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Comparative Statics And Perfect Foresight
in Infinite Horizon Economies

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Abstract

This paper considers whether infinite horizon economies have determinate perfect foresight equilibria. We consider stationary pure exchange economies with both finite and infinite numbers of agents. When there is a finite number of infinitely lived agents, we argue that equilibria are generically determinate. This is because the number of equations determining the equilibria is not infinite, but is equal to the number of agents minus one and must determine the marginal utility of income for all but one agent. In an overlapping generations model with infinitely many finitely lived agents, this reasoning breaks down. We ask whether the initial conditions together with the requirement of convergence to a steady state locally determine an equilibrium price path. In this framework there are many economies with isolated equilibria, many with continua of equilibria, and many with no equilibria at all. With two or more goods in every period not only can the price level be indeterminate but relative prices as well. Furthermore, such indeterminacy can occur whether or not there is fiat money in the economy. Equilibria may be pareto efficient or inefficient regardless of whether they are determinate or not.

Comparative Statics and Perfect Foresight

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by

Timothy J. Kehoe and David K. Levine*

1. INTRODUCTION

Finite economies have the same number of equations as unknowns. Imposing assumptions of differentiability on such economies allows us to do regularity analysis. Almost all economies have equations that are locally independent at equilibria. This is important because it enables us to do comparative statics: First, equilibria are locally determinate. Second, small perturbations in the underlying parameters of the economy displace an equilibrium only slightly, and the displacement can be approximately computed by inverting a matrix of partial derivatives.

This paper considers whether infinite horizon economies have determinate perfect foresight equilibria. This question is of crucial importance. If instead equilibria are locally indeterminate, not only are we unable to make comparative static predictions, but the agents in the model are unable to determine the consequences of unanticipated shocks. The idea underlying perfect foresight is that agents' expectations should be the actual future sequence predicted by the model; if the model does not make determinate predictions, the concept of perfect foresight is meaningless.

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We restrict our attention to stationary pure exchange economies. No production, including the storage of goods between periods, can occur. These models are unrealistic, but are the easiest to study. We consider economies with both infinitely lived traders and with overlapping generations.

When there is a finite number of infinitely-lived trades, we argue that equilibria are generically determinate. This is because the effective number of equations determining equilibria is not infinite, but equal to the number of agents minus one and must determine the marginal utility of income for all but one agent. Generically, near an equilibrium, these equations are independent and exactly determine the unknowns.

When there are infinitely many overlapping generations, this reasoning breaks down: An infinite number of equations is not necessarily sufficient to determine an infinite number of unknowns. We consider whether the initial conditions together with the requirement of convergence to a nearby steady state locally determine an equilibrium price path. Examples in which they do and examples in which they do not have been constructed by Calvo (1978) in a related model.

We consider two alternative types of initial conditions. In the first the old generation in the initial period has nominal claims on the endowment of the young generation. In the second the old generation has real claims. In the terminology of Samuelson (1958), the first situation is one with fiat money and the second is one without. In both cases there are many economies with isolated equilibria, many with continua of equilibria, and many with no equilibria at all. With two or more goods in every period not only can the price level be indeterminate but relative prices can be as well. Interestingly, indeterminacy has little to do with pareto efficiency: Equilibria may be pareto efficient or inefficient regardless of whether they are determinant or not.

We also consider an alternative conceptual experiment in which agents use a forecast rule that depends only on current prices to predict next period prices. If the steady state is stable, and if we rule out a certain peculiar case, a perfect foresight forecast rule exists. If there is a continuum of equilibria, there may be a continuum of forecast rules. Even so, the derivative of such a rule, evaluated at the steady state prices, is locally determinate. This makes it possible to do comparative statics in a neighborhood of the steady state despite the local non-uniqueness of equilibrium.

2. THE FINITE AGENT MODEL

We begin by analyzing a pure exchange economy with a finite number of agents who consume over an infinite number of time periods. In each period there are n goods. Each of the m different consumers is specified by a utility function of the form $\sum_{t=0}^{\infty} \gamma_i^t u_i(x_t^i)$ and a vector of initial endowments w^i that is the same in every period. Here $1 > \gamma > 0$ is a discount factor. We make the following assumptions on u_i and w^i:

(a.1) (Differentiability) $u_i : R^n_{++} \to R$ is C^2.

(a.2) (Strict concavity) $D^2u_i(x)$ is negative definite for all $x \in R^n_{++}$.

(a.3) (Monotonicity) $Du_i(x) > 0$ for all $x \in R^n_{++}$.

(a.4) (Strictly positive endowments) $w^i \in R^n_{++}$, $i = 1,\ldots,m$.

(a.5) (Boundary) $||Du_i(x^k)|| \to \infty$ as $x^k \to x$ where some $x_j = 0$. $Du_i(x)x$ is bounded, however, for all x in any bounded subset of R^n_{++}.

It should be possible to extend our analysis to the more general types of preferences that do not require additive separability described by Koopmans, Diamond, and Williamson (1964). We do not attempt to do so here.

Let $p_t = (p_t^1, \ldots, p_t^n)$ denote the vector of prices prevailing in period t. When faced with a sequence $\{p_0, p_1, \ldots\}$ of strictly positive price vectors, agent i chooses a sequence of consumption vectors $\{x_0^i, x_1^i, \ldots\}$ that solves the problem

(2.1) $$\max \sum_{t=0}^{\infty} \gamma_i^t \, u_i(x_t^i)$$

$$\text{subject to } \sum_{t=0}^{\infty} p_t' x_t^i \leq \sum_{t=0}^{\infty} p_t' w^i$$

$$x_t^i \geq 0.$$

The purpose of assumptions a.1 - a.5 is to ensure that, for any price sequence, this problem has a solution that is strictly positive and satisfies the budget constraint with equality. a.5 ensures that consumers' indifference curves become parallel to the coordinate hyperplane as we move towards the boundary of the positive orthant. It is this assumption that rules out corner solutions. The necessary and sufficient conditions for $\{x_0^i, x_1^i, \ldots\}$ to solve 2.1 are

(2.2) $$\gamma_i^t \, Du_i(x_t^i) = \mu_i p_t' \text{ for some } \mu_i > 0, \; t = 0,1,\ldots$$

(2.3) $$\sum_{t=0}^{\infty} p_t' x_t^i = \sum_{t=0}^{\infty} p_t' w^i.$$

A (perfect foresight) equilibrium of this economy is defined to be a price sequence $\{p_0, p_1, \ldots\}$ and a sequence of consumption vectors $\{x_0^i, x_1^i, \ldots\}$ for each agent, $i = 1, \ldots, m$, that satisfies the following conditions:

(e.1) For each agent i $\{x_0^i, x_1^i, \ldots\}$ solves 2.1.

