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Working Paper 01-40
October 2001

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Efficiency Pricing, Tenancy Rent Control and Monopolistic Landlords

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Abstract

We consider a model of ‘tenancy rent control’ where landlords are not allowed to raise the rent on sitting tenants nor to evict them, though they are free to set the nominal rent when taking on a new tenant. If there is any inflation in the economy, landlords prefer to take short-staying tenants. Assuming that there is no way for landlords to tell a tenant’s type, an adverse selection problem arises. If in this context, landlords have monopoly power—which, as we argue, is indeed pervasive—then the housing market equilibria can exhibit some unexpected properties. Most strikingly, landlords may prefer not to raise the rent even when there is excess demand for housing. Such rents are labeled “efficiency rents” in this paper and their existence shows that tenancy rent control can give rise to equilibria which look as if there were traditional rent control in which the rent of each unit has a flat ceiling. In other words, tenancy rent control may not achieve the flexibility, which it was expected to impart, to the system of traditional rent control.

(JEL classification numbers: D40, K10, L12, L51, R31)

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Efficiency Pricing, Tenancy Rent Control and Monopolistic Landlords

I. Introduction

Standard, or first generation, rent control places a ceiling on the rents that a landlord can charge. Hence, under standard rent control, excess demand for housing is common.

Arguably, more common than standard rent control is ‘tenancy rent control,’ which allows a landlord to set the rent freely when leasing to a new tenant (subject to, of course, the tenant’s right not to accept), but thereafter the landlord is not allowed to raise the rent, nor can he evict the tenant.¹ It will be shown in this paper that tenancy rent control can, through the workings of the market, result in an outcome which looks as if there is standard rent control. That is, in equilibrium it may be in the landlord’s interest to keep the rent for new tenants so low that there is excess demand for housing at that rent. The landlord voluntarily behaves as if there was a legal ceiling on the rental rate.

This surprising result has analogues in the theory of efficiency wage (Leibenstein, 1957; Shapiro and Stiglitz, 1984) or the theory of efficiency interest rate (Stiglitz and Weiss, 1981).

It should be clarified here that our model is one where there is tenancy rent control and landlords have monopolistic power. In fact what we consider in this paper is the case of a monopolist landlord – which is the other polar extreme to the case modeled in Basu and Emerson (2000). We do not claim that this case is in any way closer to the reality of housing markets than the perfect competition assumption. The reality is surely

¹ In reality, landlords are often only limited in their ability to raise rents on, and evict, sitting tenants. For the present analysis it is only necessary that the allowable rental increases are insufficient to keep up with inflation, which is generally the case (see Basu and Emerson (2000) for a discussion).
somewhere in between the two models. We do believe that it is essential to understand both polar cases so that we can better comprehend the reality of modern rent control systems. The standard result of monopoly (with fixed supply) where price is raised above the market clearing level and some of the product (housing, in this case) remains unsold (vacant) arises in our model. Of course a monopolist with a fixed supply may not sell all of its good, this is also standard and arises in our model as well. But, what is surprising is that under certain parametric conditions, we can have the opposite case where the landlord sets the rent so low as to give rise to excess demand for housing.

The assumption of perfectly competitive or monopolistically competitive housing suppliers is a common one in the theoretical literature on rent control (see Arnott, 1995, for a thoughtful review of the literature). But there is a fair amount of empirical evidence that suggests that many rental housing markets are far from competitive. Cronin (1983), for example, notes that in the Washington, D.C. suburbs of Virginia, on average 70% of all units in each submarket are controlled by one owner and that the average number of rental housing firms in each submarket is slightly over four. Mollenkopf and Pynoos (1973) noted that in Cambridge, Massachusetts, 6% of the city's households controlled 70% of the rental housing units. In addition, they estimated that 90% of the apartment owners in the city belong to an association of property owners 700 strong, and that within that organization 20 owners account for 40% of the rental housing stock. Hence even when there are many landlords there is scope for monopoly behavior through collusion. Appelbaum and Glasser (1982; as cited in Gilderbloom, 1989) found that, in Isla Vista, California, 27 owners controlled 50% of the rental housing stock. Close by in Santa Barbara, Linson (1978; as cited in Gilderbloom, 1989) reported that over 50% of the
rental housing was owned by only 60 owners and that seven of them could account for 20% of the rental stock. Finally, Gilderbloom and Keating (1982; as cited in Gilderbloom, 1989) found that in Orange, New Jersey, just ten owners control close to one third of the rental housing and that in Thousand Oaks, California, just one owner controls over 30% of the rentals. In an empirical study of supply side concentration of the rental housing market in Boston, Cherry and Ford (1975) find that housing prices are significantly determined by concentration of within market segments.

