working paper department of economics

ASSORTATIVE MATCHING AND SEARCH

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Assortative Matching and Search[*†]

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February, 1996

Abstract

This paper reprises Becker's (1973) neoclassical marriage market model, assuming search frictions: There is a continuum of heterogeneous agents with types $x \in \mathbb{R}$; match (x, y) yields flow output $f(x, y) \equiv f(y, x)$, and utility is transferable; foregone output is the only cost of search.

We characterize equilibrium and constrained efficient matchings, and prove equilibrium existence. We then compare matching patterns with Becker's benchmark frictionless allocation, where positively or negatively assortative matching arises as agents' types are complements or substitutes in production. We formalize a notion of *assortative matching* with search frictions in the spirit of affiliation, and demonstrate that *Becker's condition no longer suffices* — e.g. $f(x, y) = (x + y)^2$ produces highly nonassortative matching. Fortunately, we prove that assortative matching does extend to both search settings — for all search frictions and type distributions — under stronger assumptions: *supermodularity* of not just f, but also $\log f_x$ and $\log f_{xy}$. Examples illustrate the necessity of these conditions. We show that our assortative matching notion in fact implies that everyone matches with a *convex* set of types; as a biproduct, this paper also provides a theory of convex matching sets.

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*We are grateful to Sheldon Chang (M.I.T. Math Dept.) for detecting a serious flaw in a key proof, and for spending the time to outline a fix for it. We wish to thank Daron Acemoglu, Susan Athey, Peter Diamond, Bart Lipman, Andreu MasColell, Andrew McLennan, and Charles Wilson provided useful comments on this paper or its prequel, as well as participants at the MIT Theory Lunch, and seminars at NYU and Princeton. We have also benefited from frequent help received via `sci.math.research`. Marcos Chamon provided valuable simulations assistance.

†This paper answers questions originally posed in a 1994 version of our mimeo "Matching, Search, and Heterogeneity." That unfinished work now focuses on a host of macro topics, such as the cross-sectional nature of search inefficiency, and has a more general model in some respects than this one, which is honed for a micro audience.

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1. INTRODUCTION

In this paper, we revisit what is arguably the classic insight of the neoclassical matching literature — when assortative matching arises — in a setting with search frictions. We consider a simple model of synergistic matching inspired by Becker's (1973) seminal analysis of the marriage market. There is a continuum of heterogeneous agents with characteristics x in $[0,1]$, who can only produce in pairs: The flow return of a match between agents x and y is $f(x,y)$, where f is a symmetric production function — for instance, $f(x,y) = xy$. Everyone is impatient and must search for partners, with potential matches arriving at a Poisson hazard rate. Matching precludes further search. Everyone seeks to maximize the present value of her future payoffs, and we ask who agrees to match with whom in a dynamic steady state. Such matching decisions turn on how the match output is shared. Smith (1996) considers exogenous sharing rules, as he studies the model where utility is not transferable (NTU). This paper instead posits *transferable utility* (TU).

We first study *search equilibrium* — the decentralized solution when match output is shared according to the Nash bargaining rule. We provide what we believe to be the first general proof of existence of such an equilibrium with heterogeneous types.[1] It is of independent value, as it parses the logical argument into a three point recipe for application elsewhere — especially the key continuous map from strategies to measures of unmatched agents. There are externalities in equilibrium, since search is costly, and the decision of two agents to match precludes others from matching with either. As a result, the present value of output in the economy is not maximized. We therefore next examine the constrained efficient outcome. Here, all matching decisions are made by a social planner seeking to maximize the present value of output, but unable to circumvent the search frictions faced by individuals.

The central theme of this paper, and our most innovative contribution, is that the equilibrium and constrained optimal matching outcomes are united by important cross-sectional structure. To introduce this, recall Becker's main insight into the core allocation of a frictionless market: If types are complementary to the match output (namely, $f_{xy} > 0$), then other things equal, agents should engage in *positively assortative matching.* As usual, search frictions create temporal matching rents, and thus an acceptable match need not be an ideal one. Still, the ease with which Becker

[1]But see the text for a discussion of other special case proofs.

derives his result suggests that there might be a simple extension of this neoclassical insight to a model with search frictions: Namely, agents might be willing to match with all others who are not too different than themselves. In this paper, we first dispel any hopes of such a free lunch. Instead, we demonstrate the negative result that *for an open class of complementarity production functions, individuals may only be willing to match with types who are quite different.* An especially salient example is afforded by the production function $f(x,y) = (x+y)^2$. The resulting matching pattern, depicted in Figure 2 (page 16), is in no way assortative.

Fortunately, all is not lost: *Assortative matching does extend to a search setting.* We find a simple open subclass of complementary production functions for which it arises *for any search frictions or type distributions.* Our definition of assortative matching is also quite natural, and essentially asks that matching sets be *affiliated.* For types in $\mathbb{R}$, this asks that any two agreeable matches can be severed, and then the greater two and lesser two types agreeably rematched. As testimony to the richness of this notion, we show that it in fact implies that everyone matches with a convex set of types in $\mathbb{R}$. *This paper therefore also simultaneously provides a theory of convex matching sets.* We consider this an interesting application of the supermodularity research program,[2] insofar as one must *combine* super/sub-modularity of (*i*) the production function (as in Becker) and also of (*ii*) its log marginal product $\log f_x$, *and* (*iii*) log cross partial derivative $\log f_{xy}$. The latter condition implies a key *single crossing property* of matching preferences.

We use these assumptions to prove that any individual z has a quasiconcave match surplus function $s(z,y) = f(z,y) - w(z) - w(y)$, where $w(y)$ is the type y's unmatched option value. This immediately produces convex matching sets, and then it is simply a matter of orienting whether matching sets are increasing or decreasing by Becker's condition (*i*). In fact, since one's current value is essentially an option on future surplus, or $w(z) = k\int_0^1 \max(0, f(z,y) - w(z) - w(y))u(y)dy$, where the mass of unmatched type-y agents $u(y)$ is unaffected by a single agent, one view of our descriptive theory is that it exclusively develops and exploits the properties of such implicit integral equations.

For an intuitive overview of the quasiconcavity logic, think of an economy with search frictions so severe that any match is acceptable. Surplus quasiconcavity entails comparing derivatives $f_y(z,y)$ and $w'(y)$. The time cost of search renders

[2] A good reference will be Athey, Milgrom, and Roberts (1996).

$w'(y)$ a multiple $\gamma < 1$ of the expectation over unmatched agents x of $f_y(x,y)$. Thus, the slope of z's surplus function is $f_y(z,y) - \gamma E_x f_y(x,y)$. Under (i), this is positive for $z = 1$, and thus 1's surplus function is quasiconcave, simply because $f_y(1,y) \geq f_y(x,y) > \gamma f_y(x,y)$ for all x. So by continuity, the frictionless logic of Becker's result carries through for 'high' types with search frictions.

But at the opposite extreme $z = 0$, this argument fails, because $f_y(0,y)$ and $\gamma f_y(x,y)$ are incomparable under (i) alone. Here is where (ii) comes into play, for it guarantees that $f_y(x,y)/f_y(0,y)$ is nondecreasing in y. Then rewriting $f_y(0,y) - \gamma E_x f_y(x,y) \geq 0$ as $1/\gamma \geq E_x(f_y(x,y)/f_y(0,y))$, we see that 0's surplus function is increasing for low partners y and decreasing for large y, i.e. quasiconcave. Again, by continuity, 'low' types have quasiconcave surplus functions under (ii).

Finally, we use (iii) to prove a crucial single-crossing property (SCP): For a given y, if $\bar{z}$ solves $f_y(\bar{z},y) = E_x f_y(x,y)$, then $f_y(\bar{z},y') \leq E_x f_y(x,y')$ for all $y' \geq y$. Loosely, this guarantees that everyone is either a 'high' or 'low' type. For z's surplus function is quasiconcave if her surplus is falling after any extremum: $s_y(z,y) = 0$ implies $s_y(z,y') \leq 0$ for $y' > y$. In other words, the marginal product $f_y(z,y)$ adjusts *proportionately* less than the marginal value $w'(y)$, and so (if everyone matches), less than the expected marginal product, or $f_y(z,y')/f_y(z,y) \leq w'(y')/w'(y) = E_x f_y(x,y')/E_x f_y(x,y)$. By the SCP, this holds at $\bar{z}$, and so is true at all $z \leq \bar{z}$ by (ii). Finally, by (i), our quasiconcavity premise is false for types $z \geq \bar{z}$, as $s_y(z,y) \equiv f_y(z,y) - \gamma E_x f_y(x,y) > f_y(\bar{z},y) - E_x f_y(x,y) \equiv 0$, given the definition of $\bar{z}$.

The only other paper that we are aware that considers matching in a TU search model with ex ante heterogeneity is Sattinger (1995). That paper does not touch on our most striking findings — the link with models in the traditional frictionless matching literature, as well as existence of a search equilibrium.

In section 2, we summarize the relevant results of the frictionless matching literature. Our model with search frictions is described in section 3. In sections 4 and 5, we define and characterize search equilibria and social optima. In section 6, we ask who matches with whom. We first derive the conditions that guarantee convex equilibrium and optimal matching sets. We establish the necessity of our new conditions with some illustrative examples. This sets the stage for us to define and explore (both positively and negatively) assortative matching. Section 7 establishes the existence of a steady state equilibrium. We appendicize the less intuitive proofs.

2. THE FRICTIONLESS MATCHING MODEL

Consider a frictionless matching model, with an atomless continuum of agents. Everyone in our economy is indexed by her publicly observable type, a number $x \in [0,1]$. An agent's type is fixed, and determines her productivity while matched.

Normalize the mass of agents to unity, and let the distribution of types be $L : [0,1] \mapsto [0,1]$. The fraction/mass of agents with type at most y is $L(y)$. We assume throughout that L is differentiable, with Borel measurable type density ℓ. The existence proof alone also requires that ℓ be boundedly finite and positive: $0 < \underline{\ell} < \ell(x) < \bar{\ell} < \infty$ for all x. One view of our set-up is that there is implicitly a continuum of every type of agent, with individuals belonging to the graph $\{(x,i)|x \in [0,1], 0 \leq i \leq \ell(x)\}$, where i is an index number of the type x agent.

When two individuals of types x and y — agents x and y — are matched together, they produce a flow output that is purely a function of their types, $f : [0,1]^2 \mapsto \mathbb{R}$. We shall later need to refer collectively to a set of assumptions:

A0 (Regularity Conditions). *The production function f is strictly increasing and strictly positive when $x, y > 0$. It is also symmetric, or $f(x,y) \equiv f(y,x)$, continuous, and twice differentiable with a uniformly bounded first partial derivative.*[3]

Thus, everyone prefers to be matched with someone rather than to be unmatched, and, everything else equal, prefers to be matched with higher types of agents.

In the core allocation, wages allocate the scarce resource, namely high productivity agents. When does *positively assortative matching* or *self-preference* obtain, where everyone matches exclusively with others of the same type? A sufficient condition is that characteristics be complementary inputs in the production function:

A1-Sup (Strict Supermodularity). *The own marginal product of any agent $x > 0$ is strictly increasing in her partner's type. Equivalently, the production function f is strictly supermodular in the positive quadrant: $f_{xy}(x,y) > 0$ when $x, y > 0$.*

If A1-Sup obtains, then high productivity agents enjoy the highest marginal product when they are matched with other high productivity agents. In the core allocation, matching must be (almost surely) positively assortative. To see this, simply note that any allocation in which some positive measure of agents of type x match with agents of type $y \neq x$ admits a simple Pareto-improvement. If we reassign

[3]More exactly, this *follows* from continuity of the first partials, and compactness of $[0,1]$.

all such agents to another of her own type, output rises, since $f(x, x) + f(y, y) > 2f(x, y)$ whenever $x \neq y$ given A1-Sup. So the unique value-maximizing allocation entails assortative matching, with everyone just matching with his own type.

Constructing an equilibrium with positively assortative matching provides a useful benchmark for the remainder of this paper. With a market 'wage' $w^0(x)$ for every type x, symmetry requires that each agent receive half of the output from her match: $w^0(x) = f(x, x)/2$.[4] We define for later use the *surplus function* for x: $s^0(x, y) \equiv f(x, y) - w^0(x) - w^0(y)$ measures how much the output of a partnership with y exceeds the sum of the wages of the two partners. If this is negative for all $x \neq y$, then we have constructed market wages to decentralize this allocation.

For a given x, an increase in her partner's type yields marginal surplus equal to

$$s_y^0(x, y) = f_y(x, y) - f_x(y, y)/2 - f_y(y, y)/2 = f_y(x, y) - f_y(y, y)$$

since f is symmetric. If A1-Sup obtains, then $f_y(x, y) \gtrless f_y(y, y)$ as $x \gtrless y$, i.e. for a given x, the surplus function is increasing or decreasing in her partner's type y as $x \gtrless y$. So it is strictly quasi-concave, with maximum value 0 when matched with a like partner x. Hence, under A1-Sup, the unique equilibrium of this model has positively assortative matching, and coincides with the *socially optimal* allocation.

A symmetric result obtains in the frictionless matching model when the marginal productivity of an agent is a decreasing function of her partner's type.

A1-Sub (Strict Submodularity). *An agent's own marginal product is decreasing in her partner's type. Or, the production function f is strictly submodular: $f_{xy} < 0$.*

Under A1-Sub, the social optimum and unique core allocation entails *negatively assortative matching*: Each agent x matches with her 'opposite' type $y(x)$, where $L(x) + L(y(x)) \equiv 1$. For by A1-Sub, $f(x_1, y_2) + f(x_2, y_1) > f(x_1, y_1) + f(x_2, y_2)$ whenever $x_1 < x_2$ and $y_1 < y_2$. It follows that if there are four agents, $z_1 < z_2 \leq z_3 < z_4$, the allocation in which z_1 and z_4 are matched and z_2 and z_3 are matched, Pareto dominates the two other possible allocations in which these four agents match in pairs. As before, we can prove the existence of the required core allocation by demonstrating that the surplus function for each agent x is a strictly quasi-concave function of her partner's type, achieving its maximum when her partner is $y(x)$.

[4] The 0 superscript denotes 'zero search frictions'.

◊ Remark 1. Throughout the paper, we shall generally maintain either A1-Sup or A1-Sub. This excludes, for instance, the knife-edge case of *additivity*, where the own marginal product of an agent is independent of her partner's type, $f(x,y) = g(x) + g(y)$, and for which any allocation is efficient. Also excluded are production functions for which own marginal product is not monotonic in partner's type — for example, $f(x,y) = \max\langle x^2y, xy^2\rangle$ — as Kremer and Maskin (1995) exploit.[5]

3. A MODEL OF MATCHING WITH SEARCH

In this section, we develop a continuous time, infinite horizon model of matching with search frictions. As per usual, frictions mean that finding and meeting other agents is time-consuming process.[6] Crucially, with an atomless continuum of agents, if agent x uses a rule like 'only match with a type y agent', then she will almost surely never match. Rather, everyone must be willing to match with *sets* of agents.

⋆ **Action Sets.** At any instant in continuous time, one is either *matched* or *unmatched.* Only the unmatched engage in (costless) search for a new partner. When two unmatched agents meet, their types are perfectly observable to each other. Either may veto the proposed match. If both approve, it is consummated and remains so until nature destroys the match. This event occurs with a constant flow probability $\delta > 0$, i.e. after an elapse time of t with chance $e^{-\delta t}$. At the moment the match is destroyed, both agents instantaneously re-enter the pool of searchers.

◊ Remark 2. In steady state, a match that is profitable to accept is profitable to sustain; therefore, to simplify our analysis, we disallow match quits.

◊ Remark 3. Match dissolutions is one way to maintain a steady-state. One could instead assume an inflow of entrants, and that matches are eternal. Our choice is moot for the descriptive theory, but is standard in the TU matching literature.

⋆ **Preferences.** Agents maximize their lifetime expected present discounted payoffs, using the interest rate $r > 0$. We assume *transferable utility* (in a sense specified later), as the benchmark frictionless model Becker (1973) is TU and not

[5]For such production functions, matching sets are not convex; some agent may be willing to match either with y_1 or y_3, yet unwilling to pair up with $y_2 \in (y_1, y_3)$. This provides a foretaste of section 6, where we establish that convexity and assortative matching are kindred concepts.

[6]See, for instance, Diamond (1982), Mortensen (1982), and Pissarides (1990).

NTU. As such, the flow income of x when matched with y comes from an endogenously determined payoff $\pi(x|y)$. These payoffs derive from the output of a match, so that $\pi(x|y) + \pi(y|x) = f(x,y)$ for all x and y. Unmatched agents earn nothing.

⋆ **Unmatched Agents and Search.** Let $u \leq \ell$ be the *unmatched density function,* i.e. $\int_X u(x)dx$ is the mass of unmatched agents with types $x \in X \subseteq [0,1]$.

Our search frictions are captured by the following simple story. *Were it possible,* an unmatched individual *would* meet another unmatched or matched agent according to a Poisson process with constant *rendezvous rate* $\rho > 0$. But it is presumed technically infeasible to meet someone who is already matched — or equivalently, the other individual is already engaged, and so misses any meeting. Thus, the flow probability that agent x meets any $y \in Y \subseteq [0,1]$ is $\rho \int_Y u(y)dy$. Simply put, the rate at which one meets others with types in any subset Y is in direct proportion to the mass of those *unmatched* in Y. Our cross-sectional results extend well beyond this (*quadratic*) search technology, as we underscore in section 8.

