being measured [17]. In some cases, there is an inter-frequency bias from the receiver hardware being used. This bias is caused by electrical path length differences between frequencies. It is constant for all satellites over the mission time and is at most a few meters. If ignored by the user, it behaves as a small clock error and does not affect the navigation solution [18]. It will, however, affect the estimation of the ionospheric group delay. Including such a bias, the two-frequency pseudo-range difference equation becomes:

\[
\rho_{L2} - \rho_{L1} = \frac{A}{f_{L2}^2} - \frac{A}{f_{L1}^2} + B_{L2} + \xi \rho_{L2} - \xi \rho_{L1} \\
= A \left( \frac{1}{f_{L2}^2} - \frac{1}{f_{L1}^2} \right) + B_{L2} + \xi \rho_{L2} - \xi \rho_{L1} \\
= A \left( \frac{f_{L1}^2}{f_{L1}^2 f_{L2}^2} - \frac{f_{L2}^2}{f_{L1}^2 f_{L2}^2} \right) + B_{L2} + \xi \rho_{L2} - \xi \rho_{L1} \\
= A \left( \frac{f_{L1}^2 - f_{L2}^2}{f_{L1}^2 f_{L2}^2} \right) + B_{L2} + \xi \rho_{L2} - \xi \rho_{L1} 
\] 

(3.2.1-8)

For which A is the non-frequency-dependent ionospheric delay of the GPS signal. The convention of subtracting L1 from L2 is used since L1 is the higher frequency, making the L1 pseudo-range the smaller of the two pseudo-ranges and yielding a positive difference of the two (ignoring the effects of the bias which could be positive or negative). To get the ionospheric delay in terms of the L1 frequency:

\[
\Delta I_{L1} + \ldots \approx A \left( \frac{f_{L1}^2 - f_{L2}^2}{f_{L1}^2 f_{L2}^2} \right) + \left( B_{L2} + \xi \rho_{L2} - \xi \rho_{L1} \right) \left( \frac{f_{L2}^2}{f_{L1}^2 - f_{L2}^2} \right) \\
= A \left( \frac{1}{f_{L1}^2} \right) + \left( B_{L2} + \xi \rho_{L2} - \xi \rho_{L1} \right) \left( \frac{f_{L2}^2}{f_{L1}^2 - f_{L2}^2} \right) \\
\approx \left( \rho_{L2} - \rho_{L1} \right) \left( \frac{f_{L2}^2}{f_{L1}^2 - f_{L2}^2} \right) 
\] 

(3.2.1-9)

The multiplier can be simplified:
\[
\frac{f_{L2}^2}{f_{L1}^2 - f_{L2}^2} = \left( \frac{f_{L1}^2 - f_{L2}^2}{f_{L2}^2} \right)^{-1} = \left( \frac{f_{L1}^2}{f_{L2}^2} - 1 \right)^{-1} = \frac{1}{(w_{L2} - 1)}
\]

So:
\[
\left( \rho^{L2} - \rho^{L1} \right) \left( \frac{1}{(w_{L2} - 1)} \right) = \Delta t^{L1} + (B_{L2} + \xi \rho^{L2} - \xi \rho^{L1}) \left( \frac{1}{(w_{L2} - 1)} \right)
\]

With the term, \(w_{L2}\), defined in table 3.2.1-1.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>(154 * 10.23e6 \text{Hz} = 1575.42 \text{MHz})</td>
</tr>
<tr>
<td>L2</td>
<td>(120 * 10.23e6 \text{Hz} = 1227.60 \text{MHz})</td>
</tr>
<tr>
<td>L2 – L1 conversion</td>
<td>(w_{L2} = \left( \frac{f_{L1}}{f_{L2}} \right)^2 \approx 1.6469)</td>
</tr>
</tbody>
</table>

Table 3.2.1-1: Frequency Values and Conversion Term

Isolating the measurement part of equation 3.2.1-11 for use in the Kalman filter:
\[
\left( \rho^{L2} - \rho^{L1} \right) = (w_{L2} - 1)\Delta t^{L1} + B_{L2} + \xi \rho^{L2} - \xi \rho^{L1}
\]

Delta-range is defined as:
\[
\delta^{L1}(t_i, t) = \frac{c}{f_{L1}} \int_{t_i}^{t} (f_t - f_{L1}) dt = \sum_{t_i} \delta^{L1}_{t_i}
\]

Raw delta-range can be written as:
\[
\delta(t_i, t) = C_r - C_{t_i} - \Delta L_i + \Delta L_{t_i} + \xi \delta_t
\]

It is important to note that delta-range is usually accumulated coherently. This means that the accumulation (summing) of delta-ranges for which the end time of one corresponds to the start
time of the next does not produce a random walk. This is most easily understood by looking at
the phase of the carrier signal. If we kept track of the integer wavelengths as well as phase, this
would look like:

\[
\phi_k = C_k - \Delta I_k + \Delta A_k + \xi \phi_k
\]

\[
\phi_{k+1} = C_{k+1} - \Delta I_{k+1} + \Delta A_{k+1} + \xi \phi_{k+1}
\]

\[
\phi_{k+2} = C_{k+2} - \Delta I_{k+2} + \Delta A_{k+2} + \xi \phi_{k+2}
\]

(3.2.1-15 : 3.2.1-17)

Delta-range is the difference between successive “phase” measurements:

\[
\delta(k, k + 1) = \phi_{k+1} - \phi_k
\]

\[
= C_{k+1} - \Delta I_{k+1} + \Delta A_{k+1} + \xi \phi_{k+1}
\]

\[
- C_k + \Delta I_k - \Delta A_k - \xi \phi_k
\]

(3.2.1-18)

\[
\delta(k + 1, k + 2) = \phi_{k+2} - \phi_{k+1}
\]

\[
= C_{k+2} - \Delta I_{k+2} + \Delta A_{k+2} + \xi \phi_{k+2}
\]

\[
- C_{k+1} + \Delta A_{k+1} - \Delta A_k - \xi \phi_{k+1}
\]

(3.2.1-19)

Summing the two:

\[
\delta_{k+1} + \delta_{k+2} = C_{k+2} - \Delta I_{k+2} + \Delta A_{k+2} + \xi \phi_{k+2}
\]

\[
- C_k + \Delta I_k - \Delta A_k - \xi \phi_k
\]

(3.2.1-20)

Like pseudo-range differencing, differencing delta-ranges from two frequencies over the same
integration period cancels out the common mode terms, leaving the ionospheric delays and noise:

\[
\delta^{L1}(t_1, t_2) - \delta^{L2}(t_1, t_2) = -2 \Delta I^{L1}_{t_1} + \Delta I^{L2}_{t_1} - \Delta I^{L1}_{t_2} + \Delta I^{L2}_{t_2} + \xi \delta^{L1}_{t_1} - \xi \delta^{L2}_{t_1}
\]

(3.2.1-21)

Applying the same logic as in equations 3.2.1-9 through 3.2.1-12 to convert the equation to be
solely in terms of the L1 ionospheric group delay, equation 3.2.1-21 becomes:

\[
\delta^{L1}(t_1, t_2) - \delta^{L2}(t_1, t_2) = (w_{L2} - 1) \Delta I^{L1}_{t_1} + (1 - w_{L2}) \Delta I^{L1}_{t_2} + \xi \delta^{L1}_{t_1} - \xi \delta^{L2}_{t_1}
\]

(3.2.1-22)

A third type of measurement is available from the raw pseudo-range and delta-range data:
single-frequency ionospheric rate measurements. These measurements make use of the carrier
phase advance to provide information about the rate of change of the ionospheric group delay
from a reference time to the current time. These rate measurements do not, however, provide
information about the initial delay, only the change from some initial value [16]. For L1:

\[
\rho^{L1}_t - \left( \rho^{L1}_{t_1} + \delta^{L1}(t_1, t) \right) = 2 \Delta I^{L1}_t - 2 \Delta I^{L1}_{t_1} + \xi \rho^{L1}_t - \xi \rho^{L1}_{t_1} - \xi \delta^{L1}_t
\]

(3.2.1-23)

Rate measurements for L2 require the conversion term defined in table 3.2.1-1:
3.2.2 Discrete, Linear, Time-Invariant Kalman Filter

Most physical systems are non-linear. Although a version of the Kalman filter, the extended Kalman filter (EKF), was developed for estimation of non-linear systems, it has some drawbacks. Besides added complexity, the most salient drawback is the EKF is not guaranteed to converge. A poor choice of system model would leave the user with a rather useless EKF and no way to determine the EKF was not correctly estimating the states. Many non-linear systems can be sufficiently represented by a linear model. Such models can be continuous or discrete. Most models are of discretely sampled continuous systems. This is the version that will be considered here. The continuous, linear, time-invariant (LTI) model takes the form:

\[
x = Ax + B_d u + B_s w
\]
\[y = Cx + D_d u + D_s w + v
\]

For which \( A \) is the continuous model dynamics matrix, \( B_d \) is the continuous model deterministic input matrix, \( B_s \) is the continuous model stochastic input matrix, \( C \) is the continuous model observation matrix, \( D_d \) is the continuous model deterministic coupling matrix and \( D_s \) is the continuous model stochastic coupling matrix. \( x \) is the continuous model state vector, \( u \) is the continuous model deterministic input vector, \( w \) is continuous process noise vector and \( v \) is the continuous measurement noise vector. The time-varying system takes a similar form except each matrix has a time dependency (i.e. \( A(t) \) instead of \( A \)). The discrete, LTI model takes the form:

\[
x_{k+1} = \Phi x_k + G_d u_k + G_s w_k
\]
\[y_k = H x_k + \Gamma_d u_k + \Gamma_s w_k + v_k
\]

For which \( \Phi \) is the state transition matrix and “transitions” the state estimate from one point in time to another, \( G_d \) is the discrete model deterministic input matrix, \( G_s \) is the discrete model stochastic input matrix, \( H \) is the discrete observation matrix and relates the states to the “observed” measurements, \( \Gamma_d \) is the discrete model deterministic coupling matrix and \( \Gamma_s \) is the discrete model stochastic coupling matrix. \( x_k \) is the discrete model state vector at time \( k \), \( u_k \) is the discrete model deterministic input vector, \( w_k \) is the discrete process noise vector and \( v_k \) is the discrete measurement noise vector. For the sake of brevity, only this version of the Kalman filter will be described as this is the version that will be used. Filtering can also be optimal or sub-optimal. The optimal version will be employed here since computation demand is not a consideration (sub-optimal filtering is often used in flight software for which the demand on the processor is of concern). Again, for brevity, the proof will not be presented here, however, the Kalman filter can be shown to provide the optimal estimate of the states as the state and state error from the previous time are uncorrelated:

\[
E[e_k^+] = 0 \quad \text{and} \quad E[\hat{x}_{k+1}^+, e_k^+]= 0
\]
With:

\[ e_k^* = x_k - \hat{x}_k^* \]  

(3.2.2-7)

[19] The measurement noise is modeled as zero-mean, Gaussian:

\[ E[v_k] = 0 \]  

(3.2.2-8)

\[ E[v_k v_i^T] = \begin{cases} R_k & i = k \\ 0 & i \neq k \end{cases} \]  

(3.2.2-9)

\[ E[v_k x_{k+1}^T] = 0 \]  

(3.2.2-10)

**R** is known as the measurement covariance matrix. The process noise is also modeled as zero-mean, Gaussian:

\[ E[w_k] = 0 \]  

(3.2.2-11)

\[ E[w_k w_i^T] = \begin{cases} Q_k & i = k \\ 0 & i \neq k \end{cases} \]  

(3.2.2-12)

\[ E[x_k w_{k+1}^T] = 0 \]  

(3.2.2-13)

**Q** is known as the process covariance matrix. For the case considered here, the process noise and measurement noise will be considered uncorrelated:

\[ E[v_k w_{k+1}^T] = 0 \]  

(3.2.2-14)

[20] The Kalman filter employed here will be used to passively estimate the ionospheric delay of GPS signals. This model has no deterministic input and there will be no modeled coupling between stochastic inputs and measurements. With these simplifications, our system becomes:

\[ \dot{x} = Ax + w \]  

\[ y = Cx + v \]  

(3.2.2-15 : 3.2.2-16)

In discrete form:

\[ x_{k+1} = \Phi x_k + w_k \]  

\[ y_k = H x_k + v_k \]  

(3.2.2-17 : 3.2.2-18)

To convert a continuous system into a discrete representation, we start with the uncontrolled, discrete state in general terms:
For the system considered in this thesis:

\[
x_{k+1} = \Phi(t_{k+1}, t_k)x_k + \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau)B_s(\tau)w(\tau)d\tau
\]  
(3.2.2-19)

With:

\[
\Phi(t_{k+1}, t_k) = e^{A(t_{k+1} - t_k)} = e^{A\Delta t}
\]  
(3.2.2-21)

The discrete, stochastic input is:

\[
w_k = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau)B_s(\tau)w(\tau)d\tau
\]  
(3.2.2-22)

The process covariance matrix, \(Q\), (from equation 3.2.2-12) becomes:

\[
Q_k = E\left[\begin{bmatrix}
\int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \gamma)B_s(\gamma)w(\gamma)d\gamma \\
\int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \eta)B_s(\eta)w(\eta)d\eta
\end{bmatrix}\right]^T
\]  
(3.2.2-23)

\[
= \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \gamma)E[w(\gamma)w^T(\eta)]B_s^T(\eta)\Phi^T(t_{k+1}, \eta)d\gamma d\eta
\]

For our case:

\[
Q_k = E\left[\begin{bmatrix}
\int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \gamma)w(\gamma)d\gamma \\
\int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \eta)w(\eta)d\eta
\end{bmatrix}\right]^T
\]  
(3.2.2-24)

\[
= \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \gamma)E[w(\gamma)w^T(\eta)]\Phi^T(t_{k+1}, \eta)d\gamma d\eta
\]

\(E[w(\gamma)w^T(\eta)]\) is a matrix of Dirac delta functions the values of which are usually known for the continuous system. We can then define:
\[ Q = \int_{t_i}^{t_{i+1}} \Phi(t_{k+1}, \eta) N \Phi(t_{k+1}, \eta)^T d\eta \quad (3.2.2-25) \]

For which:

\[ N = E[ww^T] \quad (3.2.2-26) \]

Note that \( N \) is the covariance for the continuous system. \( Q \) is that of the discrete. The Kalman filter propagation equations for the state vector and state covariance matrix are, respectively:

\[ \dot{x}_{k+1} = \Phi \hat{x}_k \]

\[ P_{k+1} = \Phi P_k \Phi^T + Q_k \quad (3.2.2-28) \]

For which:

\[ P_k = E[e^{-T}e_k e_k^T] \quad (3.2.2-29) \]

\( P \) is the state covariance matrix, the diagonals of which give the variances of the state estimates. The Kalman update equations are:

\[ K_k = P_k H^T (HP_k H^T + R)^{-1} \]

\[ \dot{x}_k = \hat{x}_k + K_k (z_k - H_k \hat{x}_k) \]

\[ P_k^+ = (I - K_k H)P_k^- (I - K_k H)^T + K_k R K_k^T \quad (3.2.2-32) \]

For which \( K \) is the optimal Kalman gain [19]. To ensure the symmetry of the state covariance matrix (asymmetry occurs due to computer round-off error):

\[ P_k^+ = \frac{P_k^- + (P_k^+)^T}{2} \quad (3.2.2-33) \]

The state covariance update in equation 3.2.2-32 is called the “Joseph form” of the update. If the optimal Kalman gain is used at each recursion (and only if the optimal gain is used), an equivalent form is:

\[ P_k^+ = (I - K_k H)P_k^- \quad (3.2.2-34) \]
Although this form is equivalent to the form in equation 3.2.2-32 for the optimal Kalman gain, numerical error (error introduced from computer round-off) is greater for the version in equation 3.2.2-34. For this reason, equation 3.2.2-32 will be used for the state covariance update.

In any Kalman filter, the issue of state observability arises. Observability refers to the ability of the filter to estimate linearly independent states. The rank of the observability matrix gives the number of linearly independent states that can be estimated. The observability matrix is defined as:

$$Q_{obs} = [H | H\Phi | H\Phi^2 | H\Phi^3 | ... | H\Phi^{n-1}]$$

(3.2.2-35)

For a constant observation matrix. n is the number of states. For filters in which the observation matrix is not constant (usually if a measurement is missing):

$$Q_{obs} = [H_k | H_{k+1}\Phi | H_{k+2}\Phi^2 | H_{k+3}\Phi^3 | ... | H_{k+n-1}\Phi^{n-1}]$$

(3.2.2-36)

Although it is not necessarily a “bad” filter if the rank of the observability matrix is less than the number of states, the user must be careful to note which states are not observable by themselves and what linear combinations of the states are observable.

### 3.2.3 Single-State Filter

Most terrestrial receivers use single-state filters to estimate the ionospheric delay. Although the exact formulations of such filters are proprietary, the concept is the same in all of them. Single-state filters with long time-constants are used to essentially average ionosphere-induced delay estimates. For a terrestrial receiver, the ionospheric signal delays do not change very rapidly. The user is not moving quickly enough nor is the ionosphere changing rapidly enough to cause high rates of change in the delay. A large source of error for a terrestrial receiver is multipathing. The best way to “correct” for multipathing effects is to simply average them out. [1]

### 3.2.4 Five-State Filter

Recently, Charles Stark Draper Laboratory designed, implemented and is in the process of validating a real-time Kalman filter for vehicles which experience rapid changes in ionospheric delays of GPS signals. This filter estimates the L1 ionospheric group delay, delay rate and delay second derivative for the purpose of ridding the GPS signal of ionospheric group delay effects. Three measurements were used as input to the five-state filter: dual-frequency pseudo-range differencing and single-frequency rate measurements for both frequencies used (L1 and L2). From section 3.2.1, the measurement equations are:
\[
\begin{align*}
    z_{1,t} &= 2 \Delta I_{11} - 2 \Delta I_{11} + \xi \xi_{11} - \xi \xi_{11} - \xi \delta_{11} \\
    z_{2,t} &= 2 w_{L1} \Delta I_{11} - 2 w_{L1} \Delta I_{11} + \xi \xi_{12} - \xi \xi_{12} - \xi \delta_{12} \\
    z_{3,t} &= (w_{L2} - 1) \Delta I_{11} + B_{L2} + \xi \xi_{12} - \xi \xi_{11}
\end{align*}
\] (3.2.4-1 : 3.2.4-3)

The five-state filter models the ionospheric delay dynamics as the third integral of white noise. Two constant states (sometimes referred to as Kalman delay states) are required to retain data from the reference times, data which are necessary to process the rate measurements. The five-state filter requires a careful re-set of states and the state covariance matrix when one or more of the three measurements are lost as the reference times (and corresponding data) for the rate measurements change.

Seven possible states for the filter are listed in table 3.2.4-1. The dynamics for this alternate filter are presented because doing so will aid the reader in understanding the derivation of the five-state filter.

<table>
<thead>
<tr>
<th>State</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1^a)</td>
<td>(\Delta I_{11}) m</td>
</tr>
<tr>
<td>(x_2^a)</td>
<td>(dx_1^a/dt) (\Delta I_{11}) m/s</td>
</tr>
<tr>
<td>(x_3^a)</td>
<td>(dx_2^a/dt) (\Delta I_{11}) m/s²</td>
</tr>
<tr>
<td>(x_4^a)</td>
<td>constant (\Delta I_{11}) m</td>
</tr>
<tr>
<td>(x_5^a)</td>
<td>constant (\xi \xi_{11}) m</td>
</tr>
<tr>
<td>(x_6^a)</td>
<td>constant (\Delta I_{11}) m</td>
</tr>
<tr>
<td>(x_7^a)</td>
<td>constant (\xi \xi_{12}) m</td>
</tr>
</tbody>
</table>

Table 3.2.4-1: Alternate State Definitions

Various reference pseudo-range data are needed for measurements involving accumulated delta-ranges. The reference times are defined in table 3.2.4-2.

<table>
<thead>
<tr>
<th>Reference Time</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t_1)</td>
<td>Start of L1 rate measurement interval</td>
</tr>
<tr>
<td>(t_2)</td>
<td>Start of L2 rate measurement interval</td>
</tr>
</tbody>
</table>

Table 3.2.4-2: Reference Time Definitions
These states are the simplest way to define this problem and the easiest way to understand the equations behind it, but they are not necessarily the best choices. The states described here also assume no or a negligible inter-frequency bias.

