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CONTRACTS AND THE DIVISION OF LABOR

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Contracts and the Division of Labor*

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Abstract

We present a tractable framework for the analysis of the relationship between contract incompleteness, technological complementarities and the division of labor. In the model economy, a firm decides the division of labor and contracts with its worker-suppliers on a subset of activities they have to perform. Worker-suppliers choose their investment levels in the remaining activities anticipating the ex post bargaining equilibrium. We show that greater contract incompleteness reduces both the division of labor and the equilibrium level of productivity given the division of labor. The impact of contract incompleteness is greater when the tasks performed by different workers are more complementary. We also discuss the effect of imperfect credit markets on the division of labor and productivity, and study the choice between the employment relationship versus an organizational form relying on outside contracting. Finally, we derive the implications of our framework for productivity differences and comparative advantage across countries.

JEL Classification: D2, J2, L2, O3

Keywords: incomplete contracts, division of labor, productivity

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1 Introduction

This paper develops a simple framework for the analysis of the relationship between contracting and the division of labor. While a major strand of the economics literature, following Adam Smith (1776) and Allyn Young (1928), emphasizes the importance of the extent of the market as the major constraint on the division of labor, the recent literature has recognized the role of the costs of designing and managing organizations consisting of many individuals each performing different tasks. Naturally, greater division of labor will necessitate both more complicated (contractual or implicit) relationships and greater coordination between the firm and its workers. It is thus difficult to imagine a society reaching the division of labor we observe today without the important advances in accounting and use of information within organizations.[1] This perspective has been emphasized, for example, by Becker and Murphy (1992), who suggest that the division of labor is not limited by the extent of the market, but by "coordination costs" inside the firm. Nevertheless, they do not provide a microeconomic model of how and why these costs vary across societies and industries. This is one of our objectives in this paper.

Our approach builds on Grossman and Hart's (1986) and Hart and Moore's (1990) seminal papers, which develop an incomplete-contracting model of the internal organization of the firm. The key insight is to think of the ownership structure and the size of the firm as chosen partly in order to encourage investments by managers, suppliers and workers. Hart and Moore show how this approach can be used to determine the boundaries of the firm, but do not investigate how the degree of contract incompleteness and the nature of the production technology affect the division of labor. We extend Hart and Moore's framework by considering an environment with partially incomplete contracts and varying degrees of technological complementarities (i.e., complementarities between different production tasks). Using this framework, we investigate how the degree of contracting incompleteness (for example, determined by the contracting institutions of the society) and technological complementarity impact on the equilibrium degree of division of labor.

In our baseline model, a firm decides the range of tasks that will be performed and the division of labor, recognizing that a greater division leads to greater productivity. However, together with the greater division of labor comes the need to contract with multiple workers (suppliers) that are undertaking relationship-specific investments.[2] A fraction of the activities that workers have to undertake are contractible, while the rest, as in the work by Grossman-Hart-Moore, are nonverifiable and noncontractible. The fraction of contractible activities is our measure of the quality of

[1] See Pollard (1965) on the advances in accounting and other information-collection technologies during the 19th century, Michaels (2004) and Yates (1989) on the role of clerks and advances in information processing during the early 20th century in the reorganization of production, and Chandler (1977) on the rise of the modern corporation and the relationship between this process and increasing informational demands of organizations. A different perspective, which we do not pursue in this paper, would build on Marglin (1974) and emphasize the benefits of division of labor in monitoring workers.

[2] Throughout, we focus on the relationship between a firm and its "worker-suppliers," and simplify the terminology by referring to these as workers. Since our model is sufficiently abstract, it can also be applied to the relationship between a downstream firm and multiple upstream suppliers.

contracting institutions.[3] While workers are contractually obliged to perform their duties in the contractible activities, they are free to choose their investments and can withhold their services in noncontractible activities. and this leads to an ex post multilateral bargaining problem. As in Hart and Moore (1990), we use the Shapley value to determine the division of ex post surplus between the firm and the workers. We derive an explicit solution for the Shapley value of the firm and the workers, which enables us to provide a closed-form characterization of the equilibrium.

Workers' expected surplus in this bargaining process determines their willingness to invest in the noncontractible activities. Since they are not the full residual claimants of the productivity gains derived from their investments, they tend to underinvest. Our first major result is that greater contracting incompleteness reduces worker investments, and via this channel, limits the profitability and the equilibrium extent of the division of labor. Secondly, a greater degree of technological complementarity limits the division of labor, since it further discourages investments. The reason for this is interesting; although greater technological complementarity increases equilibrium payments to each worker, it also makes their payments less sensitive to their noncontractible investments, and reduces investments. These lower investments make a greater division of labor less profitable. We also show that better contracting institutions are more important for firms with technologies featuring greater complementarity. These results also have natural implications for productivity. For example, better contracting institutions lead to greater productivity both because the equilibrium division of labor is higher and because, conditional on the division of labor, workers invest more and are more productive.

We also characterize the equilibrium when workers cannot make ex ante transfers to the firm to compensate for the ex post rents they receive in the bargaining process. While the major comparative static results are the same as in the case with ex ante transfers, the relationship between the equilibrium division of labor and productivity is more complex. In fact, equilibrium productivity can now be higher than both in the case with incomplete contracts and ex ante transfers and in the case with complete contracts. The reason is that, in the absence of ex ante transfers, the firm uses investments in contractible tasks as an instrument for extracting surplus from the workers, potentially inducing overinvestment. Even though overinvestment is inefficient, it increases the observed level of productivity. Consequently, without ex ante transfers, greater division of labor may be associated with lower productivity.

We also use this framework to discuss when the employment relationship, where various production tasks are performed by employees of the firm rather than by outside contractors, may emerge as an equilibrium outcome. The key observation is that agents have greater bargaining power in their relationship with the firm when they are outside contractors, which is similar to non-integrated suppliers having greater bargaining power in Grossman-Hart-Moore's theory of the firm. We show

[3] Maskin and Tirole (1999) question whether the presence of nonverifiable actions and unforeseen contingencies necessarily lead to incomplete contracts. Their argument is not central to our analysis because it assumes the presence of "strong" contacting institutions (which, for example, allow contracts to specify sophisticated mechanisms), while in our model a fraction of activities may be noncontractible not because of "technological" reasons but because of weak contracting institutions. In fact, we interpret the fraction of noncontractible activities as a measure of the quality of contracting institutions.

that with ex ante transfers, the equilibrium always involves outside contractors, because this organizational form increases agents' bargaining power and investment.[4] The employment relationship can arise as an equilibrium arrangement in the absence of ex ante transfers, however. The reason is that agents may obtain too much rents as outside contractors, and the employment relationship enables the firm to reduce these rents. Interestingly, in this case, the employment relationship also makes the division of labor more profitable, and leads to greater division of labor. Our framework therefore leads to a theory of employment relationship complementary to that by Marglin (1974). While in his theory, given the employment relationship, the division of labor is used to extract more rents from workers, in ours, the employment relationship itself enables the firm to obtain greater rents and thus encourages a greater division of labor.[5]

A succinct way of expressing the results in our paper is that in the presence of noncontractibilities and ex post bargaining, the (equilibrium) profit function of a firm is modified to

$$AZF(N) - C(N)$$

where N represents the division of labor, $C(N)$ its cost and A is a measure of aggregate demand or the scale of the market. Naturally, $F(N)$ is an increasing function. Here Z is a constant and depends on the degree of contract incompleteness and technological complementarity. Key comparative static results follow from the response of Z to parameter changes. For example, Z is decreasing in the degree of contract incompleteness and technological complementarity.

This simple form of the equilibrium profit function enables us to embed our model in a general equilibrium framework. The interesting result here is that, because of the equilibrium resource constraint, an improvement in contracting institutions does not increase the division of labor in all sectors (firms). Instead, the division of labor increases in sectors that are more "contract-dependent" and declines in others. In our model, sectors with greater technological complementarity are more contract-dependent and experience an increase in the division of labor in general equilibrium, while those with a high degree of substitutability experience a decline in the division of labor. In the context of an open economy, the same interactions lead to endogenous comparative advantage as a function of contracting institutions. In particular, among countries with identical technologies, those with better contracting institutions will specialize in sectors with greater complementarities among inputs.

While our main focus is on the division of labor, the results in this paper are also relevant for the literature on cross-country differences in (total factor) productivity. Although there is a broad consensus that differences in total factor productivity ("efficiency") are a major part of the large

[4]This result is driven by the fact that the firm does not undertake any investments other than its technology choice.

[5]Our framework can also be used to discuss the trade-off between vertical integration and non-integration with workers interpreted as suppliers. In this context, Aghion and Tirole (1994) and Legros and Newman (2000) show how inefficient vertical integration can arise in the presence of imperfect credit markets. The interesting result in our framework, different from these papers, is that the employment relationship (or vertical integration) can be "efficient" because of its implications for the division of labor.

cross country differences in living standards (e.g., Klenow and Rodriguez, 1997, Hall and Jones, 1999, Caselli, 2004), there is no agreement on the sources of these productivity differences. Our results suggest that differences in contracting institutions can be an important source of productivity differences. Societies facing the same technological possibilities, but functioning under different contracting institutions, will choose technologies corresponding to different degrees of division of labor and will naturally have different levels of productivity. Furthermore, because of the differences in workers' investment levels, these societies will also experience different levels of productivity even conditional on the division of labor. A detailed investigation of whether this could be an important source of (total factor) productivity differences across countries is an interesting area for future research.

In addition to the papers mentioned above, our work relates to two large literatures. The first investigates the relationship between the division of labor and the extent of the market. It includes the work by Yang and Borland (1991), as well as the product-variety models of endogenous growth, such as Romer (1990) and Grossman and Helpman (1991), that can be interpreted as linking the level of (equilibrium) demand in the economy to the division of labor.

The second literature derives implications for the division of labor from the theory of the firm. Important work here includes the Grossman-Hart-Moore papers discussed above as well as their predecessors, Klein, Crawford and Alchian (1978) and Williamson (1975, 1985), which emphasize incomplete contracts and hold-up problems. The two papers by Stole and Zwiebel (1996a,b) are most closely related to our work. Stole and Zwiebel consider a relationship between a firm and a number of workers where wages are determined by ex post bargaining according to the Shapley value. They show how the firm may overemploy in order to reduce the bargaining power of the workers and discuss the implications of this framework for a number of organizational design issues.[6] Stole and Zwiebel's framework does not incorporate relationship-specific investments, which is at the center of our approach, and they do not discuss the division of labor or the effects of contract incompleteness and complementarities on the equilibrium division of labor.

Another important strand of the literature stems from the seminal work by Holmstrom and Milgrom (1991, 1994), which emphasizes issues of multi-tasking, job design and complementarity of tasks as important determinants of the internal organization of the firm. These considerations naturally lead to a theory of the division of labor, where the firm may want to separate substitutable tasks in order to prevent severe multitask distortions. Yet another approach, perhaps closer to the spirit of the contribution by Becker and Murphy (1992), explicitly models the costs of communication or information processing within the firm (e.g., Sah and Stiglitz, 1986, Geanakoplos and Milgrom, 1991, Radner 1992, 1993, Radner and Van Zandt, 1992, Bolton and Dewatripont, 1994, and Garicano, 2000). Another strand of the literature, e.g., Calvo and Wellisz (1978) and Qian (1994), links the size of the firm to other informational problems, such as monitoring or selection. Finally, a number of recent papers, most notably Levchenko (2003), Costinot (2004), Nunn (2004)

[6] Bakos and Brynjolfsson (1993) develop a related model to study the effects of improvements in information technology on the number of suppliers that firms choose to contract with.

and Antràs (2005), apply the insights of these literatures to generate endogenous comparative advantage from differences in contracting institutions. Among these papers, Costinot (2004) is most closely related since he also focuses on the division of labor (though using a more reduced-form approach).

The rest of the paper is organized as follows. Section 2 introduces the basic partial equilibrium framework and characterizes the equilibrium with complete contracts. Section 3 characterizes the equilibrium with incomplete contracts and contains the main comparative static results. Section 4 investigates how the results change when there are no ex ante transfers. Section 5 uses our framework to study the choice between employment relationship and outside contracting. Section 6 embeds the partial equilibrium model in a general equilibrium framework and discusses how differences in contracting institutions generate endogenous comparative advantage. Section 7 concludes, while Appendix A contains the main proofs.

2 Model

In the next four sections, we present our model of the firm and derive the main partial equilibrium results, and later turn to general equilibrium analysis.

2.1 Technology

Consider a firm, facing a demand curve for its product of the form $q = Ap^{-1/(1-\beta)}$, where q denotes quantity and p denotes the price, and $\beta \in (0,1)$. $A > 0$ corresponds to the level of aggregate demand in the economy and is taken as given by the firm. This form of demand can be derived from a constant elasticity of substitution preference structure for the consumers over many products (see Section 6). It implies that the firm in question generates a revenue of

$$R(q) = A^{1-\beta} q^{\beta} \tag{1}$$

from producing a quantity q.

The firm decides the technology of production which specifies a continuum range of tasks $[0, N]$ that will be performed, and the number of worker-suppliers that will perform these tasks, M. For now, we do not distinguish whether these worker-suppliers are employees of the firm or outside contractors, and simply refer to them as workers (see, however, Section 5). Each worker will perform $\varepsilon = N/M$ tasks, and we denote the services of tasks performed by worker j by $X(j)$ for $j = 1, 2, ...M$. The production technology, which can only be operated by the firm, is:

$$q = F\left(\left[X(j)\right]_{j=1}^{M} \mid M, N\right) \equiv N^{\kappa+1-1/\alpha} \left[\sum_{j=1}^{M} \frac{N}{M} X(j)^{\alpha}\right]^{1/\alpha}, \quad 0 < \alpha < 1, \ \kappa > 0. \tag{2}$$

A number of features of this production function are worth noting. First, since $\alpha > 0$, the elasticity of substitution between tasks, $1/(1-\alpha)$, is always greater than one and determines the degree of complementarity of the technology. Second, we follow Benassy (1998) in introducing the term $N^{\kappa+1-1/\alpha}$, which allows us to disentangle the elasticity of substitution between tasks from the elasticity of output with respect to the level of technology. To see this, suppose that $X(j) = X$ for all j. In this case, when technology is N, $q = N^{\kappa+1}X$; so greater N translates into greater productivity, and the parameter κ determines the relationship between N and productivity.[7]

In the text, we simplify the analysis further by considering the case in which $M \to \infty$, and work with the production function

$$q = F^N\left([X(j)]_{j=0}^{N}\right) \equiv N^{\kappa+1-1/\alpha}\left[\int_0^N X(j)^\alpha \, dj\right]^{1/\alpha}, \tag{3}$$

where the only "technology" choice of the firm is N.[8] In Appendix B, we show that with complete contracts the equilibrium will feature $M \to \infty$ as long as there are "diseconomies of scope" in assigning tasks to workers, and we also provide conditions under which $M \to \infty$ with incomplete contracts. To simplify the discussion further, it is useful to suppose, from now on, that there is a continuum of workers and that every task is performed by a separate worker. Under these circumstances N stands for the measure of tasks as well as the measure of workers, and we refer to N as the division of labor of the technology.[9]

Each worker assigned to a task needs to undertake relationship-specific investments in a unit measure of (symmetric) activities, each entailing a marginal cost c_x. The services of the task in question is then a Cobb-Douglas aggregate,

$$X(j) = \exp\left[\int_0^1 \ln x(i,j)\, di\right], \tag{4}$$

where $x(i,j)$ denotes the level of investment in activity i performed by the worker. We adopt this formulation to allow for a tractable parameterization of contract incompleteness, whereby a subset of the investments necessary for production will be nonverifiable and thus noncontractible. We also assume that the cost of investments is non-pecuniary (e.g., "effort" cost, see equation (38) in Section 6).

Finally, we denote the costs of division of labor by $C(N)$, which could include potential (exogenous) coordination costs, costs of investment in technology, and upfront payments to cover the outside options of the workers (when these are positive, see Section 6). We assume

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[7] In contrast, with the standard specification of the CES production function, without the term $N^{\kappa+1-1/\alpha}$ in front (i.e., $\kappa = 1/\alpha - 1$), total output would be $q = N^{1/\alpha}X$, and these two elasticities are governed by the same parameter, α.

[8] Equation (3) is obtained as the limit of equation (2) when $M \to \infty$. With a slight abuse of notation, we use j to index workers in both the discrete and continuum cases.

[9] In other words, we suppose that the measure of workers performing the tasks in any interval $[a, b]$ is $b - a$. Note also that, alternatively, we could have started with the production function (3) and the assumption that every task is performed by a separate worker.

Assumption 1

(i) For all $N > 0$, $C(N)$ is twice continuously differentiable, with $C'(N) > 0$ and $C''(N) \geq 0$.

(ii) For all $N > 0$, $NC''(N)/C'(N) > [\beta(\kappa+1) - 1]/(1-\beta)$.

The first part of this assumption is standard. The second part is necessary for second-order conditions and to ensure interior solutions.

2.2 Payoffs

The firm and the workers maximize expected returns.[10] Suppose that the payment to worker j consists of two parts: an ex ante payment, $\tau(j)$, and a payment after the $x(i,j)$s have been delivered, $s(j)$. Then, the payoff to worker j is

$$\pi_x(j) = \mathbb{E}\left[\tau(j) + s(j) - \int_0^1 c_x x(i,j)\, di\right],$$

where $\mathbb{E}$ denotes the expectations operator, which applies if there is uncertainty regarding the ex post payment, $s(j)$, which will be the case in the bargaining game. As stated above, we assume that if a worker does not participate in the relationship with the firm, then she has an outside option of zero. Similarly, the payoff to the firm is

$$\pi = \mathbb{E}\left[A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}\left[\int_0^N \left(\exp\left(\int_0^1 \ln x(i,j)\, di\right)\right)^\alpha dj\right]^{\beta/\alpha} - \int_0^N [\tau(j) + s(j)]\, dj - C(N)\right],$$

which follows by substituting (3) and (4) into (1), and by subtracting the payments to workers and the costs of division of labor.

2.3 Division of Labor with Complete Contracts

As a benchmark, consider the case of complete-contracts (the "first best" from the viewpoint of the firm), which corresponds to the case in which the firm has full control over all investments and pays each worker her outside option. In analogy to our treatment of the division of labor with incomplete contracts below, consider a game form where the firm chooses a technology with division of labor N and makes a contract offer, $\{x(i,j)\}_{i\in[0,1],j\in[0,N]}, \{s(j), \tau(j)\}_{j\in[0,N]}$, to a set of potential workers. If a worker accepts this contract, she is obliged to supply $\{x(i,j)\}_{i\in[0,1]}$ as stipulated in her contract. The subgame perfect equilibrium of this game with complete contracts

[10] We use risk neutrality only in the calculation of Shapley values to express the payoff of a player as the expected (average) value of his or her marginal contributions to all possible coalition structures.

corresponds to the solution to the following maximization problem:

$$\max_{N,\{x(i,j)\}_{i,j},\{s(j),\tau(j)\}_j} A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}\left[\int_0^N\left(\exp\left(\int_0^1 \ln x(i,j)\,di\right)\right)^{\alpha}dj\right]^{\beta/\alpha} - \int_0^N [\tau(j)+s(j)]\,dj - C(N)$$

subject to

$$s(j)+\tau(j)-c_x\int_0^1 x(i,j)\,di \geq 0 \text{ for all } j\in[0,N].$$

Since the firm has no reason to provide rents to the workers in this case, the constraints bind, and substituting for these to eliminate $\tau(j)$ and $s(j)$, which are perfect substitutes in this case, the program simplifies to

$$\max_{N,\{x(i,j)\}_{i,j}} A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}\left[\int_0^N\left(\exp\left(\int_0^1 \ln x(i,j)\,di\right)\right)^{\alpha}dj\right]^{\beta/\alpha} - c_x\int_0^N\int_0^1 x(i,j)\,didj - C(N).$$

Put differently, the firm's profits equal revenue minus the cost of investments of workers and the cost of division of labor. Moreover, since this is a convex problem in the investment levels $x(i,j)$ and these investments are all equally costly, the firm will choose the same investment level x for all activities in all tasks and we can simplify the program to

$$\max_{N,x} A^{1-\beta}N^{\beta(\kappa+1)}x^{\beta} - c_x Nx - C(N). \quad (5)$$

The first-order conditions of this problem are:[11]

$$\beta(1+\kappa)A^{1-\beta}x^{\beta}N^{\beta(\kappa+1)-1} - c_x x - C'(N) = 0,$$

$$\beta A^{1-\beta}x^{\beta-1}N^{\beta(\kappa+1)} - c_x N = 0,$$

and yield a unique solution (N^*, x^*), which is implicitly defined by:

$$(N^*)^{\frac{\beta(\kappa+1)-1}{1-\beta}} A\kappa\beta^{1/(1-\beta)}c_x^{-\beta/(1-\beta)} = C'(N^*), \quad (6)$$

$$x^* = \frac{C'(N^*)}{\kappa c_x}. \quad (7)$$

Equations (6) and (7) can be solved recursively. Given Assumption 1, equation (6) yields a unique solution for N^*, which, together with (7), yields a unique solution for x^*.