⋆ **Strategies.** A steady state strategy $A(x)$ for agent $x \in [0,1]$ specifies a Borel measurable set of agents with whom x is willing to match.[7] (That all agents of the same type employ the same strategy follows from our later analysis.) In the steady state of this model, A is time-invariant[8] and depends only on the unmatched density function u, the payoff-relevant state variable. Next define an agent's *matching set,* $\mathcal{M}(x) \equiv A(x) \cap \{y \,|\, x \in A(y)\}$, the set of types y with whom x is willing to match, *and* who are willing to match with x. A match (x,y) is *mutually agreeable* iff $y \in \mathcal{M}(x)$ (and so by symmetry, iff $x \in \mathcal{M}(y)$). We also identify each matching set with its *match indicator function* α: $\alpha(x,y) = 1$ if $y \in \mathcal{M}(x)$ and 0 otherwise.

⋆ **Steady State.** In steady state, the flow creation and flow destruction of matches for every type of agent must exactly balance. The density of matched agents $x \in [0,1]$ is $\ell(x) - u(x)$; these agents' matches exogenously dissolve with flow probability δ. The flow of matches created by unmatched agents of type x is $\rho u(x) \int_{\mathcal{M}(x)} u(y)\,dy$. Putting this together, in steady state for any type $x \in [0,1]$,

$$\delta(\ell(x) - u(x)) = \rho u(x) \int_{\mathcal{M}(x)} u(y)\,dy = \rho u(x) \int_0^1 \alpha(x,y) u(y)\,dy \tag{1}$$

[7]We can express strategies as acceptance sets given an atomless type distribution. With type atoms, equilibrium existence demands mixed strategies: either probabilistic matching or quitting.

[8]WLOG, we restrict attention to *stationary* acceptance sets. Since no single agent can affect any future state of the economy through her choices, if an alternative acceptance set is optimal at time t it remains optimal at time $t+s$.

4. THE DECENTRALIZED ECONOMY

We now characterize the *steady state search equilibria* (SE) of this model. Search equilibrium requires that (*i*) everyone maximizes her present discounted payoffs, taking all other strategies as given, and (*ii*) a match is consummated iff both parties accept it. Yet this criterion has insufficient cutting power. For example, if everyone chooses to match with no one, then no one has a strict incentive to deviate. This outcome does not seem sensible to us, as it doesn't survive the possibility of small strategy 'trembles' by others. So we also insist that everyone also choose a best response to *any possible strategy* one might face at any moment. Equivalently, (*ii'*) in a SE all matches with strictly positive mutual gains are accepted.

4.1 Search Equilibrium

⋆ **Value Equations.** Let $w(x)$ denote the expected *average* present value for an unmatched agent x, assuming x maximizes her expected present value.[9] Similarly, let $w(x|y)$ be the expected average present value for x while matched with y. We solve for a SE by defining the implicit equations solved by these Bellman values.

While unmatched, x earns nothing, but at flow rate $\rho \int_{\mathcal{M}(x)} u(y)dy$, she meets and matches with some $y \in \mathcal{M}(x)$, enjoying a capital gain $\left(w(x|y) - w(x)\right)/r$.

$$w(x) = \rho \int_{\mathcal{M}(x)} \frac{w(x|y) - w(x)}{r} u(y)\, dy \tag{2}$$

Similarly, x enjoys a *flow payoff* $\pi(x|y)$ when matched with y. With flow probability δ, her match is destroyed, and she suffers a capital loss $\left(w(x|y) - w(x)\right)/r$. Hence,

$$w(x|y) = \pi(x|y) - \delta\big(w(x|y) - w(x)\big)/r \tag{3}$$

We can eliminate $w(x|y)$ from equations (2) and (3), and obtain a simple expression for $w(x)$ as a function of the equilibrium payoffs $\pi(x|\cdot)$.

$$w(x) = \rho \int_{\mathcal{M}(x)} \frac{\pi(x|y) - w(x)}{r + \delta} u(y)\, dy \tag{4}$$

The average unmatched value of x equals the rate that x finds matches times the

[9] Here, w is analogous to the frictionless wage schedule w^0.

expected discounted present value of the capital gain over her unmatched value — where discounting incorporates both impatience and match impermanence.

⋆ **Acceptance Sets.** In a SE, one accepts any match that increases one's expected present value.

$$w(x|y) \gtrless w(x) \Rightarrow \begin{cases} y \in A(x) \\ y \notin A(x) \end{cases} \tag{5}$$

In the borderline case $w(x|y) = w(x)$, $y \in A(x)$ is neither necessary nor precluded.

⋆ **Wage Determination.** Search frictions create temporal bilateral monopoly, since match output less the agents' outside options (i.e. their unmatched values), is generally positive: *match surplus* $s(x,y) \equiv f(x,y) - w(x) - w(y) > 0$. This shifts the question of wage determination into the realm of bargaining theory.

We abstain from innovating on this front, and simply follow a number of authors, more recently Pissarides (1990), in assuming the Nash bargaining solution,[10] i.e. the *instantaneous match surplus* — the excess of wages over average unmatched values $(w(x|y) + w(y|x)) - (w(x) + w(y))$ — is divided equally. The wage schedule satisfies

$$w(x|y) - w(x) = w(y|x) - w(y) \tag{6}$$

So by the optimality condition (5), this yields $y \in A(x)$ iff $x \in A(y)$, except possibly in the borderline case $w(x|y) - w(x) = w(y|x) - w(y) = 0$. By definition, $A(x)$ and $\mathcal{M}(x)$ coincide, except possibly in this borderline case.

The definition of $w(x|y)$ from asset value equation (3) and the Nash bargaining schedule (6) yield

$$\pi(x|y) - w(x) = \pi(y|x) - w(y) \tag{7}$$

Since the payoffs must somehow divide the output of the match, $\pi(x|y) + \pi(y|x) = f(x,y)$, and so (7) yields the equivalent Nash bargaining payoff schedule:

$$\pi(x|y) = w(x) + \tfrac{1}{2}\big(f(x,y) - w(x) - w(y)\big) \tag{8}$$

In other words, the flow payoff for an agent x is equal to her flow value from being unmatched plus half the flow surplus produced by the match.

⋆ **Synthesis.** To describe the equilibrium, observe that (3) implies that $w(x|y) \gtrless w(x)$ as $\pi(x|y) \gtrless w(x)$, while (8) says $\pi(x|y) \gtrless w(x)$ as $f(x,y) - w(x) - w(y) \gtrless 0$.

[10] See Coles and Wright (1994) for the microfoundations of Nash bargaining in dynamic settings.

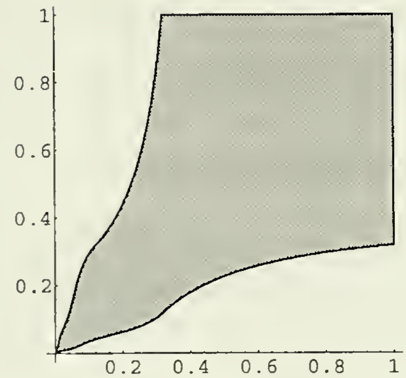


Figure 1: **Equilibrium Matching Sets.** This depicts the equilibrium matching sets for $f(x,y) = xy$, with $\rho = 50r$, $\delta = r/2$, and a uniform distribution of agents. If $x \in \mathcal{M}(y)$ and $y \in \mathcal{M}(x)$, then the point (x, y) is shaded in the graph.

Combining this with (5) yields the mutual optimality condition.

$$f(x,y) - w(x) - w(y) \gtrless 0 \quad \Rightarrow \quad \begin{cases} y \in \mathcal{M}(x) \\ y \notin \mathcal{M}(x) \end{cases} \tag{9}$$

where we have used the fact that $y \in A(x)$ and $x \in A(y)$ implies $y \in \mathcal{M}(x)$.

Next, substituting (8) into (4) yields an implicit system for unmatched values:

$$w(x) = \rho \int_{\mathcal{M}(x)} \frac{f(x,y) - w(x) - w(y)}{2\,(r+\delta)} u(y)\, dy \tag{10}$$

Note that equation (10) is well-defined, even though we have not specified whether $y \in \mathcal{M}(x)$ when $f(x,y) - w(x) - w(y) = 0$.

A SE may intuitively be fully described by specifying: (*i*) *who is matching with whom* (the matching sets $\mathcal{M}$); (*ii*) *the mass of each type searching* (the unmatched density u); and (*iii*) *how much everyone's time is worth* (the unmatched value w).

Proposition 1 (SE Characterization). *A SE can be represented as a triple* $(w, \mathcal{M}, u)$ *where:* w *solves the implicit system* (10), *given* $(\mathcal{M}, u)$; $\mathcal{M}$ *meets the optimality condition* (9) *given* w; *and* u *solves the steady state equation* (1) *given* $\mathcal{M}$.

► Example. Figure 1 graphically depicts the equilibrium matching sets for the production function $f(x,y) = xy$ as well as a particular choice of search frictions, impatience, and type distribution. Since this production function satisfies A1-Sup, in the frictionless benchmark agents are only willing to match with their own type, $\mathcal{M}(x) = \{x\}$. As one might expect, with search frictions, everyone is willing to accept a range of possible partners. In section 6 we find conditions on the production function that ensure equilibrium matching sets have approximately this shape.

4.2 Properties of the Value Function

Before proceeding, we must first derive some basic properties of the value function. Our analysis throughout the paper repeatedly applies the following inequality:

♦ Observation. For any agent x and any set $M \subseteq [0,1]$,

$$w(x) \geq \rho \int_M \frac{f(x,y) - w(x) - w(y)}{2\,(r+\delta)}\, u(y)\, dy \tag{11}$$

If this were not true, then the implicit equation (10) would yield a y such that either (*i*) $y \in \mathcal{M}(x)$, $y \notin M$, and $f(x,y) - w(x) - w(y) < 0$, or (*ii*) $y \notin \mathcal{M}(x)$, $y \in M$, and $f(x,y) - w(x) - w(y) > 0$. Either possibility contradicts (9). This leads us to

Lemma 1 (Monotonicity). *Given* A0, *the value $w \geq 0$ is increasing in a SE.*

Intuitively, since higher agents are always more productive than lower agents, they can simply imitate the matching decision of lower agents and do better. If they optimize, they will do still better. The appendicized proof formalizes this argument.

The logic underlying monotonicity also buttresses continuity: Anyone can do almost as well as a slightly more productive agent, simply by imitating her matching decision. Consequently, the value function cannot jump, as proven in the appendix.

Lemma 2 (Continuity). *Given* A0, *the value function w is continuous in a SE.*[11]

We often refer to the derivative of the unmatched value function. This is justified:

Lemma 3 (Differentiability). *Given* A0, *the value function w is a.e. differentiable in a SE, and its derivative is a.e. given by:*

$$w'(x) = \frac{\rho \int_{\mathcal{M}(x)} f_x(x,y)\, u(y) dy}{2\,(r+\delta) + \rho \int_{\mathcal{M}(x)} u(y) dy} \tag{12}$$

Monotonic functions are a.e. differentiable, and so the first part of the claim follows from Lemma 1. When the matching set is (suitably) differentiable in x, the resulting formula for $w'(x)$ is a straightforward application of the Fundamental Theorem of

[11] The appendicized proof actually establishes that w is Lipschitz.

Calculus: surplus vanishes all along the boundary of the matching set; therefore, we can safely ignore the effect on $w'(x)$ of changes in the matching set, and simply differentiate (10) under the integral sign. The difficult appendicized proof carefully argues that (*i*) this logic is still valid when matching sets are merely continuous and the value function Lipschitz at x, and (*ii*) both these conditions hold a.e. in x.

Using these lemmata, existence of SE is proved. For expositional ease, and as it is an important in its own right, we defer analysis of this complex issue to section 7.

5. CONSTRAINED EFFICIENT MATCHING

In this section, we investigate dynamic steady states where matching decisions are constrained efficient: Everyone matches so as to maximize the global present discounted value of output, rather than her own personal value. It helps to introduce a hypothetical social planner who is entrusted with all matching decisions. True to our stated goal, we assume that the planner cannot bypass the search frictions that render matching opportunities infrequent. We look for a stationary *social optimum* (SO), the steady state of the optimal dynamic program of a constrained planner.[12]

We characterize necessary conditions for a SO, by considering only stationary deviations from steady states. That is, we do not allow the planner (*i*) to destroy existing matches, or (*ii*) to employ nonstationary match acceptance strategies. These provisos are required by our model and strategy space, and reflect our steady-state spirit. But they sidestep a *very* deep issue: It is theoretically possible, if hard to fathom, that for any initial condition, a nonstationary policy may beat any stationary one the planner could devise. To establish a steady state SE, we safely ignored nonstationary deviations, as no single agent could affect the future course of the decentralized economy; since a social planner most certainly can affect the future, this is no longer WLOG, and stationary SO existence is much more problematic.[13]

Let $\mathcal{M}^*(x)$ denote the set of all agents whom the social planner lets match with agent x. By construction, $y \in \mathcal{M}^*(x)$ iff $x \in \mathcal{M}^*(y)$. We will call $\mathcal{M}^*(\cdot)$ an agent's *optimal matching set*, and let α^* be its indicator function: $\alpha^*(x, y) \equiv 1$ iff $x \in \mathcal{M}^*(y)$.

[12]This is analogous to the modified golden rule in the economic growth literature. The golden rule, by way of contrast, is the planner's best steady state if she can, at an initial time, *costlessly* jump to her prefered steady state, at which she must remain. The two notions coincide for an infinitely patient planner ($r = 0$), since the transition path is costless.

[13]Shimer and Smith (1996) will address SO existence allowing for all dynamic strategies.

Let $m : [0,1]^2 \mapsto \mathbb{R}_+$ be the density of matches, so that $\int_S m(x,y)$ is the mass of matches for $(x,y) \in S$. This satisfies a steady state constraint:

$$\delta m(x,y) = \rho\alpha^*(x,y)\, u(x)u(y) \tag{13}$$

In steady state, the flow dissolution of matches (x,y) must equal the flow creation of such matches. Then, being careful not to double-count output, the steady state average present value of output in the economy is $\int_0^1 \int_0^1 \frac{1}{2} f(x,y)m(x,y)\, dxdy$.

Thus a pair $(\mathcal{M}^*, m)$ is a SO only if the matching sets $\mathcal{M}^*$ maximize the present value of output among stationary matching rules, given the initial matched rates m; and $\mathcal{M}^*$ implies that the state of the economy remains at m. We can think of the planner maximizing the present value of output, subject to the steady state relationship (13). To solve for a SO, we could write the planner's problem as a Lagrangian, insert ε-variations on the state variables $m_\varepsilon \equiv m + \varepsilon\bar{m}$, differentiate with respect to ε, and reach (à la calculus of variations) a pointwise conclusion given the arbitrary nature of $\bar{m}$. We instead recast this as a current-valued Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\int_0^1 \int_0^1 f(x,y)m(x,y) + p(x,y)\big(\rho\alpha(x,y)\, u(x)u(y) - \delta m(x,y)\big)dxdy \tag{14}$$

where $p(x,y)/2$ is the multiplier on the steady state relationship (13), half the planner's value of a match (x,y) (again not double-counting). We describe the necessary first order conditions using the notational shortcuts $\mathcal{H}_{\alpha(x,y)}$ and $\mathcal{H}_{m(x,y)}$.

The first condition, a short-cut to placing multipliers on the constraint that $\alpha(x,y) \in [0,1]$, reflects the Kuhn-Tucker complementary slackness requirements.[14]

$$p(x,y) \gtrless 0 \Leftrightarrow \mathcal{H}_{\alpha(x,y)} \gtrless 0 \Rightarrow \begin{cases} \alpha(x,y) = 1 \\ \alpha(x,y) = 0 \end{cases} \tag{15}$$

In this 'bang-bang' control, when the shadow value of a match of x with y is positive, the planner will match them; when the shadow value is negative, she won't.

The other (steady state) first order condition determines p: $\mathcal{H}_{m(x,y)} = \frac{1}{2} r p(x,y)$. As the density of unmatched agents x is $u(x) = \ell(x) - \int_0^1 m(x,z)\, dz$, we have

$$2\mathcal{H}_{m(x,y)} = f(x,y) - \delta p(x,y) - \rho \int_0^1 (\alpha(x,z)p(x,z) + \alpha(y,z)p(y,z))\, u(z)\, dz \tag{16}$$

[14] Optimality only requires that the FOC hold almost everywhere. We assume they always hold.

Let $v(x) \equiv \rho \int_{\mathcal{M}^*(x)} p(x,z)u(z)dz = \rho \int_0^1 \alpha(x,z)p(x,z)u(z)dz$. As $p(x,z)$ is the planner's value of a match (x,z), we interpret $v(x)$ as the *social unmatched value* of agent x — namely, the flow rate at which she creates new social present value. Combining this with $2\mathcal{H}_{m(x,y)} = rp(x,y)$ and (16) yields:

$$p(x,y) = \big(f(x,y) - v(x) - v(y)\big)/(r+\delta) \tag{17}$$

Therefore, we can rewrite condition (15) as

$$f(x,y) - v(x) - v(y) \gtrless 0 \Rightarrow \begin{cases} x \in \mathcal{M}^*(y) \\ x \notin \mathcal{M}^*(y) \end{cases} \tag{18}$$

Combining (17) with the definition of v yields an implicit equation for the social unmatched value:

$$v(x) = \rho \int_{\mathcal{M}^*(x)} \frac{f(x,y) - v(x) - v(y)}{r+\delta} u(y)\, dy \tag{19}$$

Proposition 2 (SO Characterization). *A SO can be represented as* $(v, \mathcal{M}^*, u)$ *where: v solves the implicit system* (19), *given* $(\mathcal{M}^*, u)$*; $\mathcal{M}^*$ meets the optimality condition* (18) *given v; and u solves the steady state equation* (1) *given* $\mathcal{M}^*$.