The dynamics equations are:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= w
\end{align*}
\]

(3.2.4-4 : 3.2.4-6)

The process dynamics matrix for this filter is:

\[
A^* = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(3.2.4-7)

Taking \( \Phi(\Delta t, 0) = e^{A\Delta t} \), the state transition matrix becomes:

\[
\Phi^*(\Delta t, 0) = \begin{bmatrix}
1 & \Delta t & \frac{1}{2} \Delta t^2 & 0 & 0 & 0 & 0 \\
0 & 1 & \Delta t & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(3.2.4-8)

The three process noise values are considered uncorrelated with each other, meaning the outer product of the vector of the three noise values is a diagonal matrix. The diagonal terms are listed in table 3.2.4-3:
Table 3.2.4-3: Process Noise Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{11}$</td>
<td>0 m²/Hz</td>
</tr>
<tr>
<td>$N_{22}$</td>
<td>0 (m/s)²/Hz</td>
</tr>
<tr>
<td>$N_{33}$</td>
<td>2e-7 (m/s)²/Hz</td>
</tr>
</tbody>
</table>

The process covariance matrix can then be defined as:

\[ Q = \int_{t_k}^{t_{k+1}} \Phi N \Phi^T \text{ or } Q = \int_{t_k}^{t_{k+1}} \Phi N_1 \Phi^T + \int_{t_k}^{t_{k+1}} \Phi N_2 \Phi^T + \int_{t_k}^{t_{k+1}} \Phi N_3 \Phi^T \]

\((3.2.4-9 : 3.2.4-10)\)

For which $N_m$ is a matrix of all zeros except for the $N_{mm}$ element which has the value listed in table 3.2.4-3.

From equation 3.2.4-9 or 3.2.4-10, $Q$ becomes:

\[
Q^*(7 \times 7) = \begin{bmatrix}
N_{11} \Delta t + N_{22} \frac{\Delta t^3}{3} + N_{33} \frac{\Delta t^5}{20} & \frac{N_{22} \Delta t^2}{2} + N_{33} \frac{\Delta t^4}{8} & N_{33} \frac{\Delta t^3}{6} & 0 & \cdots \\
N_{22} \frac{\Delta t^2}{2} + N_{33} \frac{\Delta t^4}{8} & N_{22} \Delta t + N_{33} \frac{\Delta t^3}{3} & N_{33} \frac{\Delta t^2}{2} & 0 & \cdots \\
N_{33} \frac{\Delta t^3}{6} & N_{33} \frac{\Delta t^2}{2} & N_{33} \Delta t & 0 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix}
\]

\((3.2.4-11)\)

With the states in table 3.2.4-1, the observability matrix (equation 3.2.2-35 or 3.2.2-36) is rank deficient even if all measurements are available over the period in consideration. For this filter, the linearly independent states (with all measurements available) are defined in table 3.2.4-4.

If it is suspected that the inter-frequency bias is not negligible, either it can be calibrated out (as discussed briefly in section 4.2) or it can be included in the filter states. The bias cannot be separated from the ionospheric delay if a separate filter is used for each satellite. To maintain the separate filter arrangement, the ionospheric delay plus L2 bias is estimated. The bias must also
be included in constant states. These states are most easily understood if written in terms of the alternate states defined in table 3.2.4-1.

<table>
<thead>
<tr>
<th>State</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
<td>(x_1^a + \frac{B_{L2}}{w_{L2} - 1}) m</td>
</tr>
<tr>
<td>(x_2)</td>
<td>(dx_1/dt) (x_2^a) m/s</td>
</tr>
<tr>
<td>(x_3)</td>
<td>(dx_2/dt) (x_3^a) m/s^2</td>
</tr>
<tr>
<td>(x_4)</td>
<td>constant (x_4^a + \frac{B_{L2}}{w_{L2} - 1} + \frac{x_5^a}{2}) m</td>
</tr>
<tr>
<td>(x_5)</td>
<td>constant (x_5^a + \frac{B_{L2}}{w_{L2} - 1} + \frac{x_7^a}{2w_{L2}}) m</td>
</tr>
</tbody>
</table>

**Table 3.2.4-4: Linearly Independent Filter States Accounting for Bias**

The addition of the L2 frequency bias to the ionospheric delay states is clear enough. The third terms in the constant states (four and five) are easily understood by examining the equations for rate measurements. For L1 rate measurements:

\[
z_{1,1} = 2x_1^a - 2x_4^a - x_5^a + \left(\xi\rho_{i1} - \xi\delta_{i1}\right) \tag{3.2.4-12}
\]

Adding and subtracting the bias term (a true bias does not affect either rate measurement) yields:

\[
z_{1,1} = 2\left(x_1^a + \frac{B_{L2}}{w_{L2} - 1}\right) - 2\left(x_4^a + \frac{B_{L2}}{w_{L2} - 1}\right) - x_5^a + \left(\xi\rho_{i1} - \xi\delta_{i1}\right)
\]

\[
= 2\left(x_1^a + \frac{B_{L2}}{w_{L2} - 1}\right) - 2\left(x_4^a + \frac{B_{L2}}{w_{L2} - 1}\right) + \frac{x_7^a}{2} + \left(\xi\rho_{i1} - \xi\delta_{i1}\right) \tag{3.2.4-13}
\]

\[
= 2x_1 - 2x_4 + \left(\xi\rho_{i1} - \xi\delta_{i1}\right)
\]

With the states defined as in table 3.2.4-4, the new measurements are:

\[
z_{1,1} = 2x_1 - 2x_2 + \left(\xi\rho_{i1} - \xi\delta_{i1}\right)
\]

\[
z_{2,1} = 2w_{L2}x_1 - 2w_{L2}x_4 + \left(\xi\rho_{i2} - \xi\delta_{i2}\right) \tag{3.2.4-14 : 3.2.4-16}
\]

\[
z_{3,1} = (w_{L2} - 1)x_1 + \left(\xi\rho_{i2} - \xi\delta_{i1}\right)
\]

The process dynamics, state transition, and process covariance matrices are identical to the ones in equations 3.2.4-7, 3.2.4-8 and 3.2.4-11 with the last two rows and columns removed. The observation matrix is:
$$H = \begin{bmatrix} 2 & 0 & 0 & -2 & 0 \\ 2w_{L2} & 0 & 0 & 0 & -2w_{L2} \\ w_{L2} - 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$ (3.2.4-17)

If measurements are occasionally missing, this does not necessarily affect the observability of the states. If measurements are missing for long periods of time (see equation 3.2.2-36), observability of certain states will be lost.

From equations 3.2.4-14 through 3.2.4-16, the measurement noise vector is:

$$v_k = \begin{bmatrix} \xi \rho^{L1}_k - \xi \rho^{L1}_k \\ \xi \rho^{L2}_k - \xi \rho^{L2}_k \\ \xi \rho^{L2}_k - \xi \rho^{L1}_k \end{bmatrix}$$ (3.2.4-18)

Defining:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V^{L1}_\rho$</td>
<td>$E[(\xi \rho^{L1}_k)^2]$</td>
<td>$(0.5m)^2$</td>
</tr>
<tr>
<td>$V^{L2}_\rho$</td>
<td>$E[(\xi \rho^{L2}_k)^2]$</td>
<td>$(0.5m)^2$</td>
</tr>
<tr>
<td>$V^{L1}_\delta$</td>
<td>$E[(\xi \delta^{L1}_k)^2]$</td>
<td>$(0.01m)^2$</td>
</tr>
<tr>
<td>$V^{L2}_\delta$</td>
<td>$E[(\xi \delta^{L2}_k)^2]$</td>
<td>$(0.01m)^2$</td>
</tr>
</tbody>
</table>

Table 3.2.4-5: Measurement Noise Values

It can be assumed that the pseudo-ranges and delta-ranges are uncorrelated with each other and with those from other frequencies as they come from different sources:

$$R = E[v_k v_k^T] = \begin{bmatrix} V^{L1}_\rho + V^{L1}_\delta & 0 & V^{L1}_\rho \\ 0 & V^{L2}_\rho + V^{L2}_\delta & V^{L2}_\rho \\ V^{L1}_\rho & V^{L2}_\rho & V^{L2}_\rho + V^{L1}_\rho \end{bmatrix}$$ (3.2.4-19)

To process measurements, certain criteria must be met. If L1 pseudo-range and delta-range are good (present, within expected limits, etc.) and there is L1 accumulated delta-range, L1 rate measurements can be used:
\[
\delta^{L1}(t_1, t_k) = \delta^{L1}(t_1, t_{k-1}) + \delta^{L1}(t_{k-1}, t_k) \quad (3.2.4-20)
\]
\[
z_{1,k} = \rho_{k}^{L1} - \rho_{k}^{L1} - \delta^{L1}(t_1, t_k) \quad (3.2.4-21)
\]

If L2 pseudo-range and delta-range are good and there is L2 accumulated delta-range, L2 rate measurements can be used:

\[
\delta^{L2}(t_2, t_k) = \delta^{L2}(t_2, t_{k-1}) + \delta^{L2}(t_{k-1}, t_k) \quad (3.2.4-22)
\]
\[
z_{2,k} = \rho_{k}^{L2} - \rho_{k}^{L2} - \delta^{L2}(t_2, t_k) \quad (3.2.4-23)
\]

If both L1 and L2 pseudo-ranges are good, dual-frequency pseudo-range measurements can be used:

\[
z_{3,k} = \rho_{k}^{L2} - \rho_{k}^{L1} \quad (3.2.4-24)
\]

The measurement vector if all measurements are used is clearly:

\[
z_k = \begin{bmatrix} z_{1,k} & z_{2,k} & z_{3,k} \end{bmatrix}^T \quad (3.2.4-25)
\]

If any of the three measurements are missing or purposely neglected, the row of the observation matrix, \( H \), corresponding to that measurement is removed. The measurement vector is easily reduced as the missing measurement is simply omitted. The measurement covariance matrix, \( R \), is reduced by removing the row and column corresponding to the missing measurement. These matrices and the measurement vector can be adjusted in this way for multiple missing measurements. The state and covariance matrices are then updated according to equations 3.2.2-30 through 3.2.2-32.

It is likely that there will be times when certain raw data is unavailable. When this happens, certain states must be re-initialized. If re-initializing because of a corrupt or missing L1 delta-range measurement, the accumulated delta-range must be reset:

\[
t_1 = t_k \quad \text{and} \quad \delta^{L1}(t_1, t_k) = 0 \quad (3.2.4-26)
\]

The state vector becomes:

\[
\hat{x}_k^* = \begin{bmatrix} \_ & \_ & \hat{x}_{1,k} \end{bmatrix}^T \quad (3.2.4-27)
\]

The dashes indicate that the element remains unchanged. The covariance matrix becomes:
This correlates the fourth state with the first state. States which are unaffected by the missing data are retained. If the L1 pseudo-range measurement is corrupt or missing, the accumulated delta-range for L1 rate measurements needs to be re-initialized, but cannot be until a good pseudo-range measurement becomes available (a reference pseudo-range is needed at the same time as the delta-range accumulation starts). When this occurs, the re-initialization process outlined above is accomplished.

Similarly, if re-initializing because of a corrupt or missing L2 delta-range measurement, the L2 accumulated delta-range must be reset:

\[ t_2 = k \quad \text{and} \quad \delta^{L2}(t_2, k) = 0 \] (3.2.4-29)

The state vector becomes:

\[ \hat{\mathbf{x}}_k = \begin{bmatrix} \hat{x}_{1,k} \\ \hat{x}_{2,k} \\ \vdots \end{bmatrix} \] (3.2.4-30)

The covariance matrix is:

\[ \mathbf{P}_k = \begin{bmatrix} - & - & - & P_{11,k}^- \\ - & - & - & 0 \\ - & - & - & 0 \\ P_{11,k}^- & 0 & 0 & P_{11,k}^- + V_L^{L2} \end{bmatrix} \] (3.2.4-31)

If the pseudo-range measurement is corrupt or missing, the accumulated delta-range for L1 rate measurements needs to be re-initialized, but cannot be until a good pseudo-range measurement becomes available.

Following any re-initialization, the state vector and covariance matrix are propagated according to equations 3.2.2-27 and 3.2.2-28 and ready for additional measurements [18].
Chapter 4

Proposed Methods of Handling Dynamic Ionospheric Delay

The following methods use the five-state filter described in section 3.2.4 as a starting point. The goal was to improve upon this filter. Three areas in particular were examined: the handling of the constant states, alternate process models and methods for improving the behavior of the filter when experiencing measurement or signal losses for extended periods of time.

4.1 Constant States

It was decided that instead of continuously accumulating delta-range and keeping track of pseudo-ranges from multiple times, the reference times and pseudo-ranges would always be those from one second prior. It was discovered that the choice of constant states in table 4.1-1 not only rid the user of the need for the state and state covariance reset logic outlined in section 3.2.4, but the states are all observable and an inter-frequency bias is easily included.
Using the same dynamics equations as used by the five-state filter:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= w
\end{align*}
\] (4.1-1 : 4.1-3)

The process dynamics matrix is:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] (4.1-4)

The state transition matrix is:
Using equation 3.2.4-10, the process covariance matrix is:

\[
\begin{bmatrix}
N_{11}\Delta t + N_{22} \frac{\Delta t^3}{3} + N_{33} \frac{\Delta t^5}{20} & N_{22} \frac{\Delta t^2}{2} + N_{33} \frac{\Delta t^4}{8} & N_{33} \frac{\Delta t^3}{6} & 0 & \ldots \\
N_{22} \frac{\Delta t^2}{2} + N_{33} \frac{\Delta t^4}{8} & N_{22}\Delta t + N_{33} \frac{\Delta t^3}{3} & N_{33} \frac{\Delta t^2}{2} & 0 & \ldots \\
N_{33} \frac{\Delta t^3}{6} & N_{33} \frac{\Delta t^2}{2} & N_{33}\Delta t & 0 & \ldots \\
0 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

\(Q(8 \times 8)\)

Similar to the reset performed in section 3.2.4, each time the state vector and covariance matrix are propagated, the appropriate correlations must be made. The following matrix when multiplied by the state vector accomplishes these correlations by setting state four equal to state one, state six equal to state five and state eight equal to state seven. States five and seven are reset to zero (expected value for the Gaussian white noise terms):

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\(\Omega\)

So the multiplication accomplishes the following:
\[
\mathbf{\Omega} \mathbf{\hat{x}}_k = \begin{bmatrix} - & - & \hat{x}_{1,k}^- & 0 & x_{5,k}^- & 0 & x_{7,k}^- \end{bmatrix}^T
\]

(4.1-8)

When the covariance matrix is pre multiplied by \( \mathbf{\Omega} \) and post multiplied by its transpose, the appropriate correlations are made in the state covariance matrix:

\[
\mathbf{\Omega} \mathbf{P}_k^{-} \mathbf{\Omega}^T = \begin{bmatrix}
- & - & - & P_{11,k}^- & 0 & P_{15,k}^- & 0 & P_{17,k}^- \\
- & - & - & P_{21,k}^- & 0 & P_{25,k}^- & 0 & P_{27,k}^- \\
- & - & - & P_{31,k}^- & 0 & P_{35,k}^- & 0 & P_{37,k}^- \\
P_{11,k}^- & P_{12,k}^- & P_{13,k}^- & P_{11,k}^- & 0 & P_{15,k}^- & 0 & P_{17,k}^- \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
P_{51,k}^- & P_{52,k}^- & P_{53,k}^- & P_{51,k}^- & 0 & P_{55,k}^- & 0 & P_{57,k}^- \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
P_{71,k}^- & P_{72,k}^- & P_{73,k}^- & P_{71,k}^- & 0 & P_{75,k}^- & 0 & P_{77,k}^- \\
\end{bmatrix}
\]

(4.1-9)

Since this multiplication forces the variances of states five and seven to zero (erroneously indicating to the filter that these states’ values are known perfectly), an additional term must be added:

\[
\mathbf{\Lambda} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & V_{\rho}^{L1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & V_{\rho}^{L2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(4.1-10)

The new state covariance has the desired form:

\[
\mathbf{\Omega} \mathbf{P}_k^{-} \mathbf{\Omega}^T + \mathbf{\Lambda} = \begin{bmatrix}
- & - & - & P_{11,k}^- & 0 & P_{15,k}^- & 0 & P_{17,k}^- \\
- & - & - & P_{21,k}^- & 0 & P_{25,k}^- & 0 & P_{27,k}^- \\
- & - & - & P_{31,k}^- & 0 & P_{35,k}^- & 0 & P_{37,k}^- \\
P_{11,k}^- & P_{12,k}^- & P_{13,k}^- & P_{11,k}^- & 0 & P_{15,k}^- & 0 & P_{17,k}^- \\
0 & 0 & 0 & 0 & V_{\rho}^{L1} & 0 & 0 & 0 \\
P_{51,k}^- & P_{52,k}^- & P_{53,k}^- & P_{51,k}^- & 0 & P_{55,k}^- & 0 & P_{57,k}^- \\
0 & 0 & 0 & 0 & 0 & 0 & V_{\rho}^{L2} & 0 \\
P_{71,k}^- & P_{72,k}^- & P_{73,k}^- & P_{71,k}^- & 0 & P_{75,k}^- & 0 & P_{77,k}^- \\
\end{bmatrix}
\]

(4.1-11)
The covariance and state vector are then propagated by equations 3.2.2-27 and 3.2.2-28. To streamline the correlation followed by propagation process, the “reset” and propagation can be combined:

\[
\hat{x}_{k+1}^{-} = \Phi \Omega \hat{x}_k^+
\]  

(4.1-12)

\[
P_{k+1}^{-} = \Phi (\Phi P_k^+ \Phi^T + \Lambda) \Phi^T + Q
\]

(4.1-13)

\[
= \Phi \Omega P_k^+ \Omega^T \Phi^T + \Phi \Lambda \Phi^T + Q
\]

So we can define a new state transition matrix as:

\[
\Phi' = \Phi \Omega =
\begin{bmatrix}
1 & \Delta t & \frac{1}{2} \Delta t^2 & 0 & 0 & 0 & 0 \\
0 & 1 & \Delta t & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(4.1-14)

And a new process covariance matrix as:

\[
Q' = \Lambda + Q =
\begin{bmatrix}
N_1 \Delta t + N_{22} \frac{\Delta t^3}{3} + N_{33} \frac{\Delta t^5}{20} & N_{22} \frac{\Delta t^2}{2} + N_{33} \frac{\Delta t^4}{8} & N_{33} \frac{\Delta t^3}{6} & 0 & 0 & 0 & 0 & 0 \\
N_{22} \frac{\Delta t^2}{2} + N_{33} \frac{\Delta t^4}{8} & N_{22} \Delta t + N_{33} \frac{\Delta t^3}{3} & N_{33} \frac{\Delta t^2}{2} & 0 & 0 & 0 & 0 & 0 \\
N_{33} \frac{\Delta t^3}{6} & N_{33} \frac{\Delta t^2}{2} & N_{33} \Delta t & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & V_{\rho}^{L_1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & V_{\rho}^{L_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(4.1-15)

Using all four possible measurements from section 3.2.1:
\[ z_{1,k} = \rho_{t_1}^L - (\rho_{t-1}^L + \delta_{t_1}^L (t-1,t)) \]
\[ = 2x_1 - 2x_4 + x_5 - x_6 - \xi \delta_{t_1}^L \]

\[ z_{2,k} = \rho_{t_2}^L - (\rho_{t-1}^L + \delta_{t_2}^L (t-1,t)) \]
\[ = 2w_{t_2}x_1 - 2w_{t_2}x_4 + x_7 - x_8 - \xi \delta_{t_2}^L \]

\[ z_{3,k} = \hat{\rho}_{t_2} (t) - \hat{\rho}_{t_1} (t) \]
\[ = (w_{t_2} - 1)x_1 - x_5 + x_7 \]

\[ z_{4,k} = \delta^L (t-1,t) - \delta^L (t-1,t) \]
\[ = (w_{t_2} - 1)x_1 + (1 - w_{t_2})x_4 + \xi \delta_{t_1}^L - \xi \delta_{t_2}^L \]

The observation matrix for these states and measurements is:
\[
H = \begin{bmatrix}
2 & 0 & 0 & -2 & 1 & -1 & 0 & 0 \\
2w_{t_2} & 0 & 0 & -2w_{t_2} & 0 & 0 & 1 & -1 \\
w_{t_2} - 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
w_{t_2} - 1 & 0 & 0 & 1 - w_{t_2} & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The measurement noise vector is:
\[
v_k = \begin{bmatrix}
-\xi \delta_{t_1}^L \\
-\xi \delta_{t_2}^L \\
\xi \delta_{t_1}^L - \xi \delta_{t_2}^L \\
\end{bmatrix}
\]

The measurement covariance matrix is:
\[
R = \begin{bmatrix}
V_{t_1}^{L_1} & 0 & 0 & V_{t_1}^{L_1} \\
0 & V_{t_2}^{L_2} & 0 & V_{t_2}^{L_2} \\
0 & 0 & 0 & 0 \\
V_{t_1}^{L_1} & V_{t_2}^{L_2} & 0 & V_{t_1}^{L_1} + V_{t_2}^{L_2} \\
\end{bmatrix}
\]

The state vector and covariance matrix are updated according to equations 3.2.2-30 through 3.2.2-32 with the new state transition matrix and process covariance matrix defined in 4.1-14 and 4.1-15.
4.2 Inter-Frequency Bias

Originally, the five-state filter defined in section 5.2.4 did not include a bias term. It was discovered shortly after work began on this filter that there was an inter-frequency bias in hardware-in-the-loop simulations. Although extensive tests were not performed, the conclusion drawn was the bias was an electrical path length difference and it was a true constant, essentially unchanging with time or environmental conditions. Although this source of error in terms of the navigation solution acts as a constant receiver clock bias and requires no additional consideration at this level, it must be taken into account by a user attempting to estimate the ionospheric delay itself.

