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Kaushik Basu[1] and Tapan Mitra[2]

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Abstract

It has been known that, in aggregating infinite utility streams, there does not exist any social welfare function, which satisfies the axioms of Pareto, inter-generational equity and continuity. We show that the impossibility result persists even without imposing the continuity axiom. Hence, the problem of accommodating inter-generational equity is more obstinate than previously supposed. The paper goes on to explore the scope for obtaining possibility results by weakening the Pareto axiom and placing restrictions on the domain of utilities.

Journal of Economic Literature Classification Numbers: D00, D70, D90.

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1 Introduction

The subject of inter-generational equity in the context of aggregating infinite utility streams has been of enduring interest to economists. Ramsey (1928) had maintained that discounting one generation's utility or income vis-a-vis another's to be "ethically indefensible", and something that "arises merely from the weakness of the imagination". Since it is generally agreed that any such process of aggregation should satisfy the Pareto principle, Ramsey's observation raises the question of whether one can consistently evaluate infinite utility streams while respecting intergenerational equity, and at least some form of the Pareto axiom.

In an important contribution to this literature, Svensson (1980) established the general possibility result (for a social welfare relation (SWR)) that one can find an ordering, that satisfies the axioms of Pareto and inter-generational equity. It is worth noting here that he obtains the (complete) ordering by non-constructive methods; specifically, he defines a partial order satisfying the two axioms, and then completes the order by appealing to Szpilrajn's lemma.

Svensson's paper builds on the earlier seminal contribution to this literature by Diamond (1965). In this paper, Diamond presents the celebrated theorem[3] that there does not exist any social welfare function (SWF) (that is, a function that aggregates an infinite stream into a real number) satisfying three axioms: Pareto, intergenerational equity and continuity (in the sup metric).

Neither contribution addresses the following question: does there exist a social welfare function satisfying intergenerational equity, and the Pareto axiom? Since continuity of the SWF in Diamond's exercise is a technical axiom (in contrast to the other two axioms), we think that this

[3]Diamond attributes the result to Yaari.

is an issue worth investigating.[4] The principal task of this paper is to provide an answer to this question. The main result that motivates this paper is the finding that the continuity axiom is not needed for the Diamond-Yaari impossibility theorem.

If we denote by Y (a subset of the reals) the set of admissible utility levels of each generation, and by X the set of infinite streams of these utility levels, an SWF is a function from X to the reals. It turns out that the answer to the question posed in the previous paragraph depends on the nature of the utility space (that is, the set, denoted by Y), and on the form of the Pareto axiom. As a consequence, our results can be classified conveniently into four parts.

First, if one adopts the standard form of the Pareto Axiom, then regardless of the specific nature of the utility space, one obtains the general impossibility theorem mentioned above: there is no SWF which satisfies the Pareto and equity axioms. In other word, any Paretian SWF is necessarily inequitable. In particular, this means that the complete orderings, satisfying the Pareto and equity axioms, obtained by Svensson, do not have real-valued representations. Indeed, Svensson (1980) goes on to show that there are complete orderings, which satisfy the Pareto and equity axioms, and in addition are continuous (in a topology weaker than the discrete topology). Clearly, the continuity of the ordering (obtained by him) is unable to overcome the representation problem. The proof of our result has an affinity with the demonstration that lexicographic preferences do not have a real-valued representation: one "runs out of real numbers" in a similar way.

Suppose we do want to use a (real-valued) SWF. How does one get around this problem of

[4] In this connection, we may note that the line of research, initiated by Koopmans (1960), and continued by Koopmans, Diamond and Williamson (1964) and Diamond (1965) establishes that Paretian social welfare relations, continuous in suitably defined metrics, necessarily exhibit "impatience" in the sense that the current generation receives more favorable treatment than future generations. Burness (1973) explores the impatience implications of continuously differentiable Paretian social welfare functions.

respecting intergenerational equity? One way to do this is to use weaker forms of the Pareto axiom. If we use the weak Pareto axiom, and restrict the utility space to be a subset of the set of non-negative integers, then one can establish a possibility result: there is a social welfare function satisfying the equity and weak Pareto axioms. Specifically, if the utility space consists of just two elements (corresponding to, say, the "good" and "bad" states, and represented by 1 and 0 respectively, so that $Y = \{0, 1\}$), the above two results would indicate that there exists an SWF satisfying equity and the weak Pareto axiom, but there is no SWF satisfying equity and the standard Pareto axiom.

