THE DEMAND FOR INDEX BONDS

Stanley Fischer

Number 132 May 1974

massachusetts institute of technology

50 memorial drive cambridge, mass.02139
THE DEMAND FOR INDEX BONDS

Stanley Fischer

Number 132 May 1974

The views expressed herein are the author's sole responsibility and do not reflect those of the Department of Economics or of the Massachusetts Institute of Technology.
The Demand for Index Bonds*

Stanley Fischer

Massachusetts Institute of Technology

Introduction

The linking of contracts to the price level appears to be a simple and desirable device to reduce the costs of unanticipated inflation. Indexed bonds have been introduced in a number of European and Latin American countries but not, with one exception, in the United States or United Kingdom, despite the recommendations of a lengthy list of distinguished economists.¹

The one exception is an issue of index bonds by the Rand Kardex Company in 1925 -- when, by no coincidence, Irving Fisher was associated with the company.² Escalator clauses in wage contracts are used in many countries,

* I am grateful to Robert C. Merton for lengthy discussions and comments, to Zvi Bod, Franklin Fisher and Paul Samuelson of M.I.T., and to Milton Friedman and other members of the University of Chicago Money Workshop for their comments and suggestions. Research support from the National Science Foundation is acknowledged with thanks.

¹ A partial list of these economists and the arguments for and against indexed bonds is contained in Ahtiala (1967); see also the highly readable book by Collier (1969). Indexed bonds have been used extensively in Finland and Israel. An account of recent Brazilian experience is contained in Fishlow (1974). Tobin (1971) advances several arguments for Federal government issuance of an index bond. The desirability of indexing has also been discussed recently in the daily press, following a suggestion by Friedman (New York Times, April 3, 1974).

² The terms of the bond are described in Pigou (1929), p. 262. The bond was a failure; Fisher attributed the failure to the unfamiliarity of the market with the new asset. See Fisher (1934). However, the indexing of Fisher's bond was peculiar in that a 10% change in the price level was required before the indexing provision became effective.
though they typically include only partial adjustment for price level changes.

The non-issuance of index bonds in the United States by either the Federal government or by corporations is something of a mystery. In the case of corporations, the tax treatment of inflation-linked interest payments essentially as dividends may be a major explanatory factor. There seem to be no mitigating circumstances for the Federal government, however; issuance of an indexed bond would, it is argued, provide portfolio-holders with an asset which guards against a risk which cannot now be fully hedged. The existence of such an asset would be particularly desirable on distributional grounds, since it would enable consumers to provide themselves with an assured standard of living for their retirements. Further, since there is an evident demand for the hedge provided by such bonds, they could be sold at a real interest rate below that on nominal bonds, so that no real costs to the Treasury would be involved in the substitution of some indexed bonds for outstanding non-indexed bonds.

The purpose of this paper is to investigate the theory of the demand for index bonds by households. The basic model, outlined in Section I, is of an infinitely lived household, receiving income only as a return on asset holdings, and able to hold a nominal bond (the nominal return on which is certain), an index bond (which has a certain real return) and equity (the real and nominal returns on which may both be uncertain). There is a single consumption good purchasable at a price $P$, which changes stochastically through time. The major result of this section is that risk aversion is not sufficient to guarantee that index bonds would be held in the portfolio at lower expected nominal returns than the nominal interest rate; the premium (positive or negative) of index bonds over nominal bonds depends positively on the risk aversion of the household's utility function and negatively on the covariance of the return on equity with the price level -- the extent to which equity is a hedge against inflation.
Section II contains a generalization of the model to the situation where there are many goods, as many indexed bonds, each indexed on one of the goods, and many equity assets. The usual mutual fund theorem for equity assets holds. In the special case of the constant relative risk aversion utility function, used extensively throughout the paper, a single real asset can be formed as a linear combination of the individual indexed bonds, but this real asset is not in general equivalent to a bond indexed on the obvious price level for that utility function.

Section III returns to the one-good case and considers the effects of the existence of wage income on the demand for indexed bonds. The wage-income/wealth ratio exerts systematic effects on portfolio demands; the effect of the existence of wage income on the demand for index bonds depends on the correlation of such income with the price level.

A number of other extensions are discussed in Section IV: the implications of the analysis for recent work on the relationship between real and nominal interest rates; the welfare effects of the introduction of index bonds; and the effects of shifts in the stochastic behavior of the rate of inflation.

Conclusions and unfinished business are summarized in Section V. An appendix contains an informal review of the continuous time dynamic processes and dynamic programming techniques used in the paper. A table of notation is also provided at the end of the paper.

3/ The stochastic calculus has been used most extensively in the economics literature by Robert C. Merton (1969, 1971, 1973).
I. The Single Good, Three Asset Case.

This section contains the basic model of the paper. The model is one of an infinitely-lived household receiving all its income as a return on tradeable assets and deriving utility from consumption. The household faces a stochastic rate of inflation and can hold a real bond, equity and a nominal bond in its portfolio. Returns are expressed in nominal terms in this section: this is done partly because generalization to the multi-good case is then straightforward, and also because, since goods are not generally used as numeraire in practice, empirical work on stock-market returns has tended to use nominal rather than real returns.

The major results of the section are contained in parts E and F: asset demands are derived and the properties of the asset demand functions are studied in part E, in particular for constant relative risk aversion utility functions; in part F there is some discussion of equilibrium rates of return on index bonds.

A. Price Dynamics

The rate of inflation is stochastic and the behavior of the price level is describable by the stochastic process:\textsuperscript{4/}

\[ \frac{dP}{P} = \Pi dt + s \, dz = \Pi dt + s \, z(t) \, \sqrt{dt} \]

\textsuperscript{4/} It is assumed that the household knows the correct distributions of the price level and returns on assets.
Section 1 of the appendix contains a brief discussion of the meaning of equations such as (1); (1) states that over a short time interval, the proportionate change in the price level is normal with mean $\Pi$ dt and variance $s^2$ dt. The stochastic component of (1) is temporally independent, no matter how short the time period considered. It is shown in section 6 of the appendix that (1) implies $P(t)$ is log-normally distributed with

\[
E_o \left[ \log \frac{P(t)}{P(o)} \right] = (\Pi - \frac{s^2}{2}) t
\]

\[
E_o \left[ \left( \log \frac{P(t)}{P(o)} - (E_o \log \frac{P(t)}{P(o)}) \right)^2 \right] = s^2 t
\]

where $E_o$ is the expectation conditional on $P(o)$. The assumption that $\Pi$ and $s$ in (1) are constant through time is, of course, very strong and its relaxation will be discussed briefly in Section IV.

B. Asset Returns.

The household can hold three assets in its portfolio: a real bond, a risky asset (equity) and a nominal bond, and can adjust its portfolio instantaneously and costlessly. There are no non-negativity constraints on the asset holdings. The real bond pays a real return of $r_1$ and a nominal return of $r_1$ plus the realized rate of inflation. Then

\[
\frac{dQ_1}{Q_1} = r_1 \, dt + \frac{dP}{P} = (r_1 + \Pi) dt + s \, dz = R_1 \, dt + s_1 \, dz_1
\]
is the equation describing the nominal return on the index bond. 5/

The nominal return on equity is: 6/

\[ \frac{dQ_2}{Q_2} = R_2 dt + s_2 dz_2 \]

where \( R_2 \) is the expected nominal return on equity per unit time and \( s_2^2 dt \) is the variance of the nominal return per unit time. It will later turn out that the covariance of the nominal return on equity with that on real bonds -- and so with the rate of inflation -- is a key variable in the demand functions for assets. By Itô's Lemma (appendix section 4) that covariance is

\[ \rho s_1 s_2 dt \]

where \( \rho \) is the instantaneous coefficient of correlation between the Wiener processes, \( dz_1 \) and \( dz_2 \) (|\( \rho \)| < 1).

The real return on equity is:

\[ \frac{d(-\bar{p})}{Q_2/\bar{p}} = (R_2 - \bar{p} - \rho s_1 s_2 + s_1^2) dt + s_2 dz_2 - s_1 dz_1 \]

\[ = r^\prime dt + s_2 dz_2 - s_1 dz_1 \]

5/ The return on the real bond may be regarded as accruing either as an increase in the price of a single unit of the bond, or as an increase in the number of units of fixed nominal value held by the household. The return on equity is most usefully thought of as an increase in the price of the unit of equity. See Merton (1973) pp. 370-1.

6/ The assumption of only one equity asset is innocuous, since a generalized separation theorem holds in the presence of many risky assets. Namely, given a nominal bond, real bond and many equity assets, each with returns which are log-normally distributed, any household will be indifferent between being given the opportunity of holding all the assets and the oppor-
Once again Ito's Lemma is used in deriving (6), in a manner similar to that shown in section 5 of the appendix. Note particularly that the expected real return on equity, \( r_2 \), is not the expected nominal return minus the expected rate of inflation. This latter relationship would hold if there were no uncertainty about the rates of inflation and return on equity, or, alternatively, if \( \rho s_2 = s_1 \), i.e. if there were zero covariance between the real returns on equity and nominal bonds (see equation (7) below). It does not hold in general, however.

The covariance of the real return on equity with the rate of inflation is

\[
\rho s_1 s_2 - s_1^2
\]

and the coefficient of correlation between the real return on equity and the rate of inflation is

\[
\rho_R = \frac{\rho s_2 - s_1}{\sqrt{s_1^2 - 2\rho s_1 s_2 + s_2^2}}
\]

The deterministic nominal return on nominal bonds is:

\[
\frac{dQ_3}{Q_3} = R_3 dt
\]
The real return on the nominal bond is shown in section 5 of the appendix to be:

$$\frac{d\left(\frac{Q_3}{P}\right)}{Q_3/P} = (R_3 - \Pi + s_1^2)dt - s_1dz_1 = r_3dt - s_1dz_1.$$  

Again, the expected real return on the nominal bond, $r_3$, is not the nominal rate minus the expected rate of inflation, though that would be the case if the rate of inflation were not stochastic.

C. Budget Constraints.

Let $w_1$, $w_2$ and $w_3$ be the proportions of the portfolio held in real bonds, equity and nominal bonds respectively. Then the stock budget constraint is obviously

$$1 = w_1 + w_2 + w_3.$$  

The flow budget constraint, giving the change in nominal wealth ($W$) is:

$$dW = \sum_{i=1}^{3} w_i R_i Wdt - PCdt + \sum_{i=1}^{2} w_is_i Wdz_i.$$  

where $C$ is the rate of consumption. Uncertainty about the change in nominal wealth arises from holdings of real bonds and equity and is described by the last two terms on the right hand side of (11). Substituting for $w_3$ from (10) into (11):

$$dW = \sum_{i=1}^{2} w_i (R_i - R_3) Wdt + (R_3 W - PC)dt + \sum_{i=1}^{2} w_is_i Wdz_i.$$
D. The Household's Choice Problem.

The household is to find

\[(13) \quad \max_{\{C, w_i\}} E \int_0^\infty U(C(t), t) dt \]

subject to (12) and \(W(0) = W_0\), and with \(U(\ )\) strictly concave in \(C\). Where it is necessary to use a specific utility function, I use utility functions with constant relative risk aversion:

\[(14) \quad U(C(t), t) = e^{-\delta t} \frac{C(t)^\mu}{\mu}, \quad 1 > \mu \]

The index of risk aversion is \(1 - \mu\); for \(\mu = 0\), the utility function is logarithmic. The parameter \(\delta\) is the rate of time discount.