Thus, there are many reasons to believe that housing markets are less than perfectly competitive; the concentration of rental property in the hands of only a handful of major owners and the collusive opportunities from the presence of landlord’s associations are two. There are theoretical bases for believing in monopoly power as well, for example Arnott (1989) hypothesizes that the indivisibility and heterogeneity of housing markets leads to monopoly power on the part of landlords.

There is also reason to believe that landlords do not always respond to housing shortages by increasing initial rents and the supply of apartments. For instance in 1998 the Ontario provincial government relaxed Toronto’s rent control laws giving landlords the right to raise rents to market levels whenever a tenant moved out. However, landlords did not respond with an expected building boom, fewer than 600 new units were built over the subsequent two years while Toronto’s population increased by more than 100,000. Vacancy rates in Toronto remain under 1%.²

Again, this does not mean that a model that assumes landlords are monopolists is any closer to reality than one that assumes that they are perfect competitors. What we are

²Barry Brown, “Apartment Shortage has Toronto Renters Searching for Space,” The Buffalo News, October 1, 2000, p. 4A.
attempting to do in the present paper is to examine the other extreme that has been
neglected in the literature and point out the aspects that are particular to the monopoly
case so that we may better understand the true nature of rent controlled housing markets.

II. The Algebra of Rent

In this paper we will consider the effects of rent control when the supply of rent
controlled housing is limited. New York, inner-city Mumbai and Delhi are examples of
this. Rent controls can however be of many kinds. It can take the form of a rent fixed by
a rent control authority or government (see Olsen, 1998, for a discussion of the different
forms of rent control) or of a law that gives landlords some or full freedom to adjust rents
when leasing out property to new tenants but then requiring the rent to be held constant
(or adjusted upward only within limits) as long as a tenant remains the lessee (and with
the landlord having no right to arbitrarily evict a sitting tenant). This latter form of rent
control, called ‘tenancy rent control,’ (see, e.g., Basu and Emerson, 2000; Nagy, 1997;
Arnott, 1995; Börsch-Supan, 1986) is quite pervasive and is the subject matter of this
paper.

Given tenancy rent control, the presence of even a small positive inflation gives
rise to an adverse selection problem. Landlords now prefer short-staying tenants to long-
staying tenants (as long-stayers impose greater costs on landlords due to the erosion of
real rents during a single tenancy) but they have no way of telling types apart.3 Long-
staying tenants know their type but have no interest in revealing this information to
prospective landlords. Curiously, the relation between rent control and inflation remains

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3 This, of course, ignores the possibility that landlords offer tenancy discounts to keep “good” tenants - those that impose relatively low costs on the landlord. In our model tenants are homogenous except for length of stay. See Hubert (1995) for an excellent analysis of markets with more and less costly tenants.
a neglected subject. We tried to develop the building blocks of a model for analyzing this in Basu and Emerson (2000).

In the present paper, we develop some of the basic theory in a continuous-time model and build into our model some elements of reality – to wit, limited supply and monopolistic power on the part of landlords – which has not been modeled thus far.4

Each (potential) tenant has an exogenously given duration of tenure \( t > 0 \). When we say that a tenant is of “type \( t \)” we mean that the tenant will move from an apartment after \( t \) periods. There is a continuum of tenants and their density function on the tenure duration, \( t \), is given by \( f(t) \), with \( F(t) \) being the corresponding distribution function. All agents are supposed to have the same discount rate \( \delta \in [0,1) \).