It is also useful to derive the productivity implications of the equilibrium with complete contracts. When all the investment levels are identical and equal to x, output equals $q = N^{\kappa+1}x$. Since N workers are taking part in the production process, a natural definition of productivity is

[11] Appendix A shows that the second-order conditions are satisfied under Assumption 1.

output divided by N, i.e., $P = N^{\kappa}x$. In the case of complete contracts this productivity level is

$$P^{*} = (N^{*})^{\kappa} x^{*}. \tag{8}$$

We will compare this productivity level to the level of productivity with incomplete contracts.[12]

The following proposition summarizes the above discussion and highlights the effects of some key parameters on the division of labor, the investment level of workers and the associated productivity levels (see Appendix A for details):

Proposition 1 Suppose that Assumption 1 holds. Then program (5) has a unique solution $x^{*} > 0$ and $N^{*} > 0$ characterized by (6) and (7), and an associated productivity level $P^{*} > 0$ given by (8). Furthermore, this solution satisfies:

$$\frac{\partial N^{*}}{\partial A} > 0,\ \frac{\partial x^{*}}{\partial A} \geq 0,\ \frac{\partial P^{*}}{\partial A} > 0,\ \frac{\partial N^{*}}{\partial \alpha} = \frac{\partial x^{*}}{\partial \alpha} = \frac{\partial P^{*}}{\partial \alpha} = 0.$$

In the case with complete contracts, consistent with Adam Smith's emphasis, the size of the market as parameterized by the demand level A affects the division of labor. Our model further illustrates how a larger market size tends to be associated with greater investments by workers and higher productivity.[13]

The other noteworthy implication of this proposition is that under complete contracts the division of labor and productivity are independent of the elasticity of substitution between tasks, $1/(1-\alpha)$. This will contrast with the equilibrium under incomplete contracts, where the elasticity of substitution influences bargaining.

3 Division of Labor with Incomplete Contracts

3.1 Contractual Structure

We now consider a world with incomplete contracting, where each worker has control over the investment levels in the various activities. In the text, we work with the production function (3), which equates N with the division of labor. Appendix B shows that the firm will indeed prefer to

[12] Notice that if $x(i,j)$s were adequately measured, the index of productivity would be $P_2 = N^{\kappa}$. Our assumption that the $x(i,j)$s correspond to "effort" naturally maps into an environment where these investments are not well measured. Moreover, P appears to be much closer to what is in fact measured in practice, especially using aggregate data, than P_2.

Note also that P measures "price-adjusted" productivity, because it is in terms of output rather than revenue. An alternative concept, often used in practice because of data limitations, would be revenue divided by N, i.e., $P_1 = A^{1-\beta}N^{\beta(\kappa+1)-1}x^{\beta}$. Throughout, none of our conclusions depend on which of these two measures of productivity are used.

[13] It is also straightforward to show that the division of labor, investment levels, and productivity all decline with the marginal cost of investment, c_x. In addition, as long as parameter values are such that $N^{*} > 1$, both N^{*} and x^{*} (and thus P^{*}) are increasing in the elasticity of output with respect to the division of labor, κ. We do not emphasize these comparative statics to save space.

have the maximal division of labor given N as long as $\beta < \alpha$.[14] Therefore, the results here may be interpreted either as treating the production function (3) as a primitive, or as derived from the production function (2) with these additional requirements satisfied.[15]

We model the imperfection of the contracting institutions by assuming that there exists $\mu \in [0,1]$ such that, for every task j, investments in activities $i \in [0,\mu]$ are observable and verifiable. Consequently, complete contracts can be written for the investment levels for activities $i \in [0,\mu]$. A contract stipulates investment levels $x(i,j)$ for $i \leq \mu$, which can be enforced by a court.[16] For the remaining $1-\mu$ activities, ex ante contracts are not possible (either because the $x(i,j)$s are nonverifiable as in Grossman and Hart, 1986, Hart and Moore, 1990, or because of the weakness of contracting institutions in this society).[17] Under these circumstances workers choose the investment on noncontractible activities in anticipation of the ex post distribution of revenue between the firm and the workers. We follow the incomplete contracts literature and assume that the ex post distribution of revenue is governed by multilateral bargaining, and we adopt the Shapley value as the solution concept for the multilateral bargaining game (more on this below). The threat point of each worker in bargaining is not to provide the services for the noncontractible activities. To start with, we assume that all parties have access to perfect credit markets, so ex ante transfers from workers to the firm, or the other way around, are possible.

The timing of events is as follows:

- The firm adopts a technology (a division of labor) with N tasks, and offers a contract, denoted by $\{[x_c(i,j)]_{i=0}^{\mu}, \tau(j)\}$, for every task $j \in [0,N]$, where the $x_c(i,j)$s are the investment levels in contractible activities $i \in [0,\mu]$, which have to be performed by worker j, and $\tau(j)$ is an upfront transfer to worker j.

- The firm chooses N workers from a pool of applicants, one for each task j.

- Workers $j \in [0,N]$ simultaneously choose investment levels $x(i,j)$ for all $i \in [0,1]$. In the contractible activities $i \in [0,\mu]$ they invest $x(i,j) = x_c(i,j)$ for every j.

- The workers and the firm bargain over the division of revenue.

- Output is produced and sold, and the revenue $R(q)$ is distributed according to the bargaining agreement.

We will characterize a symmetric subgame perfect equilibrium (SSPE for short) of this game, where the bargaining outcomes in all subgames are determined by Shapley values.

[14] Provided that a mild regularity condition on diseconomies of scope is satisfied. This regularity condition will be satisfied, for example, when there are no economies of scope, or when there are sufficient diseconomies of scope.

[15] In the general equilibrium version of the model developed in Section 6, the elasticity of substitution between varieties of final goods will be $1/(1-\beta)$. The inequality $\beta < \alpha$ thus corresponds to the case in which tasks are more substitutable than final goods.

[16] For example, the court can impose a large penalty if the employee does not deliver the contractually-specified level of services. We do not introduce these penalties explicitly to simplify the exposition.

[17] For similar reasons, we assume that revenue-sharing agreements between the firm and the workers are not enforceable.

3.2 Equilibrium Formulation

The along-the-equilibrium path behavior in the SSPE can be represented as a tuple $\left\{\tilde{N}, \tilde{x}_c, \tilde{x}_n, \tilde{\tau}\right\}$ in which $\tilde{N}$ represents the division of labor, $\tilde{x}_c$ is the investment in contractible activities, $\tilde{x}_n$ is investment in noncontractible activities and $\tilde{\tau}$ is the upfront transfer to every worker. That is, for every $j \in \left[0, \tilde{N}\right]$ the upfront payment is $\tau(j) = \tilde{\tau}$, and the investment levels are $x(i,j) = \tilde{x}_c$ for $i \in [0, \mu]$ and $x(i,j) = \tilde{x}_n$ for $i \in (\mu, 1]$. For now, we assume that there are perfect credit markets (i.e., workers have deep pockets), so $\tilde{\tau} < 0$ is possible.

The SSPE will be characterized by backward induction. Consider the last stage of the game. If N is the level of technology, x_c is investment in contractible activities and x_n is investment in noncontractible activities, then (1)-(4) imply that the available revenue is $R = A^{1-\beta}\left(N^{\kappa+1} x_c^{\mu} x_n^{1-\mu}\right)^{\beta}$. This revenue is distributed among the workers and the firm according to their Shapley values. Importantly, at this point of the game N, x_c and x_n are given. We discuss the computation of the Shapley values of this game in the next section and in Appendix A. For now, consider a situation in which all workers have invested x_c in all contractible activities and x_n in noncontractible activities. Let $s_x(N, x_c, x_n)$ denote the Shapley value of a representative worker and $s_q(N, x_c, x_n)$ denote the Shapley value of the firm. These values naturally exhaust the entire revenue, i.e., $s_q(N, x_c, x_n) + N s_x(N, x_c, x_n) = A^{1-\beta}\left(N^{\kappa+1} x_c^{\mu} x_n^{1-\mu}\right)^{\beta}$.

Now move to one stage before the bargaining takes place. At that stage all workers know the technology choice N and have to invest x_c in the contractible activities, as they have committed to do so in their contract. They are, however, free to choose the investment levels in the noncontractible activities. To characterize the symmetric equilibrium, let $\bar{s}_x[N, x_c, x_n(-j), x_n(j)]$ be the Shapley value of worker j, when the technology choice is N, x_c is the required investment in the contractible activities, $x_n(-j)$ is the investment level in the noncontractible activities by all the workers other than j (these are all the same, since we are looking for a symmetric equilibrium), and $x_n(j)$ is the investment level in every noncontractible activity by worker j.[18] The function $s_x(N, x_c, x_n)$, which was introduced above, is the along-the-equilibrium-path value of $\bar{s}_x[N, x_c, x_n(-j), x_n(j)]$, i.e., $s_x(N, x_c, x_n) = \bar{s}_x(N, x_c, x_n, x_n)$. We develop an explicit formula for this function in the next subsection.

At this stage of the game, worker j chooses $x_n(j)$ to maximize $\bar{s}_x[N, x_c, x_n(-j), x_n(j)]$. As implied by the concept of equilibrium, the firm expects the workers to choose their best response investment levels in noncontractible activities:

$$x_n \in \arg\max_{x_n(j)} \bar{s}_x[N, x_c, x_n, x_n(j)] - (1-\mu) c_x x_n(j). \tag{9}$$

This is the workers' "incentive compatibility constraint," and also imposes the symmetry require-

[18] More generally, we would need to consider a distribution of investment levels, $\{x_n(i,j)\}_{i \in [0,1]}$ for worker j, where some of the activities may receive more investment than others. It is straightforward to show that the best deviation for a worker is to choose the same level of investment in all noncontractible activities, and to save on notation, we restrict attention to such deviations.

ment (in that x_n is the best response to x_n).[19]

Next move one step backward in the game tree, to the stage in which the firm chooses N workers from a pool of applicants. This pool is empty if workers expect to receive less than their outside option. Therefore, for production to take place, the final good producer has to offer a contract that yields a net reward at least as large as the workers' outside option, that is:

$$\bar{s}_x (N, x_c, x_n, x_n) + \tau \geq \mu c_x x_c + (1 - \mu) c_x x_n \text{ for } x_n \text{ that satisfies (9).} \tag{10}$$

This is the workers' "participation constraint": given N and the contract (x_c, τ), every worker $j \in [0, N]$ expects her Shapley value plus the upfront payment to cover the cost of her investments in contractible and noncontractible activities (when both she and other workers invest x_n, as in (9), in every noncontractible activity). If the firm were to choose a division of labor N and a contract (x_c, τ) that did not satisfy this participation constraint, it would not be able to hire any workers.

The firm chooses N and designs a contract (x_c, τ) to maximize its profits. These profits consist of its Shapley value minus the upfront payments (or plus the transfers from the workers) and minus the cost of N. Consequently, it solves the following problem:

$$\max_{N, x_c, x_n, \tau} s_q (N, x_c, x_n) - N\tau - C(N) \text{ subject to (9) and (10).} \tag{11}$$

The presence of perfect credit markets implies that the participation constraint (10) can never be slack (otherwise, the firm would reduce τ). We can now solve τ from this constraint, substitute the solution into the firm's objective function, and arrive at the following simpler maximization problem:

$$\max_{N, x_c, x_n} s_q (N, x_c, x_n) + N [\bar{s}_x (N, x_c, x_n, x_n) - \mu c_x x_c - (1 - \mu) c_x x_n] - C(N) \text{ subject to (9).} \tag{12}$$

The SSPE tuple $\left\{\tilde{N}, \tilde{x}_c, \tilde{x}_n\right\}$ solves this problem, and the corresponding upfront payment, $\tilde{\tau}$, satisfies

$$\tilde{\tau} = \mu c_x \tilde{x}_c + (1 - \mu) c_x \tilde{x}_n - \bar{s}_x \left(\tilde{N}, \tilde{x}_c, \tilde{x}_n, \tilde{x}_n\right). \tag{13}$$

With a slight abuse of terminology, we refer to $\left\{\tilde{N}, \tilde{x}_c, \tilde{x}_n\right\}$ as the SSPE.

3.3 Bargaining

Before we provide a more detailed characterization of the equilibrium, we derive the Shapley values in this game (see Shapley, 1953, or Osborne and Rubinstein, 1994). Since we have a continuum of players and the original Shapley value applies to games with a finite number of players, we derive our solution as follows (see the proof of Lemma 1 in Appendix A for details). Divide the interval

[19] In general, (9) may have multiple fixed points, and there can be multiple equilibria given the choice of N and x_c by the firm. Our production function (3) and the assumption that $\alpha > 0$ ensure uniqueness (see equation (18) below).

$[0, N]$ into M equally spaced subintervals with all the tasks in each one of length N/M performed by a single worker (as in the discussion of equation (2) above). Then solve the Shapley values of the bargaining game between the $M+1$ players; M workers and the firm. The key insight that simplifies the solution is that a coalition that does not include the firm, which is the only agent that can operate the technology, has a value of zero and a coalition that includes the firm has a value that equals the revenue from the final output it can produce. The solution that we use below is the limit of the solution of this game with a finite number of players when $M \to \infty$ (see Aumann and Shapley, 1974, and Stole and Zwiebel, 1996b, for a similar derivation of Shapley values in a game with a continuum of players).

According to the Shapley value, in a bargaining game with a finite number of players each player's payoff is the average of her contribution to all coalitions that consist of players ordered below her in all feasible permutations. More explicitly, in a game with $M+1$ players, let $g = \{g(0), g(1), ..., g(M)\}$ be a permutation of $0, 1, 2, ..., M$, where player 0 is the firm and players $1, 2, ..., M$ are the workers, and let $z_g^j = \{j' \mid g(j) > g(j')\}$ be the set of players ordered below j in the permutation g. We denote by G the set of feasible permutations and by $v : G \to \mathbb{R}$ the value of the coalition consisting of any subset of the $M+1$ players. Then the Shapley value of player j is

$$s_j = \frac{1}{(M+1)!} \sum_{g \in G} \left[v\left(z_g^j \cup j\right) - v\left(z_g^j\right)\right].$$

The following result is proved in Appendix A:

Lemma 1 Suppose that $M \to \infty$, and worker j invests $x_n(j)$ in her noncontractible activities, all the other workers invest $x_n(-j)$ in their noncontractible activities, every worker invests x_c in her contractible activities, and the division of labor is N. Then the Shapley value of worker j is

$$\bar{s}_x[N, x_c, x_n(-j), x_n(j)] = (1-\gamma) A^{1-\beta} \left[\frac{x_n(j)}{x_n(-j)}\right]^{(1-\mu)\alpha} x_c^{\beta\mu} x_n(-j)^{\beta(1-\mu)} N^{\beta(\kappa+1)-1}, \tag{14}$$

where

$$\gamma \equiv \frac{\alpha}{\alpha+\beta}. \tag{15}$$

A number of features of (14) are worth noting. First, if all workers invest equally in all the noncontractible activities, $x_n(j) = x_n(-j) = x_n$, then

$$s_x(N, x_c, x_n) = \bar{s}_x(N, x_c, x_n, x_n) = (1-\gamma) A^{1-\beta} x_c^{\beta\mu} x_n^{\beta(1-\mu)} N^{\beta(\kappa+1)-1}. \tag{16}$$

In this event, the joint Shapley value of the workers, $N s_x(N, x_c, x_n)$, is equal to a fraction $1-\gamma$ of the revenue (which is equal to $A^{1-\beta} x_c^{\beta\mu} x_n^{\beta(1-\mu)} N^{\beta(\kappa+1)}$). It then follows that the firm receives a fraction γ of the revenue, or

$$s_q(N, x_c, x_n) = \gamma A^{1-\beta} x_c^{\beta\mu} x_n^{\beta(1-\mu)} N^{\beta(\kappa+1)}. \tag{17}$$

Second, the derived parameter $\gamma = \alpha/(\alpha+\beta)$ represents the bargaining power of the firm; it is rising in α and declining in β. That is, a higher elasticity of substitution between tasks increases the competition between workers at the bargaining stage (since each worker is less essential for production) and reduces their bargaining power.[20] On the other hand, an increase in β (an increase in the price elasticity of demand for the final good) has the opposite effect.[21]

Finally, the parameter α also determines the concavity of the Shapley value of a worker, $\bar{s}_x [N, x_c, x_n(-j), x_n(j)]$, with respect to noncontractible activities $x_n(j)$. The intuition for this is that in the limit as $M \to \infty$, each worker has an infinitesimal effect on total output, so she does not perceive the concavity in revenues coming from the elasticity of demand (related to β). Instead, the perceived concavity for each worker simply depends on the substitution between her task and the tasks performed by other workers, which is regulated by the parameter α. The more substitutable are the tasks, the less concave is $\bar{s}_x(\cdot)$ in each worker's investment. The impact of α on concavity will play an important role in the results below.

3.4 Equilibrium

To characterize a SSPE, we first use (14) to solve x_n in (9). This is the solution to:

$$\max_{x_n(j)} (1-\gamma) A^{1-\beta} \left[\frac{x_n(j)}{x_n(-j)}\right]^{(1-\mu)\alpha} x_c^{\beta\mu} x_n(-j)^{\beta(1-\mu)} N^{\beta(\kappa+1)-1} - c_x (1-\mu) x_n(j).$$

Relative to the producer's first-best, characterized above, we see two differences. First, the term $(1-\gamma)$ implies that the worker is not the full residual claimant of all the investments she undertakes, so she will tend to underinvest. Second, as discussed above, multilateral bargaining distorts the perceived concavity of the private return function.

The first-order condition for this optimization problem, evaluated at $x_n(j) = x_n(-j) = x_n$,

[20] To understand the effect of α on γ more precisely, recall that the Shapley value of the worker is an average of her contributions to coalitions of various sizes. The impact of α on the marginal contribution of a worker depends on the size of the coalition. Consider the marginal contribution of a worker to a coalition of n workers who cooperate with the firm when the technology choice is N. We show in Appendix A that this marginal contribution equals $(\beta/\alpha)(n/N)^{\beta/\alpha} R/n$, where $R = A^{1-\beta} N^{\beta(\kappa+1)} x_c^{\mu\beta} x_n^{(1-\mu)\beta}$ is the total revenue of the firm. This marginal contribution is increasing in α for small coalitions, i.e., when $n/N < \exp(-\alpha/\beta)$, and decreasing in α for large coalitions, i.e., $n/N > \exp(-\alpha/\beta)$. Intuitively, with only a few other workers in the coalition, a technology that requires complementary inputs generates smaller marginal revenues than one that requires inputs that are more substitutable. Conversely, with most workers in the coalition, the marginal contribution of a new worker is highest when technology features relatively high complementarity. Since the Shapley value equals a weighted average of these marginal contributions, with the weight being larger for larger coalitions (because there are more permutations leading to large coalitions containing the firm), the impact of α on the Shapley value is also a weighted average of the impact of α on the marginal contributions of a worker to coalitions of different size and the negative impact of α receives more weight, making the share of each worker, $(1-\gamma)/N = \beta/(\alpha+\beta) N$, decreasing in α.