Encouraged by our second result, one might ask whether it is always possible to obtain a SWF satisfying the equity and weak Pareto axioms (regardless of the nature of the utility space, Y). Our third result indicates that the answer is in the negative. If Y is the closed interval $[0, 1]$ (the utility space chosen by both Diamond (1965) and Svensson (1980) in establishing their results, mentioned above), one can show that there is no SWF satisfying the equity and weak Pareto axioms. This result is a generalization of Diamond's since it shows that one does not need to impose the continuity axiom to get his impossibility result. One also does not need the full strength of the Pareto axiom; weak Pareto (and equity) suffices. [Indeed, as we demonstrate, one does not even need the weak Pareto axiom; a weaker form of it, which we call the dominance axiom, suffices]. Of course, continuity is still at the heart of the demonstration of this impossibility result. The interesting observation is that the extent of continuity needed for the Diamond-Yaari technique to work is already implied by the weak Pareto axiom in the $[0, 1]$ utility space case, making a separate continuity axiom superfluous. [Specifically, the underlying argument is that if an SWF were to exist, the weak Pareto axiom would imply that it would have some monotonicity properties, and monotone functions on $[0, 1]$ are continuous almost everywhere].

Finally, as we note in our fourth result, if we weaken the Pareto axiom sufficiently to what we call "weak dominance", a general possibility result can be obtained: there exists an SWF satisfying the equity and weak dominance axioms, regardless of the nature of the utility space, Y. It is worth remarking that, when the utility space, Y , is $[0, 1]$, any SWF satisfying the equity and weak dominance axioms, obtained by applying our last result, necessarily violates the weak Pareto axiom (given our third result).

2 Paretian Social Welfare Functions are Inequitable

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N}$ the set of positive integers, and $\mathbb{M}$ the set of non-negative integers. Suppose $Y \subset \mathbb{R}$ is the set of all possible utilities that any generation can achieve. Then $X = Y^{\mathbb{N}}$ is the set of all possible utility streams. If $\{x_t\} \epsilon X$, then $\{x_t\} = (x_1, x_2, ...)$, where, for all $t \epsilon \mathbb{N}$, $x_t \epsilon Y$ represents the amount of utility that the t^{th} generation (that is, the generation of period t) earns. For all $y, z \epsilon X$, we write $y \geq z$ if $y_i \geq z_i$, for all $i \epsilon \mathbb{N}$; we write $y > z$ if $y \geq z$ and $y \neq z$; and we write $y >> z$, if $y_i > z_i$, for all $i \epsilon \mathbb{N}$.

If Y has only one element, then X is a singleton, and the problem of ranking or evaluating infinite utility streams is trivial. Thus, without further mention, the set Y will always be assumed to haveat least two distinct elements.

A social welfare function (SWF) is a mapping

$$W : X \rightarrow \mathbb{R}$$

Consider now some of the axioms that we may want the SWF to satisfy. The first axiom is the standard Pareto condition.

Pareto Axiom: For all $x, y \in X$, if $x > y$, then $W(x) > W(y)$.

The next axiom is the one that captures the notion of 'inter-generational equity'. We shall call it the 'anonymity axiom'.[5] It is equivalent to the notion of 'finite equitableness' (Svensson, 1980) or 'finite anonymity' (Basu, 1994).[6]

Anonymity Axiom: For all $x, y \in X$, if there exist $i, j \in \mathbb{N}$ such that $x_i = y_j$ and $x_j = y_i$, and for $k \in \mathbb{N} - \{i, j\}$, $x_k = y_k$, then $W(x) = W(y)$.

The principal result of this section is that there is no social welfare function which satisfies both axioms.

Theorem 1 *There does not exist any SWF satisfying the Pareto and Anonymity Axioms.*

Proof. Assume, on the contrary, that there is a social welfare function, $W : X \longrightarrow \mathbb{R}$, which satisfies the Pareto and Anonymity Axioms. Since $Y \subset \mathbb{R}$ contains at least two distinct elements, there exist real numbers y^0 and y^1 in Y, with $y^1 > y^0$. Without loss of generality, and to ease the writing, we may suppose that $y^1 = 1$ and $y^0 = 0$.