◊ Remark 4. A triple $(v, \mathcal{M}^*, u)$ solving (1), (18), and (19) is not necessarily a SO. It may be dominated by nonstationary paths; or it may be a local, but not a global, maximum of the planner's problem.

◊ Remark 5. Replacing r by $2\hat{r}+\delta$ leaves (18)–(19) identical to (9)–(10), with the same steady-state equation (1).[15] Then v must share the key properties of w:

Lemma 4 (Value Properties). *The social unmatched value v is nonnegative, strictly increasing, continuous, and a.e. differentiable, with derivative a.e. given by*

$$v'(x) = \frac{\rho \int_{\mathcal{M}^*(x)} f_x(x,y)\, u(y)dy}{r + \delta + \rho \int_{\mathcal{M}^*(x)} u(y)dy}$$

Also, we can prove the existence of a triple solving (1), (18), and (19). This follows from existence of a SE (Proposition 5, page 27). However, this does not establish existence of a SO, as explained in Remark 4 above.

[15]While this may admit $\hat{r} < 0$, SE in fact only requires that $\hat{r} + \delta > 0$, which is still true.

6. DESCRIPTIVE THEORY

As seen in section 2, supermodularity (A1-Sup) alone ensures that there is positively assortative matching without search frictions in the core allocation/social optimum — agent x always matches with another agent x. Moreover, the surplus loss from not matching with one's ideal partner is increasing in the mismatch: If $x < y < z$, then x strictly prefers a match with y over z. In the frictional setting, everyone still has an ideal partner, but since individuals are willing to match with sets of agents, mismatch is the rule in equilibrium or at a constrained optimum. Curiously, the form that this mismatch takes is quite unpredictable. For example, x may prefer to match with z rather than with $y \in (x, z)$ — and sometimes would rather match with z than with another x! This significant violation of positive assortative matching motivates our new conditions on the production function.

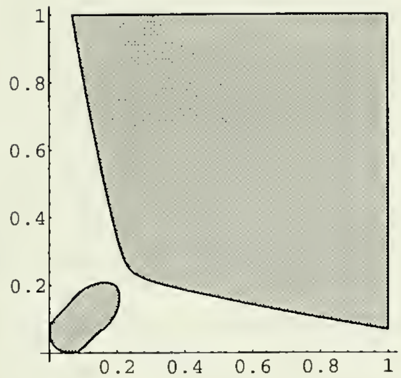
◊ Remark 6. Throughout this section we simply refer to properties of *matching sets* — as they *are equally true of any SE or SO*. For indeed, both SE and SO have identical value properties, which drive all results reported here. But for brevity and clarity, *our language and notation pertain to SE*. Note that while search equilibrium depends on the exact specification of how surplus is divided within a match, SO does not suffer from this ambiguity. This section nonetheless unites these two disparate concepts, which serves as a further robustness check on our descriptive theory.

6.1 Convexity

First on our agenda is *convexity*: If x is willing to match with y_1 and y_3, will she also agree to match with $y_2 \in (y_1, y_3)$? Optimality condition (9) tells us that if x's surplus function $s(x, y) \equiv f(x, y) - w(x) - w(y)$ is strictly quasiconcave in y, then the answer to this question is 'yes'.[16] Section 2 proved that A1-Sup or A1-Sub ensures a quasiconcave surplus function in the frictionless benchmark model.

► Example. Surprisingly, neither A1-Sup nor A1-Sub suffices with search frictions. With the supermodular production function $f(x, y) = (x + y)^2$ in Figure 2, there is positively assortative matching in the frictionless case. But with search frictions, equilibrium matching sets are not convex. Indeed, any $x \in (0.20, 0.21)$ won't match with her own type, but will match both with higher and lower types.

[16] Of course, convexity may obtain even if the surplus function is not quasiconcave; however, we see no general proof of convexity that does not rely on quasiconcavity.



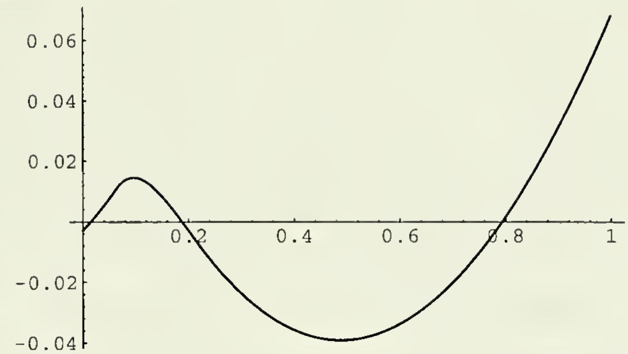


Figure 2: **Non-Convex Matching and Nonquasiconcave Surplus Function.** The left panel depicts the equilibrium matching sets for $f(x, y) = (x + y)^2$, with $\rho = 40r$, $\delta = r/2$, and a uniform distribution of types. All points (x, y) with $y \in \mathcal{M}(x)$ are shaded in the graph. The equilibrium matching sets of any $x \in (0.07, 0.21)$ are not convex. The right panel depicts the non-quasiconcave flow surplus $s(0.1, y)$ for all matches with agent 0.1.

Also, when $f(x, y) = (x + y)^2$, everyone has a strictly (quasi-)concave surplus function in the frictionless benchmark, with maximum surplus when each type x matches with another x. Indeed, the wage is $w^0(x) = 2x^2$ and surplus $s^0(x, y) = -(x - y)^2$. One might imagine that search frictions simply reduce everyone's value function in such a way as to preserve the shape of the surplus function, but shifting it up. The example disproves this conjecture. The surplus functions of some agents are clearly not quasiconcave — for instance, $x = 0.1$ has two local surplus maxima.

6.1.1 Conditions for Convex Matching Sets

Despite this example, there are restrictions on the production function f that ensure a quasiconcave surplus function, and therefore convex matching sets. Throughout this section, we impose either A2-Sup and A3-Sup, or A2-Sub and A3-Sub below:

A2-Sup. *The first partial derivative of the production function is log-supermodular: for all $x_1 \leq x_2$ and $y_1 \leq y_2$, $f_x(x_1, y_1) f_x(x_2, y_2) \geq f_x(x_1, y_2) f_x(x_2, y_1)$.*

A3-Sup. *The cross partial derivative of the production function is log-supermodular: for all $x_1 \leq x_2$ and $y_1 \leq y_2$, $f_{xy}(x_1, y_1) f_{xy}(x_2, y_2) \geq f_{xy}(x_1, y_2) f_{xy}(x_2, y_1)$.*

A2-Sub. *The first partial derivative of the production function is log-submodular: for all $x_1 \leq x_2$ and $y_1 \leq y_2$, $f_x(x_1, y_1) f_x(x_2, y_2) \leq f_x(x_1, y_2) f_x(x_2, y_1)$.*

A3-Sub. *The cross partial derivative of the production function is log-submodular: for all $x_1 \leq x_2$ and $y_1 \leq y_2$, $f_{xy}(x_1, y_1) f_{xy}(x_2, y_2) \leq f_{xy}(x_1, y_2) f_{xy}(x_2, y_1)$.*

Finally, let A2-Sup* (A2-Sub*) impose *strict inequality* in A2-Sup (A2-Sub) when $x_1 < x_2$ and $y_1 < y_2$. The above assumptions are interrelated. For example, the ratio $f_x(x_1, y)/f_x(x_2, y)$ is independent of y — and so A2-Sup and A2-Sub both hold — precisely for production functions of the form $f(x, y) \equiv c_1 + c_2\,(g(x) + g(y)) + c_3\,g(x)g(y)$, for some constants c_1, c_2, and c_3, and function $g : [0, 1] \mapsto \mathbb{R}$. A slightly weaker condition asks that $f(x, y) \equiv c_1 + c_2\,(g(x) + g(y)) + c_3\,h(x)h(y)$ for some constants c_1, c_2, and c_3, and functions $g : [0, 1] \mapsto \mathbb{R}$ and $h : [0, 1] \mapsto \mathbb{R}$. This is true iff $f_{xy}(x_1, y)/f_{xy}(x_2, y)$ is independent of y, so that A3-Sup and A3-Sub both obtain. Thus A2-Sup and A2-Sub jointly imply A3-Sup and A3-Sub, but not conversely. Some of the most obvious functions that one would write down, such as the Cobb-Douglas $f(x, y) = (xy)^a$, satisfy all four weak assumptions.

Proposition 3 (Convex Matching). *Posit* A0.
(*a*) *Given* A1-Sup, A2-Sup, *and* A3-Sup, *the matching set* $\mathcal{M}(z)$ *is convex for all* $z \in (0, 1]$, *and* $\mathcal{M}(0)$ *can be chosen convex. With* A2-Sup*, $\mathcal{M}(0)$ *must be convex.*
(*b*) *Given* A1-Sub, A2-Sub, A3-Sub, $\mathcal{M}(z)$ *is convex* $\forall z \in [0, 1]$.

Observe *Proposition 3 is wholly independent of (monotonic transformations of) the type distribution*: For instance, if we label each agent x by her type's percentile $L(x)$, and let $\hat{f}(L(x), L(y)) \equiv f(x, y)$, then f satisfies any of the assumptions in Proposition 3 iff $\hat{f}$ does. This is comforting, as *convexity in* $\mathbb{R}$ *is scale-independent.* That 'units don't matter' is a robustness check, suggesting that there are no weaker conditions on the production function that ensure convex matching sets.

◊ Remark 7. Extrapolating this logic, *biconvexity* (convex coordinate slices), and not simply convexity, is the scale-independent extension of our theory for productive types in $\mathbb{R}^n$. Since it adds little to our theory, we have not pursued this complication.

6.1.2 Proof of Convexity

We establish convexity of matching sets by proving that surplus functions are *strongly quasiconcave*. As seen in figure 3, a function is strongly quasiconcave if it is strictly so except perhaps for a flat global maximum.

Theorem 1 (Quasiconcavity). *Posit* A0 *and fix* z.
(*a*) *Given* A1,A2,A3-Sup *or* A1,A2,A3-Sub, $s(z, y)$ *is strongly quasiconcave in* y.
(*b*) *Given also* A2-Sup* *or* A2-Sub*, $s(z, y)$ *is strictly quasiconcave in* y.

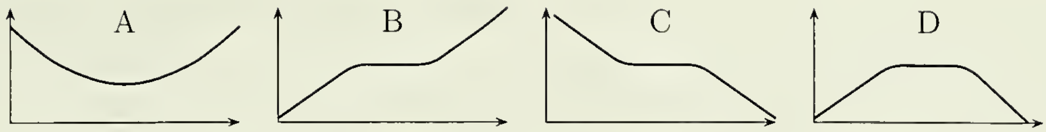


Figure 3: **Quasiconcavity.** The surplus function in panel A is not quasiconcave, while those in panels B and C are quasiconcave, but have flat portions that aren't global maxima. Neither is strongly quasiconcave. The surplus function in panel D is strongly but not strictly quasiconcave.

That A1-Sup or A1-Sub is a key ingredient ensuring a quasiconcave surplus function follows from the analysis of the frictionless model in section 2. For example, under A1-Sup, the slope of x's frictionless surplus function, $f_y(x,y) - f_y(y,y)$, is positive iff $x > y$. We illustrate the 'importance' of our other assumptions (i.e. that they are sometimes necessary conditions) through two examples now, and one later.

► Example. (*Importance of* A2-Sup *or* A2-Sub) The production function $f(x,y) = (x+y)^2$, which generated the non quasiconcave surplus functions in Figure 2, satisfies A1-Sup, A2-Sub, and *both* A3-Sup and A3-Sub. Similarly, for the production function $f(x,y) = x^2 + y^2 + x + y - xy$ satisfying A1-Sub, A2-Sup, and A3-Sup and A3-Sub, the surplus function is sometimes not quasiconcave.

► Example. (*Importance of* A2-Sup* *or* A2-Sub*) Consider the class of production functions $f(x,y) = xy+c(x+y)$, which satisfy A1-Sup, A2-Sup, and A3-Sup, but not A2-Sup*. Putting $c = \bar{x}(\sqrt{\delta^2 + 4\delta\rho} - \delta)/4(r+\delta)$, where $\bar{x} = \int_0^1 z\,\ell(z)dz$ is the mean population type, neatly yields $w(x) \equiv cx$ in equilibrium; thus $s(x,y) \equiv xy$ — i.e. all matches are weakly agreeable. But type $x = 0$ produces exactly zero surplus in all matches, and may elect an arbitrary nonconvex matching set.

Lemma 5 asserts: If the own-marginal product of a given type is the same for a sure match with $\bar{z}$ as for a random match from some set M, then any higher type has a lower own-marginal product from the match with $\bar{z}$ than with M.

Lemma 5 (Single-Crossing Property). *Posit* A0. *Assume* A1-Sup *and* A3-Sup *or* A1-Sub *and* A3-Sub. *Choose any* $y_1 \in [0,1]$ *and subset* $M \subseteq [0,1]$. *Let* $\bar{z}$ *solve*

$$f_y(\bar{z}, y_1) \equiv \frac{\int_M f_y(x,y_1)u(x)\,dx}{\int_M u(x)\,dx} \tag{20}$$

Then $\bar{z}$ *is uniquely defined, and for all* $y_2 \geq y_1$,

$$f_y(\bar{z}, y_2) \leq \frac{\int_M f_y(x,y_2)u(x)\,dx}{\int_M u(x)\,dx} \tag{21}$$

◇ Remark 8. One can verify that (21) binds for production functions of the form $f(x,y) \equiv c_1 + c_2\,(g(x) + g(y)) + c_3\,h(x)h(y)$ — i.e. where both A3-Sup and A3-Sub obtain, as $f_{xy}(x_1,y)/f_{xy}(x_2,y)$ is independent of y. So Lemma 5 is quite tight; in fact, (21) is reversed if A1-Sup and A3-Sub (or A1-Sub and A3-Sup) both obtain.

Next, for the formal proof of Theorem 1, we give a convenient characterization of quasiconcave functions that are almost everywhere differentiable.

Q-1. *If $\sigma(y) > \sigma(x)$ and $\sigma'(x)$ is defined, then $y \gtrless x$ implies $\sigma'(x) \gtrless 0$, $\forall x, y$.*

Q-2. *If $\sigma(y) \geq \sigma(x)$ and $\sigma'(x)$ is defined, then $y \gtrless x$ implies $\sigma'(x) \gtrless 0$, $\forall x, y$.*

Lemma 6. *A continuous and almost everywhere differentiable map $\sigma : [0,1] \mapsto \mathbb{R}$ is (a) strongly quasiconcave under* Q-1, *and (b) strictly so under* Q-2.

If σ is differentiable on $(0,1)$, this lemma admits a simple proof. Suppose, for example, that σ is not strictly quasiconcave but Q-2 holds. Then there exists $y_1 < y_2 < y_3$, with $\sigma(y_2) \leq \min\langle\sigma(y_1), \sigma(y_3)\rangle$. Since $\sigma(y_1) \geq \sigma(y_2)$ and $y_1 < y_2$, $\sigma'(y_2) < 0$ by Q-2. Similarly, $\sigma(y_3) \geq \sigma(y_2)$ implies $\sigma'(y_2) > 0$ by Q-2. This is a contradiction. We appendicize the general proof, for which $\sigma'(y_2)$ need not be defined.

Proof of Theorem 1. For fixed z, the surplus function $s(z,y)$ is continuous by A0 and Lemma 2. Also, $s_y(z,y)$ is defined for a.e. y, since f is differentiable by A0 and $w'(y)$ is defined for a.e. y by Lemma 3. We establish quasiconcavity using the characterization in Lemma 6. Namely, we prove that for all z and $y_1 < y_2$, ($\bigstar$) holds: if $s_y(z,y_1)$ exists and $s(z,y_2) > s(z,y_1)$, then $s_y(z,y_1) > 0$, under supermodularity.[17]

• Step 1: Conclusion of ($\bigstar$) is Valid For 'High' Types. We show that surplus is strictly increasing at y_1 for large enough z. Choose y_1 with $w'(y_1)$ defined. Define $\bar{z}$ as in Lemma 5 with $M = \mathcal{M}(y_1)$ and use differentiability Lemma 3:

$$f_y(\bar{z},y_1) \equiv \frac{\int_{\mathcal{M}(y_1)} f_y(x,y_1)u(x)\,dx}{\int_{\mathcal{M}(y_1)} u(x)\,dx} > \frac{\rho\int_{\mathcal{M}(y_1)} f_y(x,y_1)u(x)\,dx}{2\,(r+\delta) + \rho\int_{\mathcal{M}(y_1)} u(x)\,dx} = w'(y_1) \qquad (22)$$

So $0 < f_y(\bar{z},y_1) - w'(y_1) \leq f_y(z,y_1) - w'(y_1) = s_y(z,y_1)\ \forall z \geq \bar{z}$, by A1-Sup.