(e.2) $\sum_{i=1}^m x_t^i = \sum_{i=1}^m w^i, \quad t = 0, 1, \ldots$

To find the equilibria of this economy we utilize an approach developed by Negishi (1960) and Mantel (1971) for a model with a finite number of goods. Letting λ_i, $i = 1, \ldots, m$, be some strictly positive welfare weights, we set up the welfare maximization problem

$$(2.4) \quad \begin{aligned} &\max \sum_{i=1}^m \lambda_i \sum_{t=0}^\infty \gamma_i^t u_i(x_t^i) \\ &\text{subject to } \sum_{i=1}^m x_t^i < \sum_{t=1}^m w_i^i \quad t = 0, 1, \ldots \\ &\qquad\qquad x_t^i > 0. \end{aligned}$$

Again a.1 - a.5 guarantee that this problem has a solution that is strictly positive and satisfies the feasibility constraint with equality. The necessary and sufficient conditions for a solution are

(2.5) $\lambda_i \gamma_i^t Du_i(x_t^i) = p_t', \quad i = 1, \ldots, m$, for some $p_t > 0$, $t = 0, 1, \ldots$

(2.6) $\sum_{i=1}^m x_t^i = \sum_{i=1}^m w^i, \quad t = 0, 1, \ldots$

An allocation sequence is pareto optimal if and only if it solves 2.4. Notice that e.2 and 2.6 are the same and, furthermore, if we set $\lambda_i = \frac{1}{\mu_i}$, that 2.2 and 2.5 are equivalent. In other words, a pareto optimal allocation and associated lagrange multipliers $\{p_0, p_1, \ldots\}$ satisfy all of our equilibrium conditions except, possibly, 2.3. The problem of finding an equilibrium therefore becomes one of finding the right welfare weights λ_i, $i = 1, \ldots, m$, so that 2.3 is satisfied.

Let $p_t(\lambda)$ and $x_t^i(\lambda)$ be the solutions to 2.5 and 2.6. The strict concavity of u_i ensures that p_t and x_t^i are uniquely defined and continuous. For each agent we define the excess savings function

$$(2.7) \qquad s_i(\lambda) = \textstyle\sum_{t=0}^{\infty} p_t(\lambda)'(w^i - x_t^i(\lambda)).$$

If $s_i(\lambda)$ is to be well-defined, we must show that the infinite sum in (2.7) converges for $\lambda > 0$. If we can show in addition that the sum converges uniformly on compact subsets of R_{++}^m, it will follow that s_i is continuous on R_{++}^m. Suppose that $\gamma_1 > \gamma_i$, $i = 2, \ldots, m$, so that the first agent exhibits the greatest degree of patience. The consumption of any agent for whom $\gamma_i < \gamma_1$ asymptotically approaches zero in every commodity. The boundary assumption a.5 guarantees that an agent for whom $\gamma_i = \gamma_1$ must asymptotically consume positive amounts of all commodities. The sequence $\{x_0^1, x_1^1, \ldots\}$ involved in the solution to 2.4 cannot converge to any point on the boundary of R_+^n. Furthermore, for any $\lambda > 0$ the vector $x_t^1(\lambda)$ is bounded: $0 < x_t^1 < \sum_{i=1}^m w^i$. Since Du_1 is continuous and x_t^1 remains in a compact subset of R_{++}^n, $||Du_1(x_t^1)||$ is also bounded.

Since $0 < \gamma_1 < 1$, this implies that the sum

$$(2.8) \qquad \sum_{t=0}^{\infty} p_t' = \lambda_1 \sum_{t=0}^{\infty} \gamma_1^t Du_1(x_t^1(\lambda))$$

converges uniformly on compact subsets of R_{++}^m. Since x_t^i is bounded, this enables us to conclude that $s_i(\lambda)$ is well-defined and continuous.

It is easy to verify that the functions $s_i(\lambda)$ are homogeneous of degree one and sum to zero. In fact, the functions $\frac{1}{\lambda_i} s_i(\lambda)$ have mathematical properties identical to the excess demand functions of a pure exchange with m goods. Standard arguments imply the existence of a vector of welfare weights λ such that

$$(2.9) \qquad s(\lambda) = 0.$$

We call this vector λ an equilibrium since our above arguments ensure that when we solve the welfare maximization problem 2.4 using λ for welfare weights the solution is an equilibrium allocation. Conversely, any equilibrium is associated with such a vector λ.

To illustrate some of these concepts, we can consider a simple example of an economy with two agents, and one good in every period. Suppose that $w^1 = w^2 = 1$ and $u_1(x) = u_2(x) = \log x$. The only difference between the two consumers is in their discount rates, $1 > \gamma_1 > \gamma_2 > 0$. In this example the welfare maximization problem 2.4 is

$$(2.11) \qquad \max \ \lambda_1 \sum_{t=0}^{\infty} \gamma_1^t \log x_t^1 + \lambda_2 \sum_{t=0}^{\infty} \gamma_2^t \log x_t^2$$

$$\text{subject to } \ x_t^1 + x_t^2 \leq 2, \ t = 0,1,\ldots$$

$$x_t^i \geq 0.$$

Solving conditions 2.5 and 2.6, we obtain

(2.12) $$x_t^1(\lambda) = \frac{2\,\lambda_1\gamma_1^t}{\lambda_1\gamma_1^t + \lambda_2\gamma_2^t}$$

(2.13) $$x_t^2(\lambda) = \frac{2\,\lambda_2\gamma_2^t}{\lambda_1\gamma_1^t + \lambda_2\gamma_2^t}$$

(2.14) $$p_t = \frac{1}{2}\,(\lambda_1\gamma_1^t + \lambda_2\gamma_2^t)$$

The savings functions are

(2.15) $$s_1(\lambda) = \textstyle\sum_{t=0}^{\infty} p_t(\lambda)(1 - x_t^1(\lambda)) = \frac{\lambda_2}{1-\gamma_2} - \frac{\lambda_1}{1-\gamma_1}$$

(2.16) $$s_2(\lambda) = \frac{\lambda_1}{1-\gamma_1} - \frac{\lambda_2}{1-\gamma_2}\,.$$

As promised, the savings functions are continuous, are homogeneous of degree one, and sum to zero. Imposing the restriction $\lambda_1 = 1$, we can solve 2.16 to find the equilibrium welfare weights

(2.17) $$\lambda_1 = 1, \quad \lambda_2 = \frac{1-\gamma_2}{1-\gamma_1}\,.$$

We can substitute back into 2.12 - 2.14 to find the equilibrium values of p_t, x_t^1, and x_t^2.

3. REGULARITY ANALYSIS

We have reduced the equilibrium conditions for the model with a finite number of consumers to a finite number of equations in the same finite number of unknowns: The homogeneity of s implies that one of the variables λ_i is redundant. That the $s_i(\lambda)$ sum to zero, however, implies that we can ignore one of the equations $s_i(\lambda) = 0$. To do regularity analysis we must be able to ensure that s is continuously differentiable. To do this as simply as possible, we impose the following restriction on u_i:

(a.6) $\quad Du_i D^2 u_i^{-1}$ is bounded on bounded subsets of R^n_{++}.