The solution to the choice problem is found by solving for the derived utility function

\[J(W, P, t) = \max_{\{C, w_i\}} \int_0^\infty U(C(t), t) dt \]

The general method of solving for \(J(\ )\) is stated in sections 7 and 8 of the appendix; in this particular case one begins by maximizing the function

---

\[7/\]

As is well known, closed form solutions to the choice problem (13) are obtainable for a wider class of utility functions than (14); those utility functions all imply consumption functions linear in wealth. See Fischer (1969) and Merton (1971).
(15) \[ \phi(C,w_1,W,P,t) = \mathcal{U}(C,t) + J_t + J_w \left( \sum_{i=1}^{2} w_i (R_i - R_3) W + R_3 N - PC \right) \]

\[ + \frac{1}{2} J_{WW}^2 [w_1 s_1^2 + 2w_1 w_2 \rho s_1 s_2 + w_2^2 s_2^2] + J_{PP} \]

\[ + \frac{1}{2} J_{PP} s_1^2 p^2 + J_{WP} W [w_1 s_1^2 + w_2 \rho s_1 s_2] \]

with respect to \( C \) and \( w_1 \), and then solving the resulting differential equation to find the \( J(\cdot) \) function, using the fact that the maximized value of \( \phi(\cdot) \) in (15) is zero. \( J_{\cdot\cdot} \) indicates partial derivatives of \( J(\cdot) \) in the usual way.

Maximizing \( \phi(\cdot) \) with respect to \( C, w_1 \) and \( w_2 \), we obtain the first order conditions:

(16) \[ 0 = U_C(C,t) - PJ_w \]

(17) \[ 0 = J_w (R_1 - R_3) + J_{WW} W [w_1 s_1^2 + w_2 \rho s_1 s_2] + J_{WP} p s_1^2 \]

(18) \[ 0 = J_w (R_2 - R_3) + J_{WW} W [w_1 \rho s_1 s_2 + w_2 s_2^2] + J_{WP} p s_1 s_2 \]

Equation (16) is simply the condition that the marginal utility of consumption per dollar is equated to the marginal utility, in the derived utility function, of nominal wealth. Equations (17) and (18) are similar to the usual portfolio equations except for the terms in \( J_{WP} \).

E. Asset Demands.

Solving (17) and (18) for \( w_1 \) and \( w_2 \):

(19) \[ w_1 = - \frac{J_w}{J_{WW} W} \left[ \frac{R_1 - R_3}{s_1^2} - \frac{\rho (R_2 - R_3)}{s_1 s_2 (1 - \rho^2)} \right] - \frac{J_{WP} p}{J_{WW} W} \]
$$w_2 = -rac{J_{w}}{J_{WW}} \left[ \frac{r_2 - r_3}{s_2^2 (1-\sigma^2)} - \frac{\sigma (r_2 - r_3)}{s_1 s_2 (1-\sigma^2)} \right]$$

In interpreting these equations we consider first the term involving $J_{WP}$ in (19). Differentiating the first order condition (16) with respect to $P$, we obtain:

$$u_{cc} \frac{\partial c}{\partial P} = J_{w} + P J_{WP}.$$  

Then, substituting into (19), and using equations (6) and (8) for the real returns on assets:

$$w_1 = -rac{J_{w}}{J_{WW}} \left[ \frac{r_1 - r_3}{s_1^2 (1-\sigma^2)} - \frac{\sigma (r_2 - r_3)}{s_1 s_2 (1-\sigma^2)} \right] - u_{cc} \frac{\partial c}{\partial P}$$

$$w_2 = -rac{J_{w}}{J_{WW}} \left[ \frac{r_2 - r_3}{s_2^2 (1-\sigma^2)} - \frac{\sigma (r_1 - r_3)}{s_1 s_2 (1-\sigma^2)} \right]$$

The last term in (19') is a "hedging" term which reflects the demand for index bonds as a means of hedging against price level changes; it is in fact equal to unity.\(^3\)

\(^3\) To prove this, differentiate (16) to get $P J_{WW} = u_{cc} \frac{\partial c}{\partial P}$ and then, since it can be shown that consumption is a function of real wealth, note that $\frac{\partial c}{\partial P} = -\frac{\partial c}{\partial Y} \frac{\partial Y}{\partial P}$. It is possible to move from (19') and (20') to demand functions for assets expressed in terms of variances of real, rather than nominal returns; the demand functions so obtained are, obviously, the same as those that would be derived if the whole analysis were done using consumption goods as the numeraire. The reasons for doing the analysis in nominal terms are discussed above.
Next consider two special values of \( \rho \), namely \( \rho = 0 \) and \( \rho = \frac{s_1}{s_2} \), i.e. zero correlation of nominal and real returns respectively of equity with the rate of inflation. We add also the demand function for nominal bonds. For \( \rho = 0 \):

\[
(21) \quad w_1 = -\frac{J_W}{J_{WW}} \left[ \frac{r_1-r_3-s_1^2}{s_1^2} \right] + 1 = 1 - \frac{J_W}{J_{WW}} \left[ \frac{r_1-r_3}{s_1^2} \right]
\]

\[
(22) \quad w_2 = -\frac{J_W}{J_{WW}} \left[ \frac{r_2-r_3}{s_2^2} \right] = -\frac{J_W}{J_{WW}} \left[ \frac{r_2-r_3}{s_2^2} \right]
\]

\[
(23) \quad w_3 = \frac{J_W}{J_{WW}} \left[ \frac{r_1-r_3}{s_1^2} + \frac{r_2-r_3}{s_2^2} \right]
\]

The demand for the index bond depends in an intuitively plausible way on the degree of relative risk aversion \( -\frac{J_W}{J_{WW}} \), the difference between expected real returns on index bonds and nominal bonds \( (r_1-r_3) \), the variance of the rate of inflation, and the hedging term (reflecting the fact that \( s_1^2 \) and \( s_2^2 \) are variances of nominal returns).

If the covariance of real returns on equity and nominal bonds is zero, i.e. \( s_1 = \rho s_2 \), then:

\[
(21') \quad w_1 = 1 - \frac{J_W}{J_{WW}} \left[ \frac{r_1-r_3}{s_1^2 (1-\rho^2)} + \frac{r_2-r_3}{s_2^2 (1-\rho^2)} \right]
\]

\[
(22') \quad w_2 = -\frac{J_W}{J_{WW}} \left[ \frac{r_2-r_1}{s_2^2 - s_1^2} \right]
\]

\[
(23') \quad w_3 = -\frac{J_W}{J_{WW}} \left[ \frac{r_3-r_1}{s_1^2} \right]
\]
For $s_1 = \rho s_2$, $s_2^2 - s_1^2$ is just the variance of the real return on equity.

Thus we obtain the usual demand functions for assets in real terms in (21') - (21')

By differentiation of (19) and (20), it can be shown that an increase in $\rho$, the correlation coefficient between nominal equity returns and inflation, reduces the demand for index bonds and increases the demand for equity, if $R_2 > R_3 > 1$. An increase in the correlation between the returns on equity and the price level makes equity a better hedge against inflation, and if equity has a higher expected nominal rate of return than nominal bonds, which in turn have a higher expected nominal return than index bonds, then there is a shift away from index bonds towards equity.

The discussion of the demands for assets so far has been for general utility functions. To make more precise statements about these demand functions, we move now to the specific utility function (14) with constant relative risk aversion. The derived utility function, $J(W, P, t)$ is then of the form (see part G below):

$$J(W, P, t) = e^{-\delta t} \frac{K}{\mu} \left( \frac{W}{P} \right)^{\mu}, \quad K > 0$$

where $K$ is a constant. Taking the appropriate derivatives in (24) and substituting into (19) and (20), one has

---

$9/$. However, it should be noted that the condition $R_2 > R_3$ is not necessarily implied by the simultaneous holding of both types of bond by the household, as will be shown below.
The properties of the demand functions for the three assets are specified in Tables I and II. The demand for each asset is positively related to its own expected nominal rate of return; equity and the index bond are substitutes in that the demand for each is negatively related to the expected nominal return on the other (for \( \rho > 0 \)). The other derivatives of the demand functions are in general of ambiguous sign. If the covariance of real returns on equity and nominal bonds is positive (i.e. \( s_1 - \rho s_2 > 0 \)) then equity and nominal bonds are substitutes with respect to expected nominal rates of return.

The sources of the indeterminacies in the asset demand functions can be isolated with the help of Table II. If \( \rho = 0 \), the assets are substitutes with respect to expected nominal rates of return (except for equity and the index bond which are borderline between being complements and substitutes for each other). Thus the source of the ambiguity in the effects of changes in the nominal interest rate on the demands for index bonds and equity is revealed to be the correlation between the nominal returns on those assets, which permits equity to serve as a hedge against inflation (in terms of expected nominal returns). Similarly, for \( s_1 = \rho s_2 \), the assets are substitutes with respect to expected nominal returns (except that now equity and nominal bonds are borderline).

The ordering of expected returns assumed in Case C of Table II (viz. \( R_2 > R_3 > R_1 \)) removes the ambiguities in the effects of \( s_1 \), \( s_2 \) and \( \rho \) on the demands for the index bond and equity. However, note that although this ordering seems natural, it is not necessarily implied by the household's
Table I: Properties of the Asset Demand Functions,*

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Asset</th>
<th>Index Bond</th>
<th>Equity</th>
<th>Nominal Bond</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>R₁</td>
<td>R₂</td>
<td>s₁</td>
</tr>
<tr>
<td>w₁</td>
<td></td>
<td>-1/ξs₁ - ρ</td>
<td>0</td>
<td>0/ξs₁s₂</td>
</tr>
<tr>
<td>w₂</td>
<td></td>
<td>0/ξs₁s₂</td>
<td>-1/ξs₂</td>
<td>0/ξs₁s₂</td>
</tr>
<tr>
<td>w₃</td>
<td></td>
<td>0/ξs₁s₂</td>
<td>0/ξs₁s₂</td>
<td>-1/ξs₁s₂</td>
</tr>
</tbody>
</table>

Notes:
(i) \(\xi = (1-\mu)(1-\rho^2)\)
(ii) \(\rho\) is assumed positive.
(iii) \(a = R₁ - R₃\)
(iv) \(b = R₂ - R₃\)

* Entries in each cell are the partial derivatives of the demand for the asset (row) with respect to a parameter (column). Plus and minus signs indicate sign of the derivative. Derivative is of ambiguous sign in other cases.
With respect to the parameters $\xi$, $\zeta$, $\eta$, $\gamma$, $\delta$, $\epsilon$, $\zeta$, and $\eta$, the signs of the partial derivatives of the asset demands $\gamma$ and $\eta$ are ambiguous. The parentheses indicate the signs of the partial derivatives of the asset demands.

<table>
<thead>
<tr>
<th>Case</th>
<th>ii</th>
<th>i</th>
<th>i</th>
<th>+</th>
<th>0</th>
<th>-</th>
<th>+</th>
<th>-</th>
<th>-</th>
<th>-</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case A</td>
<td>i</td>
<td>i</td>
<td>0</td>
<td>$\eta$</td>
<td>0</td>
<td>i</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>0</td>
</tr>
<tr>
<td>Case B</td>
<td>i</td>
<td>i</td>
<td>i</td>
<td>+</td>
<td>0</td>
<td>i</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Case C</td>
<td>i</td>
<td>i</td>
<td>i</td>
<td>+</td>
<td>0</td>
<td>i</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

Table II: Properties of the asset demand functions for special cases.
owning positive amounts of all three assets. In Case D of Table II, for both \( \rho = 0 \) and \( R_2 > R_3 > R_1 \), all but one of the signs are determinate and readily explicable.

F. Rates of Return on Index Bonds.

We proceed now to discuss the rates of return, real and nominal, required on index bonds if the household is to hold any of the asset. To do so we examine the portfolio demand functions at the point \( w_1 = 0 \), and focus on the returns differentials between real and nominal bonds. In discussing rates of return required for a particular household to hold the assets, we are also implicitly discussing equilibrium relative rates of return in a market with portfolio holders of identical tastes and fixed supplies of assets.