• Step 2: Implication ($\bigstar$) is Valid For 'Low' Types. Fix $y_2 > y_1$. From

[17] For the proof of $y_2 < y_1 \Rightarrow s_y(z,y_1) < 0$, one instead shows that the premise $s(z,y_2) > s(z,y_1)$ fails for high types in step 1, while the implication is valid for low types in step 2. Proofs under submodularity are totally analogous.

the implicit equation (10) and inequality (11):

$$w(y_2) - w(y_1) \geq \frac{\rho \int_{\mathcal{M}(y_1)} (f(x,y_2) - f(x,y_1))\, u(x)\, dx}{2\,(r+\delta) + \rho \int_{\mathcal{M}(y_1)} u(x)\, dx}$$

Divide through by $w'(y_1)$ and its definition from Lemma 3:

$$\frac{w(y_2) - w(y_1)}{w'(y_1)} \geq \frac{\int_{\mathcal{M}(y_1)} (f(x,y_2) - f(x,y_1))\, u(x)\, dx}{\int_{\mathcal{M}(y_1)} f_y(x,y_1) u(x)\, dx} \geq \frac{f(z,y_2) - f(z,y_1)}{f_y(z,y_1)} \tag{23}$$

By A2-Sup, the final inequality is true iff $z < \underline{z}$, appropriately defined. For by A2-Sup, $f_y(z,y')/f_y(z,y_1)$ is increasing in z for $y' > y_1$, as is its integral over $y' \in [y_1, y_2]$. For some such z, suppose $s(z,y_2) > s(z,y_1) \Leftrightarrow w(y_2) - w(y_1) < f(z,y_2) - f(z,y_1)$. Then $s_y(z,y_1) = f_y(z,y_1) - w'(y_1) > 0$. Under A2-Sup*, the final inequality in (23) is strict if $z < \underline{z}$, so $s(z,y_2) = s(z,y_1)$ implies $s_y(z,y_1) > 0$ as well. Apply Lemma 6.

• STEP 3: EVERY TYPE IS EITHER 'HIGH' OR 'LOW'. Here, the key ingredient is A3-Sup: Integrate inequality (21) over $y_2 \in [y_1, \bar{y}_2]$, and divide through by (20). Put $M = \mathcal{M}(y_1)$ in the numerator and denominator, and replace $\bar{y}_2$ by y_2, to get

$$\frac{f(\bar{z},y_2) - f(\bar{z},y_1)}{f_y(\bar{z},y_1)} \leq \frac{\int_{\mathcal{M}(y_1)} (f(x,y_2) - f(x,y_1))\, u(x)\, dx}{\int_{\mathcal{M}(y_1)} f_y(x,y_1) u(x)\, dx}$$

So $\bar{z}$ satisfies inequality (23); if $z < \bar{z}$ then $z < \underline{z}$ also. So for any (y_1, y_2), the $\underline{z}$ associated with that pair is larger than the $\bar{z}$ associated with y_1 alone. □

Proof of Proposition 3. Under the basic assumptions, by Theorem 1, each agent z has a strongly quasiconcave surplus function. If $s(z,y) > 0$ for some y, Theorem 1 establishes that (*i*) the set of agents y for whom $s(z,y) > 0$ is convex and (*ii*) there are at most two types who produce exactly zero surplus with z. Whether z matches with these boundary types does not affect the convexity of $\mathcal{M}(z)$.

If $s(z,y) \leq 0$ for all y, $w(z) = 0$ by (10), and so $z = 0$ from Lemma 1. Agent 0 could choose a non-convex matching set if a positive mass of matches happen to yield zero surplus; even that cannot happen under A2-Sup*, for then each agent has a strictly quasiconcave surplus function. Finally, if A1,A2,A3-Sub obtain, $s(0,y)$ is increasing in y, and so is trivially strictly quasiconcave. □

► EXAMPLE. (*Importance of* A3-Sup *or* A3-Sub) Finding examples of production functions associated with non-quasiconcave surplus functions obeying A1-Sub and

A2-Sub (or A1-Sup and A2-Sup) is rather problematic. This difficulty arises because every type may be either 'high' or a 'low', even if it is analytically impossible to verify this. For instance, step 1 yielded strict inequality, so the surplus function is increasing below $\bar{z}$. Likewise, any failure of convexity must therefore be *a hole in the center of the acceptance set,* which we have not been able to produce.

One production function that satisfies A1-Sup and A2-Sup is:

$$f(x,y) = \begin{cases} 2626 + 260x + 260y + 480xy & \text{if } (x,y) \in [0,1/2] \times [0,1/2] \\ 2378 - 248x + 756y + 1496xy & \text{if } (x,y) \in [0,1/2] \times (1/2,1] \\ 2378 + 756x - 248y + 1496xy & \text{if } (x,y) \in (1/2,1] \times [0,1/2] \\ 2627 - 746x - 746y + 4500xy & \text{if } (x,y) \in (1/2,1] \times (1/2,1] \end{cases}$$

To see that this production function does not satisfy A3-Sup, consider any $z_1 < 0.5 < z_2$, $f_{xy}(z_1,z_1)f_{xy}(z_2,z_2) - f_{xy}(z_1,z_2)^2 = 480 \cdot 4500 - 1496^2 = -78016 < 0$. The function also does not satisfy A0, because it is not differentiable when x or y is $1/2$. Differentiability can be restored, however, suitably smoothing out the creases.

If $\delta = r$, $\rho = 156r$, and the type distribution is uniform, one can prove that in equilibrium, all matches are acceptable, and that

$$w(x) = \begin{cases} 1056.62 + 470.25x & \text{if } x \in [0,1/2] \\ 586.92 + 1409.63x & \text{if } x \in (1/2,1] \end{cases}$$

Most importantly, the equilibrium surplus functions of types $x \in [0.437, 0.438]$ are minimized when x matches with 0.5, and hence are not quasiconcave.

6.1.3 Bound Functions

Nonempty, convex matching sets are almost fully described by *lower* and *upper bound functions* $a, b : [0,1] \mapsto [0,1]$, namely $a(x) \equiv \inf\{y \mid y \in \mathcal{M}(x)\}$ and $b(x) \equiv \sup\{y \mid y \in \mathcal{M}(x)\}$. Bound functions do not encode whether 'boundary' matches are consummated (e.g. whether x matches with $a(x)$). But since the boundaries have zero measure, value functions and unmatched rates are unaffected by this choice. Also, by continuity of f and w, boundary partners are 0 or 1, or provide zero surplus.

◇ Remark 9. Under A0, we impose with trivial loss of generality that matching sets are nonempty. If ever $\mathcal{M}(x) = \varnothing$, then $w(x) = 0$ by (10). For $x > 0$, this contradicts Lemma 1, and is impossible. Next, if $w(0) = 0$, then $s(0,0) = f(0,0) \geq 0$, and we can adopt the convention $0 \in \mathcal{M}(0)$ with no other effect on the equilibrium.

Theorem 2 (Matching Set Bounds). *Assume matching sets are convex and nonempty. Then a is quasiconvex and b quasiconcave.*

Proof. If b is not quasiconcave then there exists y with $\{x|b(x) \geq y\}$ not convex, i.e. for some $x_1 < x_2 < x_3$ such that $b(x_1) \geq y$, $y > b(x_2)$, and $b(x_3) \geq y$. We may choose y large enough that it is also true that $y \in \mathcal{M}(x_1)$ and $y \in \mathcal{M}(x_3)$. So $x_1 \in \mathcal{M}(y), x_2 \notin \mathcal{M}(y)$, $x_3 \in \mathcal{M}(y)$, violating convexity. Similarly for a. □

Next, quasiconcave surplus functions imply continuous matching set bounds:

Theorem 3 (Matching Set Continuity). *Posit* A0. *Given* A1,A2,A3-Sup *or* A1,A2,A3-Sub, *a and b are continuous on* $(0,1)$, *and a bound function is constant on an interval only if its value is* 0 *or* 1 *on that interval.*

Proof. Suppose $a(y) = \bar{a} \in (0,1)$ for all $y \in [y_1, y_2]$. Then since s is continuous by A0 and Lemma 2, $s(\bar{a}, y) = 0$ for all $y \in [y_1, y_2]$. Since $\bar{a}$'s surplus function can only be flat at its maximum (by strong quasiconcavity, Theorem 1), $s(\bar{a}, \cdot)$ is nonpositive. So $w(\bar{a}) = 0$ by (10), contradicting $\bar{a} > 0$. A similar argument establishes that b is not constant. Finally, since matching sets are symmetric when reflected across the 45° line, a discontinuity in a bound function at $x \in (0,1)$ corresponds to an interval of types who are indifferent about matching with x, which we just ruled out. □

6.2 Assortative Matching

6.2.1 The Meaning of Assortative Matching

Because matching sets $\mathcal{M}$ are not singletons with search frictions, an appropriate definition of assortative matching is not a priori obvious. For example, A1-Sup or A1-Sub imposes an either-or form of assortative matching satisfied even by Figure 2.

♦ Observation. Given are agents $x_1 < x_2$, $y_1 < y_2$. Under A1-Sup, if $y_1 \in \mathcal{M}(x_2)$ and $y_2 \in \mathcal{M}(x_1)$, then either $y_1 \in \mathcal{M}(x_1)$, *or* $y_2 \in \mathcal{M}(x_2)$, or both. Under A1-Sub, if $y_1 \in \mathcal{M}(x_1)$ and $y_2 \in \mathcal{M}(x_2)$, then either $y_1 \in \mathcal{M}(x_2)$, *or* $y_2 \in \mathcal{M}(x_1)$, or both.

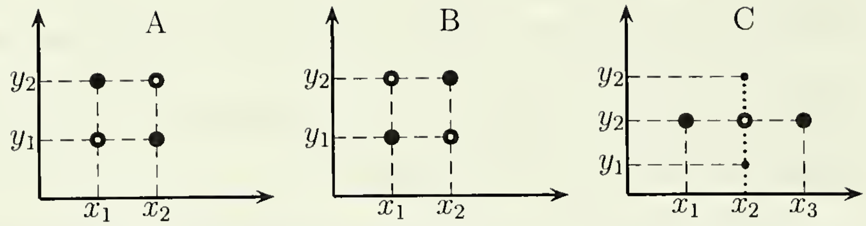


Figure 4: **Assortative Matching.** Panels A and B depict graphically the definition of positively and negatively assortative matching, respectively. If the pairs indicated by filled dots match, then the pairs indicated by hollow dots match as well. Panel C depicts the proof of Lemma 4. If low and high types (x_1 and x_3) match with some agent, then so must middle types (x_2), given positively or negatively assortative matching.

To see why, note that $y_1 \in \mathcal{M}(x_2)$ and $y_2 \in \mathcal{M}(x_1)$ imply $s(x_2, y_1) \geq 0$ and $s(x_1, y_2) \geq 0$, respectively. Summing yields $f(x_2, y_1)+f(x_1, y_2) \geq w(x_1)+w(x_2)+w(y_1)+w(y_2)$. Also A1-Sup implies $f(x_1, y_1) + f(x_2, y_2) > f(x_2, y_1) + f(x_1, y_2)$. Together these inequalities yield $s(x_1, y_1) > 0$, or $s(x_2, y_2) > 0$, or both, as required.

This observation speaks to a weaker notion of assortative matching than what we intend. To formalize our idea, let $\vec{\beta} \wedge \vec{\beta}'$ be the component-wise minimum, and $\vec{\beta} \vee \vec{\beta}'$ the maximum, of vectors $\vec{\beta}$ and $\vec{\beta}'$. Call matching positively assortative if the matching indicator function α is *affiliated*:[18] $\alpha(\vec{\beta} \wedge \vec{\beta}')\alpha(\vec{\beta} \vee \vec{\beta}') \geq \alpha(\vec{\beta})\alpha(\vec{\beta}')$ for any given matching pairs $\vec{\beta} = (x, y)$ and $\vec{\beta}' = (x', y')$. Matching is negatively assortative if the reverse (negative affiliation) inequality obtains.[19] Equivalently,

Definition. Let $x_1 < x_2$ and $y_1 < y_2$. Matching is *positively assortative* if $y_1 \in \mathcal{M}(x_2)$ and $y_2 \in \mathcal{M}(x_1)$ imply $y_1 \in \mathcal{M}(x_1)$ *and* $y_2 \in \mathcal{M}(x_2)$. Matching is *negatively assortative* if $y_1 \in \mathcal{M}(x_1)$ and $y_2 \in \mathcal{M}(x_2)$ imply $y_1 \in \mathcal{M}(x_2)$ *and* $y_2 \in \mathcal{M}(x_1)$.

These natural definitions are depicted graphically in panels A and B of Figure 4. Positively (resp. negatively) assortative matching describes a preference for matching with similar (resp. opposite) types. Our new definitions generalize the traditional frictionless matching ones, as given in section 2: The presumed contrary matches don't exist with singleton matching sets, and so the definitions vacuously hold.

6.2.2 Characterization of Assortative Matching

With assortative matching, matching sets are convex, and bounds a, b well-defined.

[18]Milgrom and Weber (1982) is the classic (auction-theory) economic application of this concept.

[19]This abstract formulation of it has the advantage of easily extending to discrete types: The probability-of-matching function must be affiliated. Likewise, the extension to $\mathbb{R}^n$ is immediate.

Theorem 4 (Assortative Matching and Convexity). *Given positively or negatively assortative matching,* $\mathcal{M}(x) \neq \varnothing\ \forall x \in (0,1)$ *implies* $\mathcal{M}(x)$ *convex for all* x.[20]

Proof. The two cases are symmetric, so assume positively assortative matching. Take any $x_1 < x_2 < x_3$ and $y_2 \in [0,1]$, with $x_1 \in \mathcal{M}(y_2)$ and $x_3 \in \mathcal{M}(y_2)$. Since $\mathcal{M}(x_2) \neq \varnothing$, there exists $y' \in \mathcal{M}(x_2)$. If $y' > y_2$, like the point y_3 in panel C of Figure 4, then $x_2 \in \mathcal{M}(y')$ and $x_3 \in \mathcal{M}(y_2)$ imply $x_2 \in \mathcal{M}(y_2)$, assuming positively assortative matching. If $y' < y_2$, like the point y_1 in panel C, then $x_2 \in \mathcal{M}(y')$ and $x_1 \in \mathcal{M}(y_2)$ imply $x_2 \in \mathcal{M}(y_2)$. Finally, if $y' = y_2$, obviously $x_2 \in \mathcal{M}(y_2)$. □

This very close relationship between our two notions of matching set cohesiveness, convexity and assortative matching, is further evidence that we have indeed found the appropriate definition of assortative matching. And, like convexity, assortative matching is independent of the type distribution.

Assortative matching implies convexity (Theorem 4) and thus well-defined bound functions (Theorem 2), To formalize a link between the parallel notions of assortative matching and monotonic bound functions, call a function into $[0,1]$ *strongly monotonic* if it is weakly monotonic, and strictly so when valued in $(0,1)$.

Lemma 7 (Assortative Matching and Monotonic Bound Functions).
(a) Posit convex matching sets. If a and b are strongly increasing, $0 \in \mathcal{M}(0)$, *and* $1 \in \mathcal{M}(1)$, *then matching is positively assortative, while if a and b are strongly decreasing and* $1 \in \mathcal{M}(0)$, *then matching is negatively assortative.*
(b) Conversely, under A0, *a and b are nondecreasing (nonincreasing) with positively (negatively) assortative matching.*

Part (a) captures one interpretation of our research program, which is that matching set convexity plus Becker's condition A1 yields assortative matching.

6.2.3 Conditions For Assortative Matching

We have shown matching sets are not necessarily convex when there are search frictions. Theorem 4 thus implies that matching is not necessarily assortative in this environment; however, the assumptions that ensure convex matching sets are almost sufficient for assortative matching. We state our main descriptive result:

[20] While the matching set assumption is endogenous, this theorem makes almost no assumptions. From Lemma 1, we know that under A0, $w(x) > 0$ and so $\mathcal{M}(x) \neq \varnothing$ when $x > 0$.

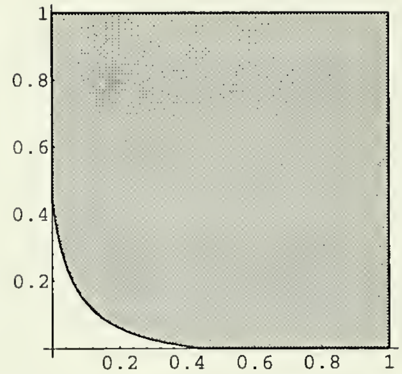


Figure 5: **Low Types Match with High Types, But Not with Low Types.** This depicts the equilibrium matching sets (with (x, y) shaded iff $y \in \mathcal{M}(x)$) for $f(x, y) = x+y+xy$, with $\rho = 50r$, $\delta = r/2$, and a uniform type distribution. The lower bound $a(x)$ is falling on $(0, 0.4)$.

Proposition 4 (Assortative Matching). *Posit* A0, *and let* $k \geq 0$.