Suppose, for example that u_i is homogeneous of degree $0 < \alpha_i < 1$. Then a.6 is satisfied since $Du_i D^2 u_i^{-1} = (\alpha_i - 2)x^i_t$. Notice, however, that a.6 allows substantially more general preferences.

PROPOSITION 3.1: If the economy $((u_i, \gamma_i, w^i), i = 1,\ldots,m)$ satisfies a.1 - a.6, then s is continuously differentiable for all $\lambda > 0$.

PROOF: Differentiating 2.7 implies that

$$(3.1) \qquad Ds_i(\lambda) = \Sigma^{\infty}_{t=0} \, (p_t(\lambda)'Dx^i_t(\lambda) + (w^i - x^i_t(\lambda))'Dp_t(\lambda)).$$

Using the implicit function theorem, and 2.5 and 2.6, to compute Dx^i_t and Dp_t period by period, we can write

(3.2) $$\frac{\partial s_i}{\partial \lambda_i} = \lambda_i^{-1} \Sigma_{t=0}^{\infty} ((Du_i D^2 u_i^{-1} + (w^i - x_t^i)')(\Sigma_{k=1} (\lambda_k \gamma_k^t D^2 u_k)^{-1})^{-1}$$

$$-\lambda_i \gamma_i^t Du_i) D^2 u_i^{-1} Du_i'$$

(3.3) $$\frac{\partial s_i}{\partial \lambda_j} = \lambda_j^{-1} \Sigma_{t=0}^{\infty} (Du_i D^2 u_i^{-1} + (w - x_t^i)') (\Sigma_{k=1}^{m} (\lambda_k \gamma_k^t D^2 u_k)^{-1})^{-1} D^2 u_i^{-1} Du_i'$$

for all i, j = 1,...,m, j≠i. Here all partial derivatives are evaluated at the appropriate $x_t^i(\lambda)$. To demonstrate the proposition we must demonstrate that the sums on the right converge uniformly for all λ in any compact subset of R_{++}^m. Since x_t^i is bounded, $0 < x_t^i < \Sigma_{k=1}^m w^k$, a.6 implies that we can prove that the infinite sums in 3.2 and 3.3 converge uniformly if we can prove that

(3.5) $$\Sigma_{t=0}^{\infty} (\Sigma_{k=1}^{m} (\lambda_k \gamma_k^t D^2 u_k)^{-1})^{-1} = \Sigma_{t=0}^{\infty} \gamma_1^t (\Sigma_{k=1}^{m} (\lambda_k (\frac{\gamma_k}{\gamma_1})^t D^2 u_k)^{-1})^{-1}$$

converges uniformly and absolutely. (Once again we suppose that $\gamma_1 > \gamma_i$, i = 2,...,m.). To do this, we show the existence of a bound b > 0 such that $||(\Sigma_{k=1}^m (\lambda_k (\frac{\gamma_k}{\gamma_1})^t D^2 u_k)^{-1})|| < b$. Since the matrix in question is symmetric and negative definite, we need only show that

(3.6) $$-\frac{1}{b} > \sup_{||y||=1} y' (\Sigma_{k=1}^{m} (\lambda_k (\frac{\gamma_k}{\gamma_1})^t D^2 u_k)^{-1}) y.$$

This, however, would follow immediately from

$$(3.7) \qquad -\frac{1}{b} > \sup_{||y||=1} y'(\lambda_1 D^2 u_1(x_t^1))^{-1} y.$$

since the other matrices in the sum are also negative definite. But x_t^1 remains in a compact subset of R_{++}^n and D^2u_1 is continuous. Since D^2u_1 is strictly negative definite, this implies 3.6.

Q.E.D.

A <u>regular economy</u> $((u_i, \gamma_i, w^i), i = 1,\ldots,m)$ is defined to be one that satisfies a.1 - a.6 and the additional restriction

(r) $Ds(\lambda)$ has rank $m - 1$ at every equilibrium λ.

This concept of regular economy is analogous to that developed by Debreu (1970) for pure exchange economies with a finite number of goods. If an economy is regular, then the inverse function theorem implies that it has a finite number of isolated equilibria. The implicit function theorem implies that these equilibria vary continuously with the parameters of the economy.

Furthermore, the topological index theorem introduced into economies by Dierker (1972) provides a valuable tool for counting the number of equilibria of such economies: Let $\bar{J}$ be the $(m - 1) \times (m - 1)$ matrix formed by deleting any row and corresponding column from $Ds(\lambda)$. Define index $(\lambda) = \text{sgn}(\det[-\bar{J}])$. The index theorem says that Σindex $(\lambda) = +1$, where the sum is over all the equilibria. This implies that there exists an equilibrium, that the number of equilibria is odd, and that the equilibrium is unique if and only if $\det[-\bar{J}] > 0$

at every equilibrium.

The appeal of the concept of regularity is enhanced by its genericity: Almost all economies are regular. Suppose we parameterize the space of economies $((u_i, \gamma_i, w^i), i = 1,\ldots,m)$ by allowing the endowments to vary while keeping their sum, $w = \Sigma_{i=1}^m w^i$, constant, but fixing the utility functions and discount factors.

PROPOSITION 3.2: Regular economies form an open dense set of full measure in the space of economies parameterized by endowments.

PROOF: Fix w^i and let $v^i \varepsilon R^n$, $i = 1,\ldots,m - 1$, be vectors that satisfy $w^i + v^i > 0$, $i = 1,\ldots,m - 1$, and $w^m - \Sigma_{i=1}^{m-1} v^i > 0$. We can think of an economy as an element of $R^{n(m-1)}$, $v = (v^1,\ldots,v^{m-1})$. The advantage of keeping the sum of endowments fixed is that the solution to welfare maximization problem 2.4 does not vary as we vary endowments, but the derived savings functions do.

$$(3.8) \qquad s_i(\lambda, v) = \Sigma_{t=0}^{\infty} p_t(\lambda)'(w^i + v^i - x_t^i(\lambda)), \; i = 1,\ldots,m - 1.$$

$$(3.9) \qquad s_m(\lambda, v) = \Sigma_{t=0}^{\infty} p_t(\lambda)'(w^m - \Sigma_{i=1}^{m-1} v^i - x_t^m(\lambda)).$$

The transversality theorem of differential topology says that regularity conditions such as r are satisfied by any function $s(\lambda, v)$ for a set of v of full measure if we can vary s in a sufficient number of directions using v.

In this case the requirement is that the $m \times n(m-1)$ matrix of partial derivatives $D_v s(\lambda, v)$ must have rank $m - 1$ (see Guillemin and Pollack (1974, pp. 67 - 69)). Differentiating 3.8 and 3.9 we obtain

$$(3.10) \qquad D_v s(\lambda, v) = \begin{bmatrix} \Sigma p_t' & 0 & \cdots & 0 \\ 0 & \Sigma p_t' & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \Sigma p_t' \\ -\Sigma p_t' & -\Sigma p_t' & \cdots & -\Sigma p_t' \end{bmatrix}$$

Since the first $m - 1$ rows of this matrix are obviously linearly independent, the set of regular economies has full measure, which implies it is also dense. Openness follows immediately from the continuity of the derivatives of s.