[21] The intuition behind the negative effect of β on γ is similar. An increase in β increases the price elasticity of demand for the firm and makes revenues less concave in output. This decreases the marginal contribution of a worker to a coalition of relatively small size and increases her marginal contribution to a coalition of relatively large size. In particular, this marginal contribution is decreasing in β whenever $n/N < \exp(-\alpha/\beta)$, and increasing in β whenever $n/N > \exp(-\alpha/\beta)$. As before, the Shapley value assigns larger weights to coalitions of larger size, so that the positive impact of β dominates and $1-\gamma = \beta/(\alpha+\beta)$ is increasing in β.

yields a unique solution:

$$x_n = \bar{x}_n(N, x_c) \equiv \left[\alpha(1-\gamma)(c_x)^{-1} x_c^{\beta\mu} A^{1-\beta} N^{\beta(\kappa+1)-1}\right]^{1/[1-\beta(1-\mu)]}. \tag{18}$$

This choice of investment in noncontractible activities rises with the investment level in the contractible activities, because different activities performed by the worker are complements (i.e., the marginal productivity of an activity rises with investments in other activities).[22]

Another important implication of equation (18) is that investments in noncontractible activities are increasing in α. Mathematically, this simply follows from the fact that $\alpha(1-\gamma) = \alpha\beta/(\alpha+\beta)$ is increasing in α. Economically, it is the outcome of two offsetting forces. As noted above, $(1-\gamma)$, which enters the Shapley value of each worker, is decreasing in α. This implies that the level of payments to workers is decreasing in α, because greater substitution between workers reduces their bargaining power. However, the effect of α on the concavity of $\bar{s}_x(\cdot)$ makes x_n increasing in α, and this effect dominates the impact of α through $(1-\gamma)$. In essence, though a greater α reduces the level of payments to workers, it also makes these payments more sensitive to their investments, encouraging greater equilibrium investments.

Now using (16), (17) and (18), the firm's optimization problem (12) boils down to

$$\max_{N, x_c} A^{1-\beta}\left[x_c^{\mu}\bar{x}_n(N, x_c)^{1-\mu}\right]^{\beta} N^{\beta(\kappa+1)} - c_x N\mu x_c - c_x N(1-\mu)\bar{x}_n(N, x_c) - C(N), \tag{19}$$

where $\bar{x}_n(N, x_c)$ is defined in (18). Substituting for $\bar{x}_n(N, x_c)$ and differentiating with respect to N and x_c,[23] we obtain two first-order conditions, which can be manipulated to yield a unique solution $\left(\tilde{N}, \tilde{x}_c\right)$ to (19) (see Appendix A for details):

$$\tilde{N}^{\frac{\beta(\kappa+1)-1}{1-\beta}} A\kappa\beta^{\frac{1}{1-\beta}} c_x^{-\frac{\beta}{1-\beta}} \left[\frac{1-\alpha(1-\gamma)(1-\mu)}{1-\beta(1-\mu)}\right]^{\frac{1-\beta(1-\mu)}{1-\beta}} \left[\beta^{-1}\alpha(1-\gamma)\right]^{\frac{\beta(1-\mu)}{1-\beta}} = C'\left(\tilde{N}\right), \tag{20}$$

$$\tilde{x}_c = \frac{C'\left(\tilde{N}\right)}{\kappa c_x}. \tag{21}$$

The unique SSPE is then given by $\left\{\tilde{N}, \tilde{x}_c, \tilde{x}_n\right\}$ such that

$$\tilde{x}_n = \frac{\alpha(1-\gamma)[1-\beta(1-\mu)]}{\beta[1-\alpha(1-\gamma)(1-\mu)]}\frac{C'\left(\tilde{N}\right)}{\kappa c_x}. \tag{22}$$

[22] The effect of N on the level of x_n is ambiguous, however. In particular, investment in noncontractible activities declines with the division of labor when $\beta(\kappa+1) < 1$ and rises with the division of labor in the opposite case. Intuitively, an increase in N has opposite effects on the incentives to invest for workers. On the one hand, because technology features a "love for variety," a higher measure of tasks increases the marginal product of noncontractible investments. On the other hand, the bargaining share of a worker, $(1-\gamma)/N$, is decreasing in N. When $\kappa \to 0$, the first effect disappears and an increase in N necessarily reduces the investment x_n.

[23] We show in Appendix A that the second-order conditions of this problem are satisfied under Assumption 1.

Comparing (7) to (21), we can see that for a given level of N, the implied level of investment in contractible activities under incomplete contracts, $\tilde{x}_c$, is identical to the investment level with complete contracts, x^*. This highlights that distortions in the investments in contractible activities are related to the distortion in the division of labor. In fact, comparing (6) to (20), we see that $\tilde{N}$ and N^* differ only because of the two bracketed terms on the left-hand side of (20). These represent the distortions created by bargaining between the firm and its workers. Intuitively, the choice of the division of labor will be distorted because incomplete contracts and bargaining distort the level of investments in contractible and noncontractible activities, which are complementary to the division of labor. Notice that as $\mu \longrightarrow 1$, both of these bracketed terms tend to 1, and we have $\left(\tilde{N}, \tilde{x}_c\right) \longrightarrow (N^*, x^*)$. That is, as the measure of noncontractible activities becomes close to zero, the incomplete contracts equilibrium $\left(\tilde{N}, \tilde{x}_c\right)$ converges to the complete contracts equilibrium (N^*, x^*).[24]

To examine the effect of incomplete contracting on productivity, notice also that productivity is now equal to:

$$\tilde{P} = \tilde{N}^{\kappa} \tilde{x}_c^{\mu} \tilde{x}_n^{1-\mu}. \tag{23}$$

The next proposition summarizes the key features of the equilibrium and the main comparative statics (proof in Appendix A):

Proposition 2 Suppose that Assumption 1 holds. Then there exists a unique SSPE $\left\{\tilde{N}, \tilde{x}_c, \tilde{x}_n\right\}$ given by (20), (21) and (22), with $\tilde{N}, \tilde{x}_c, \tilde{x}_n > 0$, and an associated productivity level $\tilde{P} > 0$ given by (23). Furthermore, $\left(\tilde{N}, \tilde{x}_c, \tilde{x}_n, \tilde{P}\right)$ satisfies

$$\tilde{x}_n < \tilde{x}_c$$

and

$$\begin{aligned}
\frac{\partial \tilde{N}}{\partial A} &> 0, \; \frac{\partial \tilde{x}_c}{\partial A} \geq 0, \; \frac{\partial \tilde{x}_n}{\partial A} \geq 0, \; \frac{\partial \tilde{P}}{\partial A} > 0, \\
\frac{\partial \tilde{N}}{\partial \mu} &> 0, \; \frac{\partial \tilde{x}_c}{\partial \mu} \geq 0, \; \frac{\partial \left(\tilde{x}_n / \tilde{x}_c\right)}{\partial \mu} > 0, \; \frac{\partial \tilde{P}}{\partial \mu} > 0, \\
\frac{\partial \tilde{N}}{\partial \alpha} &> 0, \; \frac{\partial \tilde{x}_c}{\partial \alpha} \geq 0, \; \frac{\partial \left(\tilde{x}_n / \tilde{x}_c\right)}{\partial \alpha} > 0, \; \frac{\partial \tilde{P}}{\partial \alpha} > 0,
\end{aligned}$$

and also

$$\frac{\partial^2 \tilde{N}}{\partial \alpha \partial \mu} < 0.$$

We therefore find that workers always invest less in noncontractible activities than in contractible activities, and that the division of labor, investment levels, and productivity are all increasing in the

[24] Interestingly, however, it can be verified that as $\mu \longrightarrow 1$, $\tilde{x}_n \nrightarrow x^*$, because the effect of distortions on the noncontractible activities does not disappear as $\mu \longrightarrow 1$.

size of the market, the fraction of contractible activities and the elasticity of substitution between tasks.

The first result follows from dividing equation (22) by equation (21) to obtain

$$\frac{\tilde{x}_n}{\tilde{x}_c} = \frac{\alpha(1-\gamma)[1-\beta(1-\mu)]}{\beta[1-\alpha(1-\gamma)(1-\mu)]} < 1, \tag{24}$$

where the inequality follows from $\alpha(1-\gamma) = \alpha\beta/(\alpha+\beta) < \beta$ (recall (15)). Intuitively, the firm is the full residual claimant of the investments in contractible activities and dictates these investments in the terms of the contract. In contrast, investments in noncontractible activities are decided by the workers, and as highlighted by (16), they are not the full residual claimants of the revenues generated by these investments.

The analysis of comparative statics is facilitated by the fact that, given the second-order conditions (which are ensured by Assumption 1), any change in A, μ or α that increases the left-hand side of (20) will also increase $\tilde{N}$. As in the case of complete contracts, we find that the left-hand side of (20) is increasing in A, and thus the division of labor again rises with the size of the market. Appendix A shows that the left-hand side of (20) is also increasing in μ and α, so that the division of labor is greater when there is less contract incompleteness and less technological complementarity. Both results are intuitive. Incomplete contracts imply that workers underinvest in noncontractible activities. An increase in μ increases the degree of contractibility and thus the profitability of further investments in technology (division of labor). The positive effect of α stems from the effect of this parameter on the concavity of the Shapley value of workers, $\bar{s}_x(\cdot)$, discussed above. Greater α increases workers' investments, making division of labor more profitable for the firm.

The other comparative static results in Proposition 2 then follow from equations (21), (23), and (24). In particular, from equation (23), equilibrium productivity is also increasing in the size of the market, the degree of contractibility and the substitutability of tasks, because both the division of labor and — given the division of labor — the levels of investment in contractible and noncontractible activities are increasing in these variables.[25]

It is noteworthy that the effect of α on productivity is of the opposite sign as its effect in standard Dixit-Stiglitz formulations of technology that do not include the Benassy (1998) term. In these formulations, the larger is the complementarity in production, the larger is the response of productivity to an increase in the division of labor. Our specification of technology in (3) neutralizes this effect and helps isolate a new channel by which technological complementarity affects productivity.

Another result, which will play an important role in Section 6, is $\partial^2\tilde{N}/\partial\alpha\partial\mu < 0$. It implies that better contracting institutions have a greater effect on the division of labor when there are greater technological complementarities. The intuition is that contract incompleteness has a more detrimental effect on the division of labor when there is greater complementarity between tasks,

[25] Also, as noted in footnote 13, the division of labor, investment levels, and productivity are decreasing with c_x and increasing in κ as long as $N^* \geq 1$.

because the investment distortions are larger in this case.

Finally, it is useful, both to further illustrate the intuition for the results and for future reference, to note that combining (18), (19), and the first-order condition with respect to x_c, the profit function of the firm can be expressed as (see Appendix A):

$$\pi = AZ(\alpha, \mu) N^{1+\frac{\beta(\kappa+1)-1}{1-\beta}} - C(N), \tag{25}$$

where

$$Z(\alpha, \mu) \equiv (1-\beta)\beta^{\frac{\beta\mu}{1-\beta}}[\alpha(1-\gamma)]^{\frac{\beta(1-\mu)}{1-\beta}}\left[\frac{1-\alpha(1-\gamma)(1-\mu)}{1-\beta(1-\mu)}\right]^{\frac{1-\beta(1-\mu)}{1-\beta}}(c_x)^{-\frac{\beta}{1-\beta}} \tag{26}$$

and as before, $\gamma \equiv \alpha/(\alpha+\beta)$. $Z(\alpha, \mu)$ is thus a term that captures "distortions" arising from incomplete contracting.

The comparative static results regarding the effect of α and μ on the division of labor simply follow from the fact that $Z(\alpha, \mu)$ is increasing in both variables. This (equilibrium) reduced-form profit function will become useful in our general equilibrium analysis. It also makes other potential applications of this framework quite tractable.

The above analysis also enables us to derive a straightforward comparison of equilibria with and without complete contracts. Recall that the incomplete contract equilibrium converges on the complete contract equilibrium when $\mu \to 1$. Together with Proposition 2, this implies

Proposition 3 Suppose that Assumption 1 holds. Let $\left\{\tilde{N}, \tilde{x}_c, \tilde{x}_n\right\}$ be the unique SSPE with incomplete contracts and $\{N^*, x^*\}$ be the equilibrium with complete contracts. Let $\tilde{P}$ and P^* be the productivity levels associated with each of these equilibria. Then

$$\tilde{N} < N^*,\ \tilde{x}_n < \tilde{x}_c \leq x^*, \text{ and } \tilde{P} < P^*.$$

This proposition implies that, with incomplete contracts, the equilibrium division of labor is always suboptimal.[26] Furthermore, investments in both contractible and noncontractible activities are also lower than those under complete contracts. As a result of these effects, the level of productivity with incomplete contracts is also lower than that with complete contracts.

Before concluding this section, and in order to facilitate the transition to the analysis of imperfect credit markets, consider the implications of the equilibrium for the ex ante transfer $\tilde{\tau}$. If $\tilde{\tau}$ is positive, the firm has to make an upfront payment to every worker, and if it is negative, the firm

[26] It is useful to contrast this result with the overemployment result in Stole and Zwiebel (1996a,b). There are two important differences between our model and theirs. First, in our model, there are investments in noncontractible activities, which are absent in Stole and Zwiebel. Second, Stole and Zwiebel assume that if a worker is not in the coalition in the bargaining game, then she receives no payment to compensate for her outside option. Since in our model whether the worker provides the services in the noncontractible tasks is not verifiable, whether or not she is in the coalition cannot affect her wage except through bargaining, and consequently she will continue to receive the payments for her contractible investments. The treatment of the outside option is essential for the overemployment result in Stole and Zwiebel.

ought to receive an upfront payment from every worker. We use (13) to calculate the size of this transfer in the SSPE.

First note that $\bar{s}_x\left(\tilde{N}, \tilde{x}_c, \tilde{x}_n, \tilde{x}_n\right) = (1-\gamma)\tilde{R}/\tilde{N}$, where $\tilde{R} = A^{1-\beta}\tilde{N}^{\beta(\kappa+1)}\tilde{x}_c^{\beta\mu}\tilde{x}_n^{\beta(1-\mu)}$ is the equilibrium revenue, i.e., every worker receives an ex post payment equal to the share $(1-\gamma)/\tilde{N}$ of the revenue (her Shapley value). Second note that (18) implies $c_x x_n = \alpha(1-\gamma)\tilde{R}/\tilde{N}$ as a result of the worker's optimal choice of investment in noncontractible activities. Combining these observations with (13) and (24) implies that

$$\tilde{\tau} = \left[\mu \frac{\beta - \alpha(1-\gamma)}{(1-\gamma)\left[1-\beta(1-\mu)\right]} + \alpha - 1\right](1-\gamma)\frac{\tilde{R}}{\tilde{N}}.$$

When $\gamma < 1-\beta$, this transfer is negative ($\tilde{\tau} < 0$) for all $\mu \in [0,1]$ and workers make upfront payments to the firm to compensate for the ex post rents they earn in the bargaining game. However, when $\gamma > 1-\beta$, $\tilde{\tau}$ is no longer necessarily negative. It can be shown that it will be negative if and only if contracts are sufficiently incomplete, i.e.. $\mu < (1-\alpha)(1-\beta)/(\beta/(1-\gamma) - (\alpha+\beta) + \alpha\beta)$. When contracts are more complete than this, $\tilde{\tau} \geq 0$. This result stems from the fact that investments in contractible activities are chosen by the firm and set at a level greater than what workers would have chosen themselves. As a result $s_x(\cdot) - \mu c_x x_n - (1-\mu) c_x x_c$ can become negative, in which case the firm has to make an upfront payment in order to secure workers' participation. This case is more likely when a greater fraction of activities are contractible.

4 Imperfect Credit Markets

The above analysis presumed that workers and firms had access to ex ante transfers. However, the same weak contracting institutions that make certain activities performed by the worker-suppliers noncontractible also create credit market frictions and other capital market imperfections, which may prevent such transfers (or "bonding" arrangements).

We now analyze the same model with severe credit market imperfections that make ex ante transfers from workers to the producer of the final good impossible:[27]

Assumption 2 Ex ante transfers from workers to firms are not possible, i.e., $\tilde{\tau} \geq 0$.

This extreme form of credit market imperfections is useful to focus on the interaction between incomplete contracts and imperfect credit markets. Since we assume that the costs of investments x_n and x_c are non-pecuniary, credit market imperfections do not directly prevent or restrict these investments. This assumption is therefore in line with our interpretation of the $x(i,j)$s as potentially nonverifiable "efforts".

Recall that at the end of the Section 3 we provided a necessary and sufficient condition for $\tilde{\tau} < 0$. If this condition is not satisfied, then workers are not required to make ex ante transfers. Then, as long as the firm has deep pockets, the equilibrium from the previous section prevails.[28]

[27] Given Assumption 3 below, whether or not the firm has deep pockets is not relevant.
[28] And if the firm cannot make ex ante transfers either, then the equilibrium characterized in this section applies.

We now proceed to analyze the equilibrium with binding credit constraints. This requires

Assumption 3

$$\mu < \frac{(1-\alpha)(1-\beta)}{\frac{\beta}{1-\gamma} - (\alpha+\beta) + \alpha\beta}.$$

Given Assumptions 1-3, we again look for the symmetric subgame perfect equilibrium. The main difference is that in terms of the optimization problem (11), we now impose the additional constraint $\tau \geq 0$, which is (only) binding under Assumption 3. Therefore, the participation constraint becomes $\bar{s}_x(N, x_c, x_n, x_n) \geq \mu c_x x_c + (1-\mu) c_x x_n$. The maximization problem of the firm then becomes

$$\max_{N, x_c, x_n} s_q(N, x_c, x_n) - C(N)$$
$$\text{subject to (9) and } \bar{s}_x(N, x_c, x_n, x_n) \geq \mu c_x x_c + (1-\mu) c_x x_n.$$

The problem can be further simplified by noting that the participation constraints will always be binding. To see this, note that $s_q(N, x_c, x_n)$ is strictly increasing in x_c, so if the participation constraint were slack, the firm could increase x_c and thus $s_q(N, x_c, x_n)$. Therefore, using (16), (17) and (18) and imposing the participation constraint as an equality, we obtain the simpler maximization problem

$$\max_{N, x_c} \gamma A^{1-\beta} x_c^{\beta\mu} x_n^{\beta(1-\mu)} N^{\beta(\kappa+1)} - C(N) \quad \text{subject to} \tag{27}$$
$$\alpha(1-\gamma) A^{1-\beta} x_c^{\beta\mu} x_n^{\beta(1-\mu)} N^{\beta(\kappa+1)-1} = c_x x_n,$$
$$(1-\gamma) A^{1-\beta} x_c^{\beta\mu} x_n^{\beta(1-\mu)} N^{\beta(\kappa+1)-1} = \mu c_x x_c + (1-\mu) c_x x_n.$$

The two constraints of this problem can be used to solve the investment levels in contractible and noncontractible activities as functions of the technology choice N. These investment levels are

$$x_n = \left[c_x^{-1} \alpha (1-\gamma) A^{1-\beta} N^{\beta(\kappa+1)-1} \left(\frac{1-\alpha+\alpha\mu}{\alpha\mu} \right)^{\beta\mu} \right]^{1/(1-\beta)}, \tag{28}$$

$$x_c = \left[c_x^{-1} \alpha (1-\gamma) A^{1-\beta} N^{\beta(\kappa+1)-1} \left(\frac{1-\alpha+\alpha\mu}{\alpha\mu} \right)^{\beta\mu} \right]^{1/(1-\beta)} \left(\frac{1-\alpha+\alpha\mu}{\alpha\mu} \right). \tag{29}$$

An immediate implication is that

$$\frac{\check{x}_n}{\check{x}_c} = \frac{\alpha\mu}{1-\alpha+\alpha\mu}, \tag{30}$$

where $\check{x}_c$ and $\check{x}_n$ are, respectively, the equilibrium investments in contractible and noncontractible activities. Interestingly, this ratio does not depend on β and approaches zero as $\mu \to 0$. Intuitively, as $\mu \to 0$, there are only a few activities that the firm can use to extract the surplus from the workers, and there will be greater overinvestment in these activities.