For the states as listed in table 4.1-1, the inclusion of an inter-frequency bias is very simple. When considering this bias in the measurement equations, only measurement three, pseudo-range differencing, is affected as was demonstrated in section 3.2.1.

\[ p_L^2 - p_L^1 = (w_{l2} - 1)\Delta I_l^1 + B_{l2} + \xi p_L^2 - \xi p_L^1 \]  
(4.2-1)

As stated earlier, unless either the bias or the ionospheric delay is known, it is not possible to separate the two using the filters considered in this thesis. If such a filter is initialized at a high enough altitude that the ionospheric delay can be considered negligible (close to zero and not quickly increasing), the bias can be estimated. If not, the states can be altered slightly so the filter estimates ionospheric delay plus the bias. Equation 4.2-1 can be re-arranged:

\[ p_L^2 - p_L^1 = (w_{l2} - 1)\left(\Delta I_l^1 + \frac{B_{l2}}{w_{l2} - 1}\right) + \xi p_L^2 - \xi p_L^1 \]  
(4.2-2)

State one becomes the ionospheric delay plus the bias divided by a conversion term. Looking at the other measurements, it can be seen that setting state four equal to the ionospheric delay at the previous time plus the bias divided by the conversion term is the only other change that needs to be made to the states to include the bias:

\[ x_4' = \Delta I_{l-1}^1 + \frac{B_{l2}}{w_{l2} - 1} \]  
\[ x_4' = \Delta I_{l-1}^1 + \frac{B_{l2}}{w_{l2} - 1} \]  
(4.2-3 : 4.2-4)

The bias terms in each of the remaining measurement equations fall out, leaving the rest of the measurement definitions unchanged.
The remaining constant states stay the same and since the bias is not changing, the rate terms in states two and three also remain the same.

Terrestrial receivers can roughly estimate an inter-frequency bias by comparing ionospheric delays from satellites at high and low elevations angles. Since the receivers considered in this paper are moving at high velocities through the ionosphere, this may not be as good of a solution as it is for terrestrial receivers.

4.3 Dynamics

4.3.1 Markov Processes

In this section, generalizations about various orders of Markov processes are made. Some generalizations which were fairly simple to verify were made for processes up to (inclusive) a fifth-order Markov process. These were made for ease in coding the Kalman filters for testing. For these generalizations, it was not possible to establish a definite pattern by considering only lower orders. Although the equations developed for first through fifth-order Markov processes may very well apply to higher orders, the limits of validity of these equations (if there are such limits) were not determined because high-order Markov processes are not commonly used and such a mathematical proof is beyond the scope of this thesis. Instead, it is simply stated that the generalizations have been demonstrated for first through fifth-order Markov processes.

Table 4.3.1-1 contains spectral properties of various Markov processes. Table 4.3.1-2 contains differential representations of these Markov processes [21].
### Table 4.3.1-1: Markov Process Properties

<table>
<thead>
<tr>
<th>Order of Markov Process</th>
<th>Power Spectral Density (PSD)</th>
<th>Autocorrelation Function</th>
<th>Correlation Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{2\beta_1 \sigma^2}{\omega^2 + \beta_1^2}$</td>
<td>$\sigma^2 e^{-\beta_1</td>
<td>\tau</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{4\beta_2^3 \sigma^2}{(\omega^2 + \beta_2^2)^2}$</td>
<td>$\sigma^2 e^{-\beta_2</td>
<td>\tau</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{16\beta_3^5 \sigma^2}{3(\omega^2 + \beta_3^2)^3}$</td>
<td>$\sigma^2 e^{-\beta_3</td>
<td>\tau</td>
</tr>
<tr>
<td>n</td>
<td>$\frac{(2\beta_n)^{2n-1} \sigma^2}{(2n-2)! (\omega^2 + \beta_n^2)^n}$</td>
<td>$\sigma^2 e^{-\beta_n</td>
<td>\tau</td>
</tr>
<tr>
<td>$n \rightarrow \infty$</td>
<td>$2\pi\sigma^2 \delta(\omega)$</td>
<td>$\sigma^2$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

### Table 4.3.1-2: Markov Process Differential Representations

<table>
<thead>
<tr>
<th>Order of Markov Process</th>
<th>Differential Equation</th>
<th>State-Space Formulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\dot{x} + \beta_1(t)x = w$</td>
<td>$\dot{x} = \beta_1(w - x)$</td>
</tr>
<tr>
<td>2</td>
<td>$\ddot{x} + 2\beta_2(t)\dot{x} + \beta_2^2(t)x = w$</td>
<td>$\dot{x}_1 = \beta_1(w - x_1)$ $\dot{x}_2 = \beta_2(x_1 - x_2)$</td>
</tr>
<tr>
<td>3</td>
<td>$\dddot{x} + 3\beta_3\ddot{x} + 3\beta_3^2(t)\dot{x} + \beta_3^3(t)x = w$</td>
<td>$\dot{x}_1 = \beta_1(w - x_1)$ $\dot{x}_2 = \beta_2(x_1 - x_2)$ $\dot{x}_3 = \beta_3(x_2 - x_3)$</td>
</tr>
<tr>
<td>n</td>
<td>$\frac{d^n x}{dt^n} + \beta_n^n(t)x + \sum_{k=1}^{n-1} n\beta_n^{n-k} \frac{d^k}{dt^k}x = w$</td>
<td>$\dot{x}_1 = \beta_n(w - x_1)$ $\dot{x}_2 = \beta_n(x_1 - x_2)$ $\dot{x}_3 = \beta_n(x_2 - x_3)$ $\ddot{x}<em>n = \beta_n(x</em>{n-1} - x_n)$</td>
</tr>
</tbody>
</table>

Appendix A shows the equivalence of the differential and state space representations. If different time constants are used for each state in column three, the dynamics model becomes a “cascade” of first-order Markov processes. This gives the user more parameters to adjust for various situations. Cascades of Markov processes are discussed in the next section.
First-Order Markov Processes:

From table 4.3.1-2, the dynamics model for the first-order Markov process is:

\[ \dot{x} = \beta (w - x) \]  

(4.3.1-1)

With the state-space representation:

\[ A = -\beta \]  

(4.3.1-2)

The state transition matrix is:

\[ \Phi(\Delta t, 0) = e^{-\beta \Delta t} \]  

(4.3.1-3)

With the variance of the white noise driving the Markov process equal to \( N \) and using equation 3.2.4-10, the process covariance matrix can be calculated (assuming the same integration interval as used for the state transition matrix):

\[ Q = \mathcal{F} e^{-\beta \Delta t} \mathcal{F}^T \]  

(4.3.1-4)

The steady-state process covariance matrix is:

\[ Q_{ss} = \frac{N}{2\beta} \]  

(4.3.1-5)

The variance on the state is simply equal to the state covariance matrix.

\[ E[\Delta I^2] = P \]  

(4.3.1-6)

Second-Order Markov Processes:

The dynamics model for the second-order Markov process is:

\[ \dot{x}_1 = \beta (w - x_1) \]  
\[ \dot{x}_2 = \beta (x_1 - x_2) \]  

(4.3.1-7 : 4.3.1-8)

The process dynamics matrix is:
\[ A = \begin{bmatrix} -\beta & 0 \\ \beta & -\beta \end{bmatrix} \]  

(4.3.1-9)

The state transition matrix is:

\[
\Phi(\Delta t, 0) = \begin{bmatrix} e^{-\beta \Delta t} & 0 \\ \beta \Delta t e^{-\beta \Delta t} & e^{-\beta \Delta t} \end{bmatrix}
\]  

(4.3.1-10)

The only process noise term enters in the first state. \( N_{11} \) is simply the variance of the white noise driving the Markov process:

\[ N = \begin{bmatrix} N_{11} & 0 \\ 0 & 0 \end{bmatrix} \]  

(4.3.1-11)

From equation 3.2.4-10:

\[
Q = \int \begin{bmatrix} e^{-\beta \Delta t} & 0 \\ \beta \Delta t e^{-\beta \Delta t} & e^{-\beta \Delta t} \end{bmatrix} \begin{bmatrix} N_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\beta \Delta t} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \Delta t e^{-\beta \Delta t} & e^{-\beta \Delta t} \\ 0 & 0 \end{bmatrix} d\Delta t
\]  

(4.3.1-12)

With the integration interval dropped for simplicity. The process covariance matrix is:

\[
Q = \begin{bmatrix} N_{11} e^{-2\beta \Delta t} & N_{11} \beta \Delta t e^{-2\beta \Delta t} \\ N_{11} \beta \Delta t e^{-2\beta \Delta t} & N_{11} \beta^2 \Delta t^2 e^{-2\beta \Delta t} \end{bmatrix}
\]  

(4.3.1-13)

Clearly, the process covariance matrix becomes considerably more complicated for higher-order Markov processes. There is, however, a pattern for the elements of the matrix. The equation for any element, \( Q_{ij} \), for any order of Markov process up to and including the fifth-order is:

\[
Q_{ij} = N_{11} \frac{\beta^{(i-1)(j-1)}}{(i-1)!(j-1)!} \left[ \Psi[(i-1)+(j-1),-2\beta,\Delta t] - \Psi[(i-1)+(j-1),-2\beta,0] \right]
\]  

(4.3.1-14)

For which:

\[
\Psi(n, a, x) = \sum_{k=0}^{n} (-1)^k \frac{n!}{a^{k+1}(n-k)!} x^{n-k} e^{ax}
\]  

(4.3.1-15)
The derivation of these equations can be found in appendix B.

The steady-state covariance matrix is:

$$Q_{ss} = \begin{bmatrix} N_{11} & N_{11} \\ 2\beta & 4\beta \\ N_{11} & N_{11} \\ 4\beta & 4\beta \end{bmatrix}$$  \hspace{1cm} (4.3.1-16)

For the formulation in equations 4.3.1-7 and 4.3.1-8, the second state represents the ionospheric delay in terms of the L1 frequency:

$$\Delta l = x_2$$

$$\Delta \dot{l} = \beta(x_1 - x_2)$$  \hspace{1cm} (4.3.1-17 : 4.3.1-18)

The variance of the delay and the delay rate are, respectively:

$$E[\Delta l^2] = P_{22}$$

$$E[\Delta \dot{l}^2] = E[\beta(x_1 - x_2) \cdot \beta(x_1 - x_2)]$$

$$= E[\beta^2(x_1^2 - 2x_1x_2 + x_2^2)]$$

$$= \beta^2(E[x_1^2] - 2E[x_1x_2] + E[x_2^2])$$

$$= \beta^2(P_{11} - 2P_{12} + P_{22})$$  \hspace{1cm} (4.3.1-19 : 4.3.1-20)

Third-Order Markov Processes:

The dynamics model for the third-order Markov process is:

$$\dot{x}_1 = \beta(w - x_1)$$

$$\dot{x}_2 = \beta(x_1 - x_2)$$

$$\dot{x}_3 = \beta(x_2 - x_3)$$  \hspace{1cm} (4.3.1-21 : 4.3.1-23)

Or:

$$A = \begin{bmatrix} -\beta & 0 & 0 \\ \beta & -\beta & 0 \\ 0 & \beta & -\beta \end{bmatrix}$$  \hspace{1cm} (4.3.1-24)

The state transition matrix is:
The variance of the continuous process noise vector is:

\[
N = \begin{bmatrix}
N_{11} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

(4.3.1-26)

From equation 3.2.4-10, process covariance matrix is:

\[
Q = \int \begin{bmatrix}
e^{-\beta \Delta t} & 0 & 0 \\
\beta \Delta t e^{-\beta \Delta t} & e^{-\beta \Delta t} & 0 \\
\frac{\beta^2 \Delta t^2}{2} e^{-\beta \Delta t} & \beta \Delta t e^{-\beta \Delta t} & e^{-\beta \Delta t}
\end{bmatrix} \begin{bmatrix}
N_{11} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}\begin{bmatrix}
e^{-\beta \Delta t} & 0 & 0 \\
\beta \Delta t e^{-\beta \Delta t} & e^{-\beta \Delta t} & 0 \\
\frac{\beta^2 \Delta t^2}{2} e^{-\beta \Delta t} & \beta \Delta t e^{-\beta \Delta t} & e^{-\beta \Delta t}
\end{bmatrix}
\]

\[
= \int \begin{bmatrix}
N_{11} e^{-2\beta \Delta t} & N_{11} \beta \Delta t e^{-2\beta \Delta t} & N_{11} \frac{\beta^2 \Delta t^2}{2} e^{-2\beta \Delta t}
\\
N_{11} \beta \Delta t e^{-2\beta \Delta t} & N_{11} \beta^2 \Delta t^2 e^{-2\beta \Delta t} & N_{11} \frac{\beta^3 \Delta t^3}{2} e^{-2\beta \Delta t}
\\
\frac{N_{11} \beta^2}{2} \Delta t^2 e^{-2\beta \Delta t} & \frac{N_{11} \beta^3}{2} \Delta t^3 e^{-2\beta \Delta t} & \frac{N_{11} \beta^4}{4} \Delta t^4 e^{-2\beta \Delta t}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
N_{11} \beta^2 \Delta t^2 e^{-2\beta \Delta t} & N_{11} \beta^3 \Delta t^3 e^{-2\beta \Delta t} & N_{11} \beta^4 e^{-2\beta \Delta t}
\\
N_{11} \beta^{12} \Delta t^2 e^{-2\beta \Delta t} & N_{11} \beta^{13} \Delta t^3 e^{-2\beta \Delta t} & N_{11} \beta^{14} e^{-2\beta \Delta t}
\\
\frac{N_{11} \beta^3}{2} \Delta t^2 e^{-2\beta \Delta t} & \frac{N_{11} \beta^4}{2} \Delta t^3 e^{-2\beta \Delta t} & \frac{N_{11} \beta^5}{4} \Delta t^4 e^{-2\beta \Delta t}
\end{bmatrix}
\]

(4.3.1-27)

Which can be solved with equation 4.3.1-15. The steady-state process covariance matrix is:

\[
Q_{ss} = \begin{bmatrix}
N_{11} & N_{11} & N_{11} \\
\frac{2\beta}{4} & \frac{4\beta}{8} & \frac{8\beta}{16} \\
N_{11} & N_{11} & 3N_{11} \\
\frac{4\beta}{4} & \frac{4\beta}{16} & \frac{16\beta}{16} \\
N_{11} & 3N_{11} & 3N_{11} \\
\frac{8\beta}{8} & \frac{16\beta}{16} & \frac{16\beta}{16}
\end{bmatrix}
\]

(4.3.1-28)
If a higher-order Markov process is used and the final state is the ionosphere delay (as it is in the formulations described in table 4.3.1-2), it is not intuitive what the other states represent nor are they particularly useful. Instead, what would be useful is to find the estimates of ionosphere delay rate and the delay second derivative in terms of the other states. From equations 4.3.1-21 through 4.3.1-23, the delay and delay rate are:

\[ \Delta I = x_3 \]
\[ \Delta \dot{I} = \beta(x_2 - x_3) \]

(4.3.1-29 : 4.3.1-30)

With some algebra (which can also be found in appendix A), the second derivative of the delay is:

\[ \Delta \ddot{I} = \beta^2 x_1 - 2\beta^2 x_2 + \beta^2 x_3 \]

(4.3.1-31)

From these equations for the derivatives of the ionospheric delay, the variance can also be determined. For the delay, the variance is:

\[ E[\Delta I^2] = P_{33} \]

(4.3.1-32)

For the delay rate, the variance is:

\[
E[\Delta \dot{I}^2] = E[\beta(x_2 - x_3) \cdot \beta(x_2 - x_3)] \\
= E[\beta^2(x_2^3 - 2x_2x_3 + x_3^3)] \\
= \beta^2(E[x_2^3] - 2E[x_2x_3] + E[x_3^3]) \\
= \beta^2(P_{22} - 2P_{23} + P_{33})
\]

(4.3.1-33)

For the delay second derivative, the variance is:

\[
E[\Delta \ddot{I}^2] = E[\beta^2(x_1 - 2x_2 + x_3) \cdot \beta^2(x_1 - 2x_2 + x_3)] \\
= E[\beta^4(x_1^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3 + 4x_2^2 + x_3^2)] \\
= \beta^4(E[x_1^2] - 4E[x_1x_2] + 2E[x_1x_3] - 4E[x_2x_3] + 4E[x_2^2] + E[x_3^2]) \\
= \beta^4(P_{11} - 4P_{12} + 2P_{13} - 4P_{22} + 4P_{22} + P_{33})
\]

(4.3.1-34)

4.3.2 Cascades of First-Order Markov Processes

Cascades of Markov processes provide the user with additional parameters to adjust and more flexibility than higher-order Markov processes.
Two Cascades of First-Order Markov Processes:

For two cascades of first-order Markov processes, the dynamics model is:

\[
\begin{align*}
\dot{x}_1 &= \beta_a(w - x_1) \\
\dot{x}_2 &= \beta_b(x_1 - x_2)
\end{align*}
\]  

(4.3.2-1 : 4.3.2-2)

Or:

\[
A = \begin{bmatrix}
-\beta_a & 0 \\
\beta_b & -\beta_b
\end{bmatrix}
\]  

(4.3.2-3)

The state transition matrix is:

\[
\Phi(\Delta t, 0) = \begin{bmatrix}
e^{-\beta_a \Delta t} & 0 \\
\frac{\beta_b (e^{-\beta_a \Delta t} - e^{-\beta_b \Delta t})}{\beta_a - \beta_b} & e^{-\beta_b \Delta t}
\end{bmatrix}
\]  

(4.3.2-4)

Setting:

\[
\alpha = \frac{\beta_b}{\beta_a - \beta_b}
\]  

(4.3.2-5)

\[
A = e^{-\beta_a \Delta t} \quad B = e^{-\beta_b \Delta t}
\]  

(4.3.2-6 : 4.3.2-7)

\[
\Phi = \begin{bmatrix}
A & 0 \\
\alpha(B - A) & B
\end{bmatrix}
\]  

(4.3.2-8)

Then the process covariance matrix becomes:

\[
\begin{align*}
Q &= \int \begin{bmatrix}
A & 0 & N_{11} & 0 & A & \alpha(B - A) \\
\alpha(B - A) & B & 0 & 0 & 0 & B
\end{bmatrix}
\end{align*}
\]

\[
= \int \begin{bmatrix}
N_{11} A^2 & \alpha N_{11} (AB - A^2) \\
\alpha N_{11} (AB - A^2) & \alpha^2 N_{11} (B^2 - 2AB + A^2)
\end{bmatrix}
\]  

(4.3.2-9)

Keeping in mind the dependence of variables on time (all integrals are with respect to \(\Delta t\)):

\[
Q = \begin{bmatrix}
N_{11} \int A^2 & \alpha N_{11} \left( \int AB - \int A^2 \right) \\
\alpha N_{11} \left( \int AB - \int A^2 \right) & \alpha^2 N_{11} \left( \int B^2 - 2 \int AB + \int A^2 \right)
\end{bmatrix}
\]  

(4.3.2-10)
If $M$ and $N$ are defined such that:

$$M = e^{-\beta_a \Delta t} \quad \text{and} \quad N = e^{-\beta_b \Delta t} \quad (4.3.2-11 : 4.3.2-12)$$