Q.E.D.

This proof can easily be extended to a proof that regular economies form an open dense subset of the space of economies where the only restrictions are a.1 - a.6 if we are careful about giving this space a topological structure.

4. THE OVERLAPPING GENERATIONS MODEL

We now analyze an economy with an infinite number of finitely lived agents, a stationary overlapping generations model that generalizes that introduced by Samuelson (1958). Again there are n goods in each time period. Each generation $0 < t < \infty$ is identical and consumes in periods t and $t + 1$. The consumption and savings decisions of the (possibly many different types of) consumers in generation t are aggregated into excess demand functions $y(p_t, p_{t+1})$ in period t and $z(p_t, p_{t+1})$ in period $t + 1$. The vector $p_t = (p_t^1, \ldots, p_t^n)$ denotes the prices prevailing in period t. We focus our attention on strictly positive price pairs that lie in a set Q formed by deleting the origin from a closed convex cone in R_+^{2n}. Q is assumed to have a non-empty interior and a boundary that is smooth (that is, C^1) except at the origin. We assume that excess demands satisfy the following assumptions:

(A.1) (Differentiability) y, z : Q $\to R^n$ are smooth functions.

(A.2) (Walras's law) $p_t'y(p_t, p_{t+1}) + p_{t+1}'z(p_t, p_{t+1}) = 0$.

(A.3) (Homogeneity) y and z are homogeneous of degree zero.

A.1 has been shown by Debreu (1972) and Mas-Colell (1974) to entail relatively little loss of generality. A.2 implies that there is some means of contracting between generations so that each consumer faces an ordinary budget constraint in the two periods of his life. As we show later, this means the economy is one with a constant (possibly zero or negative) stock of fiat money.

In addition to A.1 - A.3, which are familiar from the finite model, we make two boundary assumptions.

(A.4) The vector (y, z), viewed as a tangent vector to Q at q, points into Q on its boundary.

(A.5) If, for fixed $p \in R_{++}^n$, $(p, \overline{\beta}p) \in Q$ but $(p, \beta p) \notin Q$ for any $\beta < \overline{\beta}$, then $p'y(p, \beta p) < 0$. Similarly, if $(p, \beta p) \notin Q$ for any $\beta > \overline{\beta}$, then $p'y(p, \overline{\beta}p) > 0$.

A.4 ensures that if the price of a good falls low enough then there is excess demand for that good. A.5 ensures, for example, that if prices in the second period fall low enough savings, $-p'y(p, \beta p)$, become positive. A.4 and A.5 are used to guarantee the existence of interior steady states. Although the theory can be extended to analyze free goods, we do not attempt to do so here.

Note that we consider only pure exchange economies and two period lived consumers. We do, however, allow many goods and types of consumers, and the multi-period consumption case can easily be reduced to the case we consider: If consumers live m periods, we simply redefine generations so that consumers born in periods 1, 2,..., m - 1 are generation 1, consumers born in periods m, m + 1,..., 2m - 2 are generation 2, and so on. In this reformulation each generation overlaps only with the next generation. Notice that the number of goods in each newly defined period, and the number of consumers in each newly defined generation, increase by a factor of m - 1.

The economy begins in period 1. The excess demand of old people (generation 0) in period 1 is $z_0(a, p_1)$ where a is a vector of parameters representing the past history of the economy. A (perfect foresight) equilibrium of an economy (z_0, y, z) starting at a is defined to be a price sequence $\{p_1, p_2, \ldots\}$ that satisfies the following conditions:

(E.1) $\quad z_0(a, p_1) + y(p_1, p_2) = 0.$

(E.2) $\quad z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = 0, \quad t > 1.$

Once p_1 and p_2 are determined E.2 acts as a non-linear difference equation determining all future prices. Our major focus is on the extent to which E.1 determines initial prices p_1 and p_2. Subsequent sections study the role of initial conditions z_0 and a. Let us now ignore E.1, however, and focus attention on the difference equation E.2.

We define a <u>steady state</u> of E.2 to be a price vector $(p, \beta p) \varepsilon Q$ that satisfies

$$(4.1) \qquad z(p, \beta p) + y(\beta p, \beta^2 p) = z(p, \beta p) + y(p, \beta p) = 0.$$

In other words, if the prices p prevail forever and the price level grows by β each period, markets would always clear. Here $1/\beta - 1$ is the steady state rate of interest. In the generic case Kehoe and Levine (1982c) show that up to price level indeterminacy there are finitely many steady states.

Our interest in this paper is in what happens near a steady state. Let $(p, \beta p)$ be a steady state, and let $U \quad Q$ be an open cone that contains $(p, \beta p)$. It is convenient to define $q_t = (p_t, p_{t+1})$ and view E.2 as the first order difference equation

$$(4.2) \qquad z(q_{t-1}) + y(q_t) = 0, \quad t > 1$$

We call a path $\{q_1, q_2, \ldots\}$ that satisfies E.1 and E.2 locally stable with respect to $q = (p, \beta p)$ and U if $q_t \varepsilon U$ and $\lim_{t\to\infty} q_t/||q_t|| = q/||q||$.
The question we are trying to answer is whether or not there is a determinate price path that satisfies E.1 and E.2 and is locally stable.

One reason for this restriction is that it is the easiest case to study. Stable price paths are also the most plausible perfect foresight equilibria. If prices are converging to a nearby steady state, then traders can compute future prices by using only local information. If prices are not going to the steady state, then traders need global information and very large computers to compute future prices.

Note that, if equilibrium is indeterminate in our restricted sense so that a continuum of equilibria converge to the steady state from a single initial condition, it is indeterminate in the broader sense as well. On the other hand, even if equilibrium is determinate in the restricted sense there may be a continuum of equilibria which leave the neighborhood of the steady state.

We can linearize E.2 around a steady state $(p, \beta p)$ as

$$(4.3) \qquad D_1 z(p_{t-1} - \beta^{t-1} p) + (D_2 z + \beta^{-1} D_1 y)(p_t - \beta^t p) + \beta^{-1} D_2 y(p_{t+1} - \beta^{t+1} p) = 0.$$

Here all derivatives are evaluated at $(p, \beta p)$ and we use the fact that the derivatives of excess demand are homogeneous of degree minus one. Our homogeneity assumption A.2 allows us to rewrite 4.3 as

$$(4.4) \quad D_1 z p_{t-1} + (D_2 z + \beta^{-1} D_1 y) p_t + \beta^{-1} D_2 y p_{t+1} = 0.$$

If the following regularity condition is satisfied, then 4.4 defines a second order linear difference equation.