Let Z denote the open interval $(0, 1)$, and let r_1, r_2, r_3, ... be an enumeriation of the rational numbers in Z. Let p be an arbitrary number in Z. We define the set:

$$E(p) = \{n \in \mathbb{N} : r_n < p\}$$

[5] In informal discussions throughout the paper, the terms "equity" and "anonymity" are used interchangeably.

[6] The Anonymity Axiom figures prominently in the social choice theory literature, where it is stated as follows: the social ordering is invariant to the information regarding individual orderings as to who holds which preference ordering. Thus, interchanging individual preference profiles does not change the social preference profile. See May (1952) and Sen (1977).

Clearly, $E(p)$ is an infinite set. We now define a sequence $a(p) = (a(p)_1, a(p)_2, ...)$ as follows:

$$a(p)_n = \begin{cases} 1 & if\ n \in E(p) \\ 0 & otherwise \end{cases}$$

Note that the sequence will have an infinite number of $1's$ and an infinite number of $0's$. Define the sequence $b(p) = (b(p)_1, b(p)_2, ...)$ as the same as the sequence $a(p)$ except that the first 0 appearing in the $a(p)$sequence is replaced by a 1. By the Pareto axiom, we have $W(b(p)) > W(a(p))$. We denote the closed interval $[W(a(p)), W(b(p))]$ by $I(p)$.

Now, let q be an arbitrary real number in Z, with $q > p$. Clearly, we must have $E(p) \subset E(q)$, for if $n \in E(p)$, then $r_n < p$, and since $p < q$, we must have $r_n < q$, so that $n \in E(q)$. Further, there are an infinite number of rational numbers in the interval (p, q). Thus, comparing the sequence $a(p)$ with the sequence $a(q)$, we note that:

$$(i) \qquad if\, n \in \mathbb{N},\ and\ \ a(p)_n = 1,\ then\ \ a(q)_n = 1 \tag{1}$$

and:

$$(ii) \qquad there\ are\ an\ infinite\ number\ of\ n\ \in\ \mathbb{N}$$
$$for\ which\ a(p)_n = 0\ and\ a(q)_n = 1 \tag{2}$$

We now proceed to compare the sequence $a(q)$ with the sequence $b(p)$. Let $m \in \mathbb{N}$ be the index for which the seqence $a(p)$ differs from the sequence $b(p)$; that is, $a(p)_m = 0$, and $b(p)_m = 1$. There are two cases to consider: (A) $a(q)_m = 1$, (B) $a(q)_m = 0$. In case (A), we clearly have

$a(q) > b(p)$, and so:

$$W(a(q)) > W(b(p)) \tag{3}$$

In case (B), we proceed as follows. Let M be the smallest integer for which $a(p)_M = 0$ while $a(q)_M = 1$. By observation (2) above, such an M exists. Also, clearly $M \neq m$, for $a(q)_M = 1$ while $a(q)_m = 0$. Since $b(p)$ differs from $a(p)$ in only the index m, we have $b(p)_M = 0$. Now, define $b'(p)$ as follows: $b'(p)_m = 0$, $b'(p)_M = 1$, $b'(p)_n = b(p)_n$ for all $n \in \mathbb{N}$ such that $n \neq m, n \neq M$. Since $b(p)_m = 1$, and $b(p)_M = 0$, the Anonymity axiom implies that:

$$W(b'(p)) = W(b(p)) \tag{4}$$

Comparing $b'(p)$ with $a(q)$, we note that $b'(p)_m = 0 = a(q)_m$, $b'(p)_M = 1 = a(q)_M$, and for all $n \in \mathbb{N}$ such that $n \neq m, n \neq M$, we have $b'(p)_n = b(p)_n = a(p)_n \leq a(q)_n$ by observation (1). By observation (2), we must therefore have $a(q) > b'(p)$, so that by the Pareto Axiom:

$$W(a(q)) > W(b'(p)) \tag{5}$$

Combining (15) and (16), we get:

$$W(a(q)) > W(b(p)) \tag{6}$$

Thus, in both cases (A) and (B), we have $W(a(q)) > W(b(p))$. This means that the interval $I(q) = [W(a(q)), W(b(q))]$ is disjoint from the interval $[W(a(p)), W(b(p))]$, the latter interval lying entirely to the left of the former interval on the real line.