(a) *Given* A1,A2,A3-Sup *and* $f(0, y) \equiv k$*, there is positively assortative matching.*

(b) *Given* A1,A2,A3-Sub*, there is negatively assortative matching.*

Proof of (a). From A1,A2,A3-Sup, all matching sets except possibly $\mathcal{M}(0)$ are convex (Proposition 3). Hence they are described bound functions a and b, constant only when valued 0 or 1 (Theorem 3). As $w(1) > 0$, $\mathcal{M}(1)$ is nonempty. Then $s(1, y)$ increasing in y (step 1 of the proof of Theorem 1) implies $1 \in \mathcal{M}(1)$, whence $b(1) = 1$. Then since b is quasiconcave (Theorem 2), it is strongly increasing. To prove $\mathcal{M}(0)$ is convex, $0 \in \mathcal{M}(0)$, and a is strongly increasing, we use the assumption that $f(0, y) = k$ for all y. This implies $s(0, y)$ is decreasing in y, and so $\mathcal{M}(0)$ is convex, $0 \in \mathcal{M}(0)$, and $a(0) = 0$. Since a is quasiconvex (Theorem 2), it is strongly increasing. Lemma 7 implies matching is positively assortative. $\square$

Proof of (b). A1,A2,A3-Sub imply all matching sets are convex, and that $s(0, y)$ is increasing in y. Quite easily $w(0) > 0$, for if $w(0) = 0$, $s(0, 0) \geq 0$; from this follows $s(0, y) > 0$ for all $y > 0$, a contradiction. So $\mathcal{M}(0)$ is nonempty; $s(0, y)$ increasing in y implies $1 \in \mathcal{M}(0)$. It follows that $a(1) = 0$ and $b(0) = 1$. Then since a and b are quasiconvex and quasiconcave, respectively (Theorem 2), and strongly monotonic (Theorem 3), they are strongly decreasing. Apply Lemma 7. $\square$

► Example. (*Importance of* $f(0, y) = k$) Consider a production function that satisfies A1-Sup, A2-Sup, and A3-Sup, but that has $f(0, 0) = 0$, and $f(0, y) > 0$ for $y > 0$. Figure 5 illustrates the matching sets for one such function, $f(x, y) = x + y + xy$. With search frictions, some agents are willing to match with 0, since

$f(0,y) > 0$ when $y > 0$; therefore, (10) implies that $w(0) > 0 = f(0,0)/2$, and so $s(0,0) < 0$. By continuity, very low types won't match with each other.

The proof of Proposition 4 establishes an important characteristic of matching set boundaries, which we state here for emphasis.

Corollary. *Posit* A0. *Bounds a and b are strongly increasing given* A1,A2,A3-Sup *and $f(0,\cdot) \equiv k$, and strongly decreasing given* A1,A2,A3-Sub.

In summary, we have found a powerful generalization of negatively and positively assortative matching to a frictional setting. *Under the three submodularity assumptions, matching sets are decreasing, convex, and continuous, while under the three supermodularity assumptions, and the proviso that type* 0 *agents have a zero marginal product, matching sets are increasing, convex, and continuous.*

6.3 Ideal Partners

In the frictionless world of section 2, every individual x only matches with her 'ideal partner', that type y who maximizes $s^0(x,y)$. Assortative matching without frictions means that this ideal partner graph is increasing. By way of comparison, let's define the set of *ideal partners* $p^*(x)$ of type x, those partners yielding the greatest surplus: $y^* \in p^*(x)$ iff $s(x,y^*) \geq s(x,y)$ for all y. Continuity of the ideal partner correspondence and quasiconcavity of the surplus function (and hence convexity of matching sets) are closely linked, further reinforcing our approach.

Theorem 5 (Ideal Partners and Quasiconcavity). *Posit* A0.
(*a*) *Given $s(x,\cdot)$ strongly (strictly) quasiconcave for all x,*[21] *$p^* : [0,1] \rightrightarrows [0,1]$ is nonempty, convex-valued and upper hemicontinuous (single-valued and continuous).*
(*b*) *Conversely, assume $p^*(x)$ is nonempty, convex-valued, and upper hemicontinuous (single-valued and continuous) for all x. Given* A1-Sup *and $f(0,\cdot) \equiv k$, or* A1-Sub *and $f(1,\cdot) \equiv k$, $s(x,\cdot)$ is strongly (strictly) quasiconcave for all x.*

Proof of (*a*). Clearly, $p^* \neq \varnothing$ as s is continuous by A0 and Lemma 2, and $[0,1]$ is compact. Since the set of maximizers of a quasiconcave function is convex, p^* is convex-valued, while u.h.c. follows from Berge's Maximum Theorem. For strictly quasiconcave surplus functions with single-valued maximizers, p^* is continuous. □

[21] Since A1,A2,A3 imply each surplus function is strongly quasiconcave (Theorem 1), the conclusions of the first part of Theorem 5 also follow from these primitive assumptions.

Proof sketch of (b). We prove in the appendix that everyone is someone's ideal partner. Then we show that under A1, if y^* is a local minimum of $s(x, y)$ — a violation of quasiconcavity — then y^* can never be someone's ideal partner. For instance, under A1-Sup, if x prefers y' to y^*, then $x' \gtrless x$ prefers $y' \gtrless y^*$ to y^*. □

Finally, we show that the ideal partner correspondence is weakly monotonic under A1 alone, an intuitive generalization of assortative frictionless matching from section 2. Strong monotonicity of p^* (strict monotonicity, except when valued 0 or 1), however, obtains only under our stronger assumptions A1,A2,A3.

Theorem 6 (Monotonic Ideal Partners). *Posit* A0.
(*a*) *Given* A1-Sup (A1-Sub), *p^* is nondecreasing (nonincreasing).*
(*b*) *Given* A1,A2,A3-Sup (A1,A2,A3-Sub), *p^* is strongly increasing (decreasing).*

Proof of (a). Let $x_1 < x_2$, with $y_1 \in p^*(x_1)$, $y_2 \in p^*(x_2)$. By definition of p^*, $s(x_1, y_1) \geq s(x_1, y_2)$ and $s(x_2, y_2) \geq s(x_2, y_1)$. Summing yields $f(x_1, y_1) + f(x_2, y_2) \geq f(x_1, y_2) + f(x_2, y_1)$. So $y_1 \leq y_2$ under A1-Sup, and $y_1 \geq y_2$ under A1-Sub. □

We prove part (*b*) in the appendix, showing that one of the above inequalities is strict given A1,A2,A3. Note that a still stronger claim, p^* is strictly monotonic, doesn't obtain under any reasonable assumptions: The highest (lowest) type has an increasing surplus function under A1,A2,A3-Sup (A1,A2,A3-Sub), and by continuity, so do nearby types; hence $p^*(x) = \{1\}$ for an open interval of types near 1 (near 0).

7. SEARCH EQUILIBRIUM EXISTENCE

We now return to the question of the existence of SE.

Proposition 5 (SE Existence). *Given* A0 *and* A1-Sup *or* A1-Sub, *a SE exists.*

The technical difficulties with both a continuum of agents and a continuous strategy space are well-known. For the larger picture for our proof, note that as a *matching-theoretic* model, goods are individual-specific, and insignificant (measure zero) changes in matching sets can have drastic consequences: If everyone opts not to match with x, her fortunes are sunk. To embed the requisite anonymity into the players' actions, we look for a fixed point of the map from players' value functions into itself.[22] Since values here play an analogous role to prices in Walrasian models,

[22]While a slight shift in all values may still lead everyone not to match with some agent x, those matches must have furnished little surplus, and this must only slightly affect the value of x.

this corresponds to the usual equilibrium in price space. Even though a search equilibrium is a *triple,* our value function program works because values essentially encode all needed information for matching sets $\mathcal{M}$ and unmatched densities u.

But in *search-theoretic* matching models, an additional hurdle is present with no analogue in the Walrasian setting, and its definitive resolution constitutes a major contribution of this paper. Since matches must be sought out in a time costly fashion, their availability must be continuous in the values. That matching sets continuously follow from value functions is not hard to show given the right space (Lemma 8). Much more subtle, however, is the fact that matching sets continuously determine unmatched measures (Lemma 9) — our 'fundamental matching lemma'. This generally proves the key continuity result for the quadratic search technology.

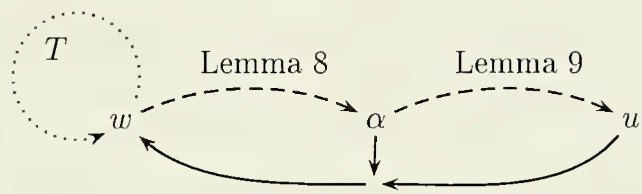


Figure 6: **Existence of Search Equilibrium: A Recipe.** The two maps: value functions w to match indicator functions α, and then α to the unmatched density u, are well-defined and continuous, respectively by lemmata 8 and 9; the composite map $w \mapsto u$ is thus well-defined and continuous. Our fixed point proof proceeds using the value function best-response map T.

◊ REMARK 10. A plausible alternative approach, taking the limits of finite agent approximations, would entail studying a largely different model — for then mixed strategies would be needed in equilibrium. We thus develop the general methodology for continuum agent search-theoretic matching models that can be applied elsewhere.

◊ REMARK 11. There are a few previous existence theorems for heterogeneous agent two-sided matching search models. Some assume for simplicity an exogenous type distribution, and thus entirely eschew the trickier aspects of search theory. All, however, take advantage of threshold matching sets to prove existence directly with them. In this class of existence proofs, Morgan (1995) and Burdett and Coles (1995) are the best. It must be underscored that with such easily structured strategy sets, our fundamental matching lemma (Lemma 9), as it were, is easily verified.

By Lemma 2, we consider value functions w as elements of the space $C[0,1]$ of continuous functions on $[0,1]$ with the sup norm: $\|w\| = \sup_{x\in[0,1]} |w(x)|$. Instead of matching sets $\mathcal{M}$, we work with the associated match indicator functions α, and the weak topology: $\alpha \in \mathcal{L}^1([0,1]^2)$, with norm $\|\alpha\|_{\mathcal{L}^1} = \int_0^1\int_0^1 |\alpha(x,y)|dx\,dy < \infty$.

Lemma 8. *Posit* A0 *and* A1-Sup *or* A1-Sub. *Any Borel measurable map* $w \mapsto \alpha(w)$ *from value functions to match indicator functions is continuous.*

This result is proven in the appendix. It is the only place where A1-Sup or A1-Sub is needed for existence. Either rules out an atom of zero surplus matches — for that would preclude a continuous map $w \mapsto \alpha(w)$. To apply our proof methodology elsewhere, this no knife-edge atoms property must be verified on an ad hoc basis.

Next, unmatched densities u are given a weak topology, $\|u\|_{\mathcal{L}^1} = \int_0^1 |u(x)|dx$.

Lemma 9 (Fundamental Matching Lemma). *The map* $\alpha \mapsto u(\alpha)$ *from match indicator functions to the unmatched density implied by the steady-state equation* (1) *for the quadratic search technology is both well-defined and continuous.*

The proof of this result is most nontrivial, but its underlying idea is not. Consider the steady-state equation (1), say $\Gamma(\alpha, u(\alpha)) = 0$. A fixed point argument shows the existence of a solution $u = u(\alpha)$ for each α, while ad hoc reasoning proves uniqueness. To prove continuity of $u(\alpha)$, it is natural to apply the Implicit Function Theorem (IFT). For this, Γ and its derivative Γ_u must be continuous, and Γ_u invertible. Here we happen upon the remarkable fact that with our quadratic search technology, Γ_u is in fact positive definite, and hence invertible. As it so happens, we can't apply the IFT, for Γ_u is never *simultaneously* invertible and continuous (later, in Remark 13). So we instead take inspiration from the IFT contraction-mapping proof, and modify it for our context with convex but not open neighborhoods of α in $\mathcal{L}^2([0,1]^2)$.

Proof of Proposition 5. Given values w in $C[0,1]$, we follow the Schauder Fixed Point Theorem program in §17.4 of Stokey and Lucas (1989).

• Step 1: The Best Response Value: Consider the map $T : C[0,1] \mapsto C[0,1]$ given by

$$Tw(x) \equiv \frac{\rho \int_0^1 \max\langle f(x,y) - w(y), w(x)\rangle \nu_w(dy)}{2(r+\delta) + \rho\bar{u}}$$

where ν_w is the unmatched measure implied by the value function w, as in Lemmas 8 and 9, and $\bar{u} \equiv \int_0^1 u(z)dz$ is the mass of unmatched agents. By Proposition 1, a fixed point of the mapping $Tw = w$ is a SE.[23]

[23]This conclusion is also true of two *much* more natural candidate best response mappings: $Tw(x)$ equal to the RHS of (10) or the implied value of $w(x)$ from solving (10). But neither mapping is closed on a small enough family $\mathcal{G}$.

To establish the existence of a fixed point in the mapping T, we need a nonempty, closed, bounded convex subset $\mathcal{G} \subseteq C[0,1]$ such that *(i)* $Tw \in \mathcal{G}$ when $w \in \mathcal{G}$; *(ii)* $T(\mathcal{G})$ is an equicontinuous family (which by the Arzela-Ascoli Theorem, yields the compactness of $T(\mathcal{G})$ needed by Schauder); *(iii)* T is continuous as an operator.

• STEP 2: THE FAMILY $\mathcal{G}$: Let $\mathcal{G}$ be the space of Lipschitz functions w on $[0,1]$ satisfying $0 \leq w(x) \leq \sup_y f(x,y)$ and $0 \leq w(x_2) - w(x_1) \leq (x_2 - x_1)\kappa$ for $x_2 > x_1$, where $\kappa \equiv \sup_{x,y} f_x(x,y)$, as in the proof of Lemma 2. This subset of $C[0,1]$ is clearly nonempty, closed, bounded, and convex. We can also immediately verify that if $w \in [0, \sup_y f(\cdot, y)]$, then so is Tw. Also, since $w(x_1) \leq w(x_2)$ when $x_1 < x_2$, and by assumption $f(x_1,y) - w(y) \leq f(x_2,y) - w(y)$ for any y, $\max\langle f(x,y) - w(y), w(x)\rangle$ is nondecreasing in x. So is its integral over y, establishing weak monotonicity of Tw. Now, since $\max\langle A,B\rangle - \max\langle C,D\rangle \leq \max\langle A-C, B-D\rangle$, for $x_2 > x_1$:

$$Tw(x_2) - Tw(x_1) \leq \frac{\rho \int_0^1 \max\langle f(x_2,y) - f(x_1,y), w(x_2) - w(x_1)\rangle \nu_w(dy)}{2\,(r+\delta) + \rho\bar{u}}$$

Then since $f(x_2,y) - f(x_1,y) = \int_{x_1}^{x_2} f_x(x,y)dx \leq (x_2 - x_1)\kappa$, and $w(x_2) - w(x_1) \leq (x_2 - x_1)\kappa$ by assumption, Tw shares the Lipschitz bound κ as well. This establishes that if $w \in \mathcal{G}$, then $Tw \in \mathcal{G}$.

• STEP 3: EQUICONTINUITY OF $T(\mathcal{G})$: We prove that for all $\varepsilon > 0$, there is an $\eta > 0$ so that for all $x \in [0,1]$ and $w \in \mathcal{G}$, if $|x - x'| < \eta$, then $|Tw(x) - Tw(x')| < \varepsilon$. To see this, first note that $|\max\langle A,B\rangle - \max\langle C,D\rangle| \leq |A-C| + |B-D|$ implies[24]

$$\begin{aligned}|Tw(x) - Tw(x')| &\leq \frac{\rho \int_0^1 \left(|f(x,y) - f(x',y)| + |w(x) - w(x')|\right) \nu_w(dy)}{2\,(r+\delta) + \rho\bar{u}} \\ &< \sup_y |f(x,y) - f(x',y)| + |w(x) - w(x')|\end{aligned}$$

Now, f is uniformly continuous, so we may choose x' close enough to x so that $\sup_y |f(x,y) - f(x',y)|$ is arbitrarily small. Also, $w'(x) \leq \sup_y f_x(x,y)$, implies that $|w(x) - w(x')| \leq \sup_y f_x(x,y)|x - x'|$. This proves that $T(\mathcal{G})$ is equicontinuous.

• STEP 4: CONTINUITY OF T: This is an appendicized algebraic exercise. □

[24]Here is a proof of this claim: Assume without loss of generality that $A \geq B$. Then if $C \geq D$, $|\max\langle A,B\rangle - \max\langle C,D\rangle| = |A - C|$, and the result is immediate. If $C < D$, we consider two more cases. First assume $A \geq D$. Then $|\max\langle A,B\rangle - \max\langle C,D\rangle| = A - D < A - C = |A - C|$. Second assume $A < D$. Then $|\max\langle A,B\rangle - \max\langle C,D\rangle| = D - A \leq D - B = |D - B|$.

8. CONCLUSION

⋆ **Robustness.** The model to investigate is the one that can be fully solved. As such, we have focused on the quadratic search technology, where we can rigorously establish existence of a search equilibrium.[25] Still, our core descriptive theory applies to any *anonymous* search technology: By this we mean that the rate that a searcher meets others is proportional to their presence in the unmatched pool. This rate may well depend, for instance, on the mass $\bar{u} \equiv \int_0^1 u(y)dy$ of unmatched agents, as it does in the *linear* technology (where $\rho = \bar{\rho}/\bar{u}$).

We have assumed for simplicity only one 'class' of agents in the economy, with many types within this class (and hence the production function is symmetric). Just as do Becker's, we fully expect that our descriptive results obtain in a model with two distinct classes of agents like workers/firms or men/women. But this would essentially double the notation, and is therefore left as a future robustness exercise.

⋆ **Summary.** This paper has pushed Becker's matching insights into a quite plausible search setting. Exploiting the implicit integral equation (10), we gave a thorough characterization of cross-sectional matching patterns in the equilibrium and constrained social optimum of a frictional matching model. In so doing, we have developed important new techniques for establishing existence, as well as extended the reach of the supermodularity research program into a wide new arena.

APPENDICES: OMITTED PROOFS

A. VALUE FUNCTION PROPERTIES

⋆ **Monotonicity: Proof of Lemma 1.** First we establish weak monotonicity. From equation (10) and inequality (11), for any $x_1 < x_2$,

$$w(x_2) - w(x_1) \geq \rho \int_{\mathcal{M}(x_1)} \frac{f(x_2,y) - f(x_1,y) - w(x_2) + w(x_1)}{2\,(r+\delta)}\, u(y)\, dy \qquad (24)$$

Solve for $w(x_2) - w(x_1)$. As $f(x_2,y) > f(x_1,y)$ for all $y > 0$, $w(x_2) - w(x_1) \geq 0$.