The general solution to integral of the product of the two terms is:

$$\int MN = \int e^{-\beta_a \Delta t} e^{-\beta_b \Delta t}$$

$$= \int e^{-(\beta_a + \beta_b) \Delta t}$$

$$= \frac{1}{-(\beta_a + \beta_b)} e^{-(\beta_a + \beta_b) \Delta t} \bigg|_0^\Delta t$$

$$= \frac{1}{(\beta_a + \beta_b)} \left( 1 - e^{-(\beta_a + \beta_b) \Delta t} \right)$$

$$= \eta(M, N) \quad (4.3.2-13)$$

For the integral of a term squared:

$$\int M^2 = \int e^{-\beta_a \Delta t} e^{-\beta_b \Delta t}$$

$$= \int e^{-2\beta_a \Delta t}$$

$$= \frac{1}{-2\beta_a} e^{-2\beta_a \Delta t} \bigg|_0^\Delta t$$

$$= \frac{1}{2\beta_a} \left( 1 - e^{-2\beta_a \Delta t} \right)$$

$$= \eta(M, M) \quad (4.3.2-14)$$

Which is identical to equation 4.3.2-13 when $n = m$. So we can define:

$$Q = \begin{bmatrix}
N_{11}, \eta(A, A) & \alpha N_{11} (\eta(A, B) - \eta(A, A)) \\
\alpha N_{11} (\eta(A, B) - \eta(A, A)) & \alpha^2 N_{11} (\eta(B, B) - 2\eta(A, B) + \eta(A, A))
\end{bmatrix} \quad (4.3.2-15)$$

The steady-state process covariance matrix is:

$$Q_{ss} = \begin{bmatrix}
\frac{N_{11}}{2\beta_a} & \frac{N_{11} \beta_b}{2\beta_a} \\
\frac{N_{11} \beta_b}{2\beta_a} & \frac{N_{11} \beta_b}{2\beta_a (\beta_a + \beta_b)} \left( \frac{\beta_a^2 - 4\beta_a \beta_b - \beta_b^2}{\beta_a + \beta_b} \right)
\end{bmatrix} \quad (4.3.2-16)$$
The ionospheric delay and the derivative of the delay are clearly:

\[ \Delta I = x_2 \]
\[ \Delta \dot{I} = \beta_b (x_1 - x_2) \]  

So the variances on the ionospheric delay and rate are:

\[ E[\Delta I^2] = P_{22} \]
\[ E[\Delta \dot{I}^2] = E[\beta_b (x_1 - x_2) \cdot \beta_b (x_1 - x_2)] \]
\[ = E[\beta_b^2 (x_1^2 - 2x_1x_2 + x_2^2)] \]
\[ = \beta_b^2 [E[x_1^2] - 2E[x_1x_2] + E[x_2^2]] \]
\[ = \beta_b^2 (P_{11} - 2P_{12} + P_{22}) \]  

Three Cascades of First-Order Markov Processes:

For three cascades of first-order Markov processes, the dynamics model is:

\[ \dot{x}_1 = \beta_a (w - x_1) \]
\[ \dot{x}_2 = \beta_b (x_1 - x_2) \]
\[ \dot{x}_3 = \beta_c (x_2 - x_3) \]  

Or:

\[ A = \begin{bmatrix} -\beta_a & 0 & 0 \\ \beta_b & -\beta_b & 0 \\ 0 & \beta_c & -\beta_c \end{bmatrix} \]
The state transition matrix is:

\[
\Phi(\Delta t, 0) = \begin{bmatrix}
    e^{-\beta_c \Delta t} & 0 & 0 \\
    \frac{\beta_b (e^{-\beta_c \Delta t} - e^{-\beta_b \Delta t})}{\beta_a - \beta_b} & e^{-\beta_a \Delta t} & 0 \\
    \frac{\beta_b \beta_c (e^{-\beta_b \Delta t} - e^{-\beta_c \Delta t}) + \beta_b (e^{-\beta_c \Delta t} - e^{-\beta_b \Delta t}) + \beta_c (e^{-\beta_b \Delta t} - e^{-\beta_c \Delta t})}{(\beta_a - \beta_b)(\beta_c - \beta_b)(\beta_c - \beta_a)} & \frac{\beta_c (e^{-\beta_c \Delta t} - e^{-\beta_a \Delta t})}{\beta_c - \beta_b} & e^{-\beta_c \Delta t}
\end{bmatrix}
\]

(4.3.2-25)

Using the same definitions in equations 4.3.2-6 and 4.3.2-7 while including:

\[
C = e^{-\beta_c \Delta t}
\]

(4.3.2-26)

And:

\[
\alpha_1 = \frac{\beta_b}{\beta_a - \beta_b}
\]

\[
\alpha_2 = \frac{\beta_c}{\beta_c - \beta_b}
\]

(4.3.2-27 : 4.3.2-29)

\[
\alpha_3 = \frac{\beta_b \beta_c}{(\beta_a - \beta_b)(\beta_c - \beta_b)(\beta_c - \beta_a)}
\]

The state transition matrix is:

\[
\Phi = \begin{bmatrix}
    A & 0 & 0 \\
    \alpha_1 (B - A) & B & 0 \\
    \alpha_3 (\beta_a (C - B) + \beta_b (A - C) + \beta_c (B - A)) & \alpha_2 (B - C) & C
\end{bmatrix}
\]

(4.3.2-30)

Defining:

\[
c_1 = \beta_b - \beta_c \quad c_2 = \beta_c - \beta_a \quad c_3 = \beta_a - \beta_b
\]

(4.3.2-31 : 4.3.2-33)

So \(\Phi\) becomes:

\[
101
\]
The process covariance matrix is then:

\[
\Phi = \begin{bmatrix}
A & 0 & 0 \\
\alpha_1(B-A) & B & 0 \\
\alpha_3(c_1 A + c_2 B + c_3 C) & \alpha_2(B-C) & C
\end{bmatrix}
\]  

(4.3.2-34)

The 2x2 block in the upper left hand corner of the process covariance matrix is identical to that for two cascades of first-order Markov processes (equation 4.3.2-15). The remainder of the terms for this case are:

\[Q_{33} = \int N_{11} \alpha_3 \left( c_1 A^2 + c_2 AB + c_3 AC \right) d\Delta t \]

\[= \int N_{11} \alpha_3 \left( c_1 A^2 + c_2 AB + c_3 AC \right) d\Delta t \]

(4.3.2-36)

\[Q_{23} \text{ and } Q_{32} = \int N_{11} \alpha_3 \left( (c_1 - c_2) AB + c_2 B^2 + c_3 BC - c_1 A^2 - c_3 AC \right) d\Delta t \]

\[= \int N_{11} \alpha_3 \left( (c_1 - c_2) AB + c_2 B^2 + c_3 BC - c_1 A^2 - c_3 AC \right) d\Delta t \]

(4.3.2-37)

\[Q_{33} = \int N_{11} \alpha_3^2 \left( c_1^2 A^2 + 2c_1 c_2 AB + 2c_1 c_3 AC + c_2^2 B^2 + 2c_2 c_3 BC + c_3^2 C^2 \right) d\Delta t \]

\[= \int N_{11} \alpha_3^2 \left( c_1^2 A^2 + 2c_1 c_2 AB + 2c_1 c_3 AC + c_2^2 B^2 + 2c_2 c_3 BC + c_3^2 C^2 \right) d\Delta t \]

(4.3.2-38)

The steady-state process covariance matrix is:
Determining the ionospheric delay, delay rate and delay second derivative is not difficult:

\[ \Delta i = \beta_c (x_2 - x_3) \]  
\[ \Delta \dot{i} = \beta_c (x_2 - x_3) \]  
\[ \Delta \ddot{i} = \beta_c (x_2 - x_3) \]

The variances of the delay and its derivatives are:

\[ E[\Delta i^2] = P(3,3) \]  
\[ E[\Delta j^2] = E[\beta_c (x_2 - x_3) \cdot \beta_c (x_2 - x_3)] \]  
\[ = E[\beta_c^2 (x_2^2 - 2x_2x_3 + x_3^2)] \]  
\[ = \beta_c^2 (E[x_2^2] - 2E[x_2x_3] + E[x_3^2]) \]  
\[ = \beta_c^2 (P_{22} - 2P_{23} + P_{33}) \]  
\[ E[\Delta j^2] = E[\beta_c^2 \beta_c (x_1 - (\beta_c + \beta_c^3)x_2 + \beta_c^2 x_3)] \]  
\[ = E[\beta_c^2 \beta_c \beta_c^4 x_1^3 + 2(\beta_c^2 + \beta_c^3 + \beta_c^5) x_2x_3 + 2\beta_c^4 \beta_c^5 x_2^2 + 2\beta_c^6 \beta_c^7 x_3^2] \]  
\[ = \beta_c^2 \beta_c^4 E[x_1^3] + 2(\beta_c^2 + \beta_c^3 + \beta_c^5) E[x_2x_3] + 2\beta_c^6 \beta_c^7 E[x_2^2] + 2\beta_c^8 \beta_c^9 E[x_3^2] \]

\[ = \beta_c^2 \beta_c^4 P_{11} + 2(\beta_c^2 + \beta_c^3 + \beta_c^5) P_{12} + 2\beta_c^6 \beta_c^7 P_{22} + 2\beta_c^8 \beta_c^9 P_{33} \]

4.3.3 Integrals of Markov Processes

Integral of First-Order Markov Process:

The dynamics model for the integral of a first-order Markov process is:
\[
\dot{x}_1 = \beta (w - x_1) \tag{4.3.3-1 : 4.3.3-2}
\]
\[
\dot{x}_2 = x_1
\]

The process dynamics matrix is:

\[
A = \begin{bmatrix}
-\beta & 0 \\
1 & 0
\end{bmatrix} \tag{4.3.3-3}
\]

The state transition matrix is:

\[
\Phi(\Delta t, 0) = \begin{bmatrix}
e^{-\beta \Delta t} & 0 \\
\frac{1}{\beta} (1 - e^{-\beta \Delta t}) & 1
\end{bmatrix} \tag{4.3.3-4}
\]

The process covariance matrix is:

\[
Q = \begin{bmatrix}
\frac{N_{11}}{\beta} \int e^{-2\beta t} & \frac{N_{11}}{\beta} \int e^{-\beta t} - \frac{N_{11}}{\beta} \int e^{-2\beta t} \\
\frac{N_{11}}{\beta^2} \int 1 - \frac{2N_{11}}{\beta^2} \int e^{-\beta t} + \frac{N_{11}}{\beta^2} \int e^{-2\beta t}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{N_{11}}{2\beta} (1 - e^{-2\beta t}) & \frac{N_{11}}{\beta^2} (1 - e^{-\beta t}) - \frac{N_{11}}{2\beta^2} (1 - e^{-2\beta t}) \\
\frac{N_{11}}{\beta^2} (1 - e^{-\beta t}) - \frac{N_{11}}{2\beta^2} (1 - e^{-2\beta t}) & \frac{N_{11}}{\beta^2} (1 - e^{-\beta t}) - \frac{N_{11}}{2\beta^2} (1 - e^{-2\beta t})
\end{bmatrix} \tag{4.3.3-5}
\]

The steady-state process covariance matrix is:

\[
Q_{ss} = \begin{bmatrix}
\frac{N_{11}}{2\beta} & \frac{N_{11}}{\beta^2} \\
\frac{N_{11}}{2\beta^2} & \infty
\end{bmatrix} \tag{4.3.3-6}
\]

Note that although the variance of the ionospheric delay tends to infinity over time, it tends to this value much slower than in the five-state model. With:

\[
\Delta l = x_2 \tag{4.3.3-7 : 4.3.3-8}
\]
\[
\Delta \dot{l} = x_1
\]

The variances of the ionospheric delay and its derivative are:
Integral of Second-Order Markov Process:

The dynamics model for the integral of a second-order Markov process is:

\[
\begin{align*}
\dot{x}_1 &= \beta(w - x_1) \\
\dot{x}_2 &= \beta(x_1 - x_2) \\
\dot{x}_3 &= x_2
\end{align*}
\]  

(4.3.3-11 : 4.3.3-13)

The process dynamics matrix is:

\[
A = \begin{bmatrix}
-\beta & 0 & 0 \\
\beta & -\beta & 0 \\
0 & 1 & 0
\end{bmatrix}
\]  

(4.3.3-14)

The state transition matrix is:

\[
\Phi(\Delta t, 0) = \begin{bmatrix}
e^{-\beta \Delta t} & 0 & 0 \\
\beta \Delta t e^{-\beta \Delta t} & 0 & 0 \\
-\Delta t e^{-\beta \Delta t} - \frac{1}{\beta} e^{-\beta \Delta t} + \frac{1}{\beta} e^{-2\beta \Delta t} + \frac{1}{\beta} & 0 & 1
\end{bmatrix}
\]  

(4.3.3-15)

The process covariance matrix, by element is:

\[
Q_{11} = N_{11} \int e^{-2\beta \Delta t}
= \frac{N_{11}}{2\beta} \left(1 - e^{-2\beta \Delta t}\right)
\]  

(4.3.3-16)

\[
Q_{12} = Q_{21} = N_{11} \beta \int \Delta t e^{-2\beta \Delta t}
= N_{11} \beta (\Psi(1, -2\beta, \Delta t) - \Psi(1, -2\beta, 0))
\]  

(4.3.3-17)

\[
Q_{13} = Q_{31} = \frac{N_{11}}{\beta} \int e^{-\beta \Delta t} - \frac{N_{11}}{\beta} \int e^{-2\beta \Delta t} - N_{11} \int \Delta t e^{-2\beta \Delta t}
= \frac{N_{11}}{\beta^2} \left(1 - e^{-\beta \Delta t}\right) - \frac{N_{11}}{2\beta^2} \left(1 - e^{-2\beta \Delta t}\right) - N_{11} (\Psi(1, -2\beta, \Delta t) - \Psi(1, -2\beta, 0))
\]  

(4.3.3-18)
\[ Q_{22} = N_{11} \beta^2 \int \Delta t^2 e^{-\beta \Delta t} \]
\[ = N_{11} \beta^2 (\Psi(2,-2\beta, \Delta t) - \Psi(2,-2\beta,0)) \]  
(4.3.3-19)

\[ Q_{23} = N_{11} \int \Delta t e^{-\beta \Delta t} - N_{11} \int \Delta t e^{-2\beta \Delta t} - N_{11} \beta \Delta t e^{-\beta \Delta t} \]
\[ = N_{11} (\Psi(1,-\beta, \Delta t) - \Psi(1,-\beta,0)) - N_{11} (\Psi(1,-2\beta, \Delta t) - \Psi(1,-2\beta,0)) \]  
\[ - N_{11} \beta (\Psi(2,-2\beta, \Delta t) - \Psi(2,-2\beta,0)) \]  
(4.3.3-20)

\[ Q_{33} = N_{11} \int \Delta t^2 e^{-2\beta \Delta t} + \frac{2N_{11}}{\beta} \int \Delta t e^{-\beta \Delta t} - \frac{2N_{11}}{\beta^2} \int \Delta t e^{-\beta \Delta t} + \frac{N_{11}}{\beta^2} \int e^{-2\beta \Delta t} - \frac{2N_{11}}{\beta^3} \int e^{-\beta \Delta t} + \frac{N_{11}}{\beta^3} \int ] \]
\[ = N_{11} (\Psi(2,-2\beta, \Delta t) - \Psi(2,-2\beta,0)) + \frac{2N_{11}}{\beta} (\Psi(1,-2\beta, \Delta t) - \Psi(1,-2\beta,0)) \]
\[ - \frac{2N_{11}}{\beta^3} (\Psi(1,-\beta, \Delta t) - \Psi(1,-\beta,0)) + \frac{N_{11}}{\beta^3} (1 - e^{-2\beta \Delta t}) - \frac{2N_{11}}{\beta^3} (1 - e^{-\beta \Delta t}) + \frac{N_{11} \Delta t}{\beta^3} \]  
(4.3.3-21)

The steady-state process covariance matrix is:

\[
Q_{ss} = \begin{bmatrix}
N_{11} & N_{11} & N_{11} \\
2\beta & 4\beta & 4\beta^2 \\
N_{11} & N_{11} & N_{11} \\
4\beta & 4\beta & 2\beta^2 \\
N_{11} & N_{11} & \infty \\
4\beta^2 & 2\beta^2 & 0
\end{bmatrix}
\]  
(4.3.3-22)

The ionospheric delay and its derivatives are:

\[ \Delta I = x_3 \]
\[ \Delta \dot{I} = x_2 \]  
(4.3.3-23 : 4.3.3-25)

\[ \Delta \ddot{I} = \dot{x}_2 \]
\[ = \beta (x_1 - x_2) \]

With the variances:

\[ E[x_3^2] = P_{33} \]
\[ E[x_2^2] = P_{22} \]  
(4.3.3-26 : 4.3.3-28)

\[ E[x_3^2] = \beta^2 (P_{11} - 2P_{12} + P_{22}) \]
Integral of Two Cascades of First-Order Markov Processes:

The dynamics model for the integral of two cascades of first-order Markov processes is:

\[
\begin{align*}
\dot{x}_1 &= \beta_a (w - x_1) \\
\dot{x}_2 &= \beta_b (x_1 - x_2) \\
\dot{x}_3 &= x_2
\end{align*}
\]

(4.3.3-29 : 4.3.3-31)

Or:

\[
A = \begin{bmatrix}
-\beta_a & 0 & 0 \\
\beta_b & -\beta_b & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

(4.3.3-32)

The state transition matrix is:

\[
\Phi(\Delta t, 0) = \begin{bmatrix}
\frac{e^{-\beta_a \Delta t}}{\beta_a - \beta_b} & 0 & 0 \\
\frac{\beta_b}{\beta_a - \beta_b} (e^{-\beta_a \Delta t} - e^{-\beta_b \Delta t}) & e^{-\beta_b \Delta t} & 0 \\
\frac{e^{-\beta_b \Delta t}}{\beta_a (\beta_a - \beta_b)} - \frac{1}{\beta_a - \beta_b} e^{-\beta_a \Delta t} + \frac{1}{\beta_a} (1 - e^{-\beta_a \Delta t}) & 1 & 0
\end{bmatrix}
\]

(4.3.3-33)

The process covariance matrix by element is:

\[
Q_{11} = N_{11} \int e^{-2\beta_a \Delta t} \\
= N_{11} \eta(A, A)
\]

(4.3.3-34)

\[
Q_{12} = \frac{N_{11} \beta_b}{\beta_a - \beta_b} \int e^{-(\beta_a + \beta_b) \Delta t} - \frac{N_{11} \beta_b}{\beta_a - \beta_b} \int e^{-2\beta_a \Delta t} \\
= N_{11} \alpha(\eta(A, B) - \eta(A, A))
\]

(4.3.3-35)

\[
Q_{13} = \frac{N_{11} \beta_b}{\beta_a \beta_a - \beta_b} \int e^{-(\beta_a + \beta_b) \Delta t} + \frac{N_{11} \beta_b}{\beta_a} \int e^{-\beta_a \Delta t} - \frac{N_{11}}{\beta_a - \beta_b} \int e^{-(\beta_a + \beta_b) \Delta t} \\
= \frac{N_{11} \alpha}{\beta_a} \eta(A, A) + \frac{N_{11}}{\beta_a^2} (1 - A) - \frac{N_{11} \alpha}{\beta_b} \eta(A, B)
\]