To summarize, we have shown that the intervals $I(p)$ associated with distinct values of $p \in$

(0, 1) are non-overlapping. But, this means that to each real number $p \in (0,1)$, we can associate a distinct rational number (in the interval $I(p)$), contradicting the countability of the set of rational numbers. ■

Remark:

The construction (used in the above proof) of an uncountable family of distinct nested sets $E(p)$, with each set containing an infinite number of positive integers, can be found in Sierpinski (1965, p. 82).

Let us now suppose that we abandon the search for an SWF and instead look for a social welfare relation (SWR), We then have the result due to Svensson (1980) that there is an SWR which satisfies the (appropriate relational versions of the) Pareto and Anonymity axioms. For reasons of completeness we briefly review Svensson's result. We do this because, the use of a variant of Szpilrajn's Theorem (due to Suzumura, 1983) allows us to give a particularly easy proof of it. Also, Svensson (1980) restricts his exercise to the case where Y is the closed interval $[0, 1]$; we state the version of his result which applies to any utility space Y.His proof, as well as ours, applies to this more general setting.

Formally, an SWR is a binary relation, $\succsim$, on X, which is complete and transitive. We use $\succ$ and $\sim$ to denote, respectively, the asymmetric and symmetric parts of $\succsim$. The properties of Pareto and Anonymity for an SWR are easy to define. We shall call these axioms $\succsim$-Pareto and $\succsim$-Anonymity to distinguish them from the axioms applied to an SWF.

$\succsim$-Pareto Axiom: For all x, $y \in X$, $x > y$ implies $x \succ y$.

$\succsim$-Anonymity Axiom: For all $x, y \in X$, if there exist $i, j \in \mathbb{N}$, such that $x_i = y_j$ and $x_j = y_i$ and for $k \in \mathbb{N} - \{i, j\}$, $x_k = y_k$, then $x \sim y$.

First, let us give a statement of Suzumura's theorem. Let Ω be a set of alternatives. If R is a binary relation on Ω and R^* an ordering on Ω, we shall say that R^* is an *ordering extension* of R if, for all $x, y \in \Omega$, $x \, R \, y$ implies $x \, R^* \, y$. We say that R is *consistent* if, for all $t \in \mathbb{N}$, and for all $x^1, x^2, ..., x^t \in \Omega$, $[x^1 R x^2$ and not $x^2 R x^1$, and for all $k \in \{2, 3, ..., t-1\}$, $x^k R x^{k+1}]$ implies not $x^t R x^1$.

Lemma 1 *Szpilrajn's Corollary [Suzumura, 1983, Theorem A(5)]: A binary relation R on Ω has an ordering extension if and only if it is consistent.*

In contrast to Theorem 1, we now have:

Theorem 2 (Svensson, 1980) *: There exists a social welfare relation satisfying the $\succsim$-Pareto and $\succsim$-Anonymity Axioms.*

Proof. Define two binary relations, P and I, on X, as follows. For all $x, y \in X$, $x > y$ implies xPy. And if there exists i, j such that $x_i = y_j$ and $x_j = y_i$ and $x_k = y_k$, for all $k \neq i, j$, then xIy. Now define the binary relation R as follows: $xRy \leftrightarrow xPy$ or xIy.

It is easy to verify that R is consistent. Hence, by Szpilrajn's Corollary, it has an ordering extension $\succsim$. Clearly $\succsim$ satisfies the $\succsim$-Pareto Axiom and the $\succsim$-Anonymity Axiom. ■

Remarks:

(i)Theorem 2 shows that there is no inherent conflict between the concept of intergenerational equity and the Pareto principle. Theorem 1 shows that a conflict arises when we try to obtain an evaluation of utility streams in terms of real numbers, while respecting these two properties.

(ii) A simple example of our set-up is one where the utility space, $Y = \{0, 1\}$.One might interpret this as follows: there are precisely two states in which each generation might find itself, a good state and a bad state. The utility obtained by each generation is 1 in the good state, and 0 in the bad state. Theorem 1 tells us that even in this simple set-up, there is no SWF which respects Anonymity and the Pareto Axiom.

3 Relaxing Pareto

If we want to work with a social welfare function *and* respect inter-generational equity or anonymity, what possibilities do we have? In the light of our Theorem 1, what we explore in this section is to relax the Pareto axiom.