Now we prove strict monotonicity. If $w(x_2) = w(x_1)$ for some $x_1 < x_2$, then $\int_{\mathcal{M}(x_1)} u(y)dy = 0$ by (24). But then $w(x_2) = w(x_1) = 0$ by (10), whence $w(x) = 0$ at

[25] Our Fundamental Matching Lemma in particular exploits a positive definite operator, and thus need not pursue a *very difficult* and messy analysis using perturbation theory for linear operators.

all $x \in [0, x_2]$, by weak monotonicity. Inequality (11) with matching set $\mathcal{M} = [0, x_2]$ for x_2 yields a contradiction: $w(x_2) \geq \rho \int_0^{x_2} f(x_2, y)\, u(y)\, dy / \left(2\,(r+\delta)\right) > 0$. □

⋆ **Continuity: Proof of Lemma 2.** Define $\kappa \equiv \max_{x,y} f_x(x,y)$. This is well-defined since f_x is a continuous function on a compact set. Then for all $x_2 > x_1$, $f(x_2, y) - f(x_1, y) = \int_{x_1}^{x_2} f_z(z, y) dz \leq \kappa(x_2 - x_1)$. Also, by (10) and (11),

$$w(x_2) - w(x_1) \leq \frac{\rho \int_{\mathcal{M}(x_2)} (f(x_2,y) - f(x_1,y) - w(x_2) + w(x_1)) u(y) dy}{2(r+\delta)}$$

Solving for $w(x_2) - w(x_1)$, and using $f(x_2, y) - f(x_1, y) \leq \kappa(x_2 - x_1)$, we have

$$w(x_2) - w(x_1) \leq \frac{\rho \int_{\mathcal{M}(x_2)} (f(x_2,y) - f(x_1,y)) u(y) dy}{2(r+\delta) + \rho \int_{\mathcal{M}(x_2)} u(y) dy} \leq \frac{\kappa(x_2 - x_1) \cdot \rho \int_{\mathcal{M}(x_2)} u(y) dy}{2(r+\delta) + \rho \int_{\mathcal{M}(x_2)} u(y) dy}$$

So w is not only continuous, but is *Lipschitz*: $w(x_2) - w(x_1) < \kappa(x_2 - x_1)$. □

⋆ **Differentiability: Proof of Lemma 3.** Define a correspondence $\mathcal{M}^+ : [0,1] \rightrightarrows [0,1]$, the set of nonnegative surplus matches, $\mathcal{M}^+(x) \equiv \{y | s(x,y) \geq 0\}$.

Let $D(B, C)$ be the *Hausdorff* distance between sets B and C: namely, $D(B,C) = \inf\{d | \forall (b,c) \in (B,C), \exists (b',c') \in (C,B), \text{with } |b - b'| < d \text{ and } |c - c'| < d\}$. Call a correspondence M is *continuous* at x if for all $\varepsilon > 0$, there is a neighorhood around x, such that if x' is in that neighorhood, $D(M(x), M(x')) < \varepsilon$.

• Step 1: $\mathcal{M}^+$ is continuous at a.e. x. First, $\mathcal{M}^+$ is nonempty-valued: if $\mathcal{M}^+(x) = \varnothing$ for some x, then $s(x,y) < 0$ for all y, and so $w(x) = 0$ by (10); then $s(x,x) = f(x,x) - 2w(x) \geq 0$, a contradiction. We now prove that $\mathcal{M}^+$ is u.h.c. and compact-valued. Take any sequence $(x_n, y_n) \to (x, y)$, with $y_n \in \mathcal{M}^+(x_n)$ for all n. Then $s(x_n, y_n) \geq 0$ for all n, and so $s(x,y) \geq 0$ as well since s is continuous by A0 and Lemma 2. Thus, $y \in \mathcal{M}^+(x)$, establishing u.h.c. Fixing $x_n = x$ for all n, $\mathcal{M}^+(x)$ is closed too. Also, $\mathcal{M}^+(x) \subseteq [0,1]$ is bounded, whence $\mathcal{M}^+$ is compact-valued.

We call x an *ε-continuity point* of a correspondence M, and say x belongs to $\mathcal{C}_M(\varepsilon)$, if for all x' sufficiently close to x, $D(M(x), M(x')) < \varepsilon$. Since $\mathcal{M}^+$ is u.h.c. and compact valued, Theorem I-B-III-4 in Hildenbrand (1974) implies that (given our nonatomic density u) for all $\varepsilon > 0$, a.e. x is an ε-continuity point of $\mathcal{M}^+$. Then for all $n = 1, 2 \ldots$, a.e. $x \in \mathcal{C}_{\mathcal{M}^+}(1/n)$, and the countable intersection $\cap_n \mathcal{C}_{\mathcal{M}^+}(1/n)$ contains a.e. x as well. That is, for a.e. x, for all n, if x' is sufficiently close to x, $D(\mathcal{M}^+(x), \mathcal{M}^+(x')) < 1/n$. So $\mathcal{M}^+$ is a.e. continuous.

• Step 2: Decomposition of the Value Function's Slope. The sets

$\mathcal{M}^+(x)$ and $\mathcal{M}(x)$ differ only at points where $s(x,y) = 0$. Thus we can define a function that everywhere coincides with w:

$$w^+(x) \equiv \rho \int_{\mathcal{M}^+(x)} \frac{f(x,y) - w(x) - w(y)}{2(r+\delta)} u(y)dy \tag{25}$$

Take any sequence $x_n \to x$. Then for each n, use (25) to expand $w^+(x_n) - w^+(x)$. Add and subtract $\rho \int_{\mathcal{M}^+(x_n)} (f(x,y) - w(x) - w(y))\, u(y)dy/2(r+\delta)$ from the resulting expression, and divide through by $x_n - x$:

$$\begin{aligned} \frac{w^+(x_n) - w^+(x)}{x_n - x} = \frac{\rho}{2(r+\delta)} \Bigg[& \int_{\mathcal{M}^+(x_n) - \mathcal{M}^+(x)} \frac{f(x_n,y) - w(x_n) - w(y)}{x_n - x} u(y)dy \\ & + \int_{\mathcal{M}^+(x)} \frac{(f(x_n,y) - f(x,y)) - (w(x_n) - w(x))}{x_n - x} u(y)dy \Bigg] \end{aligned} \tag{26}$$

where $\int_{A-B} \equiv \int_{A \setminus B} - \int_{B \setminus A}$. We prove that the first term in brackets vanishes if $\mathcal{M}^+$ is continuous at x (Step 3), and that the remaining terms reduce to the desired expression for $w'(x)$ if $\mathcal{M}(x)$ and $\mathcal{M}^+(x)$ do not differ in a significant way (Step 4).

• Step 3: Surplus vanishes at the boundary of $\mathcal{M}^+(x)$ for a.e. x. By definition, $f(x_n,y) - w(x_n) - w(y) \geq 0 > f(x,y) - w(x) - w(y)$ if $y \in \mathcal{M}^+(x_n) \setminus \mathcal{M}^+(x)$. Since also $|f(x_n,y) - f(x,y)| < \kappa |x_n - x|$ and $|w(x_n) - w(x)| < \kappa |x_n - x|$ from the proof of Lemma 2, we have $2\kappa|x_n - x| > f(x_n,y) - w(x_n) - w(y) \geq 0$. Similarly, $-2\kappa|x_n - x| < f(x_n,y) - w(x_n) - w(y) \leq 0$ for $y \in \mathcal{M}^+(x) \setminus \mathcal{M}^+(x_n)$. Thus for all n,

$$\left| \int_{\mathcal{M}^+(x_n) - \mathcal{M}^+(x)} \frac{f(x_n,y) - w(x_n) - w(y)}{x_n - x} u(y)dy \right| < 2\kappa \left| \int_{\mathcal{M}^+(x_n) - \mathcal{M}^+(x)} u(y)dy \right|$$

If $\mathcal{M}^+$ is continuous at x, then $\lim_{x_n \to x} D(\mathcal{M}^+(x_n), \mathcal{M}^+(x)) = 0$. Since $\mathcal{M}^+(\cdot)$ is closed- and nonempty-valued, Theorem I-B-II-1 in Hildenbrand (1974) then yields $\liminf \mathcal{M}^+(x_n) = \mathcal{M}^+(x) = \limsup \mathcal{M}^+(x_n)$.[26] Next, the integral of the atomless density u over $\mathcal{M}^+(x_n) - \mathcal{M}^+(x)$ must vanish as well by the sequential continuity result I-D-I-(9) in Hildenbrand (1974).

• Step 4: Integrating over $\mathcal{M}(x)$ or $\mathcal{M}^+(x)$ is equivalent for a.e. x. At x for which the first term in the brackets in (26) vanishes,

$$\lim_{x_n \to x} \frac{w^+(x_n) - w^+(x)}{x_n - x} = \lim_{x_n \to x} \rho \int_{\mathcal{M}^+(x)} \frac{f(x_n,y) - f(x,y) - w(x_n) + w(x_n)}{2(r+\delta)(x_n - x)} u(y)dy$$

[26] As usual, $\liminf_k M_k \equiv \cup_m \cap_{\ell > m} M_\ell$ and $\limsup_k M_k \equiv \cap_m \cup_{\ell > m} M_\ell$.

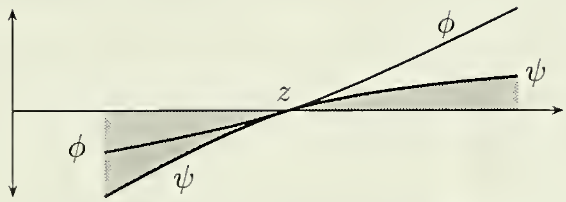


Figure 7: **Illustration of Claim 1.**

Replace w^+ by $w \equiv w^+$, solve for $\lim_{x_n \to x}(w(x_n) - w(x))/(x_n - x)$, and simplify:

$$\left(2(r+\delta) + \rho \int_{\mathcal{M}^+(x)} u(y)dy\right) w'(x) = \rho \int_{\mathcal{M}^+(x)} f_x(x,y)\, u(y)dy \tag{27}$$

We claim that for a.e. x, (27) simplifies to (12). This equivalence obtains, and can be shown through algebraic manipulation of (27), unless there is a positive measure of $y \in \mathcal{M}^+(x) \setminus \mathcal{M}(x)$ with $f_x(x,y) \neq w'(x)$ — or equivalently with $s_x(x,y) \neq 0$. Since $s(x,y) = 0$ whenever $y \in \mathcal{M}^+(x) \setminus \mathcal{M}(x)$, we will prove the claim by showing that for a.e. x: for a.e. y either $s(x,y) \neq 0$ or $s_x(x,y) = 0$.[27] Let S denote the set of pairs (x,y) with $s(x,y) = 0$ and $s_x(x,y) \neq 0$. First note that S is (Borel) measurable. For any fixed y, if $(x,y) \in S$, then any $x' \neq x$ close to x satisfies $s(x',y) \neq 0$. So every such x is isolated, and the set of such x is countable and has measure 0; therefore, for all y, for a.e. x, (x,y) is not in S. By Fubini's Theorem, S has measure 0, and so by Fubini again, we conclude that for a.e. x, for a.e. y, $(x,y) \notin S$. $\square$

B. DESCRIPTIVE THEORY

⋆ Single Crossing Property: Proof of Lemma 5.

Before proving Lemma 5, we establish the following useful input result:

Claim 1 (Density-Free Integral Comparison). *Let $\phi, \psi : [0,1] \mapsto \mathbb{R}$ both be increasing (decreasing) functions with a nondecreasing (nonincreasing) ratio ϕ/ψ on $[0,1]$. If for some density $u : [0,1] \mapsto \mathbb{R}_+$, and for some set $M \subseteq [0,1]$ of positive measure, $\int_M \psi(x)u(x)\,dx = 0$, then $\int_M \phi(x)u(x)\,dx \geq 0$.*

◊ Remark 12. Figure 7 depicts the graphical inspiration for the proof. As $\int_M \psi(x)u(x)\,dx = 0$, a threshold $z \in [0,1]$ exists such that $\psi(x) \gtrless 0$ for all $x \gtrless z$. But then also $\phi(x) \gtrless 0$ as $x \gtrless z$ if ϕ/ψ is monotonic nondecreasing (loosely, ϕ is

[27] We thank Randall Dougherty (Ohio State Univ. Math Dept.) for the clean proof below.

more convex than ψ). So if the positive and negative areas of ψ balance for some function $u > 0$, the positive area of ϕ must outweigh the negative area.

Proof. Assume WLOG that ϕ and ψ are increasing. If $M \neq [0,1]$, extend u to $[0,1]$ by defining $u(x) \equiv 0$ for all $x \notin M$. Define $I(x) \equiv \int_0^x \psi(x')u(x')\,dx'$. Clearly $I(0) = 0$, while $I(1) = 0$ by assumption. Then I is quasiconvex, as ψ is increasing, and so $I(x) \leq 0$ for all $x \in [0,1]$.

Substituting with I and the nondecreasing quotient $q(x) \equiv \phi(x)/\psi(x)$ yields

$$\int_M \phi(x)u(x)\,dx \equiv \int_0^1 q(x)I'(x)\,dx = q(1)I(1) - q(0)I(0) - \int_0^1 I(x)\,dq(x)$$

where we have integrated by parts. The first two terms in the last expression are zero, and the final term nonnegative, as $I(\cdot) \leq 0$, $dq(\cdot) \geq 0$. So $\int_M \phi(x)u(x)\,dx \geq 0$. □

Proof of Lemma 5. We just establish the supermodular case — i.e. under A1-Sup and A3-Sup. The submodular proof is analogous.

First, $\bar{z}$ is uniquely defined by A1-Sup, as the marginal product $f_y(\bar{z}, y_1)$ of a strictly supermodular production function is strictly increasing in $\bar{z}$.

Next, integrating the A3-Sup inequality over $x_1 \in [\bar{z}, x_2]$ and then $x_2 \in [x_1, \bar{z}]$, we discover that

$$f_{xy}(x, y_2)\,(f_y(x, y_1) - f_y(\bar{z}, y_1)) \geq f_{xy}(x, y_1)\,(f_y(x, y_2) - f_y(\bar{z}, y_2))$$

for all $y_2 \geq y_1$, and for all $x > \bar{z}$ *and* crucially also all $x < \bar{z}$. This is equivalent to

$$\frac{\partial}{\partial x}\frac{f_y(x, y_2) - f_y(\bar{z}, y_2)}{f_y(x, y_1) - f_y(\bar{z}, y_1)} \geq 0$$

Let $\phi(x) \equiv f_y(x, y_2) - f_y(\bar{z}, y_2)$ and $\psi(x) \equiv f_y(x, y_1) - f_y(\bar{z}, y_1)$. Then ϕ/ψ is nondecreasing, ϕ, ψ are increasing (by A1-Sup), and $\int_M \psi(x)u(x)dx = 0$ by (20); so Claim 1 implies (21). □

⋆ Characterization of Quasiconcavity: Proof of Lemma 6.

Part (a) [Strong Version]: If σ is not strongly quasiconcave, then $\sigma(y_2) \leq \min\langle\sigma(y_1), \sigma(y_3)\rangle$ and $\sigma(y_2) < \max\langle\sigma(y_1), \sigma(y_3)\rangle$ for some $0 \leq y_1 < y_2 < y_3 \leq 1$.

• STEP 1: INEQUALITY CASE. If $\sigma(y_2) < \min\langle\sigma(y_1), \sigma(y_3)\rangle$, then as σ is continuous and a.e. differentiable, there is $\hat{y}_2 \in (y_1, y_3)$ near enough y_2 that $\sigma(\hat{y}_2) <$

$\min\langle\sigma(y_1), \sigma(y_3)\rangle$ and $\sigma'(\hat{y}_2)$ exists. Q-1 then yields a contradiction: $\sigma'(\hat{y}_2) < 0$ given $\sigma(y_1) > \sigma(\hat{y}_2)$ and $y_1 < \hat{y}_2$, while $\sigma'(\hat{y}_2) > 0$ given $\sigma(y_3) > \sigma(\hat{y}_2)$ and $y_3 > \hat{y}_2$.

- Step 2: Equality Case. If $\sigma(y_2) = \min\langle\sigma(y_1), \sigma(y_3)\rangle$, assume WLOG $\sigma(y_2) = \sigma(y_1) < \sigma(y_3)$. Now, either (i) σ is constant on $[y_1, y_2]$, or (ii) $\sigma(\tilde{y}_1) > \sigma(y_2)$ for some $\tilde{y}_1 \in (y_1, y_2)$, or (iii) $\sigma(\tilde{y}_2) < \sigma(y_2)$ for some $\tilde{y}_2 \in (y_1, y_2)$. If (i), $\sigma'(x) = 0$ for all $x \in (y_1, y_2)$, but $\sigma(y_3) > \sigma(x)$ and $y_3 > x$, so $\sigma'(x) > 0$ by Q-1. If (ii) or (iii), replace y_1 by $\tilde{y}_1$ or y_2 by $\tilde{y}_2$ in the 'inequality case', yielding a contradiction.