In nominal terms, at the point \( w_1 = 0 \), the yield differential between real and nominal bonds is:

\[
(27) \quad R_1 - R_3 = s_1 [w_s_1 + (1-u)s_2 w_2]
\]

In terms of real returns:

\[
(28) \quad r_1 - r_3 = (1-u)s_1 [w_s_2 w_2 - s_1] = (1-u)s_1 [(w_2 - 1)s_2 + s_2 - s_1].
\]

Consider first the special case \( \rho = 0 \) and the nominal returns equation (27): for \( \rho = 0 \), if the household is not very risk averse (\( u > 0 \), utility functions with less risk aversion than the logarithmic), it will hold index bonds only if the expected nominal return on them exceeds the nominal interest rate. The greater the degree of risk aversion, the smaller (algebraically) the premium, in terms of the differential between expected nominal
returns, required. For the logarithmic utility function, expected nominal returns on the two bonds are equal, and for \( \mu < 0 \), the required expected nominal return on index bonds is below the nominal interest rate.

The reason for this surprising result -- that there may not be a positive premium for index bonds -- is that the expected real return on nominal bonds is not the nominal interest rate minus the expected rate of inflation (see equation (9)). In terms of expected real returns, from (28), there is always a premium for index bonds over nominal bonds (for \( \rho = 0 \)). We then have

\[
\tau_1 = \tau_3 - (1-\mu)s_1^2
\]

so that the premium is positive and greater the greater is risk aversion and the variance of the rate of inflation.

Next we take account of the correlation between the nominal return on equity and the rate of inflation. If \( \sigma > 0 \), then from (27) it is clear that \( \tau_1 \) may exceed \( \tau_3 \) even if \( \mu < 0 \); (27) says that to the extent that the household owns equity \( (w_2) \) and the more equity is a hedge against inflation (the greater \( \sigma \)), the less likely is the required expected nominal rate on index bonds to be below the nominal interest rate. And from (28), even in terms of real returns, there is not necessarily a premium for index bonds. The smaller the correlation between the nominal returns on equity and the rate of inflation, and the greater the variance of inflation, the more likely is there to be such a premium.

Alternatively, interpreting (28) in terms of the covariance of real returns on equity with the rate of inflation, using (7):

\[
(28') \quad \tau_1 - \tau_3 = (1-\mu)s_1 \bigg[ (w_2-1)\sigma s_2 + \sigma \sqrt{s_1 - 2\sigma s_1 s_2 + s_2^2} \bigg]
\]
Thus, if \( v_2 = 1 \), i.e. if the net quantities of both real and nominal bonds are each zero, then the required real rate on index bonds would be below the real rate for nominal bonds only for \( \rho_R < 0 \).

Summarizing: if \( v_2 = 1 \) and \( v_1 = 0 = v_3 \), then there is a premium for index bonds in terms of expected real returns (i.e. \( r_1 < r_3 \)) only if equity is not a hedge against inflation in real terms (i.e. only if \( \rho_R < 0 \)). However, even if there is a premium for index bonds in terms of real returns, there may not be a premium in terms of nominal returns, due to the presence of the \( s_1^2 \) term of equation (9) in translating between expected real and nominal returns on nominal bonds.

G. The Consumption Function and the Derived Utility Function.

Although the focus of the paper is on the demand for index bonds, we shall also, in Section IV, touch on the welfare implications of the existence of index bonds. Accordingly we present here the derived utility function \( J(W,P,t) \), the properties of which are closely related to the properties of the consumption function summarized in Table III.

Given the first order condition (16) and a trial solution for the \( J(\ ) \) function of the form (24), we obtain

\[
\frac{1}{\mu - 1} \frac{W}{P} = k\left(\frac{W}{P}\right)
\]

Consumption is proportional to real wealth; the factor of proportionality is obtained by solving (14) for the derived utility function. In doing so, one confirms that the trial solution is actually a solution to the differential equation for \( J(\ ) \).

The solution for \( k \) can be given in two equivalent forms:
Table III: Properties of the Propensity to Consume.

<table>
<thead>
<tr>
<th>In terms of parameters of the probability distributions</th>
<th>For $\rho = 0$</th>
<th>In terms of portfolio shares</th>
<th>At $w_1 = 0$</th>
<th>At $w_1 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$w_2 &gt; 0$</td>
<td>$w_2 = 1$</td>
</tr>
<tr>
<td>$\frac{\partial k}{\partial r_1}$</td>
<td>$\mu a^2 [\mu - \frac{s_2 a - \rho s_1 b}{s_1 s_2 (1-\rho^2)}]$</td>
<td>$-\mu a w_1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$\mu a^2 [\mu - \frac{a^2}{s_1^2}]$</td>
<td>$-\mu a w_1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\frac{\partial k}{\partial R_2}$</td>
<td>$-\mu a^2 \frac{s_1 b - \rho s_2 a}{s_1 s_2 (1-\rho^2)}$</td>
<td>$-\mu a w_2$</td>
<td>$-\mu a w_2$</td>
<td>$-\mu a$</td>
</tr>
<tr>
<td></td>
<td>$-\mu a^2 \frac{s_1^2}{s_2}$</td>
<td>$-\mu a w_2$</td>
<td>$-\mu a w_2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\frac{\partial k}{\partial R_3}$</td>
<td>$-\mu a^2 [1+ \frac{s_2^2 a - s_1^2 b}{s_1 s_2 (1-\rho^2)}]$</td>
<td>$-\mu a w_3$</td>
<td>$-\mu a w_2$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$-\mu a^2 [1+ \frac{s_2^2 a + s_1^2 b}{s_1 s_2 (1-\rho^2)}]$</td>
<td>$-\mu a w_2$</td>
<td>$-\mu a w_2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\frac{\partial k}{\partial s_1}$</td>
<td>$\mu a^2 [1- \frac{s_2 a - \rho s_1 b}{s_1 s_2 (1-\rho^2)}]$</td>
<td>$\mu a (w_2 + w_3)$</td>
<td>$\mu a$</td>
<td>$\mu a$</td>
</tr>
<tr>
<td></td>
<td>$\mu a^2 [1- \frac{a^2}{s_1^2}]$</td>
<td>$\mu a (w_2 + w_3)$</td>
<td>$\mu a$</td>
<td>$\mu a$</td>
</tr>
<tr>
<td>$\frac{\partial k}{\partial s_2}$</td>
<td>$\mu a^2 [-s_1 + \frac{a(s_2 a - \rho s_1 b)}{s_1 s_2 (1-\rho^2)}]$</td>
<td>$\mu a^2 [-s_1 + \frac{a^2 - s_1^2}{s_1^2}]$</td>
<td>$\mu a (w_2 + w_3)$</td>
<td>$\mu a (w_2 + w_3)$</td>
</tr>
<tr>
<td></td>
<td>$\mu a^2 [-s_1 + \frac{a(s_2 a - \rho s_1 b)}{s_1 s_2 (1-\rho^2)}]$</td>
<td>$\mu a^2 [-s_1 + \frac{a^2 - s_1^2}{s_1^2}]$</td>
<td>$\mu a (w_2 + w_3)$</td>
<td>$\mu a (w_2 + w_3)$</td>
</tr>
<tr>
<td></td>
<td>$\mu a^2 \frac{b^2}{s_2}$</td>
<td>$\mu a^2 \frac{b^2}{s_2}$</td>
<td>$\mu a (w_2 + w_3)$</td>
<td>$\mu a (w_2 + w_3)$</td>
</tr>
<tr>
<td>$\frac{\partial k}{\partial s_2}$</td>
<td>$\mu a^2 \frac{b}{s_2}$</td>
<td>$\mu a^2 \frac{b}{s_2}$</td>
<td>$\mu a (w_2 + w_3)$</td>
<td>$\mu a (w_2 + w_3)$</td>
</tr>
<tr>
<td>$\frac{\partial k}{\partial a}$</td>
<td>$\mu a^2 \frac{a}{s_1 s_2}$</td>
<td>$\mu a^2 \frac{a}{s_1 s_2}$</td>
<td>$\mu a (w_2 + w_3)$</td>
<td>$\mu a (w_2 + w_3)$</td>
</tr>
<tr>
<td>$\frac{\partial k}{\partial \rho}$</td>
<td>$\mu a^2 \frac{a}{s_1 s_2}$</td>
<td>$\mu a^2 \frac{a}{s_1 s_2}$</td>
<td>$\mu a (w_2 + w_3)$</td>
<td>$\mu a (w_2 + w_3)$</td>
</tr>
</tbody>
</table>

Notation: (i) $\alpha = \frac{1}{1-\mu}$
(ii) $a = R_1 - R_3$
(iii) $b = R_2 - R_3$
\[
\begin{align*}
(39) \quad k &= \frac{u}{1-u} \left[ \frac{\delta}{u} - (R_3 - \bar{r} + s_1^2) + \frac{(1-u)s_1^2}{2} \\ &- \frac{1-u}{2} \left( v_1^2 s_1^2 + 2w_1 v_2 s_1 s_2 + s_2^2 \right) \right] \\
(30') \quad k &= \frac{u}{1-u} \left[ \frac{\delta}{u} + \frac{1}{1-u} \left( \bar{r} - r_2 + u r_1 - \frac{s_1^2}{2} \\ &- \frac{1}{2 s_1^2 s_2^2} \frac{2}{(1-u)} \left[ (R_1 - R_2)^2 s_2^2 \\ &- 2(R_1 - R_2)(R_2 - R_3)s_1 s_2 \\ &+ (R_2 - R_3)^2 s_1^2 \right] \right) \right]
\end{align*}
\]

The condition \( k > 0 \) is implied by the transversality condition.

The effects of parameter changes on the propensity to consume are given in Table III. In general, the effects of a given parameter change depend on the sign of \( u \). As is well known, the propensity to consume out of wealth is independent of rates of return and their variances for the logarithmic utility function (\( u = 0 \)). If rates of return and variances are such that each of the assets are held in positive amounts in the portfolio, then increases in expected rates of return increase the propensity to consume for \( u < 0 \) and decrease it for \( u > 0 \).10/

The effects of changes in \( s_1, s_2 \) and \( \rho \) on the propensity to consume are in general more complicated. In the special case \( \rho = 0 \) and \( R_2 > R_3 > R_1 \) (\( a < 0 \),

10/ The standard interpretation in terms of income and substitution effects can be given to explain this; the substitution effect dominates for \( u < 0 \).
b > 0), an increase in the variance of the nominal return on equity reduces the 
propensity to consume and an increase in \( \rho \) increases the propensity to consume, 
for \( u < 0 \). In the even more special case where \( \rho = 0 \) and \( w_2 = 1, w_1 = 0 \), the 
effect of an increase in the variance of inflation on the propensity to consume 
is of the same sign as \(-u(1-u)\), i.e. positive for \(-1 < u < 0\) but otherwise 
negative.

The derived utility function, \( J(\ ) \), it will be recalled, is

\[
(31) \quad J(N, P, t) = e^{-\delta t} \frac{K}{u} \left( \frac{W}{P} \right)^{\mu}
\]

Thus, for any parameter, \( \beta \),

\[
\frac{\partial J(\ )}{\partial \beta} = e^{-\delta t} \frac{\mu-1}{\mu} k^{\mu-2} \frac{\partial k}{\partial \beta} \left( \frac{W}{P} \right)^{\mu}
\]

Accordingly, the effects of changes in parameters on the household's welfare 
can be obtained simply from Table III; if rates of return are such that each 
asset is held in positive quantities, then increases in rates of return in-
crease welfare.