We denote the total number of tenants in the rental market by \( N \). Hence:

\[
F(\infty) = \int_{t=0}^{\infty} f(t) \, dt = N. 
\]

Suppose a landlord leases out to tenants of only type \( t \) (that is, gets a type \( t \) tenant, after every \( t \) periods), and each time a new tenant comes he fixes the rent so that its real value is $1. Thereafter the nominal value of the rent remains fixed so long as the tenant does not leave. Let the inflation rate be such that the value of each dollar erodes in each period at the rate of \( 1 - \beta \), where \( \beta \in (0,1) \). Under these circumstances the present value of the landlord’s real income is denoted by \( v(t) \). Clearly then,

\[
v(t) = \int_0^t e^{-\beta \delta x} \, dx + e^{-\delta t} \int_0^t e^{-(\beta+\delta)x} \, dx + e^{-2\delta t} \int_0^t e^{-(\beta+2\delta)x} \, dx + \Lambda
\]

---

4 Some monopoly elements were considered in a model of tenancy by Basu (1989), but the focus of that paper was entirely on innovations and the context was that of a backward agrarian economy.
The process of adverse selection ensures that, for each rental rate, only tenants of a certain type and above will seek housing in the rent controlled market. This is due to the fact that short-staying tenants no longer find it worthwhile to rent in this market as they do not see as much benefit for the erosion of real rents than do long-stayers. For a technical description of this process, see Step 1 of the proof in Appendix A, which demonstrates this result mathematically.

Hence, the central mathematical character in such an analysis is \( \hat{v}(t) \) - the expected present value of rents (in real terms) earned by a landlord who manages to rent out his house to a tenant selected randomly from a tenant pool with tenure time \( x \geq t \), at a rent which is equal to 1 real dollar to start with and thereafter kept fixed nominally (so it erodes each period by \( 1 - \beta \)), and each time a tenant leaves the landlord repeats the above procedure. \( \hat{v}(t) \) is given by the following expression:

\[
\hat{v}(t) = \int_0^1 f(x) \left[ \int_0^x e^{-(\beta+\delta)k} dk + e^{-\delta x} \hat{v}(t) \right] dx. \tag{2}
\]

To understand this, observe that \( f(x)/(N-F(t)) \) is the probability of picking a type \( x \) tenant, conditional on tenants of type \( t \) and above being available. The expression in the square bracket is the present value of rents earned when the first tenant is of type \( x \).

Now we are ready to state and prove the one technical result on which we will build our economic analysis.
Proposition 1: If $t^* > t'$, then $\hat{v}(t^*) < \hat{v}(t')$.

Proof: See Appendix A.

With this technical result in the background, it is now easy to describe a full model of rent control. When tenants make the decision whether to lease a rent-controlled apartment the alternative they have to keep in mind is for them to find housing in a non-rent controlled area or to buy a house. Let us assume that this alternative housing arrangement costs C dollars (in present-value terms). For simplicity, we assume that C is independent of the tenant’s ‘type.’ This seems reasonable as well. In buying a house the cost will clearly be independent of whether the person is a long-stayer or a short-stayer. Likewise in renting an apartment in a non-rent controlled area, the tenant’s type is unlikely to matter because the rent can be inflation indexed or be made contingent on the length of the tenant’s stay.

Now suppose that the rent (per period) in the rent controlled housing is $R$. The life-time rental cost to a tenant of type $t$ is clearly given by $Rv(t)$. Recall that $v(t)$ is the present-value of life-time payment made by a tenant of type $t$ if the real rent at the start of tenure is set each time at 1.

Consider now a monopoly landlord, who sets the real (starting) rent equal to R. Clearly only those types of tenant, $t$, will accept this for whom $Rv(t) \leq C$. Since from Step 1 of Proposition 1 we know $v'(t) < 0$, for all $t$, it follows that all tenants types $t$ such that $t \geq v^{-1}\left(\frac{C}{R}\right)$ will accept the offer. It follows from the definition of $\hat{v}(\cdot)$ that the landlord’s expected present-value of rental earned from each apartment that is leased out is given by:
From Step 1 of Proposition 1 we know that as R rises, $v^{-1}\left(\frac{C}{R}\right)$ rises. Hence, from Proposition 1 we know that as R rises $\hat{v}\left(v^{-1}\left(\frac{C}{R}\right)\right)$ falls. It is now transparent that as R rises, V(R) may rise or fall.