Part (b) [Strict Version]: If σ is strongly but not strictly quasiconcave, we admit the additional possibility that $\sigma(y_1) = \sigma(y_2) = \sigma(y_3)$. Then either (i) σ is constant on $[y_1, y_3]$, or (ii) $\sigma(\tilde{y}) > \sigma(y_2)$ for some $\tilde{y} \in (y_1, y_3)$, or (iii) $\sigma(\tilde{y}_2) < \sigma(y_2)$ for some $\tilde{y} \in (y_1, y_3)$. If (i), $\sigma'(x) = 0$ for all $x \in (y_1, y_3)$, but $\sigma(y_1) = \sigma(x)$ and $y_1 < x$ implies $\sigma'(x) < 0$ by Q-2. If (ii) or (iii), replace y_3 or y_1 (as $\tilde{y} \gtrless y_2$) by $\tilde{y}$, or replace y_2 by $\tilde{y}_2$, in the 'equality case', yielding a contradiction. □

⋆ **Assortative Matching and Bounds: Proof of Lemma 7.**

- Step 1: Monotonic bound functions ⇒ assortative matching. As the two cases are analogous, we simply prove that strongly increasing bounds a, b imply positively assortative matching. Choose $x_1 < x_2$ and $y_1 < y_2$ with $y_1 \in \mathcal{M}(x_2)$ and $y_2 \in \mathcal{M}(x_1)$. Then we immediately have $b(x_1) \geq y_2$ and $a(x_2) \leq y_1$. But b strongly increasing implies $b(x_2) \geq b(x_1) \geq y_2$. If $b(x_2) > y_2$, then $a(x_2) \leq y_1 < y_2 < b(x_2)$, and so $y_2 \in \mathcal{M}(x_2)$, as claimed. Otherwise, $b(x_2) = b(x_1) = y_2$, which is only possible if $y_2 = 0$ or $y_2 = 1$. That $y_1 < y_2$ precludes $y_2 = 0$, so suppose $y_2 = 1$. Then $x_1 \in \mathcal{M}(y_2)$, $1 \in \mathcal{M}(y_2)$, and $\mathcal{M}(y_2)$ convex, implies $x_2 \in \mathcal{M}(y_2)$, or equivalently $y_2 \in \mathcal{M}(x_2)$. Similarly, a strongly increasing lower bound function a ensures that $y_1 \in \mathcal{M}(x_1)$, whence matching is positively assortative.

- Step 2: Assortative matching ⇒ monotonic bound functions. First, $\mathcal{M}(x) \neq \varnothing$ if $x > 0$. For by (10), $w(x) = 0$ if $\mathcal{M}(x) = \varnothing$, contrary to Lemma 1 unless $x = 0$. By Theorem 4, matching sets are convex, and thus bounds a, b well-defined.

To avoid tedious repetition among four similar cases, we simply prove that b is nondecreasing with positively assortative matching. If not, $b(x_1) > b(x_2)$ for some pair $x_1 < x_2$. Then there exists $y_1 \in \mathcal{M}(x_1)$, with $y_1 > b(x_2)$. Also choose any $y_2 \in \mathcal{M}(x_2)$. Then $y_1 > y_2$, and so $y_1 \in \mathcal{M}(x_2)$ by the definition of positively assortative matching. This contradicts the definition $b(x_2) = \sup\{y | y \in \mathcal{M}(x_2)\}$. □

⋆ **Ideal Partners and Quasiconcavity: Proof of Theorem 5(b).** We just prove the supermodular case; the submodular case in analogous.

• STEP 1: EVERYONE IS SOMEONE'S IDEAL PARTNER. Given A1-Sup, we have $p^*(1) = \{1\}$, since 1 has an increasing surplus function by step 1 of the proof of Theorem 1. Given $f(0, y) = k$, $s(0, y) = k - w(0) - w(y)$ is decreasing, and so $p^*(0) = \{0\}$. The desired result follows from the intermediate value theorem for correspondences, because $p^*(x)$ is nonempty, convex-valued, and u.h.c..

• STEP 2: SURPLUS $s(x, y)$ IS STRONGLY QUASICONCAVE IN y. Suppose not. Then there exists $0 \leq y_1 < y_2 < y_3 \leq 1$, with $s(x, y_2) \leq \min\langle s(x, y_1), s(x, y_3)\rangle$ and $s(x, y_2) < \max\langle s(x, y_1), s(x, y_3)\rangle$. WLOG, assume $s(x, y_3) > s(x, y_2)$. Since $s(x, y_1) \geq s(x, y_2)$, A1-Sup implies that $s(x', y_1) \leq s(x', y_2)$ only if $x' \geq x$, and thus $x' \in p^*(y_2)$ only if $x' \geq x$. Similarly, since $s(x, y_3) > s(x, y_2)$, $x' \in p^*(y_2)$ only if $x' < x$. Then $p^*(y_2) = \varnothing$, a contradiction. Finally, a strongly quasiconcave function (for p^* single-valued) with a unique maximizer is strictly quasiconcave. □

⋆ **Monotonic Ideal Partners: Proof of Theorem 6(b).** Observe that w is everywhere differentiable under A1,A2,A3. For the proof of Lemma 3 establishes this whenever $\mathcal{M}$ is continuous at x, which is true for all x by Theorem 3.

Since w is differentiable, so is s, and $y \in p^*(x) \cap (0, 1)$ only if $s_y(x, y) = 0$. Under A1-Sup, the surplus function of higher (lower) types is increasing (decreasing) at y: $s_y(x', y) \gtrless 0$ as $x' \gtrless x$. Since $s(x, \cdot)$ is quasiconcave by Theorem 1, $s_y(x', y') = 0$ implies $y' \gtrless y$ as $x' \gtrless x$: Higher (lower) types must choose higher (lower) partners to maximize their surplus function, or $y' \in p^*(x')$ implies $y' \gtrless y$ as $x' \gtrless x$. □

C. EQUILIBRIUM EXISTENCE

⋆ **Continuity of $w \mapsto \alpha(w)$: Proof of Lemma 8.**

• STEP 1: SURPLUS FUNCTION IS RARELY CONSTANT IN ONE VARIABLE. Define $Z_s(x) = \{y : s(x, y) = 0\}$ and $Z_s = \{(x, y) : s(x, y) = 0\}$, and let μ be Lebesgue measure on $[0, 1]$. We claim that under A1-Sup or A1-Sub, $\mu(Z_s(x)) = 0$ for a.e. x. Let $x \neq x'$ and $y \neq y'$, with $s(x, y) = s(x', y) = s(x, y') = 0$. Under A1-Sup or A1-Sub, $f(x, y) + f(x', y') \neq f(x', y) + f(x, y')$, from which $s(x', y') \neq 0$ follows. Thus, $Z_s(x) \cap Z_s(x')$ contains at most one point whenever $x \neq x'$.

Assume $\mu(Z_s(x)) > 0$ for an uncountable number of x.[28] Then for some k, there

[28]Toby Gifford (Math Dept., Washington U. in St. Louis) tightened the logic of this paragraph.

are infinitely many $\langle x_n \rangle$ with $\mu(Z_s(x_n)) > 1/k$, whereupon $\sum_{n=1}^{\infty} \mu(Z_s(x_n)) = \infty$. Since $x_{ij} = Z_s(x_i) \cap Z_s(x_j)$ contains at most one point, $N = \cup_{i,j=1}^{\infty} x_{ij}$ is countable, and so $\mu(N) = 0$. Also, $Z_s(x_i) \setminus N$ and $Z_s(x_j) \setminus N$ are disjoint for all $i \neq j$. Thus

$$1 \geq \mu\left(\bigcup_{n=1}^{\infty} Z_s(x_n) \setminus N\right) = \sum_{n=1}^{\infty} \mu(Z_s(x_n) \setminus N) = \sum_{n=1}^{\infty} \mu(Z_s(x_n)) = \infty$$

Given this contradiction, there are only countably many x with $\mu(Z_s(x)) > 0$. So $\mu(Z_s(x)) = 0$ for a.e. x, and by Fubini's theorem $(\mu \times \mu)(Z_s) = 0$.

• Step 2: Close Surplus Functions Rarely Differ in Sign. As $\eta \to 0$, the set $\Sigma_s(\eta) = \{(x, y) \text{ with } |s(x, y)| \in [0, \eta]\}$ shrinks monotonically to $\cap_{k=1}^{\infty} \Sigma_s(1/k) = s^{-1}(0) = Z_s$. By the countable intersection property of measures,

$$\lim_{\eta \to 0} (\mu \times \mu)(\Sigma_s(\eta)) = \lim_{k \to \infty} (\mu \times \mu)(\Sigma_s(1/k)) = (\mu \times \mu)(\cap_{k=1}^{\infty} \Sigma_s(1/k)) = (\mu \times \mu)(Z_s) = 0$$

Finally, let w_1 and w_2 be value functions, with $\|w_1(x) - w_2(x)\| \leq \eta/2$, and α_1, α_2 corresponding match indicator functions. If $s_1(x, y) \equiv f(x, y) - w_1(x) - w_1(y) > \eta$, then $s_2(x, y) \equiv f(x, y) - w_2(x) - w_2(y) > 0$, and so $\alpha_1(x, y) = \alpha_2(x, y) = 1$, while if $s_1(x, y) < -\eta$, then $s_2(x, y) < 0$, and $\alpha_1(x, y) = \alpha_2(x, y) = 0$. Consequently, $\{(x, y) | \alpha_1(x, y) \neq \alpha_2(x, y)\} \subseteq \Sigma_{s_1}(\eta)$, whose Lebesgue measure vanishes as $\eta \to 0$. This implies the desired continuity $\lim_{\|w_1 - w_2\| \to 0} \|\alpha_1 - \alpha_2\|_{\mathcal{L}^1} = 0$. □

⋆ **Fundamental Matching Lemma: Proof of Lemma 9.** Renormalize time WLOG so that $\delta = 1$. We first we prove that the map $\alpha \mapsto u(\alpha)$ is well-defined.

Claim 2. *For each match indicator function α, there exists a unique unmatched density $u(\alpha)$ that satisfies steady-state condition* (1).

• Step 1: Proof of Existence of $u(\alpha)$: By a suitable version of Alaoglu's Theorem, the set $\mathcal{F}$ of measurable functions u on $[0, 1]$ with $0 \leq u \leq \ell$ is weak-* compact, since ℓ is integrable. So consider the following mapping Ψ_α of $\mathcal{F}$ into itself:

$$\Psi_\alpha u(x) = \frac{\ell(x)}{1 + \rho \int \alpha(x, y) u(y) dy} \tag{28}$$

Then it is easy to check that Ψ_α is weak continuous in $u \in \mathcal{F}$ given $\alpha \in [0, 1]$. So by Schauder's Fixed Point Theorem,[29] there is a fixed point $u \in \mathcal{F}$, i.e. (1) holds.

[29] Here, we are speaking of its general version, such as Theorem 5.1.3 in Istratescu (1981).

• Step 2: Proof of Uniqueness of $u(\alpha)$: We now pursue a different tack. Suppose that there are two essentially distinct solutions u_1, u_2 to (28) in $\mathcal{F}$. Then

$$\frac{u_1(x)}{u_2(x)} = \frac{1/\rho + \int \alpha(x,y)u_2(y)dy}{1/\rho + \int \alpha(x,y)u_1(y)dy} \tag{29}$$

Let θ_1 (resp. θ_2) be the essential supremum[30] of $u_1(x)/u_2(x)$ (resp. $u_2(x)/u_1(x)$) over $x \in [0,1]$, and assume WLOG that θ_1 is weakly greater. Then $\theta_1 > 1$ since u_1 and u_2 differ on a set of positive measure. Furthermore, $\theta_2 \geq 1$ since (28) easily precludes $u_1(x) \geq u_2(x)$ for all x, with strict inequality for a positive measure of x.

Posit $\theta_1 > \theta_2$. Integrate $u_2 \leq \theta_2 u_1$ to get $\int \alpha(x,y)u_2(y)dy \leq \theta_2 \int \alpha(x,y)u_1(y)dy$, and so RHS $\leq \theta_2$ in (29). Since $\theta_1 > u_1(x)/u_2(x) > \theta_2$ for a positive measure of x, this violates (29). Yet if $\theta_1 = \theta_2$, then $\theta_2 > 1$, and so RHS $< \theta_2$ in (29), as $1/\rho > 0$. Once again, LHS $>$ RHS in (29) for a positive measure of x, a contradiction. $\square$

Claim 3. *The map $\alpha \mapsto u(\alpha)$ is continuous.*

• Step 1: A Steady-State Operator. For fixed α_0, let u_0 be the unique unmatched density from Claim 2: i.e. $\ell(x) - u_0(x) = \rho u_0(x) \int_0^1 \alpha_0(x,y)u_0(y)dy$. Defining $G(\alpha,u)(x) = u(x)(1 + \rho \int \alpha(x,y)u(y)dy) - \ell(x)$, we see that u solves (1) given α iff $G(\alpha,u) = 0$. The derivative $G_u(\alpha,u)$ is a linear operator on $\mathcal{L}^2[0,1]$:[31]

$$\begin{aligned} G_u(\alpha,u)(g) &= \lim_{t\to 0} \left(G(\alpha,u+tg) - G(\alpha,u)\right)/t = g\left(1 + \rho \int_0^1 \alpha u\right) + \rho u \int_0^1 \alpha g \\ &= g + \rho\left(g \int_0^1 \alpha u + u \int_0^1 \alpha g\right) \end{aligned}$$

Write $G_u(\alpha,u) \equiv I + \rho H$. Since α is symmetric ($\alpha(x,y) = \alpha(x,y)$), the operator H is self-adjoint, and also neatly positive-definite on the space $\mathcal{L}^2([0,1], 1/u)$ of functions square-integrable with respect to density $1/u$, because[32]

$$\begin{aligned} 2(g,Hg) &= 2\int_0^1 g(x)Hg(x)/u(x)dx \\ &= 2\int_0^1\int_0^1 (g(x)^2u(y)/u(x) + g(x)g(y))\alpha(x,y)dxdy \\ &= \int_0^1\int_0^1 [g(x)^2u(y)/u(x) + 2g(x)g(y) + g(y)^2u(x)/u(y)]\,\alpha(x,y)dxdy \\ &= \int_0^1\int_0^1 [g(x)\sqrt{u(y)/u(x)} + g(y)\sqrt{u(x)/u(y)}]^2\alpha(x,y)dxdy \geq 0 \end{aligned}$$

Since u is *boundedly* positive (given $\rho < \infty$, $\ell \geq \underline{\ell} > 0$) and finite ($u \leq \bar{\ell} < \infty$),

[30] That is, θ_1 is the least number such that $u_1(x)/u_2(x) > \theta_1$ only on a set of measure 0.
[31] To save on space, hereafter suppress x and (x,y) arguments whenever there is no ambiguity.
[32] We thank Robert Israel (UBC Math Dept.) for remarking on this fact.

$\mathcal{L}^2([0,1], 1/u) = \mathcal{L}^2[0,1]$. So H is a self-adjoint and positive definite linear operator in $\mathcal{L}^2[0,1]$, and hence its spectrum is real and nonnegative, and the spectrum of $G_u(\alpha_0, u_0)$ real and positive, and thus excludes 0: $G_u(\alpha_0, u_0)$ is invertible in $\mathcal{L}^2[0,1]$.

• STEP 2: A NEW STEADY-STATE OPERATOR.[33] For simplicity, embed ρ into α, now with range $[0, \rho]$. Let $\Gamma(\alpha, \cdot) = I - \Psi_\alpha$, i.e. $\Gamma(\alpha, u) = u - \ell/(1 + \int \alpha u)$. Then $G(\alpha, u) = (1 + \int \alpha u)\Gamma(\alpha, u)$ and by the product rule $G_u(\alpha, u)(h) = \Gamma(\alpha, u) \int \alpha h + (1 + \alpha u)\Gamma_u(\alpha, u)(h)$. So $G_u(\alpha_0, u_0)(h) = (1 + \int \alpha_0 u_0)\Gamma_u(\alpha_0, u_0)(h) = (\ell/u_0)\Gamma_u(\alpha_0, u_0)(h)$ given $\Gamma(\alpha_0, u_0) = 0$. Since (ℓ/u_0) is bounded away from 0 and ∞, the linear operator $A = \Gamma_u(\alpha_0, u_0)$ is invertible in $\mathcal{L}^2[0,1]$, since $G_u(\alpha_0, u_0)$ is.

◊ REMARK 13. Let the operator Γ map $\mathcal{X} \times \mathcal{Y} \mapsto \mathcal{Z}$, where $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are the ambient Banach spaces of α, u, and the range; thus, the derivative Γ_u maps $\mathcal{X} \times \mathcal{Y} \times \mathcal{Y} \mapsto \mathcal{Z}$. To apply the IFT, Γ and Γ_u must be continuous at (α_0, u_0). The form $\Gamma_u = I + \#$ means that $\mathcal{Z} = \mathcal{Y}$ if Γ_u^{-1} exists. But Lemma 8 demands a weak norm on $\mathcal{X}$ (not a supremum norm), and so continuity of Γ_u requires that a weaker norm on $\mathcal{Z}$ than $\mathcal{Y}$ (for instance, to apply the Holder inequality.) For instance, if $\mathcal{X} = \mathcal{Y} = \mathcal{L}^2$, then continuity demands that $\mathcal{Z} = \mathcal{L}^1$, while invertibility requires $\mathcal{Z} = \mathcal{L}^2$.