Now substituting (28) and (29) into the firm's objective function in (27), we obtain the following maximization problem:

$$\max_N \gamma \left[c_x^{-1}\alpha(1-\gamma)\right]^{\frac{\beta}{1-\beta}} \left(\frac{1-\alpha+\alpha\mu}{\alpha\mu}\right)^{\frac{\beta\mu}{1-\beta}} AN^{\frac{\beta(\kappa+1)-1}{1-\beta}+1} - C(N). \tag{31}$$

The first-order condition of this problem characterizes the equilibrium technology $\check{N}$.[29] It is given by

$$\check{N}^{\frac{\beta(\kappa+1)-1}{1-\beta}} \frac{\beta\kappa\gamma A}{1-\beta} \left(\frac{1-\alpha+\alpha\mu}{\alpha\mu}\right)^{\frac{\beta\mu}{1-\beta}} \left[c_x^{-1}\alpha(1-\gamma)\right]^{\frac{\beta}{1-\beta}} = C'\left(\check{N}\right). \tag{32}$$

Using this equation and the equations for the investment levels (28) and (29), the equilibrium investment levels in the credit constrained equilibrium can be represented by

$$\check{x}_n = \frac{\alpha(1-\gamma)}{\beta\gamma}(1-\beta)\frac{C'\left(\check{N}\right)}{\kappa c_x}, \tag{33}$$

$$\check{x}_c = \frac{\alpha(1-\gamma)}{\beta\gamma}(1-\beta)\left(\frac{1-\alpha+\alpha\mu}{\alpha\mu}\right)\frac{C'\left(\check{N}\right)}{\kappa c_x}. \tag{34}$$

Comparing these equations with (21) and (22) when $\check{N} = \tilde{N}$, we note that Assumption 3 implies $\check{x}_c > \tilde{x}_c$ and $\check{x}_n > \tilde{x}_n$. That $\check{x}_c > \tilde{x}_c$ in this case is again a consequence of the fact that the firm now extracts the surplus from the workers by forcing them to "overinvest" in contractible activities.[30] Naturally, however, the division of labor differs in the two equilibria, and this will further affect the equilibrium investment levels.

Analogously to before, we define equilibrium productivity (now without ex ante transfers) as

$$\check{P} = \check{N}^{\kappa}\check{x}_c^{\mu}\check{x}_n^{1-\mu}. \tag{35}$$

We have (proof in Appendix A):

Proposition 4 Suppose that Assumptions 1-3 hold. Then there exists a unique SSPE without ex ante transfers, $\left\{\check{N}, \check{x}_c, \check{x}_n\right\}$, given by (32), (33) and (34), with $\check{N}, \check{x}_c, \check{x}_n > 0$ and an associated productivity level $\check{P} > 0$ given by (35). Furthermore, $\left(\check{N}, \check{x}_c, \check{x}_n, \check{P}\right)$ satisfies

$$\check{x}_n < \check{x}_c,$$

[29] Assumption 1 again ensures that the second-order condition is satisfied.

[30] Moreover, it is no longer the case that as $\mu \longrightarrow 1$, $\check{x}_c \longrightarrow x_c^*$, since the firm will use the contractible activities to extract surplus for all $\mu < 1$.

and

$$
\begin{aligned}
\frac{\partial \check{N}}{\partial A} &> 0, \ \frac{\partial \check{x}_c}{\partial A} \geq 0, \ \frac{\partial \check{x}_n}{\partial A} \geq 0, \ \frac{\partial \check{P}}{\partial A} > 0 \\
\frac{\partial \check{N}}{\partial \mu} &> 0, \ \frac{\partial \check{x}_n}{\partial \mu} \geq 0, \ \frac{\partial \left(\check{x}_n / \check{x}_c\right)}{\partial \mu} > 0, \ \frac{\partial \check{P}}{\partial \mu} > 0, \\
\frac{\partial \check{N}}{\partial \alpha} &> 0, \ \frac{\partial \check{x}_n}{\partial \alpha} \geq 0, \ \frac{\partial \left(\check{x}_n / \check{x}_c\right)}{\partial \alpha} > 0,
\end{aligned}
$$

and also

$$
\frac{\partial^2 \check{N}}{\partial \alpha \partial \mu} < 0.
$$

As in the case of perfect capital markets, workers always invest less in noncontractible activities than in contractible activities. The reason for this is two-fold. First, as was the case in Proposition 2 above, workers are not the full residual claimants of the increase in productivity generated by their investments. Second, in the presence of credit constraints, contractible investments are an instrument of surplus extraction by the firm, which forces workers to overinvest in these activities.

The comparative statics in Proposition 4 related to the division of labor and to the choice of noncontractible activities are identical to those derived in Proposition 2. In particular, we find that the division of labor and the investment in noncontractible activities are increasing in the size of the market, the quality of the contractual environment, and the elasticity of substitution between tasks. Furthermore, a higher degree of contractibility and lower technological complementarities also lead to a lower gap between the investment levels in contractible and noncontractible activities. Moreover, better contracting institutions have a larger impact on the division of labor when there is greater technological complementarity.

Notice, however, that contrary to Proposition 2, equilibrium investment levels in contractible activities $\check{x}_c$ now need *not* be increasing in the quality of contracting institutions (or in the substitutability of tasks). The reason for this is again the dual role played by contractible activities; in addition to their productivity enhancing role, they are also instruments for surplus extraction by the firm. When the degree of contractibility is low, there are fewer activities in which the firm can extract surplus, so overinvestment in the remaining contractible activities may increase. Similarly, when there is limited substitution between tasks, the share of revenue accruing to workers is higher (recall that γ is increasing in α) and the incentive to set a relatively high x_c is greater.

The potential for overinvestment also has implications for productivity. Although, as before, productivity is increasing in the size of the market and the quality of the contracting institutions, it is no longer monotonic in α. This is because, as noted in the previous paragraph, a higher α reduces the share of revenue accruing to workers and thus the amount of surplus to be extracted through the choice of $\check{x}_c$. Other things equal, this lower incentive to overinvest in $\check{x}_c$ leads to a reduction in productivity. The overall effect of α on $\check{P}$ is thus ambiguous. More interestingly, the next proposition shows that productivity can be higher with imperfect credit markets than both the case with incomplete contracts and perfect credit markets and the case with complete contracts (proof in Appendix A):

Proposition 5 Suppose that Assumptions 1-3 hold. Let $\{N^*, x^*\}$ be the solution to the complete contracts problem, let $\left\{\tilde{N}, \tilde{x}_c, \tilde{x}_n\right\}$ be the unique SSPE with ex ante transfers, and let $\left\{\check{N}, \check{x}_c, \check{x}_n\right\}$ be the unique SSPE without ex ante transfers. Let P^*, $\tilde{P}$, and $\check{P}$ be the productivity attained in each of these three environments. Then

$$1 > \tilde{x}_n/\tilde{x}_c > \check{x}_n/\check{x}_c,$$

and

$$N^* > \tilde{N} > \check{N}.$$

Suppose further that $\kappa \leq 1 - \beta$. Then (i) $\check{P} > \tilde{P}$ for all $\mu \in (0,1)$; and moreover (ii) if $\gamma < 1 - \beta$, there exists a unique $\overline{\mu} \in (0,1)$, such that $\mu < \overline{\mu}$ implies $\check{P} < P^*$ and $\mu > \overline{\mu}$ implies $\check{P} > P^*$.

Credit market problems therefore increase the spread between investments in noncontractible and contractible activities and further reduce the division of labor. Intuitively, the firm is now obtaining part of the returns by inefficiently extracting the surplus from the workers by means of high levels of $\check{x}_c$ (and this is inefficient, since there are diminishing returns to x_c). This naturally reduces the "productivity" of further increases in the division of labor. Therefore, $\tilde{N} > \check{N}$.

More interestingly, because $\check{x}_c$ may be higher than $\tilde{x}_c$, it now becomes possible that $\check{P} > \tilde{P}$ even when $\check{N} < \tilde{N}$. In other words, although credit constraints always depress the equilibrium division of labor, measured productivity can be greater because of overinvestment in contractible activities. In fact, the latter will necessarily be the case whenever the productivity gains from division of labor, represented by κ, are relatively low ($\kappa \leq 1 - \beta$). Moreover, in this case, it is also possible for productivity in the equilibrium with imperfect capital markets to exceed that of the first best (i.e., $\check{P} > P^*$).[31]

5 The Employment Relationship Versus Outside Contracting

We now use our framework to discuss the choice between the employment relationship and an organizational form relying on outside contracting. Our analysis so far assumed that the threat point of each worker-supplier was not to deliver the noncontractible activities (which is equivalent to not delivering the entire $X(j)$ given the Cobb-Douglas structure in (4)). To endogenize the choice of organizational form, we assume that the threat point of a particular agent is not to deliver a fraction $1-\delta$ of their $X(j)$ and suppose that δ depends on whether she is an employee or an outside contractor (in the spirit of Grossman and Hart, 1986, and Hart and Moore, 1990). Our analysis

[31] Conversely, when κ is relatively large ($\kappa > 1-\beta$), the productivity loss associated with a limited division of labor may dominate. In fact, when κ approaches its upper bound dictated by Assumption 1, it is necessarily the case that $P^* > \tilde{P} > \check{P}$. Notice also that in the absence of perfect credit markets, we have $\lim_{\mu \to 1} \check{P} \neq P^*$, and this is what makes $\check{P} > P^*$ possible.

above therefore corresponds to the case where $\delta = 0$, but the results in this section (in particular, Lemma 3) demonstrate that all the results so far hold for any value of δ. In particular, in the case of outside (arms-length) contracting, we assume that $\delta = \delta_C \in [0, 1)$, and with an employment relationship, $\delta = \delta_E \in (\delta_C, 1)$ because the assets the agent uses are the property of the firm.[32] Moreover, to simplify the exposition we normalize $\delta_C = 0$.

We also denote the choice of organizational form by $O \in \{C, E\}$, with C corresponding to outside contracting and E corresponding to an employment relationship. We assume that the organizational form, O, is chosen concurrently with the choice of technology N.[33] The rest of the analysis continues to apply except that Lemma 1 needs to be modified to account for the fact that the Shapley values in the ex post bargaining game now depend on the choice of O.

The following generalization of Lemma 1 is proved in Appendix A:

Lemma 2 Denote $\delta_O = \delta$ and suppose that $M \to \infty$, worker j invests $x_n(j)$ in her noncontractible activities, all the other workers invest $x_n(-j)$ in their noncontractible activities, every worker invests x_c in her contractible activities, and the division of labor is N. Then the Shapley value of worker j is

$$\bar{s}_x [N, x_c, x_n(-j), x_n(j)] = (1-\gamma) A^{1-\beta} \left[\frac{x_n(j)}{x_n(-j)}\right]^{(1-\mu)\alpha} x_c^{\beta\mu} x_n(-j)^{\beta(1-\mu)} N^{\beta(\kappa+1)-1},$$

where

$$\gamma \equiv \frac{\alpha\left(1-\delta^{\alpha+\beta}\right)}{(\alpha+\beta)(1-\delta^{\alpha})}. \quad (36)$$

Imposing symmetry it is straightforward to show that γ again represents the fraction of revenue accruing to the firm in the bargaining, with $(1-\gamma)/N$ representing the fraction assigned to each of the N workers.

Several features of this result are worth mentioning. Notice first that when $\delta = 0$, the Shapley values (and thus the SSPE of the game) are identical to those described in the previous sections. When $\delta > 0$, γ is greater than $\alpha/(\alpha+\beta)$, so the firm is now more powerful in the bargaining game and obtains a larger fraction of the revenue. In the limit where δ goes to 1, the bargaining share γ also goes to 1, and the firm has full bargaining power.

The following result provides a more complete characterization of the bargaining share γ:

Lemma 3 For all $\delta \in [0, 1)$, γ is increasing in α and δ and decreasing in β. Furthermore, $\alpha(1-\gamma)$ is increasing in α and $\alpha(1-\gamma)/\beta\gamma$ is nondecreasing in α.

[32] To express these issues more formally, we could introduce $X^e(j)$ as the efficiency units of services provided by worker j. Then, in the case of disagreement with an outside contractor, $X^e(j) = \delta_C X(j)$, and in the case of disagreement when there is an employment relationship, $X^e(j) = \delta_E X(j)$.

[33] For notational simplicity, we assume that the firm chooses either to have an employment relationship for all of the tasks or for none (given the symmetry of our setup this is without loss of generality).

Using this lemma, it can be shown that all the results regarding the comparative statics with respect to μ and α in the previous sections (Propositions 2 through 5) continue to hold. In particular, these results simply depended on the following properties: γ and $\alpha(1-\gamma)$ are increasing in α and $\alpha(1-\gamma)/\beta\gamma$ is nondecreasing in α. Lemma 3 implies that these properties hold for all positive values of δ. Consequently, conditional on the equilibrium choice of organizational form, Proposition 2 through 5 continue to apply.

We next turn to discussing the equilibrium organizational form and its effect on the division of labor. Consider first the model with perfect credit markets. Remember that in this case there exists a unique SSPE $\left\{\tilde{N}, \tilde{x}_c, \tilde{x}_n\right\}$ given by (20), (21) and (22). We can now expand the definition of the SSPE to include the choice of organizational form, i.e., $\left\{\tilde{O}, \tilde{N}, \tilde{x}_c, \tilde{x}_n\right\}$, where $\tilde{O} \in \{C, E\}$. The analysis is simplified since the Envelope Theorem implies that the firm's profits in (19) are declining in γ and therefore in δ (to see this, recall that since $x_n < x_n^*$, the level of profits is increasing in x_n, which is decreasing in γ). This immediately implies that the firm prefers $\delta = 0$ to any positive δ and establishes:

Proposition 6 Suppose that Assumption 1 holds. Then in the unique SSPE with incomplete contracts and no credit market imperfections, the equilibrium organizational form involves outside contracting, i.e. $\tilde{O} = C$.

In the absence of credit constraints, it is never optimal for the firm to hire the agents as employees. This stems from the fact that increasing the bargaining share of the firm reduces the incentives of workers to invest in the noncontractible activities, which in turn reduces the marginal product of investing in contractible activities and in the division of labor.[34] In other words, because worker-suppliers are the only agents undertaking noncontractible investments, the efficient allocation of power entails giving them as much ex post bargaining power as possible. With perfect capital markets this efficient organizational form is attained with outside contracting, and an employment relationship is never optimal.[35]

Let us next turn to the case with credit constraints. In this case, though $\delta = 0$ maximizes sales revenue, the firm is constrained to obtain only a fraction γ of the revenue ex ante and may want to increase this share. The employment relationship is then potentially useful as another instrument for surplus extraction from the workers (in addition to x_c). We find that in this case the possibility to create an employment relationship may indeed enhance the division of labor.

To show this, we proceed in a manner analogous to that in the case with perfect credit markets. From problem (31) and the Envelope Theorem, it is clear that the choice of organizational form O is made to maximize $\gamma(1-\gamma)^{\beta/(1-\beta)}$. Because γ is monotonically increasing in δ, a higher level of

[34] Formally, we find that $\partial\tilde{N}/\partial\delta < 0$, $\partial\tilde{x}_c/\partial\delta \leq 0$, and $\partial\tilde{x}_n/\partial\delta < 0$. This can be established by noting that (i) the only effect of δ on the equilibrium value of $\tilde{N}$ works through γ, which is in turn increasing in δ, and (ii) the left-hand-side of equation (20) and $\tilde{x}_n/\tilde{x}_c$ are increasing in $\alpha(1-\gamma)$, and thus decreasing in δ.

[35] This result depends on the fact that the firm does not make any relationship-specific investments. Otherwise, a greater share of surplus for the workers would distort the firm's investment incentives (see, for example, Hart and Moore, 1990, Antràs, 2003, or Antràs and Helpman, 2004).

δ lowers this term whenever $\gamma \geq 1-\beta$. If this inequality holds for all $\delta \in [0,1)$ (i.e., $\alpha+\beta \geq 1$), then $\delta = 0$ maximizes profits and $\check{O} = C$ is the best strategy. Therefore, when $\gamma \geq 1-\beta$, the equilibrium organizational form involves outside contracting.

More interesting results obtain when $\gamma < 1-\beta$ for some $\delta \in [0,1)$ (i.e., $\alpha+\beta < 1$). In this case there exists a unique $\bar{\delta} > 0$ such that if $\delta_E \in (0,\bar{\delta})$, then the firm's profits are maximized when all agents are hired as employees. Under these circumstances the equilibrium organizational form involves the employment relationship between the firm and the workers. Furthermore, the threshold $\bar{\delta}$ is implicitly defined by

$$\frac{1-\bar{\delta}^{\alpha+\beta}}{1-\bar{\delta}^{\alpha}}\left[1-\frac{\alpha\bar{\delta}^{\alpha}\left(1-\bar{\delta}^{\beta}\right)}{\beta\left(1-\bar{\delta}^{\alpha}\right)}\right]^{\frac{\beta}{1-\beta}} = 1 \text{ for } \bar{\delta} > 0. \tag{37}$$

This threshold is decreasing in α (see Appendix A for details). This discussion leads to:

Proposition 7 Suppose that Assumptions 1-3 hold. Then in the unique SSPE without ex ante transfers, the equilibrium organizational form $\check{O}$ satisfies:

(i) $\check{O} = C$ when $\alpha+\beta > 1$.

(ii) If $\alpha+\beta < 1$, then there exists a unique $\bar{\delta} \in (0,1)$ such that $\check{O} = E$ for $\delta_E < \bar{\delta}$ and $\check{O} = C$ for $\delta_E > \bar{\delta}$. Furthermore, $\bar{\delta}$ is decreasing in α.

The intuition behind this result is as follows. When there is a high degree of technological complementarity (i.e., $\alpha+\beta < 1$), the Shapley value of the firm tends to be relatively low too. In such cases, the firm may be able to increase its profits (and its incentive to invest in the division of labor) by hiring the agents as employees. In doing so, the firm trades off a larger fraction of the revenue for a smaller level of revenue (due to the reduced investment incentives of workers).

A similar logic explains the last statement in Proposition 7: other things equal, the region of the parameter space in which the employment relationship emerges as an equilibrium outcome is larger when there is greater complementarity between tasks (because, with greater complementarity, the firm captures an even smaller fraction of sale revenues when dealing with outside contractors, it is more likely to prefer $\check{O} = E$).

This result relates to, but is different from, Marglin's (1974) argument that the employment relationship encourages the division of labor as a way of reducing the autonomy and bargaining power of workers. Our result implies that the employment relationship itself emerges as a way of reducing the bargaining power of the worker-suppliers, and this encourages further division of labor.[36]

[36] This result is also related to Aghion and Tirole (1994) and Legros and Newman (2000), who develop property-rights models of the firm in which parties have a limited ability to transfer surplus ex-ante. An important distinction between our approach and theirs stems from partial contractibility. In particular, the presence of partial contractibility implies that, even though vertical integration (the employment relationship) is a way of transferring resources from worker-suppliers to the firm, it emerges in equilibrium only when it is constrained-efficient and increases the equilibrium division of labor. This is different from existing results in the literature, where credit constraints lead to inefficient integration decisions.