Figure 1 represents a possible picture of V(R).

Define $\tilde{\tau}$ to be the supremum of the set $\{t|f(t) > 0\}$. In other words, and more informally, $\tilde{\tau}$ is the upper support of $f(t)$. So, $\tilde{\tau}$ is such that there are no tenants of type $t > \tilde{\tau}$, and for all $t > 0$, there exists tenants of type $t \in [\tilde{\tau} - t, \tilde{\tau}]$.

Now define $\tilde{R}$ such that $v^{-1}\left(\frac{C}{\tilde{R}}\right) = \tilde{\tau}$. Then if rent goes above $\tilde{R}$, there are no further takers among the tenants. Hence V(R) is not defined for $R > \tilde{R}$. At $\tilde{R}$, the
only takers are of type \( t \). Hence \( V(\tilde{R}) = \tilde{R}v(\tilde{t}) = \tilde{R}_v(\tilde{t}) = \tilde{R}C/\tilde{R} = C \). It is easy to see that for all \( R < \tilde{R} \), \( V(R) < V(\tilde{R}) \). This explains the shape of \( V(R) \) in Figure 1.

It is also evident that \( V(R) \) can fall over some stretches. This is especially transparent if tenant types are finite. Then over some increases in \( R \), large numbers of short-stayers can decline the rental offer, leaving the pool of tenants suddenly worse from the landlord’s point of view. This is the classic adverse selection problem (Akerlof, 1970).

### III. Excess Supply, Excess Demand and Efficiency Rent

The results are the outcome of the landlord’s optimization problem when confronted with an earnings curve, \( V(R) \). The case of many landlords who drive profits down to zero was analyzed in Basu and Emerson (2000). Here we take on the other polar end: the case of limited supply and monopoly. Rent control applied to a fixed stock of housing, such as in New York, and the evidence supporting the contention that rental housing markets are not competitive, make it worthwhile investigating this polar case.

To begin this analysis, let us derive the demand for rent-controlled housing as a function of the (per-period) rent, \( R \). From section 2, we know that given \( R \), all tenants of type \( t \geq v^{-1}(C/R) \) will want to lease rent-controlled housing. Hence, the demand for housing, \( D \), is given by

\[
D(R) = \int_{v^{-1}(C/R)} f(t)dt = N - F\left(v^{-1}\left(\frac{C}{R}\right)\right).
\]
From Step 1 of Proposition 1, \( v^{-1}(C/R) \) rises as \( R \) rises. Hence, \( D(R) \) declines, as \( R \) rises. Such a demand curve is illustrated in the lower panel of Figure 2. The upper panel is a reproduction of Figure 1.

\[ \text{Figure 2} \]

Next, draw the supply curve in the lower panel. The landlord, it is assumed, owns \( S \) units of property. For simplicity, it is assumed that the opportunity cost of leasing out
property is zero. Hence the supply curve is perfectly inelastic through the point marked S. Let \( \hat{R} \) be the rent which equates demand and supply.

To locate the landlord's optimum rent, consider all rents less than \( \hat{R} \), and locate the rent (left of \( \hat{R} \)) which maximizes \( V(R) \). This, in Figure 2, is given by \( R^* \). Since \( V(R) \) is not necessarily monotonic, there is no reason why \( R^* \) will coincide with \( \hat{R} \).

Observe that, if the landlord was restricted to selecting a rent less than or equal to \( \hat{R} \), she would choose \( R^* \). This is because for all \( R \leq \hat{R} \), she manages to lease out the same number of apartments, to wit, S, and at \( R^* \) the per-apartment earnings are maximized. Hence, the total earnings are maximized at \( R^* \).