(4.3.3-36)
\[
Q_{22} = \frac{N_{11} \beta_b^2}{(\beta_a - \beta_b)^2} \int e^{-2\beta_a \Delta t} - \frac{2N_{11} \beta_b^2}{(\beta_a - \beta_b)^2} \int e^{-(\beta_a + \beta_b) \Delta t} + \frac{N_{11} \beta_b^2}{(\beta_a - \beta_b)^2} \int e^{-2\beta_b \Delta t}
\]
\[
= N_{11} \alpha^2 (\eta(B,B) - 2\eta(A,B) + \eta(A,A))
\]

\[
Q_{23} \text{ and } Q_{32} = \frac{N_{11} \beta_b^2}{(\beta_a - \beta_b)^2} \int e^{-(\beta_a + \beta_b) \Delta t} + \frac{N_{11} \beta_b^2}{(\beta_a - \beta_b)^2} \int e^{-2\beta_b \Delta t} - \frac{N_{11} \beta_b}{\beta_a (\beta_a - \beta_b)} \int e^{-\beta_a \Delta t}
\]
\[
= \frac{N_{11} \alpha^2}{\beta_a} \eta(A,B) - \frac{N_{11} \alpha^2}{\beta_b} \eta(B,B) + \frac{N_{11} \alpha}{\beta_a \beta_b} (1 - B) + \frac{N_{11} \alpha^2}{\beta_b} \eta(A,B) - \frac{N_{11} \alpha^2}{\beta_a} \eta(A,A)
\]
\[
- \frac{N_{11} \alpha}{\beta_a} (1 - A)
\]
\[
= N_{11} \alpha^2 \left( \frac{\eta(A,B)}{\beta_a} - \frac{\eta(B,B)}{\beta_b} + \frac{\eta(A,B)}{\beta_b} - \frac{\eta(A,A)}{\beta_a} \right) + \frac{N_{11} \alpha}{\beta_a} \left( \frac{1 - B + A - 1}{\beta_b} \right)
\]

\[
Q_{33} = \frac{N_{11} \beta_b^2}{(\beta_a - \beta_b)^2} \int e^{-2\beta_a \Delta t} + \frac{2N_{11} \beta_b}{\beta_a (\beta_a - \beta_b)} \int e^{-(\beta_a + \beta_b) \Delta t} - \frac{2N_{11} \beta_b}{\beta_b (\beta_a - \beta_b)} \int e^{-\beta_b \Delta t}
\]
\[
+ \frac{N_{11} \beta_a^2}{\beta_b^2 (\beta_a - \beta_b)^2} \int e^{-2\beta_b \Delta t} + \frac{2N_{11} \beta_b}{\beta_a^2 (\beta_a - \beta_b)^2} \int e^{-(\beta_a + \beta_b) \Delta t} + \frac{N_{11} \Delta t}{\beta_a^2} \int e^{-\beta_a \Delta t}
\]
\[
= \frac{N_{11} \alpha^2}{\beta_b} \eta(B,B) - \frac{2N_{11} \alpha^2}{\beta_a \beta_b} \eta(A,B) - \frac{2N_{11} \alpha}{\beta_a \beta_b^2} (1 - B) + \frac{2N_{11} \alpha^2}{\beta_a^2 \beta_b} \eta(A,A) - \frac{2N_{11} \alpha}{\beta_b^2} \eta(A,B) - \frac{2N_{11} \alpha}{\beta_a^2} \eta(A,A)
\]
\[
+ \frac{2N_{11} \alpha}{\beta_a} \left( \frac{B - 1 + A - 1}{\beta_b^2} \right)
\]
\[
= N_{11} \alpha^2 \left( \frac{\eta(B,B)}{\beta_b^2} - \frac{2\eta(A,B)}{\beta_a \beta_b} + \frac{\eta(A,A)}{\beta_a^2} \right) + \frac{2N_{11} \alpha}{\beta_a} \left( \frac{B - 1 + A - 1}{\beta_b^2} \right)
\]
\[
+ \frac{N_{11} \Delta t}{\beta_a^2}
\]

The steady-state process covariance matrix is:

\[
Q_{ss} = \begin{bmatrix}
\frac{N_{11} \beta_b}{2\beta_a} & \frac{N_{11} \beta_b}{2(\beta_a + \beta_b)} & \frac{2N_{11} \beta_b}{N_{11} \beta_a}
\
\frac{N_{11} \beta_b}{2\beta_a (\beta_a + \beta_b)} & \frac{2N_{11} \beta_b}{N_{11} \beta_a} & \frac{N_{11} \beta_b}{2(\beta_a + \beta_b) \beta_a^2}
\
\frac{2N_{11} \beta_b}{N_{11} \beta_a} & \frac{N_{11} \beta_b}{2(\beta_a + \beta_b) \beta_a^2} & \frac{N_{11} \beta_b}{2\beta_a^2}
\end{bmatrix}
\]

(4.3.3-40)

The ionospheric delay and its derivatives are:
\[ \Delta I = x_3 \]
\[ \dot{\Delta I} = x_2 \]  \hspace{1cm} (4.3.3-41 : 4.3.3-43)
\[ \ddot{\Delta I} = \dot{x}_2 = \beta(x_1 - x_2) \]

The variances of these values of interest are:
\[ E[\Delta I^2] = P_{33} \]
\[ E[\Delta I^2] = P_{22} \]  \hspace{1cm} (4.3.3-44 : 4.3.3-46)
\[ E[\Delta I^2] = \beta^2 (P_{11} - 2P_{12} + P_{22}) \]

### 4.4 Hybrid Estimator and Model

There are benefits and drawbacks to using a model or an estimator alone to correct for ionospheric group delay. As described earlier, models alone will likely not produce the accuracy required for real-time ionosphere correction. An estimator alone may not provide accurate corrections if measurements are lost for long periods of time. To provide the benefits of both, the measurements outlined in section 3.2.1 can be altered slightly so an estimator produces estimated deviations of ionospheric group delays from a given reference model. With Markov processes or cascades of Markov processes as the dynamics model in a Kalman filter, this will result in the “estimated” delays tending to the “modeled” delays in the absence of measurements since the models are zero-mean. Using the integral of Markov processes or cascades of Markov processes as the dynamics model results in the estimated delays tending to the reference model delays plus a bias.

Recall that the measurements are defined as:

\[ z_1 = 2 \Delta I_{t+1} - 2 \Delta I_{t+1} + \xi \rho_{t+1} - \xi \rho_{t+1} - \xi \delta_{t+1} \]
\[ z_2 = 2w \Delta I_{t+1} - 2w \Delta I_{t+1} + \xi \rho_{t+1} - \xi \rho_{t+1} - \xi \delta_{t+1} \]
\[ z_3 = (w - 1) \Delta I_{t+1} + B + \xi \rho_{t+1} - \xi \rho_{t+1} \]
\[ z_4 = (w - 1) \Delta I_{t+1} + (1 - w) \Delta I_{t+1} + \xi \delta_{t+1} - \xi \delta_{t+1} \]

(4.4-1 : 4.4-4)

If using an ionosphere reference model to create “modeled measurements”:
\[
\begin{align*}
\bar{z}_1 &= 2\Delta \tilde{I}_{rL}^L - 2\Delta \tilde{I}_{nL}^L \\
\bar{z}_2 &= 2w_{L2}\Delta \tilde{I}_{rL}^L - 2w_{L2}\Delta \tilde{I}_{nL}^L \\
\bar{z}_3 &= (w_{L2} - 1)\Delta \tilde{I}_{rL}^L \\
\bar{z}_4 &= (w_{L2} - 1)\Delta \tilde{I}_{nL}^L + (1 - w_{L2})\Delta \tilde{I}_{nL}^L
\end{align*}
\] (4.4-5 : 4.4-8)

The modeled measurements can then be subtracted from the actual measurements so the first four filter states are now related to the deviation of the ionospheric delay from a reference model:

\[
\begin{align*}
z_1 - \bar{z}_1 &= 2\Delta I_{rL}^L - 2\Delta I_{nL}^L + \xi \rho_{rL}^L - \xi \rho_{nL}^L - \xi \delta_{rL}^L - \left(2\Delta \tilde{I}_{rL}^L - 2\Delta \tilde{I}_{nL}^L\right) \\
&= 2\left(\Delta I_{rL}^L - \Delta \tilde{I}_{rL}^L\right) - 2\left(\Delta I_{nL}^L - \Delta \tilde{I}_{nL}^L\right) + \xi \rho_{rL}^L - \xi \rho_{nL}^L - \xi \delta_{rL}^L \\
&= 2\Delta M_{rL}^L - 2\Delta M_{nL}^L + \xi \rho_{rL}^L - \xi \rho_{nL}^L - \xi \delta_{rL}^L \\

z_2 - \bar{z}_2 &= 2w_{L2}\Delta I_{rL}^L - 2w_{L2}\Delta I_{nL}^L + \xi \rho_{rL}^{L2} - \xi \rho_{nL}^{L2} - \xi \delta_{rL}^{L2} - \left(2w_{L2}\Delta \tilde{I}_{rL}^L - 2w_{L2}\Delta \tilde{I}_{nL}^L\right) \\
&= 2w_{L2}\left(\Delta I_{rL}^L - \Delta \tilde{I}_{rL}^L\right) - 2w_{L2}\left(\Delta I_{nL}^L - \Delta \tilde{I}_{nL}^L\right) + \xi \rho_{rL}^{L2} - \xi \rho_{nL}^{L2} - \xi \delta_{rL}^{L2} \\
&= 2w_{L2}\Delta M_{rL}^L - 2w_{L2}\Delta M_{nL}^L + \xi \rho_{rL}^{L2} - \xi \rho_{nL}^{L2} - \xi \delta_{rL}^{L2} \\

z_3 - \bar{z}_3 &= (w_{L2} - 1)\Delta I_{rL}^L + B_{L2} + \xi \rho_{L2} - \xi \rho_{L1} - \left(w_{L2} - 1\right)\Delta \tilde{I}_{rL}^L \\
&= (w_{L2} - 1)\left(\Delta I_{rL}^L - \Delta \tilde{I}_{rL}^L\right) + B_{L2} + \xi \rho_{L2} - \xi \rho_{L1} \\
&= (w_{L2} - 1)\Delta M_{rL}^L + B_{L2} + \xi \rho_{L2} - \xi \rho_{L1} \\

z_4 - \bar{z}_4 &= (w_{L2} - 1)\Delta I_{rL}^L + (1 - w_{L2})\Delta I_{nL}^L + \xi \delta_{rL}^L - \xi \delta_{nL}^L - \left((w_{L2} - 1)\Delta \tilde{I}_{rL}^L + (1 - w_{L2})\Delta \tilde{I}_{nL}^L\right) \\
&= (w_{L2} - 1)\left(\Delta I_{rL}^L - \Delta \tilde{I}_{rL}^L\right) + (1 - w_{L2})\left(\Delta I_{nL}^L - \Delta \tilde{I}_{nL}^L\right) + \xi \delta_{rL}^L - \xi \delta_{nL}^L \\
&= (w_{L2} - 1)\Delta M_{rL}^L + (1 - w_{L2})\Delta M_{nL}^L + \xi \delta_{rL}^L - \xi \delta_{rL}^L
\end{align*}
\] (4.4-9 : 4.4-12)

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Again, the idea is that in the absence of measurements, the filter will smoothly transition to a model. Note that only the measurement format and state definitions change. The rest of the filter remains exactly the same as defined above.
Chapter 5

Filter Performance

5.1 Experimental Setup

The NAISD described in section 3.1.3 was used to provide “truth” data for the purposes of this research. Actual two-frequency pseudo-ranges and delta-ranges are classified and could not be used. Simulated ionospheric and atmospheric delays are generated from the NAISD. For a given trajectory, Yuma almanac file, sample rate, and date and time of epoch, simulated data is generated and printed to a text file. This file contains the time, receiver location in ECEF, each satellite position in ECEF and ionospheric and atmospheric delays for L1 and L2 at each sample time. These values are used as truth for filter tests.

Three simulated trajectories were used as input for the NAISD: a re-entry trajectory, a launch trajectory and a low earth orbit (LEO) trajectory. A recent Yuma file was used to generate GPS satellite positions (and only the positions). The date selected for the IRI to provide monthly averaged data was November 15, 1999 at 1400 local time. The date and time selected was a period of high solar activity, resulting in an active ionosphere.

The delays generated by the NAISD were then used to construct simulated pseudo-ranges and delta-ranges which were subsequently corrupted by noise. The noise by which the true data are corrupted is modeled as zero-mean, Gaussian, white noise. The simulated noise is generated by using the randn function in Matlab and added to the “true” data. The standard deviations of the noise are listed in table 5.1-1. These values correspond to the measurement noise values assumed by the Kalman filter. A simulated inter-frequency bias can also be added to L2 pseudo-ranges.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{\rho}^{L1}$</td>
<td>$(0.5m)^2$</td>
</tr>
<tr>
<td>$V_{\rho}^{L2}$</td>
<td>$(0.5m)^2$</td>
</tr>
<tr>
<td>$V_{\phi}^{L1}$</td>
<td>$(0.01m)^2$</td>
</tr>
<tr>
<td>$V_{\phi}^{L2}$</td>
<td>$(0.01m)^2$</td>
</tr>
<tr>
<td>$B_{L2}$</td>
<td>0 – 4 m</td>
</tr>
</tbody>
</table>

Table 5.1-1: Noise and Bias Added to Truth to Create Simulated Data

Pseudo-range measurements are simulated by adding to the distance between satellite and receiver the true ionospheric and atmospheric delays in corresponding units of distance. The noise terms described above are also added:

$$
\tilde{p}_k^{L1} = \tilde{R} + \Delta\tilde{\tau}_k^{L1} + \Delta\tilde{\tau}_k^{L1} + \xi\tilde{\rho}_k^{L1}
$$

$$
\tilde{p}_k^{L2} = \tilde{R} + \Delta\tilde{\tau}_k^{L2} + \Delta\tilde{\tau}_k^{L2} + \xi\tilde{\rho}_k^{L2}
$$

(5.1-1 : 5.1-2)

Delta-range measurements are simulated by differencing successive simulated phases. The ionosphere carrier phase advance is considered exactly equal and opposite of the ionosphere signal delay (see section 2.3 for discussions of higher order effects on ionospheric group delay and phase advance):

$$
\tilde{\phi}_k^{L1} = \tilde{R} - \Delta\tilde{\tau}_k^{L1} + \Delta\tilde{\tau}_k^{L1} + \xi\tilde{\phi}_k^{L1}
$$

$$
\tilde{\delta}_k^{L1} = \tilde{\phi}_k^{L1} - \tilde{\phi}_{k-1}^{L1}
$$

(5.1-3)

$$
\tilde{\phi}_k^{L2} = \tilde{R} - \Delta\tilde{\tau}_k^{L2} + \Delta\tilde{\tau}_k^{L2} + \xi\tilde{\phi}_k^{L2}
$$

$$
\tilde{\delta}_k^{L2} = \tilde{\phi}_k^{L2} - \tilde{\phi}_{k-1}^{L2}
$$

(5.1-4)

(5.1-5)

(5.1-6)

If a measurement does not exist or is modeled as “bad” for the purpose of testing the robustness of the filter, it is assigned the value NaN. Routines are included to allow both deterministic and random loss of measurements of either frequency.

Again, to test the ability of the filter to handle the L2 bias, a simulated bias can be added to all L2 pseudo-ranges:
Tests were conducted simulating the conditions of a twelve-channel receiver allocated such that ten channels are dedicated to the primary frequency (considered to be L1 for these tests). The remaining two channels are dedicated to the secondary frequency (L2) and switch satellites every five seconds.

An attempt was made to find the “best” filter parameters for each process model using a simple local search algorithm, but the process was time-consuming and produced inadequate results. Instead, the parameters selected by the local search algorithm were used as a starting point and the final parameters selected by visually comparing the results of trials with varying parameter values. For the parameter search simulations, no noise was added to the simulated measurements and there were no measurement losses on either frequency. This is the equivalent of running the simulation enough times to average out the noise. The filter parameters were selected such that they easily accommodated the “worst case” ionospheric delays and delay rates experienced during the re-entry trajectory. In theory, then, the filters could accommodate almost any delay profile without alteration. Table 5.1-2 shows the parameter values that were used for all of the simulations.

<table>
<thead>
<tr>
<th>Filter Type</th>
<th>Selected N</th>
<th>Selected Ba</th>
<th>Selected Bb</th>
<th>Selected Bc</th>
</tr>
</thead>
<tbody>
<tr>
<td>First-Order Markov</td>
<td>1e6</td>
<td>1e-2</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>Second-Order Markov</td>
<td>10</td>
<td>1e-2</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>Third-Order Markov</td>
<td>5e3</td>
<td>1e-2</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>2 Cascades of First-Order Markov Processes</td>
<td>10</td>
<td>1e-3</td>
<td>1e-2</td>
<td>N/A</td>
</tr>
<tr>
<td>3 Cascades of First-Order Markov Processes</td>
<td>1e3</td>
<td>1e-3</td>
<td>1e-2</td>
<td>1/90</td>
</tr>
<tr>
<td>Integral of First-Order Markov</td>
<td>1e-3</td>
<td>1e-2</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>Integral of Second-Order Markov</td>
<td>10</td>
<td>1e-3</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>Integral of 2 Cascades of First-Order Markov Processes</td>
<td>10</td>
<td>1e-3</td>
<td>1/700</td>
<td>N/A</td>
</tr>
</tbody>
</table>

Table 5.1-2: Filter Parameters Used for All Simulations

For comparison purposes, the plots in each of the following sections are for the same satellite: PRN 6 for the re-entry trajectory, PRN 18 for the launch trajectory and PRN 1 for the low earth orbit trajectory. In addition, the “random” noise added to the data was seeded, so the noise values are identical for each filter.

There were negligible differences in performance from model to model when there were no measurement losses on either frequency (i.e. a 6/6 channel allocation). The variances of the

\[
\tilde{\rho}_k^{L2} = \tilde{\rho}_k^{L2} + \tilde{B}_k^{L2}
\]  

(5.1-7)
measurements are small enough that with reasonable choices in process model parameters, measurements will essentially correct for any modeling mis-matches. For this reason, the comparison of the filters with no measurement losses is not presented here. For each of the three trajectories used, the performances of filters with three, two and single dynamic states are shown. Individual filter results of the three simulations follow: performance with a 10/2 primary/secondary channel allocation with five second dwell time on the secondary frequency and no signal loss, 10/2 channel allocation with signal loss at a given time without an ionospheric delay reference model and 10/2 channel allocation with signal loss at a given time with a reference model. The ionospheric reference model used was the augmented Klobuchar model discussed in section 3.1.4.

Maximum errors discussed in comparing filters do not include errors experienced while the filter was still “settling” after initialization nor do they include errors at discontinuities in the augmented Klobuchar model (see section 3.1.4). The determination of when a filter had “settled” was somewhat subjective, however, at least one crossing of zero by the plot of delay errors was required. Such subjectivity was almost exclusively a factor in the first simulation and not the second and third simulations. Such subjectivity was considered necessary in order to give the reader an at-a-glance guide to general filter performance which omits transient errors experienced shortly after initialization.

5.2 Re-Entry Results

The re-entry trajectory used was a 464 second, simulated, ballistic descent trajectory. Figures 5.2-1 through 5.2-12 show the performances of filters with three, two and single dynamic states with no measurement losses on either frequency for PRN 6. The start of re-entry is at time $t = 0$. PRN 6 becomes visible and the filter initializes at time $t = 41$.

Filters with three dynamic states had maximum delay errors around 0.15 meters and covariance envelopes around 0.1 meters with no measurement losses. The maximum delay rate errors were around 0.03 m/s with covariance envelopes around 0.012 m/s. The maximum delay second derivative errors were around 0.009 m/s with covariance envelopes around 0.006 m/s².
Figure 5.2-1: PRN 6, Ionospheric Delay for Filter with 3 Dynamic States, 6/6 Channels, Re-Entry, No Signal Loss

Figure 5.2-2: PRN 6, Ionospheric Delay Error for Filter with 3 Dynamic States, 6/6 Channels, Re-Entry, No Signal Loss

Figure 5.2-3: PRN 6, Ionospheric Delay Rate for Filter with 3 Dynamic States, 6/6 Channels, Re-Entry, No Signal Loss

Figure 5.2-4: PRN 6, Ionospheric Delay Rate Error for Filter with 3 Dynamic States, 6/6 Channels, Re-Entry, No Signal Loss
Filters with two dynamic states also had maximum delay errors around 0.15 meters and covariance envelopes around 0.1 meters. The maximum delay rate errors increased somewhat and were around 0.05 m/s with covariance envelopes around 0.025 m/s.
Filters with a single dynamic state had slightly lower delay errors: around 0.14 meters and covariance envelopes around 0.1 meters.