6 General Equilibrium

The theory of the division of labor that we have developed is simple enough to be used in various applications, and in particular in applications that require general equilibrium interactions. We illustrate a number of such applications in what follows. To save space, we limit the analysis to the case with perfect credit markets, and we do not present the benchmark results with complete contracts. Thus throughout the section we have $0 < \mu < 1$ and $\delta = 0$. Under these circumstances the firm solves problem (19). Recall from Section 3 that equilibrium profits take the simple form $\pi = AZ(\alpha, \mu) N^{1+\frac{\beta(\kappa+1)-1}{1-\beta}} - C(N)$ (see (25)), with $Z(\alpha, \mu)$ given by (26). We now exploit this representation to embody our partial equilibrium model in a general equilibrium framework.

6.1 Basics

Assume that there exists a continuum of final goods $q(j)$, with $j \in [0, Q]$, where Q represents the number (measure) of final goods. Consumer preferences are given by the quasi-linear utility function:

$$u = \left(\int_0^Q q(z)^\beta dz\right)^{1/\beta} - c_x e, \quad 0 < \beta < 1, \tag{38}$$

where e is the total effort exerted by this individual as a worker,[37] and the elasticity of substitution between final goods, $1/(1-\beta)$, is greater than 1, so that final goods are gross substitutes in consumption.

As is well known, these preferences imply the demand function $q(z) = \left[p(z)/p^I\right]^{-1/(1-\beta)} S/p^I$, where $p(z)$ is the price of good z, S is the aggregate spending level, and $p^I = \left[\int_0^Q p(i)^{-\beta/(1-\beta)} di\right]^{-(1-\beta)/\beta}$ is the ideal price index, which we will take as numeraire, i.e., $p^I = 1$. In other words, these preferences imply the demand function $A(p(z))^{-1/(1-\beta)}$ that we have used in the derivation of the profit function, where $A = S$.

We assume that firms (sectors) differ according to their technological complementarity, α, and denote the cumulative distribution function of α by $H(\alpha)$ with support as a connected subset of $(0, 1)$. This implies that if Q products are available for consumption, a fraction $H(\alpha)$ of them are produced with elasticities of substitution smaller than $1/(1-\alpha)$.

The key general equilibrium interaction will come for competition between sectors (firms) for scarce resources. Assume that there is one scarce input, labor, in fixed supply L that is used for setting up and manning organizations. The wage rate in terms of real consumption is denoted by w and is determined in general equilibrium with all other endogenous variables. More specifically, let us assume that

$$C(N) = wC_L(N),$$

where $C_L(N)$ is the amount of labor needed to construct and operate a production facility with the division of labor N. For example, if all that a firm needs to do is to make sure that N agents

[37] In particular, the cost of effort c_x corresponds to the constant marginal rate of substitution between effort and consumption.

act as workers to perform the productive tasks, then $C_L(N) = N$. But this function can be highly nonlinear when other activities are needed to set up the organization of production. Clearly Assumption 1 now applies to $C_L(N)$.

Using this notation, the first-order condition in the maximization of (25) for a firm with substitution parameter α is

$$\frac{\beta\kappa}{1-\beta} AZ(\alpha,\mu) N(\alpha,\mu)^{\frac{\beta(\kappa+1)-1}{1-\beta}} = wC_L'[N(\alpha,\mu)], \quad \text{for all } \alpha \in (0,1), \tag{39}$$

where $N(\alpha,\mu)$ denotes the division of labor of the producer with complementarity given by α when a fraction μ of tasks are contractible.

This equation shows that the key price that needs to adjust is w/A. It is convenient for what follows to define $a \equiv A/w$, which can be thought of as an adjusted extent of the market, and rewrite (39) as:

$$\frac{\beta\kappa}{1-\beta} aZ(\alpha,\mu) N(\alpha,\mu)^{\frac{\beta(\kappa+1)-1}{1-\beta}} = C_L'[N(\alpha,\mu)], \quad \text{for all } \alpha \in (0,1).$$

Although a is endogenous in general equilibrium, each firm takes it as given. Proposition 2 implies that firms with higher elasticities of substitution choose finer divisions of labor. Therefore this equation implies a positive correlation between the degree of substitutability of tasks and the division of labor in the cross-section of firms.

Given each firm's desired level of division of labor, $N(\alpha,\mu)$, their demand for labor is $C_L([N(\alpha,\mu)]$. This implies that market clearing for labor can be expressed as[38]

$$Q\int_0^1 C_L[N(\alpha,\mu)]\, dH(\alpha) = L. \tag{40}$$

The left-hand side describes the demand for labor while the right-hand side describes its supply. This market clearing equation will generate the main general equilibrium results in this section. Proposition 2 implies that $N(\alpha,\mu)$ is rising in μ, so we may expect better contracting institutions to increase the division of labor for all producers. But the resource constraint, (40), implies that not all $N(\alpha,\mu)$s can increase simultaneously. Consequently, the endogenous variable, a, has to adjust to clear the market. We now examine the implications of the competition for labor using three alternative models.[39]

6.2 Exogenous Number of Products

In this example we assume that the number of products, Q, is given, and that every product is produced by a different firm. The labor market clearing condition is given by (40). Since the division of labor is increasing in the extent of the market a and the cost function $C_L(N)$ is increasing in N,

[38] More formally, (40) should hold with complementary slackness together with $w \geq 0$, but it is straightforward to see that the wage will always be positive.

[39] We have also worked out an endogenous growth model with expanding product variety, in which new entrants draw an α from a known distribution. In this model, the long-run rate of growth depends on the degree of contract incompleteness. We do not discuss this model in order to save space.

the demand for labor, i.e., the left-hand side of (40), is an increasing function of a, and (40) yields a unique solution for a. Using this value of a in (39) then determines the cross-sectional variation in the division of labor.

To illustrate general equilibrium interactions, and in particular, the implications of differences in contracting institutions, compare two countries that are identical, except for their μs. Differentiating the first-order condition (39) yields

$$\hat{a} + \zeta_{\mu}(\alpha, \mu) \hat{\mu} = \lambda [N(\alpha, \mu)] \hat{N}(\alpha, \mu), \tag{41}$$

where $\hat{y}$ represents the proportional rate of change of variable y, $\zeta_{\mu}(\alpha, \mu)$ is the elasticity of $Z(\alpha, \mu)$ with respect to μ and $\lambda(N)$ is the elasticity of the marginal cost curve $C_L'(N)$ minus $[\beta(\kappa + 1) - 1] / (1 - \beta)$, i.e.,

$$\lambda(N) \equiv \frac{C''(N) N}{C'(N)} - \frac{\beta(\kappa + 1) - 1}{1 - \beta}.$$

Assumption 1 implies that $\lambda(N) > 0$. Moreover, we show in the proof of Proposition 2 in Appendix A that $\zeta_{\mu}(\alpha, \mu) > 0$ and $\zeta_{\mu}(\alpha, \mu)$ is declining in α.

Obtaining the relationship between μ and a is straightforward. Differentiating (40) yields

$$\int_0^1 \sigma_L(\alpha) \hat{N}(\alpha, \mu) \, dH(\alpha) = 0, \tag{42}$$

where $\sigma_L(\alpha) = C_L'[N(\alpha, \mu)] N(\alpha, \mu)$. Substituting (41) into (42) gives a simple relationship between μ and a,

$$\hat{a} = -\frac{\int_0^1 \sigma_L(\alpha) \zeta_{\mu}(\alpha, \mu) \lambda [N(\alpha, \mu)]^{-1} dH(\alpha)}{\int_0^1 \sigma_L(\alpha) \lambda [N(\alpha, \mu)]^{-1} dH(\alpha)} \hat{\mu},$$

establishing that a is declining in μ. In other words, when contracting institutions improve, wages increase relative to expenditure and a will decline.

Equation (42) also shows that $\hat{N}(\alpha, \mu)$ (in response to $\hat{\mu}$) can only be positive for some αs and has to be negative for others, because the resource constraint implies that not all firms can improve their division of labor. The question is therefore whether the division of labor improves in high or low α firms. The answer is clear from (41): since the left-hand side is decreasing in α, it is negative for high α's and positive for low α's. Therefore, the division of labor improves in firms that are more contract dependent, i.e., low α firms, and worsens in firms that are less contract dependent, i.e., high α firms. This is a reflection of the negative cross partial $\partial \zeta_{\mu}(\alpha, \mu) / \partial \alpha$. More precisely, there exists a critical value α_{μ} such that $\hat{N}(\alpha, \mu) > 0$ for all $\alpha < \alpha_{\mu}$ and $\hat{N}(\alpha, \mu) < 0$ for all $\alpha > \alpha_{\mu}$.

We summarize this result as:

Proposition 8 Suppose Assumption 1 holds. Then there exists $\alpha_{\mu} \in (0, 1)$ such that in the general equilibrium economy with Q constant, an increase in μ raises the division of labor $N(\alpha, \mu)$ in all firms with $\alpha < \alpha_\mu$ and reduces it in all firms with $\alpha > \alpha_\mu$.

6.3 Comparative Advantage

We now extend the analysis to a world with two trading countries, indexed by $\ell = 1, 2$, to derive implications for comparative advantage as a function of contracting institutions.[40] We continue to assume that there are fixed numbers of products, and every product is distinct not only from other products produced in its own country but also from products produced in the foreign country. All products can be freely traded between the two countries.

We focus on the impact of contract incompleteness on comparative advantage. For this purpose suppose that the two countries are identical, in particular, $L^1 = L^2$, $Q^1 = Q^2$ and $H^1(\alpha) = H^2(\alpha)$ for all $\alpha \in (0,1)$, but the fraction of activities μ^ℓ that are contractible differs across countries. Without loss of generality, we assume that $\mu^1 > \mu^2$, so that country 1 has better contracting institutions.

The equilibrium condition for the division of labor, (39), holds in both countries, with different wage rates, w^ℓ, for the two countries (A is the same for both countries and equal to world expenditure, since all goods are traded freely). Defining $a^\ell \equiv A/w^\ell$ and $N^\ell(\alpha) \equiv N(\alpha, \mu^\ell)$, this implies that

$$\frac{\beta\kappa}{1-\beta} a^\ell Z\left(\alpha, \mu^\ell\right) N^\ell(\alpha)^{\frac{\beta(\kappa+1)-1}{1-\beta}} = C_L'\left[N^\ell(\alpha)\right], \quad \text{for all } \alpha \in (0,1). \tag{43}$$

In addition, the labor market clearing condition, (40), holds in both countries. This immediately implies that the country with better contracting institutions, i.e., country 1, has higher wages, thus lower a^ℓ.[41]

The pattern of trade can now be determined by comparing the revenues of firms with the same value of α (technological complementarity) in the two countries. Appendix A (see the proof of Proposition 2) shows that revenue of a producer with elasticity α in country ℓ is given by:[42]

$$R^\ell(\alpha) = AZ\left(\alpha, \mu^\ell\right) N^\ell(\alpha)^{\frac{\beta\kappa}{1-\beta}} \frac{1-\beta\left(1-\mu^\ell\right)}{(1-\beta)\left[1-\beta\left(1-\mu^\ell\right)\frac{\alpha}{\alpha+\beta}\right]}. \tag{44}$$

Revenues are increasing in total world expenditure, A, in $Z(\alpha, \mu)$, in the division of labor, but also include an additional correction which is the last term.[43]

If $R^1(\alpha)/R^2(\alpha) > \int_0^1 R^1(\alpha)\, dH(\alpha) / \int_0^1 R^2(\alpha)\, dH(\alpha)$, then country 1 is a net exporter of goods with substitution parameter α. Consequently, we simply need to determine the distribution

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[40] The link between contracting institutions and endogenous comparative advantage was previously discussed by Levchenko (2003), Costinot (2004), Nunn (2004) and Antràs (2005).

[41] Suppose not, then from (43), $N^1(\alpha) > N^2(\alpha)$ for all α (since $N(\alpha)$ is increasing in μ for given a), and this would violate market clearing.

[42] Firms use their revenue to pay for labor costs and compensation of worker-suppliers, with the rest being profits.

[43] This last term does not appear in the profit function, since it corresponds to the part of the revenue paid out to workers to compensate for their effort.

of $R^1(\alpha)/R^2(\alpha)$ across different αs. From (44), we have

$$\frac{R^1(\alpha)}{R^2(\alpha)} = \frac{Z(\alpha,\mu^1)}{Z(\alpha,\mu^2)} \left[\frac{N^1(\alpha)}{N^2(\alpha)}\right]^{\frac{\beta\kappa}{1-\beta}} \frac{1-\beta(1-\mu^1)}{1-\beta(1-\mu^2)} \frac{1-\beta(1-\mu^2)\frac{\alpha}{\alpha+\beta}}{1-\beta(1-\mu^1)\frac{\alpha}{\alpha+\beta}}. \tag{45}$$

Proposition 8 implies that there exists an α_μ such that $N^1(\alpha) < N^2(\alpha)$ for all $\alpha > \alpha_\mu$ and $N^1(\alpha) > N^2(\alpha)$ for all $\alpha < \alpha_\mu$. As a result, country 1, which has the better contracting institutions, exports some low-α products and imports some high-α products (remember that $Z(\alpha,\mu^1) > Z(\alpha,\mu^2)$). Nevertheless, when the elasticity of marginal cost $C'_L(N)$ is not constant, country 1 may also export some high-α products. On the other hand, when $\lambda(N)$ is a constant, $\hat{N}(\alpha,\mu)$ decreases further with α when μ is higher (because the cross partial $\partial\zeta_\mu(\alpha,\mu)/\partial\alpha$ is negative). This implies that $R^1(\alpha)/R^2(\alpha)$ is declining with α in equation (45). Consequently, there exists $\bar{\alpha}_\mu \in (0,1)$ such that $R^1(\alpha)/R^2(\alpha) > \int_0^1 R^1(\alpha)\,dH(\alpha) \,/\, \int_0^1 R^2(\alpha)\,dH(\alpha)$ for all $\alpha < \bar{\alpha}_\mu$ and the opposite inequality holding for all $\alpha > \bar{\alpha}_\mu$. In this event country 1, with the better contracting institutions, is a net exporter in low α sectors and a net importer in high α sectors. This result is summarized in the next proposition:

Proposition 9 Suppose Assumption 1 holds and $\lambda(N)$ is constant. Then there exists $\bar{\alpha}_\mu \in (0,1)$ such that in the two-country world equilibrium, country 1 with $\mu^1 > \mu^2$ is a net exporter of the products with $\alpha < \bar{\alpha}_\mu$ and a net importer of products with $\alpha > \bar{\alpha}_\mu$.

The implication of this result is that differences in contracting institutions have created endogenous comparative advantage for the country with better contracting institutions towards sectors that are more "contract dependent," which, in this case, are those with greater technological complementarity.

6.4 Free Entry

We have so far assumed that the number of products Q is fixed in every country. Another source of general equilibrium interactions comes from endogenizing Q through a free entry condition. To briefly discuss these issues, suppose that there is free entry into the production of final goods and the number of products is endogenously determined. In particular, suppose that an entrant faces a fixed cost of entry wf, where f is the amount of labor required for entry. This cost is born in addition to the cost of division of labor. Moreover, in the spirit of Hopenhayn (1992) and Melitz (2003), assume that an entrant does not know a key parameter of the technology prior to entry. While in their models the entrant does not know its productivity, we assume instead that it does not know α (but knows that α is drawn from the cumulative distribution function $H(\alpha)$). Since the relationship between α and productivity is determined in general equilibrium, the distribution of productivity levels is endogenous.

After entry a firm learns its α and maximizes the profit function (25). This leads to the first-order condition (39), which determines the division of labor as a function of the extent of the market

a, the degree of contract incompleteness μ and its α. Let us define

$$\Pi(\alpha, \mu, a) = aZ(\alpha, \mu) N(\alpha, \mu)^{1+\frac{\beta(\kappa+1)-1}{1-\beta}} - C_L[N(\alpha, \mu)],$$

with $Z(\alpha, \mu)$ given by (26), and $N(\alpha, \mu)$ as the profit-maximizing choice of division of labor (which we do not condition on a to simplify notation as before). In other words, $\Pi(\alpha, \mu, a)$ is the indirect profit function, with profits measured in units of labor. $\Pi(\alpha, \mu, a)$ is increasing in all three arguments.

Free entry implies that expected profits must equal the entry cost wf,

$$\int_0^1 \Pi(\alpha, \mu, a)\, dH(\alpha) = f. \tag{46}$$

Since the expected profits are increasing in the adjusted extent of the market, a, this free entry condition uniquely determines the equilibrium a (i.e., without reference to the labor market clearing condition, (40))

Now, given the equilibrium value of a, the first-order condition (39) determines the cross-sectional variation in the division of labor. Firms with larger αs choose a finer division of labor and they are more productive. Consequently, the distribution of α induces a distribution of productivity and firm size.[44]

Finally, we can use the labor market clearing condition to determine the number of entrants. Since labor is now used to cover both the costs for entry and the division of labor, the market clearing condition (40) is replaced by

$$Q\left[\int_0^1 C_L([N(\alpha, \mu)]\, dH(\alpha) + f\right] = L.$$

Using the values of $N(\alpha, \mu)$ that were determined by (39) and (46), we can use this modified labor market clearing condition to solve for the number of entrants Q.[45]

7 Conclusion

In this paper, we develop a tractable framework for the analysis of the impact of the degree of contract incompleteness (or more generally contracting institutions) and technological complementarities on the equilibrium division of labor. In our model, a firm decides the division of labor and contracts with a set of worker-suppliers on a subset of activities they have to perform. Investments

[44] The degree of dispersion of productivity depends on the degree of contract incompleteness. In particular, in countries with better contracting institutions there will be less productivity and size dispersion.

[45] Another interesting implication of this analysis is that the overall supply of labor L is not a source of comparative advantage. If two countries that differ only in L freely trade with each other, their wages are equalized, their division of labor is the same in every industry α, and they have the same distribution of productivity and firm size. The only difference is that the larger country has proportionately more final good producers. This contrasts with the case with exogenous number of products, where differences in L would generate comparative advantage.

in the remaining activities are noncontractible, so workers choose them anticipating the ex post bargaining equilibrium.

We model the ex post bargaining problem using the Shapley value and provide a characterization of the division of surplus. Using this characterization, we show that greater contract incompleteness reduces both the division of labor and the equilibrium level of productivity given the division of labor. The impact of contract incompleteness is greater when there is greater complementarities between the tasks performed by different workers. Similar result hold when credit markets are imperfect, so that ex ante transfers are not possible. Interestingly, however, imperfect credit markets may increase (measured) productivity, because the firm may force its workers to overinvest in contractible activities in order to extract the surplus they capture in the bargaining game. We also show how the firm may want to hire worker-suppliers as employees in order to change the bargaining game, and how this interacts with the equilibrium division of labor.

Finally, in the last section, we embody the partial equilibrium model in a series of general equilibrium setups and derive implications for the cross-sectional distribution of division of labor and the impact of contracting institutions on comparative advantage. The most interesting result here is that societies with better contracting institutions have comparative advantage in sectors with greater technological complementarities (which are the more "contract-dependent" sectors, since contractual problems create more inefficiency in those sectors).

The analysis of general equilibrium is tractable thanks to the reduced-form (equilibrium) profit function derived in partial equilibrium, where the degree of contract incompleteness and technological complementarities affect a single derived parameter. This property makes it relatively simple to use our framework in various applications.

A number of areas are left for future research. These include, but are not limited to, the following:

- It would be interesting to derive the implications of this model for the relationship between (contracting) institutions and productivity across countries. This would be an important area for future research partly because, despite increasing evidence that differences in (total factor) productivity are an important element of cross-country differences in income per capita, we are far from a theory of productivity differences. Our model leads to both endogenous differences in the "technology" of production (as measured by the division of labor) and in the efficiency with which a given technology is used.

- The model assumes that all activities are symmetric. An important extension is to see whether similar results hold with a more general production function, where the firm may wish to treat some workers/tasks differently than others, for example, depending on how essential they are for production.