Next consider rents greater than or equal to \( \hat{R} \). As \( R \) is raised starting from \( \hat{R} \), the earnings of the landlord must eventually (weakly) rise (since \( V(\hat{R}) = C \geq V(R) \), for all \( R \)). However even if \( V(R) \) rises, the total earnings need not rise, since demand falls below S and so more and more apartments remain vacant as \( R \) is raised. Let \( R'' \) be the rent where total earnings are maximized (subject to \( R \geq \hat{R} \)).

Let \( E \) be the same height as A, and \( R' \) the projection of \( E \) on the horizontal axis. The landlord's chosen rent will clearly be either \( R^* \) or \( R'' \). If \( R'' \in [\hat{R}, R'] \), clearly, her earnings are greater at \( R^* \), since at such an \( R'' \), per apartment earning is smaller and fewer apartments are taken. Even if \( R'' > R' \), total earnings may be smaller at \( R'' \) since at such a rent, the landlord will be unable to find tenants for all her apartments.

If the optimum turns out to be at \( R'' \), then this is a fairly typical monopoly equilibrium. The monopolist holds back supply in order to push up the price and her earnings.
The interesting case occurs when \( R^* \) turns out to be the optimum. Here demand for housing exceeds supply (see lower panel of Figure 2). Nevertheless the landlord prefers not to raise the rent. This is because a higher rent worsens the ‘quality’ (from the landlord’s point of view) of the tenant. This is rather like in models of efficiency wage (e.g. Stiglitz, 1974; Mirrlees, 1975) or efficiency interest rates (e.g. Stiglitz and Weiss, 1981). We shall therefore call \( R^* \) the ‘efficiency rent.’

Usually, we would expect this kind of a rent to prevail on the market if rent control took the form of an exogenous ceiling on rent. In such a case demand exceeding supply is compatible with equilibrium. What our model illustrates is that even if there is no ceiling on rents, tenancy rent control can result in behavior such that the market equilibrium mimics a rent ceiling.

It is interesting to note that the \( R^* \) equilibrium is the more likely outcome as \( S \) increases. This is because as \( S \) increases, at \( R'' \) the landlord’s profit is unchanged, since at \( R'' \) the number of tenants is constrained by the demand (remember at this point there is excess supply of housing) and so an increase in supply does nothing to the landlord’s income. On the other hand, at \( R^* \), the landlord’s profit is given by \( V(R^*) \) multiplied by \( S \) (see Fig 2). So an equilibrium with excess demand for housing is more likely when a large portion of the rental stock is under rent control.

IV. Conclusion

Well meaning urban policy makers of the 70’s and 80’s, attempting to correct the glaring problems of old-style rent controls that placed ceilings on rents (problems that had been illustrated quite vividly by economists), turned to a form of rent control that was
more of a tenant’s protection legislation than a unit-by-unit rent restriction. This type of
‘tenancy rent control’ simply restricted landlord’s ability to raise rents on sitting tenants
and prohibited most side payments and arbitrary eviction. This was seen to be a more
flexible program and one that was less susceptible to the inefficiencies of the old rent
control laws.

What we showed in Basu and Emerson (2000) is that this new type of rent control
brings about different kinds of inefficiencies due to the adverse selection problem
brought about by the asymmetric nature of information in these markets. What the
present paper illustrates is; in the presence of monopolistic landlords, tenancy rent control
can cause landlords to operate in such a way that mimics the old style rent control. To
wit, they hold down price, even with excess demand, to attract a better ‘quality’ tenant
(i.e. one that will not stay too long). We call this ‘efficiency rent.’

Assuming rental housing markets to be monopolistic may be an abstraction from
reality, but with all of the evidence to suggest large amounts of concentration and
possible collusion (through landlord’s associations) in many rental housing markets, it is
certainly no more of an abstraction than models that assume perfect competition (as is the
norm in the literature). We believe that, through the study of both extremes, a clearer
and more comprehensive understanding of the reality of rent control will result.