(R.1) $\quad D_2 y(p, \beta p)$ is non-singular at all steady states $(p, \beta p)$.

Again letting $q_t = (p_t, p_{t+1})$, we can write out 4.4 as the first order equation $q_t = Gq_{t-1}$ where

$$(4.5) \quad G = \begin{bmatrix} 0 & I \\ -\beta D_2y^{-1}D_1z & -D_2y^{-1}(\beta D_2z + D_1y) \end{bmatrix} .$$

Homogeneity implies that $Gq = \beta q$ where $q = (p, \beta p)$; in other words, G has an eigenvalue equal to β. Walras's law implies that $p'[-\beta D_1z \;\; D_2y]G = p'[-\beta D_1z \;\; D_2y]$; in other words, G has an eigenvalue equal to unity. Let us assume that G also satisfies the following regularity condition:

(R.2) G is non-singular and has distinct eigenvalues; furthermore, eigenvalues have the same modulus if and only if they are complex conjugates.

Consider the difference equation $q_t = (1/\beta)Gq_{t-1}$. Let n^s be the number of eigenvalues of $(1/\beta)G$ that lie inside the unit circle, that is, whose moduli are less than unity. These correspond to eigenvalues of G that lie inside the circle of radius β. A standard theorem on linear difference equations implies that the set of initial conditions q_1 such that $q_t = Gq_{t-1}$ has $\lim_{t\to\infty} q_t/||q_t|| = q/||q||$ is an $n^s + 1$ dimensional subspace V_s of R^{2n} (see Irwin (1980, pp. 151 - 154) and Kehoe and Levine (1982c)). The extra dimension shows up because of homogeneity: If q_1 is such that $\lim_{t\to\infty} q_t/||q_t|| = q/||q||$, then so is θq_1 for any $\theta \neq 0$. The subspace V_s is spanned by the n^s eigenvectors of G associated with the eigenvalues that lie inside the circle of radius β and the eigenvector q associated with the eigenvalue β.

The implicit function theorem implies that, if R.1 is satisfied, then we can solve E.2 to find a non-linear difference equation $q_t = g(q_{t-1})$ defined for an open cone U that contains q. Naturally, $Dg(q) = G$. Let W_s be the subset of initial conditions $q_1 \varepsilon U$ such that $\lim_{t\to\infty} q_t/||q_t|| = q/||q||$. In other words, given (p_1, p_2) we can find a path in U that converges to the ray proportional to $(p, \beta p)$ if and only if $(p_1, p_2) \varepsilon W_s$. The relationship between V_s and W_s is given in the following theorem:

PROPOSITION 4.1: W_s is an $n^s + 1$ dimensional manifold with tangent space at q equal to V_s.

This result is proven by Kehoe and Levine (1982c). That V_s is the tangent space of W_s at q justifies our intuition about 4.4 as a linear approximation to E.2: It says that the best linear approximation to W_s at q is affine set $V_s + \{q\}$.

To establish Proposition 4.1 we need the regularity conditions R.1 - R.2. These can be justified by showing that they hold for almost all economies, in other words, that they hold for an open dense subset of the space of economies. This is done by Kehoe and Levine (1982c). This means that any regular economy can be approximated by one that satisfies R.1 - R.2 and that any slight perturbation of an economy that satisfies R.1 - R.2 still satisfies them.

We have remarked that G has one root equal to β and one unit root. Are we justified in assuming it satisfies no other restrictions? Might it not be the case, as for example in optimal control, that half the eigenvalues of G lie inside the unit circle and half lie outside? Calvo (1978) has constructed examples in a related model for which this is not the case. More strongly, Kehoe and Levine (1982c) show that for any n^s satisfying $2n - 1 > n^s > 0$, there exists an open set of economies that have a steady state with n^s roots inside the circle of radius β and $2n - n^s - 1$ outside the circle with radius β. Furthermore, the work of Mantel (1974) and Debreu (1974) shows that for any excess demands (y, z) we can find consumers with well behaved preferences whose aggregate excess demands are exactly (y, z).

5. DETERMINACY OF EQUILIBRIUM

The excess demand of generation 0 in period 1 is $z_0(a, p_1)$. The vector a represents the history of the system. This is our conceptual experiment: Prior to $t = 1$ the economy is on some price path. Suddenly, after generation 0 has made its savings decisions, but before p_1 is determined, an unanticipated shock occurs. No further shocks occur, and hereafter expectations are fulfilled, although there is no reason why generation 0's expectations of p_1 should be. Do the equilibrium conditions E.1 and E.2 determine a unique path to the new steady state, at least locally? If so, we can do comparative statics, evaluating the impact of the unanticipated shock. If not, it is questionable that traders can deduce which of the many perfect foresight paths they would be on.

Note that this is not the only question we could ask. We might enquire whether given a perfect foresight path stretching back to minus infinity there is a unique extension to plus infinity. We believe that the answer to this question is in general yes. Or we might ask whether the price paths $\{\ldots, p_{-1}, p_0, p_1, \ldots\}$ that are perfect foresight are locally unique. We believe that there is a large set of economies for which the answer to this question is yes, and an equally large set for which it is no. We feel that the question we have posed is the most interesting one, however, and, of these questions, the only one relevant for applied work. Another relevant question is, of course, how to handle price paths that are not near steady states. As we have mentioned, however, it is not clear that perfect foresight is a good hypothesis in such cases.

With this conceptual experiment in mind, we can now see the role played by the vector a: It represents the claims on current consumption owed to old people based on their savings decisions made in period 0. Define the money supply $\mu = p_1'z_0(a, p_1)$ to be the nominal claims of old people. Observe that in equilibrium $p_1'y(p_1, p_2) = -\mu$, by Walras's law $p_2'z(p_1, p_2) = \mu$, in equilibrium $p_2'y(p_2, p_3) = -\mu$, and so forth. Consquently, μ is the fixed nominal net savings of the economy for all time; that is, we assume that there is no government intervention in money markets.

In the steady state we have $\beta^t p'z(p, \beta p) = \mu$ and $\beta^t p'y(p, \beta p) = -\mu$. There are two cases of interest. The <u>nominal</u> case has $\mu \neq 0$. In this case it must be that $\beta = 1$. Gale (1973) calls steady states of this type <u>golden rule</u> steady states. This is because for excess demand functions derived from utility maximization nominal steady states maximize a weighted sum of individual utilities subject to the constraint of stationary consumption over time. Alternatively in the <u>real</u> case $\mu = 0$. Gale refers to steady states of this type as <u>balanced</u> steady states. In this case if $\beta = 1$ then $y(p, p) + z(p, p) = 0$ and $p'y(p, p) = 0$, which are typically n equations in the n - 1 unknowns p, and $\beta = 1$ is merely coincidental. Thus, when $\mu = 0$ the most interesting case is $\beta \neq 1$. Using an index theorem, Kehoe and Levine (1982c) prove that there is generically an odd number of steady states of each type, which, of course, implies the existence of a steady state of each type.