- Another area for future study is a more systematic investigation (beyond what we have presented in Appendix B) of the joint determination of the range of tasks and the number of workers performing these tasks. Such an analysis would enable us to incorporate both the forces analyzed in this paper and those emphasized by Marglin (1974) in a single framework.

- Finally, it is important to investigate whether the relationship between contracting institutions, technological complementarities and the division of labor is fundamentally different when we use different approaches to the theory of the firm, for example, as in the work by Holmstrom and Milgrom (1991).

Appendix A: Proofs of the Main Results

Second-Order Conditions in the Complete-Contracting Case

Let $\Pi = A^{1-\beta} N^{\beta(\kappa+1)} x^{\beta} - c_x N x - C(N)$. Using the first-order conditions (6) and (7), the matrix of the second-order conditions can be expressed as:

$$\begin{pmatrix} \partial^2\Pi/\partial x^2 & \partial^2\Pi/\partial x \partial N \\ \partial^2\Pi/\partial N \partial x & \partial^2\Pi/\partial N^2 \end{pmatrix} = \begin{pmatrix} -(1-\beta)\kappa N c_x^2 (C')^{-1} & -c_x[1-\beta(1+\kappa)] \\ -c_x[1-\beta(1+\kappa)] & -N^{-1}[1-\beta(1+\kappa)](\kappa+1)\kappa^{-1}C' - C'' \end{pmatrix}.$$

The second-order conditions are satisfied if this matrix is negative definite, which requires its diagonal elements to be negative and its determinant to be positive. The first diagonal element is negative, the second diagonal element is negative if and only if

$$\frac{C''N}{C'} > \frac{[\beta(1+\kappa)-1](\kappa+1)}{\kappa},$$

and the determinant is positive if and only if

$$\frac{C''N}{C'} > \frac{\beta(1+\kappa)-1}{1-\beta}.$$

Note that if $\beta(1+\kappa) < 1$ both these conditions are satisfied, and if $\beta(1+\kappa) \geq 1$ the second inequality implies the first inequality. Therefore Assumption 1 is necessary and sufficient for the second-order conditions to be satisfied.

Proof of Proposition 1

The first part of the proposition is a direct implication of Assumption 1. The comparative statics of N^* follow from the implicit function theorem by noting that, except for β, N^* increases in response to an increase in a parameter if and only if this parameter raises the left-hand side of (6). Using the results concerning the response of N^* to changes in parameters together with (7) then implies the responses of x^* to parameter changes.

Proof of Lemma 1 and Related Results

Let there be M workers each one controlling a range $\varepsilon = N/M$ of the continuum of tasks. Due to symmetry, all workers provide an amount x_c of contractible activities. As for the noncontractible activities, consider a situation in which a worker j supplies an amount $x_n(j)$ per non-contractible activity, while the $M-1$ remaining workers supply the same amount $x_n(-j)$ (note that we are again appealing to symmetry).

To compute the Shapley value for this particular worker j, we need to determine the marginal contribution of this worker to a given coalition of agents. A coalition of n workers and the firm yields a sale revenue of

$$F_{IN}(n, N; \varepsilon) = A^{1-\beta} N^{\beta(\kappa+1-1/\alpha)} x_c^{\beta\mu} \left[(n-1)\varepsilon x_n(-j)^{(1-\mu)\alpha} + \varepsilon x_n(j)^{(1-\mu)\alpha}\right]^{\beta/\alpha}, \tag{A1}$$

when the worker j is in the coalition, and a sale revenue

$$F_{OUT}(n, N; \varepsilon) = A^{1-\beta} N^{\beta(\kappa+1-1/\alpha)} x_c^{\beta\mu} \left[n\varepsilon x_n(-j)^{(1-\mu)\alpha}\right]^{\beta/\alpha} \tag{A2}$$

when worker j is *not* in the coalition. Notice that even when $n < N$, the term $N^{\beta(\kappa+1-1/\alpha)}$ remains in front, because it represents a feature of the technology, though productivity suffers because the term in square brackets is lower.

Following the notation in the main text, the Shapley value of player j is

$$s_j = \frac{1}{(M+1)!} \sum_{g \in G} \left[v\left(z_g^j \cup j\right) - v\left(z_g^j\right)\right]. \tag{A3}$$

The fraction of permutations in which $g(j) = i$ is $1/(M+1)$ for every i. If $g(j) = 0$ then $v\left(z_g^j \cup j\right) = v\left(z_g^j\right) = 0$, because in this event the firm is necessarily ordered *after* j. If $g(j) = 1$ then the firm is ordered before j with probability $1/M$ and after j with probability $1 - 1/M$. In the former case $v\left(z_g^j \cup j\right) = F_{IN}(1, N; \varepsilon)$, while in the latter case $v\left(z_g^j \cup j\right) = 0$. Therefore the conditional expected value of $v\left(z_g^j \cup j\right)$, given $g(j) = 1$, is $\frac{1}{M} F_{IN}(1, N; \varepsilon)$. By similar reasoning, the conditional expected value of $v\left(z_g^j\right)$ is $\frac{1}{M} F_{OUT}(0, N; \varepsilon)$. Repeating the same argument for $g(j) = i$, $i > 1$, the conditional expected value of $v\left(z_g^j \cup j\right)$, given $g(j) = i$, is $\frac{i}{M} F_{IN}(i, N; \varepsilon)$, and the conditional expected value of $v\left(z_g^j\right)$ is $\frac{i}{M} F_{OUT}(i-1, N; \varepsilon)$. It follows from (A3) that

$$\begin{aligned} s_j &= \frac{1}{(M+1)M} \sum_{i=1}^{M} i \left[F_{IN}(i, N; \varepsilon) - F_{OUT}(i-1, N; \varepsilon)\right] \\ &= \frac{1}{(N+\varepsilon)N} \sum_{i=1}^{M} i\varepsilon \left[F_{IN}(i, N; \varepsilon) - F_{OUT}(i-1, N; \varepsilon)\right]\varepsilon. \end{aligned}$$

Substituting for (A1) and (A2),

$$\begin{aligned} s_j = {} & \frac{A^{1-\beta} N^{\beta(\kappa+1-1/\alpha)} x_c^{\beta\mu}}{(N+\varepsilon)N} \sum_{i=1}^{M} i\varepsilon \left\{ i\varepsilon x_n(-j)^{(1-\mu)\alpha} + \varepsilon \left[x_n(j)^{(1-\mu)\alpha} - x_n(-j)^{(1-\mu)\alpha}\right]\right\}^{\beta/\alpha} \varepsilon \\ & - \frac{A^{1-\beta} N^{\beta(\kappa+1-1/\alpha)} x_c^{\beta\mu}}{(N+\varepsilon)N} \sum_{i=1}^{M} i\varepsilon \left[i\varepsilon x_n(-j)^{(1-\mu)\alpha} - \varepsilon x_n(-j)^{(1-\mu)\alpha}\right]^{\beta/\alpha} \varepsilon \end{aligned}$$

For ε small enough, the first-order Taylor expansion gives

$$s_j \simeq \frac{A^{1-\beta} N^{\beta(\kappa+1-1/\alpha)} x_c^{\beta\mu} (\beta/\alpha)\, \varepsilon x_n(j)^{(1-\mu)\alpha}}{(N+\varepsilon)N} \sum_{i=1}^{M} (i\varepsilon) \left[i\varepsilon x_n(-j)^{(1-\mu)\alpha}\right]^{(\beta-\alpha)/\alpha} \varepsilon,$$

or

$$\frac{s_j}{\varepsilon} \simeq \frac{A^{1-\beta} N^{\beta(\kappa+1-1/\alpha)} (\beta/\alpha) \left[\frac{x_n(j)}{x_n(-j)}\right]^{(1-\mu)\alpha} x_c^{\beta\mu} x_n(-j)^{\beta(1-\mu)}}{(N+\varepsilon)N} \sum_{i=1}^{M} (i\varepsilon)^{\beta/\alpha} \varepsilon.$$

Now taking the limit as $M \to \infty$, and therefore $\varepsilon = N/M \to 0$, the sum on the right hand side of this equation becomes a Riemann integral:

$$\lim_{M\to\infty} \left(\frac{s_j}{\varepsilon}\right) = \frac{A^{1-\beta} N^{\beta(\kappa+1-1/\alpha)} (\beta/\alpha) \left[\frac{x_n(j)}{x_n(-j)}\right]^{(1-\mu)\alpha} x_c^{\beta\mu} x_n(-j)^{\beta(1-\mu)}}{N^2} \int_0^N z^{\beta/\alpha} dz.$$

Solving the integral delivers

$$\lim_{M\to\infty}(s_j/\varepsilon) = (1-\gamma)\,A^{1-\beta}\left[\frac{x_n(j)}{x_n(-j)}\right]^{(1-\mu)\alpha} x_c^{\beta\mu} x_n(-j)^{\beta(1-\mu)}\,N^{\beta(\kappa+1)-1},$$

with $\gamma = \alpha/(\alpha+\beta)$. This corresponds to equation (14) in the main text, and completes the proof of the lemma.

In addition, imposing symmetry, i.e., $x_n(j) = x_n(-j)$, the firm's payoff is

$$s_0 = A^{1-\beta}N^{\beta(\kappa+1)}x_c^{\beta\mu}x_n{}^{\beta(1-\mu)} - Ns_j = \gamma A^{1-\beta}x_c^{\beta\mu}x_n^{\beta(1-\mu)}N^{\beta(\kappa+1)},$$

as stated in equation (17) in the text.

We can also compute the marginal contribution of a worker to a coalition of size n when the technology choice is N. Imposing symmetry, i.e., $x_n(j) = x_n(-j)$, if the worker controls a range ε of tasks, her marginal contribution is:

$$\psi_\varepsilon = A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}x_c^{\beta\mu}\left\{\left[(n+\varepsilon)\,x_n^{(1-\mu)\alpha}\right]^{\beta/\alpha} - \left[nx_n^{(1-\mu)\alpha}\right]^{\beta/\alpha}\right\},$$

which taking the limit $\varepsilon \to 0$ delivers

$$\lim_{\varepsilon\to 0}\left(\frac{\psi_\varepsilon}{\varepsilon}\right) = \frac{A^{1-\beta}N^{\beta(\kappa+1)}x_c^{\beta\mu}x_n^{\beta(1-\mu)}}{n}(\beta/\alpha)\left(\frac{n}{N}\right)^{\beta/\alpha} = \frac{R}{n}(\beta/\alpha)\left(\frac{n}{N}\right)^{\beta/\alpha},$$

where $R = A^{1-\beta}N^{\beta(\kappa+1)}x_c^{\beta\mu}x_n^{\beta(1-\mu)}$. As stated in the main text, this marginal contribution is increasing in α when $n/N < \exp(-\alpha/\beta)$ and decreasing in α for $n/N > \exp(-\alpha/\beta)$.

Proof of Proposition 2

First, we verify that the second-order conditions are again satisfied under Assumption 1. To see this, note that the problem in (9) is convex and delivers a unique $x_n = \bar{x}_n(N, x_c)$, as given in (18). Plugging this expression into (19) we obtain

$$\pi = A^{\frac{1-\beta}{1-\beta(1-\mu)}}\Psi N^{1+\frac{\beta(\kappa+1)-1}{1-\beta(1-\mu)}}x_c^{\beta\mu/[1-\beta(1-\mu)]} - c_x x_c N\mu - C(N) \tag{A4}$$

where

$$\Psi \equiv \left[(c_x)^{-1}\alpha(1-\gamma)\right]^{\beta(1-\mu)/[1-\beta(1-\mu)]}\left[1-\alpha(1-\mu)(1-\gamma)\right].$$

The second-order conditions can be checked, analogously to the case with complete contracts, by computing the Jacobian and checking that it is negative definite. We present here an alternative proof which also serves to illustrate how the reduced-form profit function (25) in the main text is derived. In particular, notice that for a given level of N, the problem of choosing x_c is convex (since $\beta\mu < 1-\beta+\beta\mu$) and delivers a unique solution

$$x_c = A\left[\frac{\beta c_x^{-1}\Psi}{1-\beta(1-\mu)}\right]^{[1-\beta(1-\mu)]/(1-\beta)}N^{\frac{\beta(\kappa+1)-1}{1-\beta}}. \tag{A5}$$

Plugging this solution in (A4) then delivers

$$\pi = AZN^{1+\frac{\beta(\kappa+1)-1}{1-\beta}} - C(N)$$

where

$$Z \equiv \left[\frac{\beta c_x^{-1}}{1-\beta(1-\mu)}\right]^{\beta\mu/(1-\beta)} \Psi^{[1-\beta(1-\mu)]/(1-\beta)} \left(\frac{1-\beta}{1-\beta(1-\mu)}\right),$$

as in equation (26) in the text. This reduced-form expression of the profit function immediately implies that $\partial^2\Pi/\partial N^2 < 0$ if and only if the second part of Assumption 1 holds. Finally, from equation (A5) we also have $c_x x_c N\mu = AZN^{1+\frac{\beta(\kappa+1)-1}{1-\beta}}\beta\mu/(1-\beta)$, which combined with (15) and (24) implies that the firm's revenues are:

$$\begin{aligned} R &= \pi + C(N) + c_x x_c N\mu + c_x x_n N(1-\mu) \\ &= AZN^{1+\frac{\beta(\kappa+1)-1}{1-\beta}} \left(\frac{1-\beta(1-\mu)}{(1-\beta)\left[1-\beta(1-\mu)\frac{\alpha}{\alpha+\beta}\right]}\right), \end{aligned}$$

which will be used in Section 6 (see equation (44)).

Next, the comparative static results follow from the implicit function theorem as in the proof of Proposition 1. First, $\partial\tilde{N}/\partial A > 0$ follows immediately. To show that $\partial\tilde{N}/\partial\alpha > 0$ and $\partial\tilde{N}/\partial\mu > 0$, let us take logarithms of both sides of (20), to obtain

$$\frac{\beta(\kappa+1)-1}{1-\beta}\ln\tilde{N} + \ln\left(A\kappa\beta^{\frac{1}{1-\beta}}c_x^{-\frac{\beta}{1-\beta}}\right) + F(\vartheta,\mu) = \ln C'\left(\tilde{N}\right),$$

where

$$F(\vartheta,\mu) = \frac{1-\beta(1-\mu)}{1-\beta}\ln\left(\frac{1-\vartheta(1-\mu)}{1-\beta(1-\mu)}\right) + \frac{\beta(1-\mu)}{1-\beta}\ln\left(\vartheta\beta^{-1}\right),$$

and $\vartheta = \alpha(1-\gamma) < \beta$ is monotonically increasing in α. Simple differentiation delivers

$$\frac{\partial F(\vartheta,\mu)}{\partial\vartheta} = \frac{(1-\mu)(\beta-\vartheta)}{(1-\beta)\vartheta(1-\vartheta(1-\mu))},$$

which implies $\partial F/\partial\alpha > 0$, and establishes that $\partial\tilde{N}/\partial\alpha > 0$.

Furthermore,

$$\frac{\partial F(\vartheta,\mu)}{\partial\mu} = \frac{\vartheta-\beta+\beta(1-\vartheta(1-\mu))\left(\ln\left(\frac{1-\vartheta(1-\mu)}{1-\beta(1-\mu)}\right)+\ln\left(\frac{\beta}{\vartheta}\right)\right)}{(1-\vartheta(1-\mu))(1-\beta)}$$

and

$$\frac{\partial^2 F(\vartheta,\mu)}{\partial\mu^2} = -\frac{(\beta-\vartheta)^2}{(1-\vartheta(1-\mu))^2(1-\beta(1-\mu))(1-\beta)} < 0.$$

Thus, $\partial F(\vartheta,\mu)/\partial\mu$ reaches its minimum over the set $\mu\in[0,1]$ at $\mu=1$, in which case it equals

$$\frac{\vartheta-\beta+\beta\ln(\beta/\vartheta)}{1-\beta} > 0,$$

where the inequality follows from $\vartheta-\beta+\beta\ln(\beta/\vartheta)$ being decreasing in ϑ for $\vartheta<\beta$, and equalling 0 at $\vartheta=\beta$. We thus have shown that $\partial F(\vartheta,\mu)/\partial\mu > 0$ for all μ, so that $\partial\tilde{N}/\partial\mu > 0$.

For the cross-partial derivative in the Proposition, simply note that

$$\frac{\partial^2 F(\vartheta,\mu)}{\partial\vartheta\partial\mu} = \frac{-(\beta-\vartheta)}{(1-\vartheta(1-\mu))^2\vartheta(1-\beta)} < 0,$$

which implies $\partial^2 \tilde{N}/\partial\alpha\partial\mu < 0$.

Incidentally, note that $\partial F(\vartheta, \mu)/\partial\mu > 0$ and $\partial^2 \tilde{N}/\partial\alpha\partial\mu < 0$ imply that the elasticity $\zeta_\mu(\alpha, \mu)$ defined in equation (41) is both positive and decreasing in α.

Next, note that straightforward differentiation of (24) delivers $\partial(\tilde{x}_n/\tilde{x}_c)/\partial A = 0$, $\partial(\tilde{x}_n/\tilde{x}_c)/\partial\alpha > 0$ and $\partial(\tilde{x}_n/\tilde{x}_c)/\partial\mu > 0$. And, in light of equation (21), the effects of A, α and μ on $\tilde{x}_c$ follow directly from those on $\tilde{N}$, where the inequalities becomes strict whenever $C''(\cdot) > 0$.

Finally, given that $\tilde{x}_c$ and $\tilde{x}_n$ are nondecreasing in A, α and μ, and given that $\tilde{N}$ is increasing in these parameters, it follows that productivity $\tilde{P}$ is also necessarily increasing in these parameters.

Proof of Proposition 3

The proof follows from Proposition 2, since $\left\{\tilde{N}, \tilde{x}_c, \tilde{P}\right\}$ converge to $\{N^*, x^*, P^*\}$ as $\mu \to 1$, and $\tilde{N}$, $\tilde{x}_c$, $\tilde{x}_c$, and $\tilde{P}$ are all increasing in μ (nondecreasing in μ, in the case of the investment levels).

Proof of Proposition 4

The proof is similar to that of Proposition 2. First, it is straightforward to establish that Assumption 1 ensures that the second-order conditions are met. This is easily verified by noting that equation (31) implicitly defines a reduced-form profit function analogous to that in (25), with the same exponent on N.

As in previous proofs, to show that $\partial\check{N}/\partial A > 0$, $\partial\check{N}/\partial\alpha > 0$, and $\partial\check{N}/\partial\mu > 0$ we need only show that the left-hand side of (32) is increasing in these three parameters. This is evident for the case of the size of the market A. For the other two parameters, let us take logarithms of both sides of (32), to obtain

$$\frac{\beta(\kappa+1)-1}{1-\beta}\ln\check{N} + \ln\left(\frac{\beta\kappa A}{1-\beta}c_x^{-\frac{\beta}{1-\beta}}\right) + G(\alpha,\mu) = \ln C'\left(\check{N}\right),$$

where

$$G(\alpha,\mu) \equiv \ln\gamma + \frac{\beta\mu}{1-\beta}\ln\left(\frac{1-\alpha+\alpha\mu}{\alpha\mu}\right) + \frac{\beta}{1-\beta}\ln\left(\alpha\left(1-\gamma\right)\right).$$

It is straightforward to show that $\partial\check{N}/\partial\mu > 0$. Note simply that because γ is independent of μ, we have that

$$\frac{\partial G(\alpha,\mu)}{\partial\mu} = \frac{\left((1-\alpha+\alpha\mu)\ln\left(\frac{(1-\alpha(1-\mu))}{\alpha\mu}\right) - (1-\alpha)\right)\beta}{(1-\alpha(1-\mu))(1-\beta)}$$

and

$$\frac{\partial^2 G(\alpha,\mu)}{\partial\mu\partial\alpha} = -\frac{(1-\alpha)\beta}{(1-\beta)(1-\alpha(1-\mu))^2\alpha} < 0.$$

Thus, $\partial G(\alpha,\mu)/\partial\mu$ reaches its minimum over the set $\alpha \in [0,1]$ at $\alpha = 1$, in which case it equals 0. From this we can conclude that for $\alpha < 1$, $\partial G(\alpha,\mu)/\partial\mu > 0$ and thus $\partial\check{N}/\partial\mu > 0$. Notice also that the cross partial derivative above also implies $\partial^2\check{N}/\partial\alpha\partial\mu < 0$.