Interestingly, if \( \beta = 0 \), and rates of return are such that \( w_1 = \gamma, v_1 = 1, \)

\[
(32) \quad \frac{\partial J(\ )}{\partial s_1} = (1+\mu)s_1.
\]

In this special circumstance, increases in the variance of inflation make 
households with \( u > -1 \) better off; this phenomenon is related to the fact 
that the derived utility function is convex in prices for \( u > -1 \).

\[\text{See Fancher (1974) for a discussion of price stabilization.}\]
II. The Multi-Goods Multi-Indexed Bonds Case.

In the presence of many different goods, whose relative prices change, there is an obvious difficulty in choosing the index on the basis of which real contracts are made. No such difficulty would arise if there were as many indexed bonds as goods, each bond indexed to a single good — in effect, futures markets in each good. It is this case which we take up first.

Let the price dynamics for each good be:

\[ \frac{dp}{p_i} = \pi_i dt + s_i dz_i \quad i = 1, \ldots, m \]

and let the first \( m \) assets be indexed bonds, each paying nominal returns of:

\[ \frac{dQ}{Q_i} = r_i dt + \frac{dp}{p_i} = (r_i + \pi_i) dt + s_i dz_i = R_i dt + s_i dz_i \quad i = 1, \ldots, m \]

Note that we are not assuming that the real returns on the indexed bonds are the same for each bond.

There are \( n-m \) equity assets, with nominal returns dynamics:

\[ \frac{dQ}{Q_j} = R_j dt + s_j dz_j \quad j = m+1, \ldots, n \]

and a nominal bond, yielding returns:

\[ \frac{dQ}{Q_{n+1}} = R_{n+1} dt \]

The covariances of nominal returns on the \( i \)'th and \( i \)'th assets are given by \( \sigma_{i,i}, \sigma_{j,j} \); the variance-covariance matrix of returns on the first \( n \) assets is defined by:
The household now maximizes

\[ \mathcal{J} = \int \varphi(U(C_1, \ldots, C_m, t)) \, dt \]

where \( U(\cdot) \) is strictly concave in the \( C_i \), subject to:

\[ dW = \sum_{i=1}^{n} \omega_i (P_i^t - R) W dt + \sum_{i=1}^{m} \omega_i P_i C_i dt + \sum_{i=1}^{n} \delta_i s_i \, d\pi_i \]

The optimization method is as in Section I and involves finding the derived utility function \( J(P_1, \ldots, P_m, W, t) \). The first order conditions for a maximum at each moment of time are

\[ 0 = U_i - \lambda_i^t \]

where \( U_i \) is the partial derivative of the instantaneous utility function with respect to its \( i \)th argument. These conditions have their regular interpretation. For assets we have:

\[ 0 = J^w (P_i^w - R) + J^w W s_i \sum_{k=1}^{n} \delta_k \omega_i P_k^w s_k + s_i \sum_{k=1}^{m} \omega_k \delta_i P_k s_k \]

Now define the vectors

\[ [\vec{R}^t] = [R_1 - R \cdots R_m - R R_{n+1} - R \cdots R_n - R]^t \]

\[ [\omega] = [\omega_1^t; \omega_j^t]^t \]

\[ = [\omega_1 \cdots \omega_m \cdots \omega_{m+1} \cdots \omega_n]^t \]
and rewrite (41) as

\[
(42) \quad 0 = J_i^{\text{W}}[R-R] + J_i^{\text{W}} \otimes [w] + \tau [J_i^{\text{m}}] \]

or

\[
(43) \begin{bmatrix} w_1 \\ w_1 \end{bmatrix} = -J_i^{\text{W}} \frac{1}{J_i^{\text{m}}} \begin{bmatrix} R_1-R \\ R_1-R \end{bmatrix} - \frac{1}{J_i^{\text{m}}} \begin{bmatrix} J_i^{\text{m}} \ 0 \end{bmatrix} \]

It is immediate from inspection of (43) that, in the presence of the \( m \) real bonds, a single equity mutual fund can be formed such that, if expectations are identical, all households will be indifferent between portfolios in which they have a choice of holding the original \( n+1 \) assets or the \( m \) real bonds, the nominal bond, and the equity mutual fund.

Examining (43) we see the hedging term for each real bond in \( J_i^{\text{W}} \); the holdings of the index bonds are the usual functions of expected returns plus a term, the magnitude of which depends on the degree to which changes in that price affect the marginal utility of wealth.

A number of questions concerning the possible existence of a mutual fund formed out of the \( m \) indexed bonds suggest themselves. They are:

1. Can an individualized mutual fund for a household be formed
   
   (a) in the strong sense, that the proportions (to each other) in which the real bonds are held in the household's portfolio are independent of both its wealth and the prices of the goods
   
   (b) in the weak sense, that the proportions (to each other) in which the real bonds are held are independent of wealth but not of prices?
2. Can a general mutual fund be formed such that all households, whatever their utility functions, are indifferent between holding the original menu of assets or only three assets (the equity mutual fund, the nominal bond, and a real bond consisting of fixed proportions of the original real bonds)? Again, we can distinguish between the strong and weak senses in which a mutual fund can exist, as above.

It is shown below that for a CRRA utility function\(^{12}\) there is an individualized mutual fund in the strong sense, but that the general mutual fund does not exist, even if all households have CRRA utility functions and identical expenditure shares but differ only in the degree of their risk aversion.\(^{13}\)

\(^{12}\) To be defined.

\(^{13}\) I conjecture, but have not been able to show, that the individualized mutual fund exists only if the utility function belongs to the CRRA family. Letting \(V_i\) denote the elements of

\[ i=1 \]

we have

\[
\begin{align*}
\bar{w}_i &= \left( \sum_{k=1}^{n} \left( v_{ij} (r_k - r) + \sum_{i=1}^{J} \frac{J_{ij} P_j}{J_{ii}} \right) \right) \\
&= \left[ \sum_{i=1}^{n} \sum_{j=1}^{J} v_{ij} (r_k - r) + \sum_{i=1}^{J} \frac{J_{ij} P_j}{J_{ii}} \right]
\end{align*}
\]

Since \(J(P, W, t)\) is zero degree homogeneous in \(W\) and the vector \(P\),

\[
\sum_{j=1}^{J} J_{wp} P_j = -1 - \frac{J_{wp} P_j}{J_{ii}}
\]

Then an individualized mutual fund exists in the weak sense if
The existence of an individualized mutual fund in the strong sense for the CRRA utility function leads to a third question:

3. In the case of a CRRA utility function, is the household indifferent between holding a portfolio consisting of the original menu or assets and a portfolio consisting of nominal bonds, equity, and a bond indexed on the price index corresponding to the utility function? Somewhat surprisingly, the answer is, in general, no.

Consider now the constant relative risk aversion utility function

\[
U(C_1, C_2, t) = \frac{C_1^{\mu_1} C_2^{\mu_2}}{\mu} e^{-\delta t}
\]

with \( \mu = \mu_1 + \mu_2 < 1 \), \( \mu_1 \mu_2 > 0 \).

Nothing of significance is lost by setting \( m = 2 \). The variable \( \mu \) is unity minus the index of relative risk aversion (in wealth).

13/cont.

\[
\begin{align*}
\sum_{k=1}^{n} \left[ \frac{J_{kp}^x}{J_{w}} \right] &+ \frac{\sum_{i=1}^{m} \left[ \frac{J_{kp}^x}{J_{w}} \right]}{\sum_{j=1}^{n} \left[ \frac{J_{kp}^x}{J_{w}} \right]} = \left[ \frac{J_{kp}^x}{J_{w}} \right] \\
i = 1, \ldots, m
\end{align*}
\]

The individualized mutual fund exists in the strong sense if \( h_i(P) \) in the above equation is identically zero.
Given the utility function (44), the derived utility function is, as one would suspect:

\[ J(W, P_1, P_2, t) = \frac{K}{\mu} e^{-\delta t} \tilde{w}^\mu P_1 -^\mu_1 P_2 -^\mu_2 \quad K > 0 \]

\( K \) in (45) is not directly a function of the level of prices though it is a function of the rates of return on assets. Using (45), the expenditure shares are

\[ \begin{align*}
\frac{P_1 C_1}{P_2 C_2} &= \frac{\mu_1}{\mu_2}
\end{align*} \]

Substituting from (45) into (43) for the two-good case, one equity asset case:

\[ \begin{bmatrix}
  w_1 \\
  w_2 \\
  w_3
\end{bmatrix} = \frac{1}{1-\mu} \Omega^{-1} \begin{bmatrix}
  R_1 - R \\
  R_2 - R \\
  R_3 - R
\end{bmatrix} \begin{bmatrix}
  \mu_1 \\
  u_2 \\
  0
\end{bmatrix} \]

It is clear that a mutual fund of the first two assets can be formed. If the two indexed bonds are held in a mutual fund in the proportion

\[ \frac{w_1}{w_2} = \frac{3}{\sum_{k=1}^{3} V_{1k} (R_k - R) - u_1} \frac{3}{\sum_{k=1}^{3} V_{2k} (R_k - R) - u_2} \]

where \( V_{ij} \) are the elements of \( \Omega^{-1} \), then the household would be indifferent between choosing from the restricted three asset menu (real mutual fund, equity mutual fund and the nominal bond) and the original \( n+1 \) assets.\(^{14/}\)

\(^{14/}\)The CRRA utility function has, of course, the even stronger implication that a single individualized mutual fund can be formed since portfolio proportions are independent of wealth.
Notice that (48) implies that the shares of the two indexed bonds in the real mutual fund depend on $\mu_1$ and $\mu_2$ separately, rather than just their ratio. Thus, even if two households both had CRRA utility functions and the same expenditure shares, but their utility functions were not identical (i.e. they differed in their attitudes to risk), they would not hold the same mutual funds of indexed bonds.\footnote{15/}

It is useful to examine the demand functions for the indexed bonds more closely in this CRRA case. In particular, suppose the variance covariance matrix is diagonal so that returns on all assets are orthogonal. Then

$$
(49) \quad \nu_i = \frac{1}{1-\mu} \left[ \frac{R_i - R}{s_i^2} - \mu_i \right] \quad i = 1, 2
$$

are the demands for the indexed bonds. The two elements in the demand functions, the "asset" demand, represented by the $(R_i - R)/s_i^2$ term, and the hedging demand represented by the $\mu_i$ term appear clearly in (49). Notice, however, the phenomenon of Section I above in which the hedging demand is negative for utility functions with index of risk aversion less than unity (i.e. $\mu > 0$). Whatever the sign of the hedging demand, it is absolutely greater the greater the expenditure share on that good. The smaller in absolute value is $\mu_i$, the smaller the hedging demand; in fact, one can think of equity as being bonds indexed on commodities which are not consumed and for which there is accordingly no hedging demand. Alternatively, consider utility functions exhibiting high risk aversion, for which $\mu_i$ will be large and negative; then the major element in the demand for indexed bonds will be the hedging term, and the returns will

\footnote{15/} Unless, of course, $\sum_{k=1}^{n} V_{ik} (R_k - R) = 0 \quad i = 1, 2.$
play only a small role.