Table 5.2-1 shows the maximum errors experienced by each filter for the three simulations specified above: 10/2 channel allocation with no signal loss, 10/2 channel allocation with signal loss at 200 seconds without a reference model and 10/2 channel allocation with signal loss at 200 seconds with a reference model. The top three process models in terms of lowest maximum ionospheric delay error are highlighted in green and the bottom two process models highlighted in red. Sections 5.2.1 through 5.2.8 cover the performance of each filter in more detail.
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>First-Order Markov</td>
<td>Max Iono Delay Error (m)</td>
<td>0.8381</td>
<td>23.76</td>
<td>5.736</td>
</tr>
<tr>
<td></td>
<td>Max Iono Delay Error (m)</td>
<td>0.5293</td>
<td>21.81</td>
<td>4.922</td>
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<tr>
<td></td>
<td>Max Iono Delay Rate Error (m/s)</td>
<td>9.952e-2</td>
<td>0.2829</td>
<td>0.1040</td>
</tr>
<tr>
<td>Second-Order Markov</td>
<td>Max Iono Delay Error (m)</td>
<td>0.5393</td>
<td>5.999</td>
<td>12.99</td>
</tr>
<tr>
<td></td>
<td>Max Iono Delay Rate Error (m/s)</td>
<td>0.1502</td>
<td>0.2142</td>
<td>0.1223</td>
</tr>
<tr>
<td></td>
<td>Max Iono Delay 2\textsuperscript{nd} Derivative Error (m/s\textsuperscript{2})</td>
<td>2.122e-2</td>
<td>1.565e-2</td>
<td>1.564e-2</td>
</tr>
<tr>
<td>Third-Order Markov</td>
<td>Max Iono Delay Error (m)</td>
<td>0.5254</td>
<td>18.78</td>
<td>3.898</td>
</tr>
<tr>
<td></td>
<td>Max Iono Delay Rate Error (m/s)</td>
<td>0.1157</td>
<td>0.2636</td>
<td>0.1092</td>
</tr>
<tr>
<td>2 Cascades of First-Order Markov</td>
<td>Max Iono Delay Error (m)</td>
<td>0.4953</td>
<td>9.550</td>
<td>11.84</td>
</tr>
<tr>
<td></td>
<td>Max Iono Delay Rate Error (m/s)</td>
<td>0.1062</td>
<td>0.2004</td>
<td>0.1487</td>
</tr>
<tr>
<td></td>
<td>Max Iono Delay 2\textsuperscript{nd} Derivative Error (m/s\textsuperscript{2})</td>
<td>1.415e-2</td>
<td>9.347e-3</td>
<td>8.871e-3</td>
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<tr>
<td>Integral of First-Order Markov</td>
<td>Max Iono Delay Error (m)</td>
<td>0.5269</td>
<td>18.19</td>
<td>3.708</td>
</tr>
<tr>
<td></td>
<td>Max Iono Delay Rate Error (m/s)</td>
<td>0.1175</td>
<td>0.2590</td>
<td>0.1112</td>
</tr>
<tr>
<td>Integral of Second-Order Markov</td>
<td>Max Iono Delay Error (m/s)</td>
<td>0.4944</td>
<td>43.26</td>
<td>49.02</td>
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<td></td>
<td>Max Iono Delay Rate Error (m/s)</td>
<td>0.1051</td>
<td>0.4289</td>
<td>0.3562</td>
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<td></td>
<td>Max Iono Delay 2\textsuperscript{nd} Derivative Error (m/s\textsuperscript{2})</td>
<td>1.439e-2</td>
<td>1.106e-2</td>
<td>9.209e-3</td>
</tr>
<tr>
<td>Integral of 2 Cascades of First-Order Markov</td>
<td>Max Iono Delay Error (m)</td>
<td>0.5130</td>
<td>81.55</td>
<td>87.35</td>
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<tr>
<td></td>
<td>Max Iono Delay Rate Error (m/s)</td>
<td>0.1284</td>
<td>0.6714</td>
<td>0.6000</td>
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<tr>
<td></td>
<td>Max Iono Delay 2\textsuperscript{nd} Derivative Error (m/s\textsuperscript{2})</td>
<td>1.835e-2</td>
<td>1.187e-2</td>
<td>1.176e-2</td>
</tr>
</tbody>
</table>

Table 5.2-1: Maximum Errors for Re-Entry Trajectory

5.2.1 First-Order Markov Process

The time constant for the first-order Markov process was purposely selected to be fairly short. Very long time constants essentially created a sample and hold during periods of signal loss over the duration of a comparatively short simulation. The first-order Markov process produced noisy
results for the first simulation, the worst of the filters considered with the re-entry trajectory. The errors for PRN 6, however, stayed under 0.84 meters and the covariance envelope stayed around 0.3 meters for most of the simulation.

![Figure 5.2.1-1: PRN 6, Ionospheric Delay for First-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss](image1)

![Figure 5.2.1-2: PRN 6, Ionospheric Delay Error for First-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss](image2)

The filter produced fairly poor results in the absence of measurements and a model. Even though the trend was increasing ionospheric delays, the process model drove the estimates down to near zero quickly after loosing measurements. Errors just under 24 meters were experienced and the covariance envelope opened very quickly. The covariance envelope’s behavior is a consequence of the large noise parameter selected. Although undesirable, it was accepted as choosing a smaller noise value gave more weight to the model compared to measurements, resulting in worse estimates.
With a delay model, the filter performed fairly well. In this case, the modeled ionospheric delays were quite accurate. Had the model been poor, however, the filter would not have performed as well. Instead of tending to zero, the filter tended quickly to the model in the absence of measurements. Using this filter, PRN 6 experienced errors of less than 6 meters.

5.2.2 Second-Order Markov Process

As with the first-order process model, the time constant for the second-order Markov process was purposely selected to be fairly short so the estimate wouldn’t run away in the absence of
measurements. The second-order Markov process produced smoother ionospheric delay estimates than the first-order. After a spike in estimated delay at the start of the simulation, the error remains less than 0.53 meters with the covariance envelope around 0.2 meters. After the spike, the rate error doesn't exceed 0.1 m/s and the covariance envelope stays around 0.06 m/s.

Figure 5.2.2-1: PRN 6, Ionospheric Delay for Second-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.2-2: PRN 6, Ionospheric Delay Error for Second-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.2-3: PRN 6, Ionospheric Delay Rate for Second-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss, No Delay Model

Figure 5.2.2-4: PRN 6, Ionospheric Delay Rate Error for Second-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss, No Delay Model

The second-order Markov process offered a slight improvement over the first-order in the absence of measurements and a reference model. The estimate continues the trend of increasing ionospheric delays at first and then tends more slowly to zero. Nevertheless, the maximum delay
error during the simulation exceeds 21 meters and the maximum delay rate error exceeds 0.28 m/s.

Figure 5.2.2-5: PRN 6, Ionospheric Delay for Second-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, No Delay Model

Figure 5.2.2-6: PRN 6, Ionospheric Delay Error for Second-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, No Delay Model

Figure 5.2.2-7: PRN 6, Ionospheric Delay Rate for Second-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, No Delay Model

Figure 5.2.2-8: PRN 6, Ionospheric Delay Rate Error for Second-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, No Delay Model

With a reference model, the second-order Markov process was one of the best examined in simulation three. The delay error did not exceed five meters while the delay rate error was only slightly greater than 0.1 m/s at worst. Again, a lot of the performance must be attributed to the ionospheric delay model used.
5.2.3 Third-Order Markov Process

The third-order Markov process did not provide an improvement over the second-order Markov process during the first simulation. Although it performed only slightly worse than the second-order with the parameters selected, it was the second worst of all the models considered in the first simulation. The third-order Markov process experienced a maximum ionospheric delay error about 0.01 meters greater than the second-order Markov process. The covariance envelope remained around 0.2 meters. After the initial spike, the rate error exceeded 0.15 m/s once and by
very little with the covariance envelope remaining around 0.06 m/s. The delay second derivative error exceeded 0.021 m/s² once by very little with the covariance envelope around 0.016 m/s².

Figure 5.2.3-1: PRN 6, Ionospheric Delay for Third-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.3-2: PRN 6, Ionospheric Delay Error for Third-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.3-3: PRN 6, Ionospheric Delay Rate for Third-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.3-4: PRN 6, Ionospheric Delay Rate Error for Third-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss
The third-order Markov process was a great improvement over the second-order in the second simulation. With a maximum delay error just under 6 meters, it was the best process model for the second simulation with the re-entry trajectory. The maximum delay rate error was a little less than 0.22 m/s with the maximum second derivative error just under 0.016 m/s$^2$.
The third-order Markov process performed worse with a model than without and was on the low end performance-wise in the third simulation. The maximum delay error was just under 13 meters. The maximum delay rate error was just over 0.12 m/s and the maximum delay second derivative error was just under 0.016 m/s².
Figure 5.2.3-13: PRN 6, Ionospheric Delay for Third-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, with Delay Model

Figure 5.2.3-14: PRN 6, Ionospheric Delay Error for Third-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, with Delay Model

Figure 5.2.3-15: PRN 6, Ionospheric Delay Rate for Third-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, with Delay Model

Figure 5.2.3-16: PRN 6, Ionospheric Delay Rate Error for Third-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at $t = 200$ Seconds, with Delay Model
5.2.4 Two Cascades of First-Order Markov Processes

Two cascades of first-order Markov processes performed comparably to the second-order Markov process. With no signal outage, the maximum delay error was just over 0.52 meters with a covariance envelope around 0.2 meters. The maximum delay rate error was just under 0.12 m/s with a covariance envelope around 0.06 m/s.
The two cascades process model performed slightly better than the second-order Markov process in the second simulation. The maximum delay error was under 19 meters with the maximum rate error just over 0.26 m/s.
The two cascades process model is separated from the second-order Markov process by its performance in the third simulation. With a maximum delay error under 4 meters and the maximum delay rate error just under 0.11 m/s, it was a close second-best performer in the third simulation.
5.2.5 Three Cascades of First-Order Markov Processes

Three cascades of first-order Markov processes performed well in the first simulation; second best by about a millimeter. The maximum delay error was just under 0.5 meters and the delay rate error was just over 0.1 m/s with covariance envelopes around 0.2 meters and 0.04 m/s, respectively. The maximum delay second derivative error was around 0.014 m/s$^2$ with a covariance envelope around 0.009 m/s$^2$. 

![Figure 5.2.5-1: PRN 6, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, No Signal Loss](image1)

![Figure 5.2.5-2: PRN 6, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, No Signal Loss](image2)
The three cascades process model was also second best in the second simulation, although, by a few meters this time. The maximum delay error was around 9.5 meters, the maximum delay rate error was just over 0.2 m/s, and the maximum delay second derivative error was just under 0.01 m/s².
Figure 5.2.5-7: PRN 6, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, No Delay Model

Figure 5.2.5-8: PRN 6, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, No Delay Model

Figure 5.2.5-9: PRN 6, Ionospheric Delay Rate for Three Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, No Delay Model

Figure 5.2.5-10: PRN 6, Ionospheric Delay Rate Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, No Delay Model
The process model was average in performance in the third simulation. The maximum delay, delay rate, and delay second derivative errors were around 12 meters, 0.15 m/s and 0.009 m/s$^2$, respectively. Like the third-order Markov process, the three cascades model was worse with an ionospheric delay reference model than without.
5.2.6 Integral of First-Order Markov Process

Using the integral of a first-order Markov process as a process model yielded average results in the first simulation. The maximum delay error was just under 0.53 meters with a covariance
envelope around 0.2 meters. The maximum rate error was just under 0.12 m/s with a covariance envelope around 0.06 m/s.

The process model was one of the best performers during the second simulation. The error was still nearly double that of the three cascades model at over 18 meters. The maximum delay rate error was just under 0.26 m/s.

Figure 5.2.6-1: PRN 6, Ionospheric Delay for Integral of First-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.6-2: PRN 6, Ionospheric Delay Error for Integral of First-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.6-3: PRN 6, Ionospheric Delay Rate for Integral of First-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.6-4: PRN 6, Ionospheric Delay Rate Error for Integral of First-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss
The integral of a first-order Markov process proved the best choice by a narrow margin in the third simulation with a delay error just over 3.7 meters and a delay rate error just over 0.11 m/s. In the absence of measurements, the estimate tends to a bias from the reference ionospheric model.
5.2.7 Integral of Second-Order Markov Process

The integral of a second-order Markov process performed well in the first simulation. It was the best of the models considered. The maximum delay error was just under 0.5 meters with a covariance envelope around 0.2 meters. The maximum delay rate error was just over 0.1 m/s with a covariance envelope 0.04 m/s. The maximum delay second derivative error was around 0.014 m/s² with a covariance envelope close to 0.008 m/s².
Figure 5.2.7-1: PRN 6, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.7-2: PRN 6, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.7-3: PRN 6, Ionospheric Delay Rate for Integral of Second-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.7-4: PRN 6, Ionospheric Delay Rate Error for Integral of Second-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss
The process model did not fair well in the absence of measurements. In the second simulation, it was the second-worst process model considered with a maximum delay error over 43 meters, a maximum delay rate error of around 0.43 m/s, and a maximum delay second-derivate error around 0.011 m/s².

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**Figure 5.2.7-5:** PRN 6, Ionospheric Delay 2nd Derivative for Integral of Second-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss

**Figure 5.2.7-6:** PRN 6, Ionospheric Delay 2nd Derivative Error for Integral of Second-Order Markov Process, 10/2 Channels, Re-Entry, No Signal Loss

**Figure 5.2.7-7:** PRN 6, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, No Delay Model

**Figure 5.2.7-8:** PRN 6, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, No Delay Model
The process model performed worse with a delay model than without. The maximum delay error during the third simulation was 49 meters. The maximum delay rate error was around 0.36 m/s and the maximum delay second derivative error was just under 0.01 m/s². Recall that the parameters were tuned for use without a reference model and robustness of the filters once the parameters have been tuned is of interest.
Figure 5.2.7-13: PRN 6, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, with Delay Model

Figure 5.2.7-14: PRN 6, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, with Delay Model

Figure 5.2.7-15: PRN 6, Ionospheric Delay Rate for Integral of Second-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, with Delay Model

Figure 5.2.7-16: PRN 6, Ionospheric Delay Rate Error for Integral of Second-Order Markov Process, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, with Delay Model
5.2.8 Integral of Two Cascades of First-Order Markov Processes

The integral of two cascades of first-order Markov processes performed well in the first simulation with a maximum delay error just over 0.51 meters. The maximum delay rate error was just under 0.13 m/s and the maximum delay second derivative error was around 0.018 m/s². The respective covariance envelopes were around 0.2 meters, 0.05 m/s and 0.012 m/s².

Figure 5.2.8-1: PRN 6, Ionospheric Delay for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, No Signal Loss

Figure 5.2.8-2: PRN 6, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, No Signal Loss
The integral of two cascades suffered the same problems as the integral of a second-order Markov process when experiencing measurement losses, but to a greater extent. The integral of two cascades was the worst of the models considered in both the second and third simulations. During the second simulation, the maximum delay error was almost 82 meters, the maximum delay rate error over 0.67 m/s and the maximum delay second derivative error just under 0.012 m/s².
Figure 5.2.8-7: PRN 6, Ionospheric Delay for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at \( t = 200 \) Seconds, No Delay Model

Figure 5.2.8-8: PRN 6, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at \( t = 200 \) Seconds, No Delay Model

Figure 5.2.8-9: PRN 6, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at \( t = 200 \) Seconds, No Delay Model

Figure 5.2.8-10: PRN 6, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at \( t = 200 \) Seconds, No Delay Model
Like the integral of a second-order Markov process, the integral of two cascades of Markov processes performed worse with a delay model than without. The maximum delay error was over 87 meters with a maximum delay rate error of 0.6 m/s and a maximum delay second derivative error just under 0.012 m/s².
Figure 5.2.8-15: PRN 6, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, with Delay Model

Figure 5.2.8-16: PRN 6, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, with Delay Model

Figure 5.2.8-17: PRN 6, Ionospheric Delay 2nd Derivative for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, with Delay Model

Figure 5.2.8-18: PRN 6, Ionospheric Delay 2nd Derivative Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Re-Entry, Signal Loss at t = 200 Seconds, with Delay Model
5.3 Launch Results

The launch trajectory used was a 291 second simulated ballistic ascent trajectory. Figures 5.3-1 through 5.3-12 show the performances of filters with three, two and single dynamic states with no measurement losses for PRN 18.

Filters with three dynamic states had maximum delay errors around 0.16 meters and covariance envelopes around 0.1 meters with no measurement losses. The maximum delay rate errors were around 0.026 m/s with covariance envelopes around 0.012 m/s. The maximum delay second derivative errors were around 0.01 m/s² with covariance envelopes around 0.006 m/s².

Figure 5.3-1: PRN 18, Ionospheric Delay for Filter with 3 Dynamic States, 6/6 Channels, Launch, No Signal Loss

Figure 5.3-2: PRN 18, Ionospheric Delay Error for Filter with 3 Dynamic States, 6/6 Channels, Launch, No Signal Loss
Filters with two dynamic states had maximum delay errors around 0.16 meters and covariance envelopes around 0.1 meters. The maximum delay rate errors were around 0.064 m/s with covariance envelopes around 0.025 m/s.
Filters with one dynamic state also had maximum delay errors around 0.16 meters and covariance envelopes around 0.1 meters.
Figure 5.3-11: PRN 18, Ionospheric Delay for Filter with 1 Dynamic State, 6/6 Channels, Launch, No Signal Loss

Figure 5.3-12: PRN 18, Ionospheric Delay Error for Filter with 1 Dynamic State, 6/6 Channels, Launch, No Signal Loss

Table 5.3-1 shows the maximum errors experienced by each filter for the same three simulations as for the re-entry trajectory. The signal outage occurs at $t = 100$ seconds.
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<td>First-Order Markov</td>
<td>Max Iono Delay Error (m)</td>
<td>0.6796</td>
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<td>Max Iono Delay Rate Error (m/s)</td>
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<td>Max Iono Delay 2nd Derivative Error (m/s^2)</td>
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<td>1.460e-2</td>
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<td>Max Iono Delay Error (m)</td>
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<td>Max Iono Delay Rate Error (m/s)</td>
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<td>Max Iono Delay 2nd Derivative Error (m/s^2)</td>
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<td>6.943e-3</td>
<td>6.933e-3</td>
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<td>Integral of First-Order Markov</td>
<td>Max Iono Delay Error (m)</td>
<td>0.4442</td>
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<td>Max Iono Delay Rate Error (m/s)</td>
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<td>7.182e-2</td>
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<td>Max Iono Delay 2nd Derivative Error (m/s^2)</td>
<td>1.125e-2</td>
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<td>Max Iono Delay Rate Error (m/s)</td>
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<td>Max Iono Delay 2nd Derivative Error (m/s^2)</td>
<td>1.524e-2</td>
<td>1.068e-2</td>
<td>1.043e-2</td>
</tr>
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</table>

Table 5.3-1: Maximum Errors for Launch Trajectory

5.3.1 First-Order Markov Process

The first-order Markov process, again, proved to be noisy and the process model was the worst of those considered in the first simulation. The maximum error was around 0.68 meters with a covariance envelope around 0.3 meters.
In simulation two, the first-order Markov process was one of the best considered with a maximum delay error just over 6.5 meters. For a launch, the ionospheric delay will tend to zero as time goes on, so the first-order Markov process is better suited for such a trajectory than for a re-entry (or low earth orbit) trajectory.

The first-order Markov process was an average performer in the third simulation. The maximum delay error was around 5.3 meters, performing slightly better with an ionospheric delay model than without.
5.3.2 Second-Order Markov Process

The second-order Markov process was a very close second-best process model considered in simulation one. The maximum delay error was under 0.45 meters with a covariance envelope close to 0.2 meters. The maximum delay rate error was just over 0.075 m/s with a covariance envelope around 0.06 m/s.
The second-order Markov process was also the second-best model in simulation two. The maximum delay error was just over 3.2 meters. The maximum delay rate error was just over 0.074 m/s.
The process model was an average performer in the third simulation, but performed slightly worse with a delay model than without. The maximum delay error was just over 4.6 meters with a maximum delay rate error around 0.079 m/s.
5.3.3 Third-Order Markov Process

The third-order Markov process provided average performance in all three simulations. It was slightly worse than the second-order, but better than the first-order Markov process in the first simulation with a maximum delay error of just under 0.5 meters and a covariance envelope around 0.2 meters. The maximum delay rate error was just under 0.1 m/s with a covariance envelope around 0.06 m/s. The maximum delay second derivative error was around 0.018 m/s² with a covariance envelope around 0.016 m/s².
The third-order Markov process performed worse than either the first-order or the second-order in the second simulation with a maximum delay error just over 7.1 meters. The maximum delay rate error was around 0.11 m/s and the maximum delay second derivative error was just under 0.015 m/s².
Figure 5.3.3-7: PRN 18, Ionospheric Delay for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.3-8: PRN 18, Ionospheric Delay Error for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.3-9: PRN 18, Ionospheric Delay Rate for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.3-10: PRN 18, Ionospheric Delay Rate Error for Third-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model
The third-order Markov process performed worse than the first-order and second-order Markov processes in the third simulation, but average overall. The maximum delay error was just under 9.4 meters with a maximum delay rate error around 0.1 m/s and a maximum delay second derivative error just under 0.015 m/s². Like the second-order Markov process, the third-order performed worse with an ionospheric delay reference model than without.
5.3.4 Two Cascades of First-Order Markov Processes

Two cascades of first-order Markov processes was one of the best process models in the first simulation with a maximum delay error just over 0.46 meters. Nevertheless, the second-order Markov process performed better. The covariance envelope for the two cascades model remained around 0.2 meters. The maximum delay rate error was under 0.08 m/s with a covariance envelope around 0.06 m/s.
The two cascades model was an average performer in the second simulation, but worse than the second-order Markov process with a maximum delay error just over 10 meters. The maximum delay rate error was just under 0.12 m/s.
This process model was a close second-best choice for the third simulation with a maximum delay error just over 3.9 meters. The maximum delay rate error was just under 0.073 m/s.
5.3.5 Three Cascades of First-Order Markov Processes

The three cascades model was an average performer in the first simulation with a maximum delay error just under 0.48 meters and a covariance envelope around 0.2 meters. The maximum delay rate error was just under 0.067 m/s with a covariance envelope around 0.04 m/s. The maximum delay second derivative error was just over 0.011 m/s² with a covariance envelope.
around 0.009 m/s². These values made the three cascades model slightly better than the third-order Markov process in the first simulation.