To show that $\partial\check{N}/\partial\alpha > 0$, let us write:

$$\begin{aligned}\frac{\partial G(\alpha,\mu)}{\partial\alpha} &= \frac{\partial\ln\gamma}{\partial\alpha} + \frac{\beta}{1-\beta}\frac{\partial\ln(1-\gamma)}{\partial\alpha} + \frac{(1-\alpha)(1-\mu)\beta}{\alpha(1-\alpha(1-\mu))(1-\beta)} = \\ &= \frac{1}{\gamma}\left(1+\frac{\beta\gamma}{1-\beta-\gamma}\right)^{-1}\frac{\partial\gamma}{\partial\alpha} + \frac{(1-\alpha)(1-\mu)\beta}{\alpha(1-\alpha(1-\mu))(1-\beta)}.\end{aligned}$$

When $1-\beta-\gamma > 0$ this expression is positive, because γ increases in α. Next consider the case $1-\beta-\gamma < 0$.

Assumption 3 implies

$$1-\mu>1-\frac{(1-\alpha)(1-\beta)}{(1-\alpha)(1-\beta)-\frac{1-\beta-\gamma}{(1-\gamma)}}=\frac{\frac{1-\beta-\gamma}{(1-\gamma)}}{\frac{1-\beta-\gamma}{(1-\gamma)}-(1-\alpha)(1-\beta)}.$$

Therefore

$$\frac{\partial G(\alpha,\mu)}{\partial\alpha}>\frac{1}{\gamma}\left(1+\frac{\beta\gamma}{1-\beta-\gamma}\right)^{-1}\frac{\partial\gamma}{\partial\alpha}-\frac{(1-\beta-\gamma)}{\alpha\gamma(1-\beta)}.$$

It follows that as long as $1-\beta-\gamma<0$ we have $\partial G(\alpha,\mu)/\partial\alpha>0$ if

$$\frac{\alpha}{1-\gamma}\frac{\partial\gamma}{\partial\alpha}<1.$$

The last inequality follows since $\alpha(1-\gamma)=\alpha\beta/(\alpha+\beta)$ is increasing in α, proving that $\partial G(\alpha,\mu)/\partial\alpha>0$ and thus $\partial\check{N}/\partial\alpha>0$.

Let us next turn to the comparative statics related to the investment levels. First, note that differentiation of (30) delivers $\partial(\check{x}_n/\check{x}_c)/\partial A=0$, $\partial(\check{x}_n/\check{x}_c)/\partial\alpha>0$ and $\partial(\check{x}_n/\check{x}_c)/\partial\mu>0$. Next, in light of equation (34), the effects of μ on $\check{x}_n$ follow directly from those on $\check{N}$ (since γ is independent of μ). On the other hand, $\check{x}_n$ is nondecreasing in α provided that $\partial\check{N}/\partial\alpha>0$ (shown above) and that $\alpha(1-\gamma)/\beta\gamma$ is nondecreasing in α. The latter is clearly the case when $\gamma=\alpha/(\alpha+\beta)$, in which case $\alpha(1-\gamma)/\beta\gamma=1$. Lemma 3 states that this is also true for the general definition of γ in section 5.

Finally, consider the comparative statics related to productivity. The positive effect of A on $\check{P}$ follows directly from noting that $\check{x}_c$ and $\check{x}_n$ are nondecreasing in A, while $\check{N}$ is increasing in A. For the effect of μ on productivity note that, using (32), $\check{P}=\check{N}^{\kappa}\check{x}_c^{\mu}\check{x}_n^{1-\mu}$ can be rewritten as:

$$\check{P}=\left(\frac{1-\beta}{\beta\kappa\gamma}C'\left(\check{N}\right)\left(\frac{\check{N}}{A}\right)^{1-\beta}\right)^{1/\beta},$$

which depends on μ only through the term $C'\left(\check{N}\right)\check{N}^{1-\beta}$, thus establishing $\partial\check{P}/\partial\mu>0$.

Proof of Proposition 5

Consider first the inequality $1>\tilde{x}_n/\tilde{x}_c>\check{x}_n/\check{x}_c$. The first inequality is a direct implication of Proposition 3. As for the second inequality, straightforward manipulation of (24) and (30) indicates that $\tilde{x}_n/\tilde{x}_c>\check{x}_n/\check{x}_c$ provided that Assumption 3 holds.

Next, we note that the inequality $N^*>\tilde{N}$ was established in Proposition 3. Thus we only show that $\tilde{N}>\check{N}$. Using equations (20) and (32), and the arguments in previous propositions, it should be clear that this will be the case as long as

$$\frac{\beta^{\frac{1}{1-\beta}}\left[\frac{1-\alpha(1-\gamma)(1-\mu)}{1-\beta(1-\mu)}\right]^{\frac{1-\beta(1-\mu)}{1-\beta}}\left[\beta^{-1}\alpha(1-\gamma)\right]^{\frac{\beta(1-\mu)}{1-\beta}}}{\frac{\beta\gamma}{1-\beta}\left(\frac{1-\alpha+\alpha\mu}{\alpha\mu}\right)^{\frac{\beta\mu}{1-\beta}}\left[\alpha(1-\gamma)\right]^{\frac{\beta}{1-\beta}}}>1.$$

Taking logarithms on both sides of the inequality, and after some manipulation, we can write this condition as

$$H(\mu) \equiv \ln\left(\frac{1-\alpha+\alpha\mu}{\alpha\mu}\right)+\ln\left(\frac{\alpha(1-\gamma)}{\beta\gamma}(1-\beta)\right)+\frac{1-\beta(1-\mu)}{1-\beta}\ln\left(\frac{\frac{\beta}{(1-\gamma)}-\alpha\beta(1-\mu)}{1-\beta(1-\mu)}\frac{\mu}{1-\alpha+\alpha\mu}\right)>0.$$

Simple differentiation delivers

$$\begin{aligned} H'(\mu) &= \frac{\alpha(1-\gamma)((1-\alpha)(1-\beta)-\mu(\beta/(1-\gamma)-(\alpha+\beta)+\alpha\beta))}{(1-\alpha(1-\gamma)(1-\mu))(1-\alpha+\alpha\mu)(1-\beta)} \\ &\quad+\frac{\beta}{1-\beta}\ln\left(\frac{\frac{\beta}{(1-\gamma)}-\alpha\beta(1-\mu)}{1-\beta(1-\mu)}\frac{\mu}{1-\alpha+\alpha\mu}\right)\end{aligned}$$

which can be rewritten as

$$H'(\mu)=\frac{\beta}{1-\beta}\left(\frac{\alpha\mu}{1-\alpha+\alpha\mu}\psi-\ln(1+\psi)\right),$$

where

$$\psi \equiv \frac{(1-\gamma)((1-\alpha)(1-\beta)-\mu(\beta/(1-\gamma)-(\alpha+\beta)+\alpha\beta))}{(1-\alpha(1-\gamma)(1-\mu))\beta\mu}>0.$$

The fact that ψ is positive is implied by Assumption 3. Notice also that

$$\frac{\alpha\mu}{1-\alpha+\alpha\mu}(1+\psi)=\frac{\alpha(1-\gamma)(1-\beta(1-\mu))}{\beta(1-\alpha(1-\gamma)(1-\mu))}<1,$$

(where the inequality follows from $\beta>\alpha(1-\gamma)$), which implies (24) in the text.

We next use the fact that $b\psi-\ln(1+\psi)$ is decreasing in ψ whenever $b(1+\psi)<1$, to conclude that

$$H'(\mu)<0$$

whenever Assumption 3 holds. But note that Assumption 3 places an upper bound on μ, and thus $H(\mu)$ reaches its minimum at this upper bound, in which case it equals 0. Mathematically, we have that

$$H(\mu)>H((1-\alpha)(1-\beta)/(\beta/(1-\gamma)-(\alpha+\beta)+\alpha\beta))=0,$$

which completes the proof that $\tilde{N}>\check{N}$.

To obtain the ranking of productivity levels, combine (21), (22), (33), (34), and the definitions of $\tilde{P}$ and $\check{P}$, to obtain

$$\frac{\check{P}}{\tilde{P}}=\left(\frac{\check{N}}{\tilde{N}}\right)^{\kappa}\frac{\frac{\alpha(1-\gamma)}{\beta\gamma}(1-\beta)\left(\frac{1-\alpha+\alpha\mu}{\alpha\mu}\right)^{\mu}}{\left(\frac{\alpha(1-\gamma)[1-\beta(1-\mu)]}{\beta[1-\alpha(1-\gamma)(1-\mu)]}\right)^{1-\mu}}\frac{C'(\check{N})}{C'\left(\tilde{N}\right)}.$$

Using equations (20) and (32) to eliminate the terms $C'\left(\tilde{N}\right)$ and $C'(\check{N})$, and simplifying delivers:

$$\frac{\check{P}}{\tilde{P}}=\left(\frac{\check{N}}{\tilde{N}}\right)^{\frac{\kappa}{1-\beta}-1}\left(\frac{\alpha(1-\gamma)}{\beta}\frac{(1-\beta(1-\mu))(1-\alpha(1-\mu))}{\alpha\mu(1-\alpha(1-\gamma)(1-\mu))}\right)^{\frac{\mu}{1-\beta}}.$$

Part (i) of the proposition then simply follows from $\check{N}<\tilde{N}$ (as shown above) and the fact that, under Assumption 3, the second term in brackets is greater than one.

For part (ii) of the proposition, let us first show that $\check{P}/P^{*}$ is increasing in μ. Following a similar procedure as in the case of $\check{P}/\tilde{P}$ – but this time using (6) and (7) – we obtain:

$$\frac{\check{P}}{P^*} = \frac{\check{N}^{\kappa} C'\left(\check{N}\right)}{\left(N^*\right)^{\kappa} C'\left(N^*\right)} \frac{\alpha\left(1-\gamma\right)}{\beta\gamma} \left(1-\beta\right) \left(\frac{1-\alpha+\alpha\mu}{\alpha\mu}\right)^{\mu}.$$

Notice that this ratio is necessarily increasing in μ because $\partial\check{N}/\partial\mu > 0$, $C''\left(\check{N}\right) \geq 0$, and the last term is increasing in μ. To see the latter, take logarithms and notice that

$$\frac{\partial^2 \left(\mu \ln\left(\frac{1-\alpha+\alpha\mu}{\alpha\mu}\right)\right)}{\partial\mu\partial\alpha} = \frac{-\left(1-\alpha\right)}{\alpha\left(1-\alpha+\alpha\mu\right)^2} < 0,$$

and thus the partial derivative

$$\frac{\partial \left(\mu \ln\left(\frac{1-\alpha+\alpha\mu}{\alpha\mu}\right)\right)}{\partial\mu} = \frac{\left(1-\alpha+\alpha\mu\right)\ln\left(\frac{1-\alpha+\alpha\mu}{\alpha\mu}\right) - \left(1-\alpha\right)}{1-\alpha+\alpha\mu}$$

attains a minimum over the set $\alpha \in [0,1]$ at $\alpha = 1$, in which case it equals 0. This establishes that $\partial\left(\check{P}/P^*\right)/\partial\mu > 0$.

Next note that when $\mu \to 0$, this ratio goes to

$$\frac{\check{P}}{P^*} = \frac{\check{N}^{\kappa} C'\left(\check{N}\right)}{\left(N^*\right)^{\kappa} C'\left(N^*\right)} \frac{\alpha\left(1-\gamma\right)}{\beta\gamma} \left(1-\beta\right) < 1,$$

where the inequality follows from $\check{N} < N^*$.

Next, plugging (6) and (32) to eliminate the terms $C'\left(N^*\right)$ and $C'\left(\check{N}\right)$, and simplifying delivers

$$\frac{\check{P}}{P^*} = \left(\frac{\check{N}}{N^*}\right)^{\frac{\kappa}{1-\beta}-1} \left(\frac{\left(\frac{1-\alpha+\alpha\mu}{\alpha\mu}\right)^{\mu} \alpha\left(1-\gamma\right)}{\beta}\right)^{\frac{1}{1-\beta}}.$$

When $\mu \to 1$, this ratio becomes

$$\frac{\check{P}}{P^*} = \left(\frac{\check{N}}{N^*}\right)^{\frac{\kappa}{1-\beta}-1} \left(\frac{1-\gamma}{\beta}\right)^{\frac{1}{1-\beta}}.$$

Notice finally that when $\kappa \leq 1-\beta$ then $\left(\check{N}/N^*\right)^{\frac{\kappa}{1-\beta}-1} \geq 1$, which combined with $\gamma < 1-\beta$ implies $\check{P} > P^*$. This establishes that provided that $\kappa \leq 1-\beta$ and $\gamma < 1-\beta$, there exists a unique $\overline{\mu} \in (0,1)$, such that if $\mu < \overline{\mu}$, then $\check{P} < P^*$, but if $\mu > \overline{\mu}$, then $\check{P} > P^*$.

Proof of Lemma 2

The proof is similar to that of Lemma 1. A coalition of n workers and the firm yields a sale revenue of

$$F_{IN}\left(n, N; \varepsilon\right) = A^{1-\beta} N^{\beta(\kappa+1-1/\alpha)} x_c^{\beta\mu} \left[\left(n-1\right)\varepsilon x_n\left(-j\right)^{(1-\mu)\alpha} + \varepsilon x_n\left(j\right)^{(1-\mu)\alpha} + \left(N-n\varepsilon\right)\delta^{\alpha} x_n\left(-j\right)^{(1-\mu)\alpha}\right]^{\beta/\alpha}, \tag{A6}$$

when the worker j is in the coalition, and a sale revenue

$$F_{OUT}(n,N;\varepsilon) = A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}x_c^{\beta\mu}\left[n\varepsilon x_n(-j)^{(1-\mu)\alpha} + \varepsilon\delta^\alpha x_n(j)^{(1-\mu)\alpha} + (N-(n+1)\varepsilon)\,\delta^\alpha x_n(-j)^{(1-\mu)\alpha}\right]^{\beta/\alpha} \tag{A7}$$

when worker j is *not* in the coalition.

As in the case with $\delta = 0$, the Shapley value of worker j can be written as

$$s_j = \frac{1}{(N+\varepsilon)N}\sum_{i=1}^{M} i\varepsilon\left[F_{IN}(i,N;\varepsilon) - F_{OUT}(i-1,N;\varepsilon)\right]\varepsilon.$$

Plugging the new formulas for $F_{IN}(n,N;\varepsilon)$ and $F_{OUT}(n,N;\varepsilon)$ in (A6) and (A7) then delivers

$$\begin{aligned} s_j \;=\; & \frac{A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}x_c^{\beta\mu}}{(N+\varepsilon)N}\sum_{i=1}^{M} i\varepsilon\left\{\begin{array}{c} i\varepsilon(1-\delta^\alpha)\,x_n(-j)^{(1-\mu)\alpha} + \varepsilon\left[x_n(j)^{(1-\mu)\alpha} - x_n(-j)^{(1-\mu)\alpha}\right] \\ +N\delta^\alpha x_n(-j)^{(1-\mu)\alpha}\end{array}\right\}^{\beta/\alpha}\varepsilon \\ & -\frac{A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}x_c^{\beta\mu}}{(N+\varepsilon)N}\sum_{i=1}^{M} i\varepsilon\left[\begin{array}{c} i\varepsilon(1-\delta^\alpha)\,x_n(-j)^{(1-\mu)\alpha} + \varepsilon\left[\delta^\alpha x_n(j)^{(1-\mu)\alpha} - x_n(-j)^{(1-\mu)\alpha}\right] \\ +N\delta^\alpha x_n(-j)^{(1-\mu)\alpha}\end{array}\right]^{\beta/\alpha}\varepsilon. \end{aligned}$$

For ε small enough, the first-order Taylor expansion gives

$$\frac{s_j}{\varepsilon} \simeq \frac{A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}(\beta/\alpha)(1-\delta^\alpha)\left[\frac{x_n(j)}{x_n(-j)}\right]^{(1-\mu)\alpha} x_c^{\beta\mu}x_n(-j)^{\beta(1-\mu)}}{(N+\varepsilon)N}\sum_{i=1}^{M}(i\varepsilon)\left[(i\varepsilon)(1-\delta^\alpha)+N\delta^\alpha\right]^{(\beta-\alpha)/\alpha}\varepsilon.$$

Taking the limit as $M \to \infty$, now yields

$$\lim_{M\to\infty}\left(\frac{s_j}{\varepsilon}\right) = \frac{A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}(\beta/\alpha)(1-\delta^\alpha)\left[\frac{x_n(j)}{x_n(-j)}\right]^{(1-\mu)\alpha} x_c^{\beta\mu}x_n(-j)^{\beta(1-\mu)}}{N^2}\int_0^N z\left[z(1-\delta^\alpha)+N\delta^\alpha\right]^{(\beta-\alpha)/\alpha}dz.$$

Finally, integrating by parts delivers

$$\lim_{M\to\infty}(s_j/\varepsilon) = (1-\gamma)\,A^{1-\beta}\left[\frac{x_n(j)}{x_n(-j)}\right]^{(1-\mu)\alpha} x_c^{\beta\mu}x_n(-j)^{\beta(1-\mu)}\,N^{\beta(\kappa+1)-1},$$

where

$$\gamma \equiv \frac{\alpha\left(1-\delta^{\alpha+\beta}\right)}{(\alpha+\beta)(1-\delta^\alpha)},$$

as claimed in the Lemma.

Proof of Lemma 3

Let

$$\Lambda \equiv \ln\gamma = \ln\left(\frac{\alpha}{\alpha+\beta}\right) + \ln\left(\frac{1-\delta^{\alpha+\beta}}{1-\delta^\alpha}\right). \tag{A8}$$

From inspection of (A8), in order to prove that $\partial\Lambda/\partial\delta > 0$ it is sufficient to show that

$$\frac{\partial^2 (\ln (1-\delta^x))}{\partial\delta\partial x} = \frac{(\delta^x - x(\ln\delta) - 1)\,\delta^x}{\delta (1-\delta^x)^2} \geq 0 \tag{A9}$$

for all $x \in (0,2)$ and $\delta \in [0,1)$. But this inequality follows from the fact that $\delta^x - x(\ln\delta) - 1$ is decreasing in δ for $x \in (0,2)$ and $\delta \in [0,1)$, and equals zero when $\delta \to 1$. Hence, for $\delta \in [0,1)$, $\partial\Lambda/\partial\delta > 0$ and thus $\partial\gamma/\partial\delta > 0$. Notice in addition that using l'Hôpital rule, it is easily verified that $\lim_{\delta\to 1}\gamma = 1$.

To prove that $\partial\Lambda/\partial\beta < 0$, note that

$$\frac{\partial\Lambda}{\partial\beta} = -\frac{1}{\alpha+\beta} - (\ln\delta)\frac{\delta^{\alpha+\beta}}{1-\delta^{\alpha+\beta}},$$

while (A9) implies $\partial^2\Lambda/\partial\beta\partial\delta \geq 0$. We thus need only show that $\partial\Lambda/\partial\beta \leq 0$ when $\delta = 1$. We establish this by using l'Hôpital rule to show that

$$\lim_{\delta\to 1}\left(\frac{\delta^x \ln\delta}{1-\delta^x}\right) = -\frac{1}{x},$$

from which it follows that $\partial\Lambda/\partial\beta = 0$ when $\delta = 1$. This implies that for $\delta \in [0,1)$, $\partial\Lambda/\partial\beta < 0$ and thus $\partial\gamma/\partial\beta < 0$.