We suppose first that claims are denoted in nominal terms. Thus we cannot assume that excess demand by the old $z_0(a, p_1)$ is homogeneous of degree zero in p_1. We do assume, however, that a is an element of an open subset A of a finite dimensional vector space and that

(I.1) (Differentiability) $z_0\colon A \times R^n_{++} \to R^n$ is a smooth function.

(I.2) (Homogeneity) z_0 is homogeneous of degree zero in a and p_1.

Let $q = (p, \beta p)$ be the steady state after the shock. We assume

(I.3) (Steady state) $z_0(0, p) + y(p, \beta p) = 0$.

That is, when $a = 0$ we are at a steady state. Our goal is to analyze what happens when $||a||$ is small. Various interpretations of this assumption are possible: Previous to $t = 0$ the economy was at or near a steady state and a temporary shock displaced it. Alternatively, a permanent shock occurred and the steady state itself was slightly displaced. All that is necessary is that there be some steady state nearby.

To analyze the impact of the shock, observe that prices (p_1, p_2) are determined by E.1. Using the homogeneity of z_0, we can linearize E.1 around the steady state to find

(5.1) $$(D_2 z_0 + D_1 y)p_1 + D_1 z_0 a + D_2 y p_2 = 0.$$

R.1 implies that we can solve 5.1 for p_2 as

$$(5.2)\qquad p_2 = D_2y^{-1}(D_1z_0 + D_1y)p_1 + D_2y^{-1}D_2z_0a,$$

or, introducing, as before, $q_1 = (p_1, p_2)$,

$$(5.3)\quad q_1 = L\begin{bmatrix} a \\ p_1 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -D_2y^{-1}D_1z_0 & -D_2y^{-1}(D_2z_0 + D_1y) \end{bmatrix}\begin{bmatrix} a \\ p_1 \end{bmatrix}$$

Let $U_1 \subset R^n_{++}$ denote the natural projection of U onto its first n coordinates. The implicit function theorem implies that in a neighborhood of the steady state we get a corresponding solution of the non-linear equation E.1, $q_1 = \ell(a, p_1)$, defined for $p_1 \in U_1$, $a \in A$ with $D\ell(0, p) = L$. We ask whether for given $a \in A$, is there a unique initial $q = (p_1, p_2)$ that satisfies E.1 and has an extension to a price path $\{q_1, q_2, \ldots\}$ in U that satisfies E.2 and converges to some point on the steady state ray. The results of the last section imply that the corresponding mathematical question is whether, for given a, is there a unique p_1 such that $\ell(a, p_1) \in W_s$.

Let us consider the linear problem first. For fixed $a \in A$ 5.3 defines an n dimensional affine subspace of R^{2n}. The linearized version of W_s is V_s, which is $n^s + 1$ dimensional. We would expect, in general, that these spaces intersect in an $n + (n^s + 1) - 2n = n^s + 1 - n$ dimensional linear space. Suppose, in fact, that L satisfies

(IR.1) L has rank 2n.

Note that this requires that A be at least n dimensional, in other words, that there are at least n independent ways to shock the economy. The transversality theorem of differential topology can be translated into the following result:

PROPOSITION 5.1: Let S_a denote the set of $p_1 \in U_1$ such that $\ell(a, p_1) \in W_s$. For almost all $a \in A$ the set S_a, if it is non-empty, has dimension $n^s + 1 - n$.

In other words, what we expect in general of the linear system is almost always true of the non-linear system. Here we use almost all to mean an open dense subset of A whose complement has measure zero. If $n^s + 1 - n < 0$, this means there is no $p_1 \in U_1$ with $\ell(a, p_1) \in W_s$. If $n^s + 1 - n > 0$, however, S_a can either have this dimension or be the empty set. I.2 implies that S_0 is non-empty. If we can ensure that ℓ is transversal to W_s at q, then the structural stability of transversality would imply that S_a is non-empty for all a close enough to 0. We assume

(IR.2) The $2n \times (n + 1 + n^s)$ matrix $\left[\begin{array}{c|cccc} I & & & & \\ -D_2y^{-1}(D_1z_0 + D_1y) & q & v_1 & \cdots & v_{n^s} \end{array}\right]$ has full row rank.

The first n columns of this matrix span the tangent space of the manifold of vectors q_1 that satisfy $q_1 = \ell(a, p_1)$. The final $n^s + 1$ columns span V_s, which is the tangent space W_s. For $n^s + 1 - n > 0$ this says that ℓ is transversal to W_s at q.

Like our previous regularity conditions, IR.1 and IR.2 are generic: Given a y that satisfies R.1, the condition can easily be shown to hold for almost all

z_0. Under IR.1 and IR.2, we can distinguish three cases:

I. $n^s < n - 1$. In this case, for almost all a, S_a is empty. In other words, there are no stable paths locally. We call such a $(p, \beta p)$ an unstable steady state. For most initial conditions the asymptotic behavior of the system is to not reach the steady state. Such steady states are not very interesting; they are unreachable.

II. $n^s = n - 1$. In this case, locally stable equilibrium paths are locally unique and, in a small enough neighborhood actually unique. This is the case where we can do comparative statics and in which perfect foresight is a plausible description of behavior. This is called the determinate case.

III. $n^s > n - 1$. In this case there is a continuum of locally stable paths. Equilibrium is indeterminate. Comparative statics is impossible and perfect foresight implausible.

There are large sets of economies (non-empty open sets of economies) that have steady states of any desired type: unstable, determinate or indeterminate. Thus, none of these possibilities is in any way degenerate.

Let us consider the argument that we get indeterminacy because we ask too much: Because z_0 is not homogeneous we demand that the price level be determined by initial conditions. Is it possible that this is the only possible form of indeterminacy? No. If $n^s + 1 - n > 1$, S_a has two or more dimensions implying that there is relative price indeterminacy.

Now we turn to the case of real initial conditions. The change in conceptual experiment lies in z_0: It is homogeneous of degree zero in p_1 and satisfies Walras's law $p_1'z_0(a, p_1) = 0$. Since $\mu = 0$, the initial price vector must satisfy $p_1'y(p_1, p_2) = 0$. This restriction defines a $2n - 1$ dimensional manifold in some neighborhood of the steady state $(p, \beta p)$ if $(p, \beta p)$ is a regular point of $p_1'y(p_1, p_2)$, in other words, if $(y' + p'D_1y, p'D_2y)$ does not vanish at $(p, \beta p)$. This, however, follows immediately from R.1. We call this manifold the <u>real</u> manifold and denote it Q_r. Its tangent space at $(p, \beta p)$ is made up of the vectors (p_1, p_2) that satisfy $(y' + p'D_1y)p_1 + p'D_2yp_2 = 0$. Differentiating Walras's law with respect to p_1, we establish that $y' + p'D_1y + \beta p'D_1z = 0$ at $(p, \beta p)$. Consequently, the condition defining the tangent space of Q_r can be expressed

(5.4) $$p'[-\beta D_1z \quad D_2y] \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 0.$$

The stability of the system is determined by the roots of $(1/\beta)G$. Recall that Walras's law implies that $p'[-\beta D_1z \quad D_2y]G = p'[\beta D_1z \quad p'D_2y]$. In other words, $(1/\beta)G$ has an eigenvalue $1/\beta$ associated with a (left) eigenvector that is orthogonal to the tangent space of Q_r. Consequently, the root $1/\beta$ has no effect on the behavior of the system on Q_r. Outside of Q_r, however, the root $1/\beta$ determines the behavior of the system. If $\beta < 1$, no path with initial conditions that do not satisfy $p_1'y(p_1, p_2) = 0$ can ever approach the real steady state.