Given the pervasiveness of ‘tenancy rent control,’ it is important to understand
fully the nature of the inefficiencies it creates. What the present work illustrates (as does
Basu and Emerson, 2000) is that certain types of tenants are helped by this policy while
other types are hurt. In addition, it illustrates a type of strategic behavior on the part of
monopolistic landlords that has not been previously explored in the literature.
Appendix A: Proof of Proposition 1

To prove Proposition 1, first note that \( \hat{v}(t) \) can be simplified using (1) and (2) to be written as:

\[
\hat{v}(t) = \frac{\int_{t}^{\infty} f(x)(1 - e^{-\delta x})v(x)dx}{N - F(t) - \int_{t}^{\infty} f(x)e^{-\delta x}dx}
\]

From (3) it is clear that \( \hat{v}(t) \) is the weighted average of the different values of \( v(x) \), as \( x \) ranges from \( t \) to \( \infty \). This is obvious from the fact that if \( v(x) \) is removed (i.e. set equal to 1, for all \( x \)) from the right-hand side of (3), then the right-hand term equals 1.

The proposition is now proved in three steps.

**Step 1:** We will show that \( v(t) \) rises as \( t \) falls.

In other words, \( v'(t) < 0 \), for all \( t \).

To prove this note that:

\[
v(t) = \left( \int_{0}^{\infty} e^{-\beta x} dx \right) + \left( \int_{0}^{\infty} e^{-\beta x} e^{-\delta x} dx \right) e^{-\delta t} + \left( \int_{0}^{\infty} e^{-\beta x} e^{-\delta x} dx \right) \left( e^{-\delta t} \right)^2 + A
\]

\[
= \frac{1 - e^{-\beta t}}{\beta + \delta (1 - e^{-\delta t})}
\]

Then:

\[
v'(t) = v(t) \left( \frac{\beta + \delta}{1 - e^{-\beta t}} - \frac{\delta e^{-\delta t}}{1 - e^{-\delta t}} \right)
\]

Note that \( v(t) \) is positive. We must now show that the term in parentheses is negative, rearranging and collecting terms under a common denominator reduces our problem to showing the numerator, or
\[(\beta + \delta)e^{\delta t} - \beta - \delta e^{(\beta+\delta)t} \equiv X(t), \text{ is negative.}\]

We need to prove this for all \( t \geq 0 \). To see this note that

\[ X'(t) = (\beta + \delta)e^{\delta t}[1 - e^{\text{rt}}]. \]

Hence \( X'(t) = 0 \) if \( t = 0 \), and \( X'(t) < 0 \) if \( t > 0 \).

It follows that if \( X(0) \leq 0 \), then \( X(t) < 0, \forall t > 0 \).

Finally, note that \( X(0) = (\beta + \delta)-\beta - \delta = 0 \).

This establishes Step 1.

**Step 2:** As \( t \) falls from \( t'' \) to \( t' \), the weight on each \( v(x) \), for \( x \geq t'' \), in (3) falls.

To prove this, suppose \( x \geq t'' \). The weight on \( v(x) \) in (3), denoted by \( w(x,t) \), is given by:

\[ w(x,t) \equiv \frac{f(x)(1-e^{\delta x})}{N - F(t) - \int_{t}^{\infty} f(x)e^{-\delta x} \, dx}. \]

Note that:

\[ w(x, t') < w(x, t'') \quad (4) \]

iff

\[ N - F(t') - \int_{t'}^{\infty} f(x)e^{-\delta x} \, dx > N - F(t'') - \int_{t'}^{\infty} f(x)e^{-\delta x} \, dx \]

or

\[ \int_{t'}^{\infty} f(x)e^{-\delta x} \, dx < \int_{t'}^{\infty} f(x) \, dx \quad (5) \]

(5) is obviously true. Hence (4) is true, which establishes Step 2.
Step 3: Since the weights on $v(x)$ in (3) always sum to 1, a decline in the weights on $v(x)$, for all $x \geq t''$ implies positive weights on $v(x)$, for all $x \in [t', t'']$. Hence we know from Step 2 that as $t$ falls from $t''$ to $t'$, the weights get transferred from values of $v(x)$, where $x \geq t''$ to values of $v(x)$ where $x \in [t', t'']$. From Step 1 we know $v(x) > v(y)$ for all $x, y$ such that $x \in [t', t'']$ and $y > t''$.

It follows that $\hat{v}(t'^{+}) < \hat{v}(t')$. 
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