We next turn to prove that $\partial\Lambda/\partial\alpha > 0$. Note that

$$\frac{\partial\Lambda}{\partial\alpha} = \frac{\beta}{\alpha(\alpha+\beta)} + \frac{\left(1-\delta^\beta\right)\delta^\alpha(\ln\delta)}{\left(1-\delta^{\alpha+\beta}\right)(1-\delta^\alpha)}.$$

But letting $x = \alpha+\beta$, we can write

$$\frac{\partial^2\Lambda}{\partial\alpha\partial\beta} = \frac{\left(\delta^x - 2 - x^2\left(\ln^2\delta\right)\right)\delta^x + 1}{(1-\delta^x)^2 x^2}.$$

It suffices to show that $\partial^2\Lambda/\partial\alpha\partial\beta$ is positive, because in such case $\partial\Lambda/\partial\alpha$ would reach its minimum over the set $\beta \in [0,1]$ at $\beta = 0$, in which case $\partial\Lambda/\partial\alpha$ equals 0. We thus need only show that

$$g(x) = \delta^{2x} - 2\delta^x - x^2\left(\ln^2\delta\right)\delta^x + 1 > 0$$

for all $0 < x \leq 2$. Given that $g(0) = 0$, we prove this by showing that $g'(x) > 0$ for $x > 0$. In particular,

$$g'(x) = -\delta^x \ln(\delta)\left((1 + x(\ln\delta))^2 + 1 - 2\delta^x\right) > 0,$$

which is positive because $h(x) = (1 + x(\ln\delta))^2 + 1 - 2\delta^x$ is increasing in x and equals 0 at $x = 0$. To see that $h'(x) > 0$ note that

$$h'(x) = 2\ln(\delta)(x\ln(\delta) - \delta^x + 1),$$

and we showed above that $1 + x(\ln\delta) - \delta^x < 0$. This completes the proof that $\partial\Lambda/\partial\alpha > 0$.

We next show that $\alpha(1-\gamma)/\beta\gamma$ is nondecreasing in α. Notice that this is equivalent to showing that

$$\frac{\partial\left(\frac{(\alpha+\beta)(1-\delta^\alpha)}{\beta(1-\delta^{\alpha+\beta})} - \frac{\alpha}{\beta}\right)}{\partial\alpha} = \frac{\frac{1}{\beta}\delta^\alpha\left(1-\delta^\beta\right)}{\left(1-\delta^{\alpha+\beta}\right)^2}\left(-(\ln\delta)(\alpha+\beta) - \left(1-\delta^{\alpha+\beta}\right)\right)$$

is nonnegative. But this follows from the fact that $\delta^x - (\ln \delta) x - 1 > 0$ for all $\delta \in [0, 1)$, which was established above.

Because $\beta\gamma$ is increasing in α, it should be clear that, from $\alpha(1-\gamma)/\beta\gamma$ being nondecreasing in α, we can conclude that $\alpha(1-\gamma)$ is increasing in α. This completes the proof of the Lemma.

Proof of Proposition 7

That $\check{O} = C$ when $\alpha + \beta \geq 1$ has been proved in the main text. We thus focus on the case $\alpha + \beta < 1$. In such case, the employment relationship dominates outside contracting only if $\gamma(1-\gamma)^{\beta/(1-\beta)}$ is higher under an employment contract than outside contracting. Defining the function

$$g(\delta) \equiv \frac{\alpha\left(1-\delta^{\alpha+\beta}\right)}{(\alpha+\beta)(1-\delta^{\alpha})},$$

(where clearly $\gamma = g(\delta)$), this condition can be written as

$$g(\delta_E)\left[1-g(\delta_E)\right]^{\frac{\beta}{1-\beta}} > g(0)\left[1-g(0)\right]^{\frac{\beta}{1-\beta}} = \frac{\alpha}{\alpha+\beta}\left(\frac{\beta}{\alpha+\beta}\right)^{\frac{\beta}{1-\beta}}. \tag{A10}$$

Notice that the left-hand side of this inequality is equal to the right-hand side whenever $\delta_E = 0$, and it equals 0 when $\delta_E \to 1$ (see the proof of Lemma 3). Furthermore, straightforward differentiation indicates that the left-hand side of the inequality is increasing in δ_E when $g(\delta_E) < 1-\beta$ and decreasing in δ_E when $g(\delta_E) > 1-\beta$. Coupled with the fact that $g'(\delta_E) > 0$, this implies that $g(\delta_E)\left[1-g(\delta_E)\right]^{\frac{\beta}{1-\beta}}$ is strictly concave in δ_E and attains a unique maximum in the set $\delta_E \in [0, 1]$.

Importantly, when $\alpha + \beta < 1$ then $g(0) < 1-\beta$, and thus for sufficiently small δ_E, the left-hand side of (A10) is necessarily increasing in δ_E. This implies that the above inequality (A10) will hold for sufficiently small values of δ_E making an employment relationship optimal (i.e., $\check{O} = E$). In such case, and given that $\lim_{\delta_E \to 1} g(\delta_E) = 1$, there exists a unique $\bar{\delta} \in (0, 1)$ such that, for $\delta_E > \bar{\delta}$, the above inequality is reversed and outside contracting becomes optimal again (i.e., $\check{O} = C$).

Notice also that the threshold $\bar{\delta}$ is implicitly defined by:

$$J\left(\bar{\delta}, \alpha\right) \equiv \ln g\left(\bar{\delta}\right) + \frac{\beta}{1-\beta}\ln\left(1-g\left(\bar{\delta}\right)\right) - \ln\left(\frac{\alpha}{\alpha+\beta}\right) - \frac{\beta}{1-\beta}\ln\left(\frac{\beta}{\alpha+\beta}\right) = 0.$$

We finally use the implicit function theorem to show that $\bar{\delta}$ is decreasing in α. Notice first that $J\left(\bar{\delta}, \alpha\right)$ is necessarily decreasing in $\bar{\delta}$ since, at $\bar{\delta}$, the function $g(\delta)\left[1-g(\delta)\right]^{\frac{\beta}{1-\beta}}$ is decreasing in δ, (i.e., $g\left(\bar{\delta}\right) > 1-\beta$). On the other hand,

$$\begin{aligned}\frac{\partial J\left(\bar{\delta}, \alpha\right)}{\partial \alpha} &= \frac{\partial \ln g\left(\bar{\delta}\right)}{\partial \alpha} + \frac{\beta}{1-\beta}\frac{\partial \ln\left(1-g\left(\bar{\delta}\right)\right)}{\partial \alpha} - \frac{\beta(1-\beta-\alpha)}{\alpha(\alpha+\beta)(1-\beta)} \\ &= \left(\frac{1-\beta-g\left(\bar{\delta}\right)}{(1-\beta)\left(1-g\left(\bar{\delta}\right)\right)}\right)\frac{\partial g\left(\bar{\delta}\right)/\partial \alpha}{g\left(\bar{\delta}\right)} - \frac{\beta(1-\beta-\alpha)}{\alpha(\alpha+\beta)(1-\beta)}.\end{aligned}$$

Since $1 > g\left(\bar{\delta}\right) > 1-\beta$, $\partial g\left(\bar{\delta}\right)/\partial \alpha > 0$ (from the proof of Lemma 3) and $\alpha+\beta < 1$, we have $\partial J\left(\bar{\delta}, \alpha\right)/\partial \alpha < 0$, and thus $\partial \bar{\delta}/\partial \alpha < 0$.

Appendix B: The Division of Labor and the Scope of Firms

This Appendix analyzes the simultaneous choice of division of labor and scope of firms, i.e., M and N, using the production function (2) under complete and incomplete contracts. Recall that in this case production requires the combination of N tasks and these tasks are divided equally among M workers (so that each performs $\varepsilon = N/M$ tasks). We assume, in addition to the assumptions in the main text, that each worker has to incur a cost $\Gamma(N/M)$, where Γ is a nonnegative, increasing and convex function, with $\Gamma(0) = 0$ and $\lim_{M\to\infty} M\Gamma(N/M) = 0$. This cost captures "diseconomies of scope" at the individual level, and implies that it is more costly for a worker to deal with more tasks, for example, because it requires greater effort or training. The last requirement implies that these economies of this scope are sufficiently powerful.

Now an argument similar to that in the text implies that, with complete contracts, the profit maximization problem of the firm is

$$\max_{N,M,\{x(i,j)\}_{i,j}} A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)} \left[\sum_{j=1}^{M}\frac{N}{M}\left(\exp\left(\int_0^1 \ln x(i,j)\,di\right)\right)^{\alpha}\right]^{1/\alpha} - c_x\sum_{j=1}^{M}\frac{N}{M}\int_0^1 x(i,j)\,di - C(N) - M\Gamma\left(\frac{N}{M}\right)$$

Given this formulation, $\lim_{M\to\infty} M\Gamma(N/M) = 0$ implies that the maximum of this program will be reached as $M \to \infty$.[46] In other words, sufficient diseconomies of scope leads to maximum division of labor under complete contracts. It is also straightforward to see that the rest of the analysis applies in this case.

We next turn to study the choice of division of labor under incomplete contracts. Here M influences ex post bargaining and through this channel has a direct effect on the investments x_n. Consequently, the profit-maximizing choice of M will *not* simply minimize $M\Gamma(N/M)$. In fact, in the absence of credit constraints and the cost $\Gamma(N/M)$, M would be chosen to maximize workers' investment in noncontractible activities as a function of M, $\bar{x}_n(x_c, M)$. This is equivalent to maximizing the ex post bargaining power of workers.

To illustrate how these considerations shape the choice of M, we now compute the Shapley value when there are a finite number of workers equal to M. Suppose that each worker invests x_c in contractible activities. As for the noncontractible activities, consider a situation in which a worker j supplies an amount $x_n(j)$ per non-contractible activity, while the $M-1$ remaining workers supply the same amount $x_n(-j)$. To compute the Shapley value for this particular worker j, we need to determine the marginal contribution of this worker to a given coalition of agents. A coalition of n workers and the firm generates a revenue of

$$F_{IN}(n, M) = A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}x_c^{\beta\mu}\left[(n-1)\frac{N}{M}x_n(-j)^{(1-\mu)\alpha} + \frac{N}{M}x_n(j)^{(1-\mu)\alpha}\right]^{\beta/\alpha}, \qquad \text{(B1)}$$

when the worker j is in the coalition, and a sale revenue

$$F_{OUT}(n, M) = A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}x_c^{\beta\mu}\left[n\frac{N}{M}x_n(-j)^{(1-\mu)\alpha}\right]^{\beta/\alpha} \qquad \text{(B2)}$$

when worker j is *not* in the coalition.

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[46] To see this, it suffices that the solution to this problem will be symmetric with $x(i,j) = x$ for all i, j, and this implies that the only dependence on M comes from the last term. The assumption that $\lim_{M\to\infty} M\Gamma(N/M) = 0$ implies that this term is minimized as $M \to \infty$.

Following the notation in the main text, the Shapley value of player j is

$$s_j = \frac{1}{(M+1)!}\sum_{g\in G}\left[v\left(z_g^j \cup j\right) - v\left(z_g^j\right)\right]$$
$$= \frac{1}{(M+1)\,M}\sum_{i=1}^{M} i\left[F_{IN}(i,M) - F_{OUT}(i-1,M)\right].$$

Substituting for (B1) and (B2), we obtain

$$s_j = \frac{A^{1-\beta}N^{\beta(\kappa+1)}x_c^{\beta\mu}}{M+1}\sum_{i=1}^{M}\frac{i}{M}\left[(i-1)\frac{1}{M}x_n(-j)^{(1-\mu)\alpha} + \frac{1}{M}x_n(j)^{(1-\mu)\alpha}\right]^{\beta/\alpha} \tag{B3}$$
$$-\frac{A^{1-\beta}N^{\beta(\kappa+1)}x_c^{\beta\mu}}{M+1}\sum_{i=1}^{M}\frac{i}{M}\left[(i-1)\frac{1}{M}x_n(-j)^{(1-\mu)\alpha}\right]^{\beta/\alpha}.$$

At a symmetric equilibrium, with $x_n(-j) = x_n(j) = x_n$, this yields the following payoff for each worker

$$s_x = \phi(M,\alpha/\beta)\frac{R}{M},$$

where, as in the main text, $R = A^{1-\beta}N^{\beta(\kappa+1)}x_c^{\beta\mu}x_n^{(1-\mu)\alpha}$ is revenue, and

$$\phi(M,\alpha/\beta) \equiv \frac{M}{M+1}\left[1 - \frac{1}{M}\sum_{i=1}^{M}\left(\frac{i-1}{M}\right)^{\beta/\alpha}\right].$$

The function $\phi(\cdot)$ describes the fraction of the revenue allocated to all the workers combined. It rises with β/α, i.e., it declines with α, because $(i-1)/M$ in the sum is smaller than 1. The residual fraction, $1-\phi(\cdot)$, goes to the firm. It follows that the effect of α on these fractions is the same as in the main text. Notice also that as $M \to \infty$, we obtain

$$\lim_{M\to\infty}\frac{1}{M}\sum_{i=1}^{M}\left(\frac{i}{M}\right)^{\beta/\alpha} = \int_0^1 i^{\beta/\alpha}di = \frac{\alpha}{\alpha+\beta}, \tag{B4}$$

as before, and therefore

$$\lim_{M\to\infty}\phi(M,\alpha/\beta) = \frac{\beta}{\alpha+\beta} = 1-\gamma,$$

where γ is as defined in equation (15) in the main text.

Now consider the impact of M and x_c on investment in noncontractible activities. worker j seeks to maximize $s_j - c_x\frac{N}{M}(1-\mu)\,x_n(j)$. Using (B3), the first-order condition, evaluated at $x_n(-j) = x_n(j) = x_n$, is:

$$\bar{x}_n(x_c,M) = \left[(c_x)^{-1}x_c^{\beta\mu}A^{1-\beta}N^{\beta(\kappa+1)-1}\beta\left(\frac{1}{M+1}\sum_{i=1}^{M}\left(\frac{i}{M}\right)^{\beta/\alpha}\right)\right]^{1/[1-\beta(1-\mu)]}. \tag{B5}$$

This is the same expression as (18) in the text, except that here $\frac{1}{M+1}\sum_{i=1}^{M}\left(\frac{i}{M}\right)^{\beta/\alpha}$ replaces $\alpha/(\alpha+\beta)$. Evidently, here too investment is rising in α (because the i/M terms in the sum are smaller than 1). So we obtain once again that investments in noncontractible activities are increasing in α and the share accruing to the workers is declining in α. Moreover, from (B4) we have $\frac{1}{M+1}\sum_{i=1}^{M}\left(\frac{i}{M}\right)^{\beta/\alpha} \to \alpha/(\alpha+\beta)$ as M goes to infinity.

As argued above, ignoring the effect of M on the fixed costs associated with the diseconomies of scope, the profit-maximizing choice of M will seek to maximize (B5), or simply

$$S(M) \equiv \frac{1}{M+1}\sum_{i=1}^{M}\left(\frac{i}{M}\right)^{\beta/\alpha}.$$

The following lemma characterizes the properties of this function:

Lemma 4 The function $S(M)$ is monotonic in M. In particular, it is increasing in M for $\beta/\alpha < 1$ and decreasing in M for $\beta/\alpha > 1$. Furthermore, $S(1) = 1/2$ and $\lim_{M\to\infty} S(M) = \alpha/(\alpha+\beta)$.

Proof. The two last claims were shown above, so we focus here on proving the monotonicity of $S(M)$. First note, that $S(M) > S(1) = 1/2$ for $\beta/\alpha < 1$ and $S(M) < S(1) = 1/2$ for $\beta/\alpha > 1$ whenever $M \geq 2$. In the case $\beta/\alpha < 1$ (the case $\beta/\alpha > 1$ is symmetric), this follows from

$$S(M) = \frac{1}{M+1}\sum_{i=1}^{M}\left(\frac{i}{M}\right)^{\beta/\alpha} > \frac{1}{M+1}\sum_{i=1}^{M}\left(\frac{i}{M}\right) = \frac{1}{2} \quad \text{for } M \geq 2.$$

Next note that using

$$S(M+1) = \frac{1}{(M+2)(M+1)^{\beta/\alpha}}\left[\sum_{i=1}^{M} i^{\beta/\alpha} + (M+1)^{\beta/\alpha}\right],$$

we can write

$$S(M+1) - S(M) = \frac{1}{M+2} + \left[\frac{(M+1)M^{\beta/\alpha}}{(M+2)(M+1)^{\beta/\alpha}} - 1\right] S(M).$$

Combining this expression with the inequality $S(M) > 1/2$ for $\beta/\alpha < 1$, we can establish that for $\beta/\alpha < 1$,

$$\begin{aligned} S(M+1) - S(M) &> \frac{1}{M+2} + \left[\frac{(M+1)M^{\beta/\alpha}}{(M+2)(M+1)^{\beta/\alpha}} - 1\right]\frac{1}{2} \\ &= \frac{M}{2(M+2)}\left[\left(\frac{M+1}{M}\right)^{1-\beta/\alpha} - 1\right] > 0, \end{aligned}$$

which establishes that $S(M)$ is monotonically increasing whenever $\beta/\alpha < 1$.

The case with $\beta/\alpha > 1$ can be analyzed similarly. Using $S(M) < 1/2$, we can establish that

$$\begin{aligned} S(M+1) - S(M) &< \frac{1}{M+2} + \left[\frac{(M+1)M^{\beta/\alpha}}{(M+2)(M+1)^{\beta/\alpha}} - 1\right]\frac{1}{2} \\ &= \frac{M}{2(M+2)}\left[\left(\frac{M+1}{M}\right)^{1-\beta/\alpha} - 1\right] < 0, \end{aligned}$$

and hence $S(M)$ is monotonically decreasing from $1/2$ to $\alpha/(\alpha+\beta)$. ■

Consider the implications for this Lemma for the profit-maximizing choice of M. First, this Lemma implies that as long as $\beta < \alpha$, even in the absence of diseconomies of scope, i.e., for $\Gamma(N/M) \equiv 0$, the firm prefers $M \to \infty$, because this choice maximizes the ex post bargaining power of workers and therefore their investments. The presence of diseconomies of scope only reinforces this effect. This implies:

Proposition 10 Suppose that there are incomplete contracts and no credit constraints, and that $\beta < \alpha$. Then for all nonnegative, increasing and convex cost functions $\Gamma(N/M)$ that satisfy $\lim_{M\to\infty} M\Gamma(N/M) = 0$, the profit-maximizing choice of the firm is $M \to \infty$.

This result justifies our focus in the text on the case where the range of tasks and the division of labor are treated the same (i.e., N is equated with the division of labor). Moreover, given $M \to \infty$, the equilibrium derived in the text applies exactly.

The assumption $\lim_{M\to\infty} M\Gamma(N/M) = 0$ in this proposition can be relaxed to $\lim_{M\to\infty} M\Gamma(N/M) \leq M'\Gamma(N/M')$ for all $M' \geq 1$. Essentially, the results requires that there are no "economies of scope" and that the function $M\Gamma(N/M)$ does not have an interior strict minimum for some $M' \geq 1$. This assumption is satisfied, for example, when $\Gamma(x) = 0$ for all x.

In contrast, when $\beta > \alpha$, the choice of M will result from a trade-off between minimizing diseconomies of scope (which dictates $M \to \infty$) and maximizing ex ante incentives to invest in x_n (which dictates $M = 1$). The exact equilibrium in this case will depend on the shape of the Γ function. When $\beta > \alpha$ our analysis in the main text (which imposes $M \to \infty$) can be justified by assuming that the effect of the diseconomies of scope is large enough to dominate the outcome. Mathematically, this corresponds to assuming that $\Gamma(N/M)$ is rising sufficiently fast.

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