We let $\overline{n}^s$ be the number of roots of $(1/\beta)G$, excluding the root $1/\beta$, that lie inside the unit circle. Because of homogeneity, including that of z_0, the price level is indeterminate and we can reduce everything by one dimension by a price normalization. In this reduced space Q_r has $2n - 2$ dimensions, while the initial condition $z_0(a, p_1) + y(p_1, p_2) = 0$ generically determines an $n - 1$ dimensional submanifold. The intersection of stable manifold W_s with Q_r has dimension $\bar{n}^s$. Consequently, the intersection of the initial condition submanifold and W_s has dimension $(n - 1) + \bar{n}^s - (2n-2)$. Thus, there are the same three possibilities in the real case as in the nominal case, although in the real case $0 < \bar{n}^s < 2n - 2$ while in the nominal case $0 < n^s < 2n -1$.

So far we have assumed that D_2y is non-singular at every steady state. Suppose instead that D_2y has rank $0 < k < n$ on an open neighborhood of the steady state $(p, \beta p)$. In this situation linearizing E.2 produces an $n + k$, rather than a $2n$, dimensional first order difference equation to replace 4.5. Otherwise our analysis stays the same. In the nominal case, the determinacy condition remains $n^s = n - 1$ where now $0 < n^s < n + k - 1$. In the real case, the determinacy condition remains $\bar{n}^s = n - 1$ where now $0 < \bar{n}^s < n + k - 2$. In particular, if $k = 1$, only a one dimensional indeterminacy is possible in the nominal case, and no indeterminacy is possible in the real case. That D_2y has rank one at a steady state where $n > 2$ is true only for a closed, nowhere dense set of economies; it is a degenerate situation. Yet it is possible to think of conditions that imply it: Suppose each generation consists of one, two period lived consumer who has an intertemporally separable utility function. Then both D_2y and D_1z have at most rank one. That D_1z has rank one implies that $n - 1$ of the $n + k = n + 1$ eigenvalues are zero. The determinacy conditions for an economy of this type are therefore the same as for an economy with only one good in every period. In the case where the single, two period lived consumer has a Cobb-Douglas utility function, Balasko and Shell (1981) prove that with real initial conditions equilibria are determinant and with nominal initial conditions there is a one dimensional indeterminacy. Their model is more general than that used here in that their demand functions are not necessarily stationary nor do they require convergence to a steady state. Yet, their result is not surprising given that $D_2y_t(p_t, p_{t+1})$ and $D_1z_t(p_{t-1}, p_t)$ generically have rank one even in a non-stationary environment. Kehoe and Levine (1982b) demonstrate that these determinacy results hold for small perturbations in excess demand, that is, when consumers are "almost" identical and when when preferences are "almost" intertemporally separable. Unfortunately, this result depends crucially on the assumption of two period-lived consumers. Kehoe and Levine (1982a) have constructed robust examples of economies with a single, three period lived consumer with intertemporally separable utility and one good in every period with all of the general indeterminacy and instability possibilities.

It might be conjectured that in the case where excess demand is derived from consumer optimization over well-behaved preferences that the pareto inefficiency of paths is related to the indeterminacy of equilibrium. A moment's reflection on the real case shows this is not true. If $\beta < 1$, prices along paths converging to the steady state decline exponentially in the limit; this means that the value of every agent's endowment is finite and, by the standard arguement for finite economies in Arrow (1951), all these paths are efficient. But $\beta < 1$ implies only that no path with $\mu \neq 0$ ever approaches the real steady state; it places no restriction on $\overline{n}^s$. Thus if $n > 1$ indeterminacy is possible. Conversely, if $\beta > 1$, all convergent paths are easily shown to be inefficient, but there is still no restriction on the possible types of steady states.

Perhaps the case $\beta < 1$ is the most puzzling of all: Here if $n > 2$ we can have indeterminacy among equilibria converging to the steady state, yet all these paths are pareto efficient and all mimic the finite dimensional case in that Walras's law is satisfied even by the initial generation.

We conclude this section by noting that there are six possible types of steady states: real or nominal, each of which may be unstable, determinate or indeterminate. If there are two or more goods each period then there are open sets of economies with each possible combination. The case with one good each period, which has been studied most extensively, is exceptional however: Instability is impossible and, in the real case, indeterminacy is also impossible.

6. FORECASTING

In this section we examine the case of nominal initial conditions in more detail. We again focus on the neighborhood of a steady stable state $(p, \beta p)$ with $n^s > n - 1$, and we assume that all regularity conditions are satisfied. Our focus is on how agents forecast future prices. One possibility is that they use the dynamic equation E.2; equivalently, they forecast $q_{t+1} = g(q_t)$. Note that unless $n^s = 2n - 1$ this is actually an unstable dynamical system: Small perturbations can cause the path to depart from the steady state.

We now investigate the alternative possibility that traders forecast future prices solely as a function of current prices. This type of closed-loop forecasting leads to convergence to the steady state. Surprisingly, it also is locally determinate: This restriction on forecasting rules is sufficient to eliminate much of the indeterminacy we found in the previous section, making local comparative statics possible. Not surprisingly, such forecasting is impossible when the steady state is unstable. Here we only examine nominal initial conditions to keep the presentation as simple as possible; an analogous analysis can be done for real initial conditions.

A closed-loop forecast rule is a function $p_{t+1} = f(p_t)$ that gives prices next period as a function of current prices. We assume that f satisfies the following assumptions:

(F.1) (Differentiability) f is a smooth function defined on an open cone U that contains the steady state relative prices p.

(F.2) $f(p) = p$.

(F.3) (Homogeneity) f is homogeneous of degree one.

(F.4) (Perfect foresight) $z(p, f(p)) + y(f(p), f^2(p)) = 0$.

(F.5) (Convergence) $\lim_{t\to\infty} f^t(p)/||f^t(p)|| = p/||p||$.

Here, for example, $f^2(p)$ denotes $f(f(p))$. F.2 insists that at the steady state the forecast rule pick out the steady state. F.4 is the perfect foresight assumption: If forecasts are realized, markets indeed clear. F.5 says we are interested only in forecast rules that permit convergence to the steady state, in other words, are stable.