Figure 5.3.5-1: PRN 18, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.5-2: PRN 18, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.5-3: PRN 18, Ionospheric Delay Rate for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.5-4: PRN 18, Ionospheric Delay Rate Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss
The three cascades model was the best choice of the models considered in the second simulation with a maximum delay error just over 2.7 meters, a maximum delay rate error just under 0.069 m/s and a maximum delay second derivative error under 0.007 m/s².

Figure 5.3.5-5: PRN 18, Ionospheric Delay 2nd Derivative for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.5-6: PRN 18, Ionospheric Delay 2nd Derivative Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.5-7: PRN 18, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.5-8: PRN 18, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model
The three cascades model performed worse with an ionospheric delay model than without, but it was still one of the best choices for the third simulation. The maximum delay error was just under 4.6 meters, the maximum delay rate error, just under 0.07 m/s and the maximum delay second derivative error around 0.007 m/s².
Figure 5.3.5-13: PRN 18, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.5-14: PRN 18, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.5-15: PRN 18, Ionospheric Delay Rate for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model

Figure 5.3.5-16: PRN 18, Ionospheric Delay Rate Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at $t = 100$ Seconds, with Delay Model
5.3.5 PRN 18, Ionospheric Delay 2nd Derivative for Three Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

5.3.6 Integral of First-Order Markov Process

The integral of a first-order Markov process was the best performer in the first simulation. The maximum delay error was just over 0.44 meters with a covariance envelope around 0.2 meters. The maximum delay rate error was just over 0.076 m/s with a covariance envelope around 0.06 m/s.

5.3.6 PRN 18, Ionospheric Delay Error for Integral of First-Order Markov Process, 10/2 Channels, Launch, No Signal Loss
Figure 5.3.6-3: PRN 18, Ionospheric Delay Rate for Integral of First-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.6-4: PRN 18, Ionospheric Delay Rate Error for Integral of First-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

The process model was an average performer in the second simulation. The maximum delay error was just under 11.6 meters with a maximum delay rate error just under 0.13 m/s.

Figure 5.3.6-5: PRN 18, Ionospheric Delay for Integral of First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.6-6: PRN 18, Ionospheric Delay Error for Integral of First-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model
The integral of a first-order Markov process was also the best model considered in the third simulation with a maximum delay error just over 3.8 meters. The maximum delay rate error was just over 0.07 m/s.
5.3.7 Integral of Second-Order Markov Process

The integral of a second-order Markov process produced average results in the first simulation. The maximum delay error was just over 0.48 meters with a covariance envelope around 0.2 meters. The maximum delay rate error was around $0.067 \text{ m/s}$ with the covariance envelope remaining around 0.04 m/s. The maximum delay second derivative error was just over $0.011 \text{ m/s}^2$ with a covariance envelope around $0.009 \text{ m/s}^2$.

Figure 5.3.7-1: PRN 18, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.7-2: PRN 18, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, Launch, No Signal Loss
The integral of a second-order Markov process did not perform well in the absence of measurements. Without a reference model, it was the second-worst performer with a maximum delay error over 15 meters, a maximum delay rate error around 0.23 m/s and a maximum delay second derivative error just over 0.007 m/s².
Figure 5.3.7-7: PRN 18, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.7-8: PRN 18, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.7-9: PRN 18, Ionospheric Delay Rate for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model

Figure 5.3.7-10: PRN 18, Ionospheric Delay Rate Error for Integral of Second-Order Markov Process, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, No Delay Model
The integral of a second-order Markov process produced the second-worst estimates in the third simulation. The maximum delay error was almost 22 meters with a maximum delay rate error just over 0.28 m/s and a maximum delay second derivative error just over 0.007 m/s². As with the re-entry trajectory, the integral of a second-order Markov process performed worse with an ionospheric delay reference model than without.
5.3.8 Integral of Two Cascades of First-Order Markov Processes

The integral of two cascades of first-order Markov processes was the second-worst performer in the first simulation. The maximum delay error was over 0.53 meters with a covariance envelope around 0.2 meters. The maximum delay rate error was around 0.08 m/s with a covariance envelope around 0.05 m/s. The delay second derivative error was around 0.015 m/s² with the covariance envelope remaining around 0.012 m/s².
Figure 5.3.8-1: PRN 18, Ionospheric Delay for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.8-2: PRN 18, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.8-3: PRN 18, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss

Figure 5.3.8-4: PRN 18, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, No Signal Loss
The integral of two cascades was the worst performer in the second and third simulations. In the second simulation, the maximum delay error was over 35 meters. The maximum delay rate error was over 0.4 m/s and the maximum delay second derivative error around 0.01 m/s².
In the third simulation, the integral of two cascades of Markov processes had a maximum delay error over 43 meters. The maximum delay rate error was just over 0.48 m/s and the maximum delay second derivative error was around 0.01 m/s². As with the re-entry trajectory, this model performed better without an ionospheric delay reference model.
Figure 5.3.8-13: PRN 18, Ionospheric Delay for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.8-14: PRN 18, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.8-15: PRN 18, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model

Figure 5.3.8-16: PRN 18, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, Launch, Signal Loss at t = 100 Seconds, with Delay Model
5.4 Low Earth Orbit Results

The low earth orbit trajectory used was a single, simulated 5407 second, circular, equatorial orbit. Figures 5.4-1 through 5.4-24 show the performances of filters with three, two and single dynamic states with no measurement losses for PRN 1. This satellite is out of view from \( t = 190 \) seconds to \( t = 4200 \) seconds.

For the following descriptions of maximum errors, the determination of when a filter was settled was even more subjective than for the previous two trajectories. The filters did not seem to completely settle during the first 190 seconds of simulation, but the results should not be neglected. The most objective assessments of filter performances were made, but the interpretation of results in the first simulation should be considered carefully by the reader. Since the signal outage was at \( t = 4800 \) seconds, such subjectivity was not a factor in evaluating filter performances in the second and third simulations.

Filters with three dynamic states had maximum delay errors around 0.36 meters and covariance envelopes between 0.1 and 0.05 meters for most of the simulation. The maximum delay rate errors were around 0.033 m/s with covariance envelopes around 0.01 m/s. The maximum delay second derivative errors were around 0.01 m/s² with covariance envelopes around 0.006 m/s².
Figure 5.4-1: PRN 1, Ionospheric Delay for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4-2: PRN 1, Ionospheric Delay Error for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4-3: PRN 1, Ionospheric Delay for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4-4: PRN 1, Ionospheric Delay Error for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds
Figure 5.4-5: PRN 1, Ionospheric Delay Rate for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4-6: PRN 1, Ionospheric Delay Rate Error for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4-7: PRN 1, Ionospheric Delay Rate for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4-8: PRN 1, Ionospheric Delay Rate Error for Filter with 3 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds
Filters with two dynamic states had maximum delay errors around 0.36 meters and covariance envelopes between 0.1 and 0.05 meters. The maximum delay rate errors were around 0.067 m/s with covariance envelopes around 0.025 m/s.
Figure 5.4-13: PRN 1, Ionospheric Delay for Filter with 2 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4-14: PRN 1, Ionospheric Delay Error for Filter with 2 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4-15: PRN 1, Ionospheric Delay for Filter with 2 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4-16: PRN 1, Ionospheric Delay Error for Filter with 2 Dynamic States, 6/6 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds
Filters with a single dynamic state had maximum delay errors around 0.35 meters and covariance envelopes around 0.1 to 0.05 meters with no measurement losses.
Table 5.4-1 shows the maximum errors experienced by each filter for the same three simulations as discussed above. The signal outage occurs at $t = 4800$ seconds.
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Table 5.4-1: Maximum Errors for LEO Trajectory

5.4.1 First-Order Markov Process

As in the re-entry and launch trajectories, the first-order Markov process performed the worst in the first simulation with a maximum delay error just under 0.89 meters and a covariance envelope around 0.3 meters.
The first-order Markov process produced average results in the second simulation with a maximum delay error just under 15 meters.
The first-order Markov process was also an average performer in the third simulation with a maximum delay error just over 12.5 meters.
5.4.2 Second-Order Markov Process

The second-order Markov process produced average results in the first simulation with a maximum delay error just under 0.64 meters and a covariance envelope around 0.2 meters. The maximum delay rate error was just under 0.1 m/s with a covariance envelope around 0.06 m/s.

![Figure 5.4.2-1: PRN 1, Ionospheric Delay for Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t ≠ 0 to t = 190 Seconds](image)

![Figure 5.4.2-2: PRN 1, Ionospheric Delay Error for Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t ≠ 0 to t = 190 Seconds](image)

![Figure 5.4.2-3: PRN 1, Ionospheric Delay for Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds](image)

![Figure 5.4.2-4: PRN 1, Ionospheric Delay Error for Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds](image)
The second-order Markov process was one of the top process models in the second simulation with a maximum delay error just under 14.8 meters. The maximum delay rate error was just under 0.1 m/s.
In the third simulation, the maximum delay error was just over 12.4 meters with a delay rate error just over 0.09 m/s. The inclusion of an ionospheric delay reference model was a slight improvement over no reference model.
The discontinuity around $t = 5050$ seconds is due to the discontinuities that occur in the augmented Klobuchar model as described in section 3.1.4.
5.4.3 Third-Order Markov Process

The third-order Markov process was a slight improvement over the second-order in the first simulation. The maximum delay error was just under 0.63 meters with a covariance envelope around 0.2 meters. The maximum delay rate error was just over 0.15 m/s with a covariance envelope around 0.06 m/s. The maximum delay second derivative error was just under 0.022 m/s² with a covariance envelope around 0.016 m/s².

Figure 5.4.3-1: PRN 1, Ionospheric Delay for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.3-2: PRN 1, Ionospheric Delay Error for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.3-3: PRN 1, Ionospheric Delay for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.3-4: PRN 1, Ionospheric Delay Error for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds
Figure 5.4.3-5: PRN 1, Ionospheric Delay Rate for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.3-6: PRN 1, Ionospheric Delay Rate Error for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.3-7: PRN 1, Ionospheric Delay Rate for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.3-8: PRN 1, Ionospheric Delay Rate Error for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds
Figure 5.4.3-9: PRN 1, Ionospheric Delay 2nd Derivative for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.3-10: PRN 1, Ionospheric Delay 2nd Derivative Error for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.3-11: PRN 1, Ionospheric Delay 2nd Derivative for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.3-12: PRN 1, Ionospheric Delay 2nd Derivative Error for Third-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

This process model did not perform very well in the absence of measurements. In the second simulation, the maximum delay error was around 42 meters with a delay rate error around 0.33 m/s and a delay second derivative error around 0.022 m/s².
Figure 5.4.3-13: PRN 1, Ionospheric Delay for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.3-14: PRN 1, Ionospheric Delay Error for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.3-15: PRN 1, Ionospheric Delay Rate for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.3-16: PRN 1, Ionospheric Delay Rate Error for Third-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds
The inclusion of an ionospheric delay reference model was a slight improvement over the third-order Markov process without an ionospheric model. The maximum delay error was just under 42 meters. The maximum delay rate error was just under 0.33 m/s and the maximum delay second derivative error was around 0.022 m/s².
5.4.4 Two Cascades of First-Order Markov Processes

The two cascades model yielded average results in the first simulation with a maximum delay error of 0.65 meters and a covariance envelope around 0.2 meters. The maximum delay rate error was just over 0.1 m/s with a covariance envelope around 0.06 m/s.
Figure 5.4.4-1: PRN 1, Ionospheric Delay for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.4-2: PRN 1, Ionospheric Delay Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.4-3: PRN 1, Ionospheric Delay for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.4-4: PRN 1, Ionospheric Delay Error for Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds
The two cascades model was the second-best process model in the absence of measurements. The maximum delay error was just under 7 meters with a delay rate error just over 0.1 m/s.
The two cascades process model produced better results without an ionospheric delay reference model than with, but it was still the second-best process model in the third simulation. With a reference model, the maximum delay error was just under 7.8 meters with a maximum delay rate error just over 0.1 m/s.
Three cascades of first-order Markov processes was one of the best process models in the first simulation. The maximum delay error was just over 0.61 meters with a covariance envelope around 0.2 meters. The maximum delay rate error was just under 0.11 m/s with a 0.04 m/s...
covariance envelope and the maximum delay second derivative error was just under 0.012 m/s² with a 0.009 m/s² covariance envelope.

Figure 5.4.5-1: PRN 1, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.5-2: PRN 1, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.5-3: PRN 1, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.5-4: PRN 1, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds
Figure 5.4.5-5: PRN 1, Ionospheric Delay Rate for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.5-6: PRN 1, Ionospheric Delay Rate Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 0$ to $t = 190$ Seconds

Figure 5.4.5-7: PRN 1, Ionospheric Delay Rate for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-8: PRN 1, Ionospheric Delay Rate Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from $t = 4200$ to $t = 5407$ Seconds
The three cascades model produced average results in the absence of measurements. In the second simulation, the maximum delay error was just over 24 meters, the maximum delay rate error was just under 0.11 m/s and the maximum delay second derivative error was around 0.012 m/s².
Figure 5.4.5-13: PRN 1, Ionospheric Delay for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-14: PRN 1, Ionospheric Delay Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-15: PRN 1, Ionospheric Delay Rate for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.5-16: PRN 1, Ionospheric Delay Rate Error for Three Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds
The three cascades model performed slightly worse with an ionospheric delay reference model than without. The maximum delay error in the third simulation was just under 24.6 meters, the maximum delay rate error was just over 0.11 m/s and the maximum delay second derivative error, just under 0.012 m/s².
5.4.6 Integral of First-Order Markov Process

The integral of a first-order Markov process was the top performer in all simulations for the low earth orbit trajectory. The first simulation yielded a maximum delay error of just over 0.56 meters with a covariance envelope around 0.2 meters. The maximum delay rate error was just
over 0.1 m/s with a covariance envelope around 0.06 m/s. The reader should note that this filter was one for which it was difficult to determine when the filter had settled.

Figure 5.4.6-1: PRN 1, Ionospheric Delay for Integral of First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.6-2: PRN 1, Ionospheric Delay Error for Integral of First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.6-3: PRN 1, Ionospheric Delay for Integral of First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.6-4: PRN 1, Ionospheric Delay Error for Integral of First-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds
The maximum delay error in the second simulation was just under 1.7 meters. The maximum delay rate error was just over 0.1 m/s.
In the low earth orbit trajectory, the integral of a first-order Markov process performed better without the ionospheric delay model than with. The maximum delay error in the third simulation was 4.6 meters. The maximum delay rate error was just over 0.1 m/s. This decrease in performance with the inclusion of a reference model was a chance occurrence. In the absence
measurements and a reference model, the filter tends to a constant. Signal loss happened to occur when the true delay was changing very little.

Figure 5.4.6-13: PRN 1, Ionospheric Delay for Integral of First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.6-14: PRN 1, Ionospheric Delay Error for Integral of First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.6-15: PRN 1, Ionospheric Delay Rate for Integral of First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.6-16: PRN 1, Ionospheric Delay Rate Error for Integral of First-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds
5.4.7 Integral of Second-Order Markov Process

The integral of a second-order Markov process was the second-best performer in the first simulation with a delay error just under 0.57 meters and a covariance envelope around 0.2 meters. The maximum delay rate error was just under 0.11 m/s with a covariance envelope around 0.04 m/s. The maximum delay second derivative error was around 0.013 m/s² with a covariance envelope around 0.009 m/s².

Figure 5.4.7-1: PRN 1, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.7-2: PRN 1, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.7-3: PRN 1, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.7-4: PRN 1, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds
Figure 5.4.7-5: PRN 1, Ionospheric Delay Rate for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.7-6: PRN 1, Ionospheric Delay Rate Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.7-7: PRN 1, Ionospheric Delay Rate for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.7-8: PRN 1, Ionospheric Delay Rate Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds
The integral of a second-order Markov process was the second-worst performer in the absence of measurements (both simulations two and three). The maximum delay error in the second simulation was just under 337 meters. The maximum delay rate error was just under 0.89 m/s and the maximum delay second derivative error was just under 0.013 m/s².
Figure 5.4.7-13: PRN 1, Ionospheric Delay for Integral of Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.7-14: PRN 1, Ionospheric Delay Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.7-15: PRN 1, Ionospheric Delay Rate for Integral of Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.7-16: PRN 1, Ionospheric Delay Rate Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, No Delay Model, from $t = 4200$ to $t = 5407$ Seconds
The inclusion of an ionospheric delay reference model improved the performance of the integral of a second-order Markov process slightly. The maximum delay error was just over 333 meters with a delay rate error just over 0.88 m/s and a delay second derivative error just under 0.013 m/s².
Figure 5.4.7-21: PRN 1, Ionospheric Delay Rate for Integral of Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.7-22: PRN 1, Ionospheric Delay Rate Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.7-23: PRN 1, Ionospheric Delay 2nd Derivative for Integral of Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.7-24: PRN 1, Ionospheric Delay 2nd Derivative Error for Integral of Second-Order Markov Process, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

5.4.8 Integral of Two Cascades of First-Order Markov Processes

The integral of two cascades was the second-worst performer in the first simulation with a maximum delay error just under 0.72 meters and a covariance envelope around 0.25 meters. The maximum delay rate error was just over 0.13 m/s with a covariance envelope around 0.05 m/s.
The maximum delay second derivative error was just over 0.017 m/s² with a covariance envelope around 0.012 m/s².

Figure 5.4.8-1: PRN 1, Ionospheric Delay for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.8-2: PRN 1, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from t = 0 to t = 190 Seconds

Figure 5.4.8-3: PRN 1, Ionospheric Delay for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds

Figure 5.4.8-4: PRN 1, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from t = 4200 to t = 5407 Seconds
Figure 5.4.8-5: PRN 1, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from \(t = 0\) to \(t = 190\) Seconds

Figure 5.4.8-6: PRN 1, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from \(t = 0\) to \(t = 190\) Seconds

Figure 5.4.8-7: PRN 1, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from \(t = 4200\) to \(t = 5407\) Seconds

Figure 5.4.8-8: PRN 1, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, No Signal Loss, from \(t = 4200\) to \(t = 5407\) Seconds
The integral of two cascades yielded the largest delay errors in the second and third simulations. The maximum delay error in the second simulation was just under 781 meters with a maximum delay rate error of almost 2 m/s and a maximum delay second derivative error just over 0.017 m/s².
Figure 5.4.8-13: PRN 1, Ionospheric Delay for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.8-14: PRN 1, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.8-15: PRN 1, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.8-16: PRN 1, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds
The process model performed marginally better with an ionospheric delay reference model. The maximum delay error was just over 777 meters with a maximum delay rate error just under 2 m/s and a maximum delay second derivative error just over 0.017 m/s².