Finally we examine the demand for a bond indexed on the price level

\[(50) \quad P = [P_1^{\mu_1} P_2^{\mu_2}]^{1/\mu} \]

It is well known that the Cobb-Douglas utility function (44) implies the existence of the index (50). \(^{16/}\) A bond indexed on the price level defined by (50) would pay a nominal return of

\[(51) \quad \frac{dQ_1}{Q_1} = r_1 dt + \frac{dP}{P} \]

\[= [r_1 + \frac{\mu_1 \mu_1 + \mu_2 \mu_2}{\mu} - \frac{\mu_1 \mu_2}{2\mu} (s_1^2 - 2\rho_{12} s_1 s_2 + s_2^2)] dt \]

\[+ \frac{\mu_1}{\mu} s_1 dz_1 + \frac{\mu_2}{\mu} s_2 dz_2 \]

\[= r_1 dt + s_1 dz_1 \]

The analysis of the demand for such an indexed bond is very similar to the analysis of Section I, although terms in the covariance of the equity return with the price of each good separately enter the demand functions. However, the portfolio consisting of the single bond indexed on \(P\) in (50), the nominal bond, and equity does not provide the same opportunity set as the portfolio consisting of the original asset menu in which there are as many indexed

\(^{16/}\) See Samuelson & Swamy (1974).
bonds as goods.\footnote{17}

Since the full asset menu is less restricted than the menu with a bond indexed on the price level (50), the household would prefer the opportunity of holding the real mutual fund implied by (48) to the bond indexed on the price level (50). Thus a bond indexed on the ideal price index is not the ideal real asset. Further, since each of the individual real bonds does not necessarily command a premium over the nominal bond, nor does a bond indexed on the ideal price index.

\footnote{17} To prove this, assume the real rates on both indexed bonds in (47) are the same and equal to the real rate in (51). For simplicity, also assume a diagonal variance-covariance matrix of returns. Then, solving for portfolio demands and corresponding expected returns on the portfolio, it is easy to show that the coefficients on $\Pi_i$ in the expressions for expected portfolio returns are not the same as between the two asset menus.
III. Wage Income.

In this section we consider the effects of the existence of uncertain wage income on the demands for assets in general and for index bonds in particular. To simplify the analysis, all returns in this section are expressed in real terms; translation between real and nominal terms is easily accomplished with the aid of equations (4), (6), (7) and (9) of Section I, together with (55') below.

The behavior of the rate of inflation continues to be described by (1). The real returns on real bonds, equity and nominal bonds respectively are given, using equations (4), (6) and (9), as:

\[
\frac{d(0_1/P)}{0_1/P} = r_1 dt
\]

\[
\frac{d(0_2/P)}{0_2/P} = r_2 dt + s_2 dz_2 - s_1 dz_1 \equiv r_2 dt + \sigma_2 dx_2
\]

\[
\frac{d(0_3/P)}{0_3/P} = r_3 dt - s_1 dz_1 \equiv r_3 dt + \sigma_3 dx_3
\]

Let $\lambda$ be the coefficient of correlation between the Wiener processes $dx_2$ and $dx_3$; $\lambda = -\sigma_R$ as given in (7).

The stochastic behavior of the real wage is also modeled as an Ito process.

---

18/ In doing so, we are also providing an analysis of the effects of the existence of non-traded assets on the demands for traded assets. Siegel (1974) discusses the role of human capital in the demand for index bonds.

19/ Merton (1971) solves a problem in which the behavior of the wage is described by a Poisson process.
\[
(55) \quad \frac{dY}{Y} = \nu dt + b \, dq
\]

This implies that the nominal wage behaves according to

\[
(55') \quad \frac{d(PY)}{PY} = \left[ \eta + \nu + \frac{b^2}{2} + \eta sb + \frac{s^2}{2} \right] dt + b \, dq + s \, dz.
\]

where \( \eta \) is the coefficient of correlation between the real wage and the rate of inflation. Denote by \( \eta_2 \) and \( \eta_3 \) (= \( -\eta \)) the coefficients of correlation between the Wiener process \( dq \) and the processes \( dx_2 \) and \( dx_3 \) (equations (53) and (54)) respectively; i.e. \( \eta_2 \) is the coefficient of correlation between the real wage and real equity return and \( \eta_3 \) is the coefficient of correlation between the real wage and the real return on nominal bonds.

Letting \( w_1, w_2, w_3 \) once again be the portfolio shares of the index bonds, equity and nominal bond respectively, and with \( V \) now as real wealth, the flow budget constraint becomes

\[
(56) \quad dV = \sum_{i=2}^{3} w_i (r_i - r_1) V \, dt + (r_1 V + Y - C) \, dt + \sum_{i=2}^{3} w_i \sigma_i V \, dx_i
\]

The optimum is found by maximizing

\[
(57) \quad \phi(C, w_i; V, Y, t) = \psi(C, t) + \int_0^T [ \sum w_i (r_i - r_1) V + r_1 V + Y - C ] dt
\]

\[
+ \frac{1}{2} J_{\psi \psi} [w_2 \sigma_2^2 + 2w_2w_3 \sigma_2 \sigma_3 + w_3 \sigma_3^2] + J_{\psi \psi} V + \frac{1}{2} J_{\psi \psi} b_2 \gamma^2
\]

\[
+ J_{\psi \psi} VY[w_2 \eta_2 b_2^2 + w_3 \eta_3 b_3 \sigma_3]
\]

with respect to \( C \) and \( w_i \), leading to the usual first order condition for consumption and the asset demand functions.
(58) \( v_2 = - \frac{J_V}{J_{VV}} \left[ \frac{r_2-r_1}{\sigma_2^2(1-\lambda^2)} - \frac{\lambda(r_3-r_1)}{\sigma_2\sigma_3(1-\lambda^2)} \right] \) + \( \frac{J_{VY}Y}{J_{VV}} \frac{b}{\sigma_2(1-\lambda^2)} \) \([\eta_1 - \lambda\eta_3]\)

(59) \( w_3 = - \frac{J_V}{J_{VV}} \left[ \frac{r_3-r_1}{\sigma_3^2(1-\lambda^2)} - \frac{\lambda(r_2-r_1)}{\sigma_2\sigma_3(1-\lambda^2)} \right] \) + \( \frac{J_{VY}Y}{J_{VV}} \frac{b}{\sigma_2(1-\lambda^2)} \) \([\eta_3 - \lambda\eta_2]\)

(60) \( \nu_1 = 1 + \frac{J_V}{J_{VV}} \left[ \frac{(r_2-r_1)(\sigma_3^2-\lambda\sigma_2^2)}{\sigma_2^2\sigma_3(1-\lambda^2)} - \frac{(r_3-r_1)(\sigma_2^2-\lambda\sigma_3^2)}{\sigma_2\sigma_3^2(1-\lambda^2)} \right] \)

\[ + \frac{J_{VY}Y}{J_{VV}} \frac{b}{\sigma_2\sigma_3(1-\lambda^2)} \] \([\eta_3 - \lambda\eta_2]\sigma_2 + (\eta_2 - \lambda\eta_3)\sigma_3]\]

At first glance, comparing (58) - (60) with similar equations for the no wage income case, one is tempted to conclude that the existence of wage income affects the asset demands only through the terms in \( J_{VV} \), the hedging terms. These hedging terms arise from the non-tradeability of human capital which implies that the riskiness of wage income can be diversified away only by holding other assets. However, there is also a wealth effect produced by the existence of wage income which, appropriately capitalized, is treated as part of wealth in the asset demand functions. The nature of these two effects can be seen precisely for the CKEA case presented in (61) - (63) below.

To study the hedging effect on the demand for index bonds, consider the interpretation and sign of the term in \( J_{VY} \) in (60). From the first order condition for consumption, \( U_C = J_V \), one can show that

\[ \frac{J_{VY}}{J_{VV}} = \frac{Y \frac{\partial C}{\partial Y}}{C \frac{\partial Y}{\partial Y}} \]

so that the coefficient in the hedging terms is simply the ratio of the income elasticity to the wealth elasticity of consumption demand. It is natural to
assume that both $3C/3Y$ and $3C/3V$ are positive. Assume also, for this paragraph, that $\lambda = 0$, and, to begin, that $\eta_2 = 0$ -- that equity returns are uncorrelated with the real wage. Then the sign of the hedging term in (60) is the same as $\eta_3$ ($= -\eta$), the correlation between the real wage and the real return on index bonds. For $\eta_3 > 0$, i.e. if the real wage is negatively correlated with the rate of inflation, the hedging term is positive. Thus if the major uncertainty about the real wage arises from fluctuations in the purchasing power of a relatively stable nominal wage, the existence of wage income produces a positive hedging effect on the demand for index bonds. If the return on equity is positively correlated with the real wage, the hedging effect on the demand for index bonds is increased. Put differently, and somewhat loosely, if changes in real wages occur largely through changes in labor's share and so are negatively correlated with the returns on equity, then the existence of wage income does not necessarily increase the demand for index bonds.

The general principle for the hedging term is that the existence of wage income shifts portfolio demands in the direction of assets the holding of which offsets the riskiness of the real return on wages. To examine the wealth effect, we turn now to the CRRA utility function.

Specifically, for the CRRA utility function, we obtain the asset demand functions:

\[
(61) \quad w_2 = \frac{1}{1-u} \left[ 1 + \frac{Y}{\gamma(Y)} \right] \left[ \frac{r_2-r_1}{\sigma_2^2(1-\lambda^2)} - \frac{\lambda(r_3-r_1)}{\sigma_2\sigma_3(1-\lambda^2)} \right] \frac{Y}{\gamma(Y)^V} \frac{b}{\sigma_2(1-\lambda^2)} [\eta_2-\lambda\eta_3]
\]

\[
(62) \quad w_3 = \frac{1}{1-u} \left[ 1 + \frac{Y}{\gamma(Y)V} \right] \left[ \frac{r_3-r_1}{\sigma_3^2(1-\lambda^2)} - \frac{\lambda(r_2-r_1)}{\sigma_2\sigma_3(1-\lambda^2)} \right] \frac{Y}{\gamma(Y)^V} \frac{b}{\sigma_3(1-\lambda^2)} [\eta_3-\lambda\eta_2]
\]
\[
(63) \quad w_1 = 1 - \frac{1}{1-\mu} \left[ 1 + \frac{\gamma}{\gamma(\gamma)} \right] \left[ \frac{(r_2-r_1)(\sigma_3-\lambda\sigma_2)}{\sigma_2^2\sigma_3(1-\lambda^2)} + \frac{(r_3-r_1)(\sigma_2-\lambda\sigma_3)}{\sigma_2\sigma_3^2(1-\lambda^2)} \right] \\
+ \frac{\gamma}{\gamma(\gamma)} \frac{b}{1-\lambda^2} \left[ \frac{\eta_3-\lambda\eta_2}{\sigma_3^2} + \frac{\eta_2-\lambda\eta_3}{\sigma_2^2} \right]
\]

where \( \gamma(\cdot) \) is the discount factor applied to the capitalization of wages, the properties of which are discussed below.

From (61) - (63) it is clear that whether the existence of wage income increases or decreases the demand for index bonds depends on the two effects mentioned above: the wealth effect appears in the first set of square brackets in (63), while the hedging effect is represented by the last term in (63). If rates of return are such that \( w_2 \) and \( w_3 \) would both be positive without the existence of wage income (i.e. in the model of Section I) then the wealth effect for index bonds is negative while the sign of the hedging effect depends on the correlations of the real wage with the returns on equity and nominal bonds. Hence there is no presumption that the existence of wage income -- even if its riskiness arises primarily from price level fluctuations -- increases the demand for index bonds.

We turn now to the equilibrium yield, in real terms, on index bonds in the presence of wage income. To do so, we examine the equilibrium yield relationships at the point \( w_1 = 0 = w_3, w_2 = 1 \). Then

\[
(64) \quad r_3 - r_1 = \frac{(1-\mu)[\lambda\sigma_2\sigma_3 + \sigma_3 b\eta_3 \frac{\gamma}{\gamma(\gamma)}]}{1 + \frac{\gamma}{\gamma(\gamma)}}
\]

Thus, for \( \lambda = 0 \), if \( \eta_3 > 0 \), the existence of wage income means that there will
be a premium for real bonds over nominal bonds at the point \( w_1 = 0, w_2 = 1 \).