It is arguable that for certain philosophical and even policy purposes we do not need the full-brunt of the Pareto condition (even if we are committed Paretians) simply because all the possibilities that are technically allowed in our specification of the domain, may not arise under any eventuality. Indeed for certain ethical discourses involving the comparison of the moral worth of individual actions and universalizable rules (see Basu, 1994) it may be enough to be armed with the following weak Paretianism.[7]

Weak Pareto Axiom: For all $x, y \in X$, if there exists $j \in \mathbb{N}$ such that $x_j > y_j$, and, for all $k \neq j$, $x_k = y_k$, then $W(x) > W(y)$. For all x, $y \in X$, if $x >> y$, then $W(x) > W(y)$.

Note that if we recognize that human perception or cognition is not endlessly fine, so that sufficiently small changes in well-being go unperceived, it seems reasonable to suppose that the set of feasible utilities will be a discrete set.[8] The same is true if the benefits are measured

[7]Our Weak Pareo Axiom is somewhat different from the Weak Pareto Axiom used in social choice theory, where typically it is stated as follows: if every individual in a society is better off, then society is better off . See, for example, Sen(1977). Sometimes, though, in the social choice theory literature, this axiom is called the Pareto Axiom, and our Pareto Axiom is called the Strong Pareto Axiom. See, for example, Arrow (1963) and Sen (1969).

[8]The limits on human perception have been used in a different context by Armstrong(1939) to argue that it is implausible to suppose that indifference is a transitive relation. For a discussion of this issue in individual choice

in money and there is a well-defined smallest unit, as is true for all currencies (Segerberg and Akademi, 1976). Thus, it seems worthwhile to explore whether with $Y \subset \mathbb{M}$ (which captures this very reasonable possibility), there is a social welfare function (on X) respecting Anonymity and the Weak Pareto Axioms.[9] Our next theorem provides an interesting possibility result.

Theorem 3 *Assume* $Y \subset \mathbb{M}$ *. There exists an SWF satisfying the Weak Pareto and Anonymity Axioms.*

Proof. For each $x \in X$, let $E(x) = \{y \in X :$ there is some $N \in \mathbb{N}$, such that $y_k = x_k$ for all $k \in \mathbb{N}$, which are $\geq N\}$. Let $\Im$ be the collection $\{E : E = E(x)$ for some $x \in X\}$. Then $\Im$ is a partition of X. That is, if E and F belong to $\Im$, then either $E = F$, or E is disjoint from F; further, $\cup_{E \in \Im} E = X$.

Define a function, $f : X \to \mathbb{M}$ as follows. Given any $x \in X$, let $f(x) = \min\{x_1, x_2, ...\}$. Since $x_i \in \mathbb{M}$ for all $i \in \mathbb{N}$, the set $\{x_1, x_2, ...\}$ is a non-empty subset of the set of non-negative integers and therefore has a smallest element [Munkres, 1975, p. 32]. Thus, f is well-defined.

By the axiom of choice, there is a function, $g : \Im \to X$, such that $g(E) \in E$ for each $E \in \Im$.

Given any $x \in X$, we can denote for each $N \geq 1$, $(x_1, ..., x_N)$ by $x(N)$, and $(x_1 + ... + x_N)$ by $I(x(N))$. Now, given any x, y in $E \in \Im$, define $h(x, y) = \lim_{N \to \infty}[I(x(N)) - I(y(N))]$. Notice that h is well-defined, since given any x, y in $E \in \Im$, there is some $M \in \mathbb{N}$, such that $[\,I(x(N)) - I(y(N))]$ is a constant for all $N \geq M$. Now, given any x, y in $E \in \Im$, define $H(x, y) = 0.5[h(x, y)/[1 + |h(x, y)|]]$. Then $H(x, y) \in (-0.5, 0.5)$.

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theory, see Majumdar (1962).

[9] While our choice of Y as a subset of the set of non-negative integers is motivated by the imprecision of human perception, the mathematical technique used to obtain our possibility result applies also to the case where $Y = \{(1/n) : n \in \mathbb{N}\}$,where clearly human perception has to be considered to be sufficiently refined.

We now define $W : X \to \mathbb{R}$ as follows. Given any $x \in X$, we associate with it its equivalence class, $E(x)$. Then, using the function g, we get $g(E(x)) \in E(x)$. Next, using the functions, h and H, we obtain $h(x, g(E(x)))$ and $H(x, g(E(x)))$. Now, define $W(x) = f(x) + H(x, g(E(x)))$.