Figure 5.4.8-17: PRN 1, Ionospheric Delay 2nd Derivative for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.8-18: PRN 1, Ionospheric Delay 2nd Derivative Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, No Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.8-19: PRN 1, Ionospheric Delay for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, with Delay Model, from t = 4200 to t = 5407 Seconds

Figure 5.4.8-20: PRN 1, Ionospheric Delay Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at t = 4800 Seconds, with Delay Model, from t = 4200 to t = 5407 Seconds
Figure 5.4.8-21: PRN 1, Ionospheric Delay Rate for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.8-22: PRN 1, Ionospheric Delay Rate Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.8-23: PRN 1, Ionospheric Delay 2$^{nd}$ Derivative for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds

Figure 5.4.8-24: PRN 1, Ionospheric Delay 2$^{nd}$ Derivative Error for Integral of Two Cascades of First-Order Markov Processes, 10/2 Channels, LEO, Signal Loss at $t = 4800$ Seconds, with Delay Model, from $t = 4200$ to $t = 5407$ Seconds
5.5 Other Considerations

5.5.1 Inter-Frequency Bias

As discussed in section 4.1, an inter-frequency bias falls directly into the ionospheric delay estimate. To demonstrate this, a bias equal to \(4^{*}(w_{L2} - 1)\) is added to the secondary (L2) pseudo-range as shown in equation 5.1-7. In terms of the L1 frequency, this is a 4 meter bias. As expected, the filter estimates the ionospheric delay plus 4 meters.

![Figure 5.5.1-1: PRN 6, Re-Entry Trajectory with L2 Bias of 4*(w_{L2} - 1)](image)

![Figure 5.5.1-2: PRN 6, Re-Entry Trajectory Error with L2 Bias of 4*(w_{L2} - 1)](image)

5.5.2 Second-Order Ionospheric Effects

Ignoring second-order ionospheric effects reportedly results in errors of around 1.6 cm with a TEC of 1e18 [6]. To test this, the filter was run with measurements that were not corrupted by noise using a 6/6 channel allocation with the re-entry trajectory. The errors without the second-order ionospheric delay terms are plotted in figure 5.5.2-1 and the errors with the second-order terms are plotted in figure 5.5.2-2. The difference between the two was just over 2 centimeters at its maximum, supporting the estimate given by Bassiri and Hajj. The "equal and opposite" assumption with respect to carrier phase advance and group delay is a very good assumption.
5.5.3 Channel Allocation

Until GPS receivers with more than 12 or 24 channels become common, users must decide how to allocate the channels they have. The effects of various channel allocation schemes for a 12 channel receiver are shown in table 5.5.3-1. The dwell time on the secondary frequency is 5 seconds. There appears to be a fairly significant decrease in performance from 6/6 to 8/4 and to 9/3 channel allocations. Beyond this, there appears to be little effect.
Table 5.5.3-1: Maximum Errors for Various Channel Allocation Schemes

5.5.4 Season/Time of Day/Elevation Angle

Many factors affect the amount of ionospheric group delay that will occur along a particular path. Ionospheric activity varies from year to year with the sun’s eleven-year cycle, season to season with varying amounts of incident radiant energy at a location and from day to night. All of these factors (along with many others) affect the TEC along the line-of-sight from a GPS satellite to a receiver. Maximum ionospheric activity during the day occurs at about 1400 LT and is at a minimum around midnight. GPS signals from satellites at low elevation angles can experience significantly larger delays than those near zenith. Figures 5.5.4-1 through 5.5.4-3 show the effects of varying times of day and positions relative to a receiver. Figure 5.5.4-1 shows the ionospheric delays of GPS signals from all satellites visible during re-entry at 1400 LT. Figure 5.5.4-2 shows the delays of GPS signals from satellites visible at 0800 LT. Figure 5.5.4-3 shows delays from satellites visible at 0200 LT. Recall that the IRI artificially caps the ionosphere at 1050 km and this re-entry trajectory starts above this altitude.
Figure 5.5.4-1: Ionospheric Group Delays for Various Satellites Visible during a Re-Entry Trajectory at 1400 LT

Figure 5.5.4-2: Ionospheric Group Delays for Various Satellites Visible during a Re-Entry Trajectory at 0800 LT

Figure 5.5.4-3: Ionospheric Group Delays for Various Satellites Visible during a Re-Entry Trajectory at 0200 LT
Chapter 6

Conclusions and Future Work

6.1 Conclusions

Each process model considered had its benefits and its drawbacks. Each model considered is evaluated in terms of three criteria. First and foremost, the performance of the filter in the simulations for the three trajectories described above is considered. The second consideration is complexity in terms of dynamics equations, process covariance calculation and parameter tuning. The third, flexibility or the capacity to be easily adapted for various trajectories and environmental conditions, sometimes involves a trade-off with the first criterion.

6.1.1 First-Order Markov Process

Performance

Having two dynamic states was an improvement over having one as evidenced by the first-order Markov process’s poor performance in the first simulation, although, the process was an average performer overall. As a Markov process tends to zero in the absence of measurements, it is best used for launch trajectories and not for those for which the ionospheric delay is not expected to go to zero (if a reference model isn’t used). The use of an ionospheric delay reference model improved its performance for all three trajectories. If the first-order Markov process is used with a poor delay model, however, it can be expected to produce poor estimates in the absence of measurements.

Complexity

The first-order Markov process was the simplest of the process models considered. With a single dynamic state, the calculation of all filter matrices was simple.
Flexibility

For the number of dynamic states, the process model could be easily tailored to various trajectories and environmental conditions. The time constant determines how quickly the estimate tends to zero or a reference model in the absence of measurements.

6.1.2 Second-Order Markov Process and Two Cascades of First-Order Markov Processes

Performance

The second-order Markov process and two cascades process model performed comparably in most of the simulations. The two cascades model performed marginally better than the second-order process model in most of the simulations. The second-order Markov process clearly performed better in a launch trajectory without a reference model for the parameters selected. It performed very well in the re-entry trajectory with a reference model, in the launch trajectory without a reference model and in the LEO trajectory with or without a reference model. The two cascades model performed well in all three trajectories with a reference model and in the LEO trajectory without a reference model. Neither model performed poorly during any of the simulations.

Complexity

The second-order and two cascades models were middle-of-the-road in terms of complexity. The process covariance matrix was considerably more complex than that of the first-order Markov process model, but was still not exceptionally difficult to calculate. The two cascades model introduced an additional time constant parameter which requires tuning, but it was not difficult to do so.

Flexibility

The two cascades model offers more flexibility than the second-order Markov process. With an extra time constant, it is easier to adjust the process model for use in a different situation. The extra flexibility did not provide as much of an advantage over the second-order Markov process as was expected, but it may provide a greater advantage with more situation-specific parameter tuning.

6.1.3 Third-Order Markov Process and Three Cascades of First-Order Markov Processes

Performance

The third-order Markov process was the second-worst process model in the re-entry trajectory with no signal loss (first simulation), but the spread from best to worst was small. It was the best
choice for the re-entry trajectory during simulation two, with signal loss and no reference model. It was average in all other simulations and trajectories. The three cascades model performed well in the launch trajectory with and without a reference model. Without a reference model, it was the best choice of process models. The three cascades model also performed well in the re-entry trajectory without a reference model. It was average in the LEO trajectory. In general, the three cascades model performed better than the third-order Markov process.

**Complexity**

Both models are extremely complex. The calculation of the process covariance matrices took considerable time. The three cascades model also has four parameters that require tuning. Doing so was difficult. It was time-consuming and counter-intuitive to select appropriate combinations of parameters. More often than not, the second-order Markov process performed better than the third-order and the two cascades model performed better than the three cascades model. In the situations in which the third-order process models perform better than their second-order counterparts, it is at the discretion of the user whether or not the added complexity is acceptable.

**Flexibility**

The three cascades model is more flexible than the third-order Markov process. It is the most flexible of the process models considered. Again, the cost of its flexibility is additional complexity.

6.1.4 Integral of First-Order Markov Process

**Performance**

The integral of a first-order Markov process performed well in general. It did not perform very well during launch without a reference model, was average during re-entry with no signal loss and was a top performer for the rest of the simulations. The integral of a first-order Markov process was the best process model in the LEO trajectory for all three simulations. It was also the top performer with a reference model (simulation three) in the re-entry and launch trajectories.

**Complexity**

The process model was not very complex. It was more complex than the first-order Markov process model, but less complex than the second-order and two cascades models. Calculating the process covariance matrix was fairly easy. Parameter tuning was simple.

**Flexibility**

The process model can be easily adjusted for various situations. The time constant determined how quickly or slowly the estimate tended to an ionospheric delay reference model in the absence of measurements.
6.1.5 Integral of Second-Order Markov Process and Integral of Two Cascades of First-Order Markov Processes

Neither integral of second-order process models performed well in the simulations performed. It is possible that with additional parameter tuning, the two models can perform better, but the fact that it was not simple to choose “good” parameters is another drawback.

Performance

The integral of a second-order Markov process and integral of two cascades were poor choices of process models, performing poorly in almost all simulations and trajectories. The only time they did not perform poorly was when there was no signal loss. For the LEO trajectory, the delay errors were in the hundreds of meters. The integral of two cascades performed worse than the integral of a second-order Markov process across the board.

Complexity

The process models were not as complex as the third-order Markov process and three cascades process model, but were more complex than any of the other process models. Parameter tuning was, again, easier than for the third-order models, but not as easy as for the second-order models.

Flexibility

The integral of two cascades offers more flexibility than the integral of a second-order Markov process, but appeared to offer no benefit.

6.2 Future Work

The Kalman filter parameters used in the simulations were selected to apply to many different trajectories and environmental conditions; “catch-all” parameters. Sections 5.2 through 5.4 showed how a filter that works well for one trajectory doesn’t necessarily work well for another, and section 5.5.4 clearly shows the range of ionospheric delays that can be experienced over a twelve hour period for the same trajectory. Using elevation-angle and time-of-day dependent parameters could improve filter performance. The option of using Full Information Maximum Likelihood via Optimal Filtering (FIMLOF) to continuously adjust the filter parameters was considered, but rejected as being too computationally intensive for a filter intended to function real-time.

The next step to verifying an ionospheric delay estimator would be to subject it to further testing. Among the options for further testing: using a commercial GPS simulator which includes hardware in the loop, estimating delays with data from a rooftop antenna and post-processing data from an actual flight.
When post-processing data from an actual flight, a more robust ionosphere reference model such as the IRI can be used to generate expected ionospheric delays for the anticipated date and trajectory before the flight. These delays can be used in place of the augmented Klobuchar model.
References


Appendix A: Demonstrating the Equivalence of the Third-Order Markov Differential Equation and State-Space Representation

With a little algebra, one can prove the equivalence of the differential equation and series of differential equations. For the third-order Markov process:

\[ \begin{align*}
\dot{x}_1 &= \beta(w - x_1) \\
\dot{x}_2 &= \beta(x_1 - x_2) \\
\dot{x}_3 &= \beta(x_2 - x_3)
\end{align*} \]  

(A-1 : A-3)

Taking the derivative of equation A-3, we have:

\[ \dot{x}_3 = \beta(\dot{x}_2 - \dot{x}_3) \]  

(A-4)

Substituting equations A-1 and A-2 into this equation:

\[ \begin{align*}
\dot{x}_3 &= \beta(\beta(x_1 - x_2) - \beta(x_2 - x_3)) \\
&= \beta^2 x_1 - 2\beta^2 x_2 + \beta^2 x_3
\end{align*} \]  

(A-5)

Taking the derivative of this equation:

\[ \ddot{x}_3 = \beta^2 \dot{x}_1 - 2\beta^2 \dot{x}_2 + \beta^2 \dot{x}_3 \]  

(A-6)

Substituting equations A-1 through A-3 into this expression yields:

\[ \begin{align*}
\ddot{x}_3 &= \beta^2 \beta(w - x_1) - 2\beta^2 \beta(x_1 - x_2) + \beta^2 \beta(x_2 - x_3) \\
&= \beta^3 w - 3\beta^2 x_1 + 3\beta^2 x_2 - \beta^2 x_3
\end{align*} \]  

(A-7)

To get an expression in terms of only \( x_3 \), equations A-2 and A-3 are solved for \( x_1 \) and \( x_2 \), respectively:
\[ x_1 = \frac{\dot{x}_2}{\beta} + x_2 \]  
(A-8 : A-9)

\[ x_2 = \frac{\dot{x}_3}{\beta} + x_3 \]

Substituting equation A-8 into equation A-7, we have an expression in terms of \( x_2 \) and \( x_3 \):

\[
\ddot{x}_3 = \beta^3 w - 3 \beta^3 \left( \frac{\dot{x}_2}{\beta} + x_2 \right) + 3 \beta^3 x_2 - \beta^3 x_3 \\
= \beta^3 w - 3 \beta^2 \ddot{x}_2 - \beta^3 x_3
\]
(A-10)

We then find take the derivative of A-9 and substitute:

\[
\ddot{x}_2 = \frac{\ddot{x}_3}{\beta} + \ddot{x}_3
\]
(A-11)

\[
\dddot{x}_3 = \beta^3 w - 3 \beta^2 \left( \frac{\ddot{x}_2}{\beta} + \ddot{x}_2 \right) - \beta^3 x_3 \\
= \beta^3 w - 3 \beta \dddot{x}_3 - 3 \beta^2 \dddot{x}_3 - \beta^3 x_3
\]
(A-12)

Rearranging terms, we have:

\[
\dddot{x}_3 + 3 \beta \dddot{x}_3 + 3 \beta^2 \dddot{x}_3 + \beta^3 x_3 = \beta^3 w
\]
(A-13)

For which \( w \) now has the units of meters. In table 4.3.1-1, \( w \) has the units of meters/sec\(^3\). This form of the third-order Markov process is much more intuitive and easier to code.
Appendix B: Deriving Equations for the Process Covariance Matrix for Various Orders of Markov Processes

With the integration interval dropped for simplicity, the process covariance matrix for a third-order Markov process is:

\[
Q = \begin{bmatrix}
N_{11} \int e^{-2\beta s} \\
N_{11} \beta \int \Delta t e^{-2\beta s} \\
2 N_{11} \beta^2 \int \Delta t^2 e^{-2\beta s} \\
N_{11} \beta^3 \int \Delta t^3 e^{-2\beta s} \\
2 N_{11} \beta^2 \int \Delta t^2 e^{-2\beta s} \\
N_{11} \beta^3 \int \Delta t^3 e^{-2\beta s} \\
N_{11} \beta^4 \int \Delta t^4 e^{-2\beta s}
\end{bmatrix}
\]  
(B-1)

The elements of the matrix are then going to have the form:

\[
Q_{ij} = N_{11} C_{ij} \int f(\Delta t, e^{-2\beta})
\]  
(B-2)

For which \(ij\) is the element \((i,j)\) of the matrix \(C\). Integration by parts is performed by:

\[
\int f(x)g(x)dx = f(x)\int g(x)dx - \int (f(x)\int g(x)dx)dx
\]  
(B-3)

Exponentials are easy to integrate. For our case:

\[
\int x^n e^{ax} dx = \frac{x^n}{a} e^{ax} - \int x^{n-1} \frac{1}{a} e^{ax} dx \\
= \frac{x^n}{a} e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx \\
= \frac{x^n}{a} e^{ax} - \frac{n}{a} \left( \frac{x^{n-1}}{a} e^{ax} - \frac{n-1}{a} \int x^{n-2} e^{ax} dx \right)
\]  
(B-4)

Which continues until the power of \(x\) goes to zero. In terms of summations, this integral becomes:
\[ \int x^n e^{ax} \, dx = \sum_{k=0}^{n} (-1)^k \frac{n!}{a^{k+1}(n-k)!} x^{n-k} e^{ax} \]

\[ = \Psi(n,a,x) \]  

(B-5)

This is the indefinite integral with respect to \(x\). For our integral from 0 to \(\Delta t\):

\[ \int_{0}^{\Delta t} e^{-2\beta \Delta t} \, d\Delta t = \Psi(n,-2\beta,\Delta t) - \Psi(n,-2\beta,0) \]

Examine the pattern up to the fifth-order Markov process, the \(C\) matrix can be seen to be:

\[
C = \begin{bmatrix}
1 & 1\beta & \frac{1}{2}\beta^2 & \frac{1}{6}\beta^3 & \frac{1}{24}\beta^4 & \ldots \\
1\beta & 1\beta^2 & \frac{1}{2}\beta^3 & \frac{1}{6}\beta^4 & \frac{1}{24}\beta^5 & \ldots \\
\frac{1}{2}\beta^2 & \frac{1}{2}\beta^3 & \frac{1}{6}\beta^4 & \frac{1}{24}\beta^5 & \frac{1}{24}\beta^6 & \ldots \\
\frac{1}{6}\beta^3 & \frac{1}{6}\beta^4 & \frac{1}{12}\beta^5 & \frac{1}{24}\beta^6 & \frac{1}{576}\beta^7 & \ldots \\
\frac{1}{24}\beta^4 & \frac{1}{24}\beta^5 & \frac{1}{48}\beta^6 & \frac{1}{144}\beta^7 & \frac{1}{576}\beta^8 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \]  

(B-7)

Each element of the process covariance matrix for any-order Markov process (verified up to and including the fifth-order) can be represented as follows:

\[
Q_y = N_{11} \ast \frac{\beta^{(i-1)+(j-1)}}{(i-1)!(j-1)!} [\Psi((i-1)+(j-1),-2\beta,\Delta t) - \Psi((i-1)+(j-1),-2\beta,0)] \]

(B-8)
Appendix C: Effects of Various Channel Allocation Schemes on Filter Performance

**Figure C-1:** PRN 6, Ionospheric Delay for 6/6 Channel Allocation, Re-Entry

**Figure C-2:** PRN 6, Ionospheric Delay Error for 6/6 Channel Allocation, Re-Entry

**Figure C-3:** PRN 6, Ionospheric Delay Rate for 6/6 Channel Allocation, Re-Entry

**Figure C-4:** PRN 6, Ionospheric Delay Rate Error for 6/6 Channel Allocation, Re-Entry
Figure C-5: PRN 6, Ionospheric Delay 2nd Derivative for 6/6 Channel Allocation, Re-Entry

Figure C-6: PRN 6, Ionospheric Delay 2nd Derivative Error for 6/6 Channel Allocation, Re-Entry

Figure C-7: PRN 6, Ionospheric Delay for 8/4 Channel Allocation, Re-Entry

Figure C-8: PRN 6, Ionospheric Delay Error for 8/4 Channel Allocation, Re-Entry
Figure C-9: PRN 6, Ionospheric Delay Rate for 8/4 Channel Allocation, Re-Entry

Figure C-10: PRN 6, Ionospheric Delay Rate Error for 8/4 Channel Allocation, Re-Entry

Figure C-11: PRN 6, Ionospheric Delay 2nd Derivative for 8/4 Channel Allocation, Re-Entry

Figure C-12: PRN 6, Ionospheric Delay 2nd Derivative Error for 8/4 Channel Allocation, Re-Entry
Figure C-13: PRN 6, Ionospheric Delay for 9/3 Channel Allocation, Re-Entry

Figure C-14: PRN 6, Ionospheric Delay Error for 9/3 Channel Allocation, Re-Entry

Figure C-15: PRN 6, Ionospheric Delay Rate for 9/3 Channel Allocation, Re-Entry

Figure C-16: PRN 6, Ionospheric Delay Rate Error for 9/3 Channel Allocation, Re-Entry
Figure C-17: PRN 6, Ionospheric Delay 2\textsuperscript{nd} Derivative for 9/3 Channel Allocation, Re-Entry

Figure C-18: PRN 6, Ionospheric Delay 2\textsuperscript{nd} Derivative Error for 9/3 Channel Allocation, Re-Entry

Figure C-19: PRN 6, Ionospheric Delay for 10/2 Channel Allocation, Re-Entry

Figure C-20: PRN 6, Ionospheric Delay Error for 10/2 Channel Allocation, Re-Entry
Figure C-21: PRN 6, Ionospheric Delay Rate for 10/2 Channel Allocation, Re-Entry

Figure C-22: PRN 6, Ionospheric Delay Rate Error for 10/2 Channel Allocation, Re-Entry

Figure C-23: PRN 6, Ionospheric Delay 2nd Derivative for 10/2 Channel Allocation, Re-Entry

Figure C-24: PRN 6, Ionospheric Delay 2nd Derivative Error for 10/2 Channel Allocation, Re-Entry
Figure C-25: PRN 6, Ionospheric Delay for 11/1 Channel Allocation, Re-Entry

Figure C-26: PRN 6, Ionospheric Delay Error for 11/1 Channel Allocation, Re-Entry

Figure C-27: PRN 6, Ionospheric Delay Rate for 11/1 Channel Allocation, Re-Entry

Figure C-28: PRN 6, Ionospheric Delay Rate Error for 11/1 Channel Allocation, Re-Entry
Figure C-29: PRN 6, Ionospheric Delay 2nd Derivative for 11/1 Channel Allocation, Re-Entry

Figure C-30: PRN 6, Ionospheric Delay 2nd Derivative Error for 11/1 Channel Allocation, Re-Entry