If \( \eta_3 > 0 \), then there is a positive correlation between the real returns on nominal bonds and the real wage -- i.e. the real wage is negatively correlated with the rate of inflation; this negative correlation tends to produce a premium for real bonds over nominal bonds.

More generally, we have

\[
(65) \quad \frac{\partial (r_3 - r_1)}{\partial \frac{Y}{\gamma(Y)V}} = \frac{(1-\mu)\sigma_3 (b\eta_3 - \lambda \sigma_2)}{(1 + \frac{Y}{\gamma(Y)V})^2}
\]

so that if \( \lambda > 0 \), it is possible that an increase in the wage income/wealth ratio reduces the real premium for index bonds at the point \( w_1 = 0, w_2 = 1 \), even if the real wage is negatively correlated with the rate of inflation. The ambiguity arises, of course, from the wealth effect (represented by the \( \lambda \sigma_2 \) term) operating in the opposite direction to the hedging effect (represented by the \( b\eta_3 \) term in (65)).

Thus we conclude that if equity is a real hedge against inflation \( (\lambda > 0) \), it is possible that the existence of wage income, even if it is fixed in nominal terms, may reduce the premium of index bonds over nominal bonds. As in Section I, the correlation between equity returns and the rate of inflation is an important factor in determining the premium (positive or negative) of index bonds over nominal bonds. It is perhaps also worth repeating that it is not necessarily true that \( r_3 > r_1 \) implies \( R_3 > R_1 \) (see equation (9)): a premium of index bonds in terms of real returns is not necessarily a premium in nominal terms.

Finally in this section, although this paper is not primarily about the consumption function, we present the consumption function and the \( \gamma(\cdot) \) function
in the presence of wage income. The consumption function is

\[(66) \ C = k(\ )[V + \frac{Y}{\gamma}]\]

where \(k(\ )\) is given in equation (30) and its properties are specified in Table III.

The discount factor is the solution to

\[(67) \ 0 = \gamma^2 + \gamma[V - r_1 + \nu - \beta] - \frac{Y}{V} [r_1 - \nu + \beta + \frac{1-\mu}{2} b^2 - \theta] \]

where

\[\beta = \frac{b}{(1-\lambda^2)\sigma_2\sigma_3} [(r_2-r_1)\sigma_3(\eta_2-\lambda\eta_3) + (r_3-r_1)\sigma_2(\eta_3-\lambda\eta_2)]\]

\[\theta = \frac{(1-\mu)b^2}{2(1-\lambda^2)} [\eta_2^2 - 2\lambda\eta_2\eta_3 + \eta_3^2]\]

Although the equation appears forbidding in general, a number of special cases are interesting.

Case 1: \(b = 0\), i.e. real income from wages is deterministic.

Then

\[(68) \ \gamma = r_1 - \nu\]

This is entirely as expected a priori.

Case 2: \(\eta_2 = 0 = \eta_3\), i.e. the real wage is uncorrelated with both the rate of inflation and the returns on equity.

Then

\[(69) \ \gamma = -\left[\frac{Y}{V} - r_1 + \nu\right] \pm \sqrt{\left(\frac{Y}{V} + r_1 - \nu\right)^2 + \frac{Y^2}{V^2} \left(1-\mu\right)b^2}\]
In this case the discount factor exceeds \( r_1^{-\nu} \) by a factor which increases with \( b \), the uncertainty of real wage income.

**Case 3:** \( \eta_3 = 1, b = \sigma_3, \lambda = \eta_2 \), i.e. the nominal wage is deterministic and all uncertainty about the real wage is due to price level fluctuations. Then

\[
(70) \quad \gamma = r_3^{-\nu}
\]

In words, if wage income is like the return on a nominal asset, then it is capitalized at the real interest rate on the nominal bond. Similarly, for \( \eta_2 = 1, \eta_3 = \lambda, b = \sigma_2 \) — if the wage is perfectly correlated with the return on equity — \( \gamma = r_2^{-\nu} \).

---

20/ The transversality condition requires \( r_1 > \nu \).
IV. Extensions.

A. Asset Demands in the Absence of Indexed Bonds.

Much of the finance literature proceeds on the assumption that nominal bonds are a safe asset, yielding a known ex ante real return. An indexed bond would be one such asset in a world with a single good, but indexed bonds do not exist in the United States and there are many goods. Safe real yields in terms of single goods or a basket of goods can be obtained through storage of commodities (inventory holding), presumably at negative real interest rates. Investment in real capital, in the form of housing or equity, does not provide a safe real return.

It is well known that there are systematic portfolio effects arising from price uncertainty in the absence of a real asset, even if the multiplicity of goods is ignored. These will not be explored in detail here, but one simple case is of interest. Specifically, consider the asset menu of Section I, but without an index bond, i.e. suppose there is only a single equity asset and a nominal bond.

21/ The consequences of the absence of a safe asset have been studied by, among others, Long (1974), Merton (1973), Roll (1973), Sarnat (1973) and Tobin (1965).

22/ If there are n-m equity assets and a nominal bond, then the demands for the equity assets are given by

\[ [w_i] = \frac{Jw}{JwW} \Omega^{-1} [sp_{1i}s_{1i} - (\bar{R} - R)] + s\Omega^{-1}\rho_{1i} s_{1i} \]

where \( \Omega \) is the \((n-m)\times(n-m)\) variance-covariance matrix of nominal returns on equities, \( sp_{1i}s_{1i} \) is the \((n-m)\times1\) vector of covariances between nominal equity returns and the price level, and \( (\bar{R} - R) \) is the \((n-m)\times1\) vector of differences between the expected returns on the equity assets and the nominal interest rate.
Using the notation of Section I, the demand function for equity is then:

\[
(71) \quad w_2 = \frac{s_1s^0}{s_2} - \frac{Jw}{Jww} \left[ \frac{R_2 - R_3 - \rho s_1 s_2}{s_2} \right]^2
\]

It is not in general possible to derive from (71) the supposed equilibrium yield relationship

\[
(72) \quad R_3 - \Pi = r_2
\]

which is frequently used in calculating the expected rate of inflation from nominal interest rates.

Equation (72) is presumably based on the belief that there is in existence some safe asset yielding a real return \( r_2 \), and that there is little risk aversion in the market. However, even if there were no risk aversion in the market, so that expected real rates of return on assets were equalized, and there were a safe equity asset or an index bond, the equilibrium yield relationship would be

\[
(73) \quad R_3 - \Pi + s^2 = r
\]

where \( r \) is the safe return in question. In the absence of a safe asset and given risk aversion, (71) has to be used to obtain the equilibrium yield relationship.

An individual cannot be made worse off by an enlargement of his opportunity set. Thus, from the viewpoint of any one individual, the introduction of indexed bonds without any simultaneous changes in the returns on other assets, can at worst leave his welfare unchanged.

From the viewpoint of the community, however, matters are less simple. For instance, assume that the real and nominal expected returns on equity, and their covariance with the rate of inflation are independent of the existence of other assets. Consider now two alternative market equilibria. In the first, there exists only a nominal bond and equity. In the second there is in addition an indexed bond. Between these two equilibria the equity returns and the proportion of real wealth that is equity are the same.

Assume that all investors in the market have identical CRRA utility functions. Denote by \( J_2 \) the value of the derived utility function of the representative individual in the market equilibrium where there are only two assets. Some calculation results in:

\[
(74) \quad J_2 = \left[ \frac{\delta}{1-\mu} - \frac{\mu R_2}{1-\mu} + \frac{\mu}{2} \left( s_2^2 - 2\rho s_1 s_2 + s_2^2 \right) - \frac{\mu}{2} s_2^2 (1-w)^2 \right] \frac{\mu - 1}{\mu} \left( \frac{W}{P} \right)^{\mu}
\]

where the notation is as in Section I, with \( w \) the share of equity in the portfolio. Similarly, let \( J_3 \) be the derived utility function in the three-asset situation. Then:

\[
(75) \quad J_3 = \left[ \frac{\delta}{1-\mu} - \frac{\mu R_2}{1-\mu} + \frac{\mu}{2} \left( s_1^2 - 2\rho s_1 s_2 + s_2^2 \right) - \frac{\mu}{2} s_1^2 (1-w_1)^2 \right. \\
- \left. \frac{\mu}{2} w_1 s_1 \left( w_1 + 2\rho s_2 (w_2-1) \right) \right] \frac{\mu - 1}{\mu} \left( \frac{W}{P} \right)^{\mu}
\]
The values of $J_2$ and $J_3$ in (74) and (75) treat as data, $r_2$, $s_1$, $s_2$ and $\rho$; it is assumed that these are the same between the two situations. The rates of return on nominal bonds alone (for (74)) and for both nominal and indexed bonds (for (75)) adjust relative to $r_2$, $s_1$, $s_2$ and $\rho$ to ensure market equilibrium for the specified values of $w$ in (74) and $w_1$ and $w_2$ in (75). Assume that $w$ in (74), the proportion of equity in the portfolio, is equal to $w_2$ in (75). Then

$$J_3 - J_2 \sim w_1 s_1 \left( w_1 s_1 + 2 \rho s_2 (w_2 - 1) \right)$$

where $\sim$ means "of the same sign as".

From (76), if $w_1 = 0$, $J_3 = J_2$, i.e. if indexed bonds are introduced but priced so that none are purchased, their introduction does not increase welfare. If $w_2 < 1$ and $\rho < 0$, (negative correlation between nominal equity returns and inflation), then $J_3 > J_2$ if $w_1 > 0$. That is, if equity is not a hedge in nominal terms against inflation, the introduction of index bonds increases welfare. However, if $\rho > 0$, if equity is a hedge against inflation, the introduction of indexed bonds can reduce welfare through the changes in rates of return it induces.\textsuperscript{23/} In other words, it is possible that the substitution of indexed for non-indexed bonds could reduce the welfare of the representative investor. This cannot occur, however, unless equity is a hedge against inflation, and need not occur even then.

The above exercise has begged the question of whether, in a world of identical, infinitely lived households, it is possible to manufacture real assets. One thinks naturally of the government's doing so but it is not clear how it can if equity returns are to be left unaffected.\textsuperscript{24/}

\textsuperscript{23/} The nominal interest rate is increased by the introduction of indexed bonds if $\rho < 0$ and reduced if $\rho > 0$.

\textsuperscript{24/} These issues are related to the long-standing question of whether government bonds are net wealth for the private sector. See Barro (1974).
The introduction of index bonds by private agents can increase welfare if tastes with regard to risk differ. Similarly, differences in risk preference could produce welfare increases resulting from the introduction of indexed bonds by a government, with the return being guaranteed by the power to tax.

C. **Shifts in the Rate of Inflation.**

Up to this point it has been assumed that the parameters of the various stochastic processes used in the paper are constant. Thus the real rate of interest is constant for all time, as are the expected rate of inflation and its variance, etc. It is, however probably more reasonable to regard these parameters as changing stochastically through time.\(^\text{25/}\)

Equations of the form

\[
(77) \quad d\Pi = g \, dy
\]

where \(dy\) is a Wiener process can be added to the system set out in Section I. According to (77) the expected rate of inflation \(\Pi\), while given at any instant, changes stochastically through time. The result is to introduce systematic effects on portfolio demands, arising from attempts to offset unfavorable shifts in the opportunity set. I have not been able to establish any presumption about the direction of such effects on the demand for indexed bonds.

---

\(^\text{25/}\) Merton (1973) analyzes a system with changing opportunity set.
Summary and Conclusions.