The Anonymity Axiom can be verified as follows. If x, y are in X, and there exist i, j in $\mathbb{N}$, such that $x_i = y_j$ and $x_j = y_i$, while $x_k = y_k$ for all $k \in \mathbb{N}$, such that $k \neq i, j$, then $E(x) = E(y)$. Furthermore, denoting this common set by E, we see that $h(x, g(E)) = h(y, g(E))$, and so $H(x, g(E)) = H(y, g(E))$. Further, the set $\{x_1, x_2, ...\}$ is the same as the set $\{y_1, y_2, ...\}$, so that $f(x) = f(y)$. Thus, we obtain: $W(x) = W(y)$.

The Weak Pareto Axiom can be verified as follows. If x, y are in X, and there exists $i \in \mathbb{N}$, such that $x_i > y_i$, while $x_k = y_k$ for all $k \in \mathbb{N}$, such that $k \neq i$, then $E(x) = E(y)$. Furthermore, denoting the common set by E, we see that $h(x, g(E)) > h(y, g(E))$. This implies $H(x, g(E)) > H(y, g(E))$. Further, the smallest element of the set $\{x_1, x_2, ...\}$ is at least as large as the smallest element of the set $\{y_1, y_2, ...\}$, so that we have $f(x) \geq f(y)$. Thus, we obtain the desired inequality: $W(x) > W(y)$.

If $x, y \in X$, and $x >> y$, then $E(x) \neq E(y)$. Thus, we will not be able to compare $H(x, g(E(X)))$ with $H(y, g(E(y)))$. However, we do know that $H(x, g(E(x))) > -0.5$, and $H(y, g(E(y))) < 0.5$. Further, since $x >> y$, we have $f(x) \geq f(y) + 1$. Thus, we obtain: $W(x) = f(x) + H(x, g(E)) > f(y) + 1 - 0.5 > f(y) + H(y, g(E)) = W(y)$. ∎

Remarks:

(i) The important point to note is that our Weak Pareto Axiom demands that the SWF be positively sensitive to an increase in utility of a single generation, the utilities of other generations being unchanged (and therefore that it be positively sensitive to increases in utilities of any finite number of generations, the utilities of other generations being unchanged), and also that the SWF be positively sensitive to an increase in utilities of all generations. However, it need not be positively sensitive to an increase in utilities of an infinite number of generations, when the utilities of a (non-empty) set of generations is unchanged. This is the principal difference of our Weak Pareto Axiom from our Pareto Axiom.

(ii) Consider the simple set-up, introduced in the previous section, where $Y = \{0, 1\}$.Now, Theorem 1 implies that there is no SWF respecting the Pareto and Anonymity Axioms. And, Theorem 3 implies that there is an SWF satisfying the Weak Pareto and Anonymity Axioms. It follows that for any social welfare function, W, so obtained, it must be the case that there exist alternatives $x, y \in X$ such that $x > y$,but $W(x) < W(y)$.

4 Generalizing the Diamond-Yaari Result

The possibility result of the previous section is reason for optimism, and we might ask whether it can be extended to all utility spaces, Y. We show in this section that such an extension is not possible by demonstrating that when Y is the closed interval $[0, 1]$,then there is no SWF satisfying the Anonymity and Weak Pareto axioms. This generalizes the result of Diamond (1965) in two respects.

Diamond had shown that (when the utility space Y is $[0, 1]$), there is no SWF satisfying Anonymity, Pareto, and a continuity axiom (in the sup metric on X). Our result shows that the continuity axiom is redundant for this impossibility result. In addition, it shows that the full power of the Pareto axiom is not needed for the impossibility result; the weak Pareto axiom suffices for this purpose.

Theorem 4 *Assume* $Y = [0, 1]$. *There does not exist any SWF satisfying the Weak Pareto and the Anonymity Axioms.*

Instead of proving this theorem directly, we shall establish a more powerful theorem, of which Theorem 4 is an obvious corollary. To prove this stronger result, let us next define a very weak form of the Pareto criterion, which we call the Dominance Axiom.

Dominance Axiom: For all $x, y \in X$, if there exists $j \in \mathbb{N}$ such that $x_j > y_j$, and, for all $k \neq j$, $x_k = y_k$, then $W(x) > W(y)$. For all $x, y \in X$, if $x >> y$, then $W(x) \geq W(y)$.