We begin by asking whether, for $n^s > n - 1$, there actually exists a forecast rule that satisfies F.1 - F.5. As usual, we consider the linearized problem first. To construct a forecast rule we choose $v_1,\ldots,v_{n-1},q$ to be independent eigenvectors in V_s, the stable subspace of the linearized system. It is important that we be able to choose $v_1,\dots,v_{n-1}$ so that complex vectors appear in conjugate pairs. This can always be done if n^s is even. It can also always be done if $n^s = n - 1$ since $v_1,\dots,v_{n-1}$ includes all of the eigenvectors corresponding to eigenvalues inside the circle of radius β and such eigenvectors necessarily show up in complex conjugates. In the peculiar case where n^s is odd and there are no real eigenvalues inside the circle of radius β, and hence no real eigenvectors in V_s, we cannot make this choice of $v_1,\dots,v_{n-1}$. This is no accident: In this case there are no stable perfect foresight forecasting rules.

Let V_* be the real vector space spanned by $v_1,\dots,v_{n-1},q$; because complex vectors come in conjugate pairs, it is n dimensional. What we suggest is, for given p_t, to choose p_{t+1} so that $(p_t - \beta^t p, p_{t+1} - \beta^{t+1} p)$ is an element of V_*. From the structure of g there exists a unique choice of p_{t+1} provided

(FR) $\quad v_1^1,\dots,v_{n-1}^1,p$ are independent vectors

where v_i^1, $i = 1,\dots,n - 1$, are the first n components of the v_i. If FR holds, we can find a unique matrix F, which depends on $v_1,\dots,v_{n-1}$, so that

$$(6.1) \qquad (p_{t+1} - \beta^{t+1} p) = F(p_t - \beta^t p)$$

is our linear forecast rule.

First we check that the linearized system 6.1 satisfies the linearized versions of F.2 - F.5. Since $q \in V_*$, $(p, \beta p) \in V_*$ and, consequently, $Fp = \beta p$. Since $v_1,\dots,v_{n-1},q$ are eigenvectors of G, V_* is invariant under the dynamical system G, which implies that if $q_t \in V_*$ then $Gq_t \in V_*$. Finally, since $V_* \subset V_s$ and $(p_t, p_{t+1}) \in V_*$, we must have $\lim_{t\to\infty} p_t/||p_t|| = p/||p||$.

It is natural to conjecture that we can thus find an f with $Df(p) = F$ that satisfies F.1 - F.5; this follows from Hartmann's smooth linearization theorem in Irwin (1980, p. 117). Because g is homogeneous of degree one, f may also be chosen to be homogeneous of degree one. If $n^s = n - 1$, then f is unique. This is well known when f is linear (see, for example, Blanchard and Kahn (1980)). If, however, $n^s > n - 1$, f may not be unique nor even locally unique. Furthermore, in the case where $n - 1$ is odd and all the eigenvalues of G that lie inside the circle of radius β are complex, f does not even exist. The derivative $Df(p) = F$ at the steady state is locally unique, however, there are only finitely many possibilities. To see this write F.4 as $(f(p_t), f^2(p_t)) = g(p_t, f(p_t))$. Differentiating this at p we see that

(6.2) $$\begin{bmatrix} F \\ F^2 \end{bmatrix} = G \begin{bmatrix} I \\ F \end{bmatrix} .$$

Writing F in Jordan canonical form as $F = H\Lambda H^{-1}$, we see that

(6.3) $$\begin{bmatrix} H\Lambda \\ H\Lambda^2 \end{bmatrix} = G \begin{bmatrix} H \\ H\Lambda \end{bmatrix} .$$

R.2 implies that Λ is diagonal with diagonal entries equal to eigenvalues of G and that the columns of $\begin{bmatrix} H \\ H\Lambda \end{bmatrix}$ are the corresponding eigenvectors of G. Since G has only finitely many eigenvalues, there are only finitely many choices of F; indeed, our original construction is the only way to get solutions that satisfy the stability requirement F.5.

Notice that, if $n^s > n + 1$, there are in general many possible choices of $v_1, \ldots, v_{n-1}$ and, consequently, of F. The important fact is that there are only a finite number of choices. Furthermore, under our regularity assumptions, F varies smoothly with small changes in the parameters of (y, z). When doing comparative statics faced with a choice of finitely many forecast rules, we choose the unique F that corresponds to the forecast rule being used before the shock.

Finally, let us check on the initial condition; it is now

$$z_0(a, p_1) + y(p_1, f(p_1)) = 0. \tag{6.4}$$

We can locally solve for p_1 as

$$p_1 = -\left[D_2 z_0 + D_1 y + D_2 y F\right]^{-1} D_1 z_0 a. \tag{6.5}$$

7. CONCLUSION

We conclude by summarizing our results and indicating some possible directions for future research. When there are finitely many infinitely lived consumers we have shown equilibria are generically determinate. We have considered only a pure exchange economy, however, and assumed that preferences are additively separable between time periods. We conjecture that these results continue to hold with production and with the more general stationary impatient preferences described by Koopmans, Diamond and Williamson (1964). It would be worthwhile to verify this.

In the overlapping generations case we have shown that determinacy, indeterminacy, and instability are all possible for a wide range of economies. Again it would be worthwhile to extend this to economies with production. We also have considered only two period lived consumers. Although we have shown how to reduce the n period life case to the two period case by redefining generations our genericity results do not carry over to the n period case. This is because we have not verified that the perturbations needed to get non-degenericity can actually be generated by the n period lived consumers. The n period life case is also important because cyclical behavior can be reduced to studying steady states in an economy in which generations have been redefined: A cycle of k periods can be viewed as a steady state of a model where each newly defined generation includes k of the original generations and each newly defined time period includes k of the original time periods.

It would also be useful to know that indeterminacy can arise not only for an open set of economies, but for economies with reasonable preferences. One step in this direction is in Kehoe and Levine (1982a), which illustrates all possibilities with a single three period lived consumer with a constant-elasticity-of-substitution utility. A more general characterization of the eigenvalues in terms of assumptions on preferences would also be worthwhile.

Our results for the overlapping generations model raises other questions: We have studied the behavior of equilibrium price paths near steady states. Is it possible to say much about their behavior away from steady states? Throughout the paper we have assumed perfect foresight expectations. What theoretically attractive alternatives exist? How far do we have to depart from the perfect foresight assumption to get determinacy?

Still more interesting questions arise when we contrast the results for the two different models: Does the determinacy versus indeterminacy result depend on the finite versus infinite number of agents or on the infinite versus finite lifetimes? What mathematical properties do models with an infinite number of infinitely lived agents possess? It is well known that a bequest motive can transform a family of finitely lived agents into a single infinitely lived agent. What properties does an overlapping generations model in which some agents leave positive bequests, and some do not, possess? In other words, what are the properties of a model with a finite number of agents in any time period, some of whom are finitely lived and some are infinitely lived?

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