• STEP 3: SETTING UP A TRIPLE INDUCTION. Pick α with range $[0, \rho]$ and $\mathcal{L}^1$-close to α_0. Since $|\alpha - \alpha_0| \leq \rho$, $||\alpha - \alpha_0||_{\mathcal{L}^2} \leq \sqrt{\rho}||\alpha - \alpha_0||_{\mathcal{L}^1}$ is small too. Recursively define u_{k+1} by

$$A(u_{k+1} - u_0) = A(u_k - u_0) - \Gamma(\alpha, u_k) \tag{30}$$

for $k = 0, 1, \ldots$. Since A is invertible, $u_{k+1} \in \mathcal{L}^2[0,1]$ is well-defined. As an induction step, assume (*i*) $||u_k - u_0||_{\mathcal{L}^2} < \varepsilon$, (*ii*) $||u_k - u_{k-1}||_{\mathcal{L}^2} < d^k$, and (*iii*) $u_k > -\tau$, where $\tau, \varepsilon > 0$ and $0 < d < 1$ are determined later. To get the induction going, observe that $u_1 - u_0 = -A^{-1}\Gamma(\alpha, u_0)$ by (30). Since Γ is continuous in α, and $\Gamma(\alpha_0, u_0) = 0$, $||u_1 - u_0||_{\mathcal{L}^2}$ is vanishing with $||\alpha - \alpha_0||_{\mathcal{L}^2}$. We verify part (*iii*) with $k = 1$ later.

• STEP 4: PART 1 OF INDUCTION. The first equality below adds $0 = \Gamma(\alpha_0, u_0)$ to (30). In a recurring theme, the next parameterizes the vector from (α_0, u_0) to (α, u_k) by $(\alpha_s, u_s) = (\alpha_0, u_0) + s(\alpha - \alpha_0, u_k - u_0)$, for $0 \leq s \leq 1$, and applies (§):

[33] We thank Sheldon Chang (M.I.T. Math Dept.) for outlining the rest of the proof that follows. But any errors in executing this proof program are ours alone.

$\zeta(1) - \zeta(0) = \int_0^1 \zeta'(s)ds$, i.e. the Fundamental Theorem of Calculus.

$$\begin{aligned} A(u_{k+1} - u_0) &= A(u_k - u_0) - [\Gamma(\alpha, u_k) - \Gamma(\alpha_0, u_0)] \\ &= \Gamma_u(\alpha_0, u_0)(u_k - u_0) - \textstyle\int_0^1 [\Gamma_u(\alpha_s, u_s)(u_k - u_0) + \Gamma_\alpha(\alpha_s, u_{ks})(\alpha - \alpha_0)]ds \\ &= \textstyle\int_0^1 (\Gamma_u(\alpha_0, u_0) - \Gamma_u(\alpha_s, u_{ks}))\,(u_k - u_0) + \Gamma_\alpha(\alpha_s, u_{ks})(\alpha - \alpha_0)ds \end{aligned}$$

If we first apply $(X \pm Y)^2 \leq 2X^2 + 2Y^2$, and then the Cauchy-Schwartz inequality $(\int(\cdot)dx)^2 \leq \int(\cdot)^2 dx$, we see that $\|A(u_{k+1} - u_0)\|^2_{\mathcal{L}^2} = \int (A(u_{k+1} - u_0))^2 dx$ is at most:

$$2\textstyle\iint \left[(\Gamma_u(\alpha_0, u_0) - \Gamma_u(\alpha_s, u_{ks}))\,(u_k - u_0)\right]^2 dx + 2\int \left[\Gamma_\alpha(\alpha_s, u_{ks})(\alpha - \alpha_0)\right]^2 dx\, ds \tag{31}$$

On the first term in (31): apply $(X \pm Y)^2 \leq 2X^2 + 2Y^2$, and then parameterize the vector from (α_0, u_0) to (α_s, u_{ks}) by $\alpha_{\xi s} = \alpha_0 + \xi_s(\alpha_s - \alpha_0)$ and $u_{k\xi s} = u_0 + \xi_s(u_{ks} - u_0)$ to apply (§); finally use $(\int(\cdot)d\xi_s)^2 \leq \int(\cdot)^2 d\xi_s$ for $0 \leq \xi_s \leq 1$. In the second term in (31), substitute $\Gamma_\alpha(\alpha, u)(b) = \ell \int bu/(1 + \int \alpha u)^2$. Thus, (31) is bounded above by

$$\begin{aligned} &4\textstyle\iiint [(\Gamma_{u\alpha}(\alpha_{\xi s}, u_{k\xi s})(\alpha_s - \alpha_0)(u_k - u_0)]^2 d\xi_s\, dx\, ds \\ &+4\textstyle\iiint [\Gamma_{uu}(\alpha_{\xi s}, u_{k\xi s})(u_{ks} - u_0)(u_k - u_0)]^2 d\xi_s dx\, ds + 2\displaystyle\iint \left(\frac{\ell \int(\alpha - \alpha_0)u_{ks}}{(1 + \int \alpha_s u_{ks})^2}\right)^2 dx\, ds \end{aligned}$$

Differentiate $\Gamma_u(\alpha, u)(g) = g + \ell \int \alpha g/(1 + \int \alpha u)^2$ w.r.t. both α and u to get $\Gamma_{u\alpha}(\alpha, u)(b, g) = 2\ell \int \alpha g \int bu/(1 + \int \alpha u)^3 - \ell \int bg/(1 + \int \alpha u)^2$ and $\Gamma_{uu}(\alpha, u)(g, h) = -2\ell \int \alpha g \int \alpha h/(1 + \int \alpha u)^3$. Substituting into the previous expression yields

$$\begin{aligned} &4\iiint \left(2\frac{\ell \int \alpha_{\xi s}(u_k - u_0) \int (\alpha_s - \alpha_0)u_{k\xi s}}{(1 + \int \alpha_{\xi s} u_{k\xi s})^3} - \frac{\ell \int (\alpha_s - \alpha_0)(u_k - u_0)}{(1 + \int \alpha_{\xi s} u_{k\xi s})^2}\right)^2 dx\, d\xi_s ds \qquad (32) \\ &+4\iiint \left(2\frac{\ell \int \alpha_{\xi s}(u_{ks} - u_0) \int \alpha_{\xi s}(u_k - u_0)}{(1 + \int \alpha_{\xi s} u_{k\xi s})^3}\right)^2 dx\, d\xi_s ds + 2\iint \left(\frac{\ell \int (\alpha - \alpha_0)u_{ks}}{(1 + \int \alpha_s u_{ks})^2}\right)^2 dx\, ds \end{aligned}$$

Apply $(X \pm Y)^2 \leq 2X^2 + 2Y^2$ on the first term in (32), and replace all inner products $(\int \alpha u)^2 \leq \int \alpha^2 \int u^2$ by the Cauchy-Schwartz inequality. To bound the remaining terms, recall that $\ell < \bar{\ell} < \infty$. Also, as $(\alpha_s, u_{ks}) = (1 - s)(\alpha_0, u_0) + (\alpha, u_k)$ and $(\alpha_{\xi s}, u_{ks}) = (1 - \xi_s)(\alpha_0, u_0) + \xi_s(\alpha_s, u_{ks})$, all the α terms satisfy $0 \leq \alpha \leq \rho$; moreover, $|\alpha_{\xi s} - \alpha_0| \leq |\alpha_s - \alpha_0| \leq |\alpha - \alpha_0|$, and $|u_{k\xi s} - u_0| \leq |u_{ks} - u_0| \leq |u_k - u_0|$.

To remove the s and ξ_s integrals from (32), all denominators and a term $\int u^2_{k\xi s} dy$ must be bounded. Now, since $\|u_k - u_0\|^2_{\mathcal{L}^2} \leq \varepsilon < \infty$ by induction assumption, and $\|u_0\|_{\mathcal{L}^2} < \infty$, we have $\max\langle \int u^2_{k\xi s} dy, \int u^2_{k\xi s} dy \rangle \leq B < \infty$. Because $u_k > -\tau$

and $|\alpha| \leq \rho$, $\min\langle \int \alpha_s u_{ks}, \int \alpha_s u_{k\xi s}\rangle > -\rho\tau$, where $\tau > 0$ is small enough that $1 - \rho\tau > 1/2$. Finally, we find a new upper bound for $||A(u_{k+1} - u_0)||^2_{\mathcal{L}^2}$:

$$\left(\frac{32\bar{\ell}^2}{(1-\gamma)^6} + \frac{8\bar{\ell}^2}{(1-\gamma)^4}\right) B \iint (\alpha - \alpha_0)^2 dydx \int (u_k - u_0)^2 dx$$
$$+ \quad \frac{16\bar{\ell}^2}{(1-\gamma)^6}\left(\int (u_k - u_0)^2 dx\right)^2 + \frac{2\bar{\ell}^2 B}{(1-\gamma)^4} \iint (\alpha - \alpha_0)^2 dydx$$

Taking square roots, there must exist constants $C, J < \infty$, such that

$$\begin{aligned} ||A(u_{k+1} - u_0)||_{\mathcal{L}^2} &\leq J\left(||\alpha - \alpha_0||_{\mathcal{L}^2}||u_k - u_0||_{\mathcal{L}^2} + ||u_k - u_0||^2_{\mathcal{L}^2}\right) + C||\alpha - \alpha_0||_{\mathcal{L}^2} \\ &= J\left(||\alpha - \alpha_0||_{\mathcal{L}^2} + ||u_k - u_0||_{\mathcal{L}^2}\right)||u_k - u_0||_{\mathcal{L}^2} + C||\alpha - \alpha_0||_{\mathcal{L}^2} \end{aligned}$$

Let $0 < d < 1$, and choose $||\alpha - \alpha_0||_{\mathcal{L}^2} < d/2$ and $||u_1 - u_0||_{\mathcal{L}^2} < d(1-d)/2$. Then $||u_k - u_0||_{\mathcal{L}^2} \leq ||u_1 - u_0||_{\mathcal{L}^2}(1 + d + \cdots d^{k-1}) < d/2$ by the triangle inequality and the other induction assumption, i.e. $||\alpha - \alpha_0||_{\mathcal{L}^2} + ||u_k - u_0||_{\mathcal{L}^2} < d$. Since the inverse operator A^{-1} has a finite norm, $|A^{-1}| \cdot C||\alpha - \alpha_0||_{\mathcal{L}^2} < \varepsilon/2$ and $|A^{-1}|Jd < 1/2$ for $||\alpha - \alpha_0||_{\mathcal{L}^2}$ and d small enough; therefore, $||u_k - u_0||_{\mathcal{L}^2} < \varepsilon$ implies $||u_{k+1} - u_0||_{\mathcal{L}^2} < \varepsilon$.

• Step 5: Part 2 of Induction. Next, ($\bigstar_k$): $A(u_{k+1} - u_k) = -\Gamma(\alpha, u_k)$ by (30). Subtract $(\bigstar_k) - (\bigstar_{k-1})$, and apply (§) with $u_{kt} = u_{k-1} + t(u_k - u_{k-1})$, to get

$$\begin{aligned} A(u_{k+1} - u_k) &= A(u_k - u_{k-1}) - [\Gamma(\alpha, u_k) - \Gamma(\alpha, u_{k-1})] \\ &= \Gamma_u(\alpha_0, u_0)(u_k - u_{k-1}) - \int_0^1 \Gamma_u(\alpha, u_{kt})(u_k - u_{k-1})dt \\ &= \int_0^1 \left(\Gamma_u(\alpha_0, u_0) - \Gamma_u(\alpha, u_{kt})\right)(u_k - u_{k-1})dt \end{aligned}$$

Proceeding just as in part 1, except omitting the final Γ_α term in each expression, similarly yields a constant $K < \infty$ for which:

$$||A(u_{k+1} - u_k)||_{\mathcal{L}^2} \leq K\left(||\alpha - \alpha_0||_{\mathcal{L}^2} + ||u_k - u_{k-1}||_{\mathcal{L}^2}\right)||u_k - u_{k-1}||_{\mathcal{L}^2}$$

To prove $||u_{k+1} - u_k||_{\mathcal{L}^2} < d^{k+1}$, we use our induction assumption $||u_k - u_{k-1}||_{\mathcal{L}^2} < d^k$, as well as $|A^{-1}| \cdot K(||\alpha - \alpha_0||_{\mathcal{L}^2} + d^k) \leq d$. This latter inequality holds for $||\alpha - \alpha_0||_{\mathcal{L}^2}$ and d small enough if $k > 1$. For $k = 1$, we want $||u_2 - u_1||_{\mathcal{L}^2}/||u_1 - u_0||_{\mathcal{L}^2} \leq |A^{-1}| \cdot K\left(||\alpha - \alpha_0||_{\mathcal{L}^2} + ||u_1 - u_0||_{\mathcal{L}^2}\right)$ as small as we like.

• Step 6: Part 3 of Induction. For the last ingredient, expand ($\bigstar_k$) into

$$(u_{k+1} - u_k) + \frac{\ell \int \alpha_0(u_{k+1} - u_k)}{(1 + \int \alpha_0 u_0)^2} = -u_k + \frac{\ell}{1 + \int \alpha u_k}$$

Substituting the steady-state identity $u_0 = \ell(1 + \int \alpha_0 u_0)$ yields

$$u_{k+1} = \frac{\ell}{1 + \int \alpha u_k} - (u_0^2/\ell) \int \alpha_0 (u_{k+1} - u_0)$$

Because $u_0 \geq 0$ for $k = 1$, and more generally since $1 - \rho\tau > 1/2$, the first term is positive. Thus, $u_{k+1} \geq -(u_0^2/\ell)\|u_{k+1} - u_0\| \geq -(u_0^2/\ell)\varepsilon \geq -\tau$ for ε small enough.

• Step 7: Using the The Cauchy Limit. The induction establishes that $\langle u_k \rangle$ is a Cauchy sequence. Since $\mathcal{L}^2[0,1]$ is a Banach space, it converges to some limit u_∞ that is close to u_0. As per usual, u_∞ satisfies the recursion (30), and thus $0 = \Gamma(\alpha, u_\infty)$. So for all $\varepsilon > 0$, if u is the essentially unique feasible unmatched density solving (1) given α, then $\|u - u_0\|_{\mathcal{L}^2} < \varepsilon$ whenever $\|\alpha - \alpha_0\|_{\mathcal{L}^2}$ is small enough. Since $\|u - u_0\|_{\mathcal{L}^1} \leq \|u - u_0\|_{\mathcal{L}^2}$ by the Cauchy-Schwartz inequality, this proves the desired continuity of $\alpha \mapsto u(\alpha)$.

⋆ **Continuity of the Operator T.** We prove that for all $\varepsilon > 0$ there exists a $\eta > 0$ such that for all $w_1, w_2 \in \mathcal{G}$, if $\sup_z |w_1(z) - w_2(z)| < \eta$, then $\sup_z |Tw_1(z) - Tw_2(z)| < \varepsilon$. For any x, by the triangle inequality, the difference $|Tw_1(x) - Tw_2(x)| \leq |D_1| + |D_2| + |D_3|$, where $(2(r+\delta) + \rho\bar{u}_1)D_1$ equals

$$\begin{aligned} & \rho\left(\int_{\mathcal{M}_1(x)} (f(x,y) - w_1(y))\,\nu_{w_1}(dy) + \int_{\mathcal{M}_1^c(x)} w_1(x)\nu_{w_1}(dy)\right) \\ & -\rho\left(\int_{\mathcal{M}_1(x)} (f(x,y) - w_2(y))\,\nu_{w_1}(dy) + \int_{\mathcal{M}_1^c(x)} w_2(x)\nu_{w_1}(dy)\right) \\ = \;& \int_{\mathcal{M}_1(x)} (w_2(y) - w_1(y))\,\nu_{w_1}(dy) - \int_{\mathcal{M}_1^c(x)} (w_2(x) - w_1(x))\,\nu_{w_1}(dy) \end{aligned}$$

so that $|D_1| < \sup_z |w_1(z) - w_2(z)| < \eta$; and where

$$\begin{aligned} D_2 \;=\; & \frac{\rho\left(\int_{\mathcal{M}_1(x)} (f(x,y) - w_2(y))\,\nu_{w_1}(dy) + \int_{\mathcal{M}_1^c(x)} w_2(x)\nu_{w_1}(dy)\right)}{2(r+\delta) + \rho\bar{u}_1} \\ & - \frac{\rho\left(\int_{\mathcal{M}_1(x)} (f(x,y) - w_2(y))\,\nu_{w_2}(dy) + \int_{\mathcal{M}_1^c(x)} w_2(x)\nu_{w_2}(dy)\right)}{2(r+\delta) + \rho\bar{u}_2} \end{aligned}$$

which is uniformly small by the definition of the weak topology, as all integrands are continuous, and all the arguments of the integrals are uniformly bounded and continuous (by Lemmas 8 and 9); and finally, where $(2(r+\delta) + \rho\bar{u}_2)D_3$ equals

$$\begin{aligned} & \rho\left(\int_{\mathcal{M}_1(x)} (f(x,y) - w_2(y))\,\nu_{w_2}(dy) + \int_{\mathcal{M}_1^c(x)} w_2(x)\nu_{w_2}(dy)\right) \\ & -\rho\left(\int_{\mathcal{M}_2(x)} (f(x,y) - w_2(y))\,\nu_{w_2}(dy) + \int_{\mathcal{M}_2^c(x)} w_2(x)\nu_{w_2}(dy)\right) \end{aligned}$$

This term measures the loss x suffers with value function w_2 from using the matching set $\mathcal{M}_1(x)$ rather than $\mathcal{M}_2(x)$. As $|w_1 - w_2| < \eta$, we have $|f(x,y) - w_2(x) - w_2(y)| < 2\eta$ for all $y \in (\mathcal{M}_1(x) \cup \mathcal{M}_2(x)) \setminus (\mathcal{M}_1(x) \cap \mathcal{M}_2(x))$, the symmetric set difference between $\mathcal{M}_1(x)$ and $\mathcal{M}_2(x)$. So the third term is strictly less than 2η.

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