Note that the last inequality in the statement of this axiom is a weak inequality, unlike in the definition of the Weak Pareto Axiom. Hence, Weak Pareto is stronger than Dominance. It is not as if we wish to recommend the use of such a weak form of the Pareto condition, but since we are going to prove an impossibility result, clearly it is better to use as weak an axiom as one can. Further, our *proof* indicates that it is precisely the Dominance axiom that is needed to obtain the impossibility result.

Theorem 5 *Assume* $Y = [0, 1]$. *There does not exist any SWF satisfying the Dominance Axiom and the Anonymity Axiom.*

Proof. To establish the theorem, assume $Y = [0, 1]$ and that there exists a social welfare function, $W : X \to \mathbb{R}$, which satisfies the Dominance and Anonymity Axioms.

Denote the vector $(1, 1, 1, ...)$ in X by e. Define the sequence $\bar{u}$ in X as follows:

$$\bar{u} = (1, 1, 0, 1/2, 1, 0, 1/4, 2/4, 3/4, 1, ...) \tag{7}$$

This sequence is best understood as sequence u, defined below, with the first term changed from

0 to 1. For $s \epsilon I \equiv (-0.5, 0.5)$, define:

$$\bar{x}(s) = 0.5\bar{u} + 0.25(1+s)e \quad (8)$$

Then $(1/8)e \leq \bar{x}(s) \leq (7/8)e$, and so $\bar{x}(s) \epsilon X$ for each $s \epsilon I$.

Define the function, $f : I \to \mathbb{R}$ by:

$$f(s) = W(\bar{x}(s))$$

By the Dominance Axiom, f is monotonic non-decreasing in s on I. Thus f has only a countable number of points of discontinuity in I. Let $a \epsilon I$ be a point of continuity of the function f.

Define the sequence u in X as follows:

$$u = (0, 1, 0, 1/2, 1, 0, 1/4, 2/4, 3/4, 1, ...) \quad (9)$$

and then define:

$$x(a) = 0.5u + 0.25(1+a)e \quad (10)$$

Clearly, $x(a) \epsilon X$, and $\bar{x}_1(a) > x_1(a)$, while $\bar{x}_k(a) = x_k(a)$ for each $k \epsilon \mathbb{N}$, with $k \neq 1$. Thus, by the Dominance Axiom, we have:

$$W(\bar{x}(a)) > W(x(a))$$

We denote $[W(\bar{x}(a)) - W(x(a))]$ by θ; then $\theta > 0$.

Denote $\max(0.5 - a,\, 0.5 + a)$ by $\triangle$; then, $\triangle > 0$. Since f is continuous at a, given the θ

defined above, there exists $\delta \epsilon (0, \triangle)$, such that:

$$0 < |s - a| < \delta \quad implies \quad |f(s) - f(a)| < \theta \tag{11}$$

Note that for $0 < |s - a| < \delta$, we always have $s \epsilon I$.

We define (following Diamond), for each $k \epsilon \mathbb{N}$, a sequence u^k by starting with the sequence u, interchanging the initial 0 with the $(k + 1)$st 1 appearing in u, and then interchanging the sequence:

$$(1/2^k, 2/2^k, ..., (2^k - 1)/2^k, 0) \quad with \quad (0, 1/2^k, 2/2^k, ..., (2^k - 1)/2^k) \tag{12}$$

so that:

$$u^k = (1, 1, 0, 1/2, 1, ..., 0, 0, 1/2^k, 2/2^k, ..., (2^k - 1)/2^k, 0, ...) \tag{13}$$

Now, for each $k \epsilon \mathbb{N}$, we use u^k to define $x^k(a)$ as follows:

$$x^k(a) = 0.5u^k + 0.25(1 + a)e \tag{14}$$

Clearly, $x^k(a) \epsilon X$ for each $k \epsilon \mathbb{N}$. Comparing the expressions for $x\,(a)$ and $x^k(a)$ in (10) and (14) respectively, and using the expressions for u and u^k in (9) and (13) respectively, we see that the Anonymity Axiom yields:

$$W(x^k(a)) = W(x(a)) \qquad for\ all\ k \epsilon \mathbb{N} \tag{15}$$

Choose $K \in \mathbb{N}$ with $K \geq 2$ such that $(1/2^{K-2}) < \delta$, and define $S = (a - (1/2^{K-2}))$. We note that $0 < (a - S) < \delta$, and so $S \in I$, and:

$$W(\bar{x}(S)) = f(S) > f(a) - \theta = W(\bar{x}(a)) - \theta \tag{16}$$

We now compare the welfare levels associated with $x^K(a)$ and $\bar{x}(S)$ as follows. Notice that:

$$\begin{aligned}
x^K(a) &= 0.5u^K + 0.25(1+a)e \\
&= 0.5\bar{u} + 0.25(1+a)e - 0.5(\bar{u} - u^K) \\
&= \bar{x}(a) - 0.5(\bar{u} - u^K) \\
&\geq \bar{x}(a) - 0.5(1/2^K)e \\
&= 0.5\bar{u} + 0.25(1+a)e - 0.5(1/2^K)e \\
&= 0.5\bar{u} + 0.25(1 + a - (1/2^{K-1}))e \\
&= 0.5\bar{u} + 0.25(1 + a - (1/2^{K-2}))e + 0.25(1/2^{K-1})e \\
&>> 0.5\bar{u} + 0.25(1+S)e \\
&= \bar{x}(S)
\end{aligned}$$

Thus, by the Dominance Axiom, we have:

$$W(x^K(a)) \geq W(\bar{x}(S)) \tag{17}$$

Using (15), (16) and (17), and the definition of θ, we obtain:

$$W(\bar{x}(a)) - \theta = W(x(a)) = W(x^K(a)) \geq W(\bar{x}(S)) > W(\bar{x}(a)) - \theta$$

a contradiction which establishes our result. ■

The intuitive idea behind our proof is straight-forward. Diamond had used the axioms of Pareto and Anonymity in conjunction with another axiom concerning the continuity of W to precipitate an impossibility result. Now, as soon as we impose the Weak Pareto Axiom (or the Dominance Axiom), by a standard result in analysis, we know that W cannot be discontinuous everywhere. In other words, W must have points of continuity. And we show that that is enough to give us the impossibility result. In other words, there is no need to *assume* continuity. The minimal amount of continuity that is needed to create the impossibility arises naturally as a consequence of the Dominance Axiom.

5 Weak Dominance

What we have not yet found is an existence result where $Y = [0, 1]$ and some version of the Pareto criterion is satisfied. Theorem 5 indicates that we need to weaken the Pareto criterion further to get such a result. In fact what we need is a condition in which only the first part of the Dominance axiom is asserted. If each policy question that we consider affects the utilities of only a finite number of generations, then this "Weak Dominance" condition suffices in capturing the idea of Pareto.

Weak Dominance Axiom: For all $x, y \in X$, if for some $j \in \mathbb{N}$, $x_j > y_j$, while, for all $k \neq j$,

$x_k = y_k$, then $W(x) > W(y)$.

Theorem 6 *There exists an SWF satisfying the Weak Dominance and Anonymity Axioms.*

Proof. For each $x \epsilon X$, let $E(x) = \{y \epsilon X$: there is some $N \epsilon \mathbb{N}$, such that $y_k = x_k$ for all $k \epsilon \mathbb{N}$, which are $\geq N\}$. Let $\Im$ be the collection $\{E : E = E(x)$ for some $x \epsilon X\}$. Then, $\Im$ is a partition of X. By the axiom of choice, there is a function, $g : \Im \rightarrow X$, such that $g(E) \epsilon E$, for each $E \epsilon \Im$.

Given any x, y in $E \epsilon \Im$, define $h(x, y) = \lim_{N \rightarrow \infty}[I(x(N)) - I(y(N))]$.We now define $W : X \rightarrow \mathbb{R}$ as follows. Given any $x \epsilon X$, we associate with it its equivalence class, $E(x)$. Then, using g, we get $g(E(x)) \epsilon E(x)$, and, using h, we obtain $h(x, g(E(x)))$. Now, define $W(x) = h(x, g(E(x)))$. The Anonymity Axiom and the Weak Dominance Axiom are easily verified. ■

6 Conclusion

We summarize our results, marking out the terrain of what is possible and what is not possible, in a table:

	[0,1]	N
Pareto		Theorem 1
Weak Pareto	Theorem 4	Theorem 3
Dominance	Theorem 5	
Weak Dominance	Theorem 6	

TABLE 1

It is assumed in Table 1 that the Anonymity Axiom is satisfied. The rows represent a weakening sequence of Pareto-type axioms. The columns represent the domain restrictions, to wit, what Y is equal to. The shaded area represents the zone of impossibility (where no SWF exists) and the unshaded area the zone of possibility.

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