1. In the simplest model, there is not necessarily a premium for indexed bonds over nominal bonds, in the sense that the real interest rate required to induce a portfolio holder to hold an indexed bond is below the expected real rate on nominal bonds. There will be such a premium if real returns on equity are negatively correlated with the rate of inflation and there will not be such a premium if real equity returns are positively correlated with the rate of inflation, i.e., if equity is a hedge against inflation. However, it is possible for there not to be a premium for indexed bonds over nominal bonds in terms of nominal returns even if there is one in terms of real returns. This is because the expected real return on nominal bonds exceeds the nominal rate minus the expected rate of inflation by an amount equal to the variance, per unit time, of the rate of inflation.

2. Where there are many goods and many indexed bonds, there are two sources of demand for each bond. One is a hedging demand, related to the share of that good in the consumption basket, and the other is a speculation demand, which tends to increase the demand for bonds indexed on the prices of goods which are expected to rise relatively rapidly, given equal real rates on all bonds. The presence of the speculative demand means that the indexed bonds are not held in the portfolio in the same proportions as their expenditure shares; thus, even for utility functions for which an "ideal" price index can be computed, a

\[26/\] These statements hold exactly only at the point where net outstanding stocks of both real and nominal bonds are zero. The more general conditions are given in Section I. F. Tobin, in his classic essay (1971), suggests that real equity returns are likely to have zero covariance with the rate of inflation.

\[27/\] In a general equilibrium setting, the result would be a tendency towards equalization of expected nominal rates of return on the different indexed bonds.
bond indexed on the ideal index is not the ideal indexed bond.

3. The effects of wage income -- the return on human capital -- on the demand for indexed bonds depend on the covariances of wage income with the rate of inflation and equity returns. It is shown in Section III that if equity is not a hedge against inflation (in real terms) and if real wage income is uncorrelated with the rate of inflation, then there is no premium in terms of real returns of index bonds over nominal bonds. The premium for index bonds tends to be greater the smaller (algebraically) the covariance of real equity returns with the rate of inflation and the smaller (algebraically) the covariance of the real wage with the rate of inflation. Thus if human capital is essentially a real asset so that real wage income has zero covariance with the rate of inflation, the existence of wage income does not increase the premium of real bonds over index bonds. If human capital is in some respects a nominal asset, so that the real wage is negatively correlated with the rate of inflation, then wage income tends to increase the premium of index over nominal bonds.28/

4. It is shown in Section IV, in the context of market equilibrium, that the substitution of indexed for nominal bonds may reduce the welfare of the representative household if there are outstanding quantities of both real and nominal bonds. A necessary, but not sufficient, condition for this to happen is that nominal equity returns be positively correlated with the rate of inflation. On the other hand, if equity is not a hedge in nominal terms against inflation, then the substitution of real for indexed bonds unambiguously increases welfare.

28/ These statements are made for the point at which net quantities of real and nominal bonds are both zero; each statement assumes real equity returns have zero correlation with the rate of inflation. See equations (64) and (65) for more general conditions.
The consequences of the absence of an indexed bond—or some asset yielding a safe real return—are also discussed in Section IV. In particular, there is no basis for the familiar Fisher equation relating real and nominal interest rates, frequently used as an intermediate step in estimating expected rates of inflation.

5. These conclusions point to the key role of the correlation of equity returns with the rate of inflation in determining the market price at which indexed bonds will be sold. Essentially, indexed bonds will command a real premium if real returns on equity have negative correlation with the rate of inflation. The magnitude of that correlation is, of course, an empirical matter.²⁹/ If we assume, tentatively, zero correlation, then there is little reason to expect that indexed bonds would command a premium over nominal bonds. Nor does the analysis suggest that any great welfare gains should be expected to result from the introduction of index bonds.

However, considerations not a part of the basic model may strengthen the case for index bonds. Among these are:

1. We have assumed that expectations are correct. The real return on indexed bonds would be known ex ante, while the household has only a (possibly incorrect) probability distribution of returns on nominal bonds. To the extent that inflation is underpredicted, holders of nominal bonds are made worse off by errors in expectations, while holders of indexed bonds are not.³⁰/

²⁹/ A preliminary report on research on this question for a number of countries is contained in Cagan (1974). Cagan finds that equity appears to be a long-term, but not short-term, hedge against inflation.

³⁰/ It is easy enough to distinguish in principle between errors in expectations as defined in this paragraph—meaning that investors assume different parameters in the stochastic process (1) from the true ones—and the difference between the ex ante and ex post rates of inflation. The fact that the realized rate of inflation was different from the expected rate of inflation does not mean that expectations were incorrect. In practice, there is considerable difficulty in distinguishing between the two types of error.
2. The fact that equity assets are not directly held by the majority of households may indicate that the transactions costs of acquiring information about and purchasing equity are large enough to leave many savers without any potential inflation hedges in their portfolios. If an indexed bond were made available by the government, say in the form of indexed U.S. Savings bonds, it might be precisely those savers who would demand it, and who would most benefit from it.

3. Except in Section IV. C, it was assumed that the stochastic processes generating the rates of inflation and assets returns were stationary. The instantaneous nominal interest rate for all future periods, for instance, was assumed known. Needless to say, this is unrealistic. It is possible that too little uncertainty about the returns from holding nominal bonds and equity over long periods is reflected in the basic model of the paper, and that such uncertainty would result in portfolio holders being willing to pay a substantial premium for a long-term indexed bond.\footnote{Franco Modigliani of M.I.T. is currently working on a proposal for the introduction of indexed mortgages as a means of avoiding distortions in the time stream of repayments.}

4. Certain institutional factors may make it desirable to introduce index bonds. For example, under present mortgage repayment schedules, the percentage of income (assumed growing at the rate of inflation) devoted to mortgage payments declines more rapidly the greater the rate of inflation. If mortgage payments were indexed, this would no longer be so.\footnote{My inability to establish any such presumption when working with a shifting opportunity set is not decisive, since I could not establish the converse either.}
Finally, we note that one of the major issues connected with indexing and wage escalation -- the effects of indexing on the stability of the economy -- has not been discussed in this paper.
Bibliography

Ahtiala, Pekka

Barro, Robert J.

Cagan, Phillip

Collier, Robert P.

Cox, D., and Miller, N.

Dreyfus, Stuart

Fischer, Stanley

Fisher, Irving

Fishlow, Albert

Hanoch, Giora

Long, John B.

Merton, Robert C.


Pigou, A. C.

Roll, Richard
Samuelson, Paul A. and Swamy S. 

Sarnat, Marshall 

Siegel, Jeremy 

Tobin, James 

Appendix.

Continuous time stochastic processes have been used most extensively in the economics and finance literature by Robert C. Merton. Merton has provided several expositions of the important properties of these processes and also of the continuous time stochastic dynamic programming problem (1969, 1971, 1973). This appendix provides a non-rigorous discussion of the same points.

1. The price dynamics in Section 1 of the text is assumed to be given by the stochastic differential equation:

\[
\frac{dP}{P} = \Pi \, dt + s \, dz .
\]

The stochastic process described by (1) is called an Itô process; the stochastic part, \( dz \), can be obtained as the limiting process of a suitably defined random walk in discrete time.\(^1\)

The drift of the process, \( \Pi \), is the expected rate of inflation per unit time. It is defined by:

\[
\Pi = \lim_{h \to 0} E_t \left[ \frac{P(t+h) - P(t)}{P(t)} \right] / h
\]

where \( E_t \) is the expectation operator, conditional on the value of \( P(t) \). Similarly, the variance of the process per unit time is defined by:

\[
s^2 = \lim_{h \to 0} E_t \left\{ \left( \frac{P(t+h) - P(t)}{P(t)} - \Pi h \right)^2 \right\} / h
\]

\(^1\) See Cox and Miller pp. 203-210.
2. A discrete time difference equation which satisfies (2) and (3) is:

\[
\frac{P(t+h) - P(t)}{P(t)} = \Pi h + s y(t) \sqrt{h}
\]

where \(y(t)\) is a normal random variable with zero mean and unit variance which is not temporally correlated. The limit as \(h \to 0\) of (4) then describes a Wiener process for the variable \(s y(t) \sqrt{h}\), and the equation can be written as:

\[
\frac{dP}{P} = \Pi dt + s y(t) \sqrt{dt}
\]

\[= \Pi dt + s dz \quad \text{where} \quad dz = y(t) \sqrt{dt}.
\]

3. The important characteristics of the process described by (5) should be evident from consideration of (4). First, the increments in \(dP/P\) are independent, no matter how short the time period. Second, as the time period shrinks, the variance of the stochastic component also shrinks, since the variance is just \(s^2 h\). In fact, no jumps in \(dP/P\) are possible in the limit, even though \(dP/P\) is stochastic. For the process for \(P\) described by (5), an Itô process, \(P\) is continuous with probability one but it is, with probability one, not differentiable.

4. In solving the portfolio problem, we deal with a utility function which is a function of several random variables whose behavior can be defined by Itô processes. To handle the problem, Itô's Lemma, sometimes called the Fundamental Theorem of the Stochastic Calculus, is needed.

Suppose we have a number of stochastic processes describable by:

\[
\frac{dP_i}{P_i} = \Pi_i dt + s_i dz_i \quad i = 1, \ldots, n
\]
and with \( \rho_{ij} \) as the correlation coefficient between the Wiener processes \( dz_i \) and \( dz_j \). Then let \( F(P_1, \ldots, P_n, t) \) be a function which is at least twice differentiable which, obviously, depends on the stochastic processes. Itô's Lemma gives the rule for finding the differential of \( Y = F(P_1, \ldots, P_n, t) \).

Specifically

\[
(7) \quad dY = \sum_{i=1}^{n} \frac{\partial F}{\partial P_i} dP_i + \frac{\partial F}{\partial t} dt + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 F}{\partial P_i \partial P_j} dP_idP_j
\]

is the stochastic differential of the function \( F(\ ) \). The product \( dP_idP_j \) is defined by:

\[
(8) \quad dz_idz_j = \rho_{ij} dt \quad i, j = 1, \ldots, n \\
\quad \quad dz_idt = 0 \quad i = 1, \ldots, n 
\]

5. A simple example may help in understanding Itô's Lemma. Consider two Itô Processes:

\[
(9) \quad \frac{dP}{P} = \Pi dt + s dz \\
(10) \quad \frac{dQ}{Q} = r_N dt 
\]

We are interested in the stochastic process describing the variable \( q = Q/P \).

Now

\[
(11) \quad \frac{1}{q} d(Q/P) = \frac{1}{q} \left[ \frac{1}{P} dQ - \frac{Q}{P^2} dP - \frac{dQdP}{P^2} + \frac{Q}{P^3} dP^2 \right] 
\]

where the first two terms in brackets are the first derivative terms in (7) and the second two terms in brackets are the second derivative terms in (7).
Simplifying (11) we obtain

\[
\frac{d\varphi}{Q} = \frac{dQ}{Q} - \frac{dP}{P} - \frac{dQ}{Q} \frac{dP}{P} + \left(\frac{dP}{P}\right)^2
\]

\[= (r - \Pi + s^2) dt - s dz.\]

Ito's Lemma is used in recognizing that the term in \(\frac{dQ}{Q} \frac{dP}{P}\) in (12) disappears, and also that in the \(\left(\frac{dP}{P}\right)^2\) term we obtain a term \(s^2 dz^2\) which is just \(s^2 dt\). Note that

\[
\frac{d\varphi}{Q} \neq \frac{dQ}{Q} - \frac{dP}{P}
\]

although one accustomed to working with logarithmic derivatives would be inclined to assume so.

6. Ito's Lemma is also of use in discussing the behavior of the price level implied by (1). The solution to (1) is:

\[
(13) \quad P(t) = P(o) e^{(\Pi - \frac{s^2}{2}) t + s \int_o^t dz}
\]

[In checking the solution using Ito's Lemma, write \(P = P(t, z)\) and do not neglect the term in \(dz^2\).]

Then, taking logarithms of both sides:

\[
(14) \quad \log \frac{P(t)}{P(o)} = (\Pi - \frac{s^2}{2}) t + s \int_o^t dz
\]

Thus \(P(t)\) is log-normally distributed. Using (14):

\[
(15) \quad E[\log \frac{P(t)}{P(o)}] = (\Pi - \frac{s^2}{2}) t
\]
and (16) \( \text{Var} \left[ \log \frac{P(t)}{P(o)} \right] = s^2 t \)

Consider also the distribution of the inverse of the price level \( V = \frac{1}{P} \).

\[
\frac{dV}{V} = \frac{1}{V} \left[ -\frac{1}{P^2} dP + \frac{2}{P^3} dP^2 \right]
\]

so that

(17) \( P \left( \frac{1}{P} \right) = (-\Pi + s^2)dt - s dz \).

It follows that, conditional on \( P(o) \),

(18) \( E[\log(P(o)/P(t)))] = (-\Pi + \frac{s^2}{2})t \)

(19) \( \text{Var}[\log(P(o)/P(t))] = s^2 t \)

7. Finally we discuss the continuous time stochastic dynamic programming problem. Typically the problem is to find:

(20) \[
\text{Max} \quad E_{o} \int_{o}^{\infty} U(C(t), t) dt
\]

where \( U(C(t), t) \) is strictly concave in the vector \( C(t) \). The budget constraint is:

\[\text{Dreyfus Chap. 7 provides a lucid derivation of the equation of optimality in the stochastic continuous time problem.}\]
\[ (21) \quad dW = \sum_{i=1}^{m} w_i (R_i - R) dt + \sum_{j=1}^{n} P_j C_j dt + \sum_{i=1}^{m} w_i s_i W dz_i \]

where \( w_i \) are the portfolio shares for risky assets, \( r \) the nominal interest rate, \( P_j \) the price of the \( j \)'th consumption good, and \( R_i \) and \( s_i^2 \) are the means and variances of the nominal rates of return on the \( i \)'th asset.

The discrete time analog would be:

\[ (22) \quad \text{Max } E_{t=0}^{\infty} \sum_{t=0}^{\infty} U(C(t), t) \]

with a budget equation similar to (21), with \( C \) and \( w_i \) the portfolio shares, as the choice variables. To solve the discrete time problem one writes

\[ (23) \quad J_t(W_t, P_t) = \text{Max } E_{t}^{\infty} \sum_{t} U(C(t), t) \]

\[ \{C, w_i\} \]

\[ = U(C^*(t), t) + E_t[J_{t+1}(W_{t+1}, P_{t+1})] \]

At the optimum,

\[ o = \phi(C^*(t), w^*(t); W_t, P_t, t) = U(C^*(t), t) + E_t[J_{t+1}(W_{t+1}, P_{t+1})] - J_t(W_t, P_t) \]

In continuous time one defines

\[ (24) \quad \phi(C, w; W, P, t) = U(C, t) + L(J) \]

where \( L(J) \) is to be thought of as

\[ \lim_{h \to 0} E_t \left\{ \frac{J_{t+h}(W_{t+h}, P_{t+h}) - J_t(W_t, P_t)}{h} \right\} \]
for a given set of controls, w and C.

The basic result is given by Merton, 1971, p. 381 as Theorem 2; it states that for the $P_t$ generated by Itô Processes $3/$ and for $U(\cdot)$ concave in C, there exists a set of optimal controls $4/$ $w^*$ and $C^*$ such that

$$
(25) \quad o = \phi(C^*,w^*; W,P,t) \geq \phi(C,w; W,P,t)
$$

Thus, to find a maximum, one simply maximizes:

$$
(26) \quad \phi(C,w; W,P,t) = U(C,t) + \lim_{h \to 0} E[(J_{t+h}(W_{t+h},P_{t+h},-J_{t}(W_t,P_t))/h]
$$

with respect to C and w. The expectation in (26) is evaluated by expanding $J_{t+h}(\cdot)$ in Taylor series around $(W_t,P_t)$. If we write:

$$
(27) \quad \frac{dW}{W} = \gamma \ dt + \epsilon \ dz
$$

$$
(28) \quad \frac{dP}{P} = \gamma \ dt + \sigma \ dz
$$

where P is a vector, and $\gamma$ and $\epsilon$ are derived from the budget constraint (21), we obtain

$$
(29) \quad \phi(C,w; W,P,t) = U(C,t) + J_t + J_w \gamma W + \frac{1}{2} J_{ww} \epsilon^2 W^2
$$

$$
+ J_p \gamma P + \frac{1}{2} J_{pp} \sigma^2 P^2 + J_{wp} \rho_{PW} \sigma \epsilon W.
$$

$3/$ This is overly strong.

$4/$ In the infinite horizon case a transversality condition has also to be satisfied.
In (29) $J_{p} y P$ stands for $\sum_{i=1}^{n} J_{p_{i}} y_{i} P_{i}$, $J_{pp}^{2} p^{2}$ stands for $\sum_{i=1}^{n} \sum_{j=1}^{n} J_{p_{i} p_{j}} \sigma_{ij} P_{i} P_{j}$, etc.

8. Once the $\phi(\ )$ function is maximized, one has the differential equation for $J(\dot{W},P,t)$ - which is now a deterministic equation. Solving this provides the solution to the dynamic programming problem.
Table of Notation.

A. Subscripts.

Sections I, III, IV:  \( i = 1,2,3 \)
1. Index Bond
2. Equity
3. Nominal Bond

Section II:  \( i = 1,\ldots,n+1 \)
1,\ldots,m. Goods, and bonds indexed on those goods.
m+1,\ldots,n. Equity assets.
n+1. Nominal Bond.

B. Alphabetical.

<table>
<thead>
<tr>
<th>Letter</th>
<th>Representing</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>Variance of real wage</td>
</tr>
<tr>
<td>( g )</td>
<td>Variance of stochastic process generating shifts in expected rate of inflation</td>
</tr>
<tr>
<td>( k() )</td>
<td>Propensity to consume out of real wealth</td>
</tr>
<tr>
<td>( dq )</td>
<td>Wiener process generating real wage</td>
</tr>
<tr>
<td>( r_i )</td>
<td>Expected (some with certainty) real rates of return</td>
</tr>
<tr>
<td>( r_I )</td>
<td>Real return on indexed bond defined in Section II</td>
</tr>
<tr>
<td>( s^2 = s^2_1 )</td>
<td>Variance of rate of inflation</td>
</tr>
<tr>
<td>( s_i^2 )</td>
<td>Variances of nominal returns on assets</td>
</tr>
<tr>
<td>( s_I^2 )</td>
<td>Variance of nominal return on indexed bond defined in Section II</td>
</tr>
<tr>
<td>( V_{ij} )</td>
<td>Elements of inverse of variance - covariance matrix (( \Omega ))</td>
</tr>
<tr>
<td>( v_i )</td>
<td>Portfolio shares</td>
</tr>
<tr>
<td>( [v] )</td>
<td>([v_1, \ldots, v_n]^t)</td>
</tr>
<tr>
<td>Letter</td>
<td>Representing</td>
</tr>
<tr>
<td>---------</td>
<td>------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>$dx_i$</td>
<td>Wiener processes generating real assets returns</td>
</tr>
<tr>
<td>$dy$</td>
<td>Wiener process generating shifts in expected rate of inflation</td>
</tr>
<tr>
<td>$dz = dz_1$</td>
<td>Wiener process generating inflation rate</td>
</tr>
<tr>
<td>$dz_1$</td>
<td>Wiener processes generating nominal assets returns</td>
</tr>
<tr>
<td>$C$</td>
<td>Rate of consumption</td>
</tr>
<tr>
<td>$C_i$</td>
<td>Rate of consumption of good $i$</td>
</tr>
<tr>
<td>$J(\cdot)$</td>
<td>Derived utility function</td>
</tr>
<tr>
<td>$J_2$</td>
<td>Derived utility function when asset menu is equity and nominal bonds</td>
</tr>
<tr>
<td>$J_3$</td>
<td>Derived utility function given full asset menu of Section I</td>
</tr>
<tr>
<td>$P$</td>
<td>Price level</td>
</tr>
<tr>
<td>$P_1$</td>
<td>Prices of goods</td>
</tr>
<tr>
<td>$R_1$</td>
<td>Expected nominal returns on assets</td>
</tr>
<tr>
<td>$R_3$</td>
<td>Nominal interest rate on nominal bonds (Section I)</td>
</tr>
<tr>
<td>$R_{n+1} = R'$</td>
<td>Nominal interest rate on nominal bond (Section II)</td>
</tr>
<tr>
<td>$[R-R']$</td>
<td>$[R_1-R_{n+1}, \ldots, R_n-R_{n+1}]'$</td>
</tr>
<tr>
<td>$R_1'$</td>
<td>Expected nominal return on indexed bond of Section II</td>
</tr>
<tr>
<td>$U(\cdot)$</td>
<td>Instantaneous utility function</td>
</tr>
<tr>
<td>$V$</td>
<td>Real wealth</td>
</tr>
<tr>
<td>$W$</td>
<td>Nominal wealth</td>
</tr>
<tr>
<td>$Y$</td>
<td>Real wage</td>
</tr>
<tr>
<td>$\beta$</td>
<td>See equation (67)</td>
</tr>
<tr>
<td>$\gamma(\cdot)$</td>
<td>Capitalization factor for real wage</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Rate of time discount</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Correlation coefficient of real wage with rate of inflation</td>
</tr>
<tr>
<td>$\eta_2$</td>
<td>Correlation coefficient of real wage with equity return</td>
</tr>
<tr>
<td>$\eta_3 (=-\eta)$</td>
<td>Correlation coefficient of real wage with real return on nominal bond</td>
</tr>
<tr>
<td>Letter</td>
<td>Representing</td>
</tr>
<tr>
<td>--------</td>
<td>--------------</td>
</tr>
<tr>
<td>( \theta )</td>
<td>See Equation (67)</td>
</tr>
<tr>
<td>( \lambda(=\rho_R) )</td>
<td>Coefficient of correlation of real returns on equity and nominal bonds</td>
</tr>
<tr>
<td>( 1-u )</td>
<td>Index of relative risk aversion</td>
</tr>
<tr>
<td>( \mu_1, \mu_2 )</td>
<td>See equation (44)</td>
</tr>
<tr>
<td>( \nu )</td>
<td>Expected rate of increase of real wage</td>
</tr>
<tr>
<td>( \Pi )</td>
<td>Expected rate of inflation</td>
</tr>
<tr>
<td>( \Pi_i )</td>
<td>Expected rate of increase of price of good ( i )</td>
</tr>
<tr>
<td>( \rho )</td>
<td>Correlation coefficient of nominal returns on equity and index bonds</td>
</tr>
<tr>
<td>( \rho_R )</td>
<td>Correlation coefficient of real returns on equity and the rate of inflation</td>
</tr>
<tr>
<td>( \rho_{ij} )</td>
<td>Correlation coefficient of nominal returns on assets ( i ) and ( j )</td>
</tr>
<tr>
<td>( \sigma_2^2 )</td>
<td>Variance of real return on equity</td>
</tr>
<tr>
<td>( \sigma_3^2 = \sigma_1^2 )</td>
<td>Variance of real return on nominal bonds</td>
</tr>
<tr>
<td>( \Omega )</td>
<td>Variance-covariance matrix of nominal returns on assets</td>
</tr>
</tbody>
</table>